Title: Curvature Flows in Complex Geometry

Date: May 06, 2012 04:45 PM

URL: http://pirsa.org/12040119

Abstract: In this talk, I will discuss my recent works with J. Streets on curvature ows on Hermitian manifolds and show how they can be used to study generalized K•ahler manifolds. I will also show how they are related to the renormalization group ow coupled with B- elds. Some open problems will be discussed. In the end, I will also discuss briey a new ow which preserves symplectic structures.

Curvature Flows in Complex Geometry

Gang Tian

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In this talk, I will discuss curvature flows in complex geometry:

- Kähler-Ricci flow, i.e., Ricci flow on Kähler manifolds;
- Pluri-closed flow for Hermitian metrics (Streets-Tian).

In this talk, I will discuss curvature flows in complex geometry:

- Kähler-Ricci flow, i.e., Ricci flow on Kähler manifolds;
- Pluri-closed flow for Hermitian metrics (Streets-Tian).

M : A compact Kähler manifold, e.g., a projective algebraic manifold.

g : A Kähler metric with the Kähler form $\omega = \omega_g$.

In local coordinates $z_1, \cdots, z_n$, the metric g is given by a Hermitian positive matrix-valued function $(g_{i\bar{j}})$. Then its Kahler form can be written as

$$\omega = \sqrt{-1} \sum_{i,j=1}^{n} g_{i\bar{j}} dz_i \wedge d\bar{z}_j$$

g being Kähler $\Leftrightarrow d\omega = 0$

$$\Rightarrow \quad g_{i\bar{j}} = \frac{\partial^2 \varphi}{\partial z_i \, \partial \bar{z}_j}$$

The Ricci flow was introduced by R. Hamilton in early 80's. It preserves the Kählerian condition, that is, if $g(t)$ is a solution of the Ricci flow and $g(0)$ is Kähler, so is every $g(t)$. Hence, we can consider the Kähler-Ricci flow:

$$\begin{cases} \dfrac{\partial}{\partial t}\omega_{g(t)} = -\mathrm{Ric}(g(t)) \\ \omega_{g(0)} = \text{a given Kähler form } \omega_0 \end{cases}$$

Sharp Local Existence Theorem (Tian-Zhang):

The Kähler-Ricci flow has a unique maximal solution $g(t)$ on $[0, T)$, where T is the maximum of t such that $[\omega_0] - t\, c_1(M)$ is a Kähler class.

This theorem was proved by H.D. Cao in 1986 in the case that M has definite first Chern class $c_1(M)$ and $c_1(M) = \lambda[\omega_0]$, and proved by H. Tsuji in early 90s in the case that $K_M = -c_1(M)$ is nef and $[\omega_0] \geq -c_1(M)$. Also, a similar result was independently given by Carcini-LaNave assuming certain technical estimates.

In general, $T < \infty$, for instance, if $c_1(M) = [\omega_0] > 0$, then $g(t)$ shrinks to a point as $t \mapsto 1$.

To construct a global solution with surgery, we need to analyze $g(t)$ when $t \mapsto T < \infty$?

In general, the solution $g(t)$ may collapse to a lower dimensional variety and we need to consider how to continue the Kähler-Ricci flow on such a variety.

Conjecture 1: As $t \to T$, $(M, g(t))$ converges to a length space (X, d_T) in the Gromov-Hausdorff topology satisfying:

(1) X is a variety and there is a fibration of M over X with generic fibers being a Fano manifold;

(2) d_T is given by a smooth Kähler metric on a Zariski open subset X_0 of X such that the convergence of $g(t)$ to d_T on X_0 is in a suitable smooth topology.

Tian-Zhang: $\omega_{g(t)}$ converges to a unique current ω_T with continuous local potentials. Furthermore,
If $([\omega_0] - T\,c_1(M))^n > 0$, then $g(t)$ converges to a unique smooth metric $g(T)$ outside a subvariety. In particular, if M has non-negative Kodaira dimension, then $([\omega_0] - T\,c_1(M))^n > 0$ holds if $T < \infty$.

If M is a projective manifold and ω_0 is a rational class, then T is rational and by a result of Kawamata, $[\omega_0] - Tc_1(M)$ is semi-ample. So there is a holomorphic map $\phi : M \mapsto X \subset CP^N$ such that $\phi^*[\omega_X] = [\omega_0] - Tc_1(M)$ for some ample class $[\omega_X]$ on X. Clearly, generic fibers are Fano manifolds.

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The following conjecture tells how to extend the Ricci flow across $T < \infty$:

Conjecture 2: There is a resolution $\pi_1 : M_1 \mapsto X$ satisfying:

1. M_1 is Q-factorial, that is, any Q-divisor of X is Q-Cartier;

2. $\pi_1^* \pi_*([\omega_0] - Tc_1(X)) - (t - T)c_1(M_1)$ is ample for $t - T > 0$ small;

3. M_1 may be singular, but only $\log$-terminal singularity.

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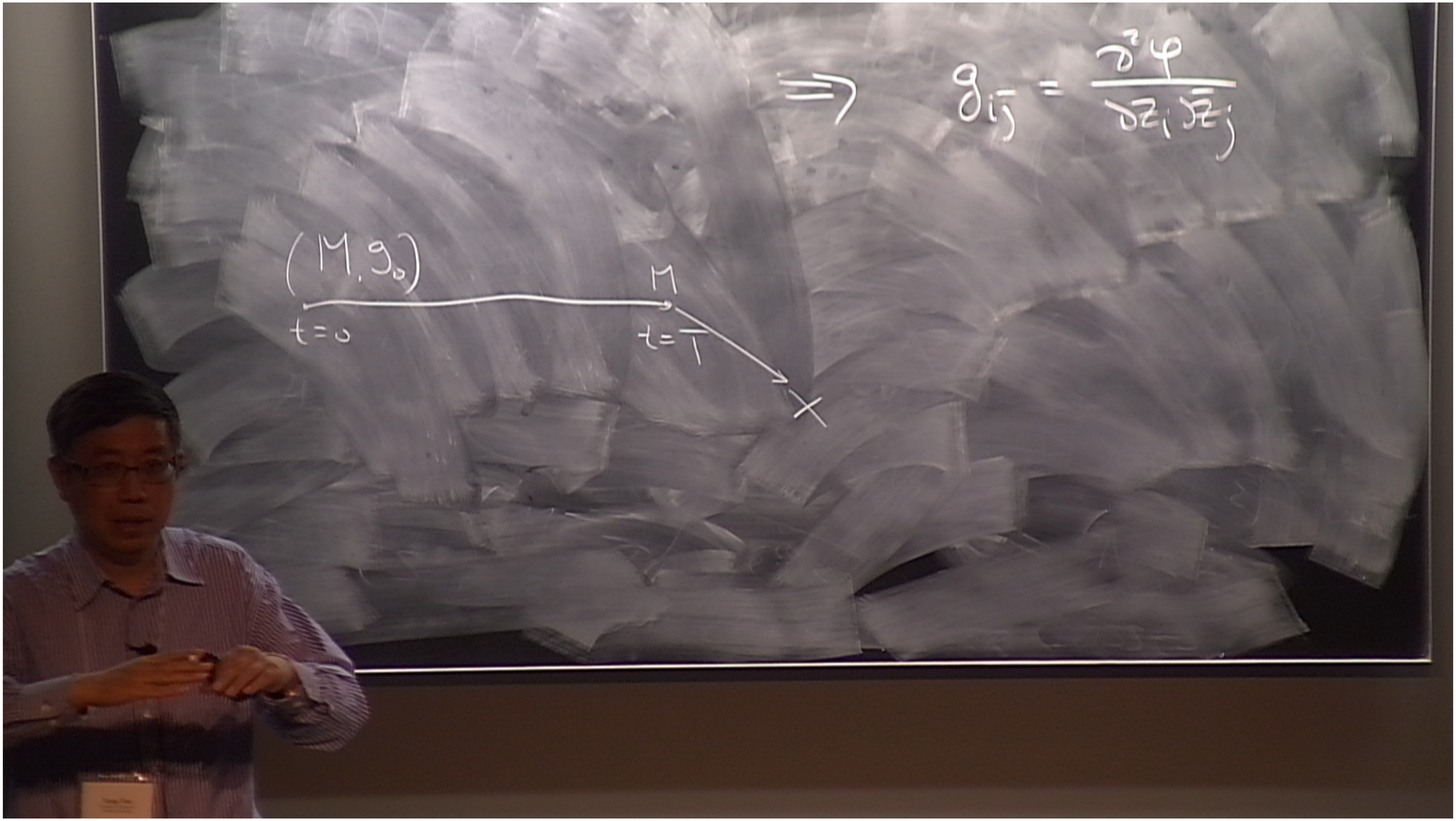
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(M, g_0)
M
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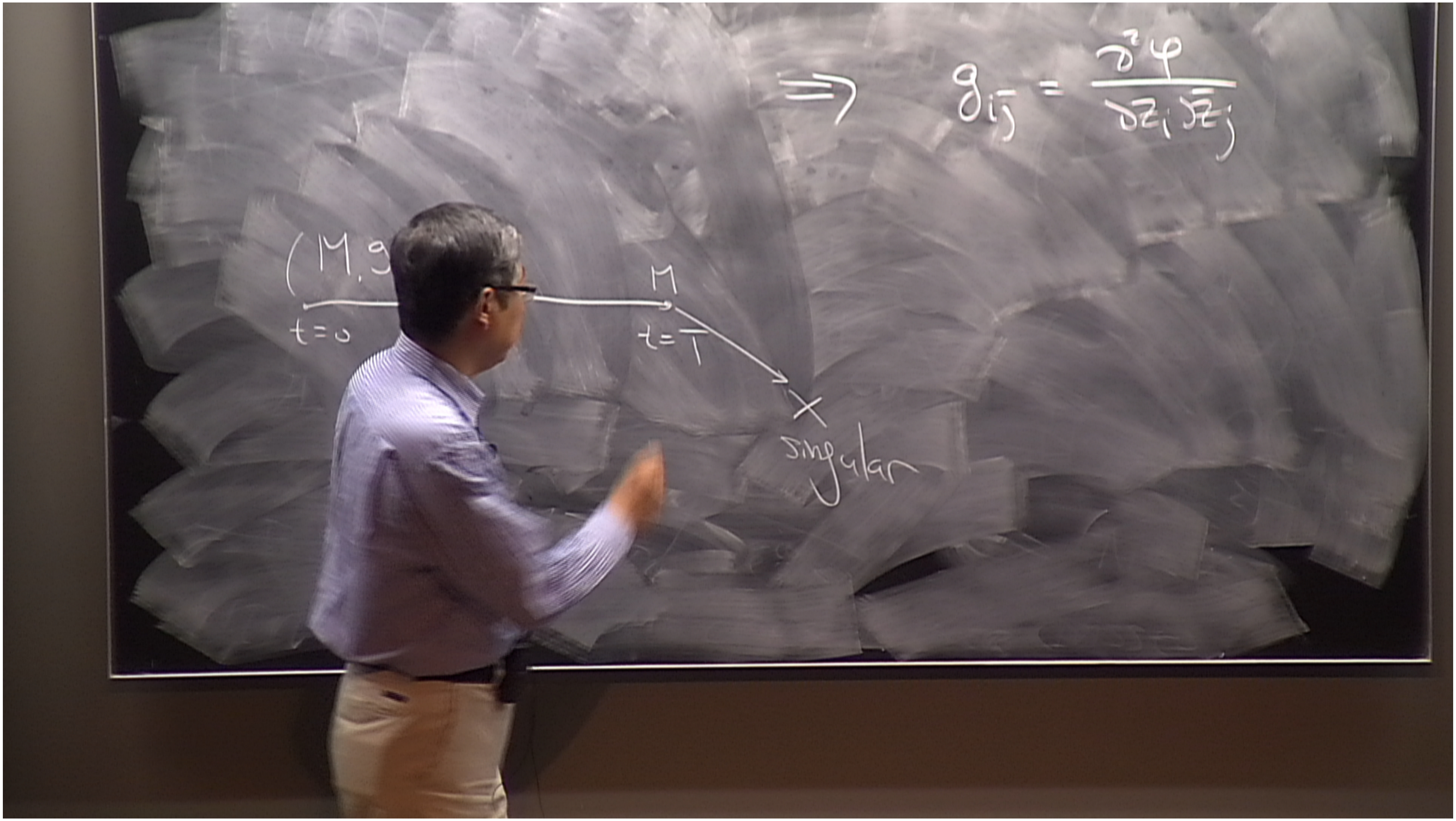
$t = T$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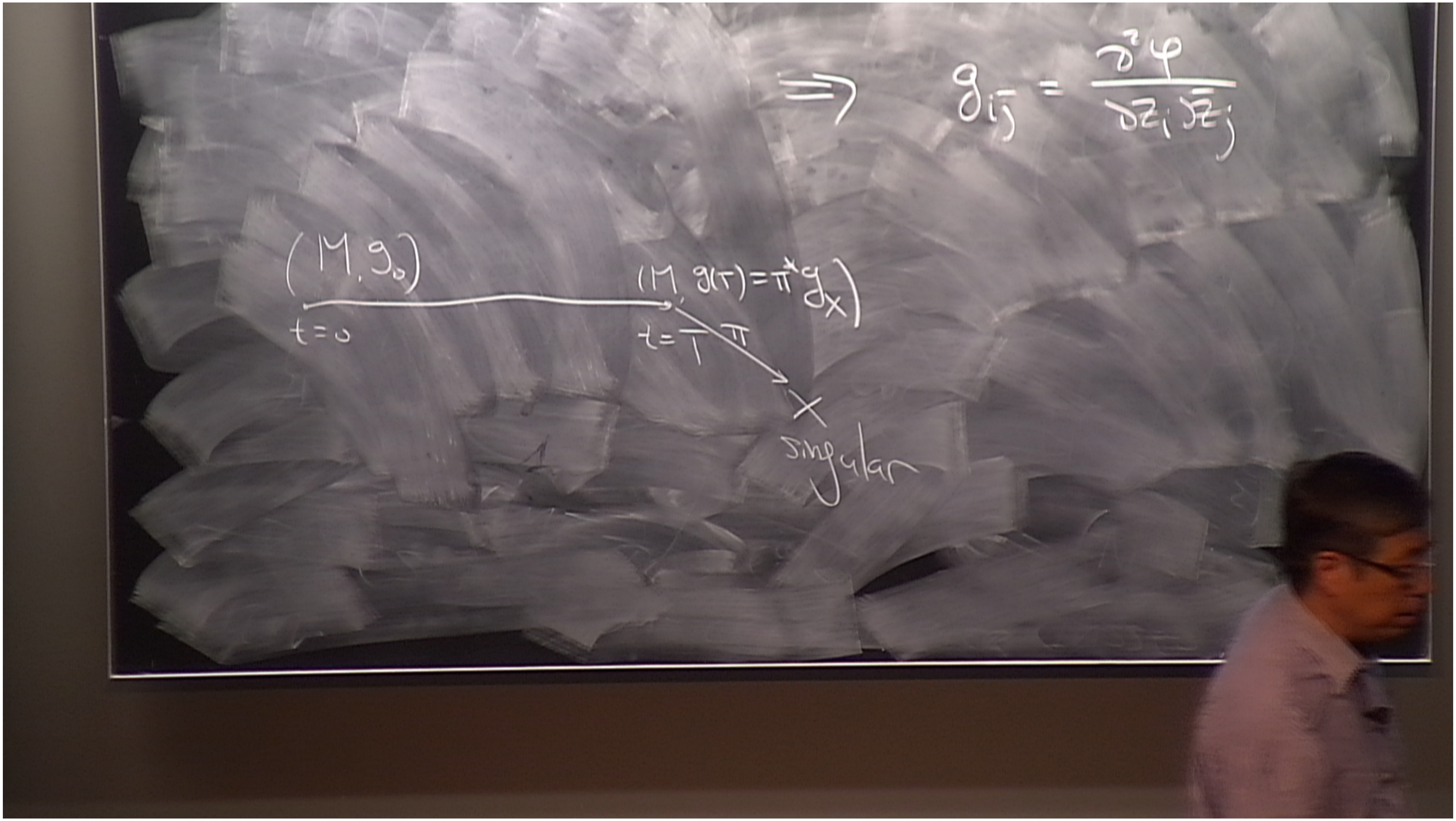
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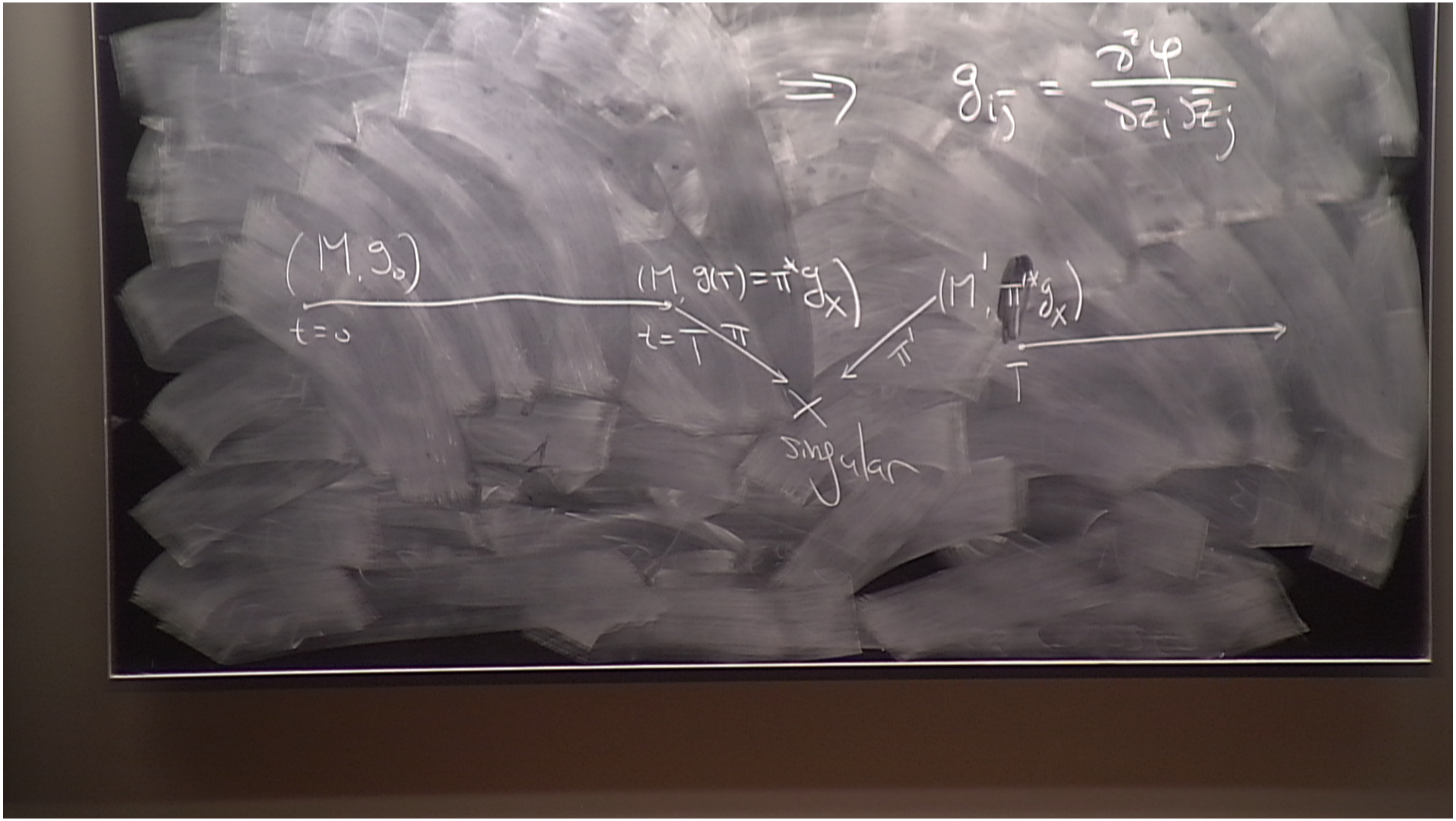
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Both Conjecture 1 and 2 have been verified for complex surfaces by purely analytic methods and projective varieties of general type by using known results in algebraic geometry in my works with J. Song et al.

Assuming the first two conjectures, we need to study how the Ricci flow extends across T. This was done by Song-Tian.

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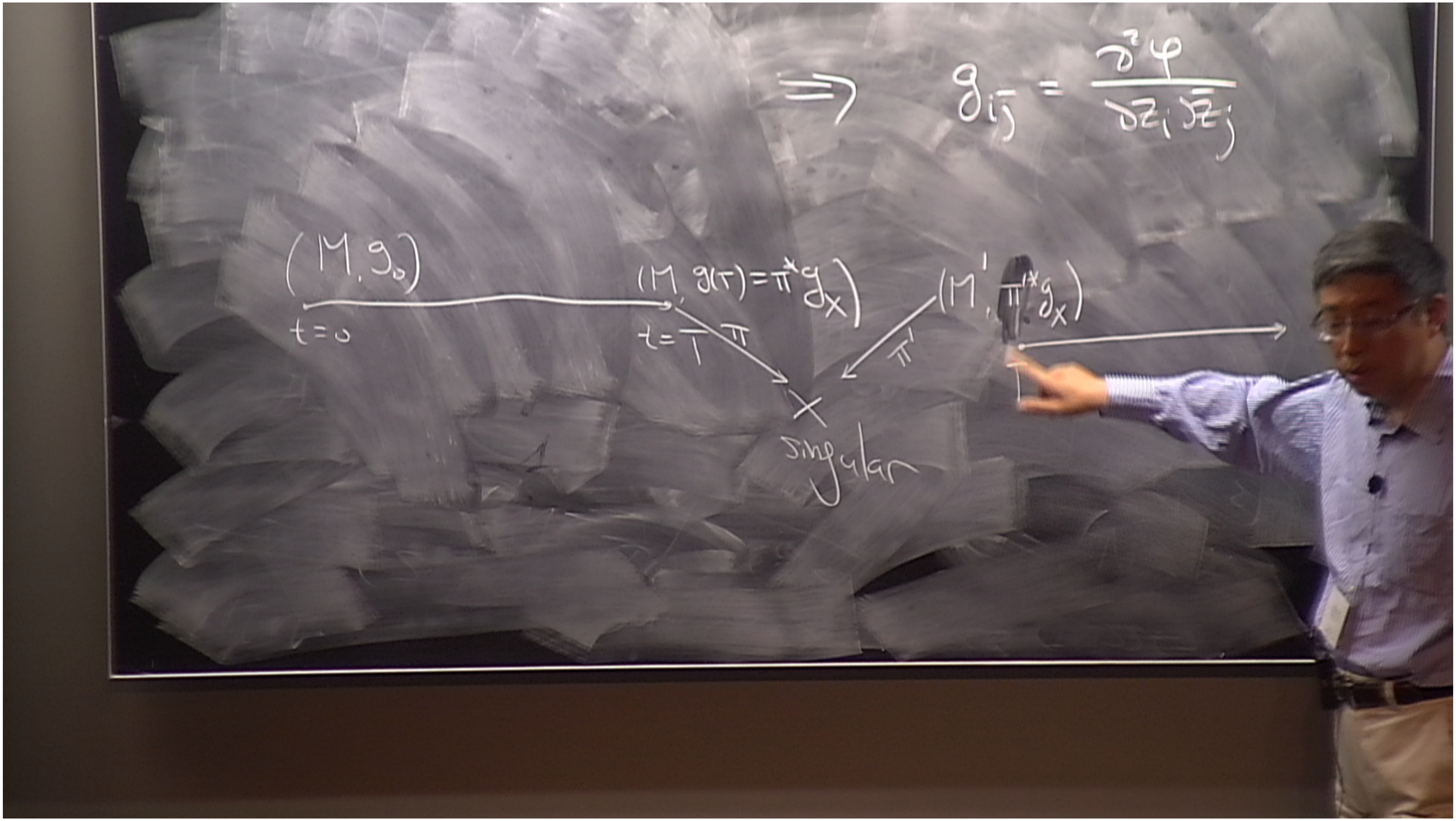
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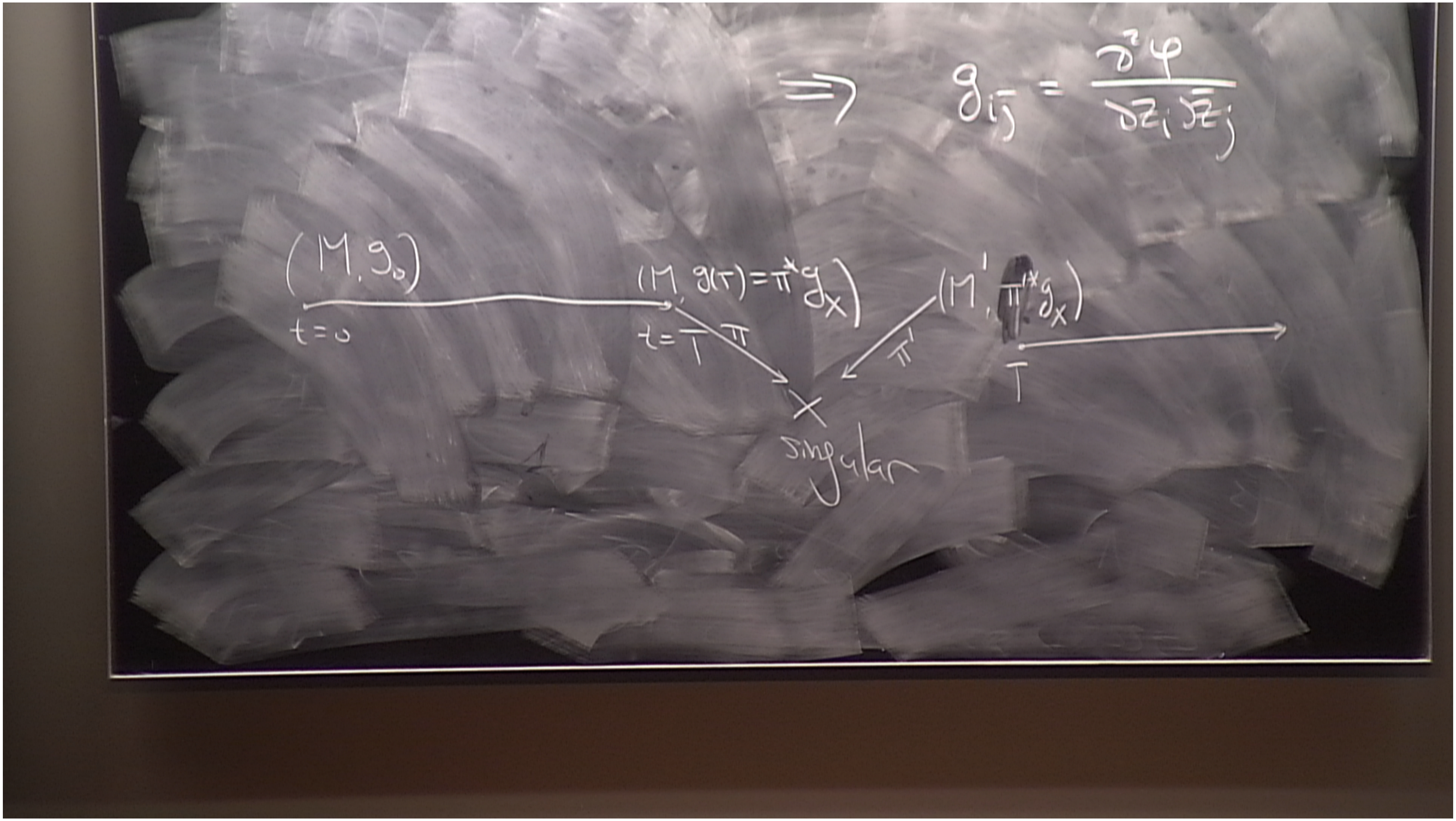
Assuming the above, we can continue the Kähler- Ricci flow on M_1 and get a solution with surgery, that is, we have $M_1, M_2, ..., M_{N-1}$ and a solution $g(t)$ of the Kähler-Ricci flow on $[0, t_N)\backslash\{t_i\}$ for $t_0 = 0 < t_1 < \cdots < t_N$ satisfying:

(1) On each interval $[t_{i-1}, t_i)$, $g(t)$ is a regular solution of the Kähler-Ricci flow on M_i ($i = 1, \cdots, N$);

(2) $(M_i, \lim_{t\to t_i+} g(t))$ is obtained from $(M_{i-1}, \lim_{t\to t_i-} g(t))$ as described above.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Conjecture 3: There is an $N < \infty$ such that M_N with $-c_1(M_N) \geq 0$ (possibly lower dimensional if collapsing occurs in previous surgeries) or M_{N-1} collapse to a point as t tends to t_N.

If the above $(M_{N-1}, g(t))$ collapses to a point as t tends to t_N, i.e., $g(t)$ on M becomes extinct at $t_N < \infty$, then we expect that M is a Fano-like manifold.

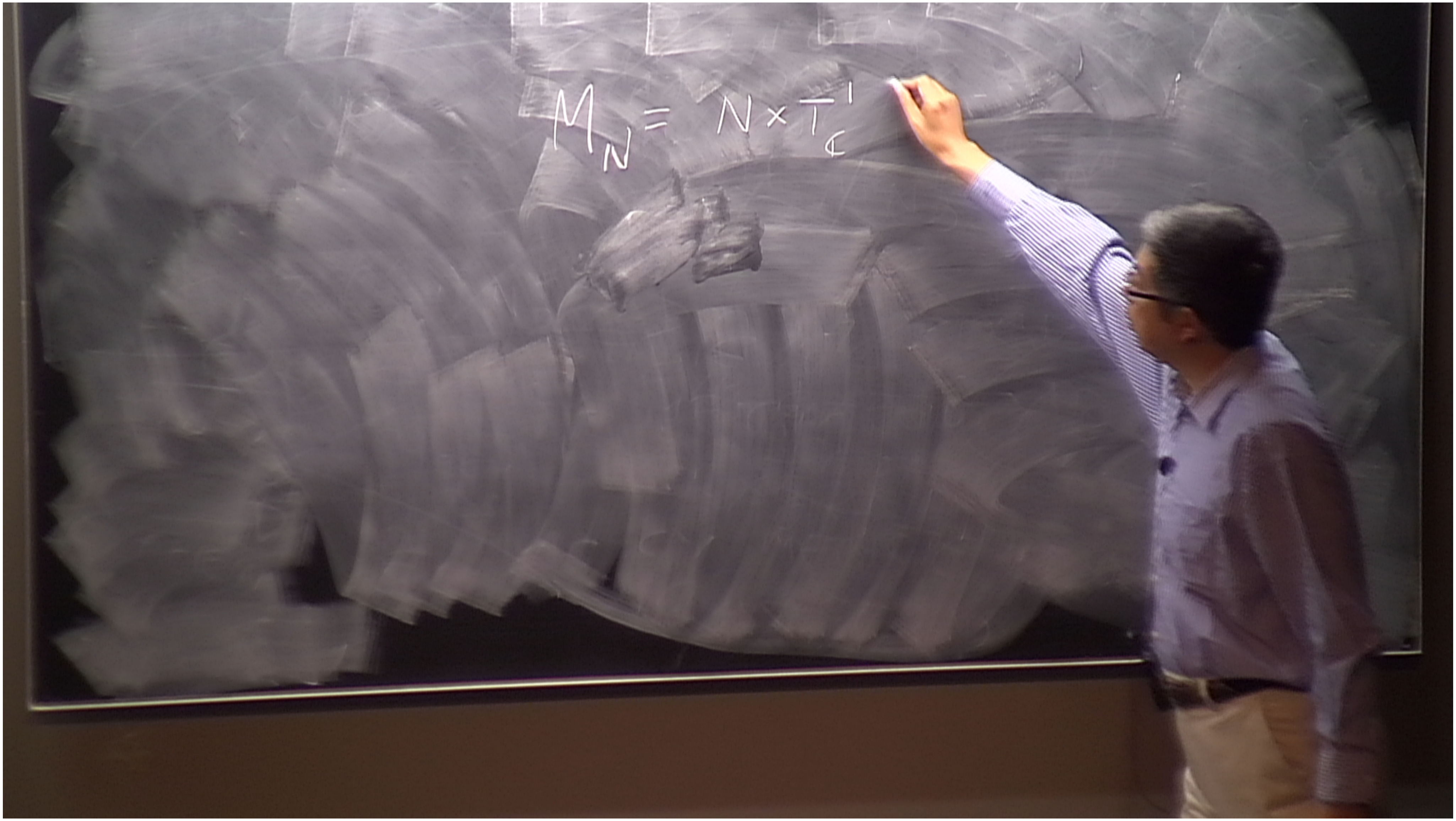
Indeed, if M is a Fano manifold, $g(t)$ becomes extinct at a finite time.

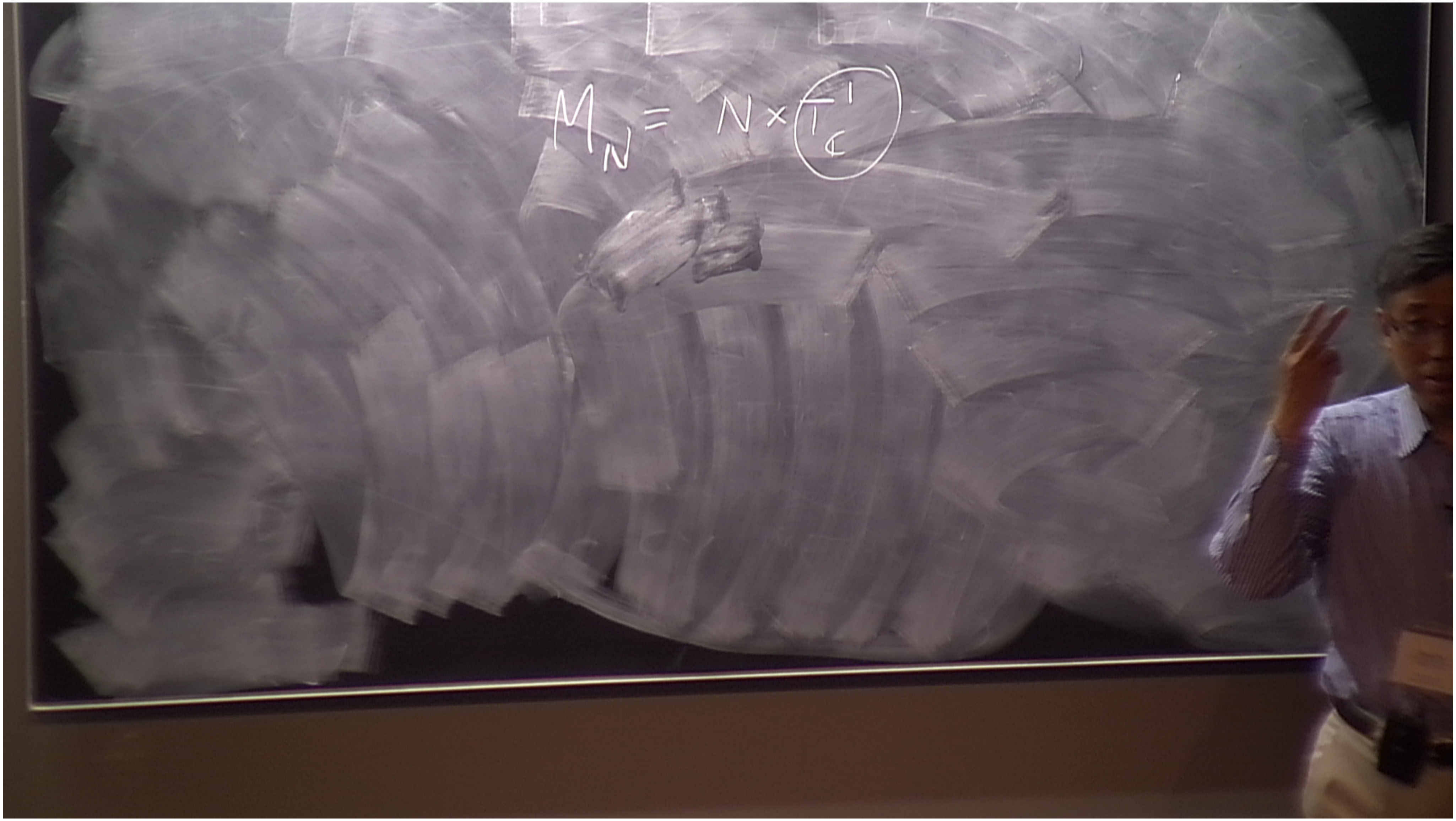
If the above M_N exists, then by Song-Tian, we get a global solution $g(t)$ for $t \geq t_N$ and consequently, a global solution with surgery for $t \geq 0$.

Next we need to study what is the limit of $(M_N, g(t))$ as $t \to \infty$.

We expect: $g(t)$ converges to the canonical metric on the canonical model of M after appropriate normalization.

This has been done in my joint works with J. Song. Of course, there are still difficult problems on how $g(t)$ converges to the canonical limit, e.g., in what topology? What behavior of $g(t)$ near singularity of the canonical limit?


$M_N = N \times T_c^1$



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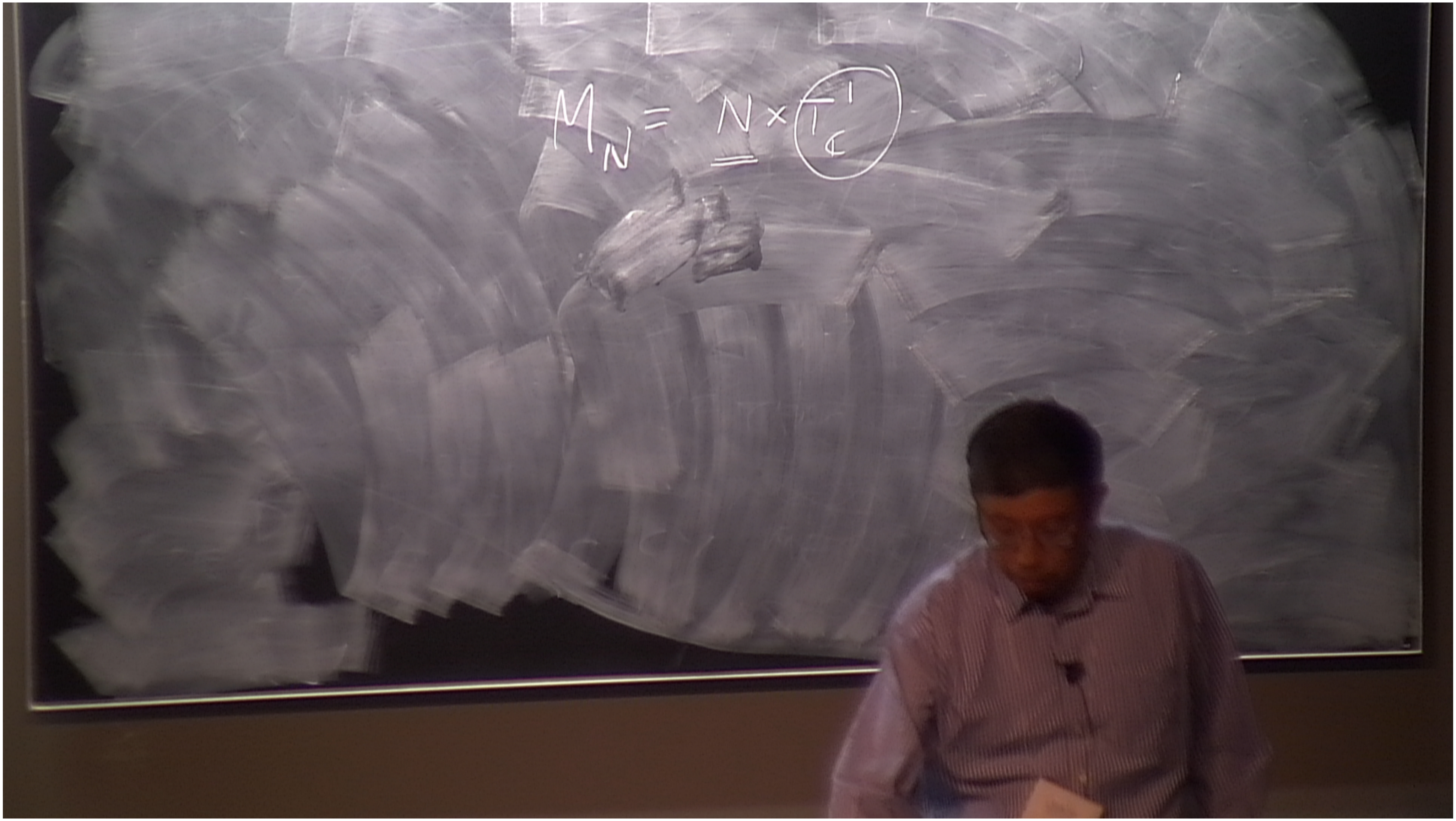
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Pluri-closed condition:

Let M be a complex manifold and g be a Hermitian metric with Kähler form ω. The metric g is pluri-closed if

$$\partial\bar{\partial}\omega = 0.$$

Gauduchon (1977): Let (M, g) be a Hermitian manifold. There exists a unique $u \in C^\infty(M)$ such that $\int_M u dV = 0$ and $(M, e^{2u}g)$ is pluri-closed.

So pluri-closed metrics always exist on complex surfaces.

The pluri-closed Flow for Hermitian Manifolds:

$$\frac{\partial \omega}{\partial t} = \partial \partial^*_\omega \omega + \bar{\partial} \bar{\partial}^*_\omega \omega + \sqrt{-1} \partial \bar{\partial} \log \det g.$$

Clearly, it preserves the pluri-closed condition, furthermore, one can show that it coincides with the Kähler-Ricci flow if the initial metric is Kähler.

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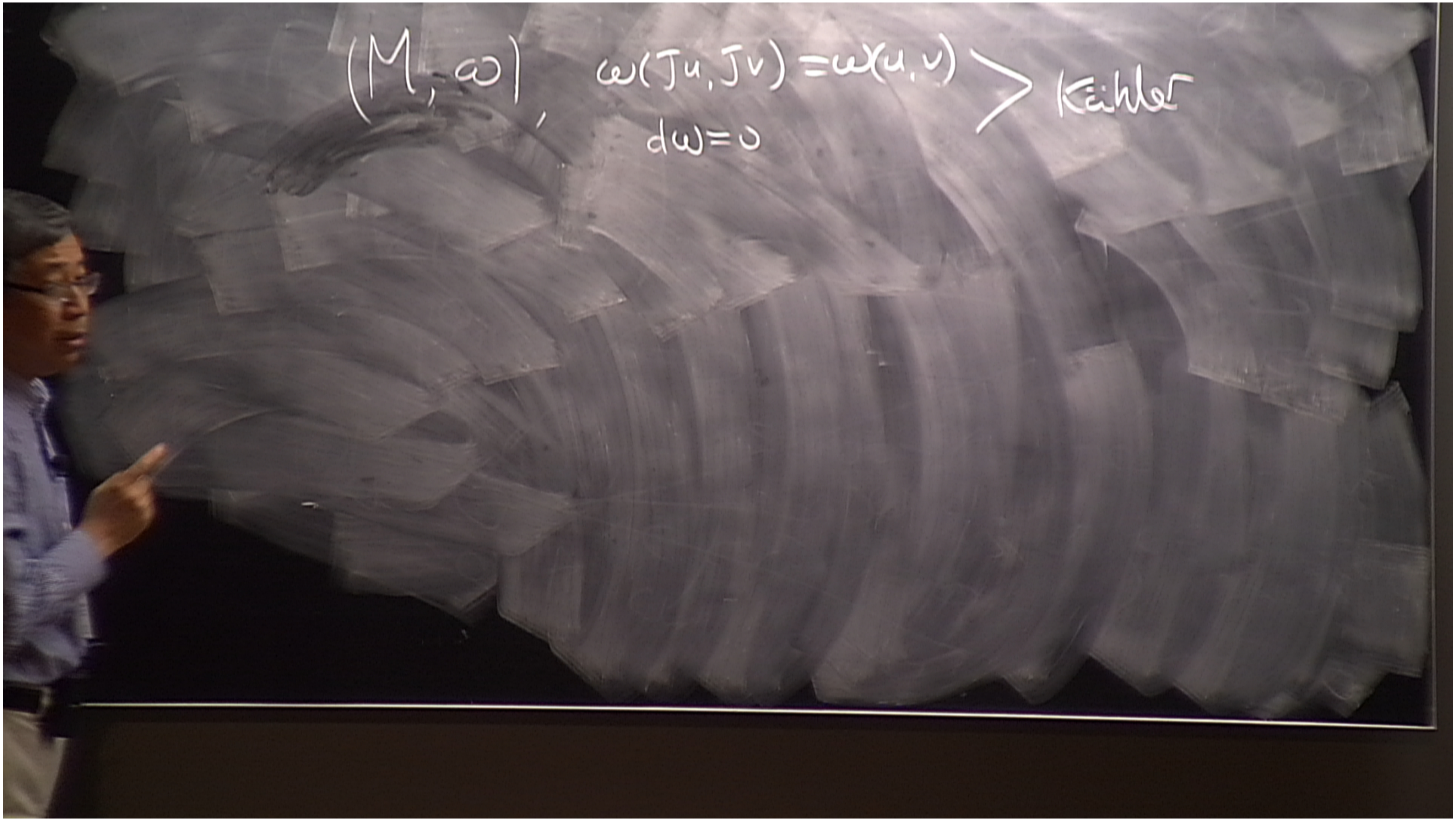
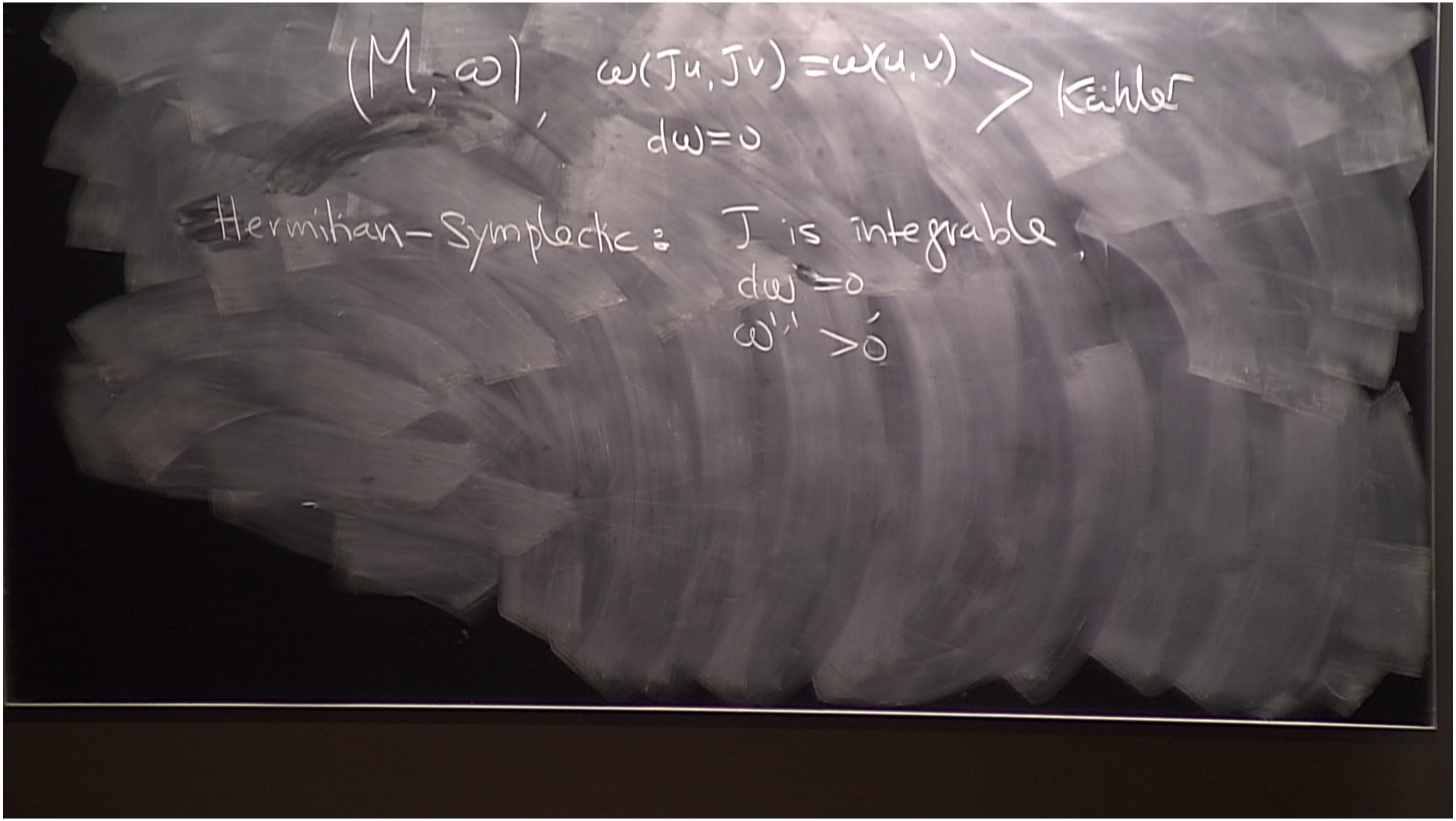
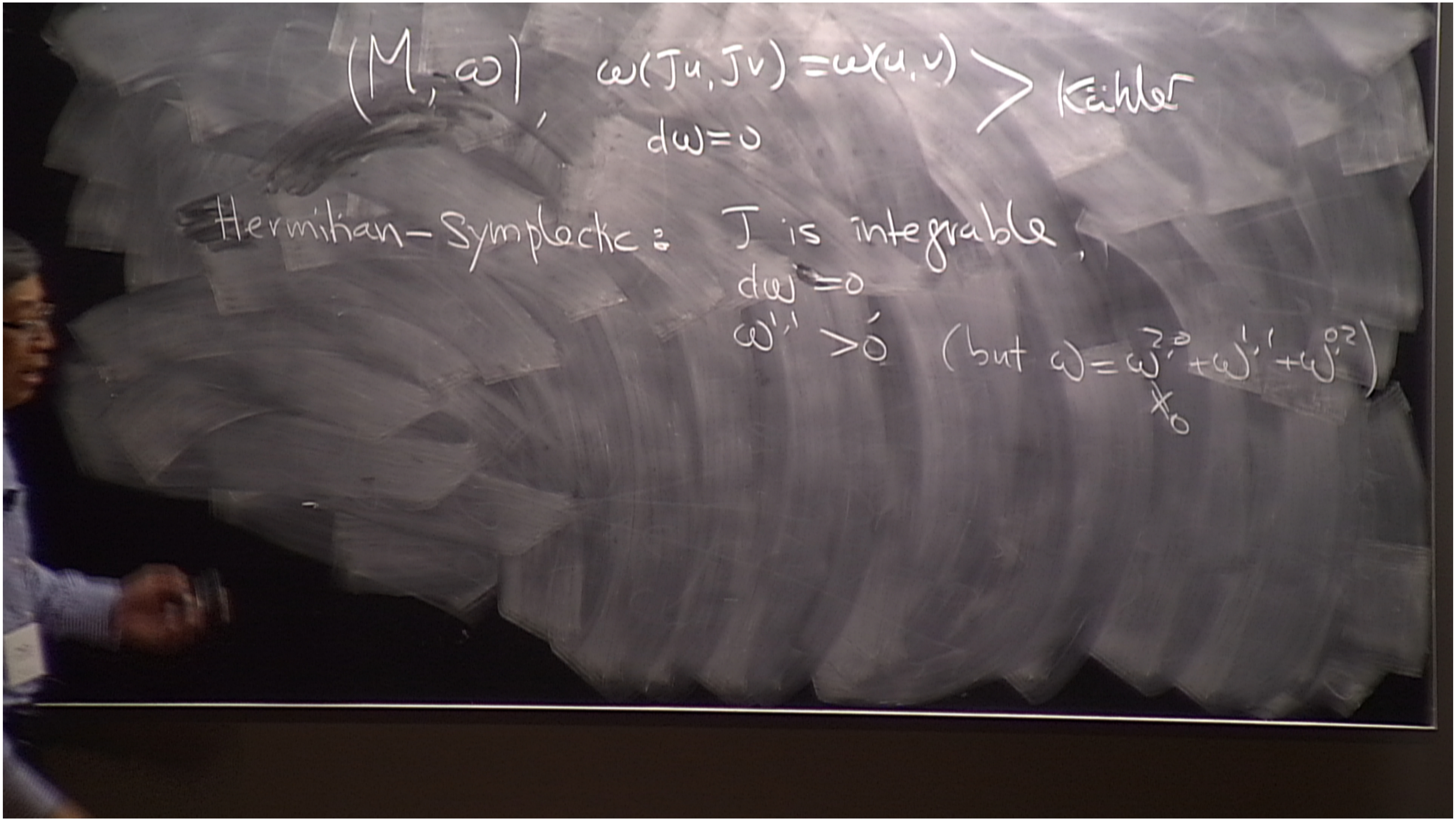

(M, ω),
ω(Ju, Jv) = ω(u, v)
dω = 0
Kähler


(M, ω), ω(Ju, Jv) = ω(u, v), dω = 0 > Kähler
Hermitian-Symplectic: J is integrable, dω = 0, ω^{1,1} > 0



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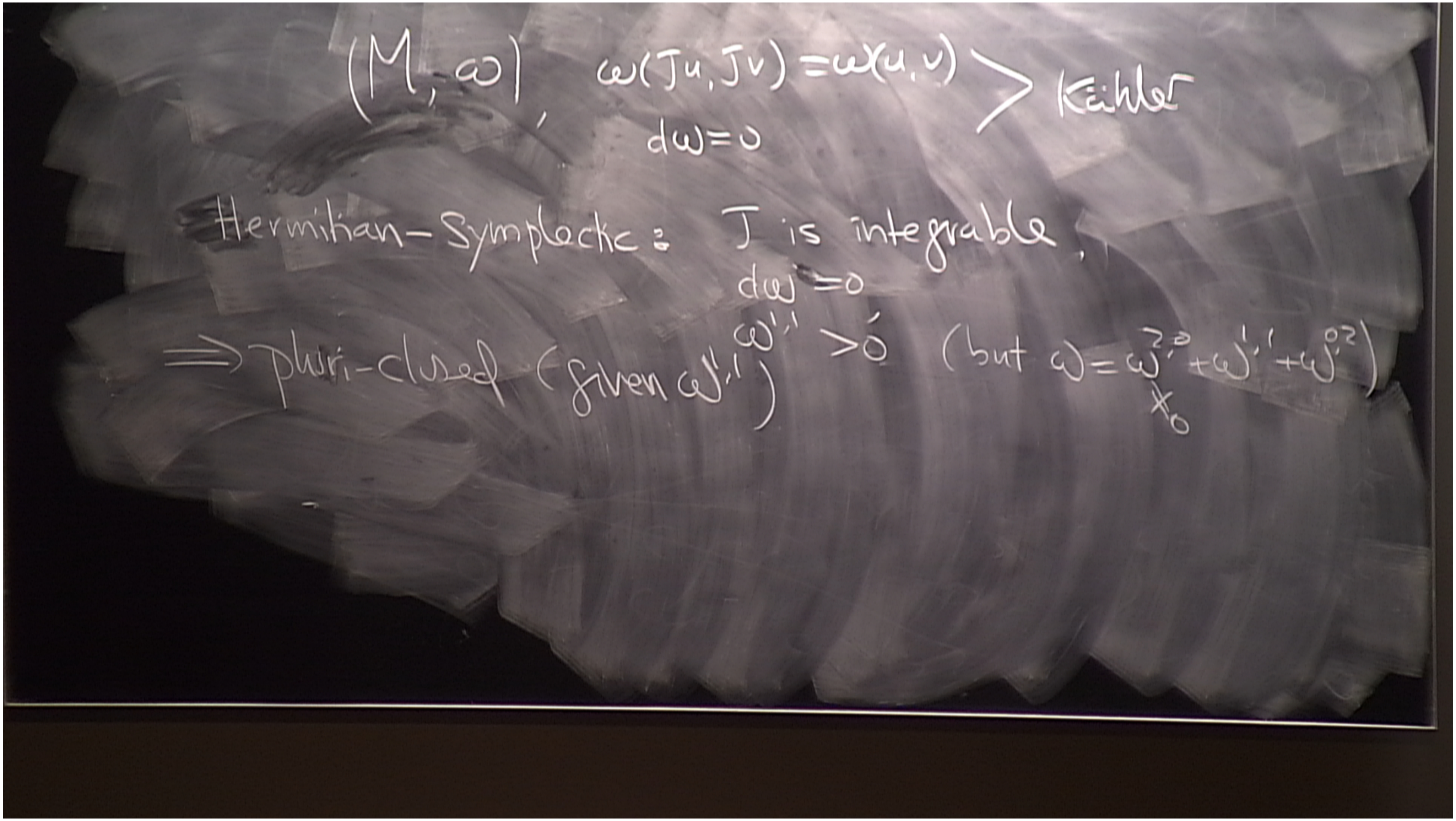
(M, ω), ω(Ju, Jv) = ω(u, v) > Kähler
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Hermitian-Symplectic: J is integrable,
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⇒ pluri-closed (given ω^{1,1}) ω^{1,1} > 0 (but ω = ω^{2,0} + ω^{1,1} + ω^{0,2})
≠ 0

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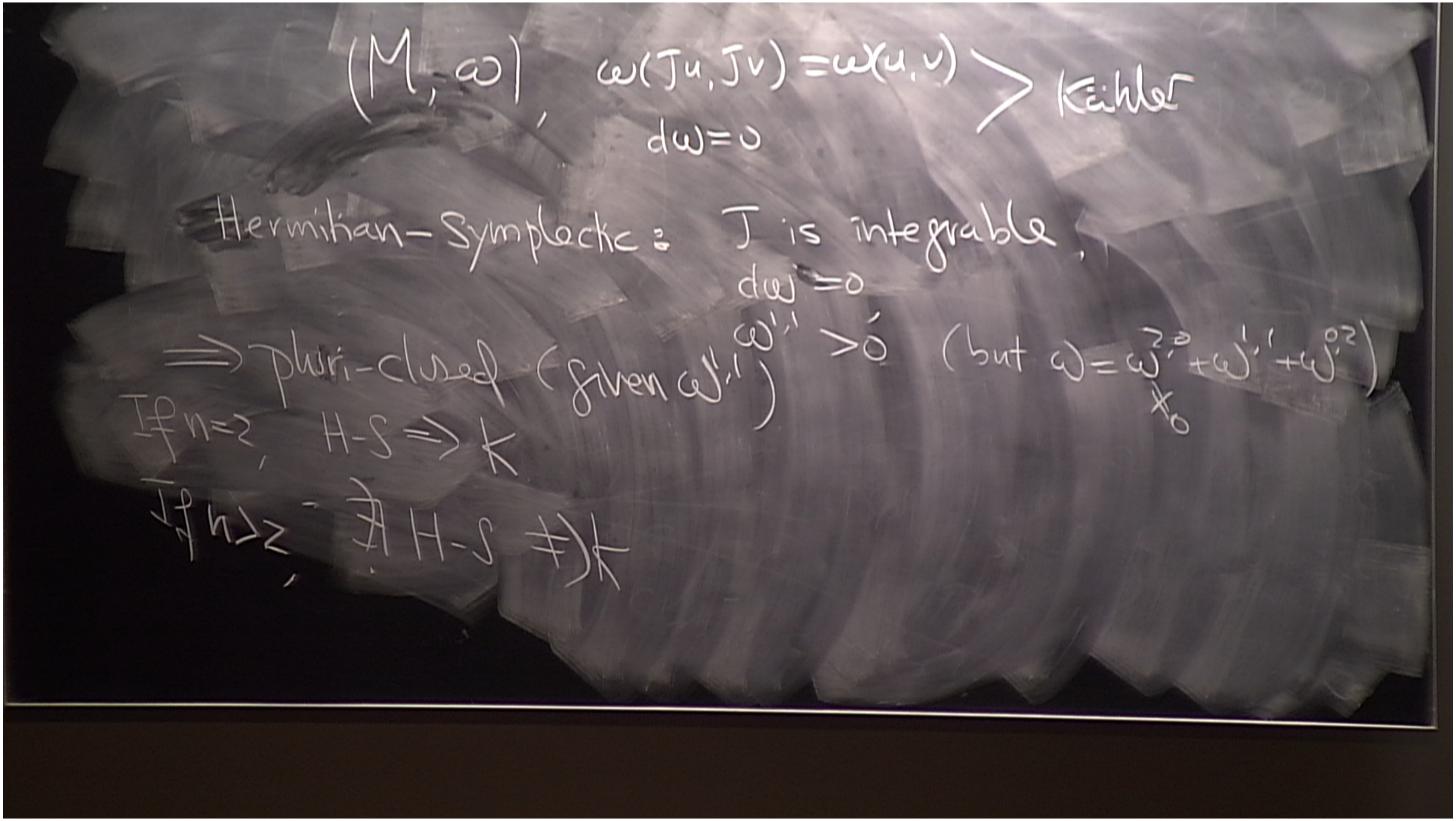
Hermitian-Symplectic: J is integrable, $d\omega = 0$, $\omega^{1,1} > 0$ (but $\omega = \omega^{2,0} + \omega^{1,1} + \omega^{0,2}$), $\omega^{2,0} \neq 0$

$\Longrightarrow$ pluri-closed (given $\omega^{1,1}$)

If $n=2$, H-S $\Longrightarrow$ K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If n=2, H-S ⟹ K
If n>2, ∄ H-S ⇏ K

The flow has other formulations: The first one is in terms of the Chern connection:

$$\frac{\partial g}{\partial t} = -S + Q,$$

where $S_{i\bar{j}} = g^{k\bar{l}}\Omega_{k\bar{l}i\bar{j}}$ is the "Ricci" curvature Ω of the Chern connection of Hermitian metric g and $Q_{i\bar{j}} = g^{k\bar{l}}g^{m\bar{n}}T_{ik\bar{n}}T_{\bar{j}\bar{l}m}$ is the square of the torsion T.

It follows that the flow is parabolic.

The second is in terms of the Bismut connection: Recall the Bismut connection is defined via

$$< \nabla_X Y, Z > = < D_X Y, Z > + \frac{1}{2} d\omega(JX, JY, JZ),$$

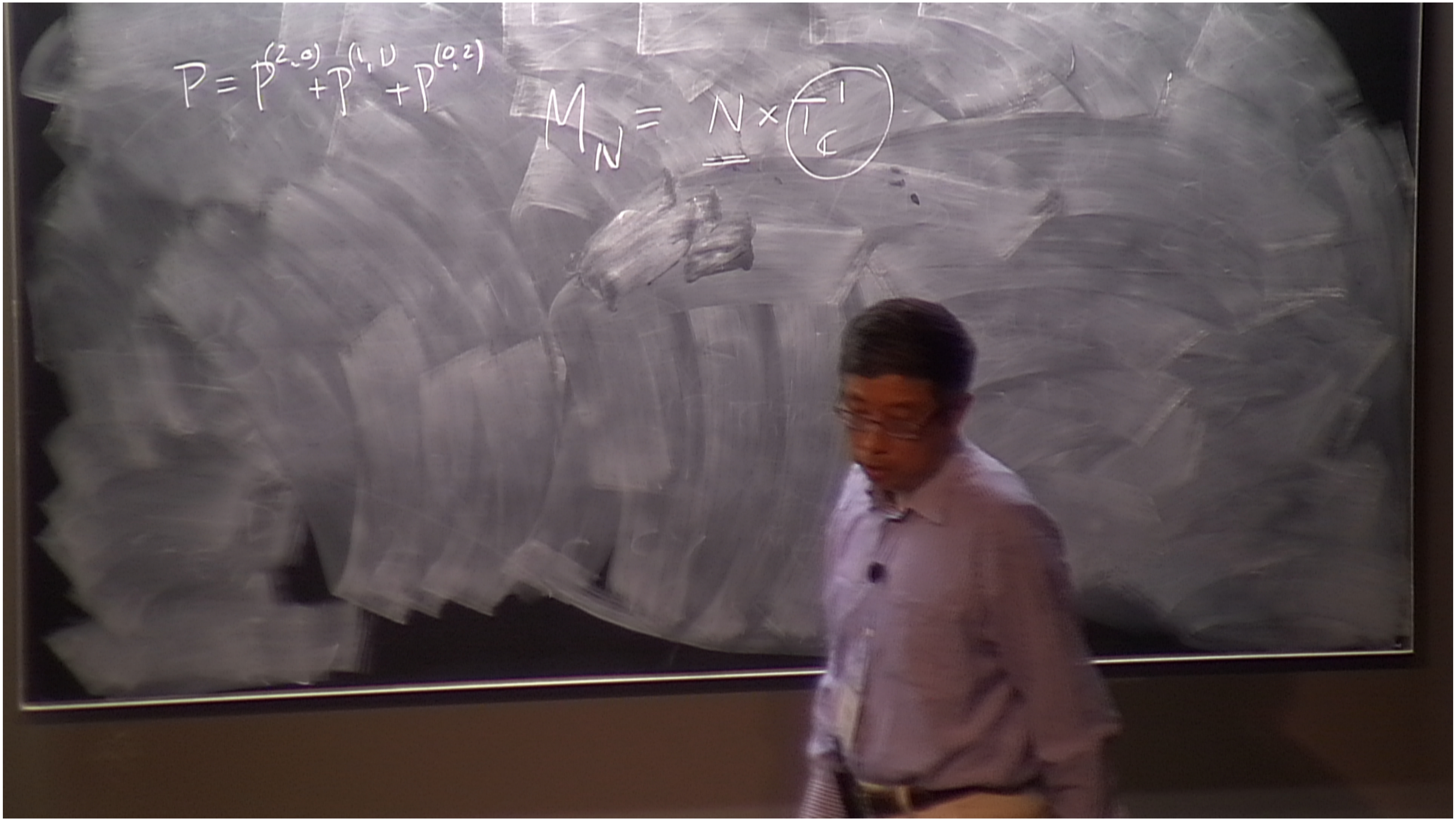
where D denotes the Levi-Civita connection of ω and J denotes the complex structure.

Let P be the Chern form of this connection, i.e. in complex coordinates

$$P_{i\bar{j}} = \Omega^{k}_{i\bar{j}k}.$$

Then the flow is the same as

$$\frac{\partial \omega}{\partial t} = -P^{1,1}.$$


$P = p^{(2,0)} + p^{(1,1)} + p^{(0,2)}$
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The torsion H of a Bismut connection is a 3-form, which actually characterizes the connection. Moreover, g is pluri-closed if and only if H is closed.

Using this, Streets and I derived a new flow on Hermitian metrics which is equivalent to the pluri-closed flow, however, the new flow allows complex structures to vary.

Let $g(t)$ be a solution of pluri-closed flow with a fixed complex structure J. Define a time-dependent vector field X by

$$i_X \omega(t) = -J d^*_{g(t)} \omega(t).$$

Let ϕ_t be its integral curve. Streets and I showed that $\bar{g}(t) = \phi_t^* g(t)$ and $\bar{J}(t) = \phi_t^* J$ satisfy:

$$\frac{\partial \bar{g}}{\partial t} = -2\mathrm{Ric}(\bar{g}) + \frac{1}{2} Q(\bar{H}), \quad \frac{\partial J}{\partial t} = \Delta J + \mathrm{Ric}^- + \bar{Q}(D\bar{J}),$$

where $Q(\bar{H})^i_j = \bar{H}_i^{pq} \bar{H}_{jpq}$, Ric^- is the (2,0)+(0,2)-part of the Ricci curvature and $\bar{Q}(D\bar{J})$ is a quadratic function of $D\bar{J}$.

- This is a parabolic flow modulo diffeomorphisms and it preserves the Hermitian property;

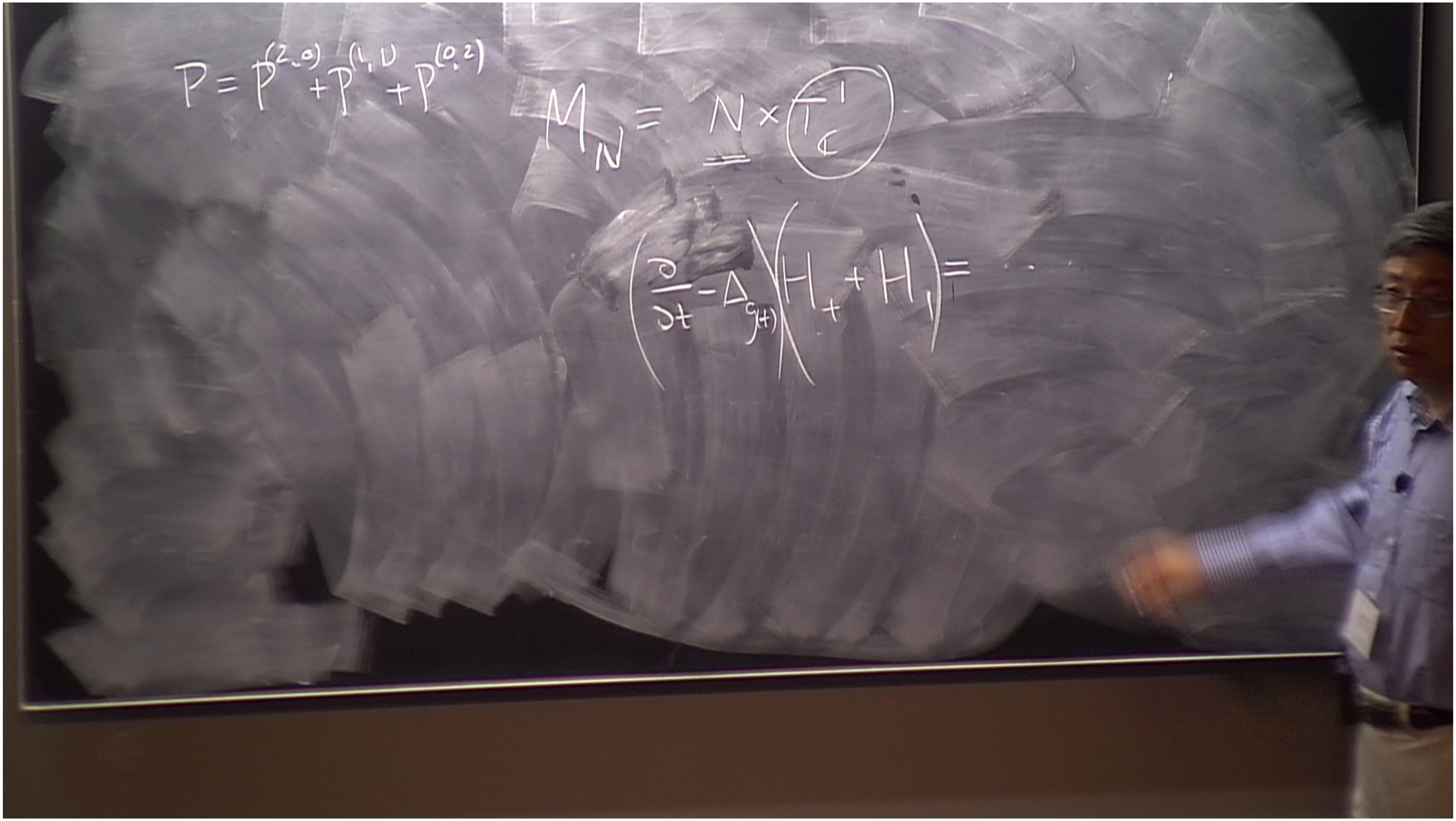
- If the initial data $(\bar{g}(0), \bar{J}(0))$ is pluri-closed, so does every $(\bar{g}(t), \bar{J}(t))$ along the flow;

- This new formulation relates the pluri-closed flow to the B-field renormalization group flow.

This new formulation also provides a new tool to studying generalized Kähler manifolds.

A generalized Kähler structure is a Riemannian metric g with two compatible complex structures J_+ and J_- such that corresponding H_+ and H_- coincide.

The pluri-closed flow preserves the generalized Kähler structures (Streets-Tian, 2011). To see this, we only need to compute the evolution equations on $H_+ - H_-$ along the pluri-closed flow and apply the Maximum principle: If the initial $H_+ - H_-$ vanishes, so does every one along the flow.


P = P(2,0) + P(1,1) + P(0,2)
(∂/∂t − Δg(t))(H+ + H1) =

If $(g(t), J(t))$ is a solution of the above flow, then $(g(t), T(t))$ solves the *renormalization group flow for the nonlinear sigma model with B-field*:

$$\frac{\partial g_{ij}}{\partial t} = -2\mathrm{Ric}_{ij} + \frac{1}{2}H_i^{pq}H_{jpq}, \quad \frac{\partial H}{\partial t} = \Delta H.$$

Oliynyk, Suneeta, Woolgar: The B-field RG flow is the gradient flow of $\mathcal{F}$, where

$$\mathcal{F}(g, H, f) := \int_M \left(R - \frac{1}{12}|H|^2 + |\nabla f|^2\right) e^{-f} dV.$$

So $\mathcal{F}$ increases along the pluri-closed flow.

We have developed an analytic theory for pluri-closed flow, including high order derivative estimates and curvature estimates.

Streets-Tian (2009): Let $(M, \omega(t), J)$ be a solution to pluri-closed flow. Let ∇ denote the associated Chern connections, and Ω and H the curvature and torsion of the Chern connection. The solution $\omega(t)$ exists on a maximal time interval $[0, T)$, and if $\tau < \infty$ then

$$\limsup_{t \to T} \{|\Omega|, |\nabla H|, |H|^2\} = \infty,$$

If $n = 2$, then $\limsup_{t \to T} |\Omega| = \infty$.

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To prove the regularity theorem, we compute the induced evolution equations for the curvature Ω and the torsion H of the Chern connections and apply the maximum principle. The arguments are similar to those in the case of Ricci flow.

In the case $n = 2$, we can drop the dependence on H because we have:

$$\frac{\partial}{\partial t}|H|^2 \leq \Delta|H|^2 + \nabla|H|^2 \star w + C|H|^2 - \frac{1}{2}|H|^4,$$

where w is a 1-form whose components are $w_i = g^{j\bar{l}} H_{ij\bar{l}}$.

In terms of the *Bismut connection*, we have:

Streets-Tian (2010): Let $(M, \omega(t), J)$ be a solution to pluriclosed flow. Suppose the maximal existence time is $T < \infty$. Then

$$\int_0^T |P_B^{1,1}| = \infty,$$

where P_B denotes the "Ricci" form of the Bismut connection.

This is analogous to a result of N. Sesum in 2004 for Ricci flow.

Let $(M, \omega(t), J)$ be a solution to pluri-closed flow, and let $\tilde{\omega}$ denote a background Hermitian metric. Let $\phi(t)$ satisfy

$$\frac{\partial \phi}{\partial t} = \Delta_\omega \phi + \mathrm{tr}_\omega \tilde{\omega} - n, \quad \phi(0) = 0.$$

Streets-Tian (2010): If both ϕ and $|H|^2$ are bounded on $[0, T)$, then the solution extends smoothly past time T.

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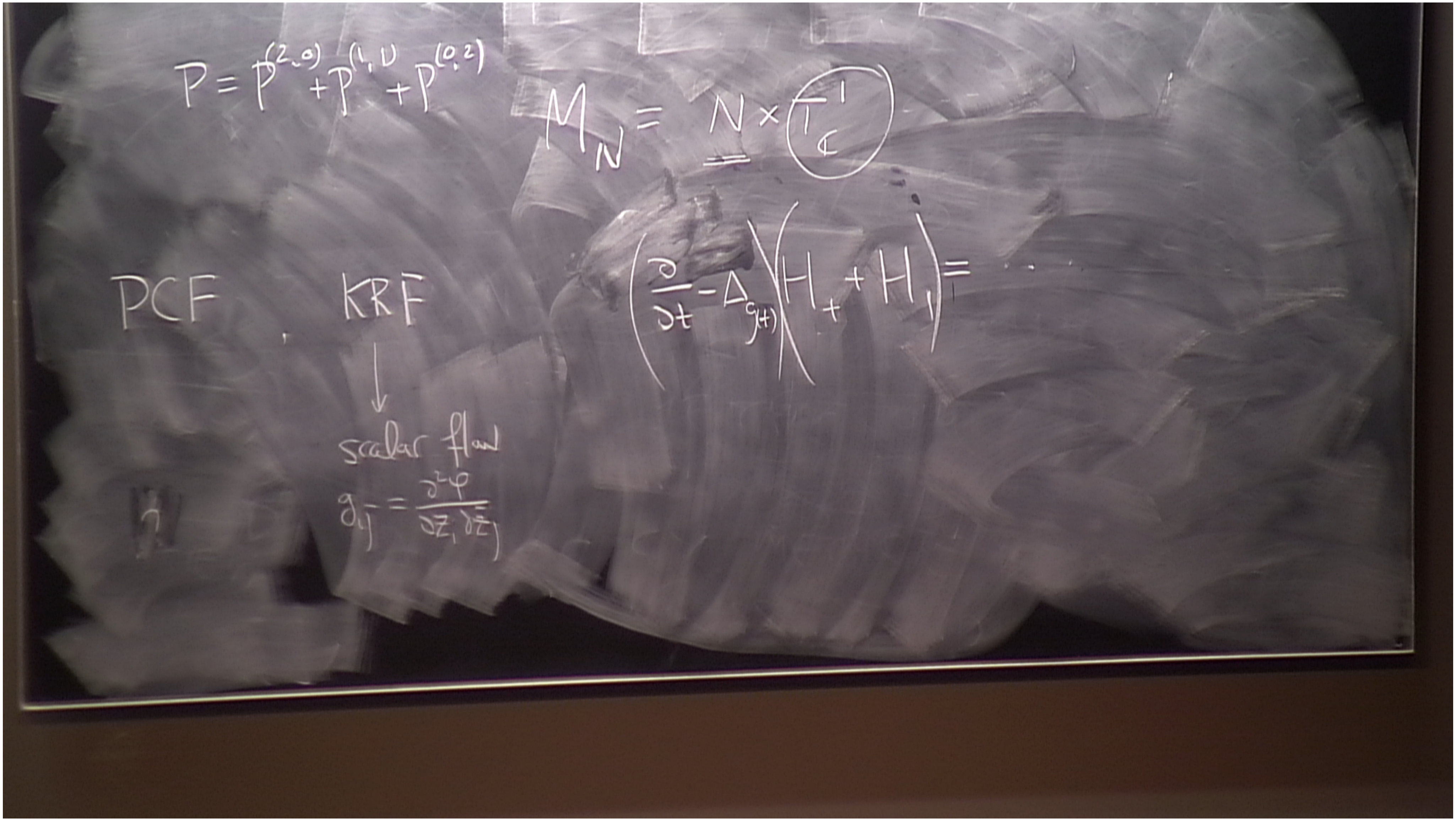
P = P^(2,0) + P^(1,1) + P^(0,2)
M_N = N × 1/T_c
PCF
KRF
scalar flow
g_ij = ∂²φ / ∂z_i ∂z̄_j
(∂/∂t − Δ_g(t)) (H_t + H_1) =

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Make an analogy with the Calabi-Yau theorem.

Kähler-Ricci flow	Pluri-closed flow
$(n + \Delta u) \leq e^{u - \inf u}$	$\mathrm{tr}_{\tilde{\omega}} \omega \leq e^{\phi + \log \frac{\omega^n}{\tilde{\omega}^n}}$
$\lvert \log \frac{\det g}{\det \tilde{g}} \rvert \leq C$	$\lvert \log \frac{\det g}{\det \tilde{g}} \rvert \leq C(T, \lvert H \rvert^2)$.
$\lvert D^3 u \rvert \leq C$	$\lvert \Gamma(g) - \Gamma(\tilde{g}) \rvert \leq C(T, \lvert H \rvert^2)$
Schauder: $\lvert u \rvert_{C^k} \leq C(\lvert u \rvert)$	Schauder: $\lvert g \rvert_{C^k} \leq C(\lvert \phi \rvert, \lvert H \rvert^2, T)$
Moser Iteration: $\lvert u \rvert \leq C$	???

Let $(M, \omega(t), J)$ be a solution to pluri-closed flow, and let $\tilde{\omega}$ denote a background Hermitian metric. Let $\phi(t)$ satisfy

$$\frac{\partial \phi}{\partial t} = \Delta_\omega \phi + \operatorname{tr}_\omega \tilde{\omega} - n, \quad \phi(0) = 0.$$

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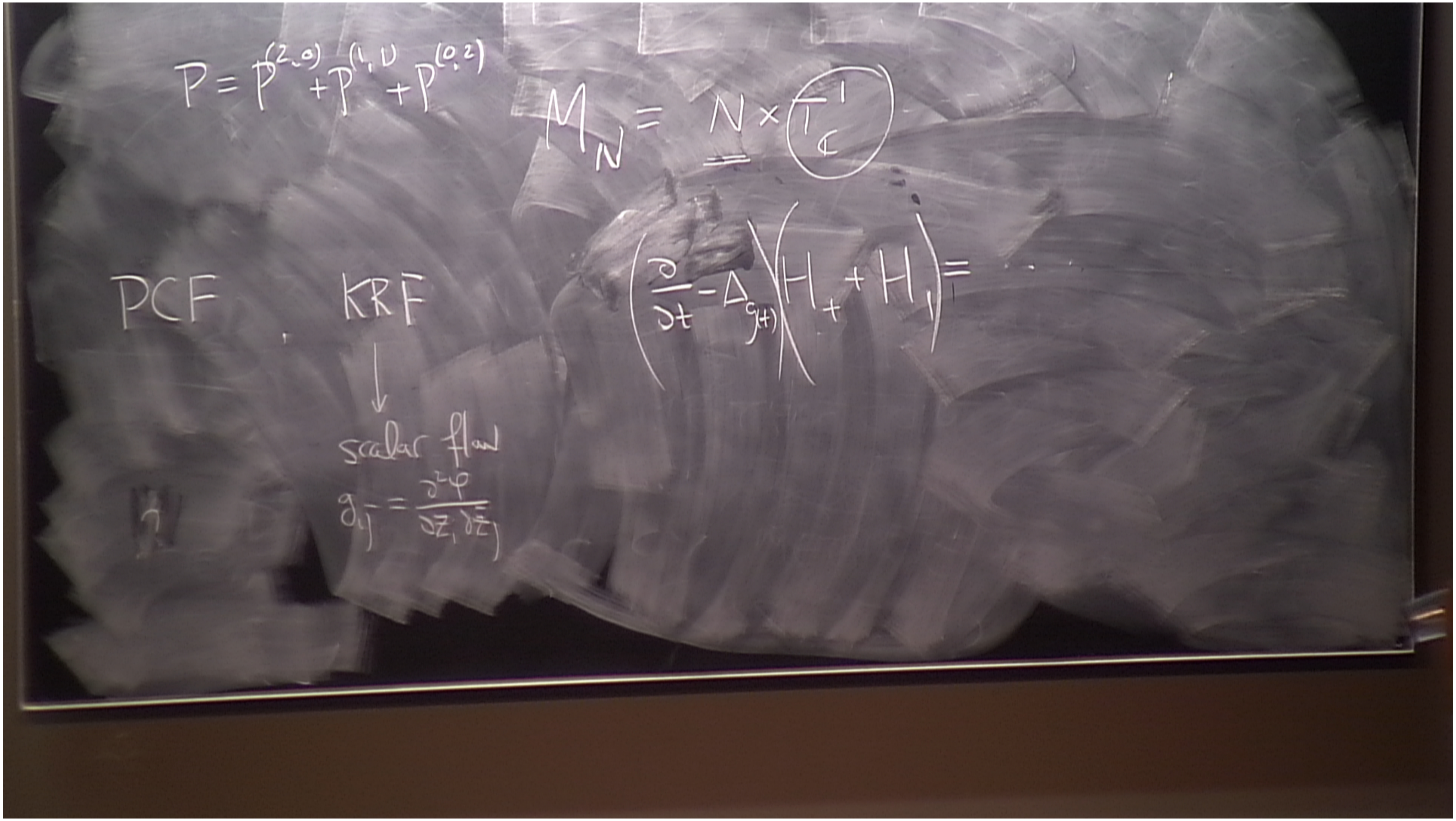
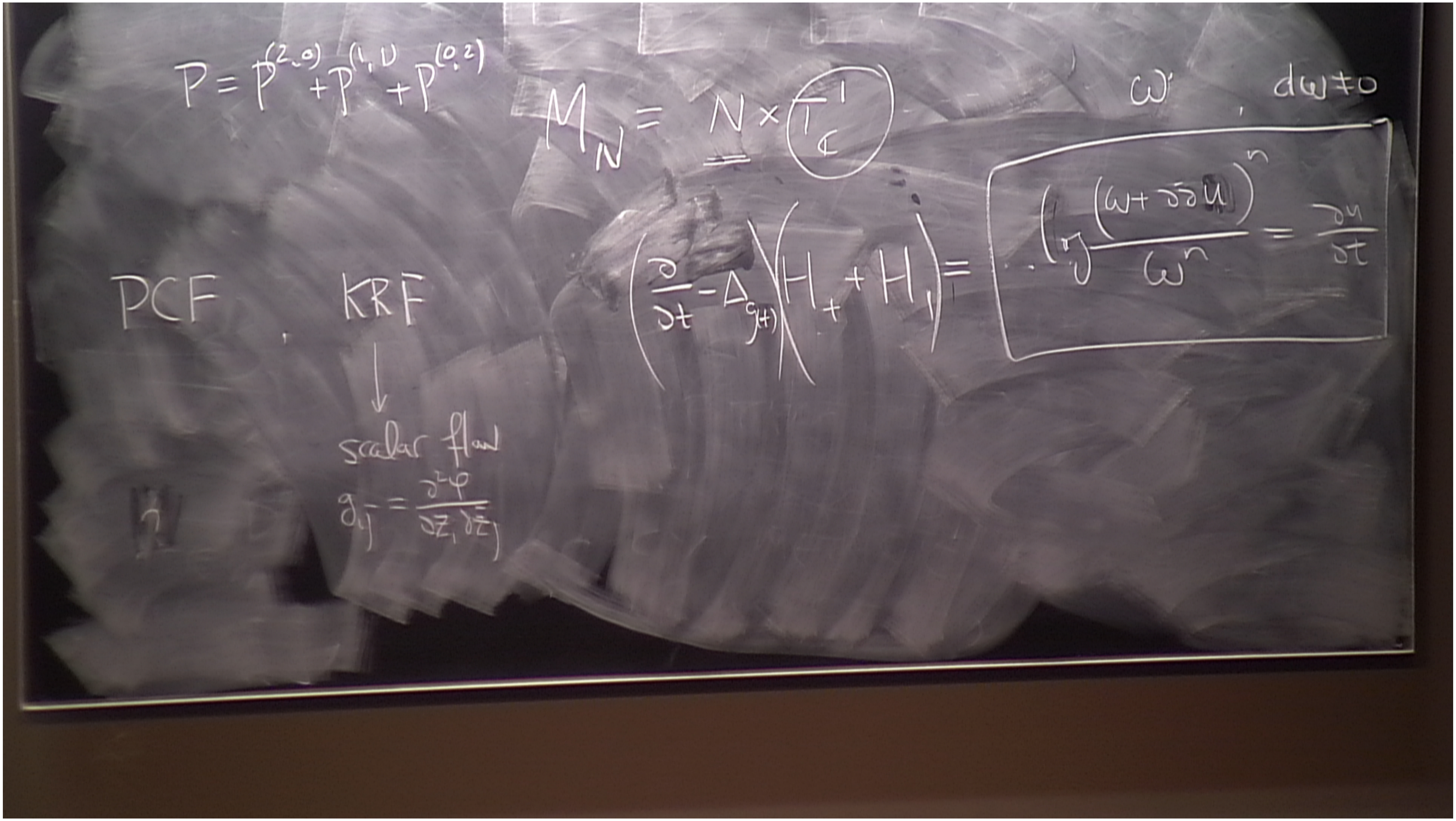

P = P(2,0) + P(1,1) + P(0,2)
M_N = N × 1/T_c
PCF
KRF
scalar flow
g_ij = ∂²φ / ∂z_i ∂z̄_j
(∂/∂t − Δ_g(t))(H_t + H_1) =


P = P^(2,0) + P^(1,1) + P^(0,2)
M_N = N × T_c^1
ω , dω ≠ 0
(∂/∂t − Δ_g(t))(H_+ + H_1) =
log (ω + ∂∂u)^n / ω^n = ∂u/∂t
PCF , KRF
scalar flow
g_ij = ∂²φ / ∂z_i ∂z_j

We expect that the pluri-closed flow shares the same properties as the Kähler-Ricci flow had.

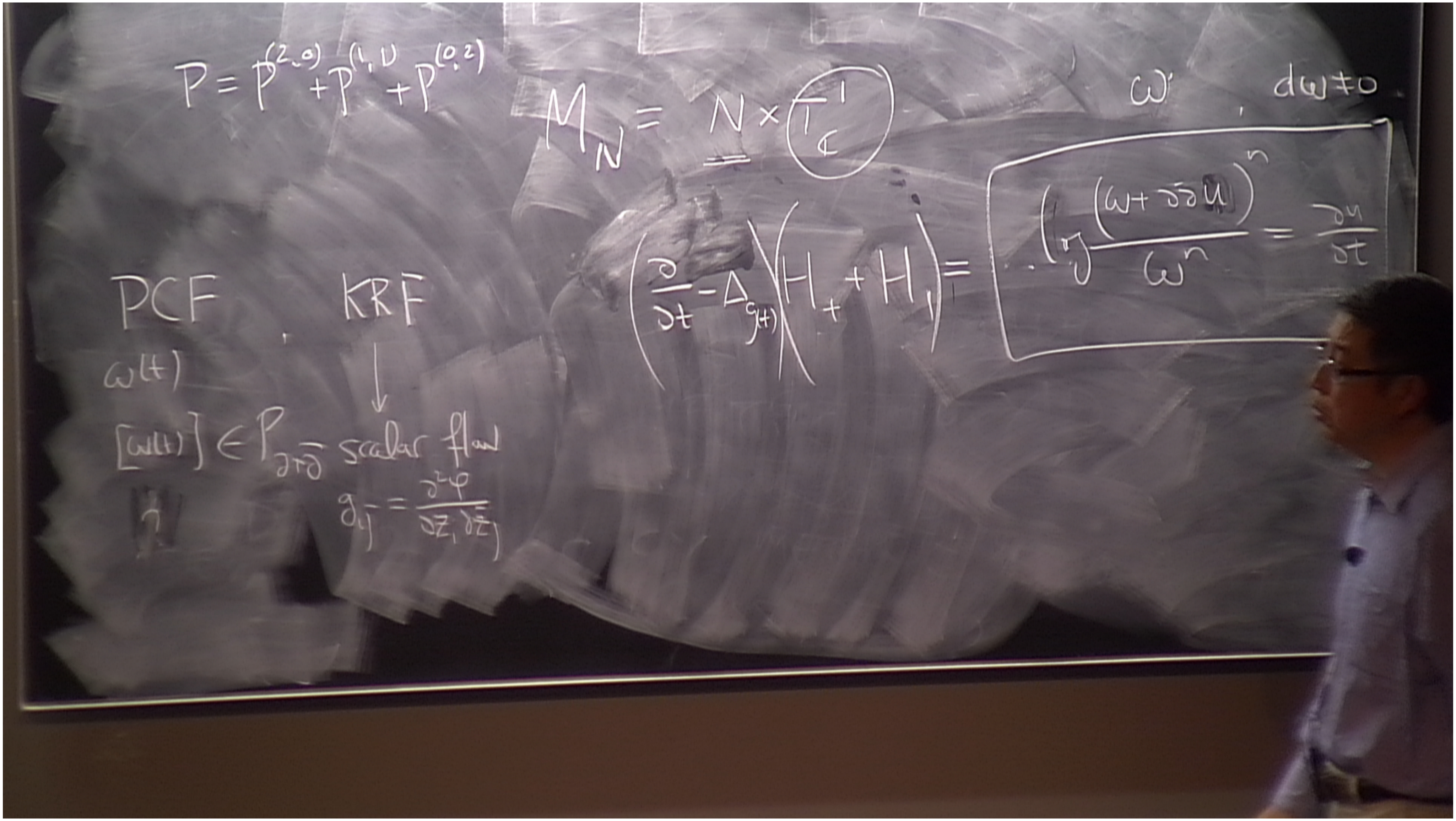
First we exam exactly when the flow stops. Define

$$\mathcal{H}^{1,1}_{\partial+\bar{\partial}} := \frac{\{\phi \in \Lambda^{1,1}_{\mathbb{R}} \mid \partial\bar{\partial}\phi = 0\}}{\{\partial\gamma + \bar{\partial}\bar{\gamma} | \gamma \in \Lambda^{0,1}\}}.$$

In analogy with the Kähler cone, let

$$\mathcal{P}_{\partial+\bar{\partial}} := \{[\phi] \in \mathcal{H}^{1,1}_{\partial+\bar{\partial}} | \exists \gamma, \phi + \partial\gamma + \bar{\partial}\bar{\gamma} > 0\}$$

Conjectural Optimal Existence: The pluri-closed flow has a solution up to $T = \sup\{t > 0 | [\omega(t)] \in \mathcal{P}_{\partial+\bar{\partial}}\}$.


PCF
KRF
scalar flow
dω ≠ 0

This pluri-closed cone can be characterized in a nice way on complex surfaces:

Streets-Tian: Let M be a complex, non-Kähler surface, and let $\phi \in \Lambda^{1,1}$ be pluriclosed. Then $\phi \in \mathcal{P}_{\partial+\bar{\partial}}$ if and only if $\int_M \phi \wedge \gamma_0 > 0$ and $\int_D \phi > 0$ for every effective divisor with negative self intersection, where γ is a fixed positive (1,1)-form.

Class VII$^+$ surfaces:

- These surfaces are the remaining unclassified complex surfaces;

- Characteristics: $b_1 = 1$, $b_2^+ = 0$, $c_1^2 = -b_2 = -n < 0$;

- Every *known* example arises by blowing up a Hopf surface n times, then changing the complex structure. In particular it is diffeomorphic to $S^3 \times S^1 \# n\overline{\mathbb{CP}}^2$;

- Conjecturally these examples are all, and by a result of Dloussky, Oeljeklaus and Toma, if (M^4, J) is of Class VII$^+$ and contains b_2 rational curves, it is of the known type.

Streets-Tian: The existence conjecture implies long time existence of the pluri-closed flow on Class VII$^+$ surfaces. Moreover, it implies the existence of a curve on these surfaces.

We have classified all the static solutions for the pluri-closed flow on Hermitian 4-manifolds, in particular, we have proved that there are no such static metrics on any Class VII$^+$ surfaces.

In view of these, we believe that a sufficiently good existence result for the pluri-closed flow will lead to a classification of Class VII$^+$ surfaces.