Title: When do Frustration-free Spin Chains Become Entangled?

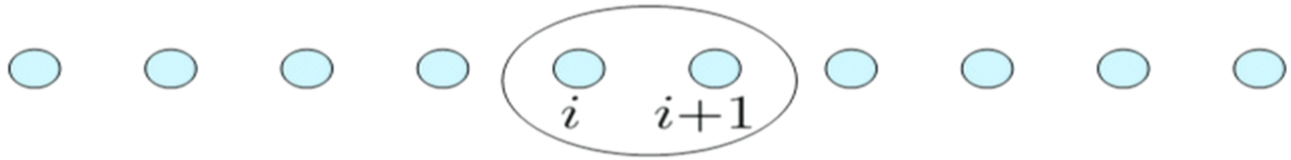
Date: Apr 12, 2012 09:00 AM

URL: http://pirsa.org/12040112

Abstract: TBA

Frustration-free spin chains

$$\Pi_{i,i+1}$$



$$(\mathbb{C}^d)^{\otimes n}$$

$$H = \sum_{i=1}^{n-1} \Pi_{i,i+1}$$

Interactions = projectors

Ground state of H = zero eigenvector of every projector

Frustration-free spin chains

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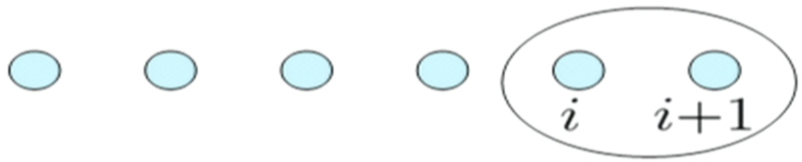

$$(\mathbb{C}^d)^{\otimes n}$$

Example 1: Heisenberg chain (d=2)

$$\Pi = |\Psi^-\rangle\langle\Psi^-|, \quad |\Psi^-\rangle \sim |01\rangle - |10\rangle$$

Ground states = symmetric subspace of $(\mathbb{C}^2)^{\otimes n}$

How entangled can be ground states of frustration-free quantum spin chains ?

Focus on `nice' spin chains:

- Small local dimension (qubits or qutrits)
- Unique ground state
- Spectral gap is not too small (polynomial in 1/n)
- Translational invariance (optional)

Ground state entanglement *S(n)* for FF qudit chains

d

21 — Create long-range singlets using courier particle
S. Irani 2010
$S(n) \sim n$

≥4 — Random FF spin chains with projectors of fixed rank = *r*, Movassagh et al 2010
Conjectured highly entangled ground states for
$d \leq r \leq d^2/4$

≥3 — AKLT state, parent Hamiltonians of PEPS and MPS
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Basis states of a qutrit

- $|0\rangle$ Vacuum
- $|l\rangle$ Left bracket [
- $|r\rangle$ Right bracket]

ain = strings of left and right brackets possibly separated by zeros

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Basis states of a chain = strings of left and right brackets possibly separated by zeros

Def. A string S over the alphabet 0,l,r is balanced iff

(i) any initial segment of S has at least as many l's as r's
(ii) total number of l's = total number of r's

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Example: balanced strings of length 4

$llrr$	$l00r$	$lrlr$
$lr00$	$l0r0$	$0lr0$
$0l0r$	$00lr$	0000

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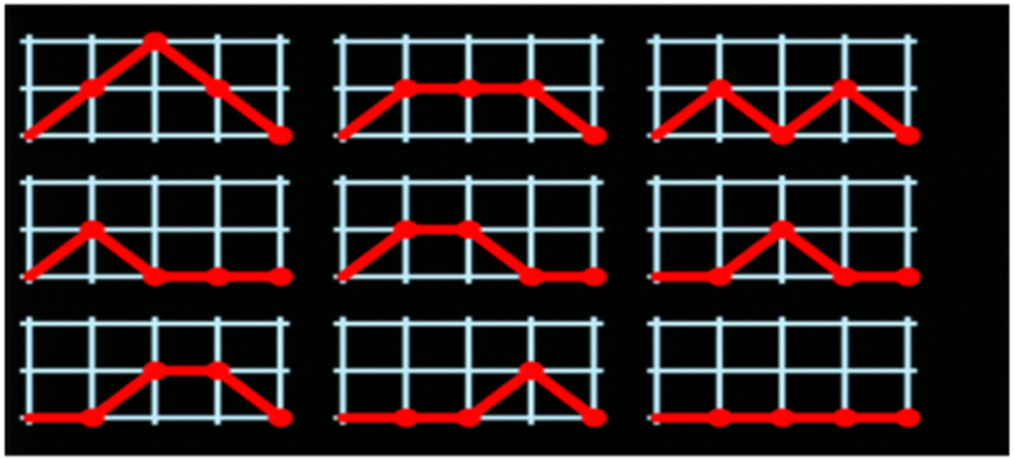
Example of unbalanced string: $\underline{l0lrrr}llr$

Balanced strings = Motzkin paths



$llrr$	$l00r$	$lrlr$
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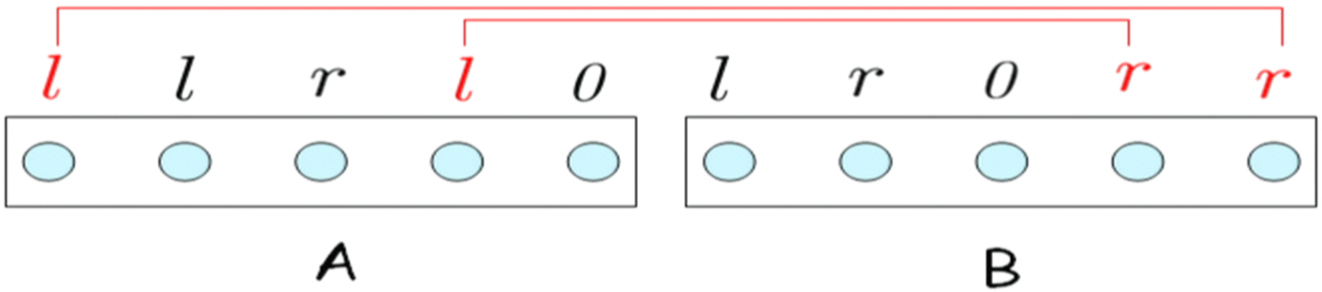
Def. The Motzkin state of n qutrits $|\mathcal{M}_n\rangle$ is the uniform superposition of all balanced strings of length n.

$$|\mathcal{M}_2\rangle \sim |00\rangle + |lr\rangle$$

$$|\mathcal{M}_3\rangle \sim |000\rangle + |lr0\rangle + |l0r\rangle + |0lr\rangle$$

$$\begin{aligned} |\mathcal{M}_4\rangle \quad \sim \quad & |0000\rangle + |00lr\rangle + |0l0r\rangle + |l00r\rangle \\ & +|0lr0\rangle + |l0r0\rangle + |lr00\rangle + |llrr\rangle + |lrlr\rangle. \end{aligned}$$

Why the Motzkin state is highly entangled ?



Entanglement between A and B stems from the locally unmatched brackets.

A and B may have p extra left and right brackets, $0 \leq p \leq n/2$.

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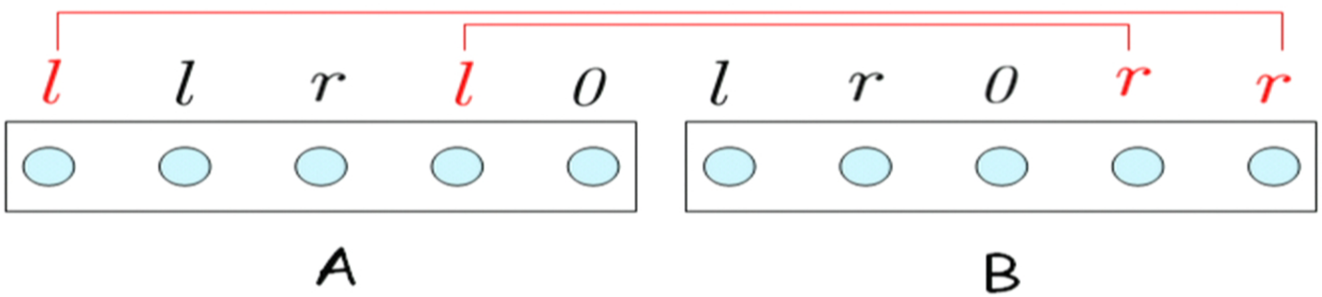

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$$
\begin{aligned}
|\mathcal{M}_4\rangle \quad \sim \quad & (|00\rangle + |lr\rangle)_A \otimes (|00\rangle + |lr\rangle)_B && \text{p=0} \\
& +(|0l\rangle + |l0\rangle)_A \otimes (|0r\rangle + |r0\rangle)_B && \text{p=1} \\
& +|ll\rangle_A \otimes |rr\rangle_B . && \text{p=2}
\end{aligned}
$$

Parent Hamiltonian

$$H = |r\rangle\langle r|_1 + |l\rangle\langle l|_n + \sum_{j=1}^{n-1} \Pi_{j,j+1}$$

Π projects onto a 3-dimensional subspace of $\mathbb{C}^3 \otimes \mathbb{C}^3$ spanned by states

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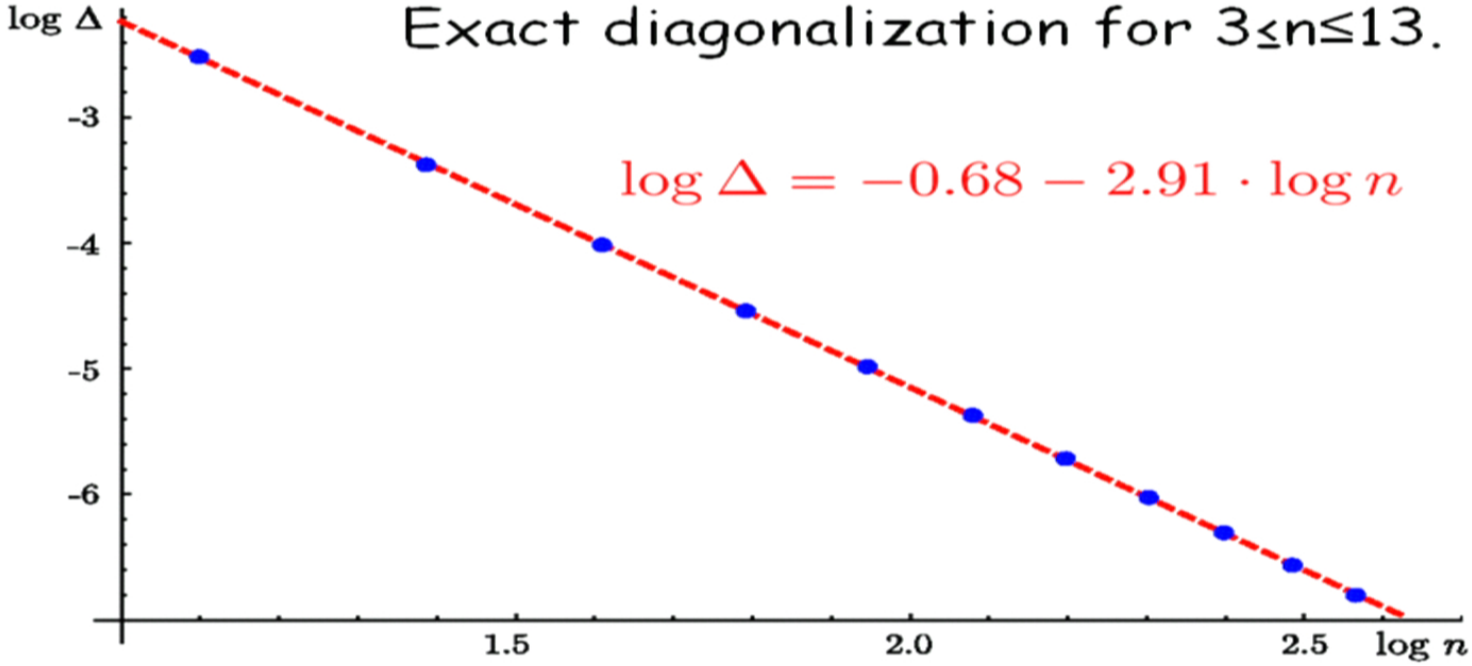
Theorem. The Motzkin state is the unique ground state of H with zero energy. The spectral gap of H is *poly(1/n)*. Entanglement entropy of one-half of the chain is

$$S(A) \approx \frac{1}{2}\log n + 0.14(5)$$

Spectral gap Δ



Rigorous bounds:

$$\Delta = O(n^{-1/2})$$

$$\Delta = \Omega(n^{-c}), \quad c \gg 1.$$

Local description of the Motzkin state

Def. Strings s and t are equivalent, s~t, iff one can go from s to t by a sequence of local moves

$$0l \leftrightarrow l0, \quad 0r \leftrightarrow r0, \quad 00 \leftrightarrow lr$$

Lemma. A string is balanced iff it is equivalent to 0^n

$\Leftarrow$ Local moves preserve balanceness

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$\Leftarrow$ Local moves preserve balanceness

$\Rightarrow$ Any balanced non-zero string must contain

$$lr \quad \text{or} \quad l0\ldots0r$$

Use local moves to annihilate the pair *lr*
Use induction in the number of brackets

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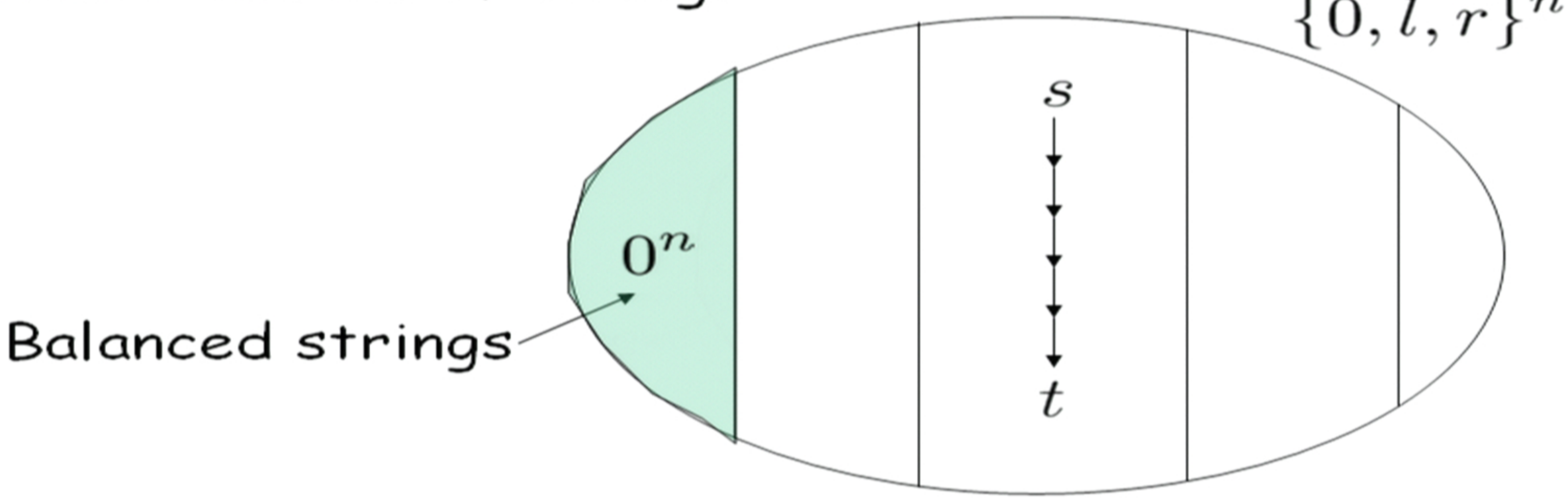
Equivalence classes of strings:



$$c_{p,q} \equiv \underbrace{r\ldots r}_{p}\underbrace{0\ldots 0}_{n-p-q}\underbrace{l\ldots l}_{q}.$$

Lemma. Any string is equivalent to one and only one string $c_{p,q}$ for some integers p,q.

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Hence any equivalence class has a form

$$C_{p,q} = \{s \in \{0, l, r\}^n \; : \; s \sim c_{p,q}\}$$

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Any string in $C_{p,q}$ can be uniquely written as

$$s = \underbrace{brbr \ldots br}_{p}\, b\, \underbrace{lblb \ldots lb}_{q}$$

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$$|C_{p,q}\rangle \sim \sum_{s \in C_{p,q}} |s\rangle$$

Contains the Motzkin state $|\mathcal{M}_n\rangle = |C_{0,0}\rangle$

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It remains to exclude the unwanted ground states with non-zero p or q.

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The class $C_{0,0}$ is the only class in which strings never start from r and never end by l.

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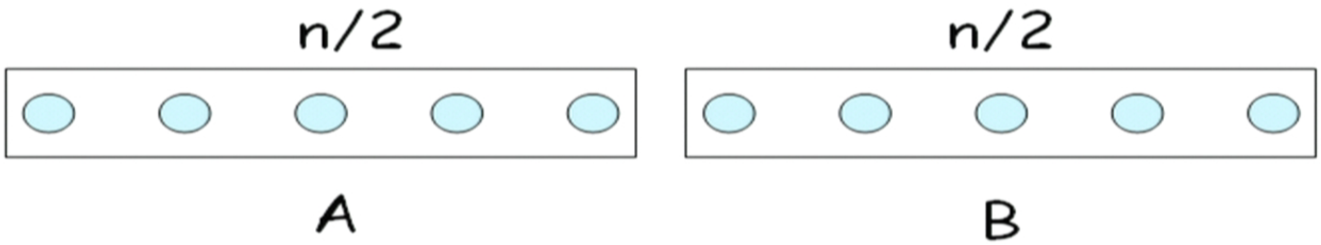
Adding energy penalty for strings starting from r or ending by l gives the desired parent Hamiltonian:

$$H = |r\rangle\langle r|_1 + |l\rangle\langle l|_n + \sum_{j=1}^{n-1} \Pi_{j,j+1}$$



Schmidt decomposition of the Motzkin state:

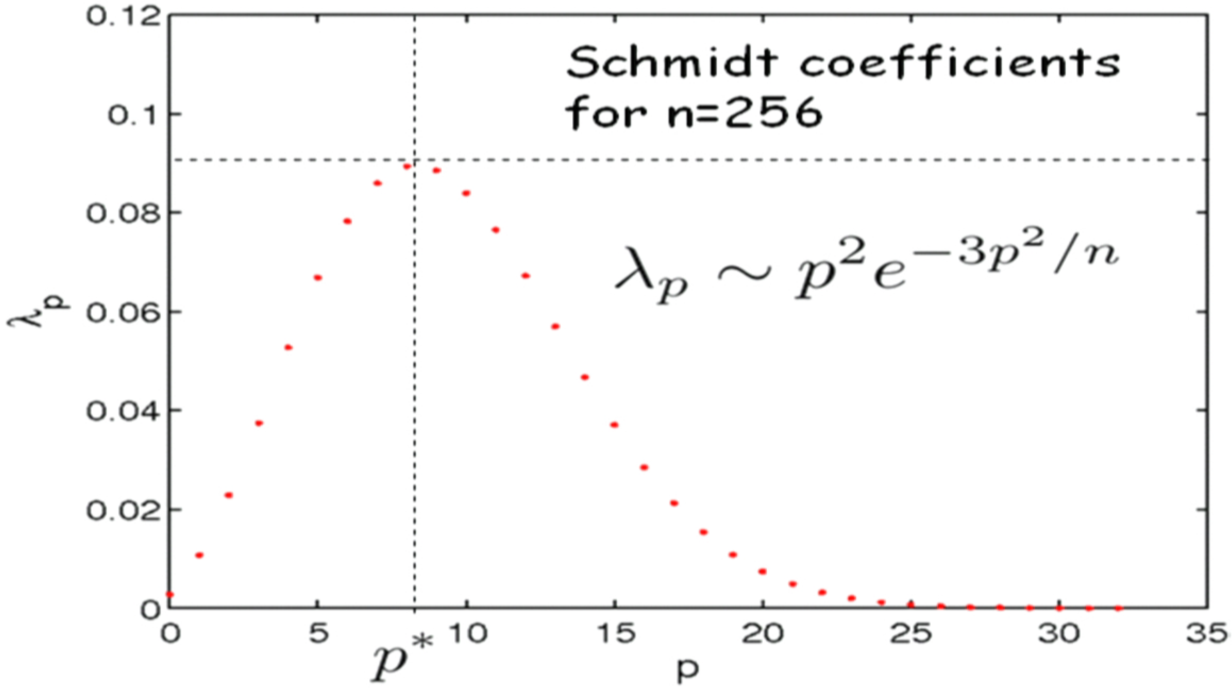
$$|\mathcal{M}_n\rangle \equiv |\hat{C}_{0,0}\rangle = \sum_{p=0}^{n/2} \sqrt{\lambda_p}\ |\hat{C}_{0,p}\rangle_A \otimes |\hat{C}_{p,0}\rangle_B$$

Here $|\hat{C}_{p,q}\rangle$ is the normalized superposition of all strings in the class $C_{p,q}$ and

$$\lambda_p = \frac{|C_{0,p}(n/2)|^2}{C_{0,0}(n)}$$ are the Schmidt coefficients

Entanglement entropy: $S(A) \approx \frac{1}{2}\log n + 0.14(5)$

$$p^* \approx \sqrt{\frac{n}{3}}$$



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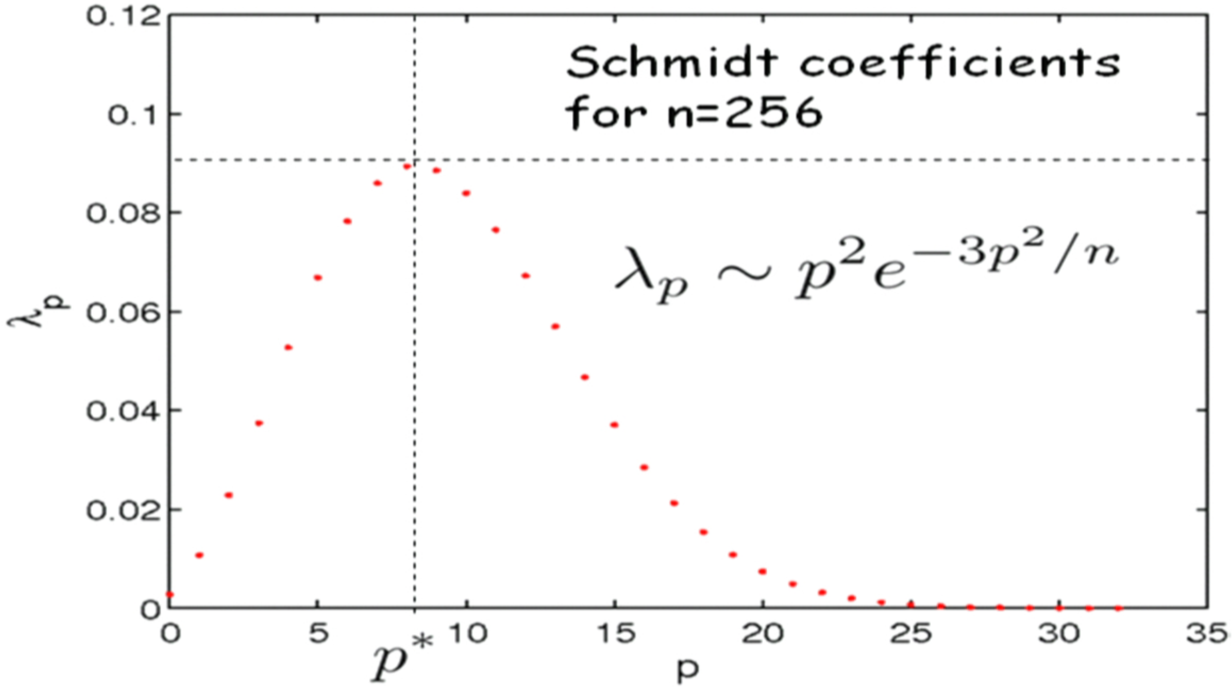

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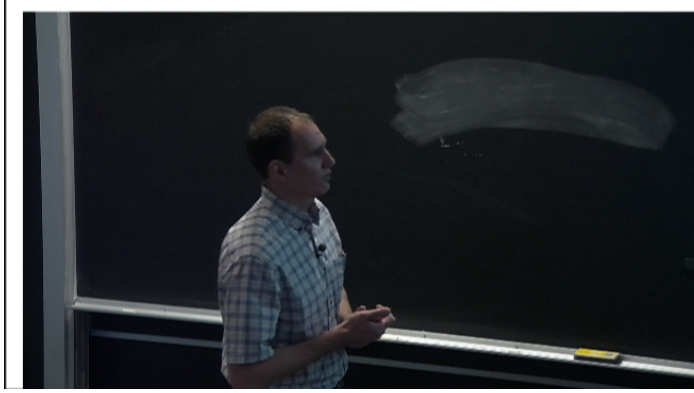
Spectral gap: lower bound

Invariant subspaces:

$$(\mathbb{C}^3)^{\otimes n} = \bigoplus_{p,q} \mathcal{H}_{p,q}$$

$\mathcal{H}_{p,q}$ is spanned by strings in the equivalence class $C_{p,q}$



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We need a polynomial lower bound for

- Spectral gap inside the balanced subspace $\mathcal{H}_{0,0}$
- Ground state energy inside any unbalanced subspace $\mathcal{H}_{p,q}$ with non-zero p or q

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We can ignore the boundary terms $|r\rangle\langle r|_1 + |l\rangle\langle l|_n$

Step 1: treat different types of local moves separately using the perturbation theory

$$H = \sum_j \Pi_{j,j+1} = H_0 + V$$

$$H_0 = \sum_j \Pi^0_{j,j+1}$$

implements local moves

$$0l \leftrightarrow l0 \quad 0r \leftrightarrow r0$$

Π^0 projects onto

$$|0l\rangle - |l0\rangle$$

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Spectral gap $1/n^2$

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Spectral gap $1/n^2$

$$V = \sum_j \Pi^{int}_{j,j+1}$$

implements local moves

$$00 \leftrightarrow lr$$

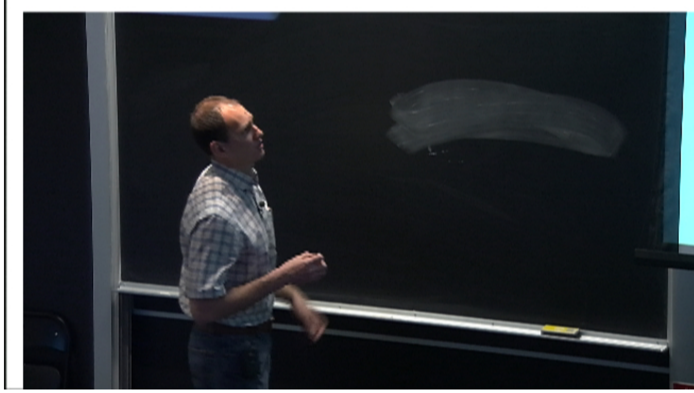
Π^{int} projects onto

$$|00\rangle - |lr\rangle$$

Define $H_\epsilon = H_0 + \epsilon V, \qquad 0 < \epsilon \leq 1$

Ground subspace of Hε does not depend on ε for ε>0

$\mathrm{gap}(H) \geq \mathrm{gap}(H_\epsilon)$ because $H \geq H_\epsilon$



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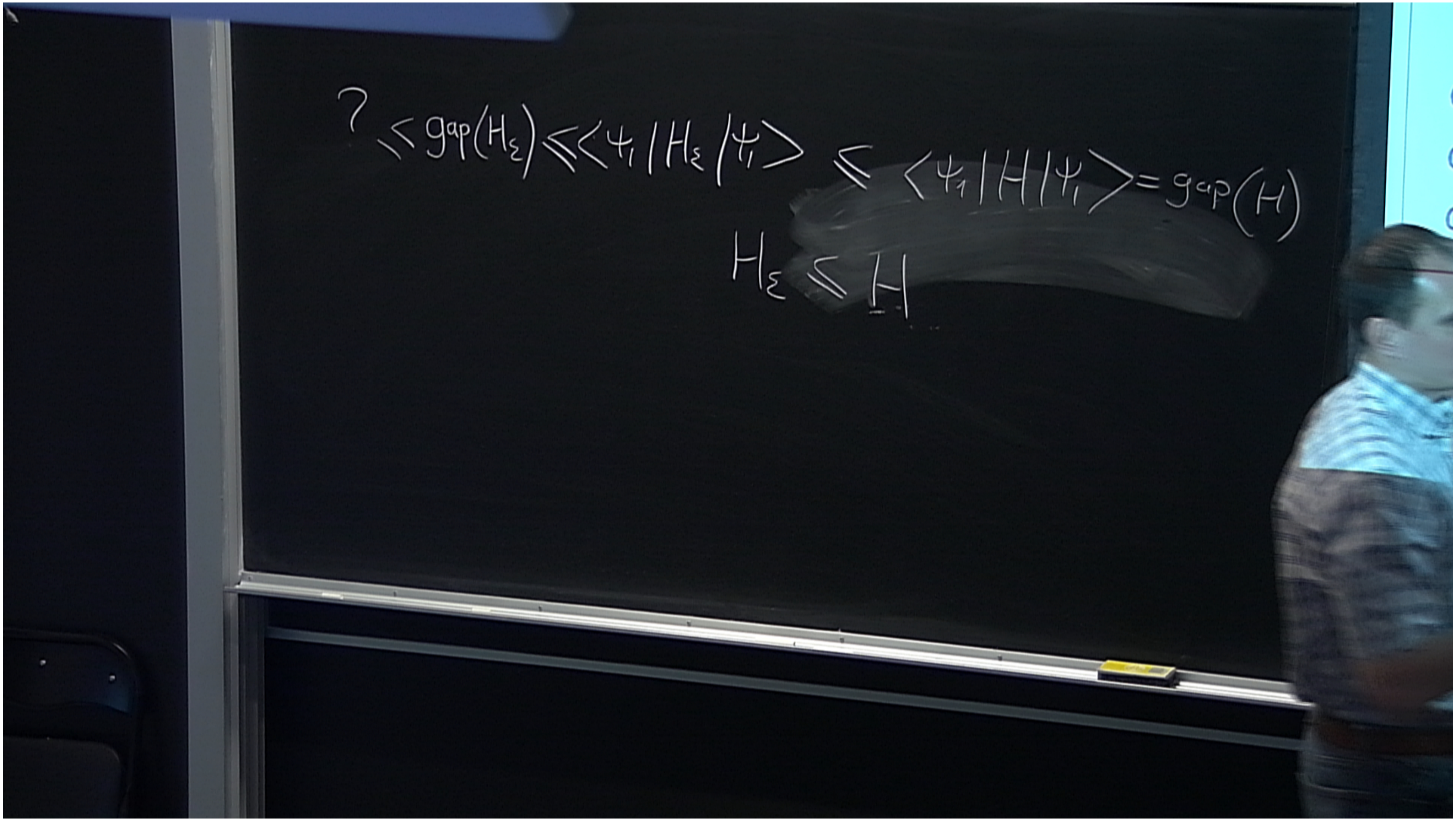
Projection Lemma [KKR 04] Define the first-order effective Hamiltonian

$$H_{\text{eff}} = \Pi_0 V \Pi_0$$

acting on the ground subspace of H_0. If the spectral gaps of H_0 and H_{eff} are polynomial in 1/n then the spectral gap of Hε is also polynomial in 1/n for sufficiently small ε.


$? \leq \mathrm{gap}(H_\varepsilon) \leq \langle \psi_1 | H_\varepsilon | \psi_1 \rangle \leq \langle \psi_1 | H | \psi_1 \rangle = \mathrm{gap}(H)$
$H_\varepsilon \leq H$

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Sufficiently small: $$\epsilon \sim \frac{1}{\|V\|} \min\left\{\mathrm{gap}(H_0), \mathrm{gap}(H_{\mathrm{eff}})\right\}$$

Ground states of H_0 = Dyck words (balanced strings of left and right brackets with no zeros)

$\boxed{\emptyset}$

$\boxed{lr}$

$\boxed{llrr}$ $\boxed{lrlr}$

$\boxed{lllrrr}$ $\boxed{llrlrr}$ $\boxed{llrrlr}$ $\boxed{lrllrr}$ $\boxed{lrlrlr}$

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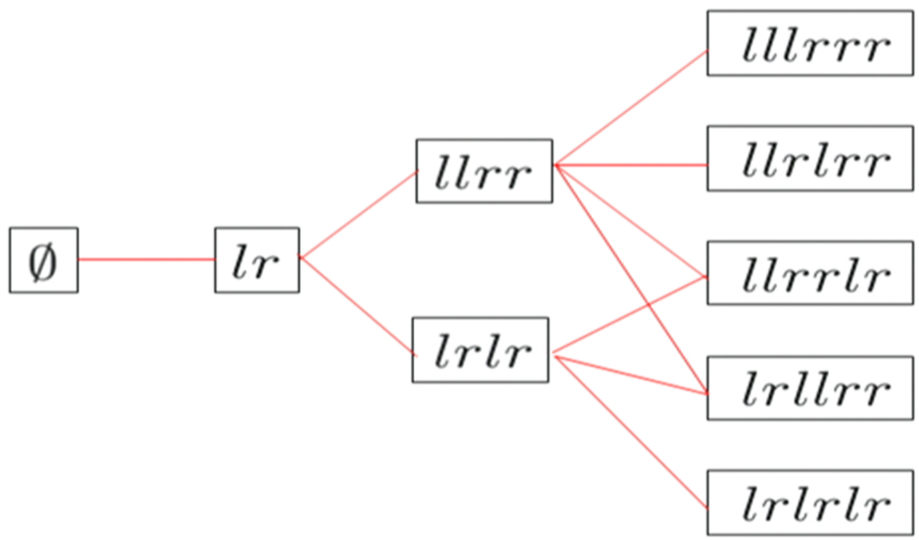

Dyck graph:

Vertices = Dyck words

Edges = insertions/removals of consecutive lr pairs

First-order effective Hamiltonian H_{eff} describes a random walk on the Dyck graph

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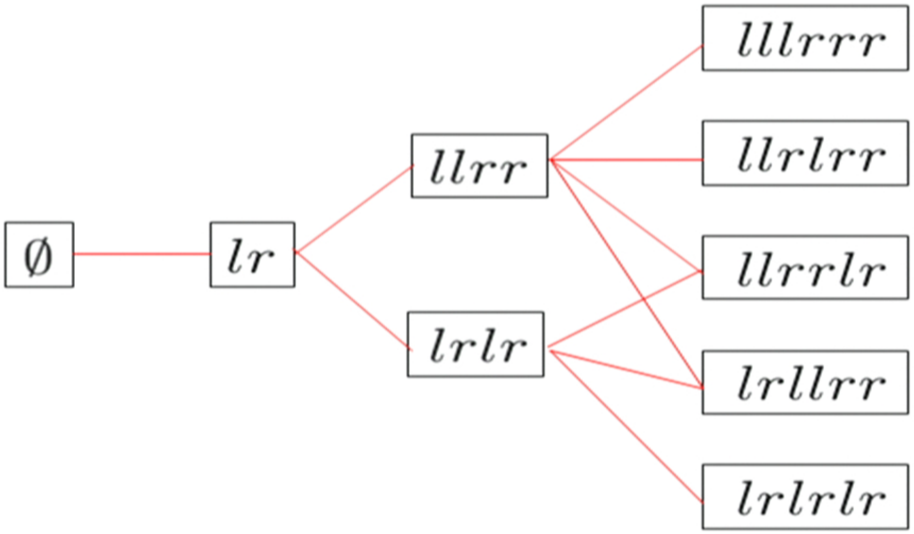

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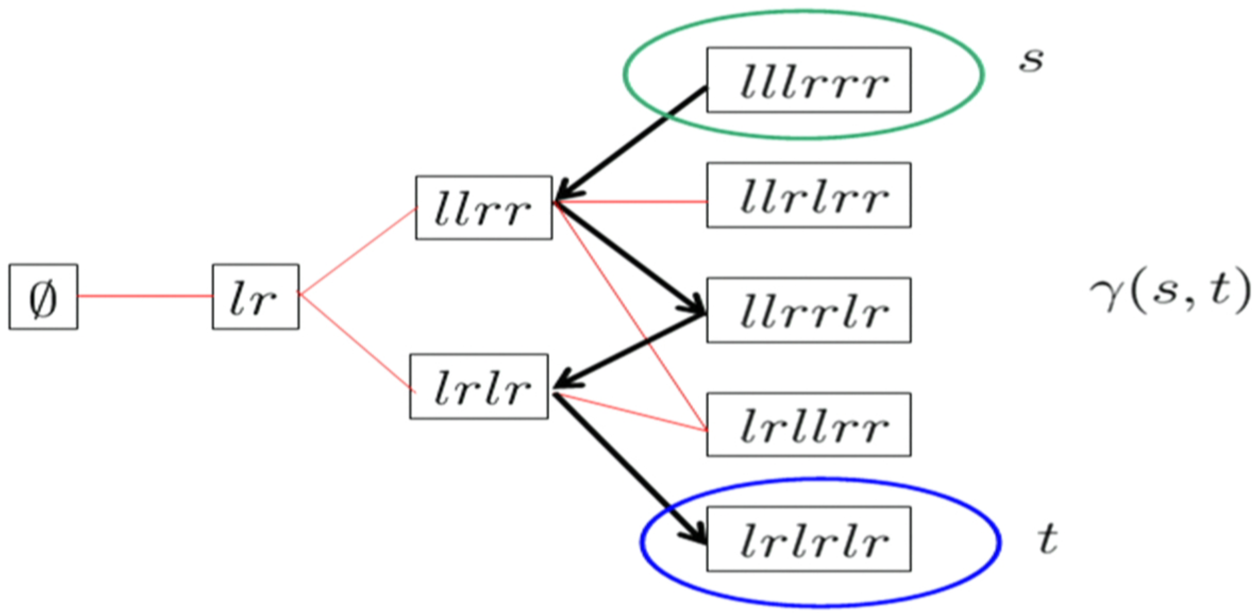
First-order effective Hamiltonian H_{eff} describes a random walk on the Dyck graph

It suffices to prove the rapid mixing property of the walk.

Step 2: Use the canonical paths theorem [Sinclair 1992] to prove the rapid mixing property.

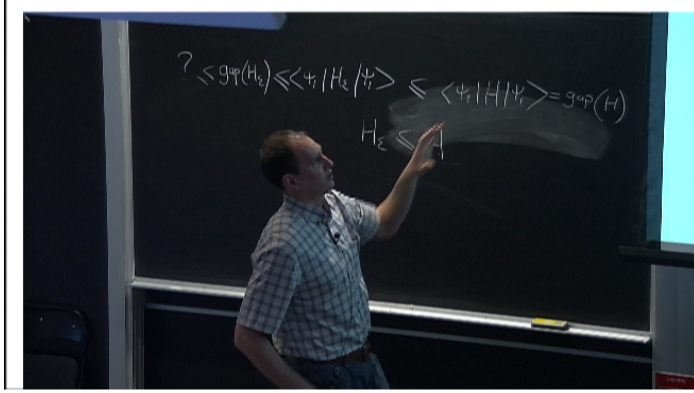
We need to connect any pair of vertices s, t on the Dyck graph by a canonical path $\gamma(s,t)$ such that no edge of the graph is used by too many paths.



Maximum edge load	$\rho = \max\limits_{(a,b)\in E} \dfrac{1}{\pi(a)P(a,b)} \sum\limits_{s,t\,:\,(a,b)\in\gamma(s,t)} \pi(s)\pi(t).$

$\pi(a)$ - steady state of the walk

$P(a,b)$ - transition probability from a to b



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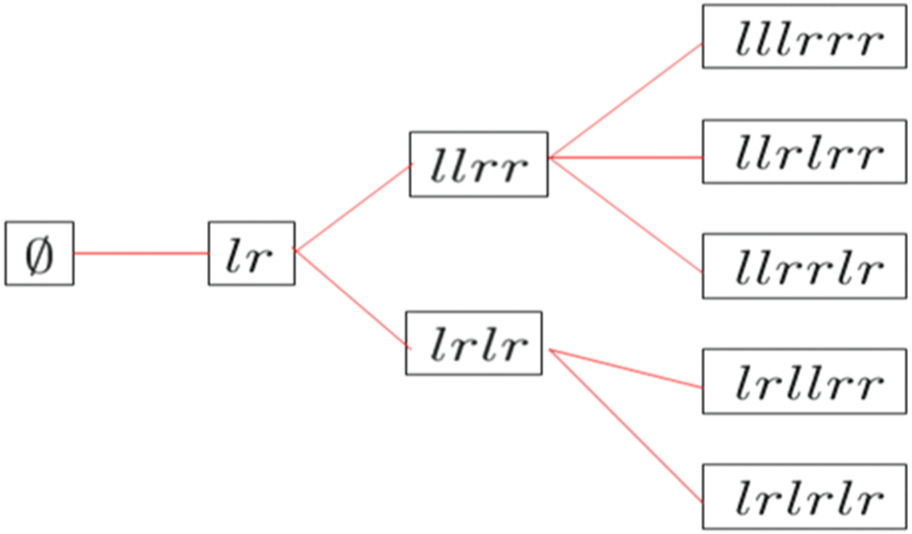
Canonical paths theorem [Sinclair 1992]
The spectral gap Δ of the walk P has a lower bound

$$\Delta \geq \frac{1}{\rho\, l_{max}}$$

where l_{max} is the maximum length of a canonical path.

New result. The Dyck graph has a spanning tree T with at most four children per node. All Dyck words of length 2k appear at the level-k of T.



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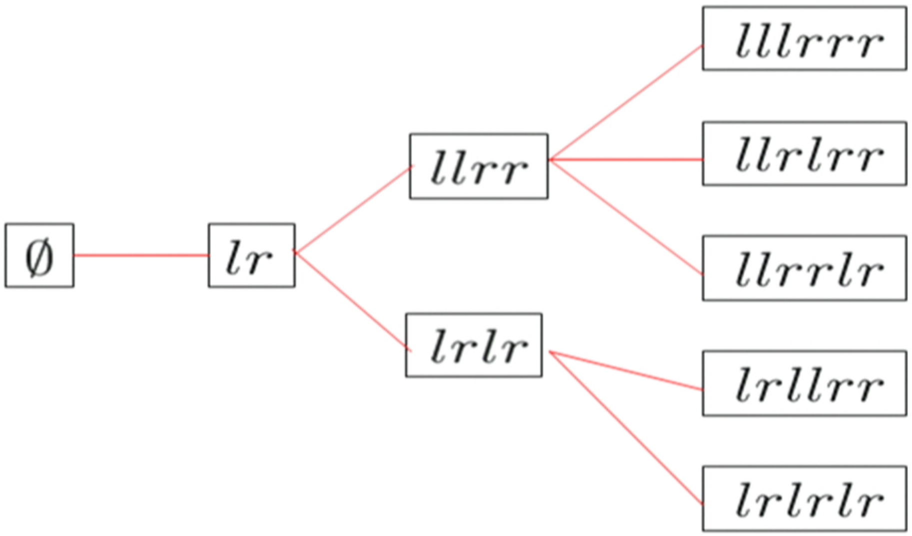

The number of Dyck words of length 2k is the Catalan number

$$C_k = \frac{1}{k+1}\binom{2k}{k} \sim \frac{4^k}{\sqrt{k}}$$

Four children per node is optimal !

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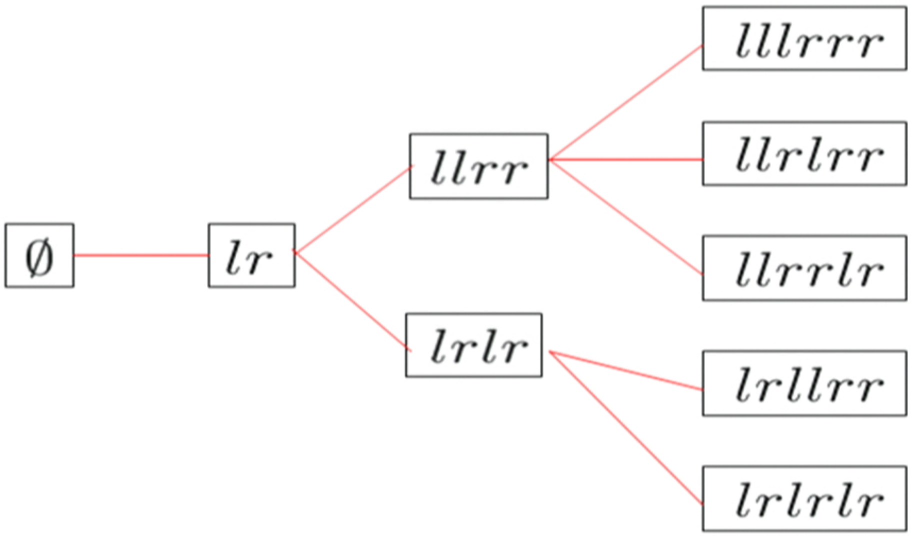

Non-constructive proof based on polyhedral description of matchings in bipartite graphs

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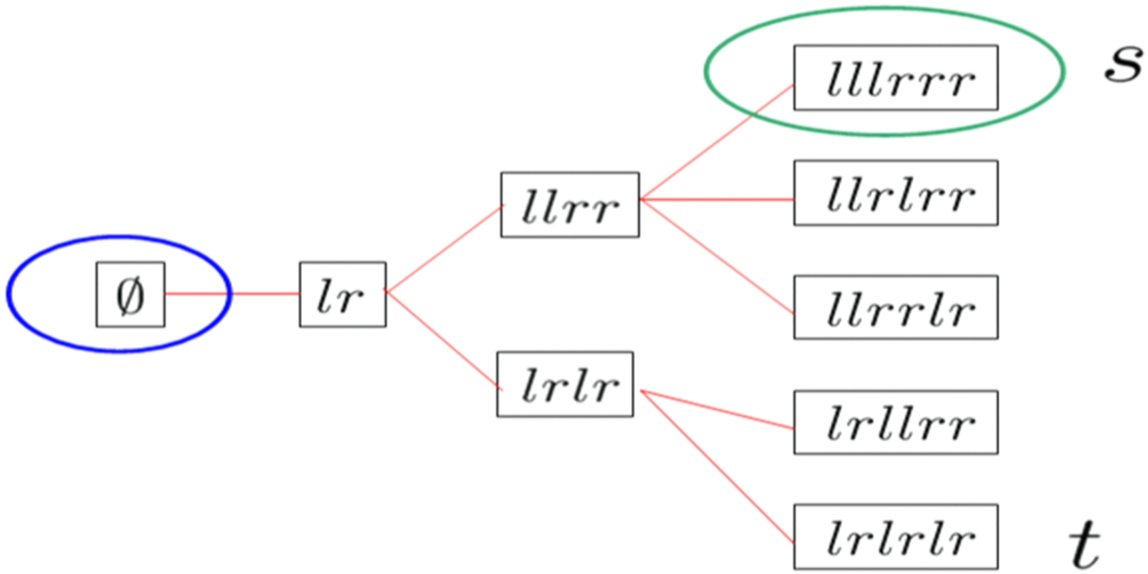
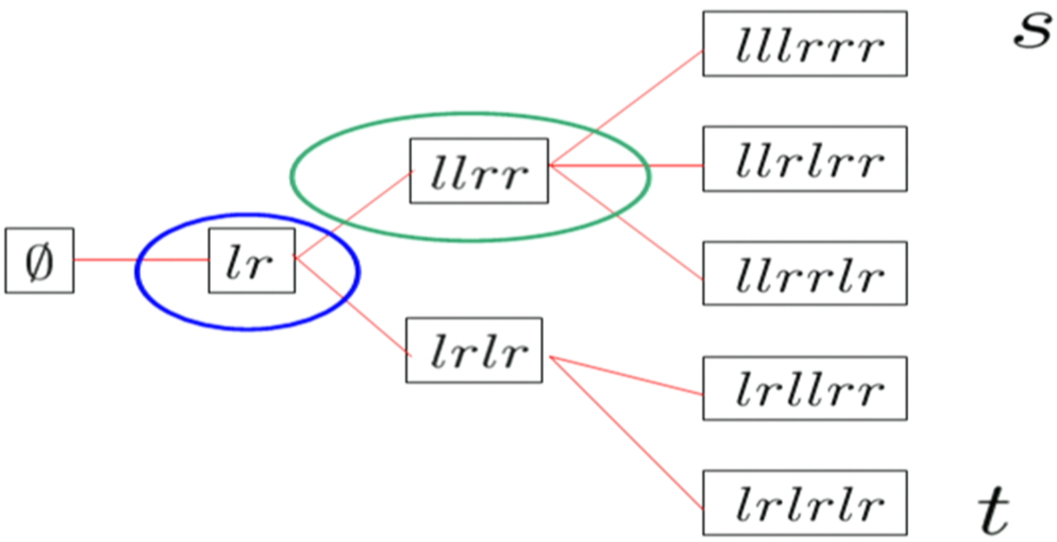
Canonical path from s to t



Start at s

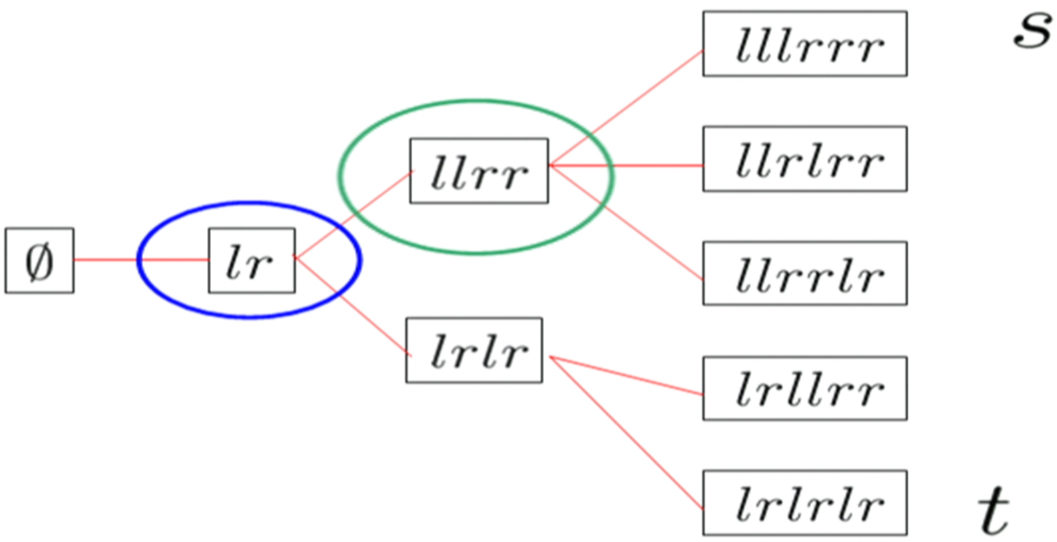
u	v
$lllrrr$	$\emptyset$

Canonical path from s to t



Grow v

Canonical path from s to t



Shrink u

Maximum edge load $$\rho = \max_{(a,b)\in E} \frac{1}{\pi(a)P(a,b)} \sum_{s,t\,:\,(a,b)\in\gamma(s,t)} \pi(s)\pi(t).$$

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For fixed length of s,t,u,v the number of terms in the sum is at most

$4^{|s|-|u|+|t|-|v|}$ (each node has at most four children)

The number of Dyck paths of length 2n is 4^n/poly(n).

$$4^{|s|-|u|+|t|-|v|} \cdot \frac{\pi(s)\pi(t)}{\pi(a)} = poly(n)$$

Conclusions

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Open problems:

Generalization to two types of brackets [ ] and { }

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http://xkcd.com/859/