Title: Twisted Lattice Supersymmetry and Applications to Quantum Gravity

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Abstract: I will review the construction of lattice theories which maintain one or more exact supersymmetries for non zero lattice spacing concentrating in particular on the case of N=4 super Yang-Mills. Such lattice theories may be studied using Monte Carlo techniques borrowed from lattice QCD and can be used to explore issues in holography. In three dimensions the same constructions can be used to formulate a topological theory of gravity which we argue is equivalent to Witten's Chern Simons theory.

New developments in lattice SUSY

Examples of the lattice construction - $\mathcal{N} = 4$ YM

Applications: holography and quantum gravity

Future

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Future

Why lattice SUSY ?

- Non-perturbative definition of supersymmetric (gauge) theories - like QCD = lattice QCD for $a \to 0$, $V \to \infty$
- New tools e.g. strong coupling, Monte Carlo for uncovering non-perturbative physics .. eg. dynamical SUSY breaking
- Connections to quantum gravity ?

Lattice SUSY - the problem

- Lattice contains no infinitessimal translations .... SUSY algebra $\{Q, \bar{Q}\} = \gamma.p$ broken at classical level
  Equivalently: Leibniz rule does not hold for difference operators
- Consequence: generic discretizations of SUSY theories lead to quantum effective actions with (many) SUSY violating terms.
- Couplings to relevant SUSY breaking ops must be fine tuned as $a \to 0$
  Hard/unnatural ...

Solution: twisted SUSY

- Can twist theory to produce nilpotent supercharge $Q^2 = 0$.
- Twisted action typically takes form $S = Q\Lambda(\mathcal{A}, \psi)$
- Thus discretization which preserves nilpotent Q will retain exact SUSY.
- Careful choice of Λ can also preserve gauge invariance and avoid fermion doubling ...
- Great deal of work in this direction: Kaplan, Ünsal, Sugino, Tsuchiya, Hanada, Damgaard, Matsuura, Kawamoto, d'Adda, Giedt, Kanamori, Catterall,...
- Physics Report: arXiv:0903.4881 (S.C, D. B. Kaplan, M. Ünsal) for more details ....

Simple Example

Example: 2D Yang-Mills with $\mathcal{Q} = 4$ SUSY:

- Contains 2 fermions λ^i_α transforming under $SO_{\rm Lorentz}(2) \times SO_{\rm Flavor}(2)$

$$\lambda^i_\alpha \to L_{\alpha\beta}\lambda^j_\beta(F^T)^{ji}$$

Under diagonal subgroup $L = F$ λ^i_α transforms like matrix - Ψ!

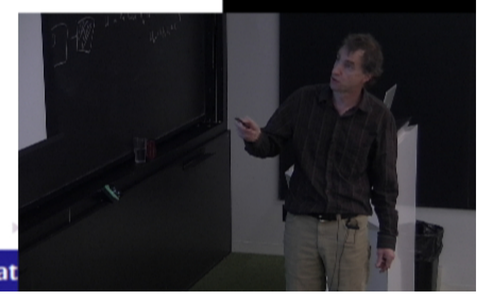
- Natural to decompose on products of γ (here Pauli) matrices

$$\Psi = \eta I + \psi_i\sigma_i + \chi_{12}\sigma_1\sigma_2$$

Integer spin p-form fields from spinors Twisting!
Appearance of scalar fermion ... implies scalar SUSY $\mathcal{Q}$.

What about the bosons ?

- Gauge field A_μ is singlet under flavor: remains vector under twisted rotation group $SO(2)' = \text{diag}(SO_L(2) \times SO_F(2))$
- Original theory had 2 scalars ϕ^1, ϕ^2 which transformed in vector rep of flavor – become components of vector B_μ under twisted rotations!!
- In fact all bosons in twisted theory get repackaged as complex gauge field $\mathcal{A}_\mu = A_\mu + iB_\mu$



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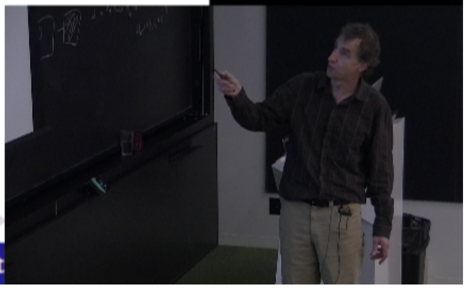
Twisted action and SUSY

Twisted form of 2d action (adjoint fields with AH generators)

$$S = \frac{1}{g^2}\mathcal{Q}\int \mathrm{Tr}\left(\chi_{\mu\nu}\mathcal{F}_{\mu\nu} + \eta[\overline{\mathcal{D}}_\mu, \mathcal{D}_\mu] - \frac{1}{2}\eta d\right)$$

$$\begin{aligned}
\mathcal{Q}\,\mathcal{A}_\mu &= \psi_\mu \\
\mathcal{Q}\,\psi_\mu &= 0 \\
\mathcal{Q}\,\overline{\mathcal{A}}_\mu &= 0 \\
\mathcal{Q}\,\chi_{\mu\nu} &= -\overline{\mathcal{F}}_{\mu\nu} \\
\mathcal{Q}\,\eta &= d \\
\mathcal{Q}\,d &= 0
\end{aligned}$$

Note: $\mathcal{D}_\mu = \partial_\mu + \mathcal{A}_\mu$, $\overline{\mathcal{D}}_\mu = \partial_\mu + \overline{\mathcal{A}}_\mu$ $\mathcal{F}_{\mu\nu} = [\mathcal{D}_\mu, \mathcal{D}_\nu]$



Untwisting

$\mathcal{Q}$-variation, integrate d:

$$S = \frac{1}{g^2} \int \mathrm{Tr} \left(-\overline{\mathcal{F}}_{\mu\nu} \mathcal{F}_{\mu\nu} + \frac{1}{2} [\overline{\mathcal{D}}_\mu, \mathcal{D}_\mu]^2 - \chi_{\mu\nu} \mathcal{D}_{[\mu} \psi_{\nu]} - \eta \overline{\mathcal{D}}_\mu \psi_\mu \right)$$

Rewrite as

$$S = \frac{1}{g^2} \int \mathrm{Tr} \left(-F_{\mu\nu}^2 + 2 B_\mu D_\nu D_\nu B_\mu - [B_\mu, B_\nu]^2 + L_F \right)$$

where

$$L_F = \begin{pmatrix} \chi_{12} & \frac{\eta}{2} \end{pmatrix} \begin{pmatrix} -D_2 - iB_2 & D_1 + iB_1 \\ D_1 - iB_1 & D_2 - iB_2 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

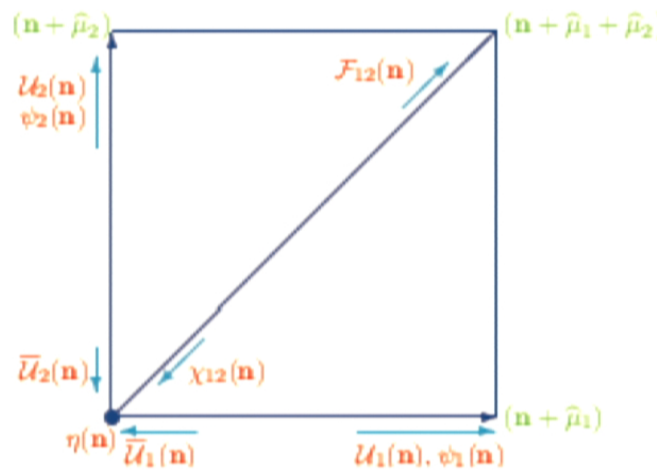
Side comment(s)

- Appearance of $\mathcal{Q}$ reminiscent of BRST gauge fixing. This is not a coincidence ...
- vevs of $\mathcal{Q}$ invariant operators independent of coupling and metric. Topological - Useful - sector of lattice theory which can be computed exactly
- Important - do not restrict to $\mathcal{Q}$-invariant states (vacuum of SUSY theory) - think of twisting as change of variables - more suitable for discretization ...

Discretization

Twisted fields assigned to links of lattice. Under GTs transform like

$$f_p(x) \to G(x) f_p(x) G^\dagger(x + \mu_p)$$



More lattice construction

- Lattice SUSY transformations same as continuum
- Precise dictionary exists to translate $\mathcal{D}_\mu$ to difference ops. Eg

$$\mathcal{D}^{(+)}_\mu \psi_\nu(x) = \mathcal{U}_\mu(x)\psi_\nu(x+\mu) - \psi_\nu(x)\mathcal{U}_\mu(x+\nu)$$

Letting $\mathcal{U}_\mu(x) = I + \mathcal{A}_\mu(x)$ yields:

$$\mathcal{D}^{(+)}_\mu \psi_\nu = \psi_\nu(x+\mu) - \psi_\nu(x) + [\mathcal{A}_\mu(x), \psi(x)] + O(a)$$

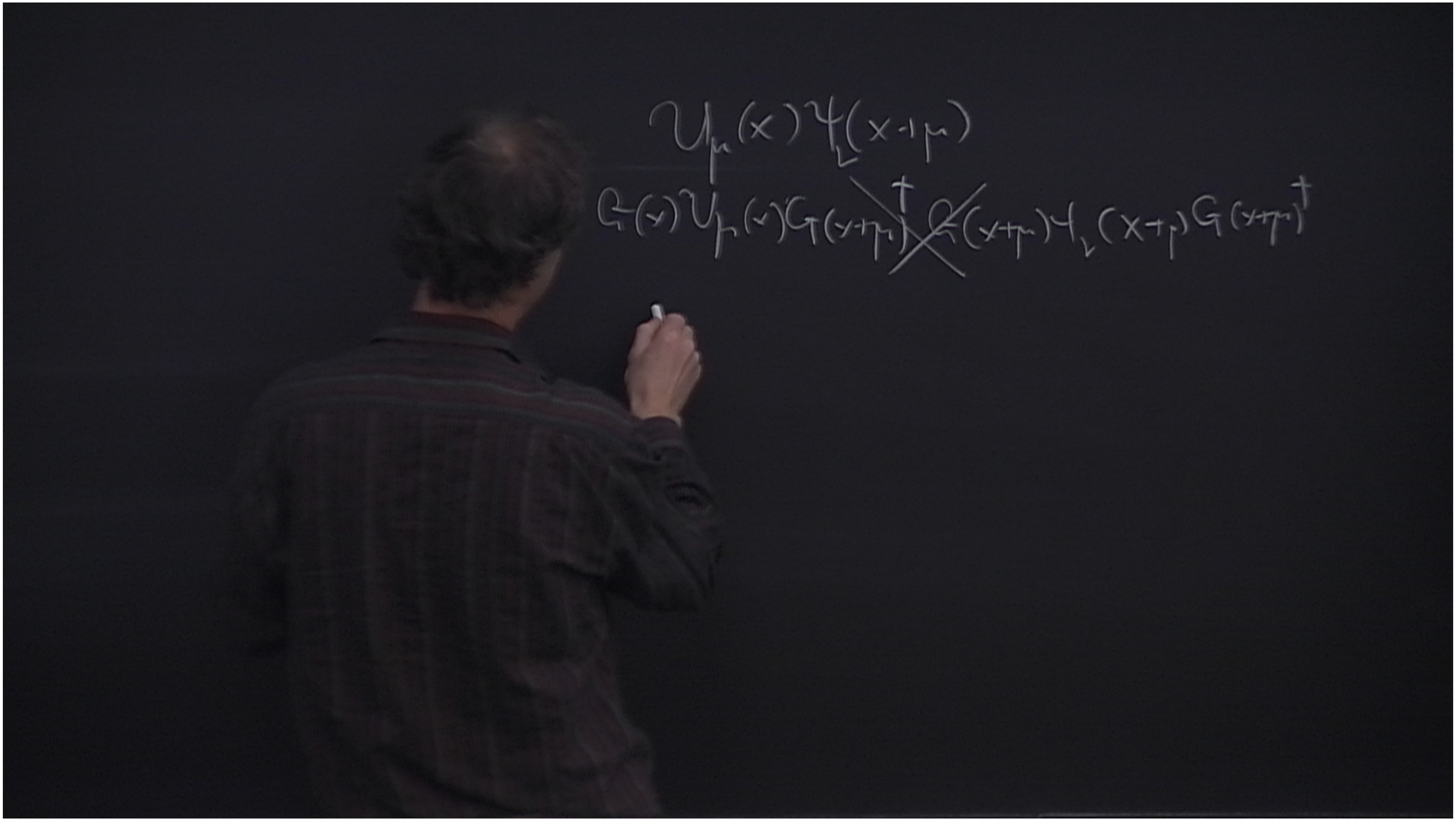
Derivative of lattice 1 form yields lattice 2 form !

$$\mathcal{F}_{\mu\nu} = \mathcal{D}^+_\mu \mathcal{U}_\nu(x)$$

- Fermion doubling evaded if (Rabin, Joos, ...)

$$\mathcal{D}_\mu \overset{curl}{\rightarrow} \mathcal{D}^+_\mu$$

$$\mathcal{D}_\mu \overset{div}{\rightarrow} \mathcal{D}^-_\mu$$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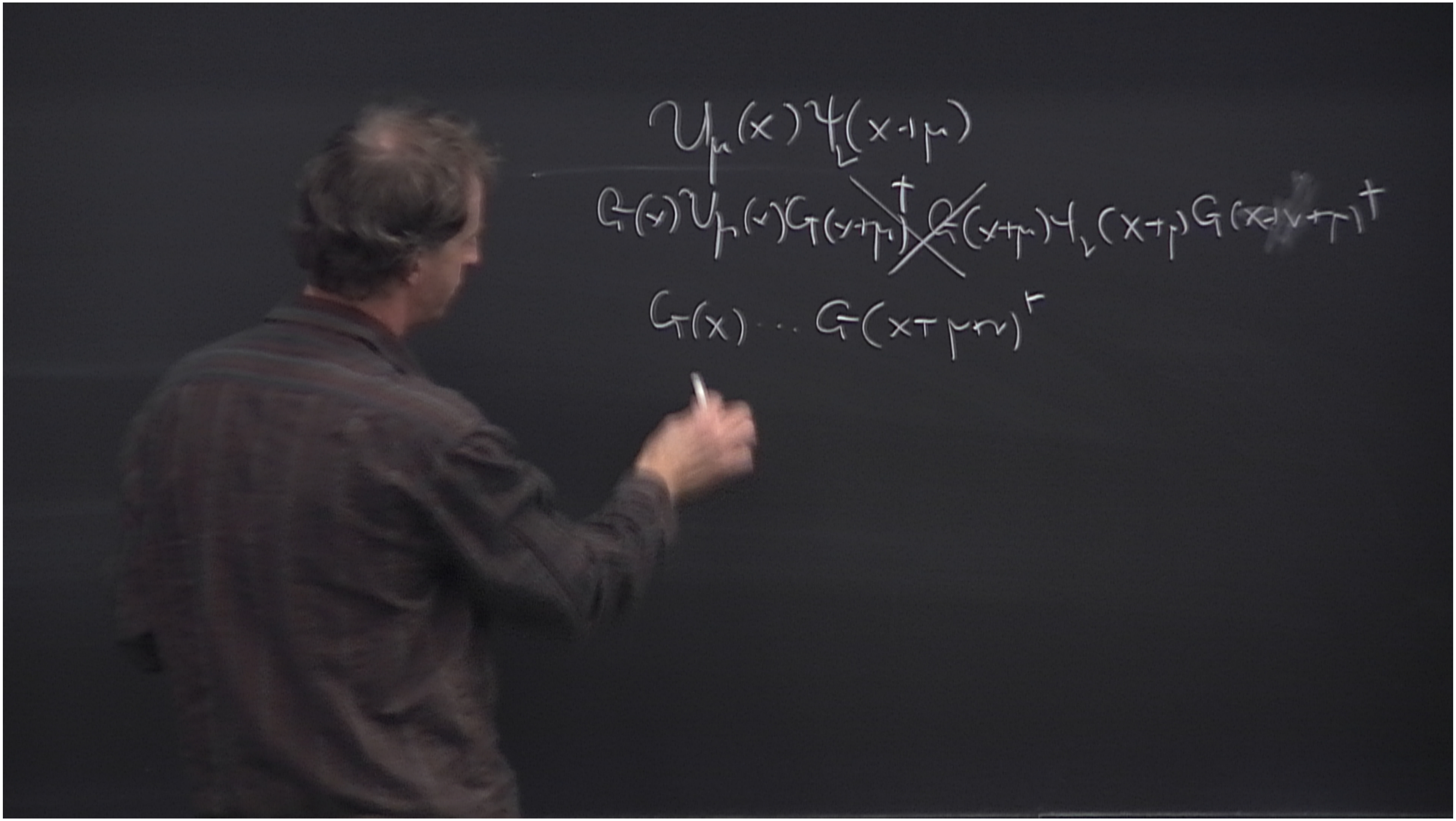

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$$U_\mu(x)\psi_\nu(x+\mu)$$

$$G(x)U_\mu(x)G(x+\mu)^\dagger G(x+\mu)\psi_\nu(x+\mu)G(x+\mu+\nu)^\dagger$$

$$G(x)\cdots G(x+\mu+\nu)^\dagger$$

transforms like lattice 2-form

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Twisted $\mathcal{N} = 4$ SYM

- Decompose fields under twisted rotation group $SO(4)' = \mathrm{diag}(SO_F(4) \times SO_{Lorentz}(4))$ – 4×4 matrix of fermions $\rightarrow$
  Twisted fermions: $(\eta, \psi_\mu, \chi_{\mu\nu}, \overline{\psi}_\mu, \overline{\eta})$. Bosons: $(A_\mu, B_\mu, \phi, \overline{\phi})$
- Compactly expressed as dimensional reduction of 5D theory
  - 16 fermions: $\Psi = (\eta, \psi_m, \chi_{mn}), m, n = 1 \ldots 5$
  - 10 bosons as 5 complex gauge fields $\mathcal{A}_m, m = 1 \ldots 5$
- Twisted scalar SUSY acts as for 2d YM
- Action $S = \mathcal{Q} \int \left(\chi_{ab} F_{ab} + \eta[\overline{\mathcal{D}}_a, \mathcal{D}_a] - 1/2\eta d\right) + S_{\rm closed}$
- $S_{\rm closed} = \frac{1}{8} \int \epsilon_{abcde} \chi_{ab} \overline{\mathcal{D}}_c \chi_{de}$

(Almost) same as 2D example !

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Lattice $\mathcal{N} = 4$ theory

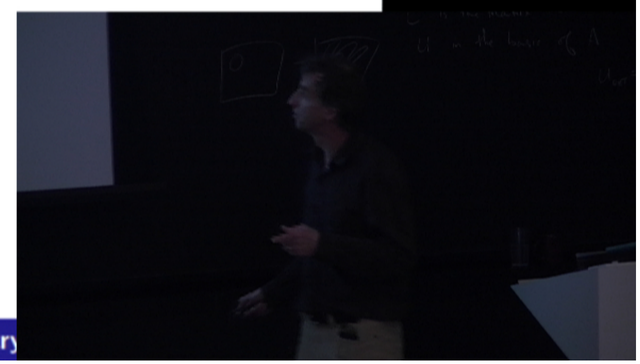
- Five complex gauge fields require a lattice with 5 (linearly dep) basis vectors - A_4^* lattice
  - $\mathcal{U}_a \quad x \to x + \mu_a$
  - $\eta \quad x \to x$
  - $\psi_a \quad x \to x + \mu_a$
  - $\chi_{ab} \quad x + \mu_a + \mu_b \to x$
- All fields transform as link objects eg: $\psi_a \to G(x)\, \psi_a\, G^\dagger(x + \mu_a)$
- Single exact lattice SUSY $\mathcal{Q}^2 = 0$
- Prescription for derivatives same as for 2d YM

Lattice Bianchi

- Supersymmetry of $\mathcal{Q}$-exact piece trivially true (as for 2d)
- Remarkably discretization of $\mathcal{Q}$-closed piece also invariant since

$$\epsilon_{abcde}\mathcal{D}^{(+)}_a\mathcal{F}_{bc} = 0$$

Exact lattice Bianchi identity !



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Conclusions

- Lattice theories with exact SUSY possible. In particular $\mathcal{N} = 4$ SYM with $U(N)$ gauge symmetry
- Key is to discretize topologically twisted form of continuum theory.
- Exact SUSY enough to ensure moduli space survives to all orders and no fine tuning at 1-loop (arXiv:1102.1725)
- Subtleties remain: necessary to add gauge invariant potential to freeze trace mode and regularize flat directions.
- Possible sign problem in $\mathcal{N} = 4$. Numerically seems safe ... theoretically - partition function is topological invariant (arXiv:1112.3588)
- Exploration of phase diagram underway...

Applications

- Holography. Clearly having a discrete regularization for $\mathcal{N} = 4$ YM allows for non-perturbative tests of AdS/CFT conjecture.
  - Finite N and λ allow for calculations of classical and quantum string corrections ..
  - Insight into nature of quantum geometry and quantum gravity ....
  - Eg. black D0, D1 branes and $(0+1)$ and $1+1$ SYM (arXiv:1008.4964,arXiv:0803.4273)
- $\mathcal{N} = 4$ theory in 3D provides an alternative description of 3D de Sitter quantum gravity equivalent to usual CS theory ...

Chern-Simons theory - quick reminder

In three dimensions can write alternative to Yang-Mills

$$S_{\rm CS} = k \int_M d^3x \, \epsilon^{\mu\nu\lambda} {\rm Tr} \left(A_\mu F_{\nu\lambda} - \frac{1}{3} A_\mu \left[A_\nu A_\lambda \right] \right)$$

- Gauge invariant: under $\delta A_\mu = D_\mu \phi$ find $\delta S_{\rm CS} = \text{boundary term}$ (Bianchi identity). (caveat: large transformations can quantize k)
- Renormalizable. In fact finite !
- $S_{\rm CS}$ independent of background metric – topological

Observables correspond to Polyakov lines

$$W(\gamma) = {\rm Tr} \mathcal{P} \int_\gamma e^{A_\mu dx^\mu}$$

3D Gravity as CS theory

Starting from CS action:

$$S_{\rm CS} = k \int d^3x \epsilon^{\mu\nu\lambda} \hat{\rm Tr} \left(A_\mu F_{\nu\lambda} - \frac{1}{3} A_\mu \left[A_\nu, A_\lambda \right] \right)$$

Choose gauge group $SO(1,3)$ (Euclidean model):

$$A_\mu = \sum_{A<B} A^{AB}_\mu \frac{1}{4} \left[\gamma^A, \gamma^B \right] \quad A, B = 1 \dots 4$$

and use modified trace

$$\hat{\rm Tr}(X) = {\rm Tr}(\gamma_5 X)$$

This has effect of contracting indices using

$$Tr(\gamma_5 \gamma_A \gamma_B \gamma_C \gamma_D) = \epsilon_{ABCD}$$

Continuing

Decompose fields into Lorentz and translational components:

$$A_\mu = \sum_{a<b} \omega_\mu^{ab}\gamma^{ab} + \frac{1}{l_C} e_\mu^a \gamma^{4a} \quad a,b = 1\ldots 3$$

$$F_{\mu\nu}^{ab} = \sum_{a<b} \left(R_{\mu\nu}^{ab} + \frac{1}{l_C^2} e_{[\mu}^a e_{\nu]}^b \right)$$

$$F_{\mu\nu}^{a} = \frac{1}{l_C} \sum_a D_{[\mu} e_{\nu]}^a$$

Plugging into CS action:

$$S_{\rm EH} = \frac{k}{l_C} \int d^3x\, \epsilon^{\mu\nu\lambda} \epsilon_{abc} \left(e_\mu^a R_{\nu\lambda}^{bc} + \frac{1}{l^2} e_\mu^a e_\nu^b e_\lambda^c \right)$$

Tetrad-Palatini formulation of 3D GR!

Why is this GR ?

- Interpret e^a_μ as dreibein with $e^a_\mu e^a_\nu = g_{\mu\nu}$ (invariant under local $SO(3)$ Lorentz symmetry)
- Corresponding gauge field is spin connection ω_μ.
- First order formulation of GR – (ω, e) independent variables.
- Classical equations of motion set $F^{AB}_{\mu\nu} = 0$ or

$$\left(R^{ab}_{\mu\nu} + \frac{1}{l_C^2} e^a_{[\mu} e^b_{\nu]}\right) = 0$$

$$T^a_{\mu\nu} = \frac{1}{l_C} D_{[\mu} e^a_{\nu]} = 0$$

- Torsion free condition $T = 0$ yields $\omega = \omega(e)$
- Then curvature equation yields constant curvature space $\mathcal{H}^3$. $R = e^\mu_a e^\nu_b R^{ab}_{\mu\nu}(\omega(e)) = -\frac{1}{l_C^2}$ where $(e^a_\mu)^{-1} = e^\mu_a$

What about diffeomorphisms ?

- The gravity theory is invariant under general coordinate transformations - where are these in CS theory ?
- Latter possesses local $SO(3)$ Lorentz symmetry plus local translations (e_μ gauge field)
- Remarkably can show that on shell the gauge symmetries are equivalent to coordinate transformations!

$$\delta_\xi e^a_\mu = D_\mu(\xi^\nu A_\nu)^a - T^a_{\mu\nu}\xi_\nu$$

- Thus on flat $SO(1,3)$ background gauge invariance yields coordinate invariance!

An Alternative

- CS are hard (impossible ?) to put on the lattice.
- But $SO(3,1)$ CS theory shares a moduli space and a set of topological observables with a twisted YM theory:

$$\boxed{\text{3D twisted } \mathcal{N} = 4 \text{ SU(2) YM}}$$

- The path integral defining this YM theory has is well defined outside of perturbation theory and possesses a lattice description which preserves the topological features
- It hence potentially constitutes a non-perturbative definition of 3d quantum gravity which can be studied numerically ...

Equivalence

Moduli space YM model:

$$\mathcal{F}_{\mu\nu} = 0$$
$$\left[\overline{\mathcal{D}}_\mu, \mathcal{D}^\mu\right] = D^\mu B_\mu = 0 \quad \mathrm{mod}\ \mathrm{SU}(2)\ \mathrm{GT}$$

Moduli space CS theory:

$$\mathcal{F}_{\mu\nu} = 0 \quad \mathrm{mod}\ \mathrm{SL}(2,\mathrm{C})\ \mathrm{GT}$$

Same! (Marcus)

Furthermore:

- Since $\mathcal{Q}\overline{\mathcal{A}}_\mu = 0$ Polyakov lines $e^{\int \overline{\mathcal{A}}.dx}$ in YM will be topological - independent of metric and coupling constant
- These are same operators as appeared in CS gravity!

Untwisting

$\mathcal{Q}$-variation, integrate d:

$$S = \frac{1}{g^2}\int \mathrm{Tr}\left(-\overline{\mathcal{F}}_{\mu\nu}\mathcal{F}_{\mu\nu} + \frac{1}{2}[\overline{\mathcal{D}}_\mu, \mathcal{D}_\mu]^2 - \chi_{\mu\nu}\mathcal{D}_{[\mu}\psi_{\nu]} - \eta\overline{\mathcal{D}}_\mu\psi_\mu\right)$$

Rewrite as

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Conjecture

Plausible:

Twisted SUSY theory gives alternative description of CS gravity

Bonus:

- YM action real, positive semi-definite.
- Only compact gauge symmetry. Measure well defined
- Gauge invariant lattice formulation possible preserving $\mathcal{Q}$-invariance
- It may hence offer an alternative non-perturbative formulation of 3d gravity.

Thoughts for the future

- Continuum CS formulations of gravity exist in any odd dim.
- Eg in 5D
  - $S = \int_{M^5} \hat{\mathrm{Tr}}\ (A \wedge F \wedge F + \ldots)$
  - Related to some twisted 5D theory ? (remember: twisted $\mathcal{N}=4$ YM embeds naturally in 5D)
- Gauge invariant CS-like lattice action exists on A_5^* lattice:

$$
\begin{aligned}
S &= \sum_x \epsilon_{abcdef} \mathcal{U}_a(x)\mathcal{U}_b(x+a)\mathcal{U}_c(x+a+b)\mathcal{U}_d(x+a+b+c) \\
&\times \mathcal{U}_e(x+a+b+d)\mathcal{U}_f(x+a+b+c+d+e)
\end{aligned}
$$

Naive continuum limit: $\int_{\mathcal{M}^5} \epsilon_{abcde} \mathcal{F}_{ab}\mathcal{F}_{cd}\mathcal{D}_e\phi = \int_{\partial\mathcal{M}} \phi\mathcal{F}\wedge\mathcal{F}$
MacDowell-Mansouri gravity on 4D boundary ...