Title: The Information-theoretic Costs of Simulating Quantum Measurements

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Abstract: Winter's measurement compression theorem stands as one of the most important, yet perhaps less well-known coding theorems in quantum information theory. Not only does it make an illuminative statement about measurement in quantum theory, but it also underlies several other general protocols used for entanglement distillation or local purity distillation. The theorem provides for an asymptotic decomposition of any quantum measurement into an "extrinsic" source of noise, classical noise in the measurement that is independent of the actual outcome, and "intrinsic" quantum noise that can be due in part to the nonorthogonality of quantum states. This decomposition leads to an optimal protocol for a sender to 1) simulate many instances of a quantum measurement acting on many copies of some state and 2) send the outcomes of the measurements to a receiver using as little classical communication as possible while still having a faithful simulation. The protocol assumes that the parties have access to some amount of common randomness, which is a strictly weaker resource than classical communication.   In this talk, we provide a full review of Winter's measurement compression theorem, detailing the information processing task, providing examples for understanding it, overviewing Winter's achievability proof, and detailing a new approach to its single-letter converse theorem. We then overview the Devetak-Winter theorem on classical data compression with quantum side information, providing new proofs of the achievability and converse parts of this theorem. From there, we outline a new protocol that we call "measurement compression with quantum side information," a protocol announced in prior work on trade-offs in quantum Shannon theory. This protocol has several applications, including its part in the "classically-assisted state redistribution" protocol, which is the most general protocol on the static side of the quantum information theory tree, and its role in reducing the classical communication cost in a task known as local purity distillation. We finally outline a connection between this protocol and recent work on entropic uncertainty relations in the presence of quantum memory.   This is joint with Patrick Hayden, Francesco Buscemi, and Min-Hsiu Hsieh.

Information-Theoretic Costs of Simulating Quantum Measurements

Mark M. Wilde

McGill University

Joint with
Patrick Hayden, **Francesco Buscemi**, and **Min-Hsiu Hsieh**

Quantum Information Seminar,
Perimeter Institute, Waterloo, Canada
April 18, 2012

Overview

- Review Winter's measurement compression protocol

- Introduce a new variation of the protocol
- Review classical data compression with QSI
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•Review Winter's measurement compression protocol

•Introduce a new variation of the protocol

•Review classical data compression with QSI

•Introduce measurement compression with QSI

•Outline some applications of MC-QSI

POVMs

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Quantum state is a positive, unit trace operator ρ

$$\rho \geq 0, \quad \mathrm{Tr}\{\rho\} = 1$$

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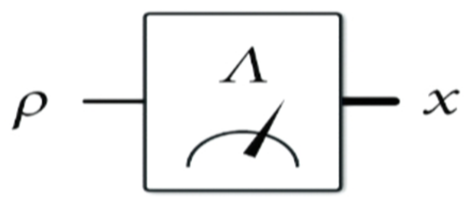
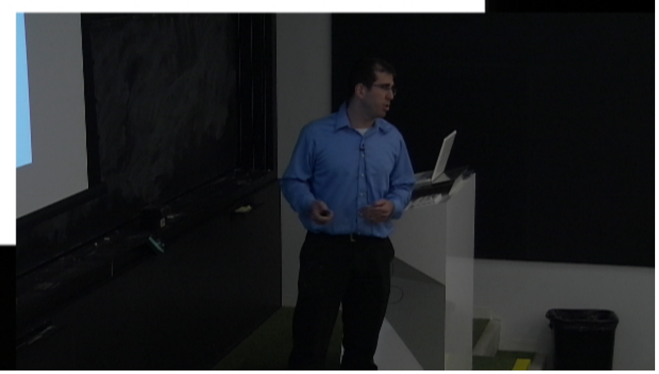


Decomposing POVMs

Just as density operators can represent **noisy quantum states**, so can POVMs represent **noisy measurements**...

Consider decomposing Λ as a **random selection** of a measurement according to M combined with a noisy post-processing $p_{X|W}(x|w)$:

$$\Lambda_x = \sum_{m,w} p_M(m)\, \Gamma_w^{(m)}\, p_{X|W}(x|w)$$



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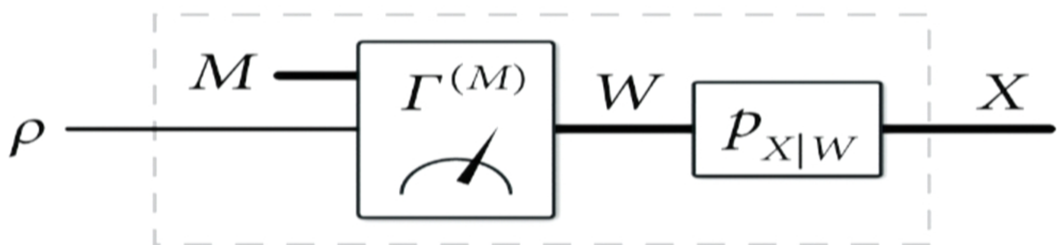

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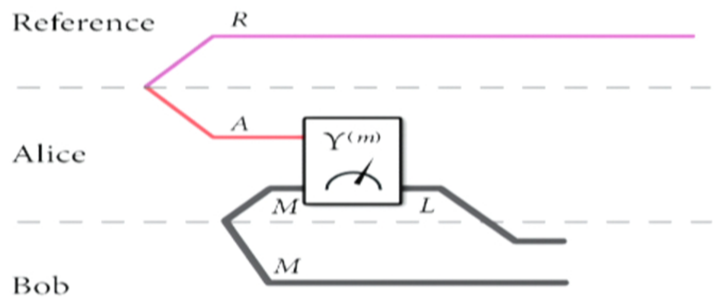
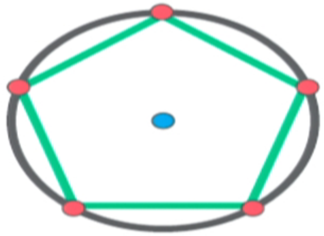


Another Example: Pentagon States

Example. Take a slice of the Bloch sphere that includes its center.
Consider 5 states that form a pentagon on the slice.
With appropriate weightings, these sum to the identity and form a POVM



Andreas Winter. "Extrinsic" and "Intrinsic" Data in Quantum Measurements. arXiv:quant-ph/0109050

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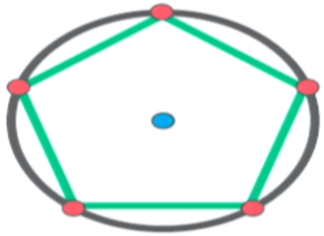

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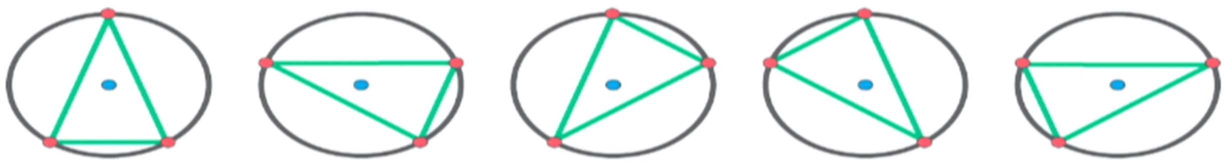

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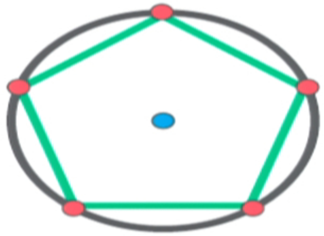

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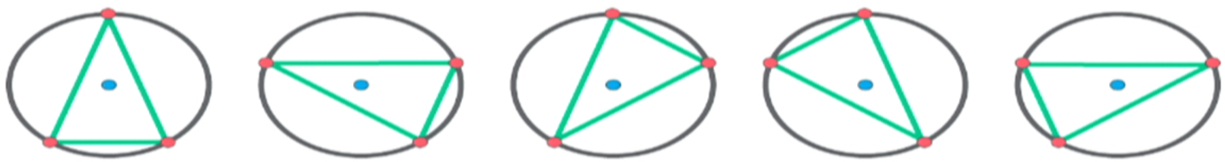

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Ideal Information Processing Task for Measurement Compression

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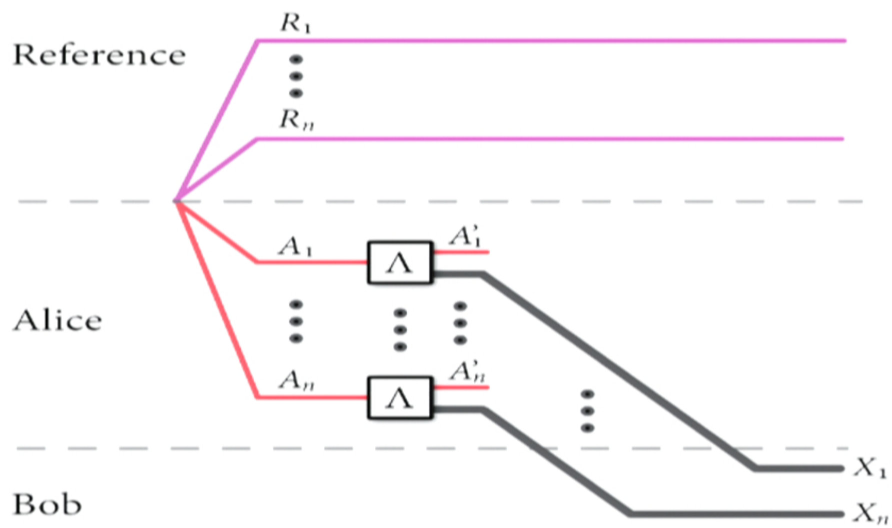

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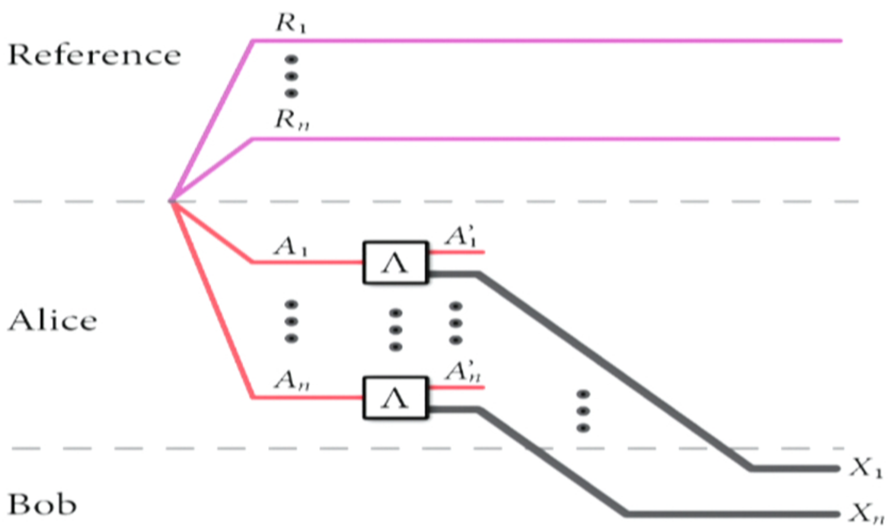

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Faithful Simulation

A measurement simulation is **faithful** if its action on an IID state is indistinguishable from the true measurement:

Definition 1 (Faithful simulation for purification) *A sequence of protocols provides a faithful simulation of the POVM Λ on the source ρ, if for a purification $|\phi_\rho\rangle$ of the source, the states on the reference and source systems after applying the measurement maps are ϵ-close in trace distance for all $\epsilon > 0$ and sufficiently large n:*

$$\left\| \left(\mathrm{id} \otimes \mathcal{M}_{\Lambda^{\otimes n}}\right)\left(\phi_\rho^{\otimes n}\right) - \left(\mathrm{id} \otimes \mathcal{M}_{\widetilde{\Lambda}^n}\right)\left(\phi_\rho^{\otimes n}\right) \right\|_1 \leq \epsilon. \quad (1)$$

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$$\mathcal{M}_{\Lambda^{\otimes n}}(\sigma) \equiv \sum_{x^n} \mathrm{Tr}\left\{\Lambda_{x^n}\sigma\right\} |x^n\rangle\langle x^n| ,$$

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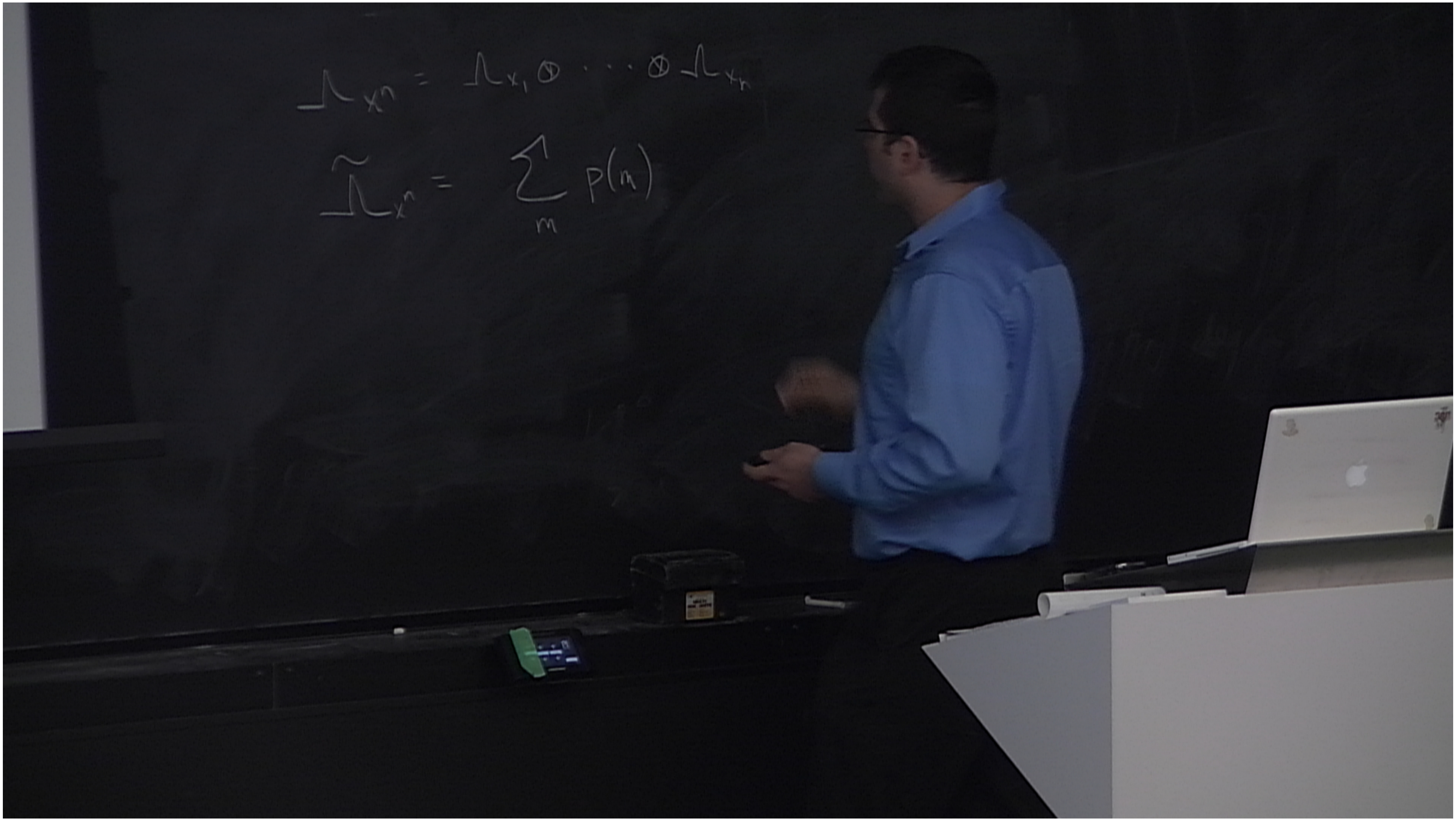
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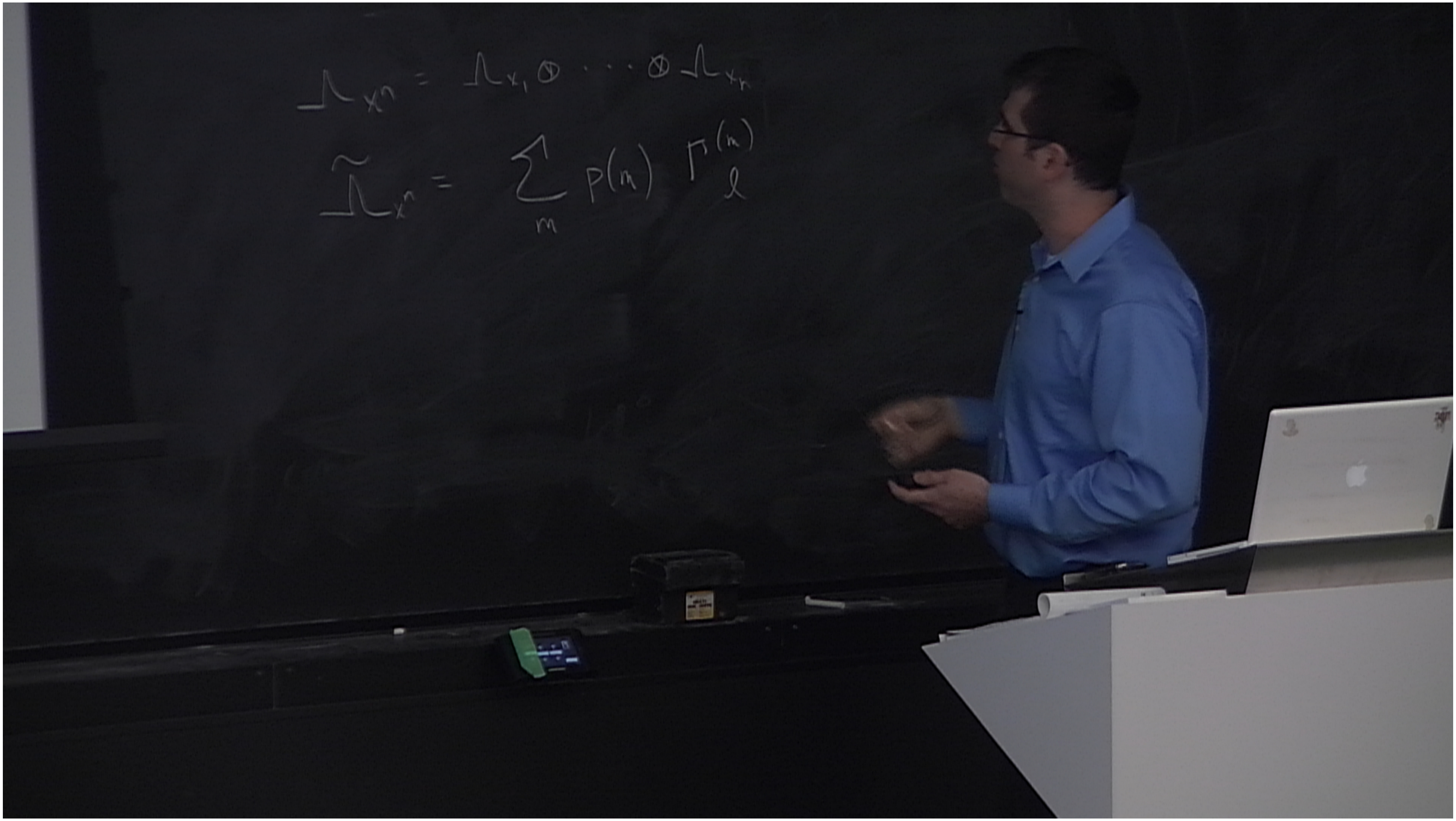
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Measurement Compression Theorem

Theorem 1 (Measurement compression theorem) *Let ρ be a source state and Λ a POVM to simulate on this state. A protocol for faithful simulation of the POVM is achievable with classical communication rate R and common randomness rate S if and only if the following set of inequalities hold*

$$R \geq I\left(X;R\right),$$
$$R+S \geq H\left(X\right),$$

where the entropies are with respect to a state of the following form:

$$\sum_x |x\rangle\langle x|^X \otimes \mathrm{Tr}_A\left\{\left(I^R \otimes \Lambda_x^A\right)\phi^{RA}\right\}, \tag{1}$$

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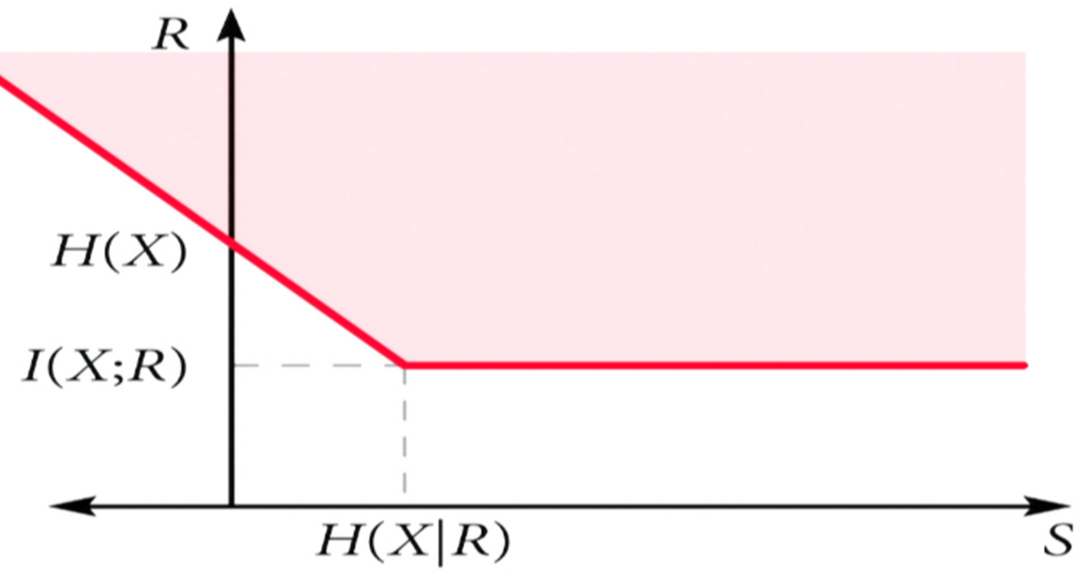
Measurement Compression Region



Notable rate pairs correspond to measurement compression and Shannon compression (*also Shannon compression combined with c. comm. to comm. rand.*)

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Achievability

Resource inequality for measurement compression:

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Can think that the goal is to “steer” the reference to be as above

Do this with $\sqrt{\text{measurement}}$ $\{\rho^{-1/2}\,p_X(x)\,\hat{\rho}_x\,\rho^{-1/2}\}$

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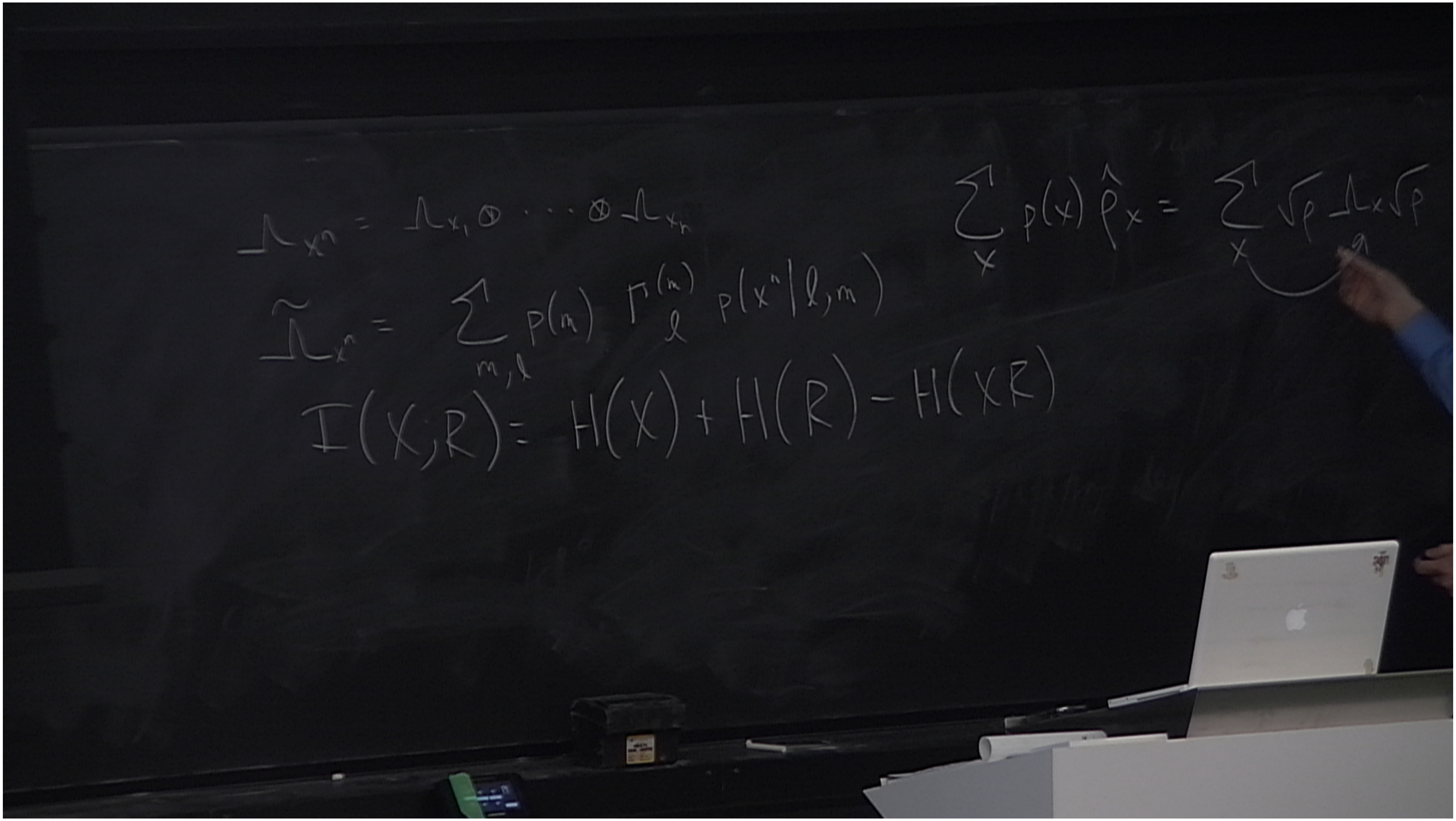
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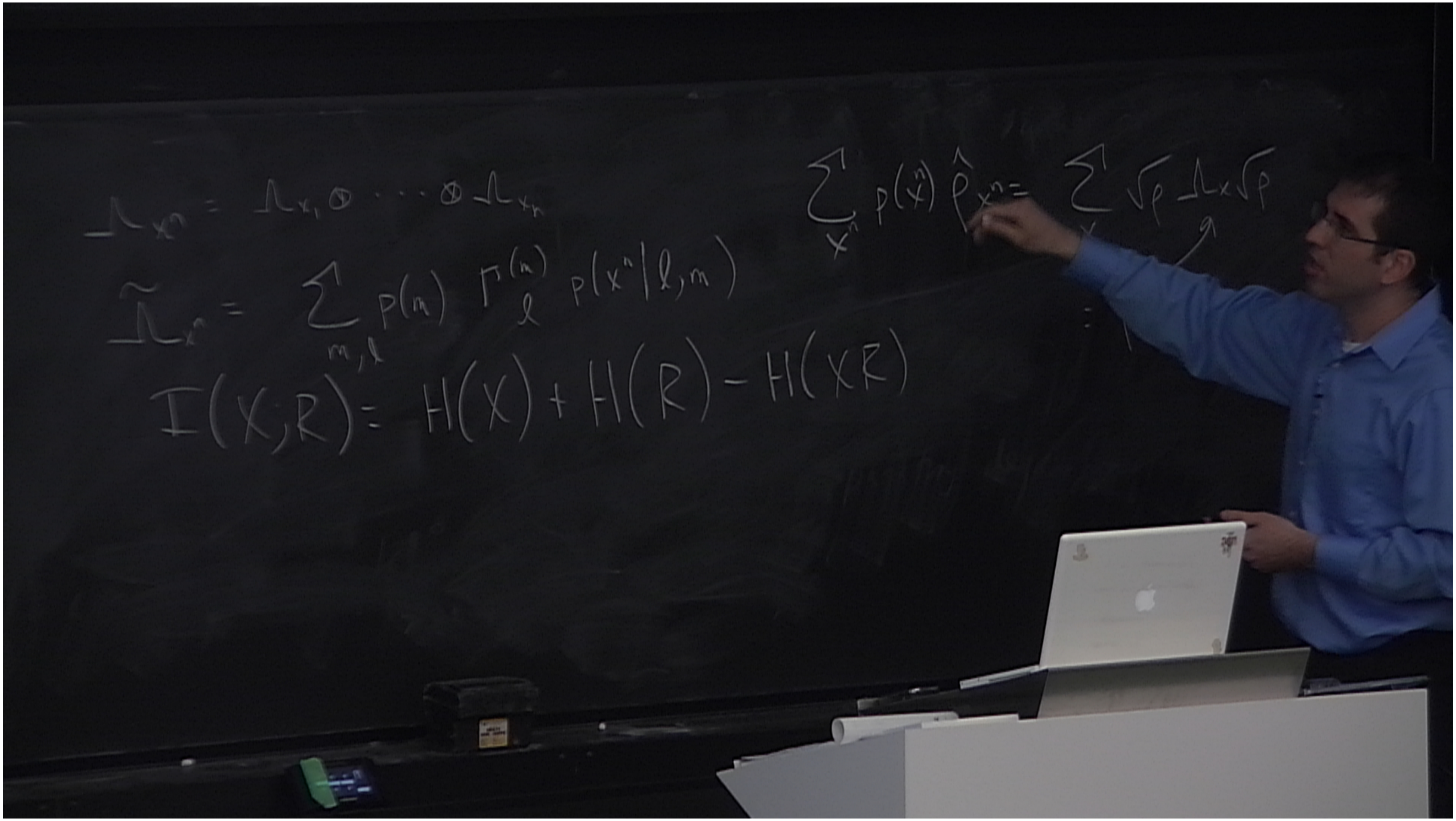
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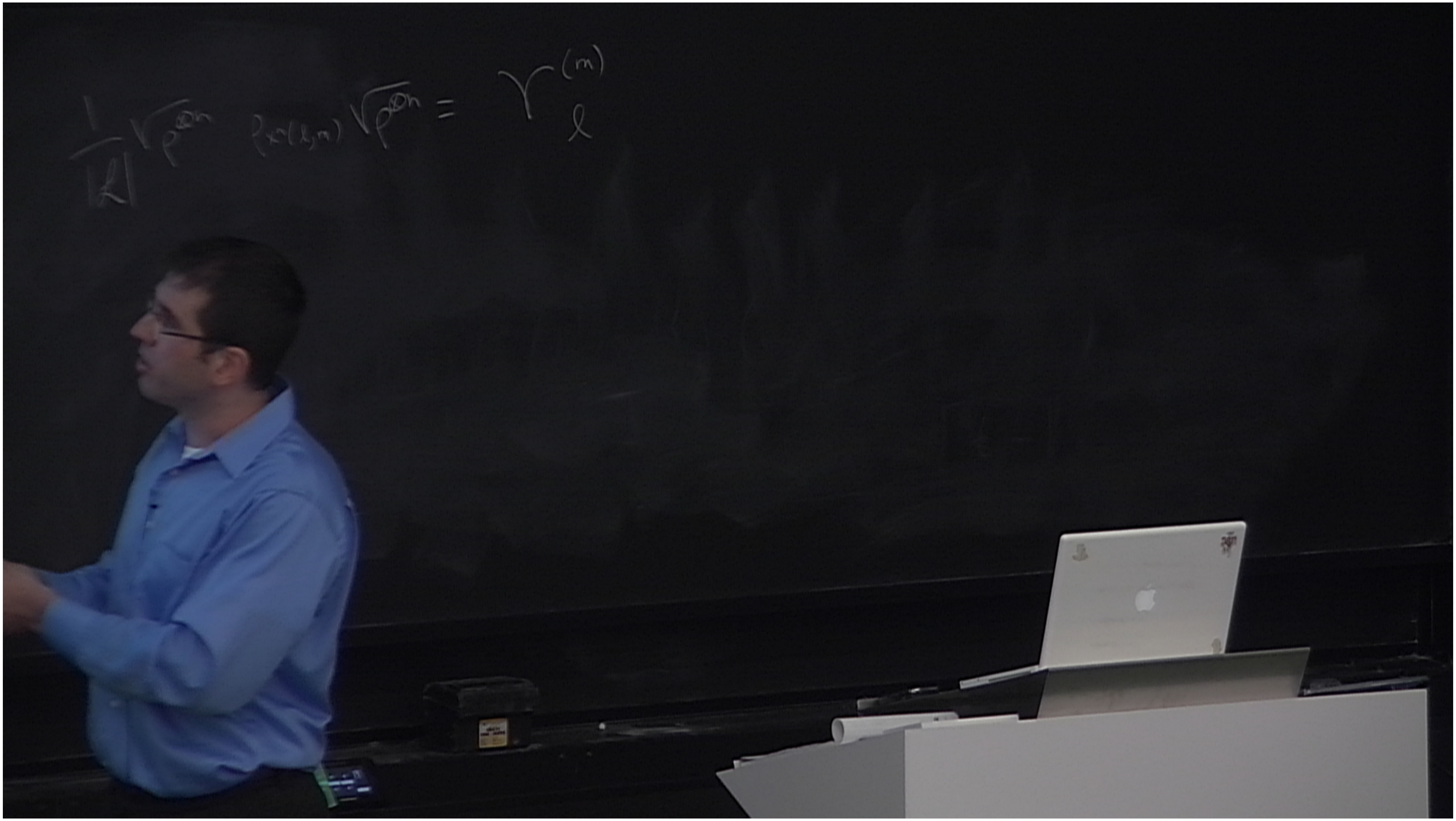
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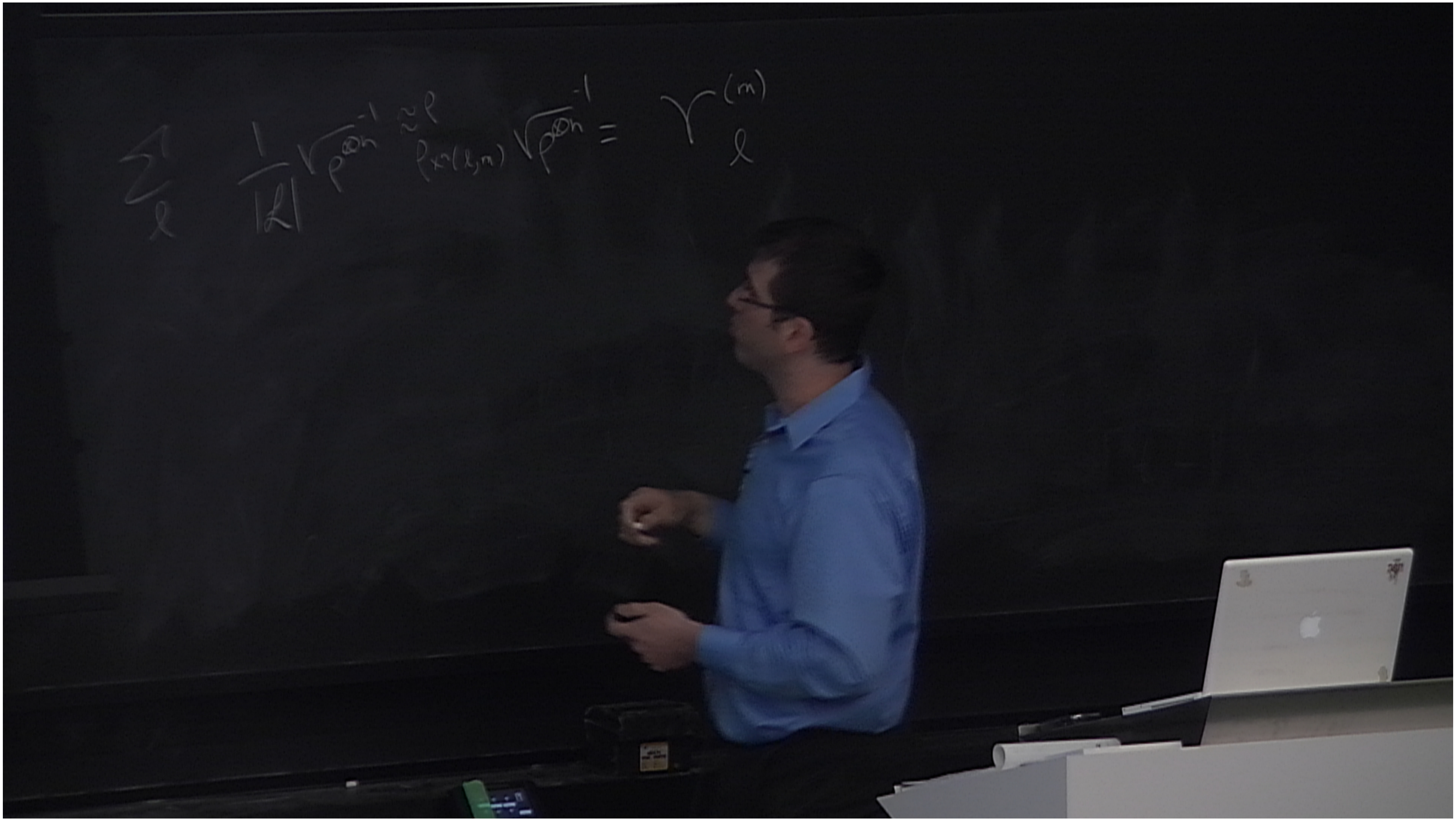
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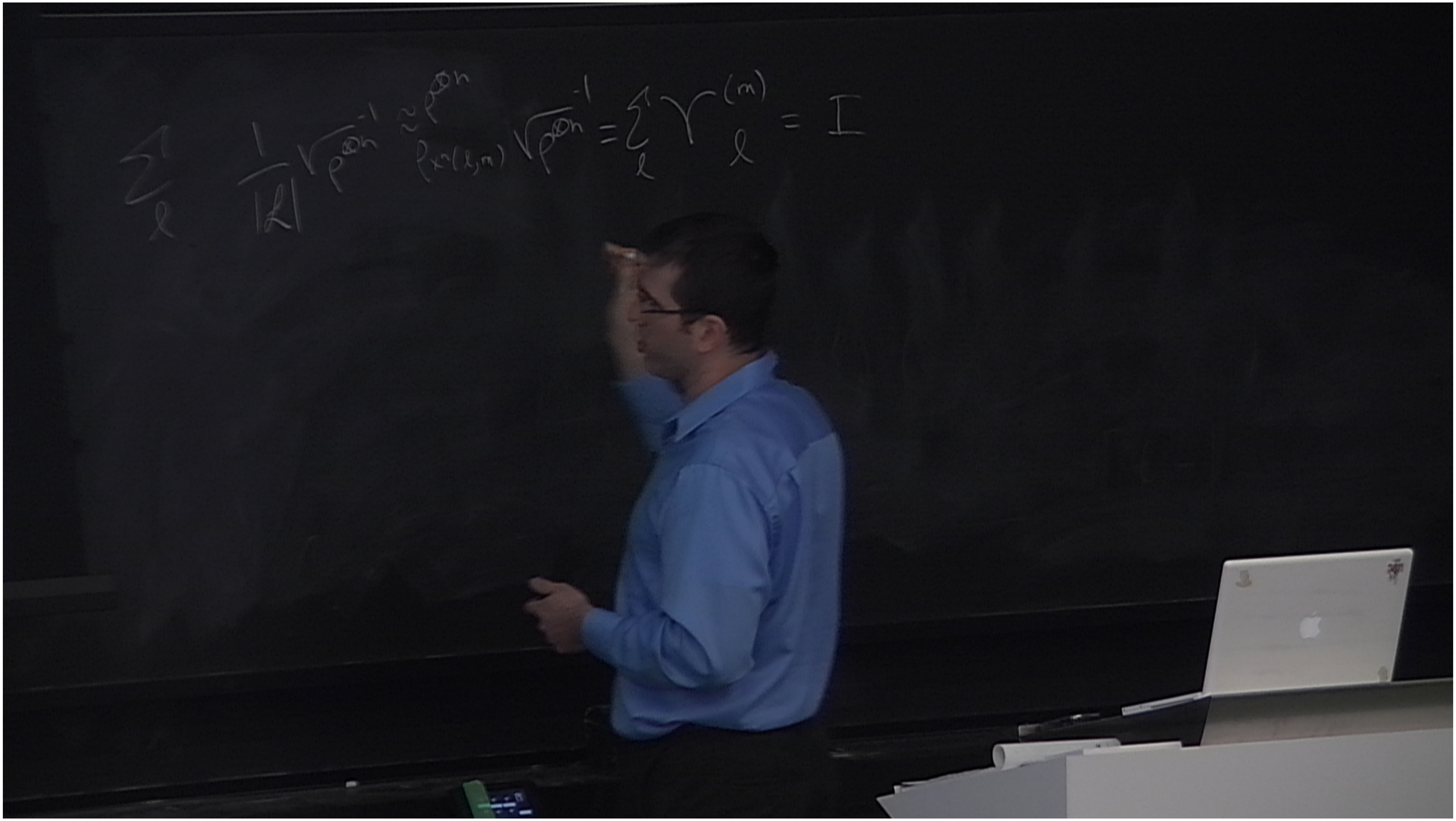
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Main steps are just to think about the most general protocol for this task and exploit **quantum data processing inequality**

Non-feedback Measurement Compression

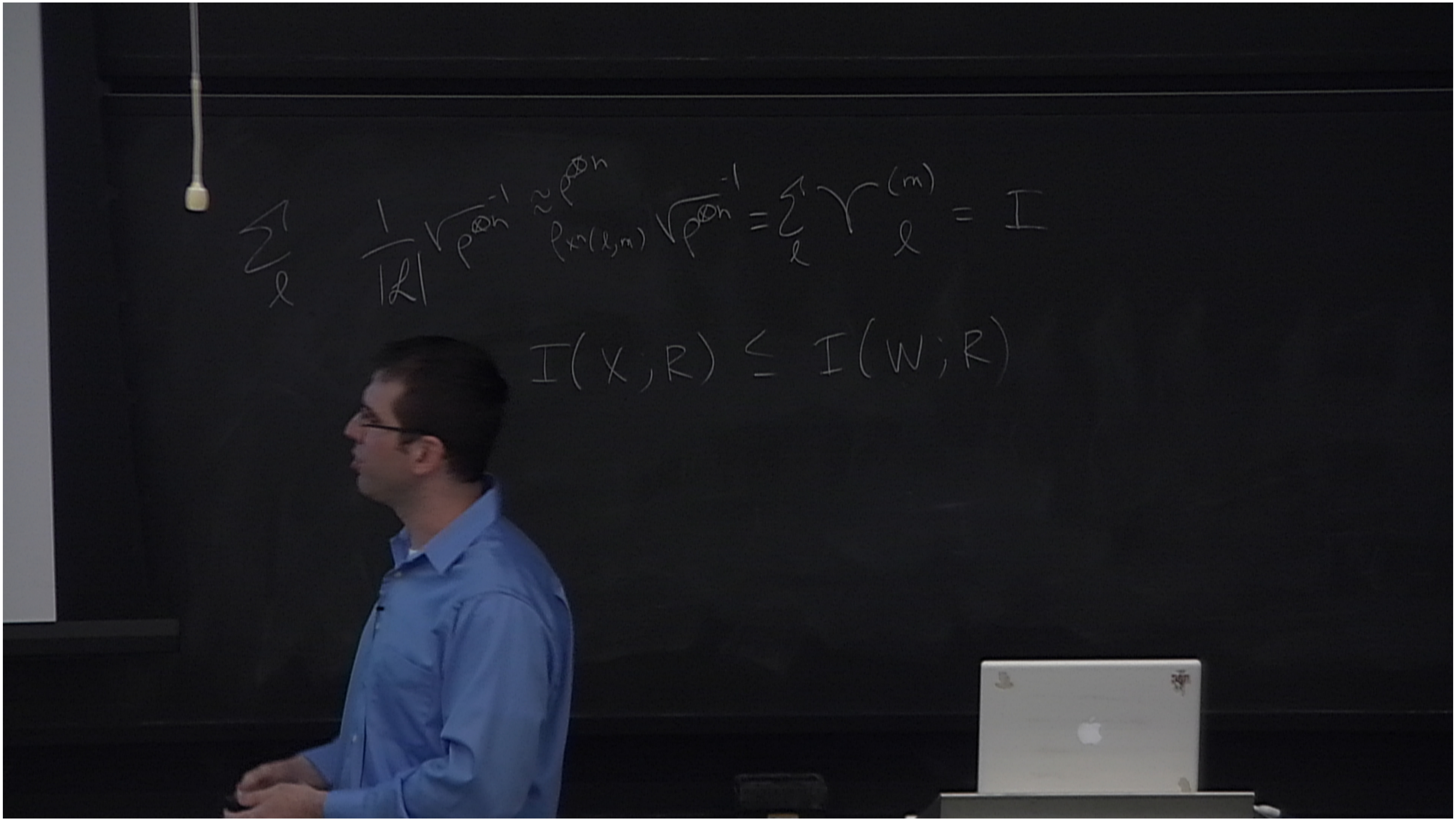
Wilde, Hayden, Buscemi, and Hsieh (2012).

Nonfeedback Measurement Compression

Suppose that POVM $\{\Lambda\}$ has a decomposition as

$$\Lambda_x = \sum_w p_{X|W}(x|w) M_w$$

Wilde, Hayden, Buscemi, and Hsieh (2012).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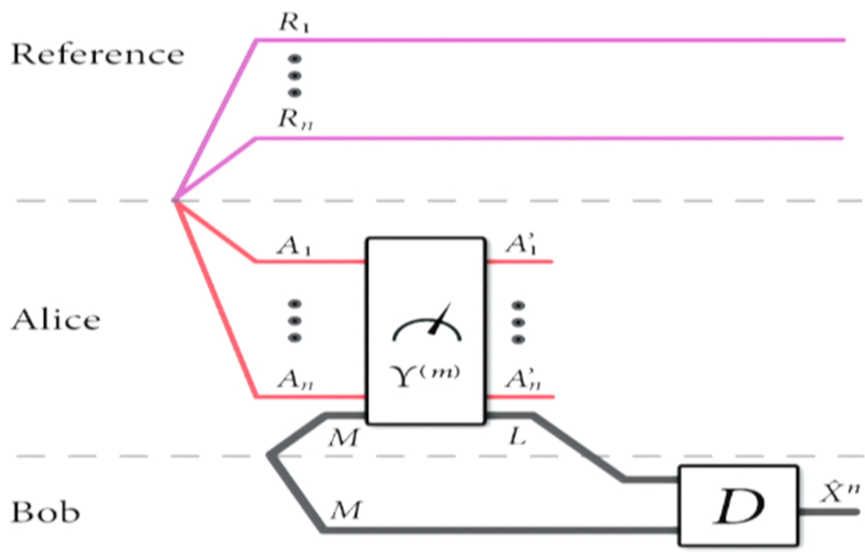
$\approx \rho^{\otimes n}$
$I(X;R) \leq I(W;R)$

Non-feedback Measurement Compression

Simulation is such that Alice does not require measurement output



Wilde, Hayden, Buscemi, and Hsieh (2012).

Nonfeedback Measurement Compression

Achievability follows by employing a variation of
Winter's measurement compression protocol:

$$I(W;R)[c \longrightarrow c] + I(W;X|R)[cc] \geq \langle \Lambda(\rho) \rangle$$

No need for as much common randomness consumption
because Bob simulates $p_{X|W}(x|w)$ locally

Total rate of **common randomness consumption**
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Single-letter converse follows from a technique similar to that of Paul Cuff (arXiv:0805.0065) adapted to the quantum case

Nonfeedback Measurement Compression

Theorem 1 (Non-feedback measurement compression theorem) *Let ρ be a source state and $\mathcal{N}$ a quantum instrument to simulate on this state:*

$$\mathcal{N}(\rho) = \sum_x \mathcal{N}_x(\rho) \otimes |x\rangle\langle x|^X .$$

A protocol for faithful simulation of the quantum instrument is achievable with classical communication rate R and common randomness rate S if and only if R and S are in the rate region given by the union of the following regions:

$$R \geq I(W;R),$$
$$R + S \geq I(W;XR),$$

where the entropies are with respect to a state of the following form:

$$\sum_{x,w} p_{X|W}(x|w)\, |w\rangle\langle w|^W \otimes |x\rangle\langle x|^X \otimes \mathrm{Tr}_A\left\{\left(I^R \otimes \mathcal{M}_w^A\right)\left(\phi_\rho^{RA}\right)\right\}, \qquad (1)$$

ϕ_ρ^{RA} is some purification of the state ρ, and the union is with respect to all decompositions of the original instrument $\mathcal{N}$ of the form:

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such that $R - W - X$ is a quantum Markov chain.

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Total cost can be lower

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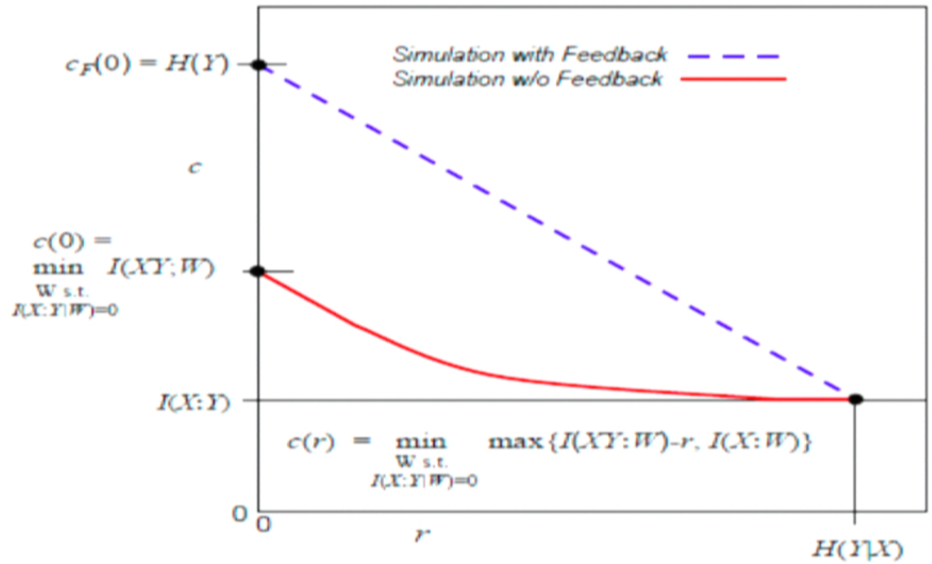
Example plot of trade-off improvement:



Charles H. Bennett, Igor Devetak, Aram W. Harrow, Peter W. Shor and Andreas Winter. 0912.5537

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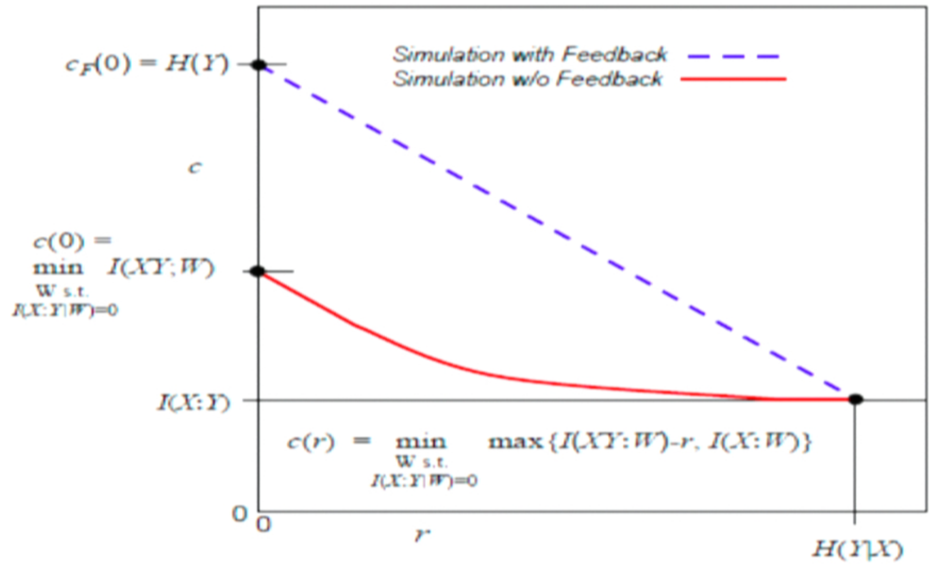

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Classical Data Compression with Quantum Side Information

Devetak and Winter. arXiv:quant-ph/0209029

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Consider an ensemble of the following form:

$$\{p_X(x), \rho_x\}$$

Suppose that an **information source** generates a classical sequence x^n and quantum state ρ_{x^n}

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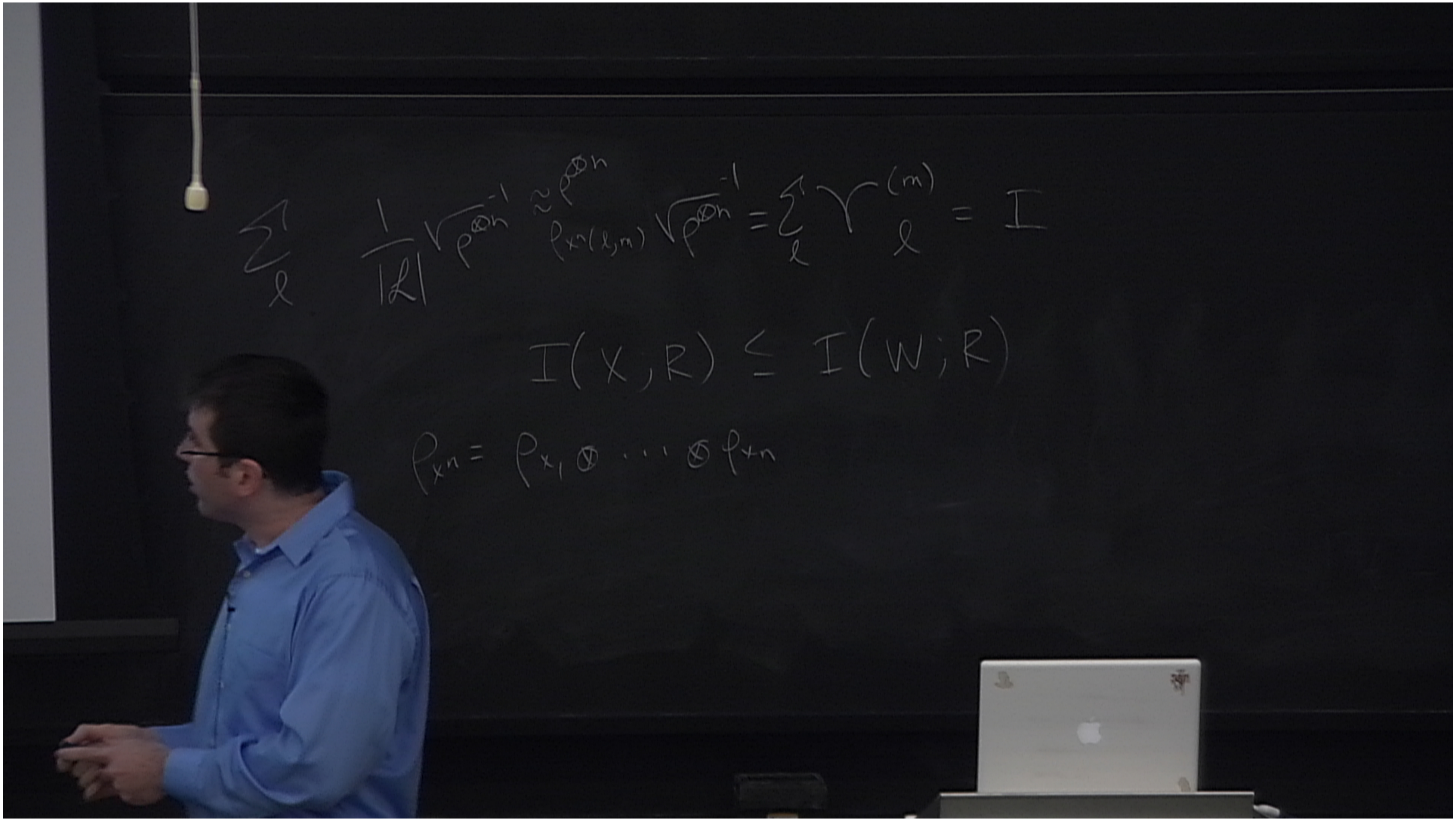
$\sum_\ell \frac{1}{|\mathcal{L}|} \sqrt{\rho^{\otimes n}}^{-1} \rho_{x^n(\ell,m)} \sqrt{\rho^{\otimes n}}^{-1} = \sum_\ell \Upsilon_\ell^{(m)} = I$
$\approx \rho^{\otimes n}$
$I(X;R) \leq I(W;R)$
$\rho_{x^n} = \rho_{x_1} \otimes \cdots \otimes \rho_{x_n}$

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It gives the classical sequence to Alice and the quantum state to Bob

Question: *How much classical communication is needed for Bob to recover x^n?*

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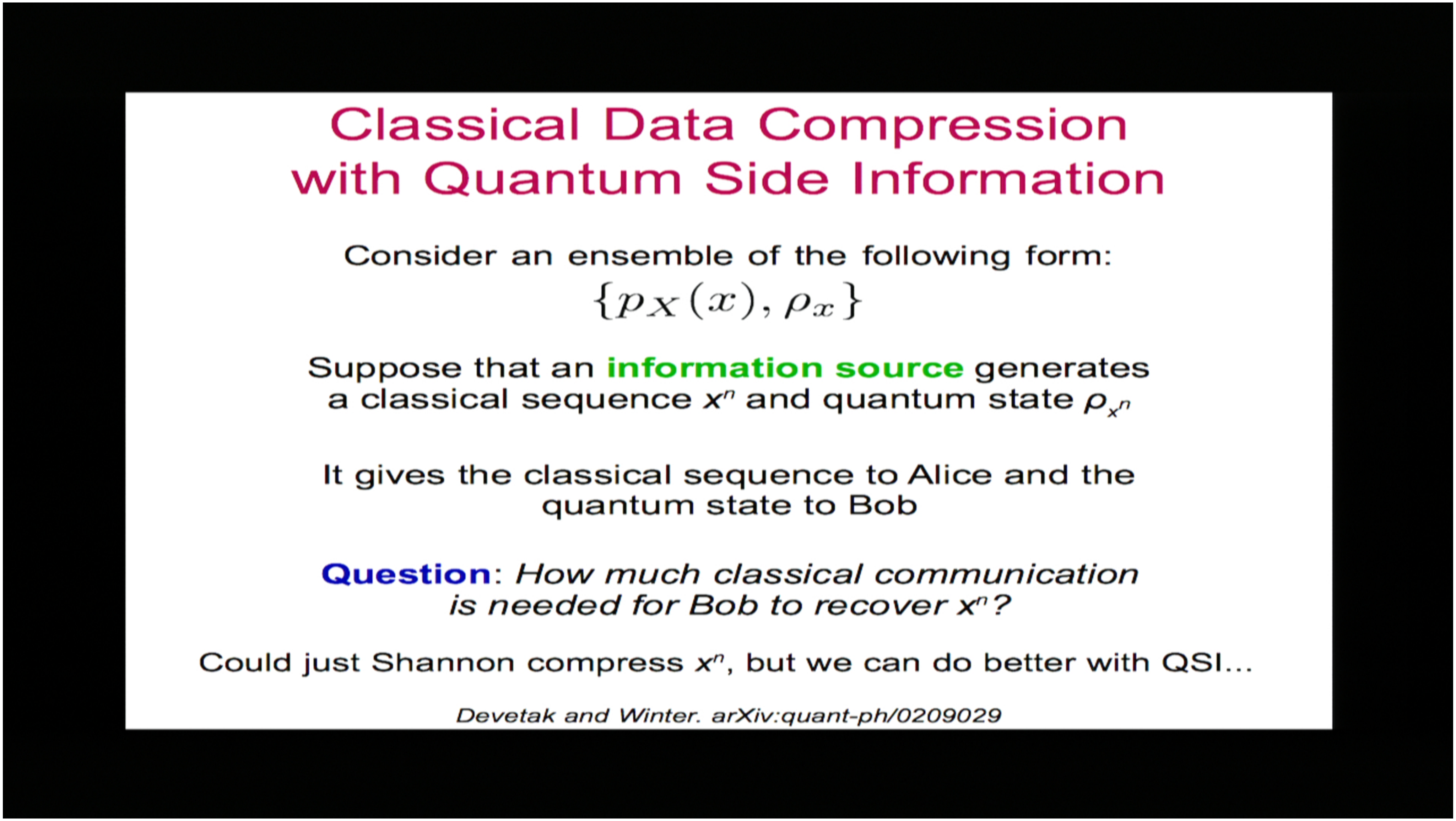
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Could just Shannon compress x^n, but we can do better with QSI...
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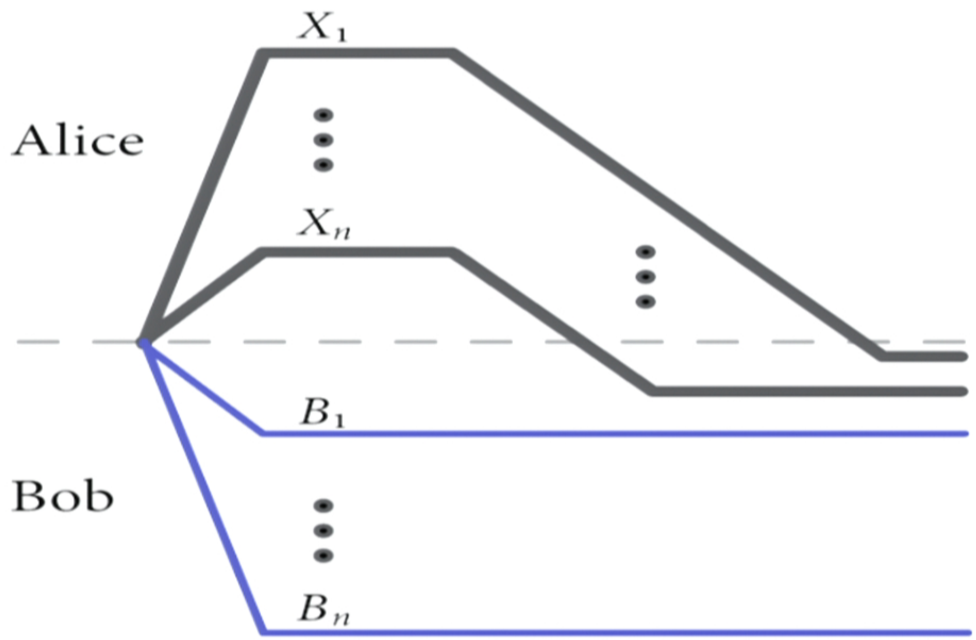
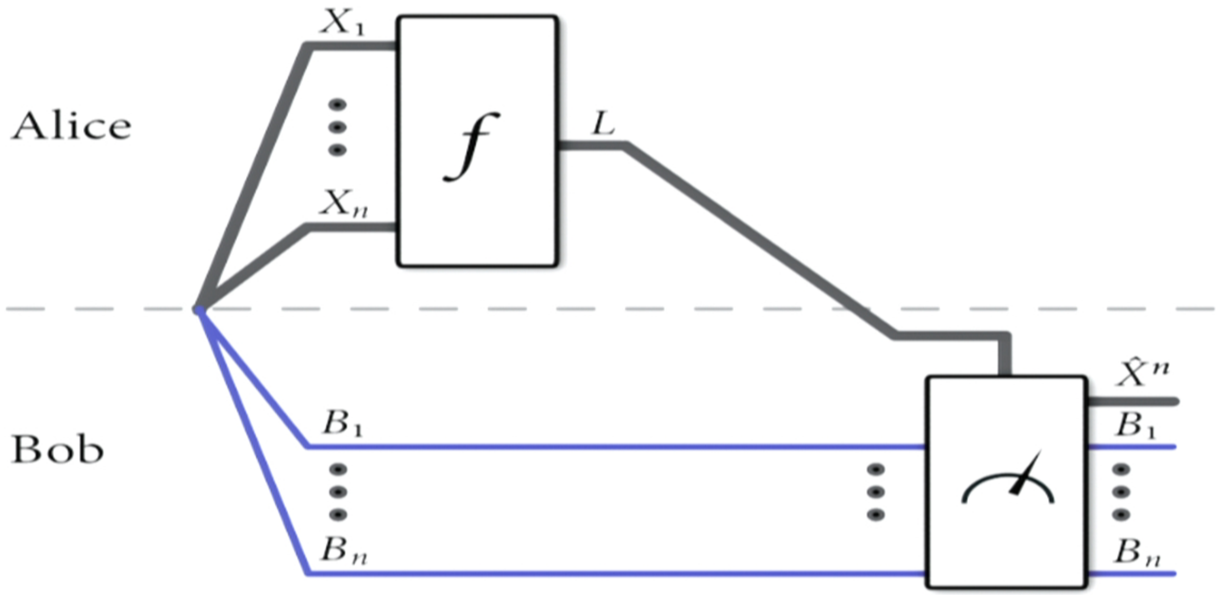
Ideal Protocol for CDC-QSI



In the ideal protocol, Alice just sends the classical sequence to Bob.

Devetak and Winter. arXiv:quant-ph/0209029

Actual Protocol for CDC-QSI



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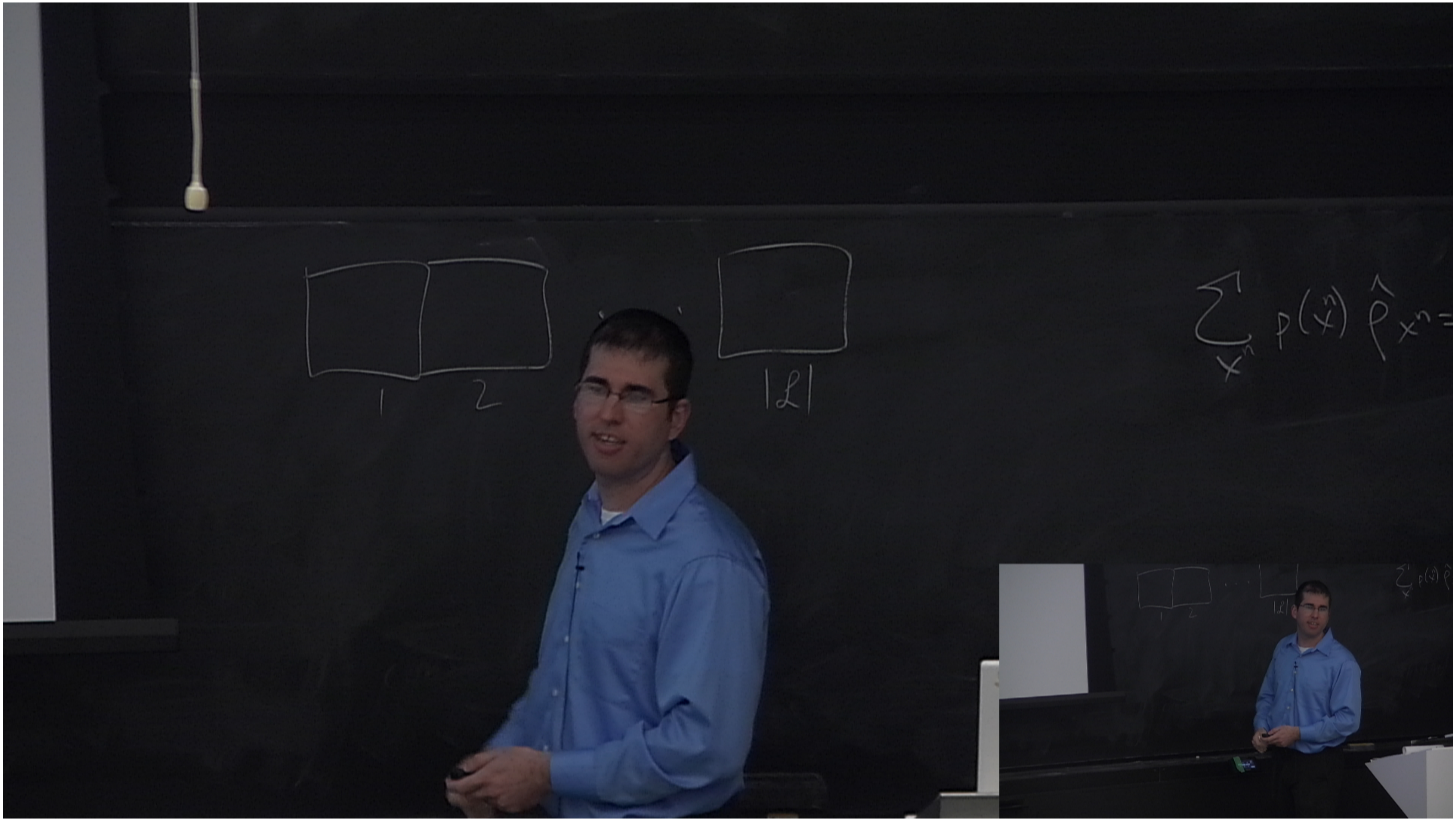
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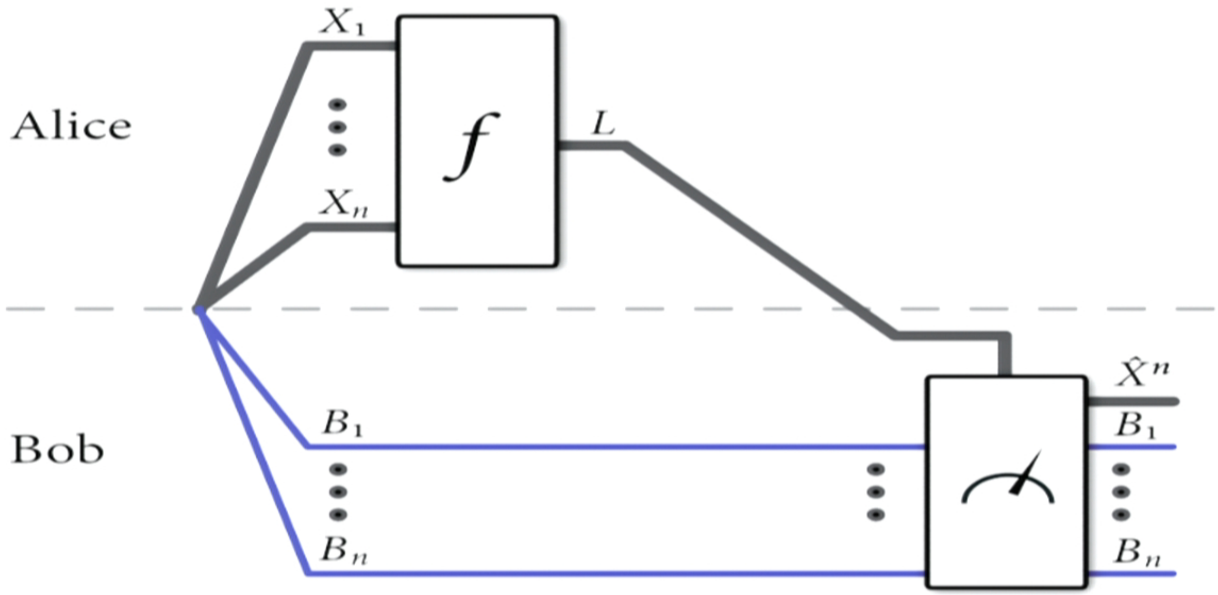

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Theorem 1 (Classical data compression with quantum side information) *Suppose that*

$$\sum_x p_X(x) |x\rangle \langle x|^X \otimes \rho_x^B$$

is a classical-quantum state that characterizes a classical-quantum source. Then the conditional von Neumann entropy $H(X|B)$ *is the smallest possible achievable rate for classical data compression with quantum side information for this source:*

$$\inf \{R \mid R \text{ is achievable}\} = H(X|B).$$

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Alice needs to send the difference $n[H(X) - I(X;B)] = nH(X|B)$

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Achievability

Resource inequality for CDC-QSI:

$$\langle \rho^{XB} \rangle + H(X|B)\,[c \rightarrow c] \geq \langle \rho^{X X_B B} \rangle$$

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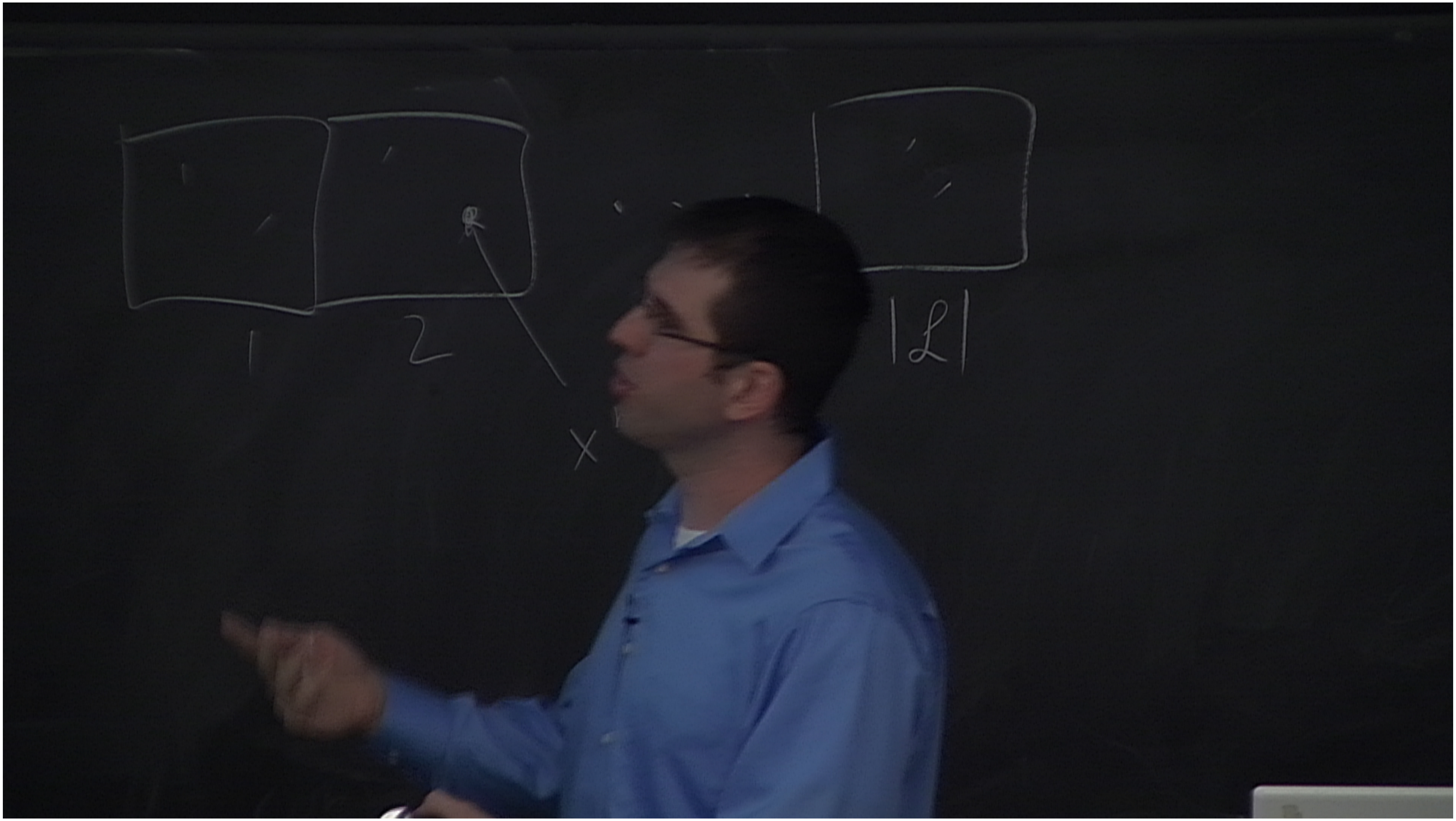
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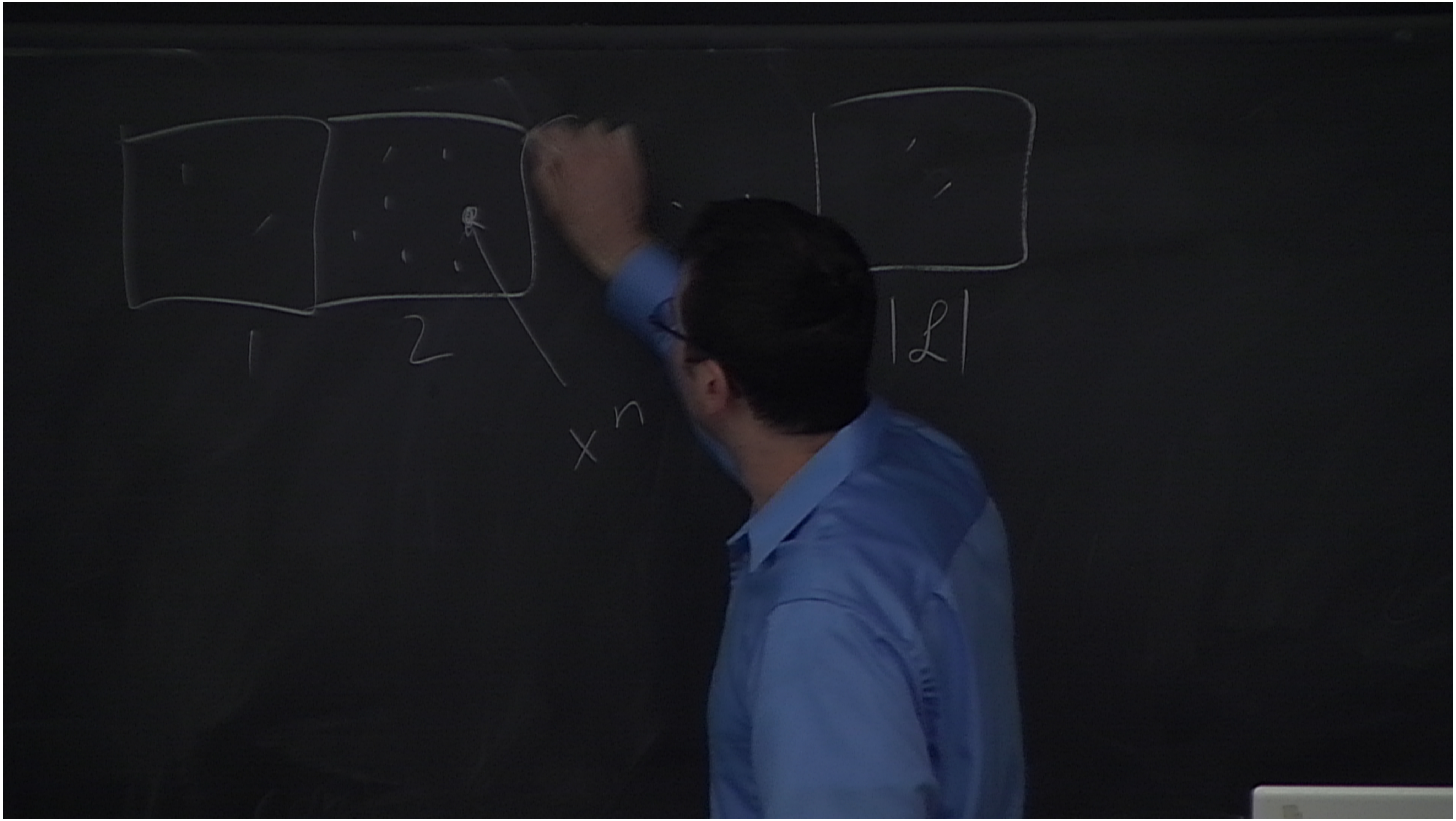
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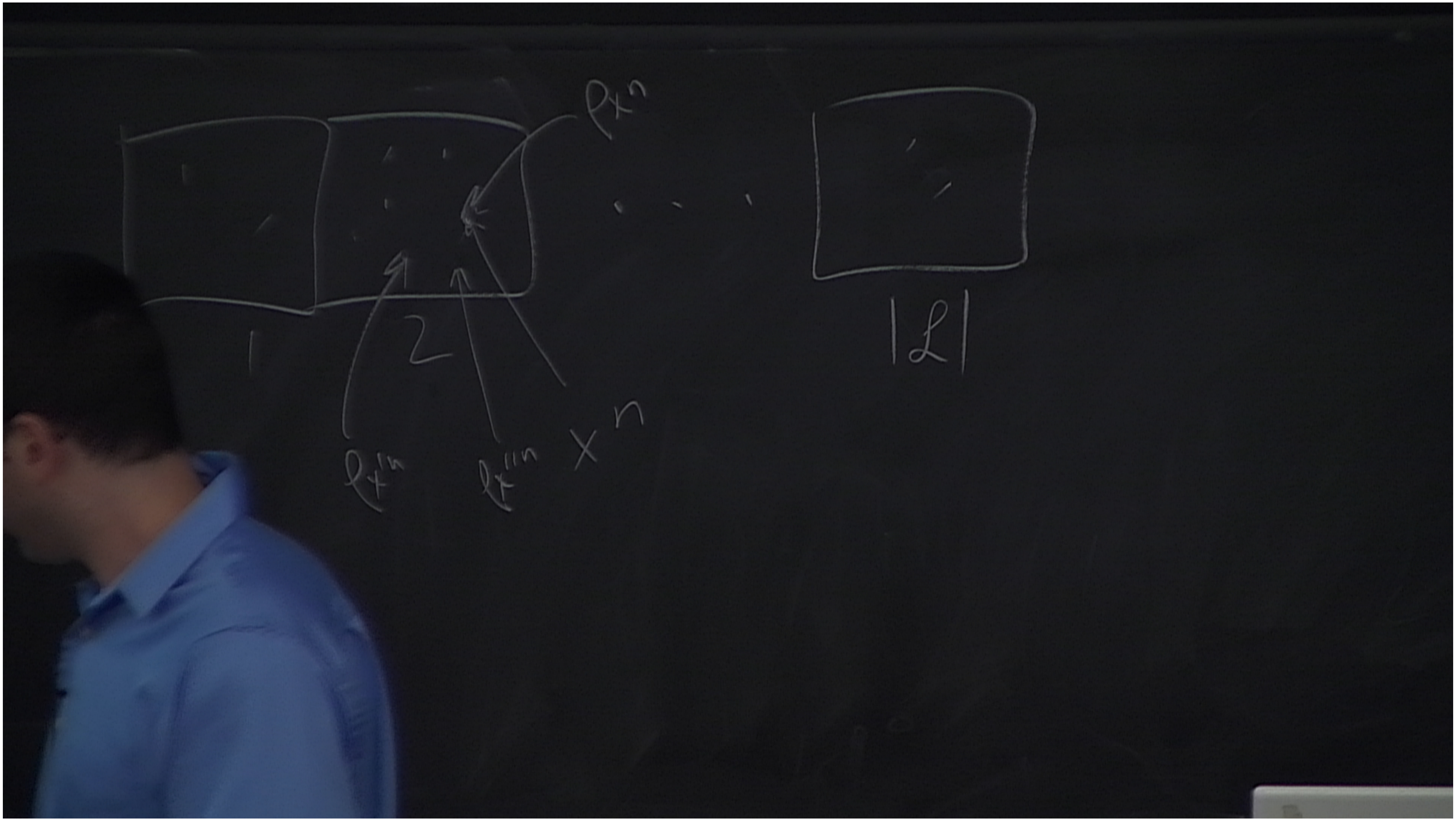
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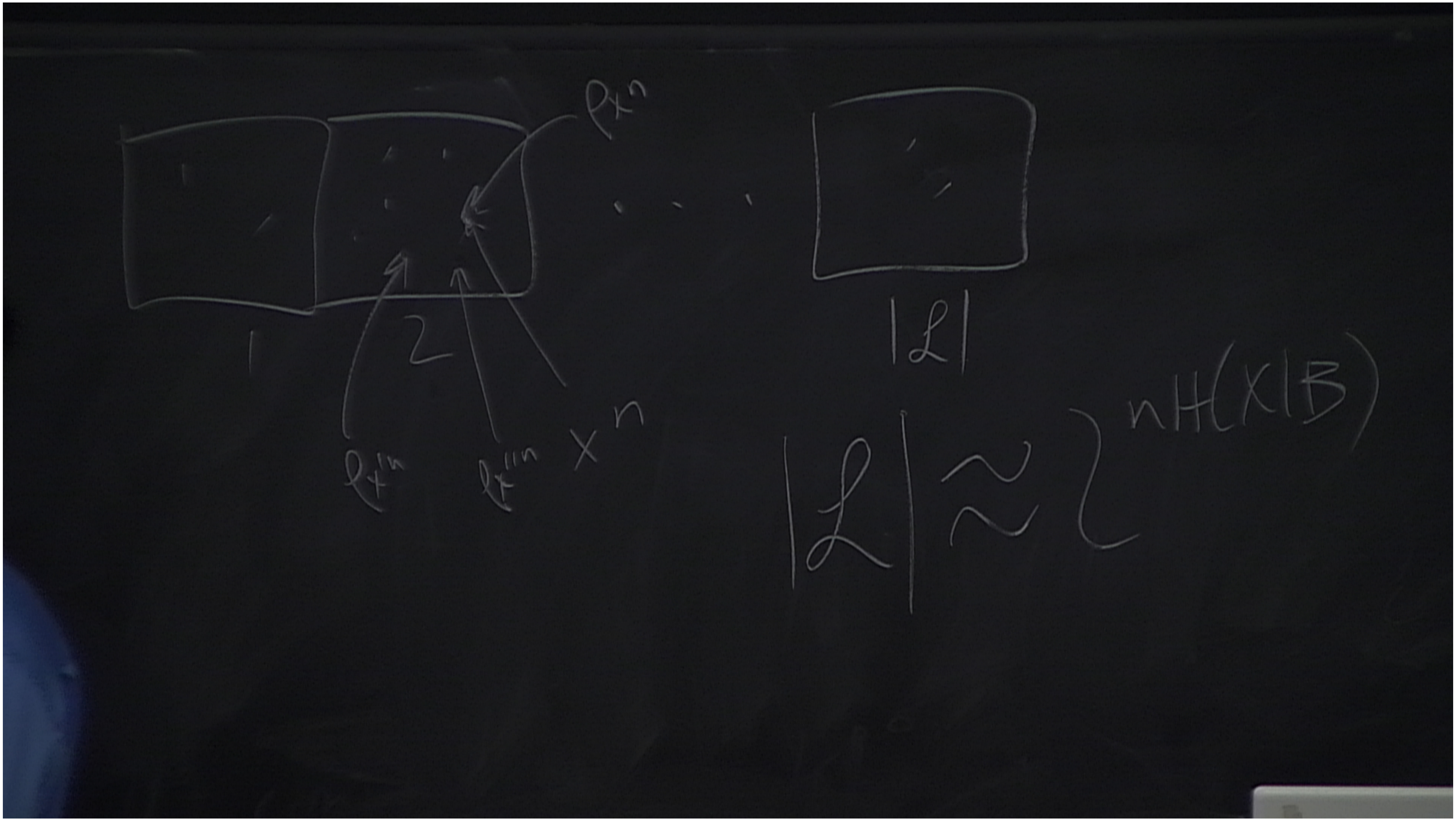
X^n



ρ_x^n
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$\rho_x'^n$
$\rho_x''^n$
x^n
$|\mathcal{L}|$
$|\mathcal{L}| \approx 2^{nH(X|B)}$

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What's different: Bob receives hash from Alice, and scans over all of the quantum states consistent with the hash value. He performs sequential binary projective measurements asking, "*Does my quantum state correspond to the m^{th} sequence consistent with the hash*?"

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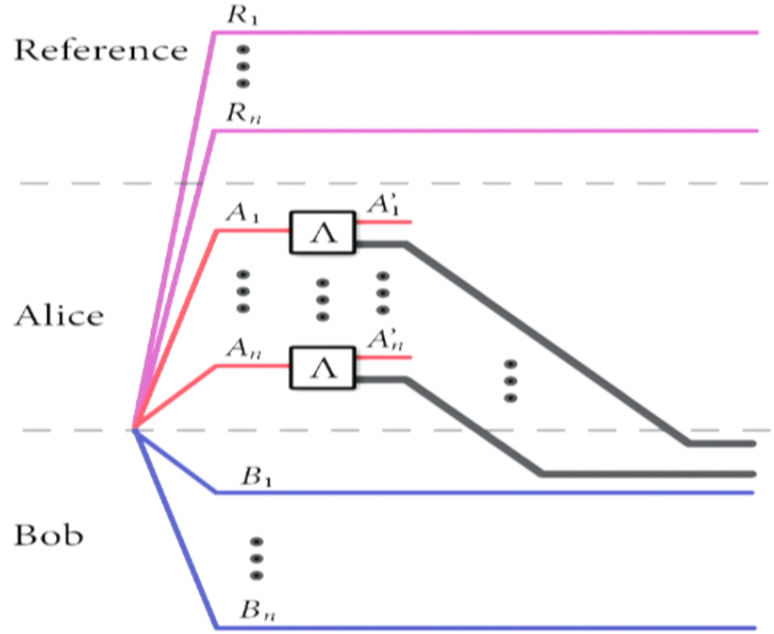
Single-Letter Converse

Main steps are just to think about the most general protocol for this task and exploit quantum data processing inequality

$$
\begin{aligned}
nR &\geq H(L) \\
&\geq H(L|B^n) \\
&\geq I(X^n;L|B^n) \\
&= H(X^n|B^n) - H(X^n|LB^n) \\
&\geq H(X^n|B^n)_\omega - H(X^n|\hat{X}^n)_\omega \\
&\geq H(X^n|B^n)_\sigma - n\epsilon' \\
&= nH(X|B) - n\epsilon'.
\end{aligned}
$$

Ideal MC-QSI Protocol



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Goal is for Alice and Bob to simulate ideal measurement and for Bob's state not to be disturbed

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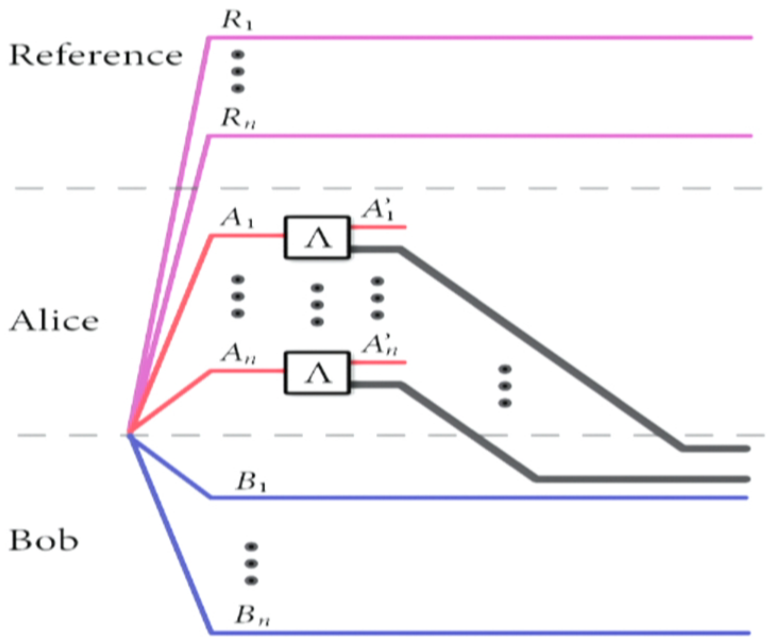

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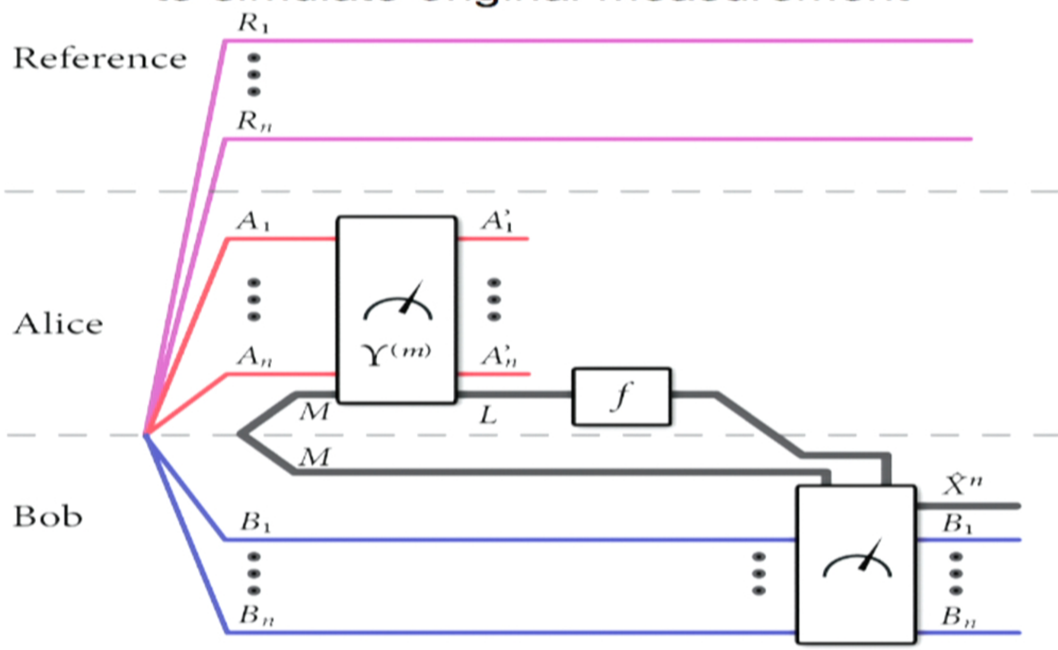
Actual MC-QSI Protocol

Use common randomness, an Alice collective measurement, classical communication, and a Bob collective measurement to simulate original measurement



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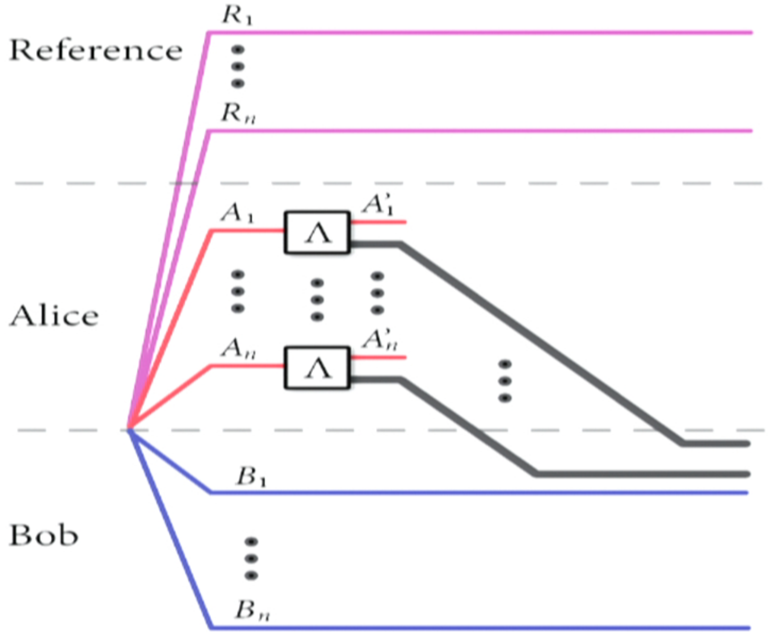

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Proof strategy combines ideas from MC and CDC-QSI

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Operation: Alice performs simulation measurement, hashes the outcome, and sends it to Bob. Bob receives hash from Alice, and scans over all of the post-measurement states consistent with the hash value. He performs sequential binary projective measurements asking, "*Does my quantum state correspond to the m^{th} measurement outcome consistent with the hash*?"

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Also, *we require that the protocol causes only a negligible disturbance to Bob's state*

Applications of MC-QSI

1) Classically assisted state redistribution

2) Quantum reverse Shannon theorem for a quantum instrument

3) Local purity distillation

Classically-assisted state redistribution

Begin with state ρ^{AB} that has purification ψ^{ABE}

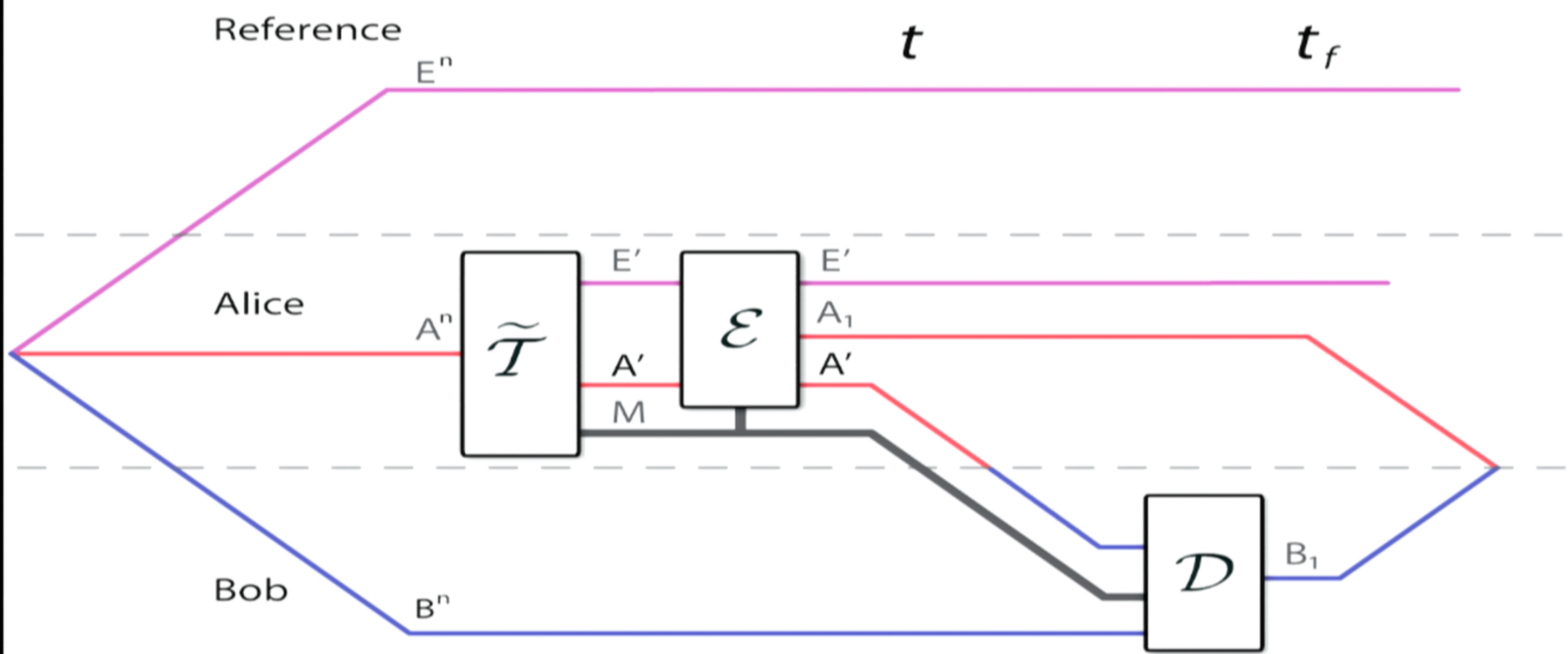
Perform **MC-QSI**

Requires $I(X_B; E|B)$ rate of classical communication

Then perform **Quantum State Redistribution** conditional on classical information

$$\langle \rho^{AB} \rangle + I\left(X_B; E|B\right)_\sigma [c \rightarrow c] + \frac{1}{2} I\left(A'; E|E' X_B\right)_\sigma [q \rightarrow q] \geq \frac{1}{2}\left(I\left(A'; B|X_B\right)_\sigma - I\left(A'; E'|X_B\right)_\sigma\right)[qq]$$

Classically-assisted state redistribution



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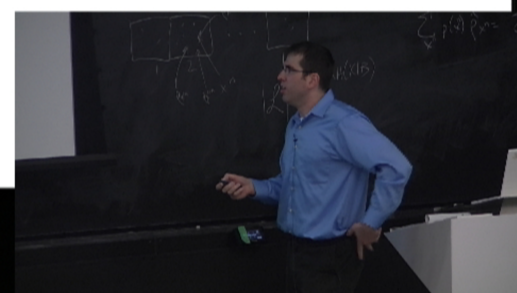
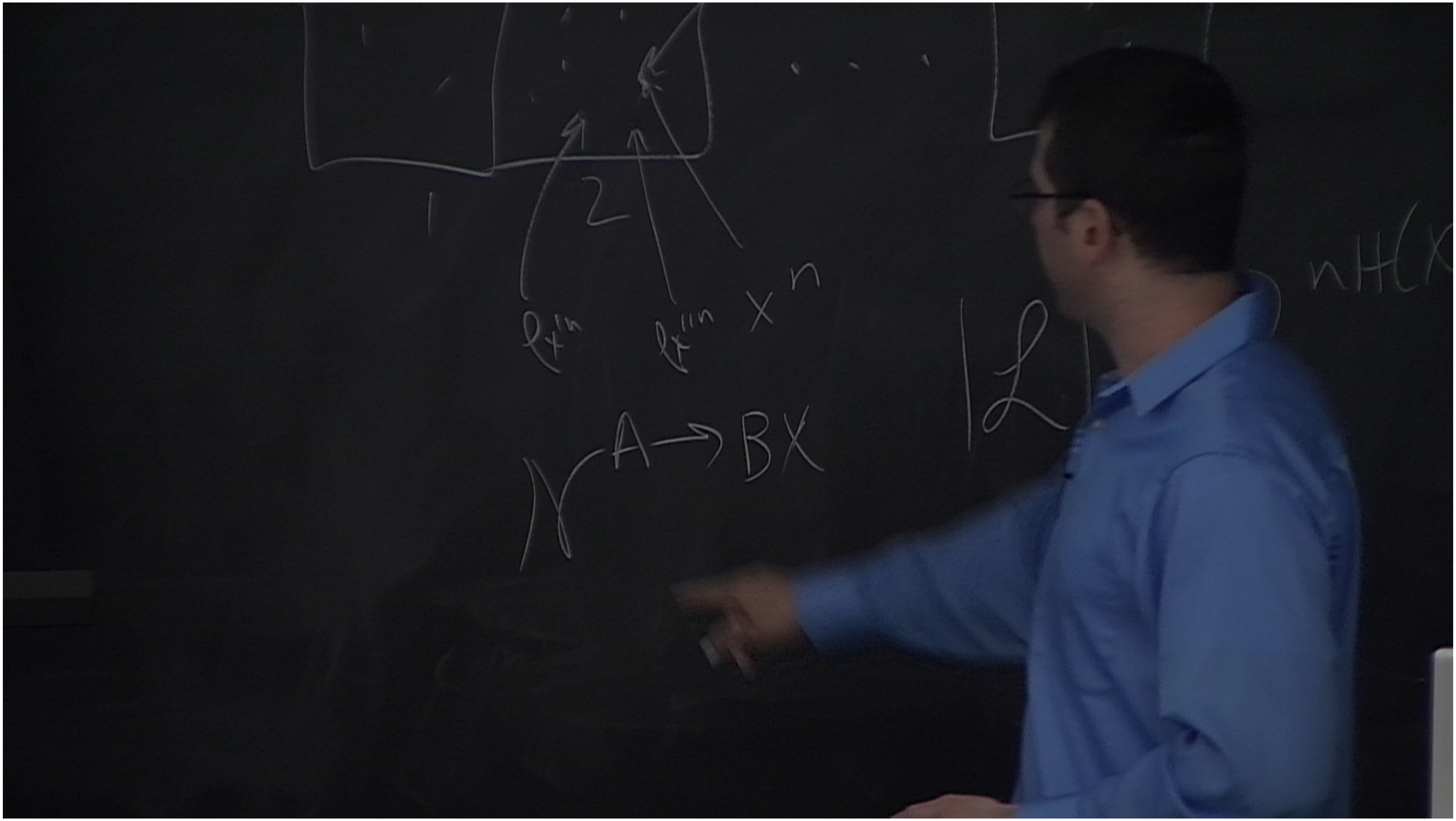
We have a reverse Shannon theorem for a POVM

What about for a quantum instrument with classical and quantum outputs?

Protocol is to perform measurement compression followed by FQRS:

$$I\left(X;R\right)[c\rightarrow c]+H\left(X|R\right)[cc]+\frac{1}{2}I\left(B';R|X\right)[q\rightarrow q]+\frac{1}{2}I\left(B';E|X\right)[qq] \geq \langle U_{\mathcal{N}}^{A\rightarrow XX_E B'E}:\rho^A\rangle.$$

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Reverse Shannon theorem when QSI is available:

$$\langle\rho^{AB}\rangle + I(X;R|B)\,[c\rightarrow c] + H(X|RB)\,[cc] + \frac{1}{2}I(B';R|BX)\,[q\rightarrow q] + \frac{1}{2}\left(I(B';E|X) - I(B';B|X)\right)[qq] \geq \langle U_{\mathcal{N}}^{A\rightarrow XX_E B'E} : \rho^{AB}\rangle$$

Local Purity Distillation

Paradigm: Alice and Bob share a state
Their goal is to distill local pure states
using classical communication and local unitaries

By using MC-QSI, we have the following improvement to Krovi-Devetak 0705.4089

Theorem 1 *The 1-way distillable local purity of the state ρ^{AB} is given by $\kappa_{\rightarrow} = \kappa^{*}_{\rightarrow}$, where*

$$\kappa^{*}_{\rightarrow}\left(\rho^{AB}, R\right) = \kappa\left(\rho^{A}\right) + \kappa\left(\rho^{B}\right) + P_{\rightarrow}\left(\rho^{AB}, R\right).$$

In the above, we have the definitions

$$\kappa\left(\omega^{C}\right) \equiv \log d_{C}$$

Lower classical comm. cost

$$P_{\rightarrow}\left(\rho^{AB}, R\right) \equiv \lim_{k\rightarrow\infty}\frac{1}{k}P^{(1)}\left(\left(\rho^{AB}\right)^{\otimes k}, kR\right)$$

and

$$P^{(1)}\left(\rho^{AB}, R\right) \equiv \max_{\Lambda}\left\{I\left(Y;B\right)_{\sigma} : I\left(Y;E|B\right) \leq R\right\},$$

$$\sigma^{YBE} \equiv \left(\mathcal{M}_{\Lambda}\otimes I^{BE}\right)\left(\psi^{ABE}\right),$$

where ψ^{ABE} is a purification of ρ^{AB}, $\mathcal{M}_{\Lambda}$ is a measurement map corresponding to the POVM Λ, and the maximization is over all POVMs mapping Alice's system A to a classical system Y.

Conclusion and Current Work

Measurement compression gives a powerful **operational way** for understanding quantum measurement

We have extended Winter's original protocol in two ways:

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