Title: Virtual Parallel Computing and a Search Algorithm Using Matrix Product States

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Abstract: We propose a form of parallel computing on classical computers that is based on matrix product states. The virtual parallelization is accomplished by evolving all possible results for multiple inputs, with bits represented by matrices. The action by classical probabilistic 1-bit and deterministic 2-bit gates such as NAND are implemented in terms of matrix operations and, as opposed to quantum computing, it is possible to copy bits. We present a way to explore this method of computation to solve search problems and count the number of solutions. We argue that if the classical computational cost of testing solutions (witnesses) requires less than O(n^2) local two-bit gates acting on n bits, the search problem can be fully solved in subexponential time. Therefore, for this restricted type of search problem, the virtual parallelization scheme is faster than Groverâ€™s quantum algorithm.

Virtual Parallel Computing with Matrix Product States

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Motivation

It is possible to simulate some complex quantum systems efficiently in a classical computer: e.g., spin chains.

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- *local interactions*
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QC can be done on a line: D. Aharonov, D. Gottesman, S. Irani, J. Kempe (2009)

Random qudit line: R. Movassagh, E. Farhi, J. Goldstone, D. Nagaj, T. J. Osborne, P. W. Shor (prep. 2010)

Matrix Product States

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$$|\Psi\rangle = \sum_{\{x\}} \mathrm{tr}\left[M_1^{(x_1)} M_2^{(x_2)} \cdots M_n^{(x_n)}\right] |x_1 x_2 \cdots x_n\rangle$$

two-body, local interactions

$$\hat{U}(t) = e^{-i\hat{H}t} \approx \left(\hat{I} - i\hat{H}\Delta t\right)^N \quad \text{with} \quad \hat{H} = \sum_{i=1}^{n-1} \hat{h}_{i,i+1}$$

⇨ evolution comes down to computing the action of $\hat{h}_{i,i+1}$ on consecutive matrices $M_i^{(x_i)} M_{i+1}^{(x_{i+1})}$.

$$M_i^{(x_i)} M_{i+1}^{(x_{i+1})} \xrightarrow{\hat{h}_{ij}} \mathcal{M}_{ij} \xrightarrow[\text{truncation}]{\text{SVD}} \tilde{M}_i^{(x_i)} \tilde{M}_{i+1}^{(x_{i+1})}$$

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SVD: singular value decomposition

any $k \times l$ matrix can be decomposed as $M = U \Lambda V$

$[U]_{k\times k}$ $[V]_{l\times l}$ *unitary matrices*

$$[\Lambda]_{k\times l} = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \dots & \\ & & & \lambda_D \end{pmatrix}$$ *positive diagonal matrix*

$D = \max\{k, l\}$

$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_D \geq 0$ *singular values*

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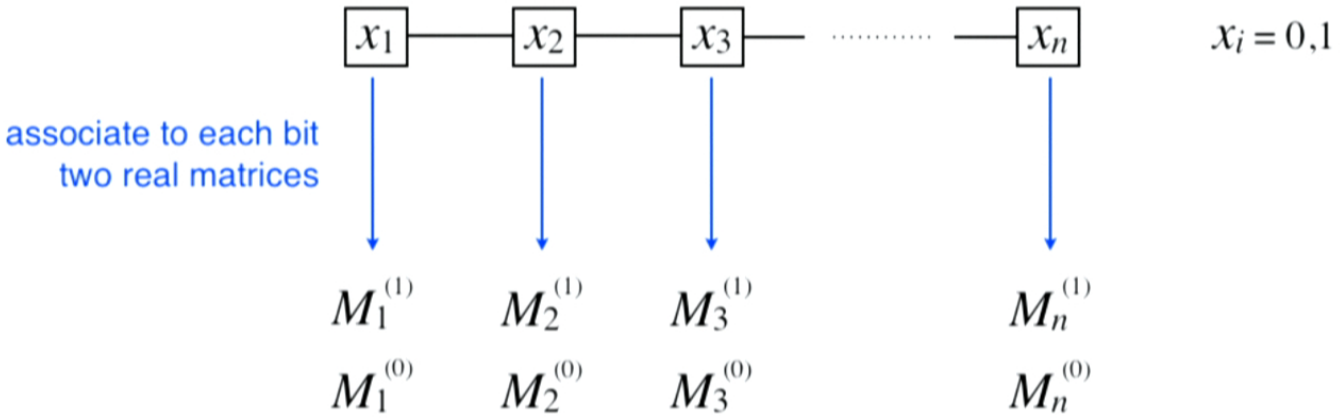
- Can we find inspiration from quantum systems to improve classical algorithms?

  - parallelism?

Our Method

Consider a string of *n* bits:



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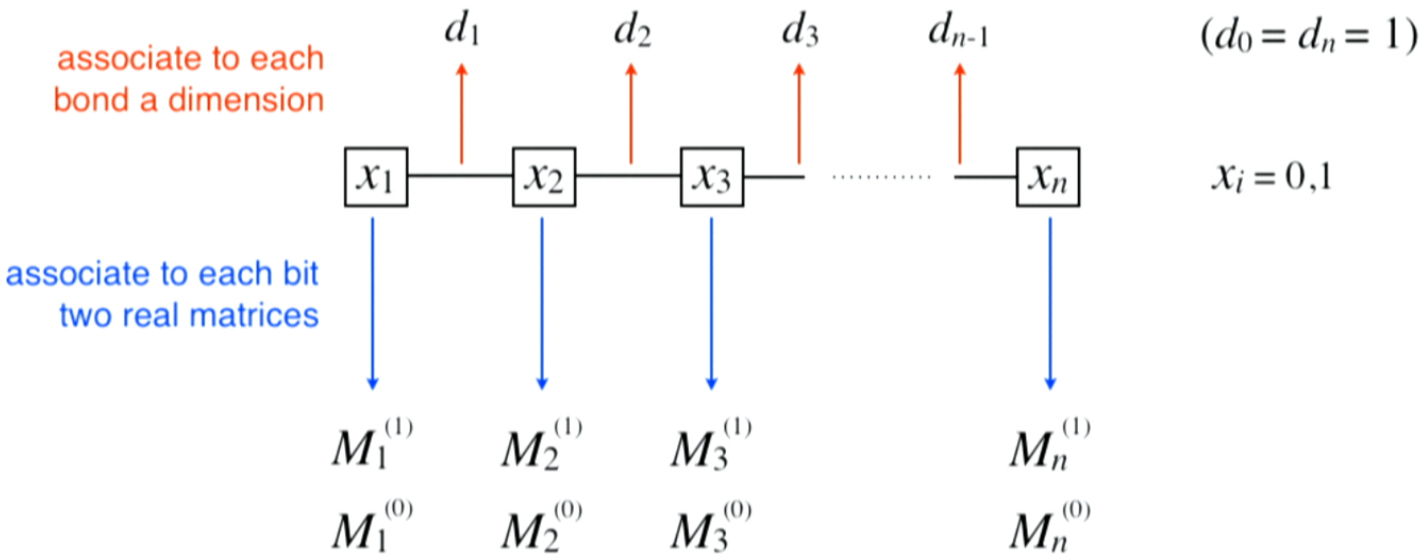

matrix dimensions:

$$\left[M_i^{(x_i)}\right]_{d_{i-1}\times d_i}$$

edge bit matrices are vectors:

$$\left[M_1^{(x_1)}\right]_{1\times d_2}$$

$$\left[M_n^{(x_n)}\right]_{d_{n-1}\times 1}$$

The probability distribution is normalized:

$$P(x_1, x_2, \ldots, x_n) = \frac{1}{Z} M_1^{(x_1)} M_2^{(x_2)} \cdots M_n^{(x_n)}$$

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$$Z = \sum_{x_1=0,1} \sum_{x_2=0,1} \cdots \sum_{x_n=0,1} M_1^{(x_1)} M_2^{(x_2)} \cdots M_n^{(x_n)}$$

$$= \left(M_1^{(0)} + M_1^{(1)}\right) \left(M_2^{(0)} + M_2^{(1)}\right) \cdots \left(M_n^{(0)} + M_n^{(1)}\right)$$

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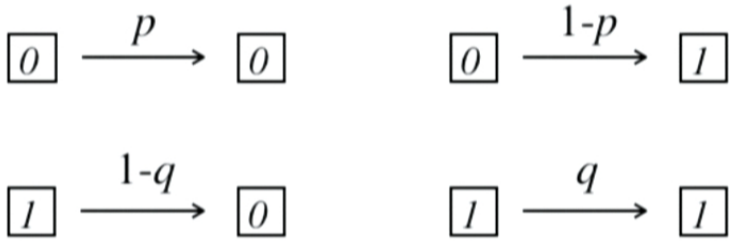
normalization constant

$$\Rightarrow \quad \sum_{x_1=0,1} \sum_{x_2=0,1} \cdots \sum_{x_n=0,1} P(x_1, x_2, \ldots, x_n) = 1$$

- As gates are applied to the bit string, $P(x_1,x_2,\ldots,x_n)$ evolves.

- Gates preserve the normalization of the probability distribution.

One-bit gates: *probabilistic*



transfer matrix

$$t^{0,0} = p$$
$$t^{0,1} = 1 - p$$
$$t^{1,0} = 1 - q$$
$$t^{1,1} = q$$

One-bit gates: *probabilistic*

$0 \xrightarrow{p} 0$ $\qquad$ $0 \xrightarrow{1\text{-}p} 1$

$1 \xrightarrow{1\text{-}q} 0$ $\qquad$ $1 \xrightarrow{q} 1$

transfer matrix

$$t^{0,0} = p$$
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$$t^{1,1} = q$$

⇒ changes the bit matrices

$$\tilde{M}_i^{(x_i)} = \sum_{x_i'=0,1} t^{x_i,x_i'} M_i^{(x_i')}$$

Examples:

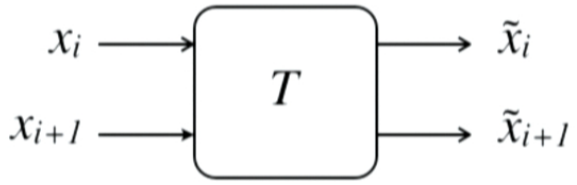
NOT gate $(p = q = 0)$

$$\tilde{M}_i^{(0)} = M_i^{(1)}$$
$$\tilde{M}_i^{(1)} = M_i^{(0)}$$

RAND gate $(p = q = 1/2)$

$$\tilde{M}_i^{(0)} = \tilde{M}_i^{(1)} = \frac{1}{2}\left(M_i^{(0)} + M_i^{(1)}\right)$$

Two-bit gates: *deterministic*



defined by two logical functions $A(a,b)$, $B(a,b)$

transfer matrix $T^{\tilde{a}\tilde{b},ab} = \begin{cases} 1, & \tilde{a} = A(a,b) \text{ and } \tilde{b} = B(a,b) \\ 0, & \text{otherwise} \end{cases}$

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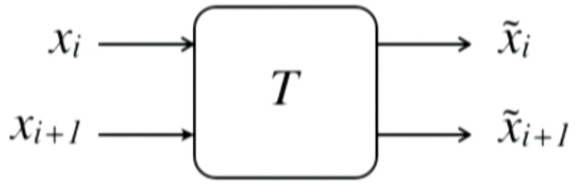

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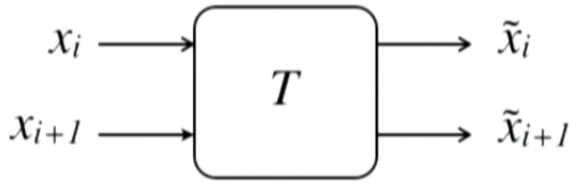

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- The new bit matrices must satisfy

$$\tilde{M}_i^{(x_i)}\, \tilde{M}_{i+1}^{(x_{i+1})} = \sum_{x'_i, x'_{i+1}=0,1} T^{x_i x_{i+1}, x'_i, x'_{i+1}}\, M_i^{(x'_i)}\, M_{i+1}^{(x'_{i+1})}$$

Example: NAND gate

$$A_{\mathrm{NAND}}(a,b) = a$$

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$$T^{01,00} = T^{01,01} = T^{11,10} = T^{10,11} = 1$$

truth table

x_i	x_{i+1}	$\tilde{x}_i$	$\tilde{x}_{i+1}$
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composed matrix

$$\mathcal{M}^{\mathrm{NAND}}_{i,i+1} = \left(\begin{array}{c|c} 0 & M_i^{(0)} M_{i+1}^{(0)} + M_i^{(0)} M_{i+1}^{(1)} \\ \hline M_i^{(1)} M_{i+1}^{(1)} & M_i^{(1)} M_{i+1}^{(0)} \end{array}\right)$$

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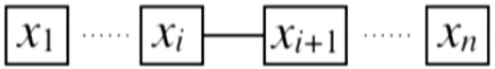
Bit insertion:



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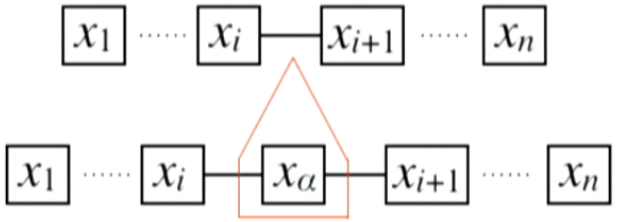

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$$M_\alpha^{(0)} = I \quad \text{identity matrix}$$

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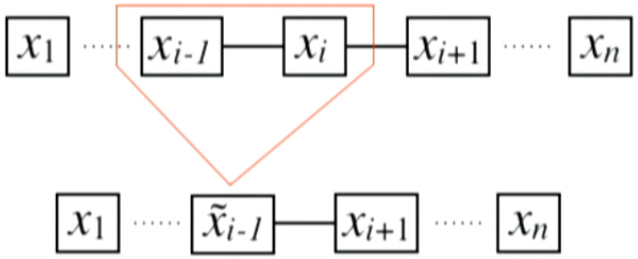

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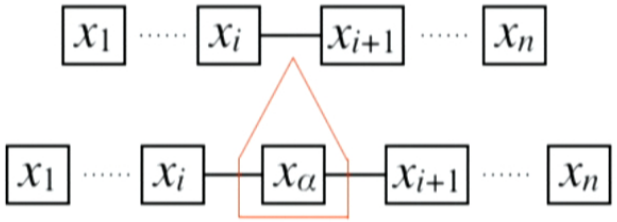

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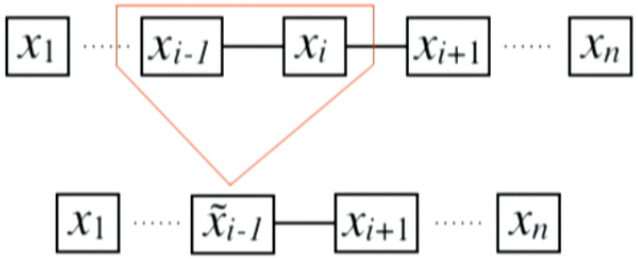

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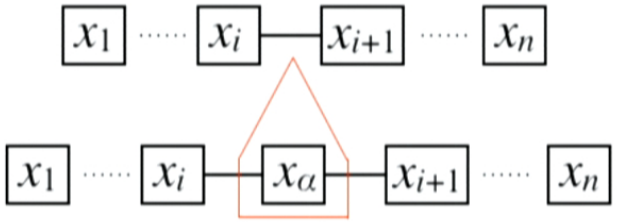

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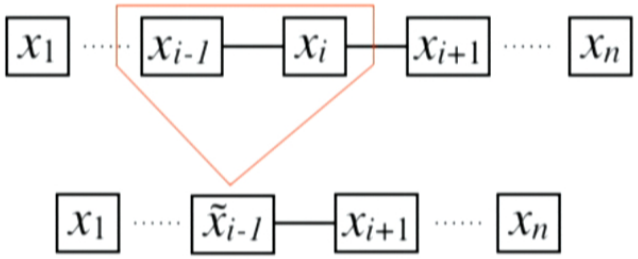

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$$\tilde{M}_{i-1}^{(x_{i-1})} = M_{i-1}^{(x_{i-1})} \left(M_i^{(0)} + M_i^{(1)} \right)$$

Search Algorithm based on Matrix Product States

Consider a function $y = f(x)$ which can be implemented with $\mathcal{O}(n^p)$ gates.

Given a certain y^* we want to find x^* such that $y^* = f(x^*)$.

Standard approach: for every value of x, compute $f(x)$ until you get y^*.

MPS approach:
- create a superposition of all values of x
- apply all gates defining $f(x)$
- evaluate $P(y^*)$
- find $P(y^*) = \frac{m}{2^n}$, where m is the number of solutions
- modify the input x bit by bit and repeat the calculation

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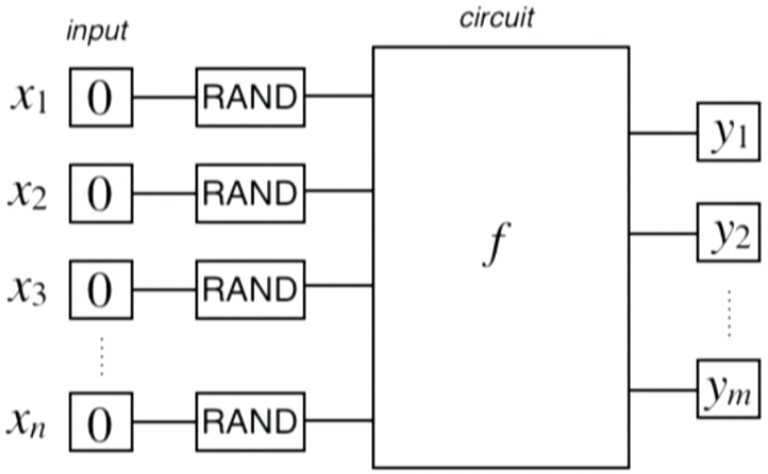

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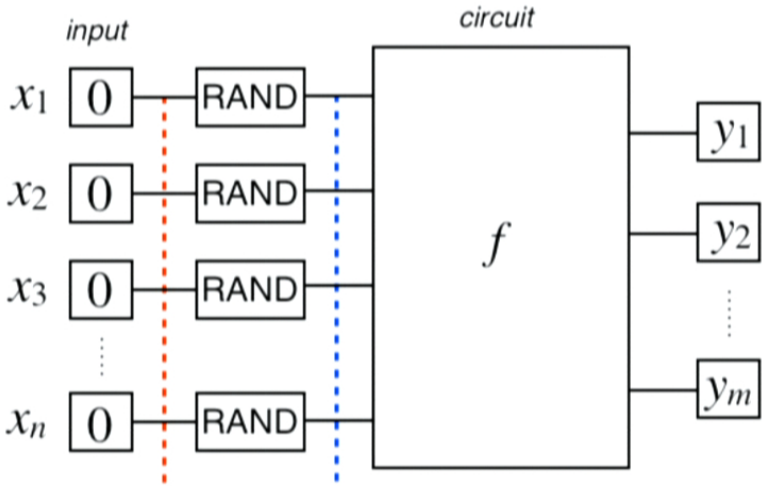

$$|\Psi\rangle = |0\rangle_1 |0\rangle_2 |0\rangle_3 \cdots |0\rangle_n$$

$$M_i^{(0)} = 1$$
$$M_i^{(1)} = 0$$

$$|\Psi\rangle = \frac{1}{2^n} \left(|0\rangle + |1\rangle\right)_1 \left(|0\rangle + |1\rangle\right)_2 \left(|0\rangle + |1\rangle\right)_3 \cdots \left(|0\rangle + |1\rangle\right)_n$$

$$M_i^{(0)} = M_i^{(1)} = \frac{1}{2}$$

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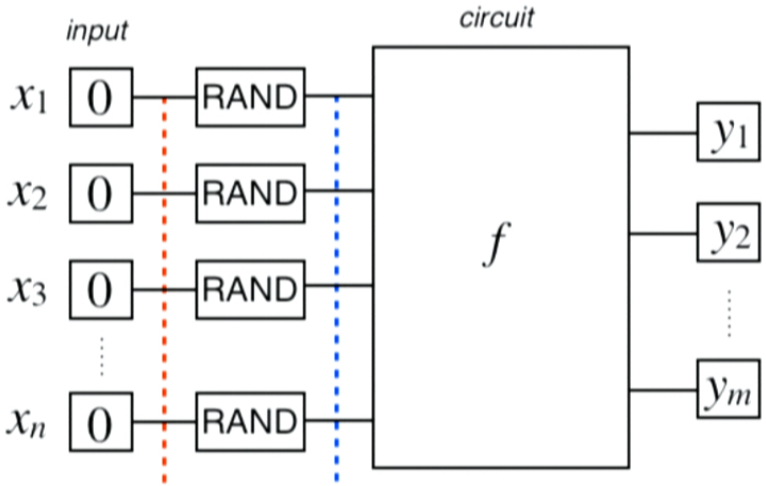

$$|\Psi\rangle = |0\rangle_1 |0\rangle_2 |0\rangle_3 \cdots |0\rangle_n \qquad M_i^{(0)} = 1 \quad M_i^{(1)} = 0$$

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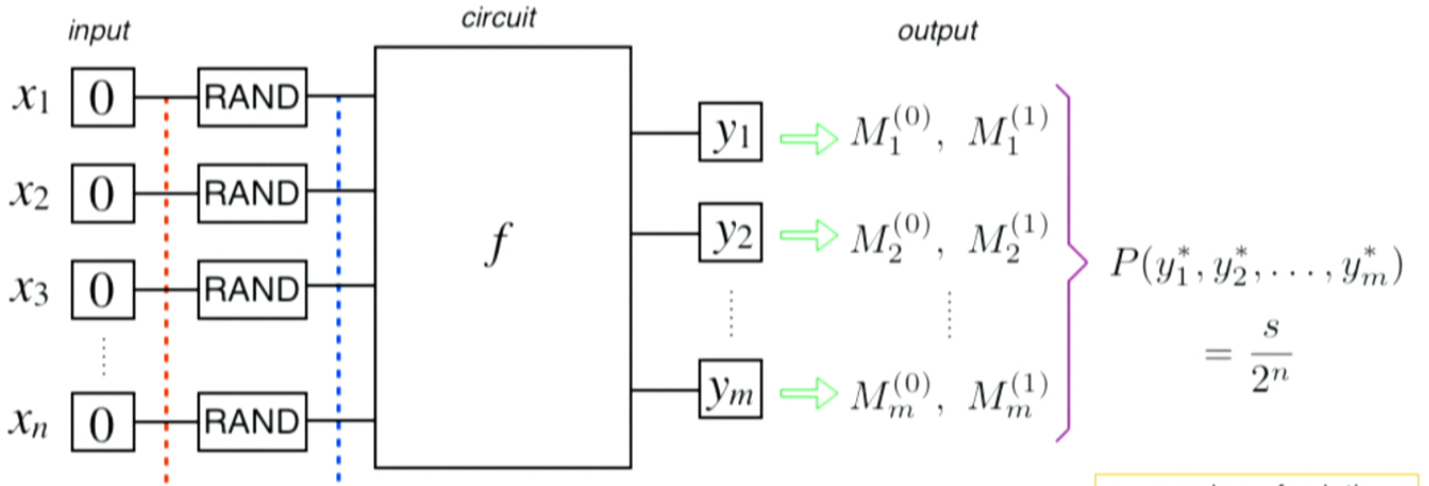

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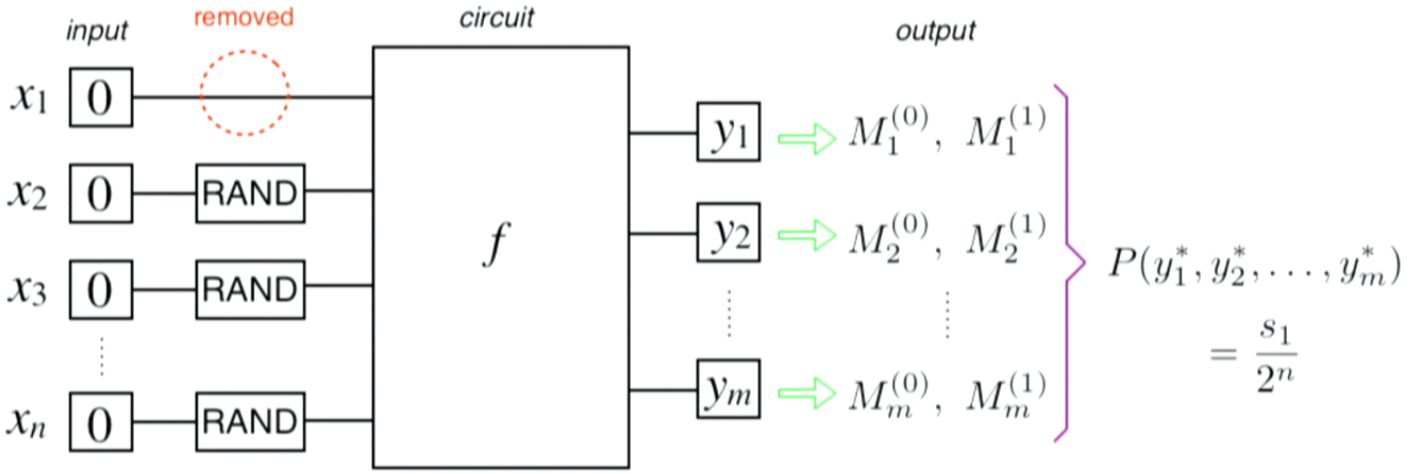

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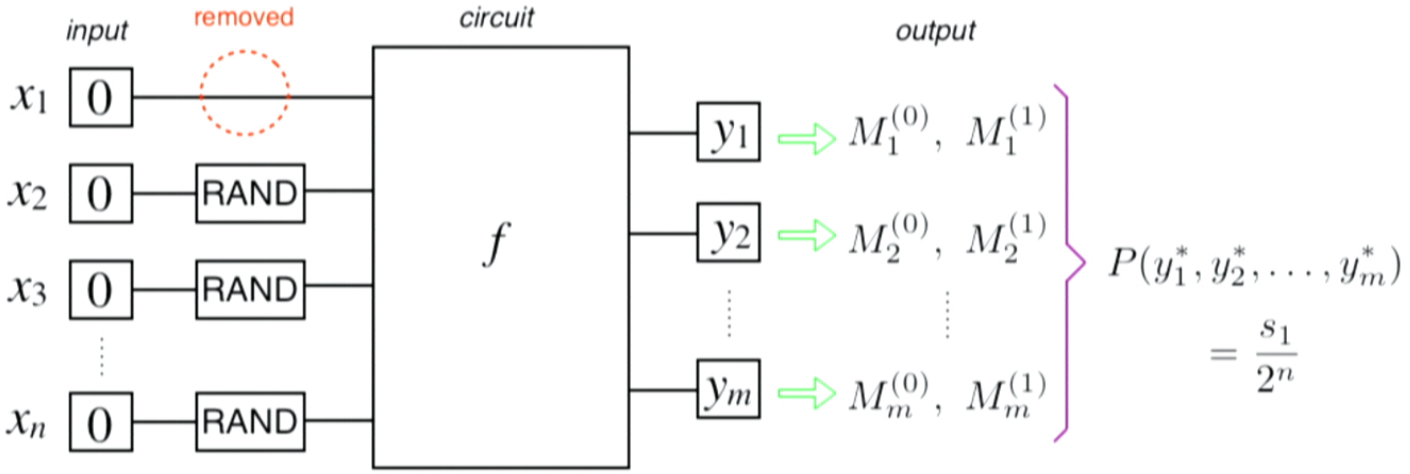

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$s_1 < 1 \Rightarrow$ set $x_1 = 1$

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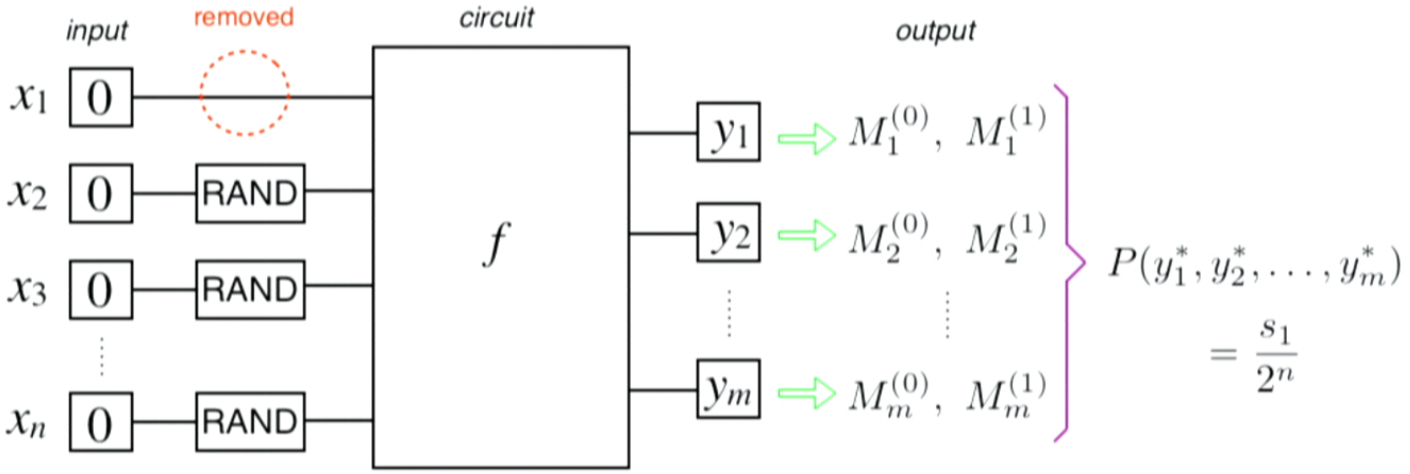

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$s_1 < 1 \Rightarrow$ set $x_1 = 1$

Repeat this procedure for all remaining n-1 bits

and find $x_1^*, x_2^*, \ldots, x_n^*$

Computational Cost

SVD $\left[M_i^{(x_i)}\right]_{d_{i-1}\times d_i}$ $\left[M_{i+1}^{(x_{i+1})}\right]_{d_i\times d_{i+1}}$ $\sim \mathcal{O}\left(d_i^3\right)$

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n_g = number of two-bit gates

D = maximum bond dimension

⇨ circuit computational cost $\sim \mathcal{O}\left(n_g \times D^3\right)$

need to reapply the circuit n times

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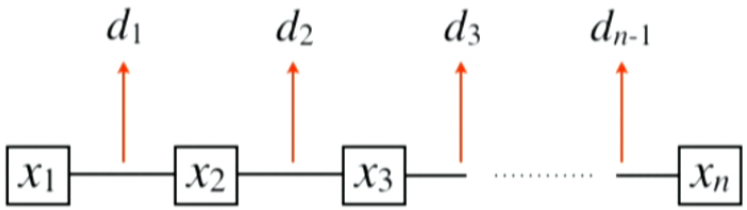
Proof:



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$x_i = 0,1$

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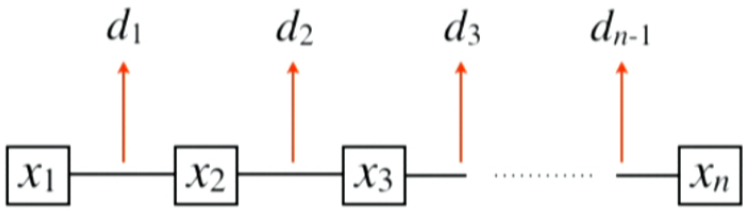

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Action of 2-bit gates

Recall, e.g., the NAND gate

truth table

x_i	x_{i+1}	$\tilde{x}_i$	$\tilde{x}_{i+1}$
0	0	0	1
1	0	1	1
0	1	0	1
1	1	1	0

composed matrix

$$\mathcal{M}^{\mathrm{NAND}}_{i,i+1} = \left(\begin{array}{c:c} 0 & M_i^{(0)}M_{i+1}^{(0)} + M_i^{(0)}M_{i+1}^{(1)} \\ \hdashline M_i^{(1)}M_{i+1}^{(1)} & M_i^{(1)}M_{i+1}^{(0)} \end{array}\right) \overset{\mathrm{SVD}}{=} \begin{pmatrix} \tilde{M}_i^{(0)} \\ \\ \tilde{M}_i^{(1)} \end{pmatrix} \begin{pmatrix} \tilde{M}_{i-1}^{(0)} & \tilde{M}_{i-1}^{(1)} \end{pmatrix}$$

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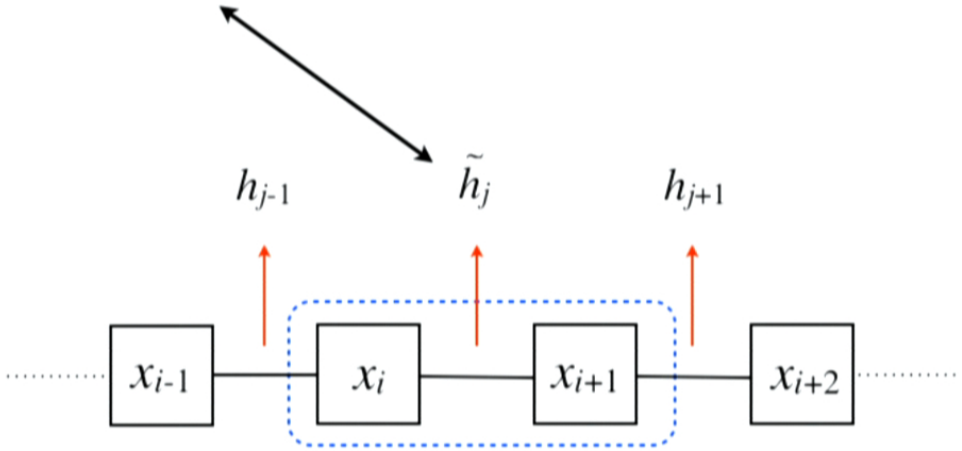
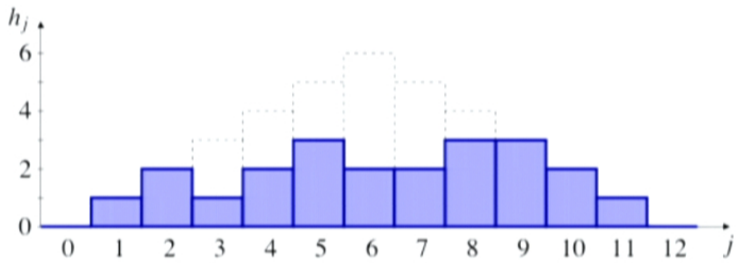
Action of 2-bit gates

$\tilde{d}_i = \min(2d_{i-1}, 2d_{i+1})$ define "heights" $\tilde{h}_i = \log_2 d_i$

$\tilde{h}_i = \min(h_{i-1}, h_{i+1}) + 1$



Evolution of heights with computational steps



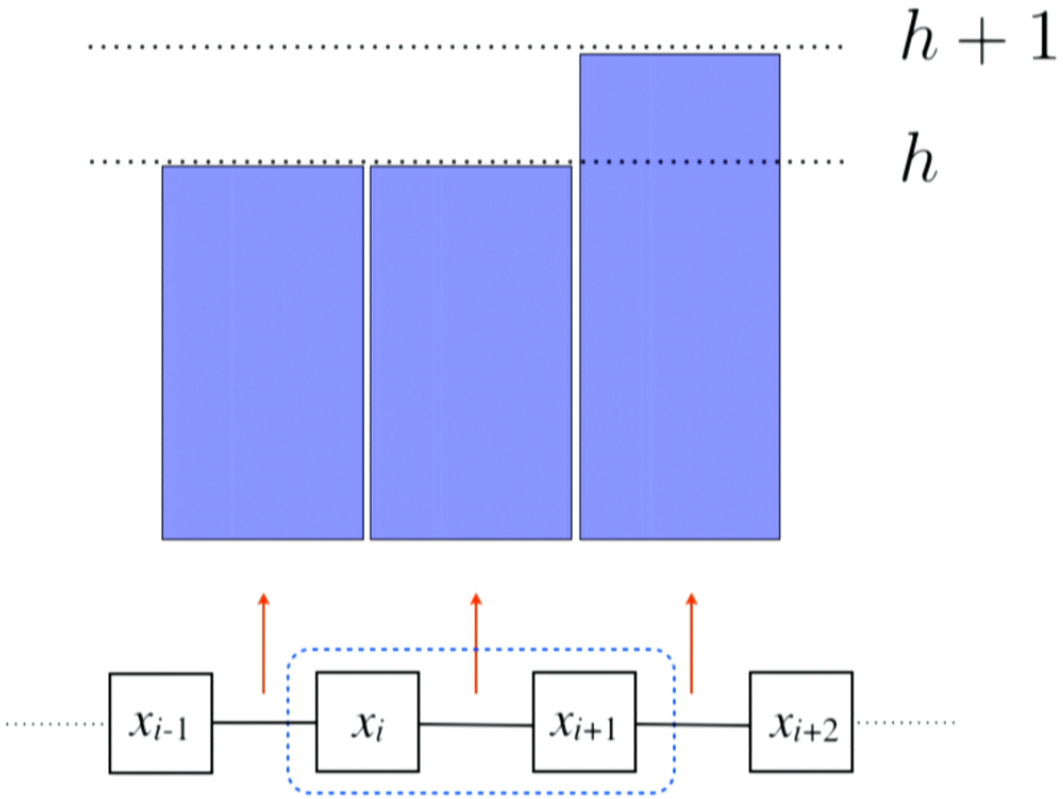
height profile satisfies $|h_i - h_{i-1}| \leq 1, \forall i$

proof by induction:

initial input is a product state $h_i = 0 \quad \forall i$

if condition is satisfied at step τ , it is satisfied at step $\tau + 1$

e.g. (1 of 9 cases)



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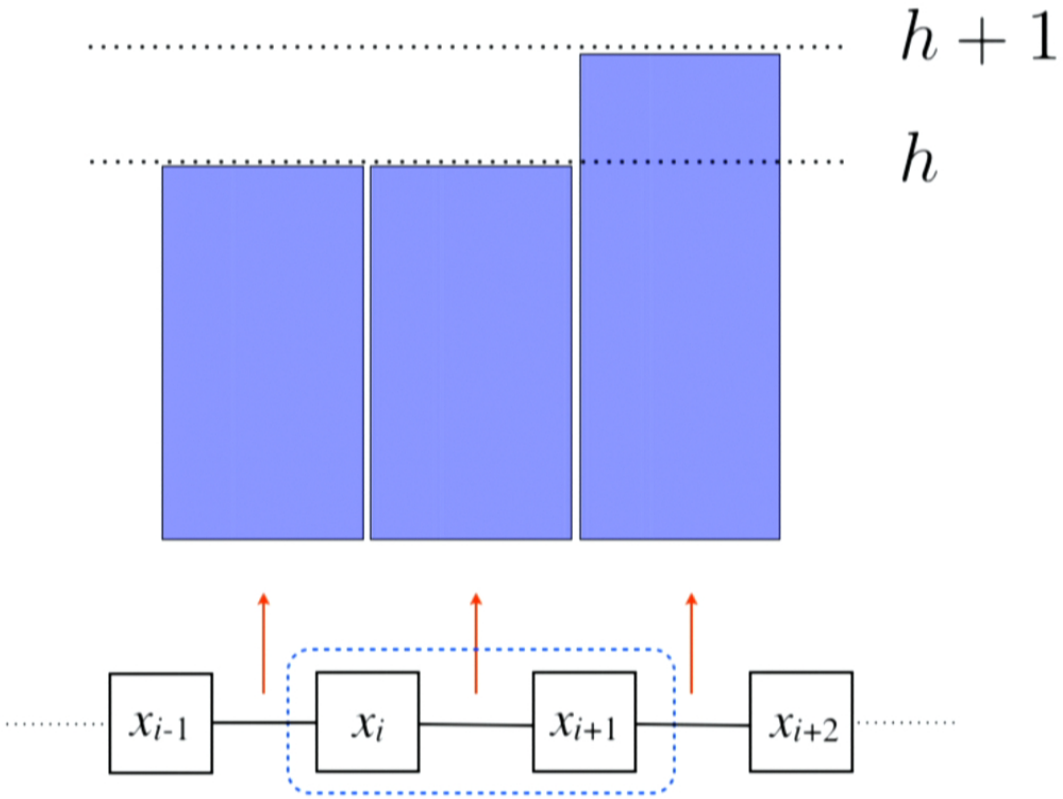

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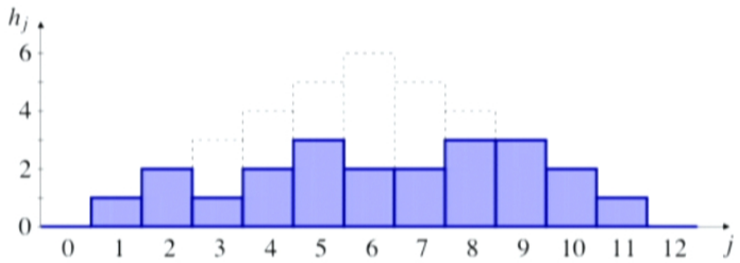

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comment: bit insertions and bit removals

$$\Delta h_i = h_i(\tau + 1) - h_i(\tau) \leq 2$$

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$$\Rightarrow \quad h_{\max} \leq \min\left(\lfloor \sqrt{2\, n_g} \rfloor, \lfloor n/2 \rfloor \right)$$

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"uniform case" $n_g \sim n \times \tau \;\rightarrow \tau \sim n \;\; (\text{or } L)$

to reach max "entanglement"

A. Hamma, S. Santra, and P. Zanardi, arXiv:1109.4391

Given $f(x)$, how does D scale with n_g and n?

$$D \leq D_{\max} = \min\left(2^{\lfloor\sqrt{2n_g}\rfloor}, 2^{\lfloor n/2\rfloor}\right)$$

Suppose $n_g \sim \mathcal{O}(n^p)$. We have two scenarios:

(i) if $p < 2 \implies D_{\max} \sim \mathcal{O}\left(2^{\sqrt{2}\, n^{p/2}}\right)$

algorithm cost $\sim \mathcal{O}\left(n^{p+1} \times 2^{3\sqrt{2}\, n^{p/2}}\right)$ *subexponential*

Critical point: $p_c = 2$

When $n_g < \mathcal{O}\left(n^2\right)$, the MPS search algorithm beats Grover's.

$\mathcal{O}\left(2^{n/2}\right)$

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catch: Grover's algorithm works for any unsorted data base, whereas the requirement $n_g < \mathcal{O}\left(n^2\right)$ may only be applicable to structured data base (we do not know for sure).

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But we can use the same method to solve other computational problems as well:

- satisfiability (2-SAT and #2-SAT) (in progress)

A topic for discussion

- Is there a “physical” interpretation to bond dimension growth?

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Complexity from physics perspective

truth table

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and remove bit i

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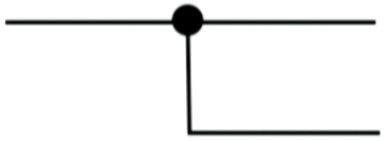
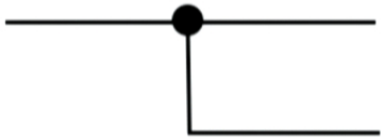


FANOUT
insert bit + copy gate

wire cross
SWAP gate

ONLY 2bit gates needed!

Complexity from physics perspective

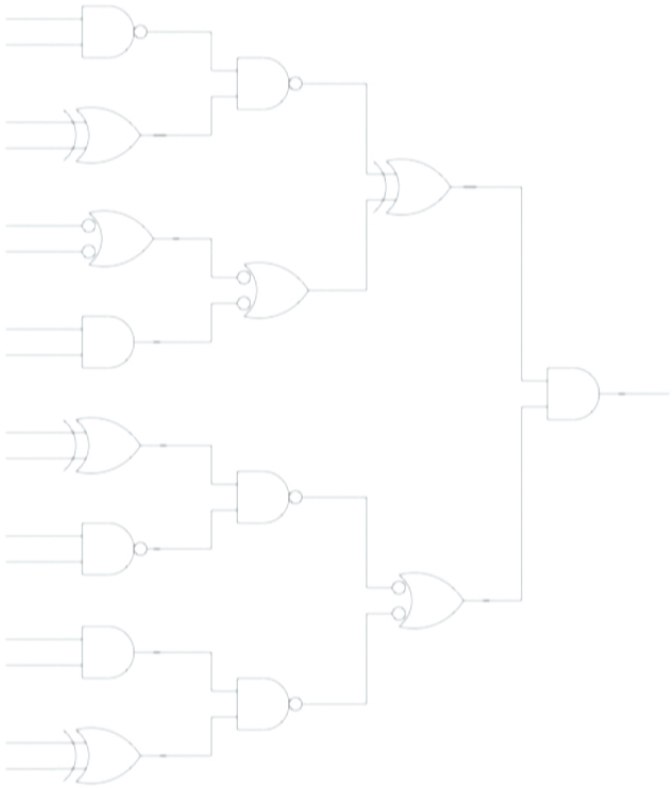


What is the physical measure of circuit complexity?

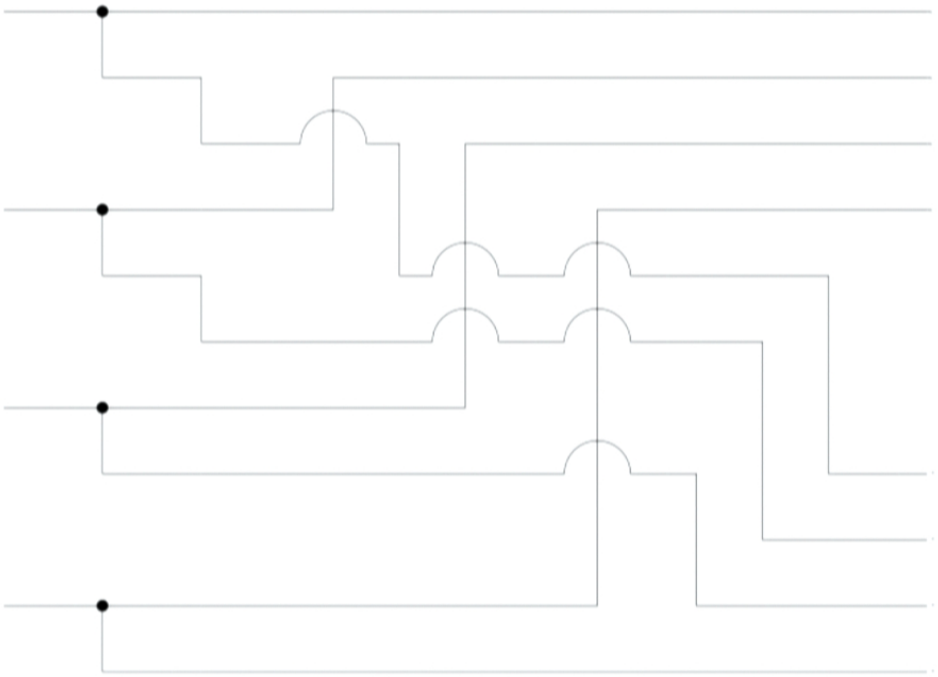
Entanglement!

LITERALLY!!

A trivial case with no entanglement



A case with a **LOT** of entanglement



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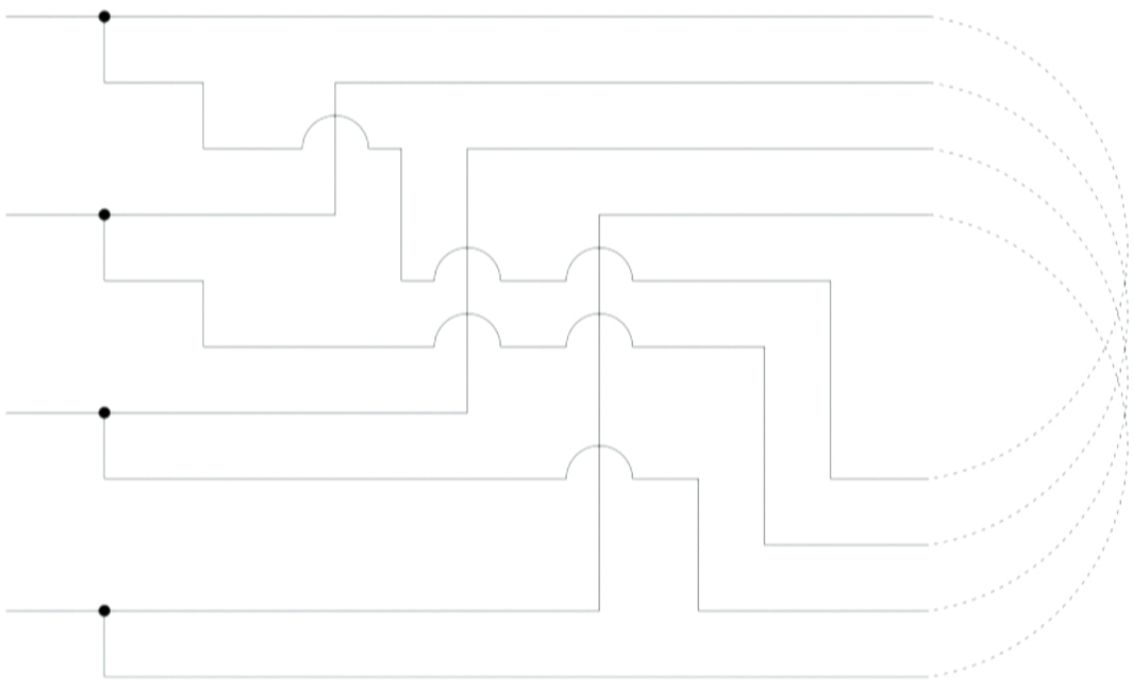

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THE END