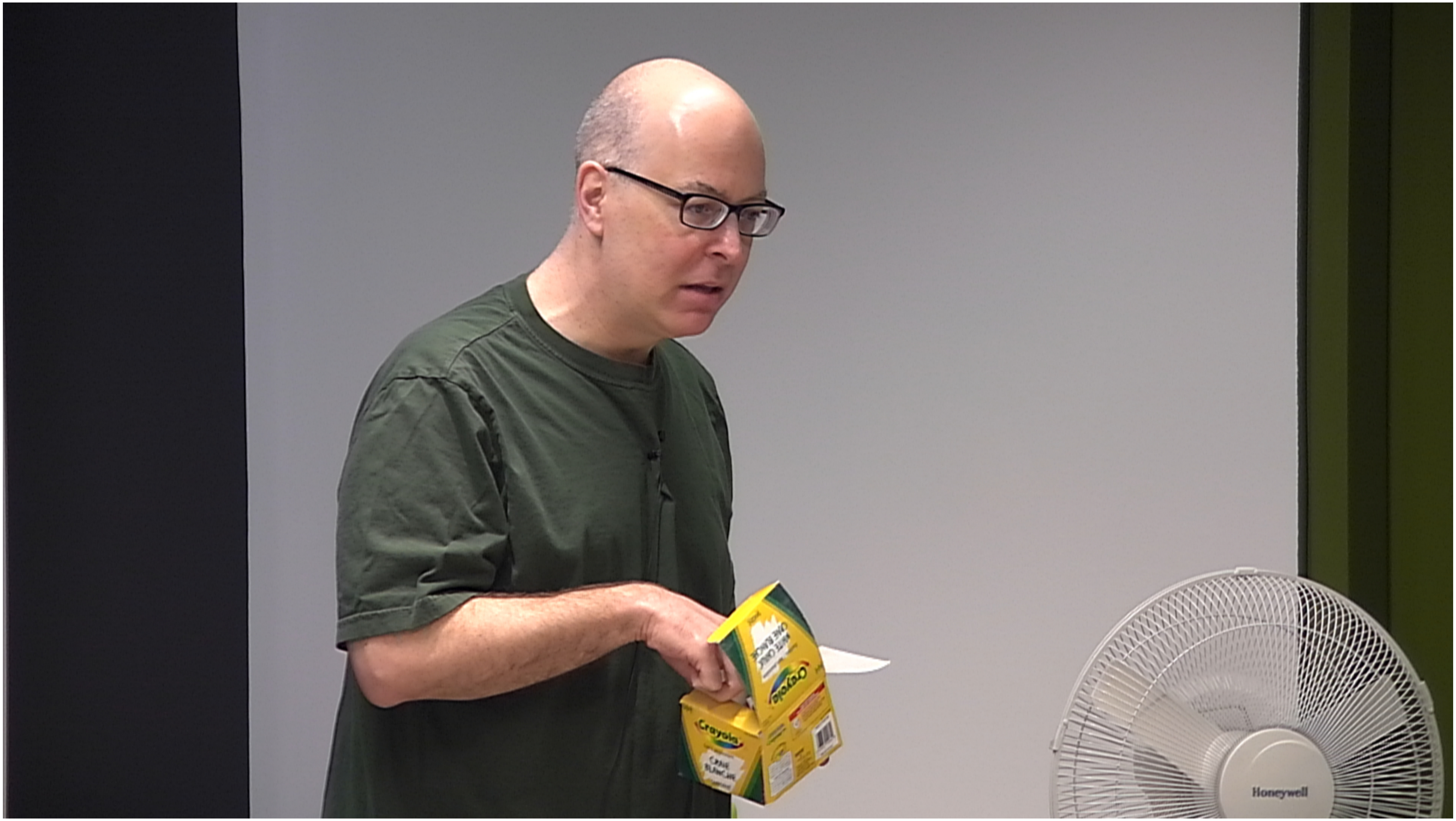
Title: Advanced General Relativity - Lecture 23

Date: Apr 04, 2012 10:00 AM

URL: http://pirsa.org/12040062

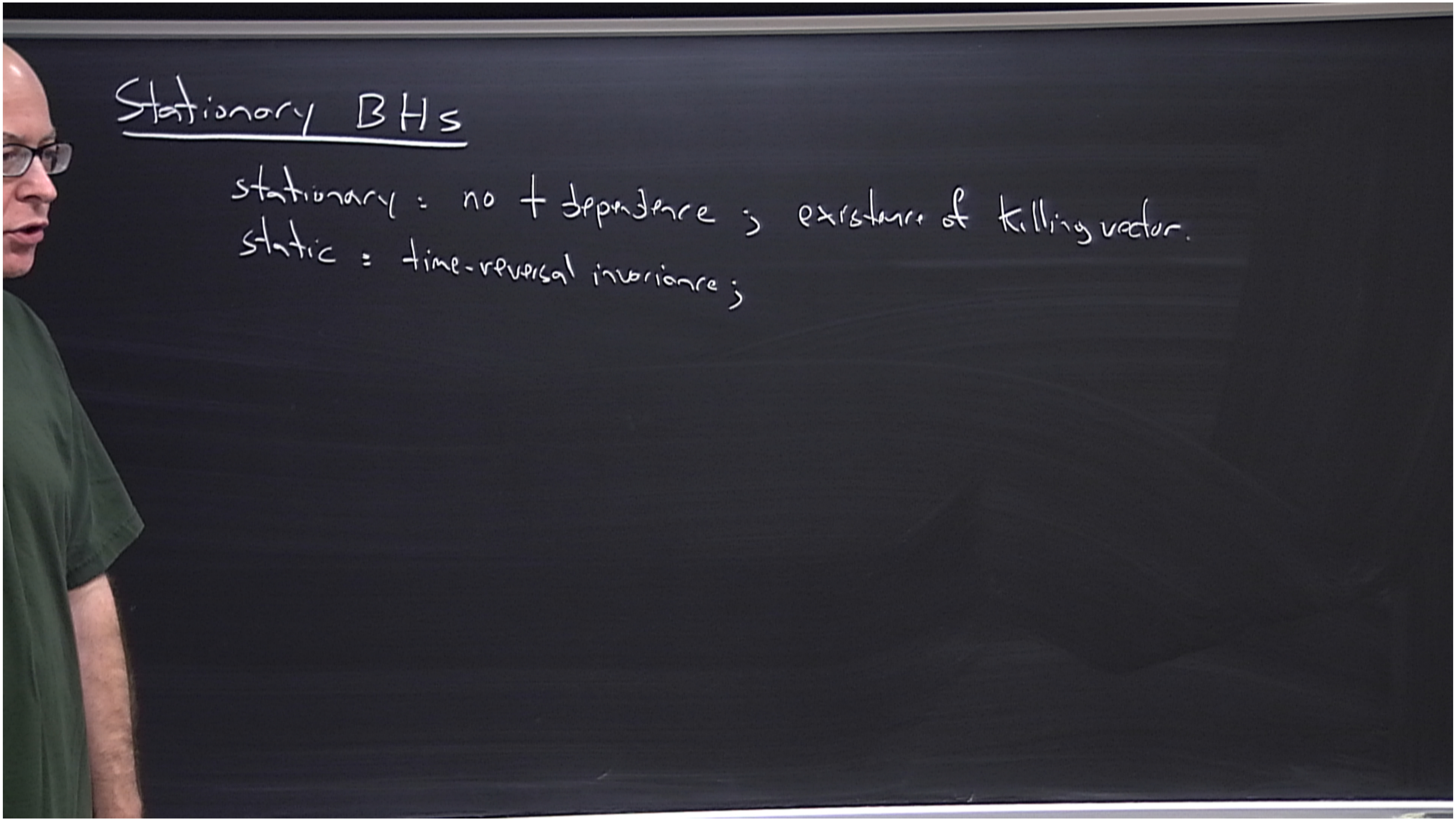
Abstract:


Crayola
Honeywell

Stationary BHs

stationary : no t dependence ; existence of killing vector.

static :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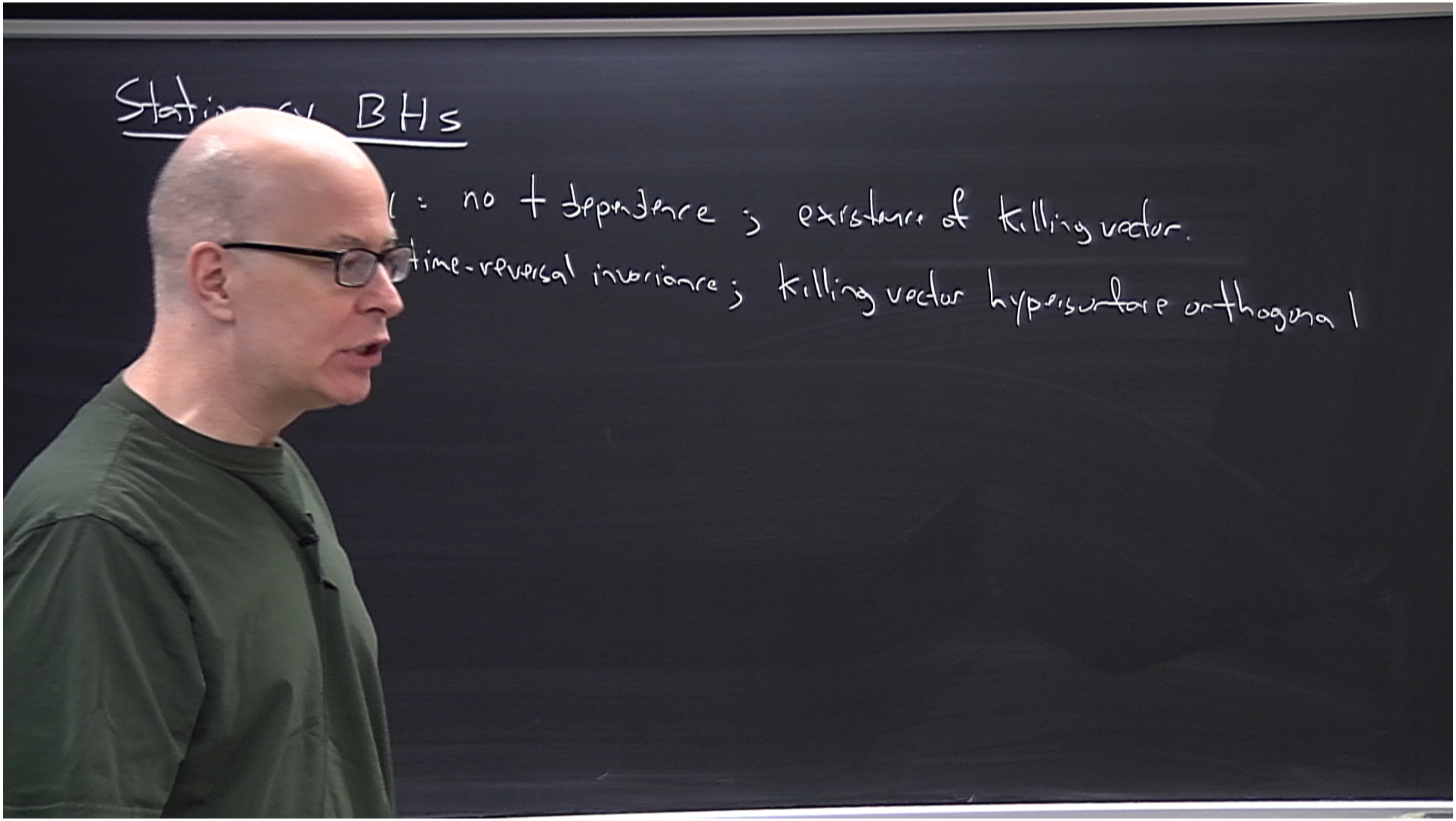
static : time-reversal invariance ;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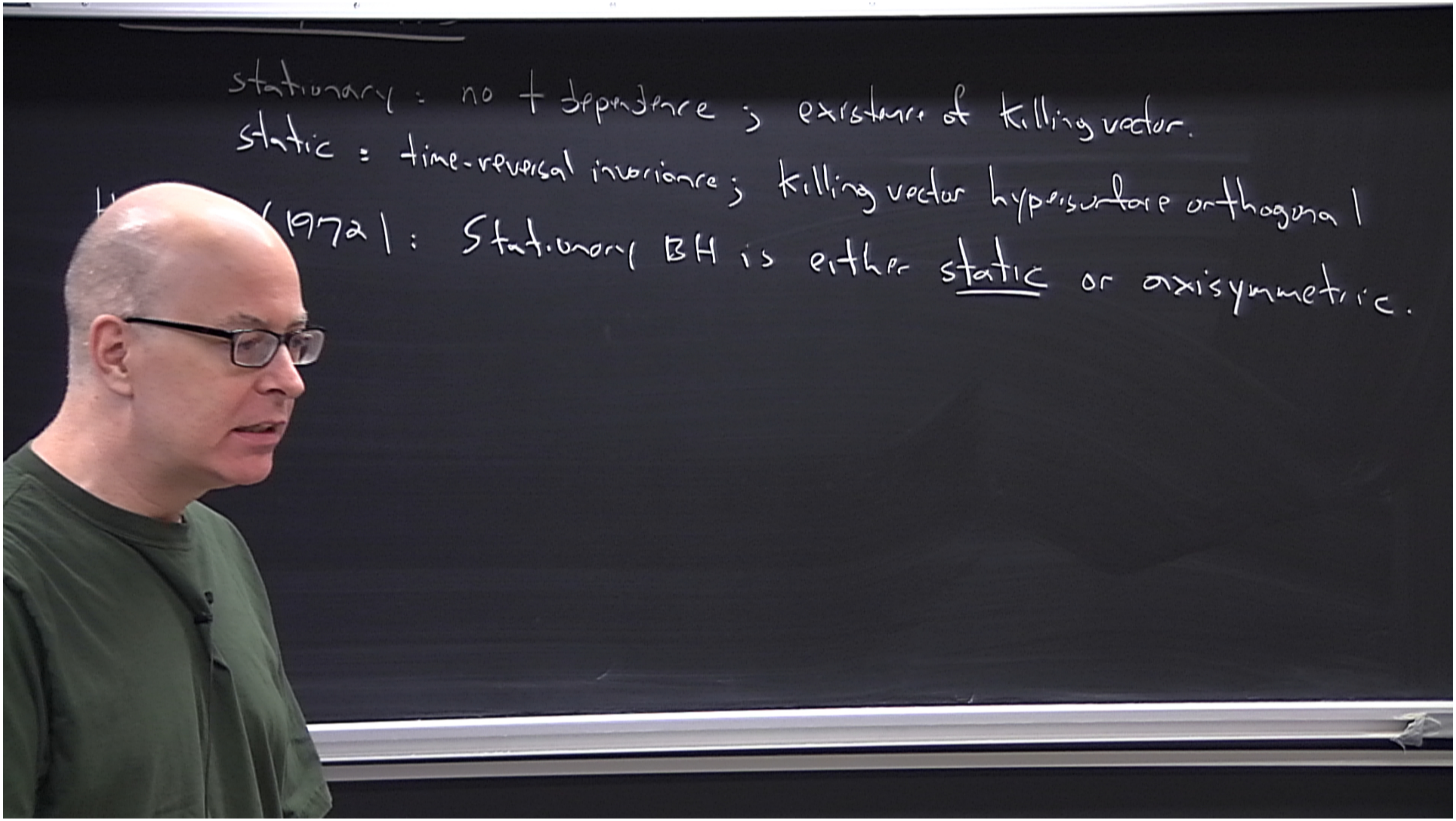
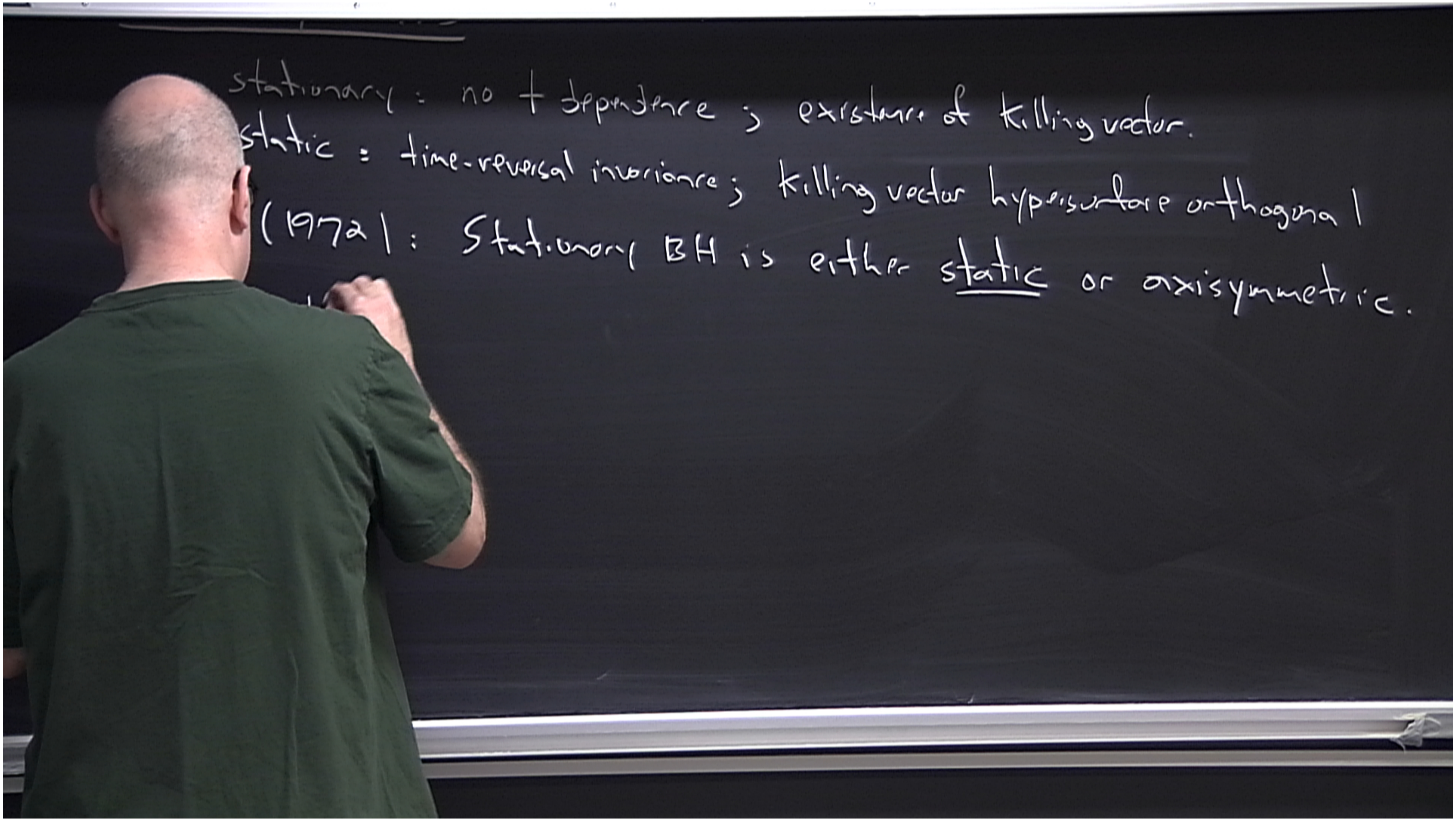

stationary : no t dependence ; existence of killing vector.
static : time-reversal invariance ; killing vector hypersurface orthogonal
(1972) : Stationary BH is either static or axisymmetric.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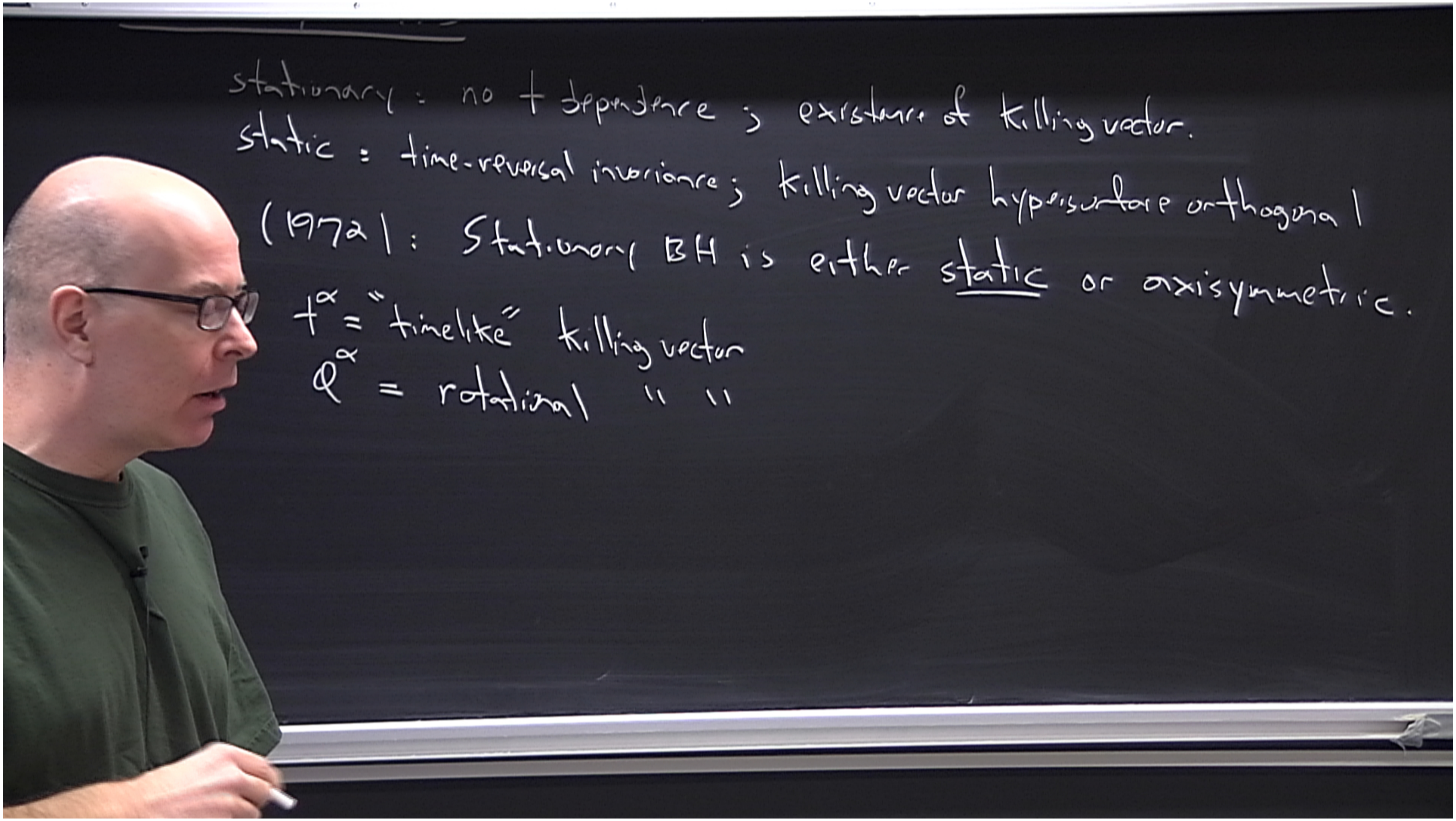
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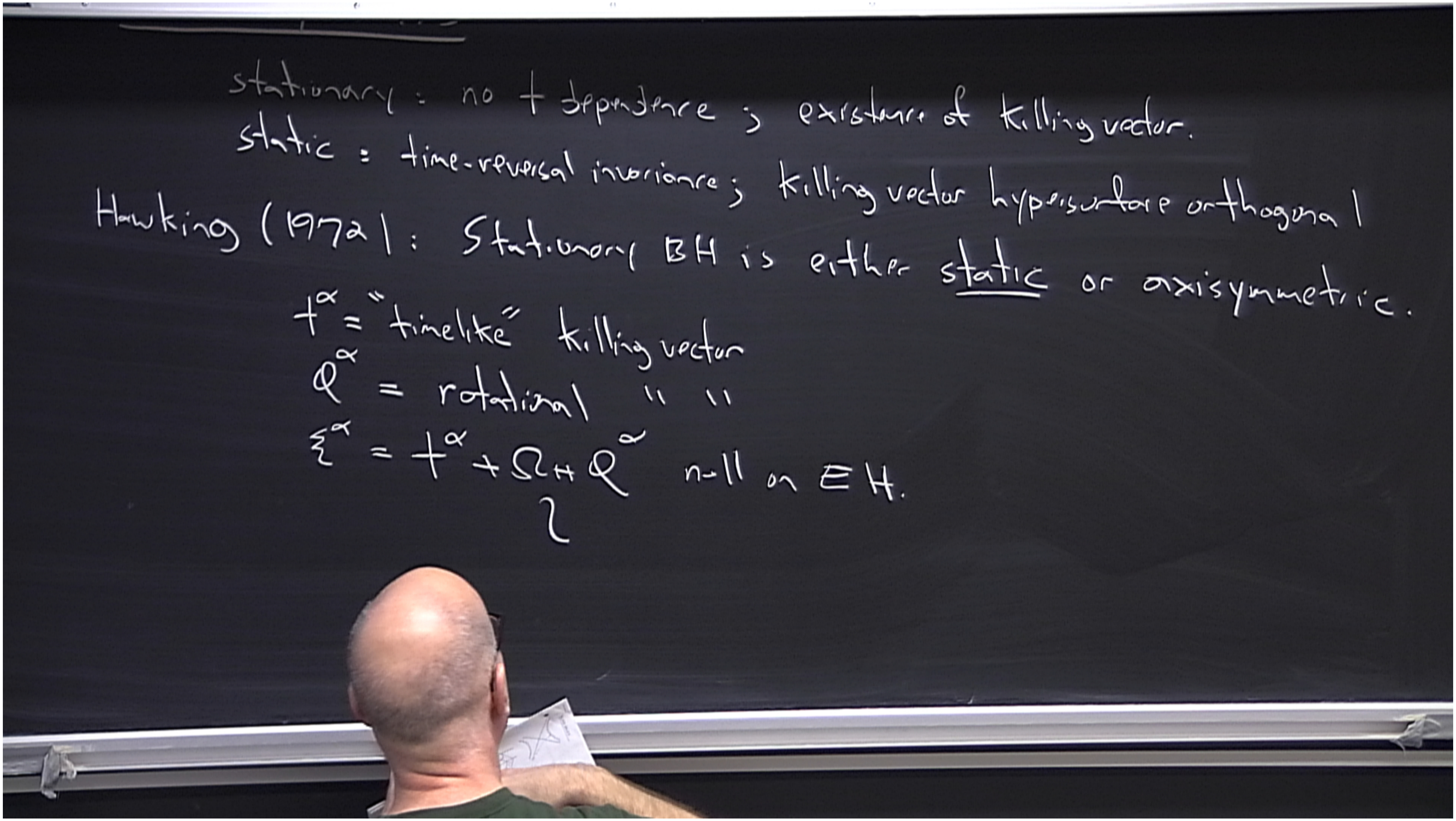
t^α = "timelike" killing vector
ϕ^α = rotational " "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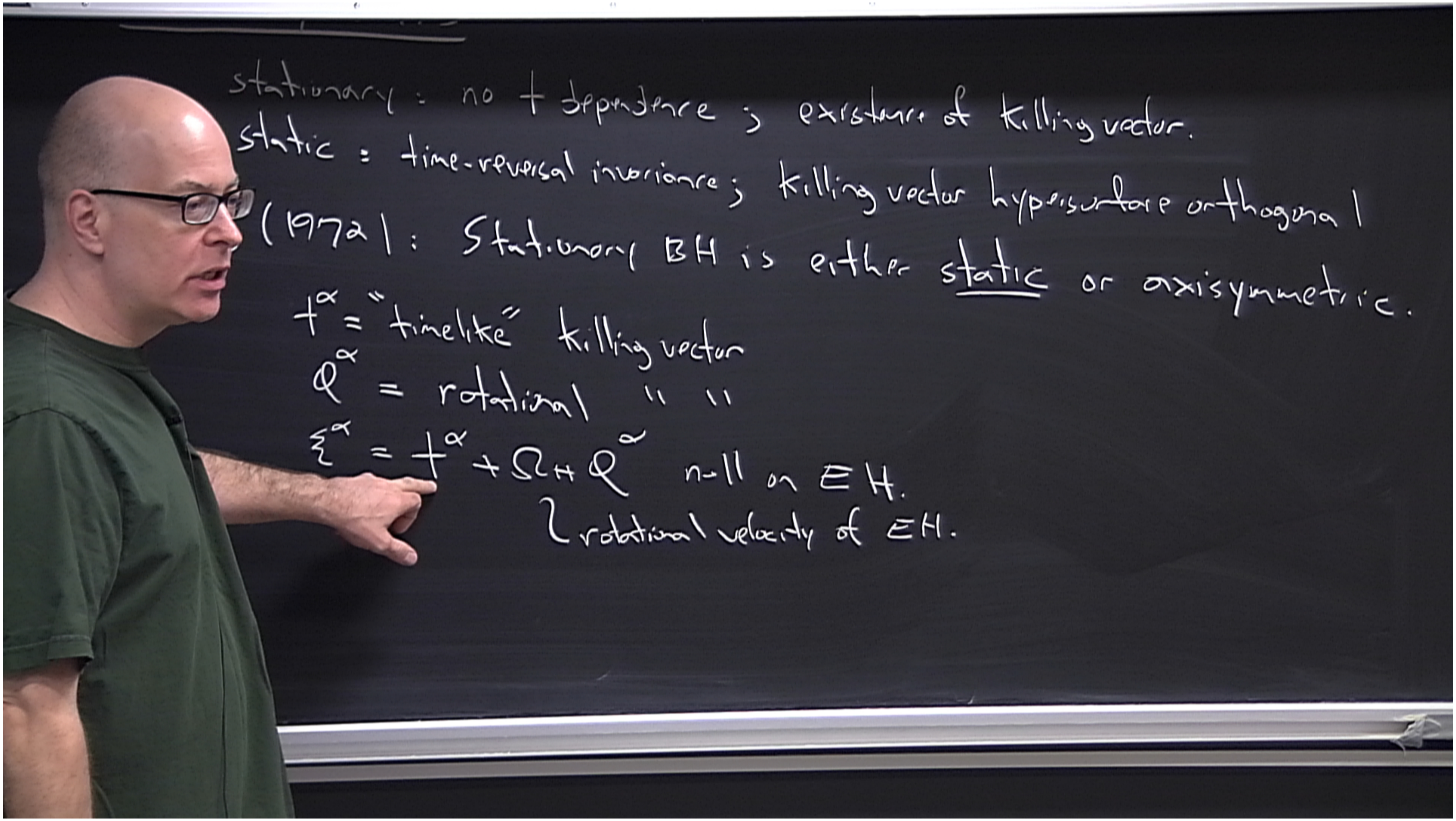
$\xi^\alpha = t^\alpha + \Omega_H \varphi^\alpha$ null on EH.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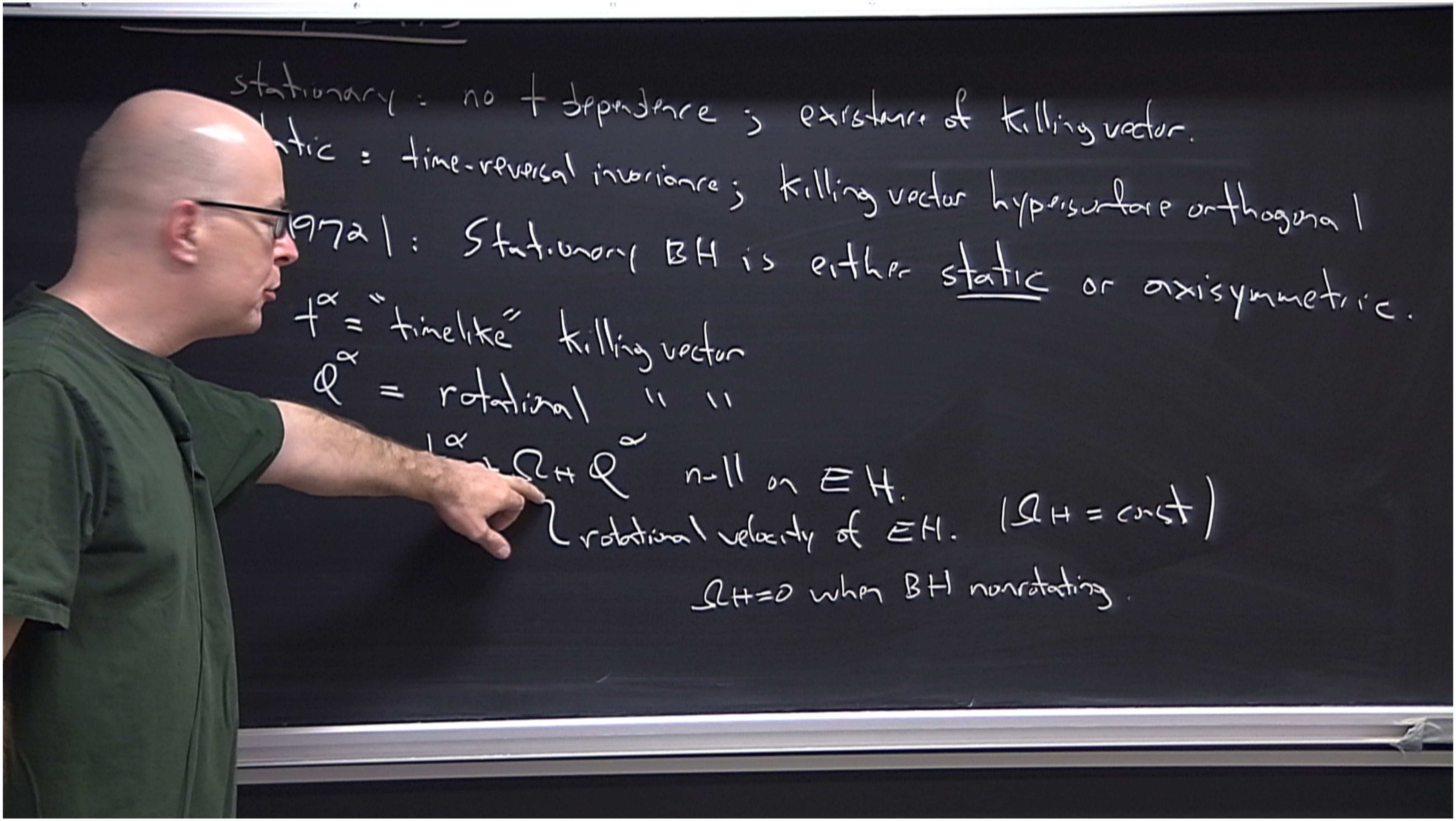
Ω_H: rotational velocity of EH.

$\Omega_H = 0$ when BH nonrotating.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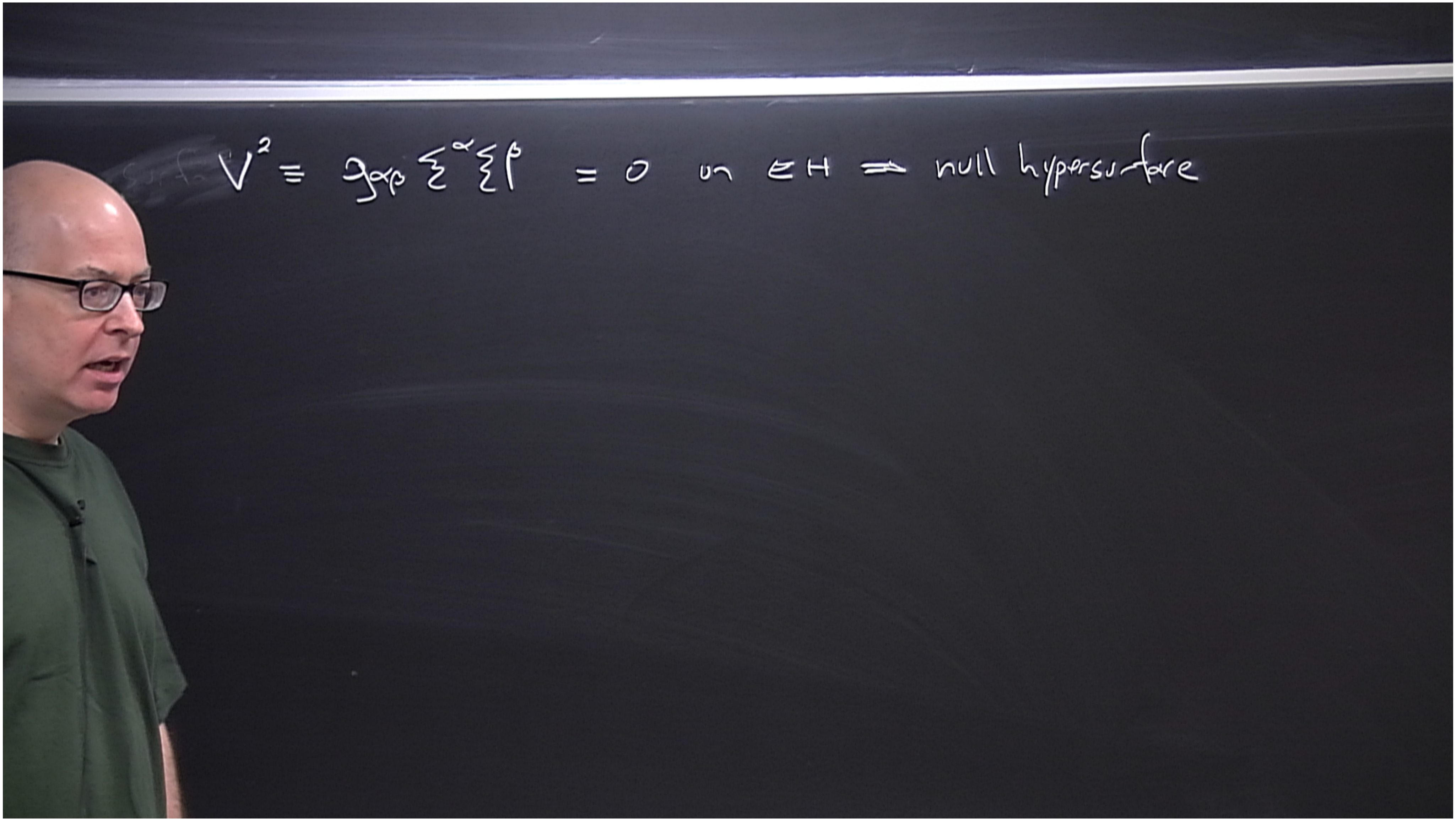
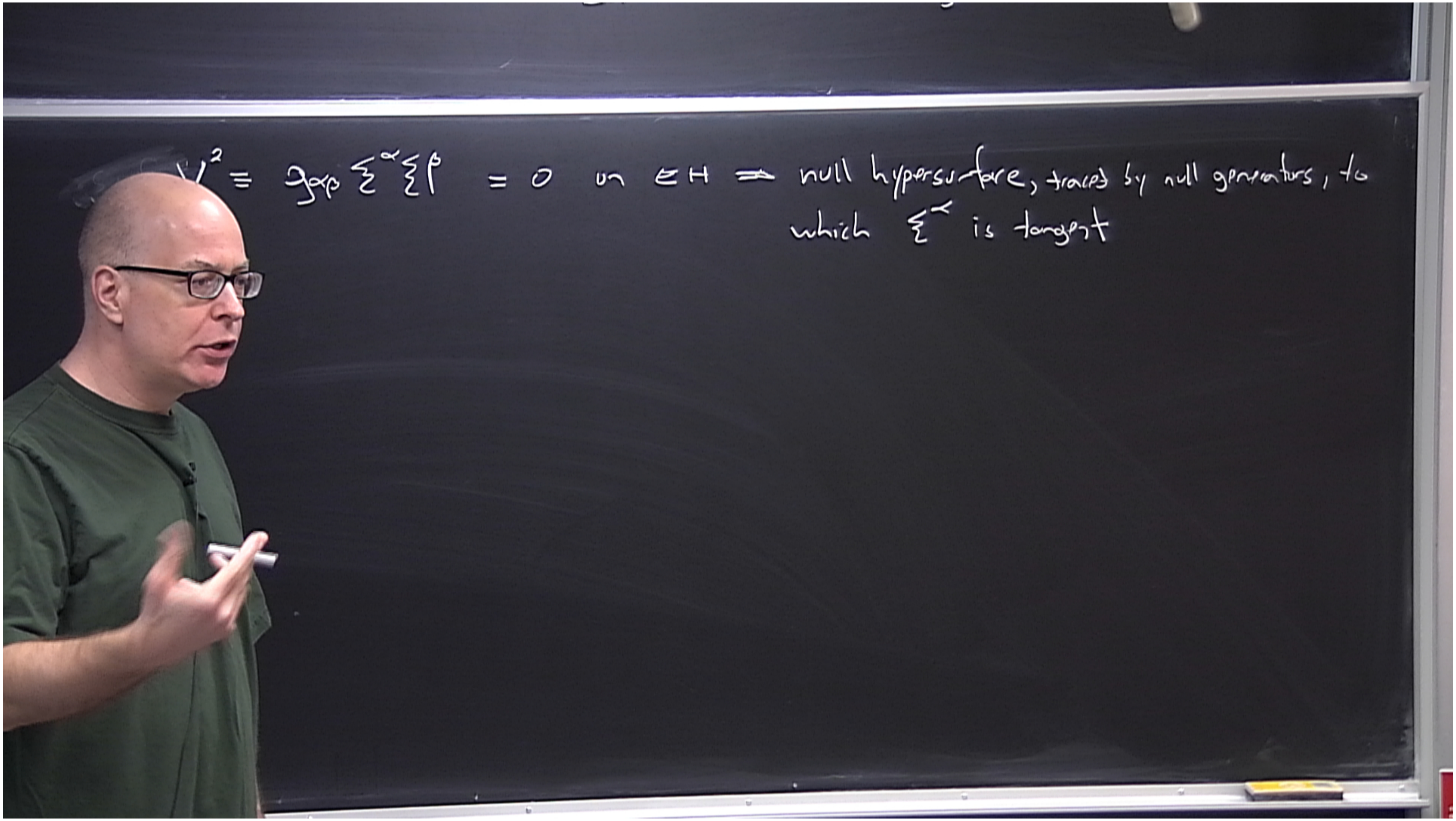

$V^2 \equiv g_{\alpha\beta} \xi^\alpha \xi^\beta = 0$ on EH $\Rightarrow$ null hypersurface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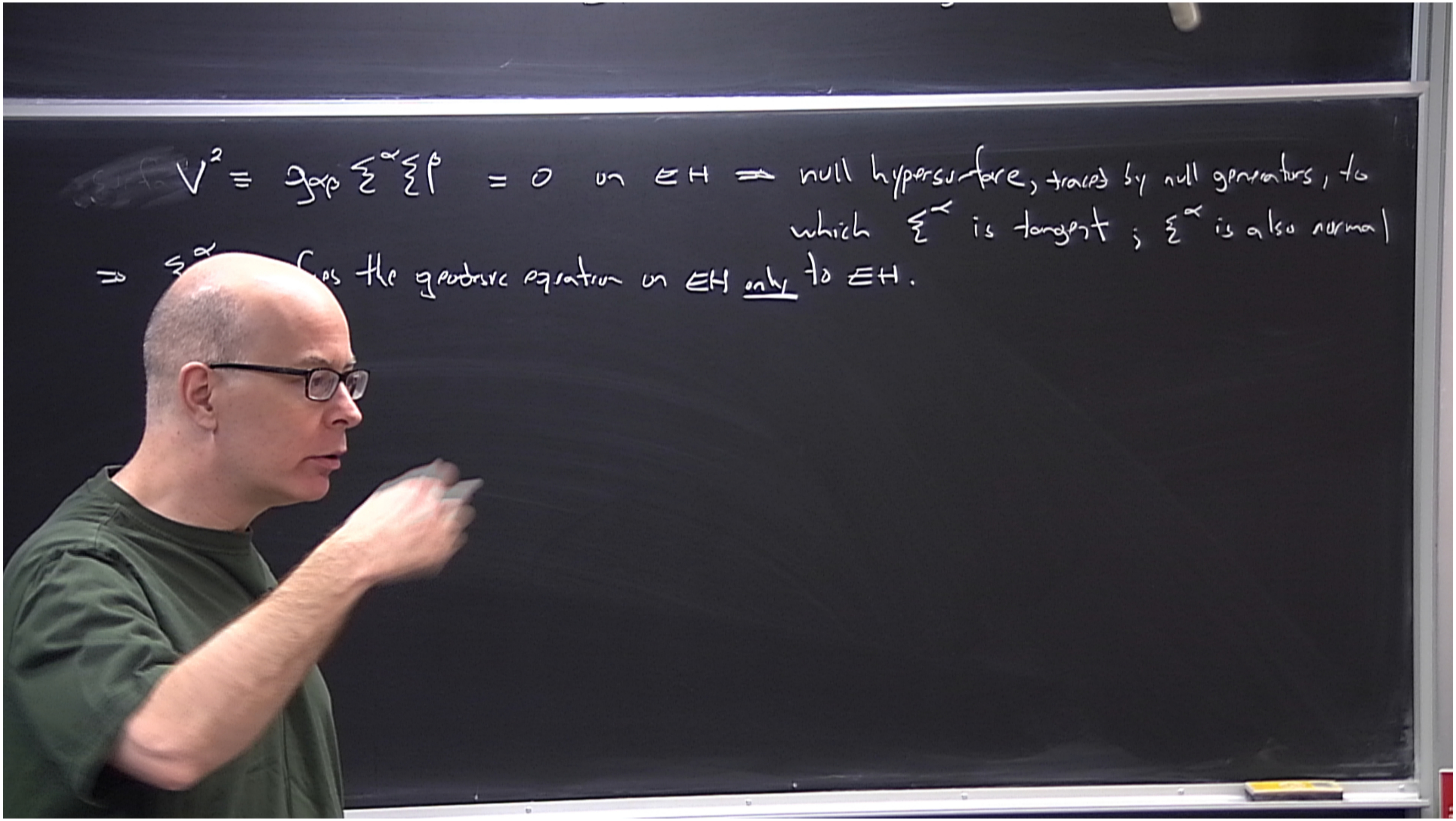
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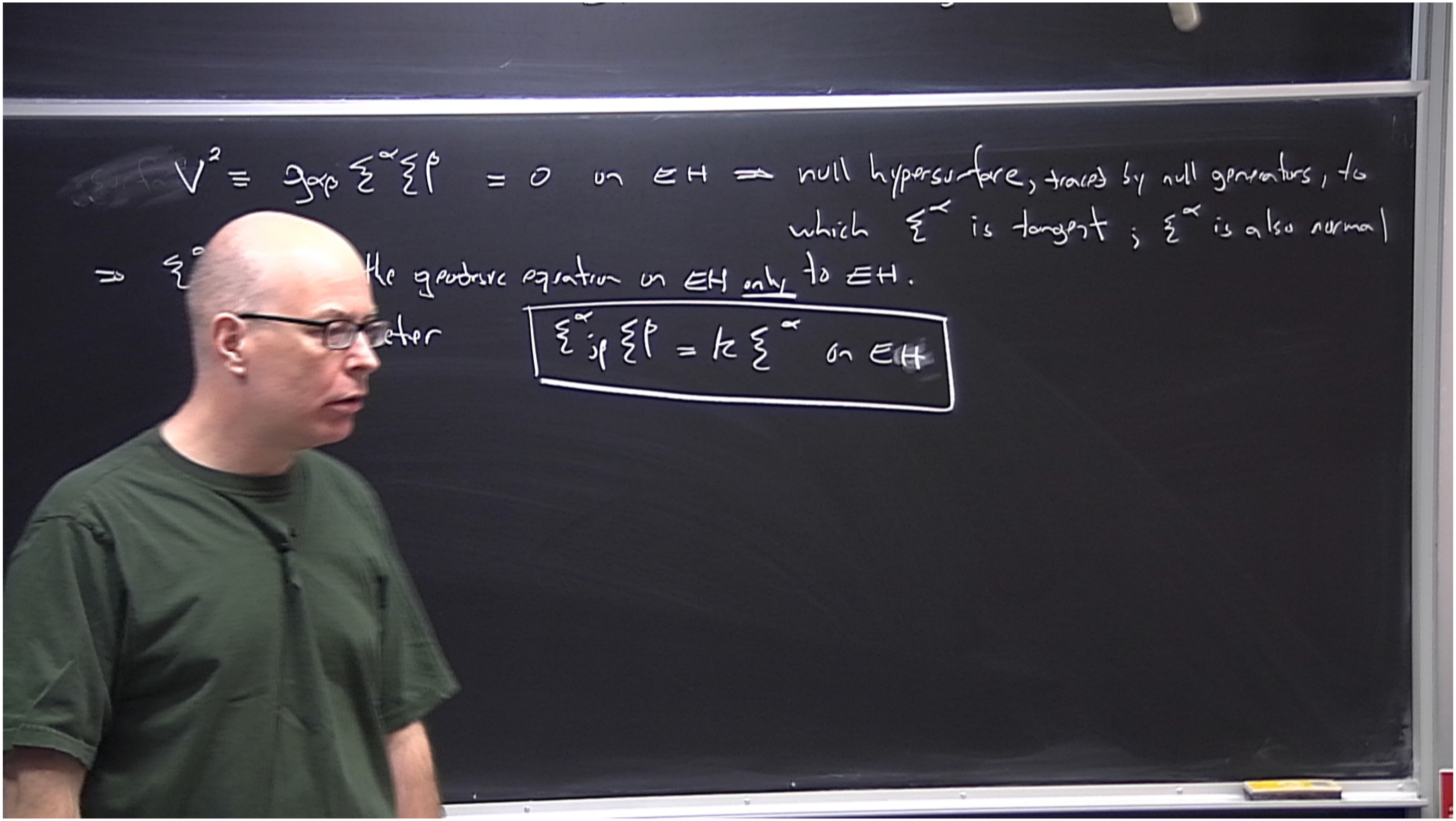
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eter
$\xi^\alpha{}_{;\beta}\xi^\beta = \kappa\xi^\alpha$ on EH

n-ll on EH

$\Phi \equiv g_{\alpha\beta}\xi^\alpha\xi^\beta = 0$ on EH $\Rightarrow$ null hypersurface, traced by null generators, to which ξ^α is tangent ; ξ^α is also normal to EH.

$\Rightarrow \xi^\alpha$ satisfies the geodesic equation on EH only

V = parameter $\boxed{\xi^\alpha{}_{;\beta}\,\xi^\beta = k\,\xi^\alpha \text{ on EH}}$

A displacement on generator : $dx^\alpha = \xi^\alpha\, dV$

Proof : $\Phi = 0$ on EH

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$\partial_\alpha \Phi$ = normal to EH $= 2k\,\xi_\alpha$

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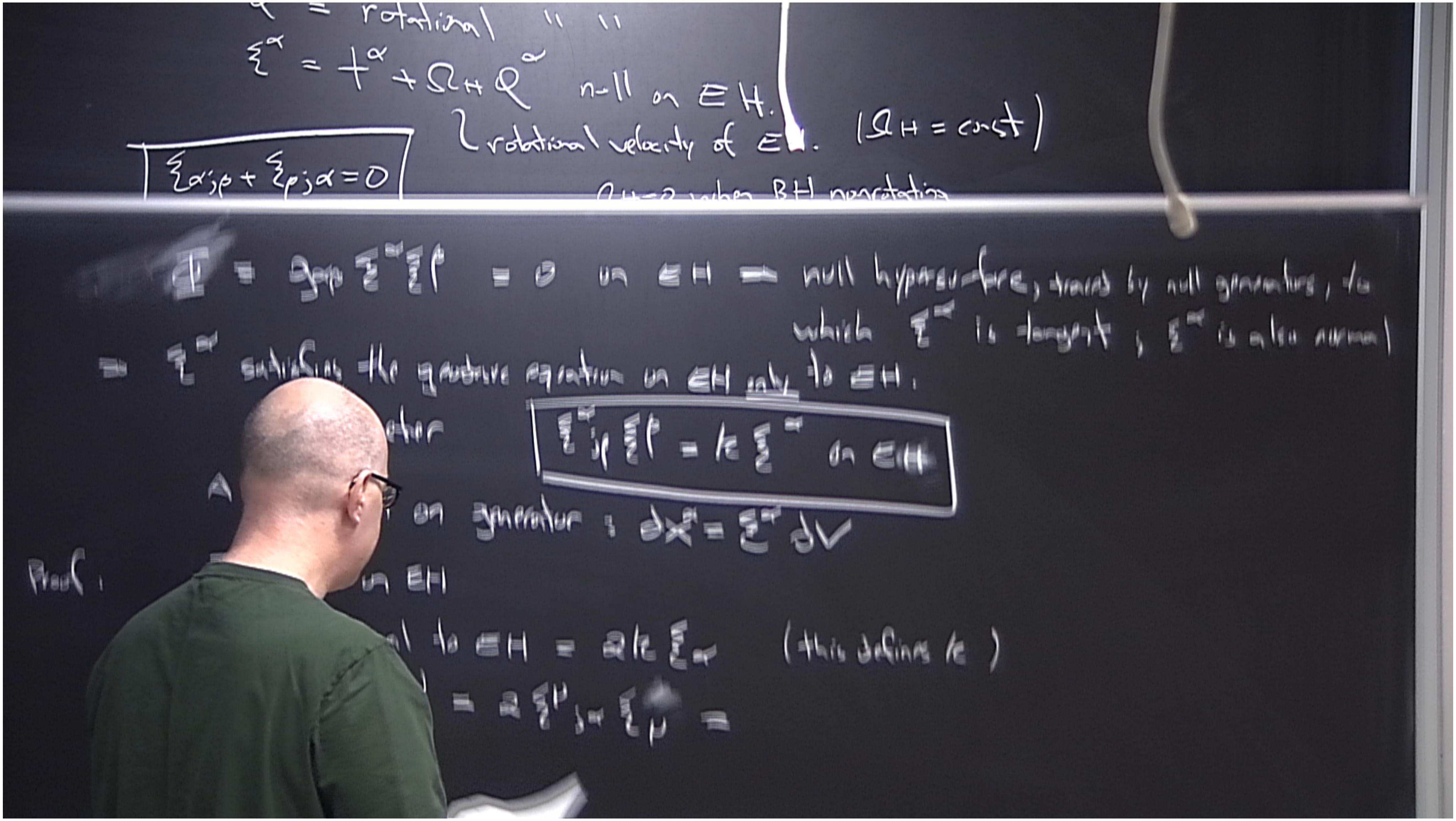
$\nabla_\alpha (g_{\mu\nu}\xi^\mu\xi^\nu) = 2\,\xi^\mu{}_{;\alpha}\,\xi_\mu$



= rotational " "
ξ^α = t^α + Ω_H Q^α null on EH.
{ rotational velocity of EH. (Ω_H = const)
ξ_α;β + ξ_β;α = 0
Ω_H = 0 when BH nonrotating
Φ = g_αβ ξ^α ξ^β = 0 on EH ⇒ null hypersurface, traced by null generators, to which ξ^α is tangent; ξ^α is also normal)
⇒ ξ^α satisfies the geodesic equation on EH only to EH.
ξ^α_;β ξ^β = κ ξ^α on EH
on generator: dx^α = ξ^α dv
Proof.
on EH
to EH = 2κ ξ_α (this defines κ)
= 2 ξ^μ_;α ξ_μ =

$\Phi \equiv g_{\alpha\beta}\,\xi^\alpha \xi^\beta = 0$ on EH $\Rightarrow$ null hypersurface, traced by null generators, to which ξ^α is tangent ; ξ^α is also normal

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$\nabla_\alpha (g_{\mu\nu}\,\xi^\mu \xi^\nu) = 2\,\xi^\mu{}_{;\alpha}\,\xi_\mu = -2\,\xi_{\alpha;\mu}\,\xi^\mu$

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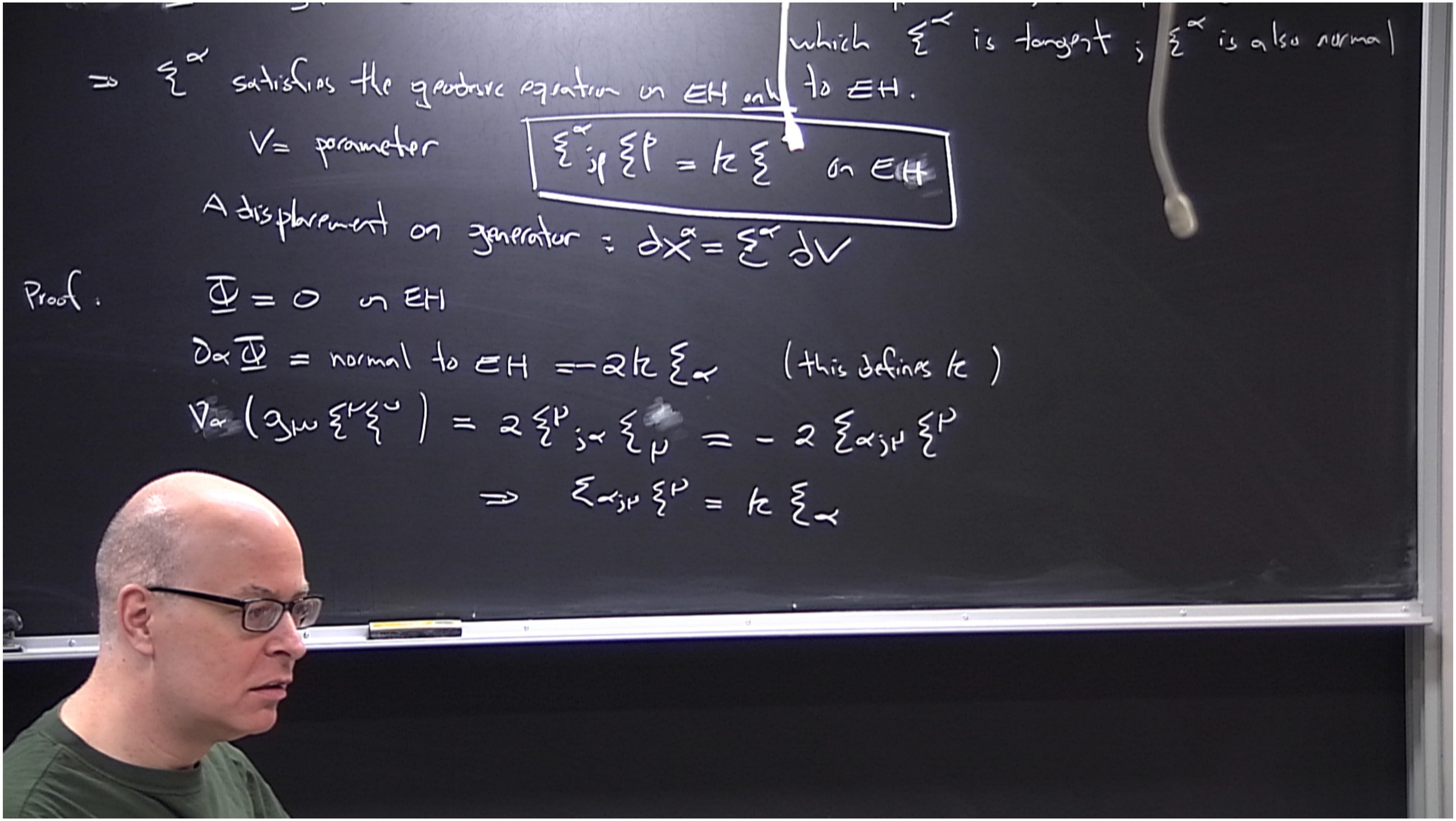
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ξ^α = rotational " "

$\xi^\alpha = l^\alpha$ ...

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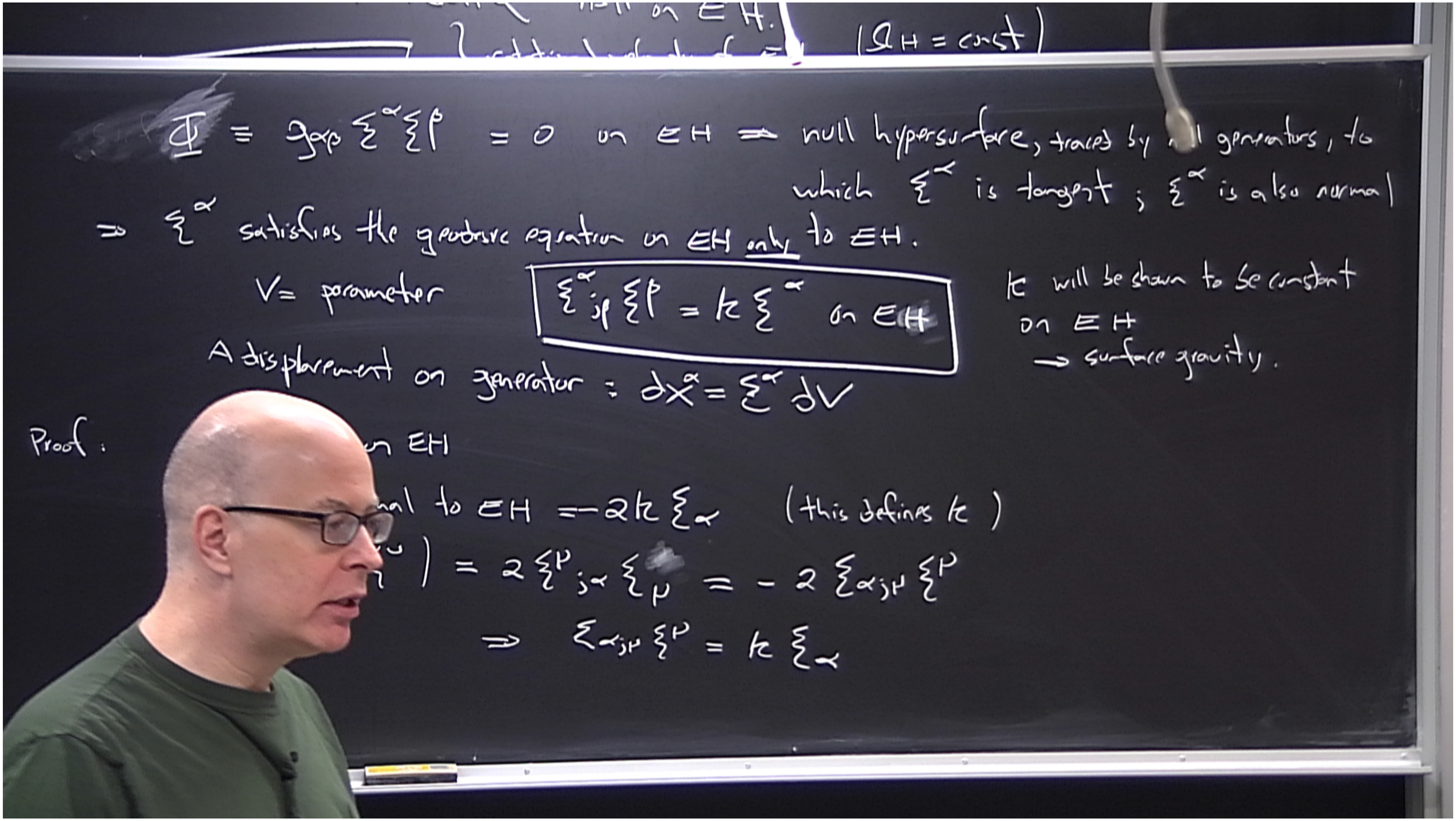
$\Phi_{,\alpha}$ = normal to EH $= -2k\,\xi_\alpha$ (this defines k)

$(g_{\mu\nu}\xi^\mu\xi^\nu)_{,\alpha} = 2\xi^\mu{}_{;\alpha}\,\xi_\mu = -2\,\xi_{\alpha;\mu}\,\xi^\mu$

$\xi_{\alpha;\mu}\,\xi^\mu = k\,\xi_\alpha$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V = parameter
ξ^α_;β ξ^β = k ξ^α on EH
k will be shown to be constant on EH
→ surface gravity.
A displacement on generator : dx^α = ξ^α dV
Proof:
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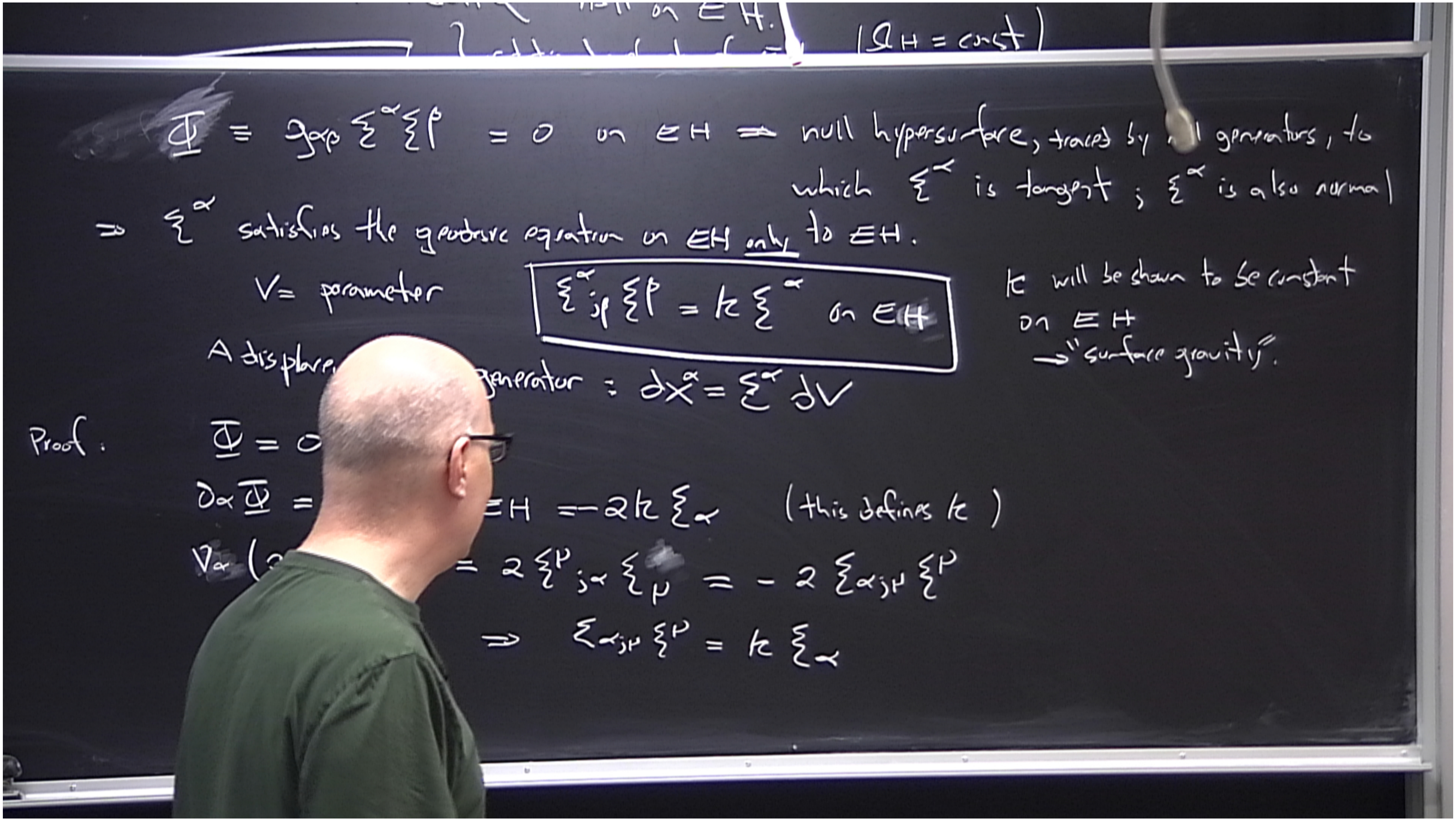
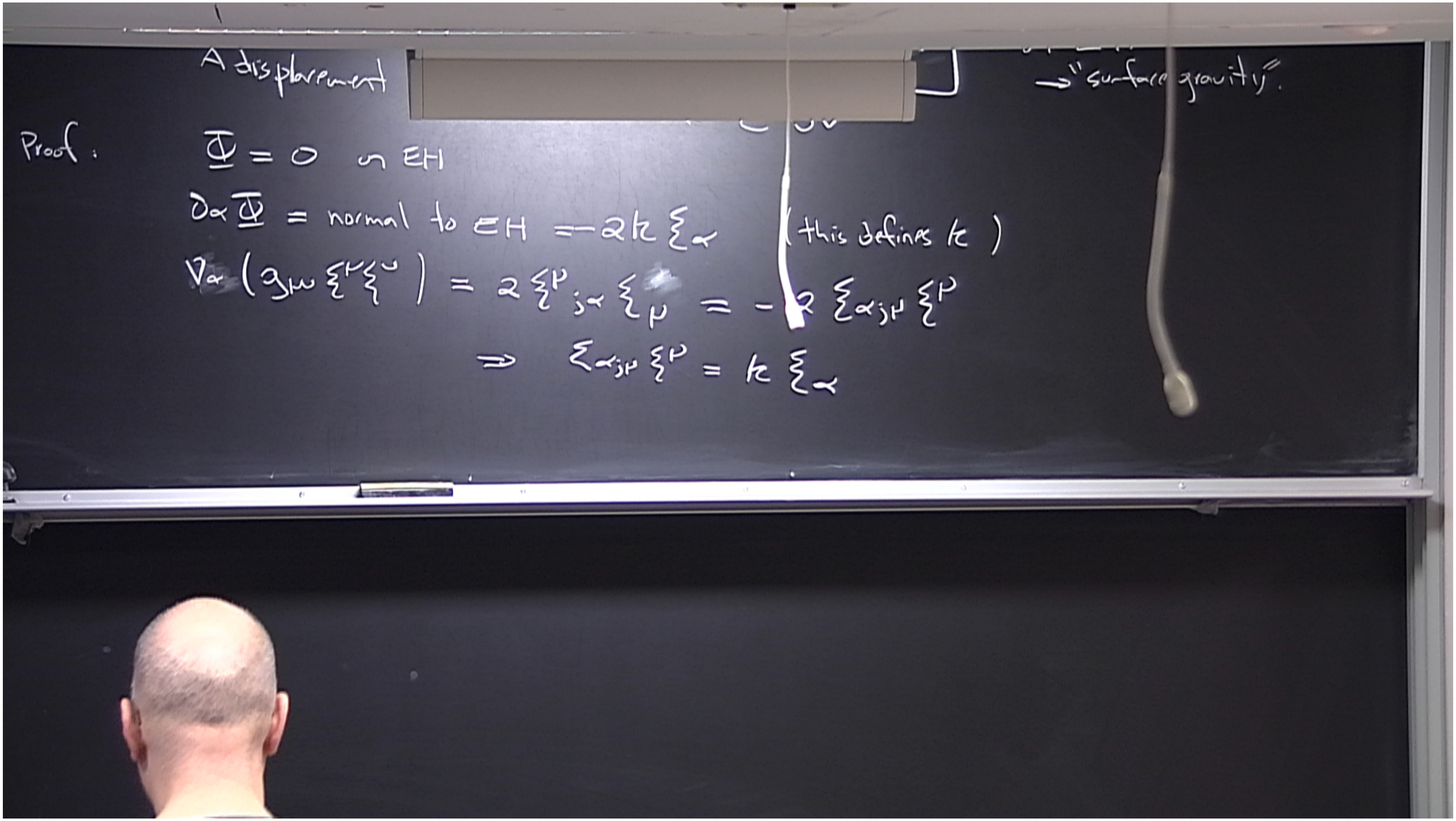
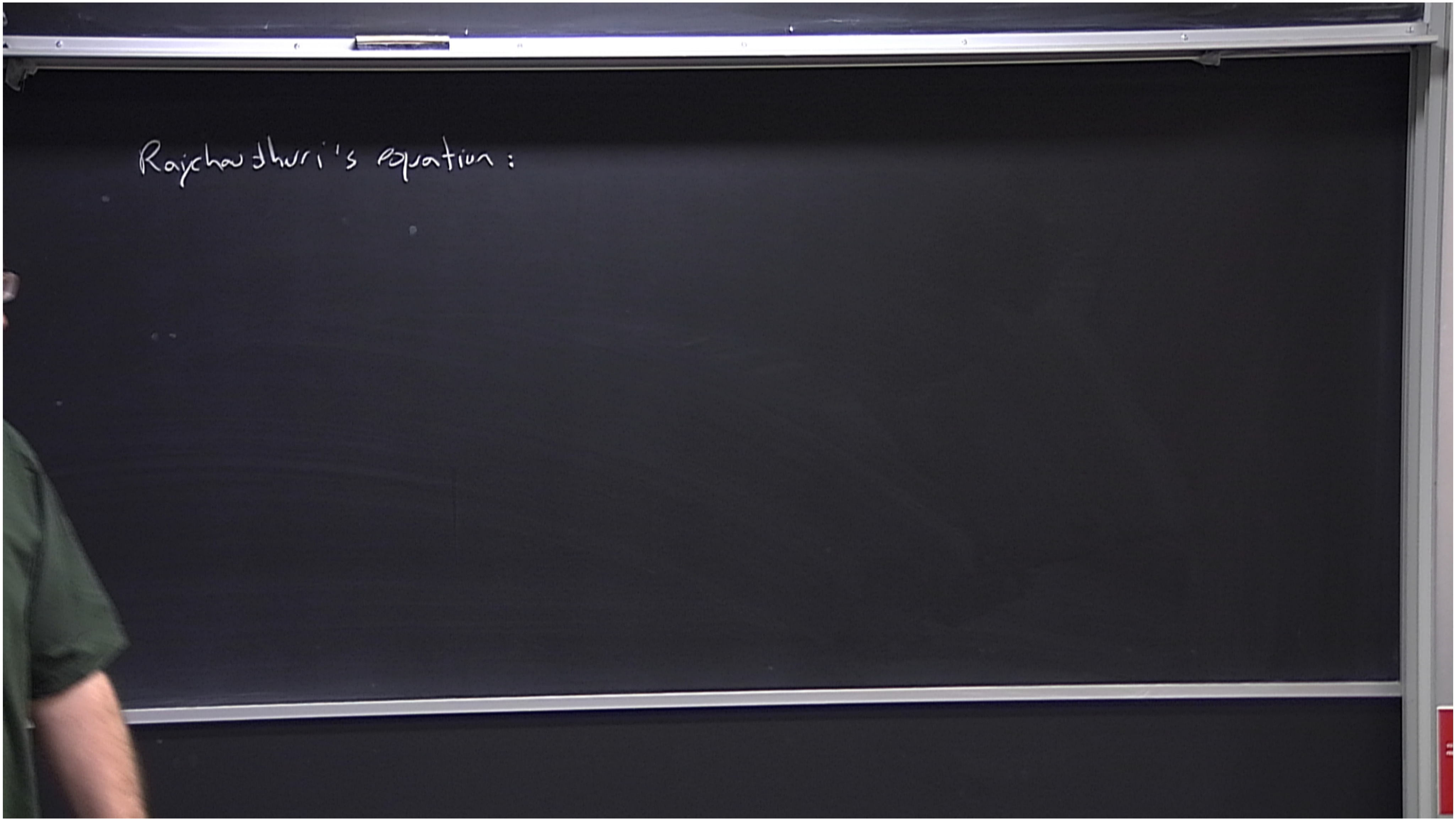
) = 2 ξ^μ_;α ξ_μ = − 2 ξ_α;μ ξ^μ
⇒ ξ_α;μ ξ^μ = k ξ_α


$(\Omega_H = \text{const})$
$\Phi = g_{\alpha\beta}\,\xi^\alpha \xi^\beta = 0$ on EH $\Rightarrow$ null hypersurface, traced by generators, to which ξ^α is tangent ; ξ^α is also normal to EH.
$\Rightarrow \xi^\alpha$ satisfies the geodesic equation on EH only
V= parameter
$\boxed{\xi^\alpha{}_{;\beta}\,\xi^\beta = \kappa\,\xi^\alpha \text{ on EH}}$
κ will be shown to be constant on EH
$\Rightarrow$ "surface gravity".
A displace generator : $dx^\alpha = \xi^\alpha\, dV$
Proof: $\Phi = 0$
$\partial_\alpha \Phi =$ EH $= -2\kappa\,\xi_\alpha$ (this defines κ)
$= 2\,\xi^\beta{}_{;\alpha}\,\xi_\beta = -2\,\xi_{\alpha;\beta}\,\xi^\beta$
$\Rightarrow \xi_{\alpha;\beta}\,\xi^\beta = \kappa\,\xi_\alpha$


A displacement
→ "surface gravity"
Proof:
$\Phi = 0$ on EH
$\partial_\alpha \Phi$ = normal to EH $= -2\kappa\, \xi_\alpha$ (this defines κ)
$\nabla_\alpha (g_{\mu\nu} \xi^\mu \xi^\nu) = 2\, \xi^\mu{}_{;\alpha}\, \xi_\mu = -2\, \xi_{\alpha;\mu}\, \xi^\mu$
$\Rightarrow \xi_{\alpha;\mu}\, \xi^\mu = \kappa\, \xi_\alpha$


Raychaudhuri's equation:

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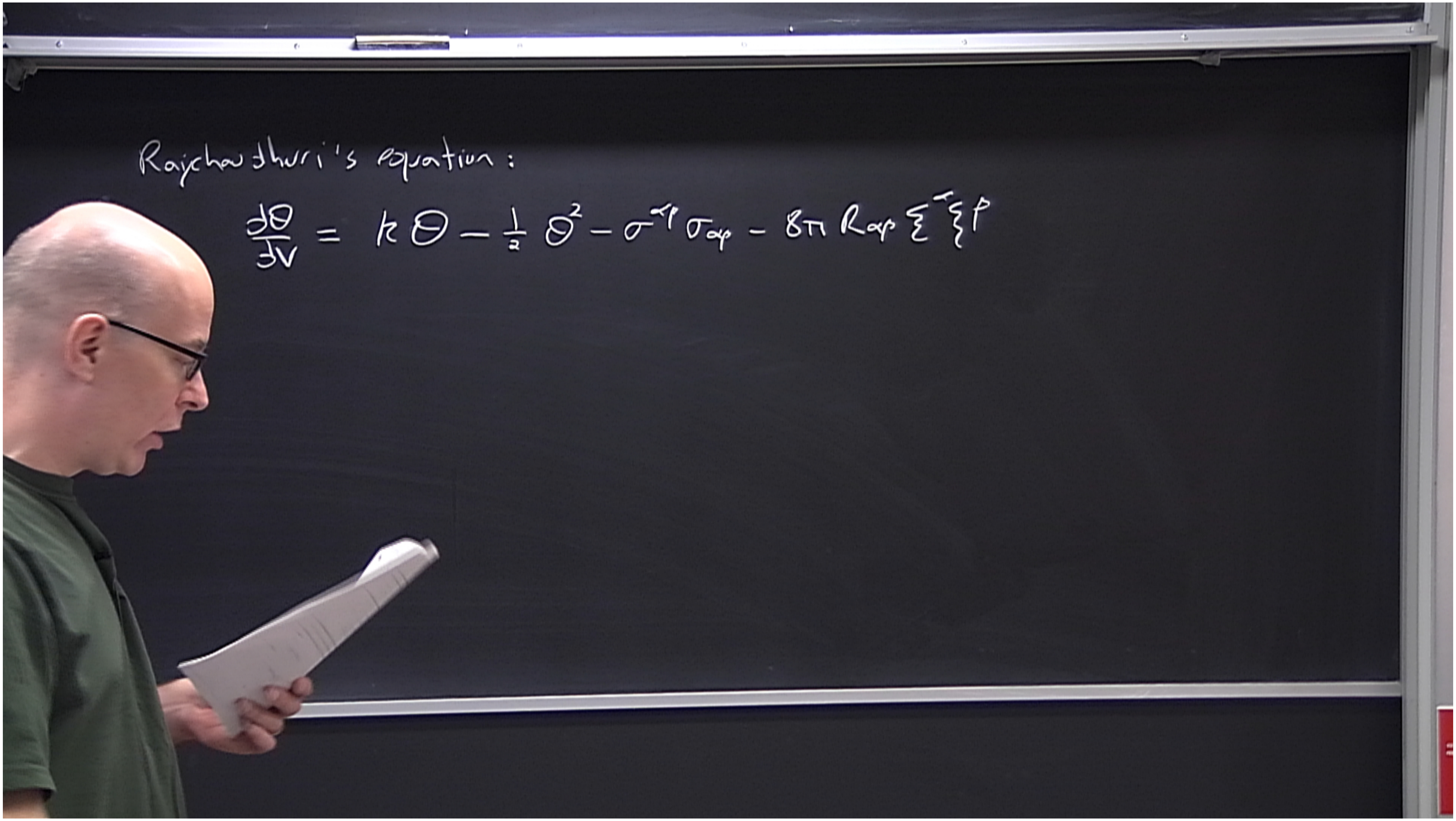
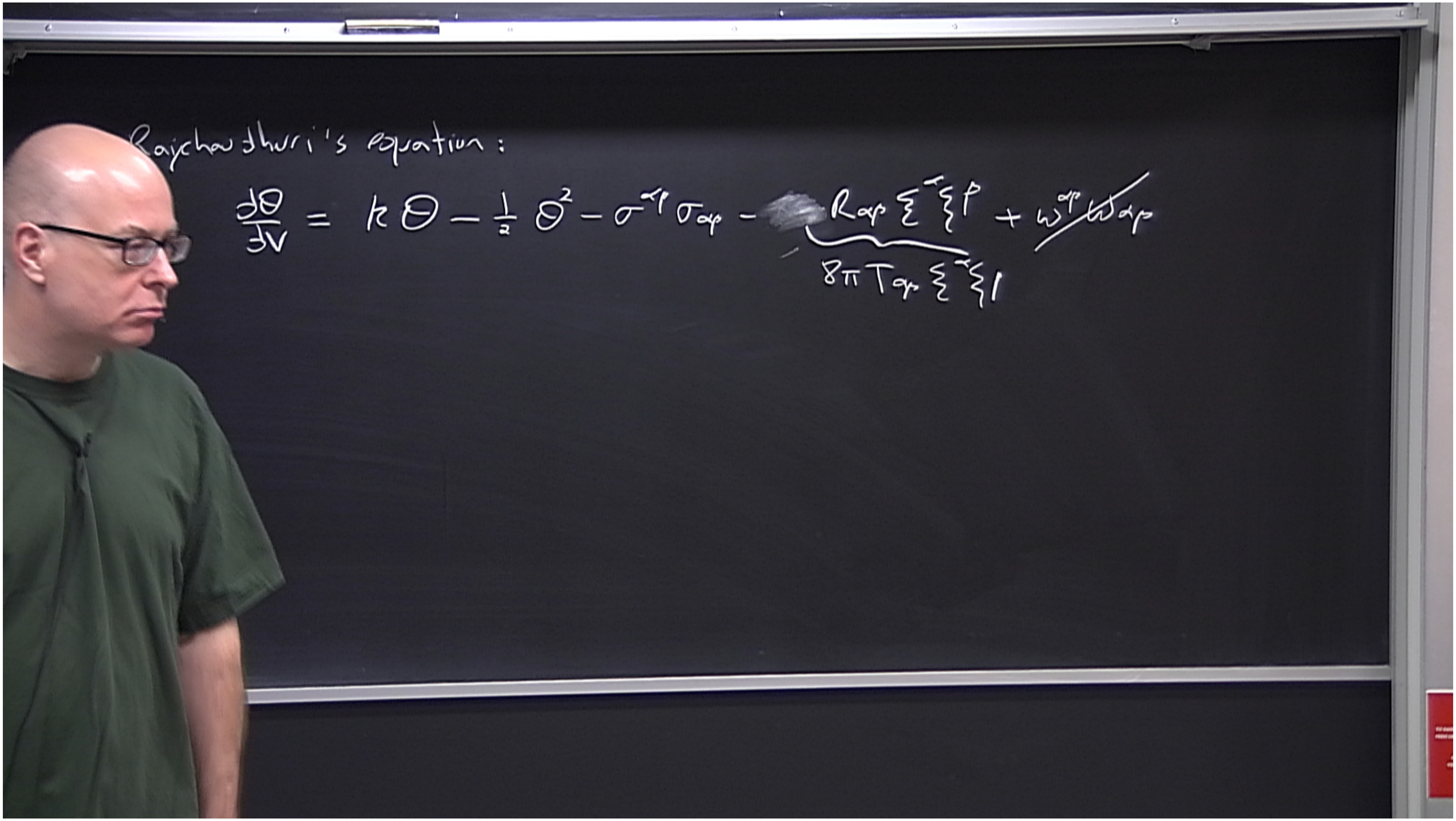
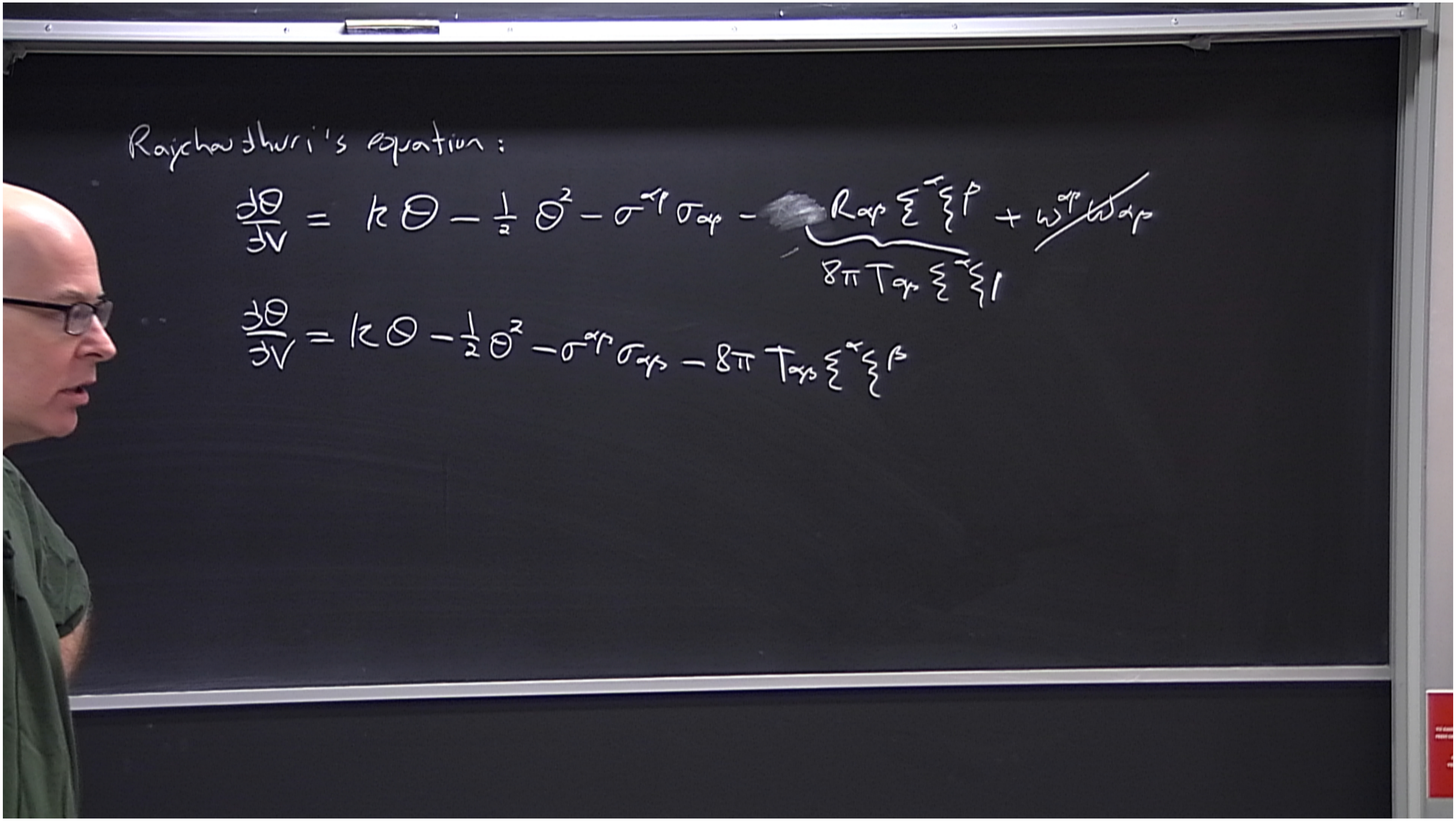
$$\frac{d\theta}{dV} = k\,\theta$$


Raychaudhuri's equation:
$\frac{d\theta}{dV} = k\theta - \frac{1}{2}\theta^2 - \sigma^{\alpha\beta}\sigma_{\alpha\beta} - 8\pi R_{\alpha\beta}\xi^\alpha\xi^\beta$


Raychaudhuri's equation:
$\frac{d\theta}{d\lambda} = k\theta - \frac{1}{2}\theta^2 - \sigma^{\alpha\beta}\sigma_{\alpha\beta} - \underbrace{R_{\alpha\beta}\xi^\alpha\xi^\beta}_{8\pi T_{\alpha\beta}\xi^\alpha\xi^\beta} + \cancel{\omega^{\alpha\beta}\omega_{\alpha\beta}}$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$\frac{d\theta}{dV} = k\theta - \frac{1}{2}\theta^2 - \sigma^{\alpha\beta}\sigma_{\alpha\beta} - 8\pi T_{\alpha\beta}\xi^\alpha\xi^\beta$

Raychaudhuri's equation:

$$\frac{d\theta}{dv} = k\theta - \frac{1}{2}\theta^2 - \sigma^{\alpha\beta}\sigma_{\alpha\beta} - \underbrace{R_{\alpha\beta}\xi^\alpha\xi^\beta}_{8\pi T_{\alpha\beta}\xi^\alpha\xi^\beta} + \cancel{\omega^{\alpha\beta}\omega_{\alpha\beta}}$$

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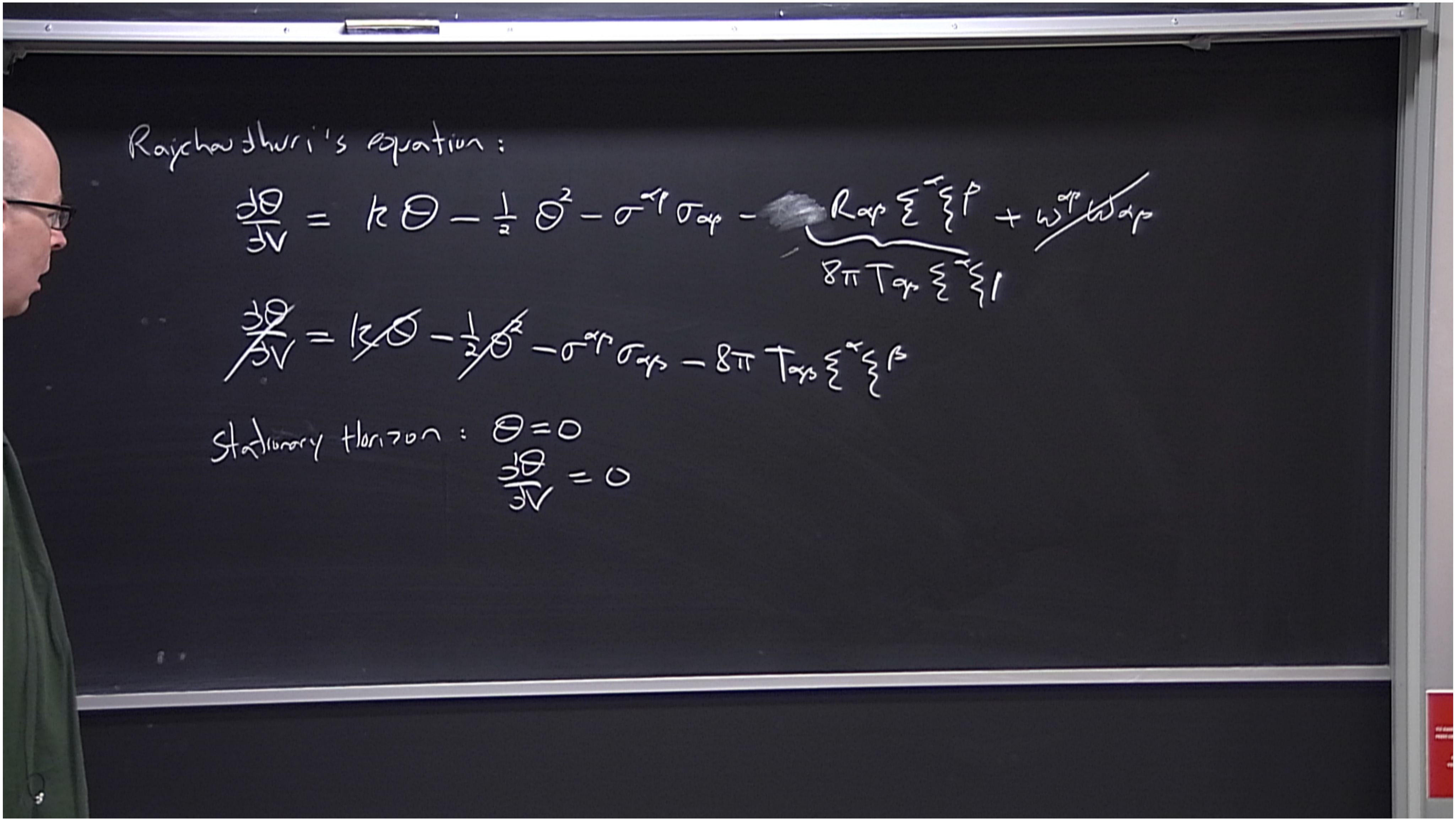
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$\cancel{\frac{d\theta}{dV}} = \cancel{\kappa\theta} - \cancel{\frac{1}{2}\theta^2} - \sigma^{\alpha\beta}\sigma_{\alpha\beta} - 8\pi T_{\alpha\beta}\xi^\alpha\xi^\beta$
Stationary Horizon: $\theta = 0$
$\frac{d\theta}{dV} = 0$

Raychaudhuri's equation:

$$\frac{d\theta}{dV} = \kappa\theta - \frac{1}{2}\theta^2 - \sigma^{\alpha\beta}\sigma_{\alpha\beta} - \underbrace{R_{\alpha\beta}\xi^\alpha\xi^\beta}_{8\pi T_{\alpha\beta}\xi^\alpha\xi^\beta} + \cancel{\omega^{\alpha\beta}\omega_{\alpha\beta}}$$

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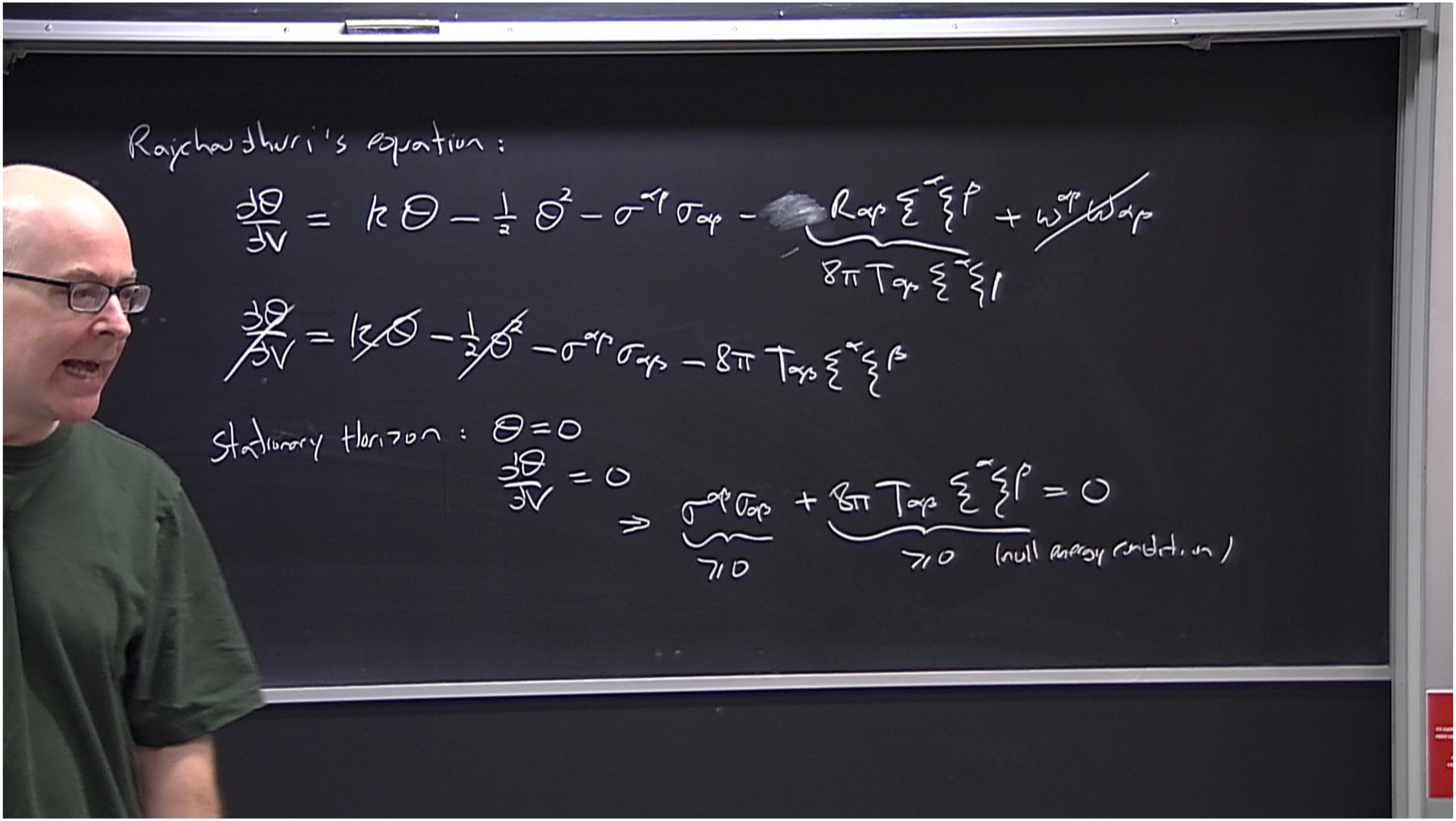
Stationary Horizon: $\theta = 0$

$$\frac{d\theta}{dV} = 0$$

$$\Rightarrow \sigma^{\alpha\beta}\sigma_{\alpha\beta} + 8\pi T_{\alpha\beta}\xi^\alpha\xi^\beta = 0$$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Stationary Horizon: $\theta = 0$
$\frac{d\theta}{dV} = 0$
$\Rightarrow \underbrace{\sigma^{\alpha\beta}\sigma_{\alpha\beta}}_{\geq 0} + \underbrace{8\pi T_{\alpha\beta}\xi^\alpha\xi^\beta}_{\geq 0} = 0$
(null energy condition)

Raychaudhuri's equation:

$$\frac{d\theta}{d\lambda} = \kappa\theta - \frac{1}{2}\theta^2 - \sigma^{\alpha\beta}\sigma_{\alpha\beta} - \underbrace{R_{\alpha\beta}\xi^\alpha\xi^\beta}_{8\pi T_{\alpha\beta}\xi^\alpha\xi^\beta} + \cancel{\omega^{\alpha\beta}\omega_{\alpha\beta}}$$

$$\cancel{\frac{d\theta}{d\lambda}} = \cancel{\kappa\theta} - \cancel{\frac{1}{2}\theta^2} - \sigma^{\alpha\beta}\sigma_{\alpha\beta} - 8\pi T_{\alpha\beta}\xi^\alpha\xi^\beta$$

Stationary Horizon: $\theta = 0$

$$\frac{d\theta}{d\lambda} = 0$$

$$\Rightarrow \underbrace{\sigma^{\alpha\beta}\sigma_{\alpha\beta}}_{\geq 0} + \underbrace{8\pi T_{\alpha\beta}\xi^\alpha\xi^\beta}_{\geq 0} = 0$$

(null energy condition)

$$\Rightarrow \boxed{\begin{array}{l} \sigma_{\alpha\beta} = 0 \\ T_{\alpha\beta}\xi^\alpha\xi^\beta = 0 \\ \quad \text{is no flux} \end{array}}$$

Raychaudhuri's equation:

$$\frac{d\theta}{dv} = k\theta - \frac{1}{2}\theta^2 - \sigma^{\alpha\beta}\sigma_{\alpha\beta} - \underbrace{R_{\alpha\beta}\xi^\alpha\xi^\beta}_{8\pi T_{\alpha\beta}\xi^\alpha\xi^\beta} + \cancel{\omega^{\alpha\beta}\omega_{\alpha\beta}}$$

$$\cancel{\frac{d\theta}{dv}} = \cancel{k\theta} - \cancel{\frac{1}{2}\theta^2} - \sigma^{\alpha\beta}\sigma_{\alpha\beta} - 8\pi T_{\alpha\beta}\xi^\alpha\xi^\beta$$

Stationary Horizon: $\theta = 0$

$$\frac{d\theta}{dv} = 0$$

$$\Rightarrow \underbrace{\sigma^{\alpha\beta}\sigma_{\alpha\beta}}_{\geq 0} + \underbrace{8\pi T_{\alpha\beta}\xi^\alpha\xi^\beta}_{\geq 0 \text{ (null energy condition)}} = 0$$

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$$\frac{d\theta}{dV} = 0$$

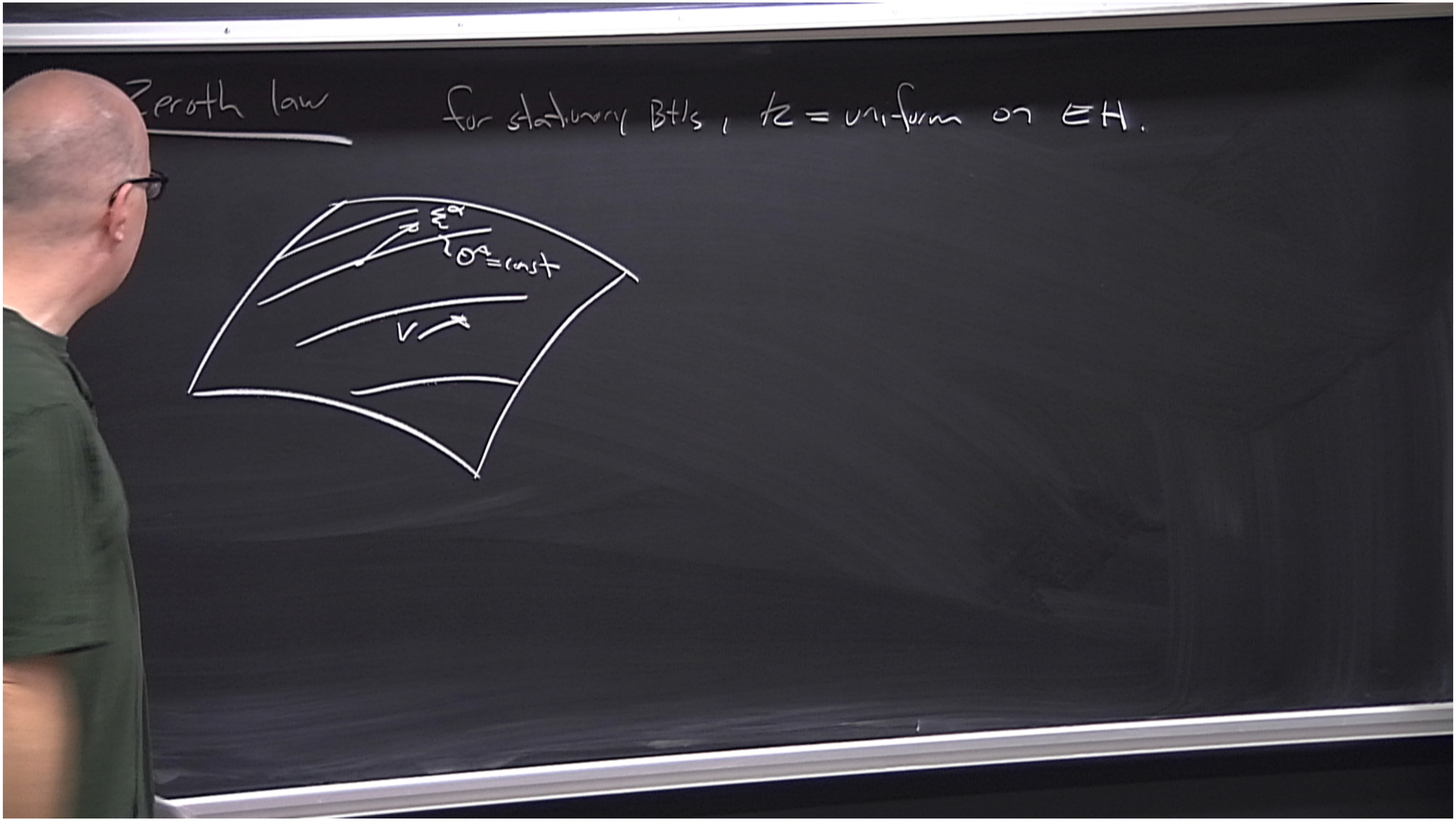
$$\Rightarrow \underbrace{\sigma^{\alpha\beta}\sigma_{\alpha\beta}}_{\geq 0} + \underbrace{8\pi T_{\alpha\beta}\xi^\alpha\xi^\beta}_{\geq 0} = 0$$

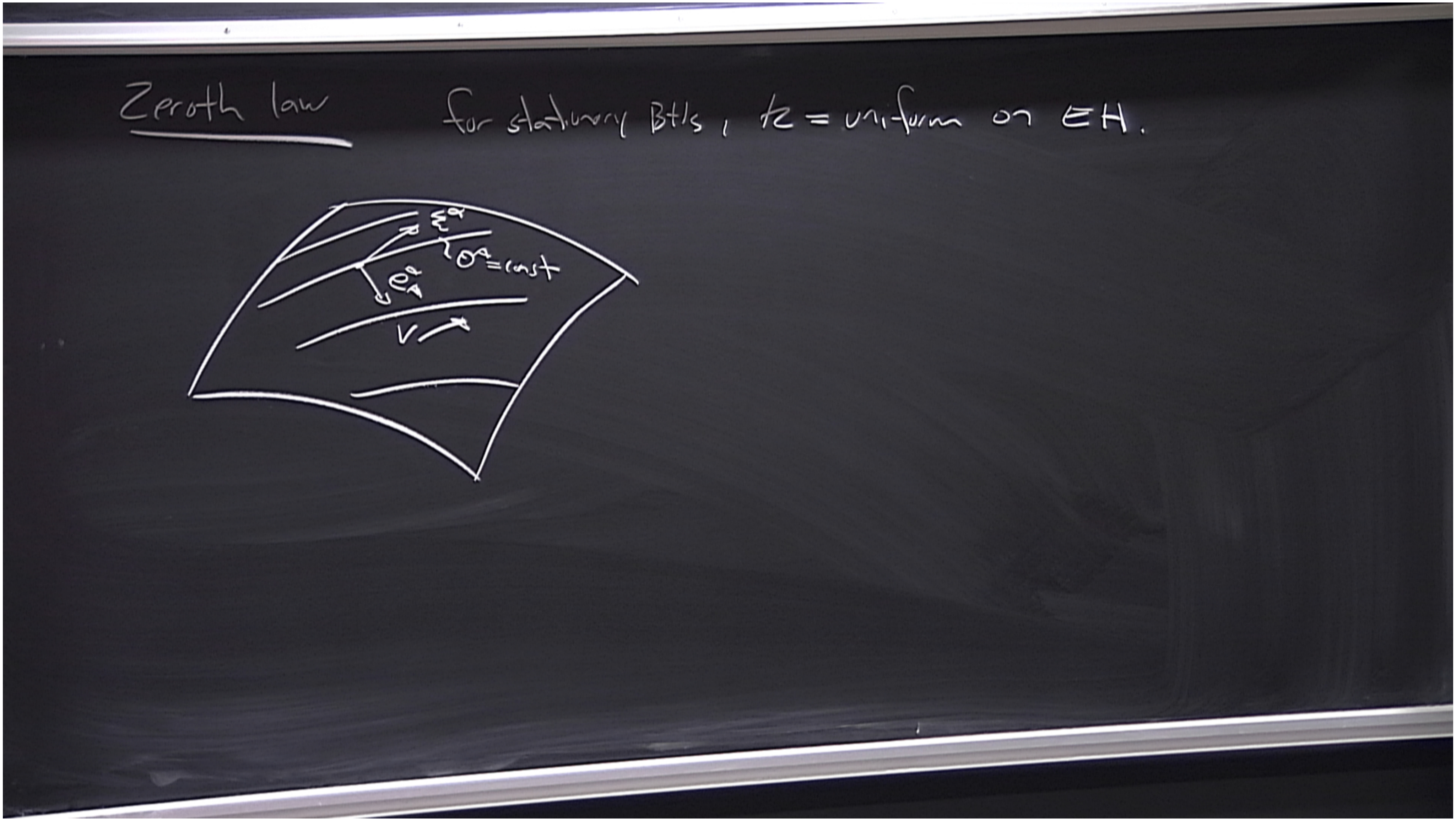
(null energy condition)

$$\Rightarrow \boxed{\begin{array}{l} \sigma_{\alpha\beta} = 0 \\ T_{\alpha\beta}\xi^\alpha\xi^\beta = 0 \\ \quad \hookrightarrow \text{no flux} \end{array}}$$

Zeroth law

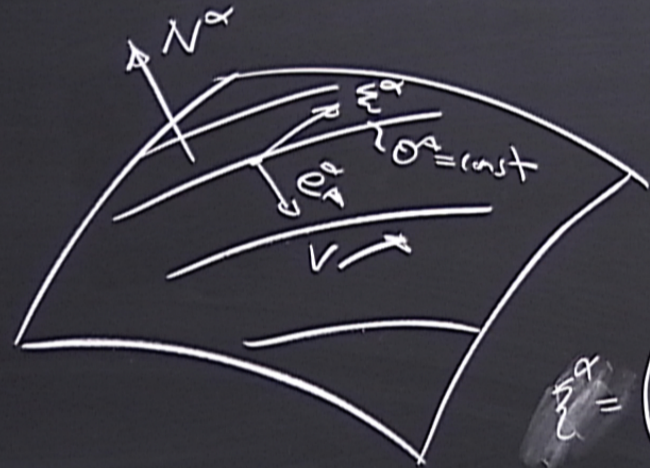
for stationary BH's, κ = uniform on EH.


Zeroth law
for stationary BH's, κ = uniform on EH.
ξ^α
θ^A = const
v


Zeroth law
for stationary BH's, κ = uniform on EH.
ξ^a
O^a = const
e^a_A
V

Zeroth law

for stationary BH's, κ = uniform on EH.



$$y^a = (v, \theta^A)$$

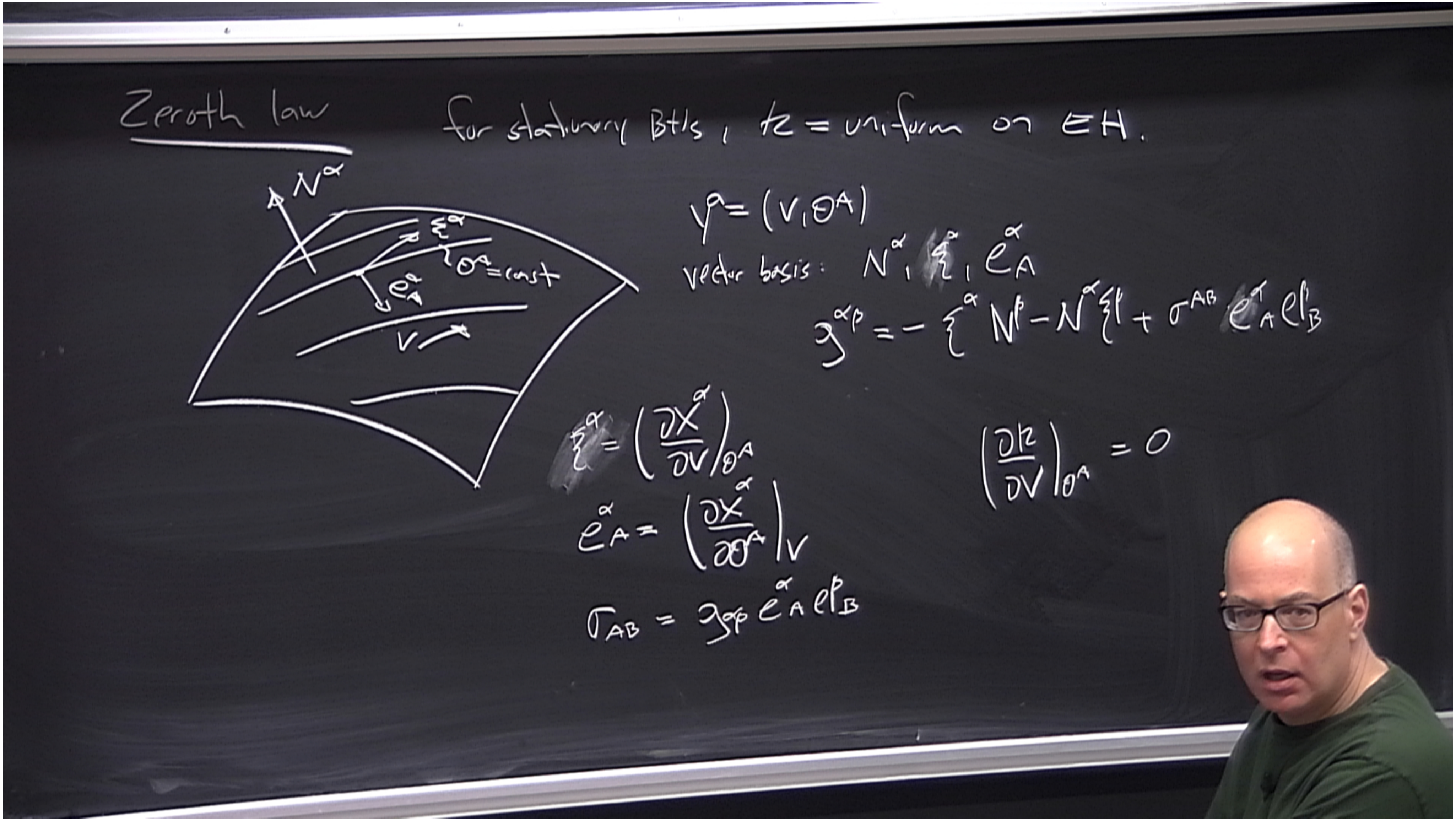
vector basis: $N^\alpha, \xi^\alpha, e^\alpha_A$

$$g^{\alpha\beta} = -\xi^\alpha N^\beta - N^\alpha \xi^\beta + \sigma^{AB} e^\alpha_A e^\beta_B$$

$$\xi^\alpha = \left(\frac{\partial x^\alpha}{\partial v}\right)_{\theta^A}$$

$$e^\alpha_A = \left(\frac{\partial x^\alpha}{\partial \theta^A}\right)_v$$

$$\sigma_{AB} = g_{\alpha\beta}\, e^\alpha_A e^\beta_B$$


Zeroth law
for stationary BHs, κ = uniform on EH.
N^α
ξ^α
$\theta^A = \text{const}$
e^α_A
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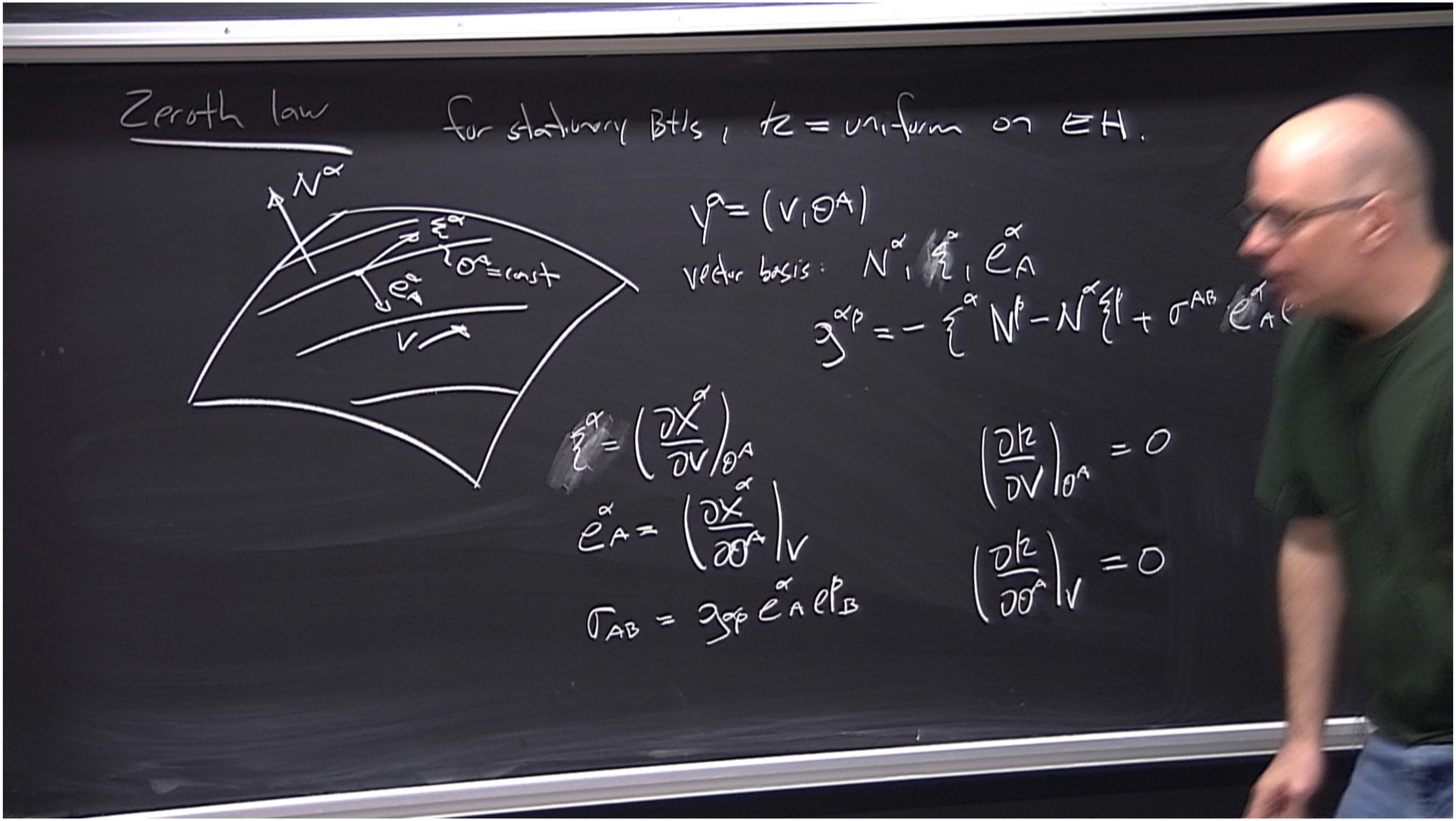
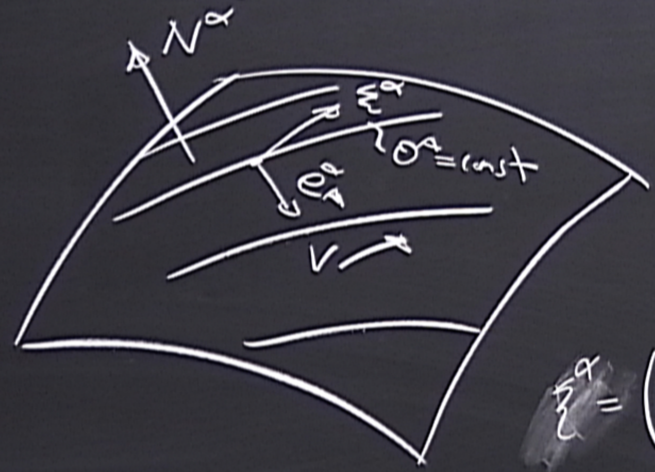
$y^\alpha = (v, \theta^A)$
vector basis: $N^\alpha, \xi^\alpha, e^\alpha_A$
$g^{\alpha\beta} = -\xi^\alpha N^\beta - N^\alpha \xi^\beta + \sigma^{AB} e^\alpha_A e^\beta_B$
$\xi^\alpha = \left(\frac{\partial x^\alpha}{\partial v}\right)_{\theta^A}$
$e^\alpha_A = \left(\frac{\partial x^\alpha}{\partial \theta^A}\right)_v$
$\sigma_{AB} = g_{\alpha\beta} e^\alpha_A e^\beta_B$
$\left(\frac{\partial \kappa}{\partial v}\right)_{\theta^A} = 0$


Zeroth law
for stationary BHs, κ = uniform on EH.
N^α
ξ^α
$\theta^A = \text{const}$
e^α_A
v
$y^a = (v, \theta^A)$
vector basis: $N^\alpha, \xi^\alpha, e^\alpha_A$
$g^{\alpha\beta} = -\xi^\alpha N^\beta - N^\alpha \xi^\beta + \sigma^{AB} e^\alpha_A e$
$\xi^\alpha = \left(\frac{\partial x^\alpha}{\partial v}\right)_{\theta^A}$
$e^\alpha_A = \left(\frac{\partial x^\alpha}{\partial \theta^A}\right)_v$
$\sigma_{AB} = g_{\alpha\beta} e^\alpha_A e^\beta_B$
$\left(\frac{\partial \kappa}{\partial v}\right)_{\theta^A} = 0$
$\left(\frac{\partial \kappa}{\partial \theta^A}\right)_v = 0$

Zeroth law

for stationary BHs, κ = uniform on EH.



$$y^\alpha = (v, \theta^A)$$

vector basis: $N^\alpha, \xi^\alpha, e^\alpha_A$

$$g^{\alpha\beta} = -\xi^\alpha N^\beta - N^\alpha \xi^\beta + \sigma^{AB} e^\alpha_A e^\beta_B$$

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(1) $\left(\frac{\partial \kappa}{\partial v}\right)_{\theta^A} = 0$

(2) $\left(\frac{\partial \kappa}{\partial \theta^A}\right)_v = 0$

$\xi_{\alpha;\beta} =$

$$\xi^\alpha = \left(\frac{\partial x^\alpha}{\partial v}\right)_{\theta^A}$$

$$e^\alpha_A = \left(\frac{\partial x^\alpha}{\partial \theta^A}\right)_v$$

$$\sigma_{AB} = g_{\alpha\beta}\, e^\alpha_A\, e^\beta_B$$

$$N_\alpha N^\alpha = 0 \; ; \; N_\alpha \xi^\alpha = -1$$

(1) $\left(\frac{\partial r}{\partial v}\right)_{\theta^A} = 0$

(2) $\left(\frac{\partial r}{\partial \theta^A}\right)_v = 0$

$$\xi_{\alpha;\beta} =$$

$$\xi^\alpha = \left(\frac{\partial x^\alpha}{\partial v}\right)_{\theta^A}$$

$$e^\alpha_A = \left(\frac{\partial x^\alpha}{\partial \theta^A}\right)_v$$

$$\sigma_{AB} = g_{\alpha\beta}\, e^\alpha_A e^\beta_B$$

(1) $\left(\frac{\partial k}{\partial v}\right)_{\theta^A} = 0$

(2) $\left(\frac{\partial k}{\partial \theta^A}\right)_v = 0$

$$N_\alpha N^\alpha = 0 \ ; \ N_\alpha \xi^\alpha = -1 \ ; \ N_\alpha e^\alpha_A = 0$$

$$\xi_{\alpha;\beta} =$$

$$\xi_{\alpha;\beta} = a\,\xi_\alpha \xi_\beta + b\,\xi_\alpha N_\beta + c^A \xi_\alpha e_{A\beta}$$
$$+ d\,N_\alpha \xi_\beta + e\,N_\alpha N_\beta + f^A N_\alpha e_{A\beta}$$
$$+ g^A e_{A\alpha} \xi_\beta + h^A e_{A\alpha} N_\beta + j^{AB} e_{A\alpha} e_{B\beta}$$

$$\xi_{\alpha;\beta} = a\,\xi_\alpha \xi_\beta + b\,\xi_\alpha N_\beta + c^A \xi_\alpha e_{A\beta}$$
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(general decomposition)

1- ξ^α is null on EH :

$$0 = (\xi^\alpha \xi_\alpha)_{;\beta}\, \xi^\beta$$

$$\xi_{\alpha;\beta} = a\,\xi_\alpha \xi_\beta + b\,\xi_\alpha N_\beta + c^A \xi_\alpha e_{A\beta}$$
$$+ d\,N_\alpha \xi_\beta + e\,N_\alpha N_\beta + f^A N_\alpha e_{A\beta}$$
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(general decomposition)

1 - ξ^α is null on EH :

$$0 = (\xi^\alpha \xi_\alpha)_{;\beta}\, \xi^\beta = 2\, \xi^\alpha \xi_{\alpha;\beta} \xi^\beta$$

$$\xi_{\alpha;\beta} = a\,\xi_\alpha \xi_\beta + b\,\xi_\alpha N_\beta + c^A \xi_\alpha e_{A\beta}$$
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$$0 = (\xi^\alpha \xi_\alpha)_{;\beta}\, e^\beta_A = 2\, \xi^\alpha \xi_{\alpha;\beta}\, e^\beta_A$$

$$\ldots N_\alpha N_\beta + f^A N_\alpha e_{A\beta}$$

$$+ g^A e_{A\alpha} \xi_\beta + h^A e_{A\alpha} N_\beta + j^{AB} e_{A\alpha} e_{B\beta}$$

(general decomposition)

1- ξ^α is null on EH :

$$0 = (\xi^\alpha \xi_\alpha)_{;\beta} \xi^\beta = 2 \xi^\alpha \xi_{\alpha;\beta} \xi^\beta$$

$$0 = (\xi^\alpha \xi_\alpha)_{;\beta} e^\beta_A = 2 \xi^\alpha \xi_{\alpha;\beta} e^\beta_A$$

$$\xi^\alpha \xi_{\alpha;\beta} =$$

$$\xi_{\alpha;\beta} = a\,\xi_\alpha \xi_\beta + b\,\xi_\alpha N_\beta + c^A\,\xi_\alpha e_{A\beta}$$
$$+ d\,N_\alpha \xi_\beta + e\,N_\alpha N_\beta + f^A N_\alpha e_{A\beta}$$
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(general decomposition)

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$$0 = (\xi^\alpha \xi_\alpha)_{;\beta}\,e^\beta_A = 2\,\xi^\alpha \xi_{\alpha;\beta}\,e^\beta_A$$

$$\xi^\alpha \xi_{\alpha;\beta} = -d\,\xi_\beta - e\,N_\beta - f^A e_{A\beta}$$

$$\xi_{\alpha;\beta} = a\,\xi_\alpha \xi_\beta + b\,\xi_\alpha N_\beta + \dots$$

$$+ d\,N_\alpha \xi_\beta + e\,N_\alpha N_\beta + f^A N_\alpha e_{A\beta}$$

$$+ g^A e_{A\alpha} \xi_\beta + h^A e_{A\alpha} N_\beta + j^{AB} e_{A\alpha} e_{B\beta}$$

(general decomposition)

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$$\xi^\alpha \xi_{\alpha;\beta} = -d\,\xi_\beta - e\,N_\beta - f^A e_{A\beta}$$

$$0 = \xi^\alpha \xi_{\alpha;\beta}\,\xi^\beta = +e$$

$$\xi_{\alpha;\beta} = a\,\xi_\alpha \xi_\beta + b\,\xi_\alpha N_\beta + c^A \xi_\alpha e_{A\beta}$$
$$+ d\,N_\alpha \xi_\beta + \cancel{e\,N_\alpha N_\beta} + f^A N_\alpha e_{A\beta}$$
$$+ g^A e_{A\alpha} \xi_\beta + h^A e_{A\alpha} N_\beta + j^{AB} e_{A\alpha} e_{B\beta}$$

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$$0 = \xi^\alpha \xi_{\alpha;\beta}\, \xi^\beta = +e$$

$$\ldots N_\alpha e_{A\beta}$$

$$+ g^A e_{A\alpha}\xi_\beta + h^A e_{A\alpha} N_\beta + j^{AB} e_{A\alpha} e_{B\beta}$$

(general decomposition)

1- ξ^α is null on EH :

$$0 = (\xi^\alpha \xi_\alpha)_{;\beta}\,\xi^\beta = 2\,\xi^\alpha \xi_{\alpha;\beta}\,\xi^\beta$$

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$$\xi^\alpha \xi_{\alpha;\beta} = -d\,\xi_\beta - e\,N_\beta - f^A e_{A\beta}$$

$$\xi^\alpha \xi_{\alpha;\beta}\,\xi^\beta = +e$$

$$\xi^\alpha \xi_{\alpha;\beta}\, e^\beta_C = -f^A$$

$$+ g^A e_{A\alpha}\xi_\beta + h^A e_{A\alpha}N_\beta + j^{AB} e_{A\alpha} e_{B\beta}$$

(general decomposition)

1- ξ^α is null on EH :

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$$\xi^\alpha\xi_{\alpha;\beta} = -d\,\xi_\beta - e\,N_\beta - f^A e_{A\beta}$$

$$0 = \xi^\alpha\xi_{\alpha;\beta}\,\xi^\beta = +e$$

$$0 = \xi^\alpha\xi_{\alpha;\beta}\,e^\beta_C = -f^A\sigma_{AC}$$

$$\xi_{\alpha;\beta} = a\,\xi_\alpha \xi_\beta + b\,\xi_\alpha N_\beta + c^A \xi_\alpha e_{A\beta}$$
$$+ d\,N_\alpha \xi_\beta + e\,\cancel{N_\alpha N_\beta} + f^A \cancel{N_\alpha e_{A\beta}}$$
$$+ g^A e_{A\alpha} \xi_\beta + h^A e_{A\alpha} N_\beta + j^{AB} e_{A\alpha} e_{B\beta}$$

(general decomposition)

1- ξ^α is null on EH :

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$$0 = \xi^\alpha \xi_{\alpha;\beta} \xi^\beta = +e$$
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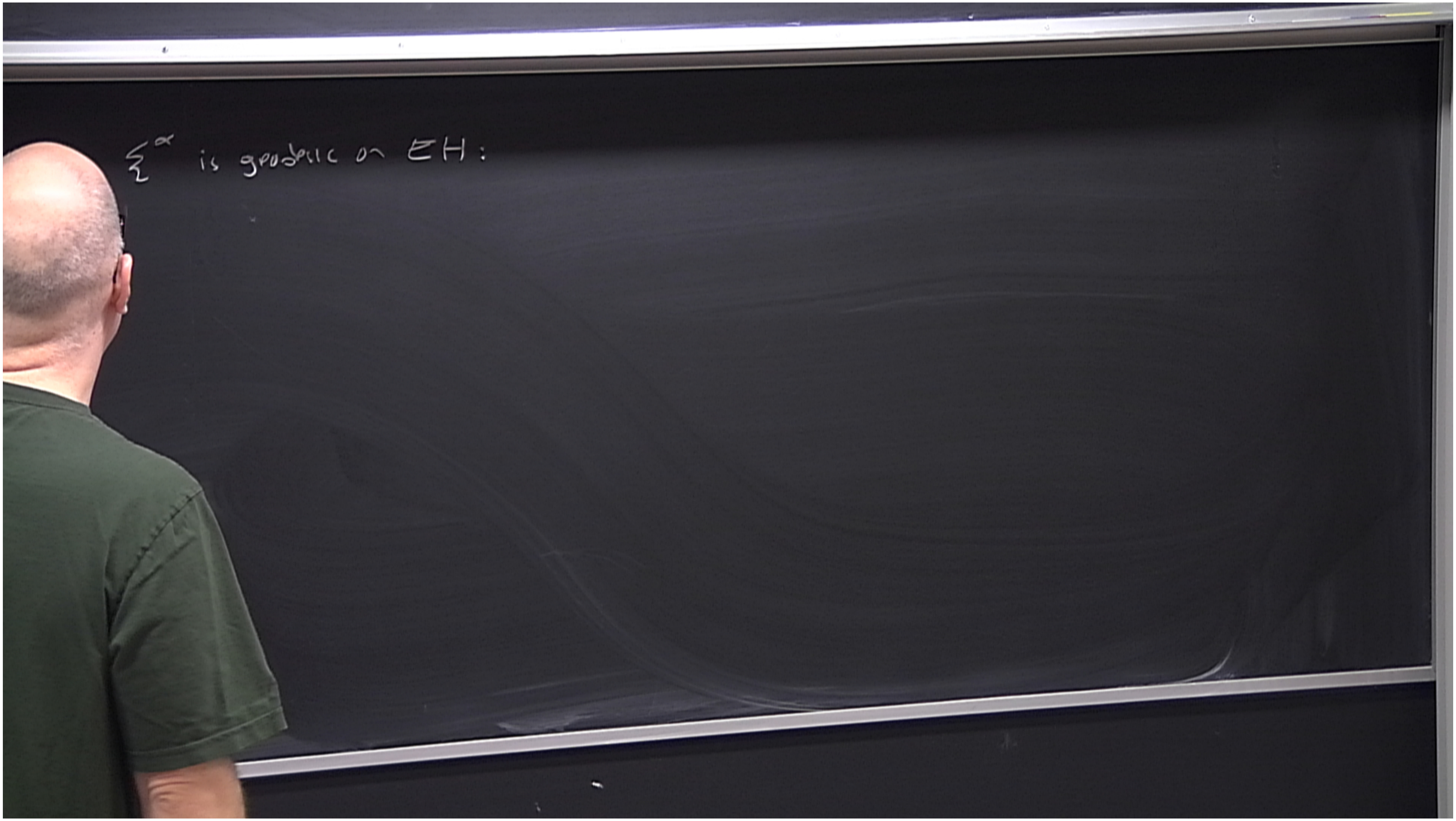
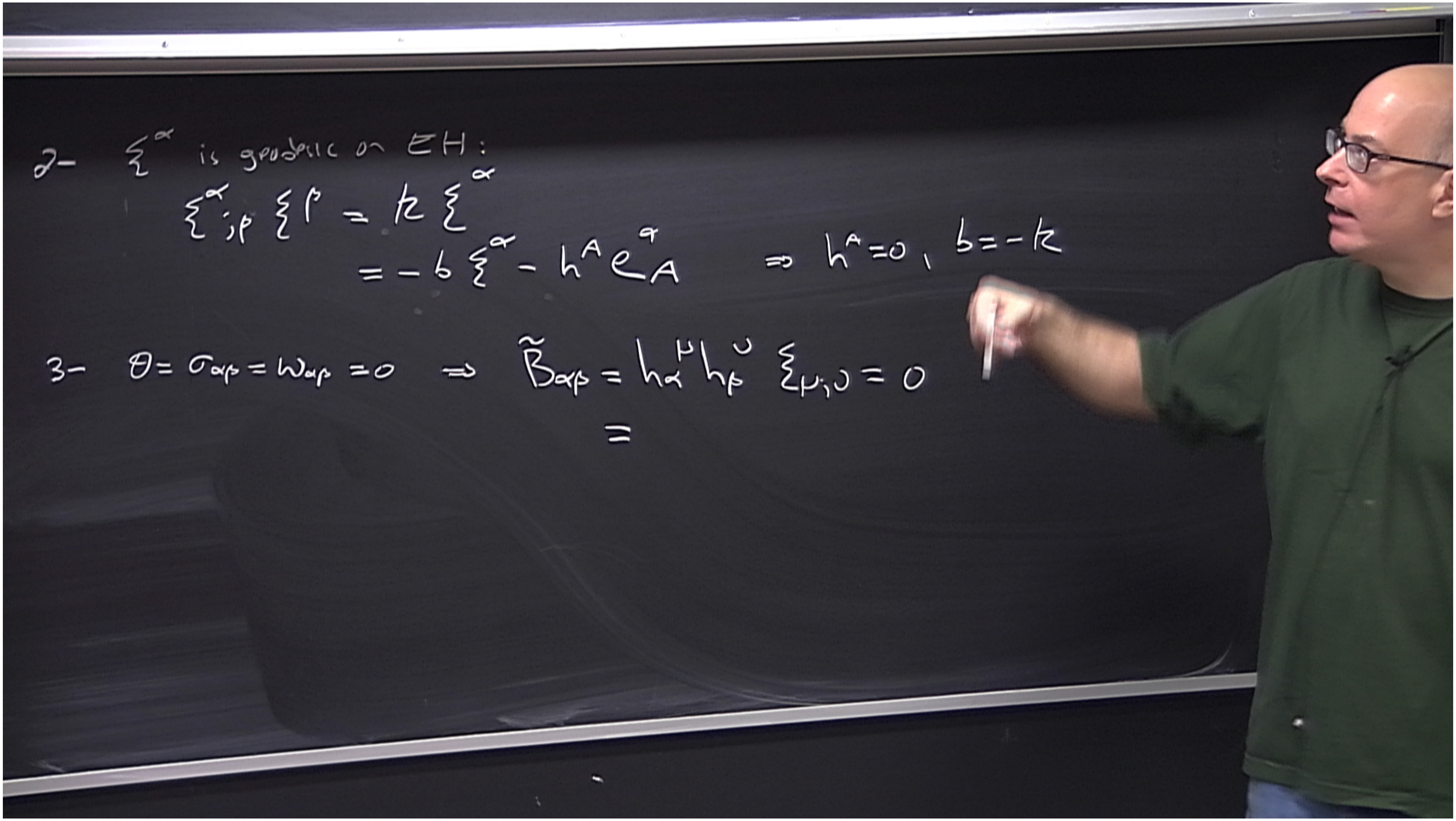

ξ^α is geodesic on EH:

2– ξ^α is geodesic on EH:

$$\xi^\alpha{}_{;\beta}\,\xi^\beta = \kappa\,\xi^\alpha$$


2- ξ^α is geodesic on EH:
$\xi^\alpha{}_{;\beta}\, \xi^\beta = k\, \xi^\alpha$
$= -b\, \xi^\alpha - h^A e^\alpha_A \Rightarrow h^A = 0,\ b = -k$
3- $\theta = \sigma_{\alpha\beta} = \omega_{\alpha\beta} = 0 \Rightarrow \tilde{B}_{\alpha\beta} = h_\alpha{}^\mu h_\beta{}^\nu\, \xi_{\mu;\nu} = 0$
$=$

$$\xi_{\alpha;\beta} = a\, \xi_\alpha \xi_\beta - k\, \xi_\alpha N_\beta + c^A \xi_\alpha e_{A\beta}$$

$$+ d\, N_\alpha \xi_\beta + e\, \cancel{N_\alpha N_\beta} + f^A \cancel{N_\alpha e_{A\beta}}$$

$$+ g^A e_{A\alpha} \xi_\beta + h^A \cancel{e_{A\alpha} N_\beta} + j^{AB} e_{A\alpha} e_{B\beta}$$

(general decomposition)

1- ξ^α is null on EH :

$$0 = (\xi^\alpha \xi_\alpha)_{;\beta}\, \xi^\beta = 2\, \xi^\alpha \xi_{\alpha;\beta}\, \xi^\beta$$

$$0 = (\xi^\alpha \xi_\alpha)_{;\beta}\, e^\beta_A = 2\, \xi^\alpha \xi_{\alpha;\beta}\, e^\beta_A$$

$$\xi^\alpha \xi_{\alpha;\beta} = -d\, \xi_\beta - e\, N_\beta - f^A e_{A\beta}$$

$$0 = \xi^\alpha \xi_{\alpha;\beta}\, \xi^\beta = +e$$

$$0 = \xi^\alpha \xi_{\alpha;\beta}\, e^\beta_C = -f^A \sigma_{AC}$$

$$0 = \xi^{\alpha} \xi_{\alpha;\beta} e^{\beta}_{C} = -f^{A} \sigma_{AC}$$

2- ξ^{α} is geodesic on EH:

$$\xi^{\alpha}{}_{;\beta} \xi^{\beta} = k \xi^{\alpha}$$

$$= -b \xi^{\alpha} - h^{A} e^{\alpha}_{A} \Rightarrow h^{A} = 0, \ b = -k$$

3- $\theta = \sigma_{\alpha\beta} = \omega_{\alpha\beta} = 0 \Rightarrow \tilde{B}_{\alpha\beta} = h_{\alpha}{}^{\mu} h_{\beta}{}^{\nu} \xi_{\mu;\nu} = 0$

$$= j^{AB} e_{A\alpha} e_{B\beta}$$

2- ξ^α is geodesic on EH:

$$\xi^\alpha{}_{;\beta}\,\xi^\beta = k\,\xi^\alpha$$

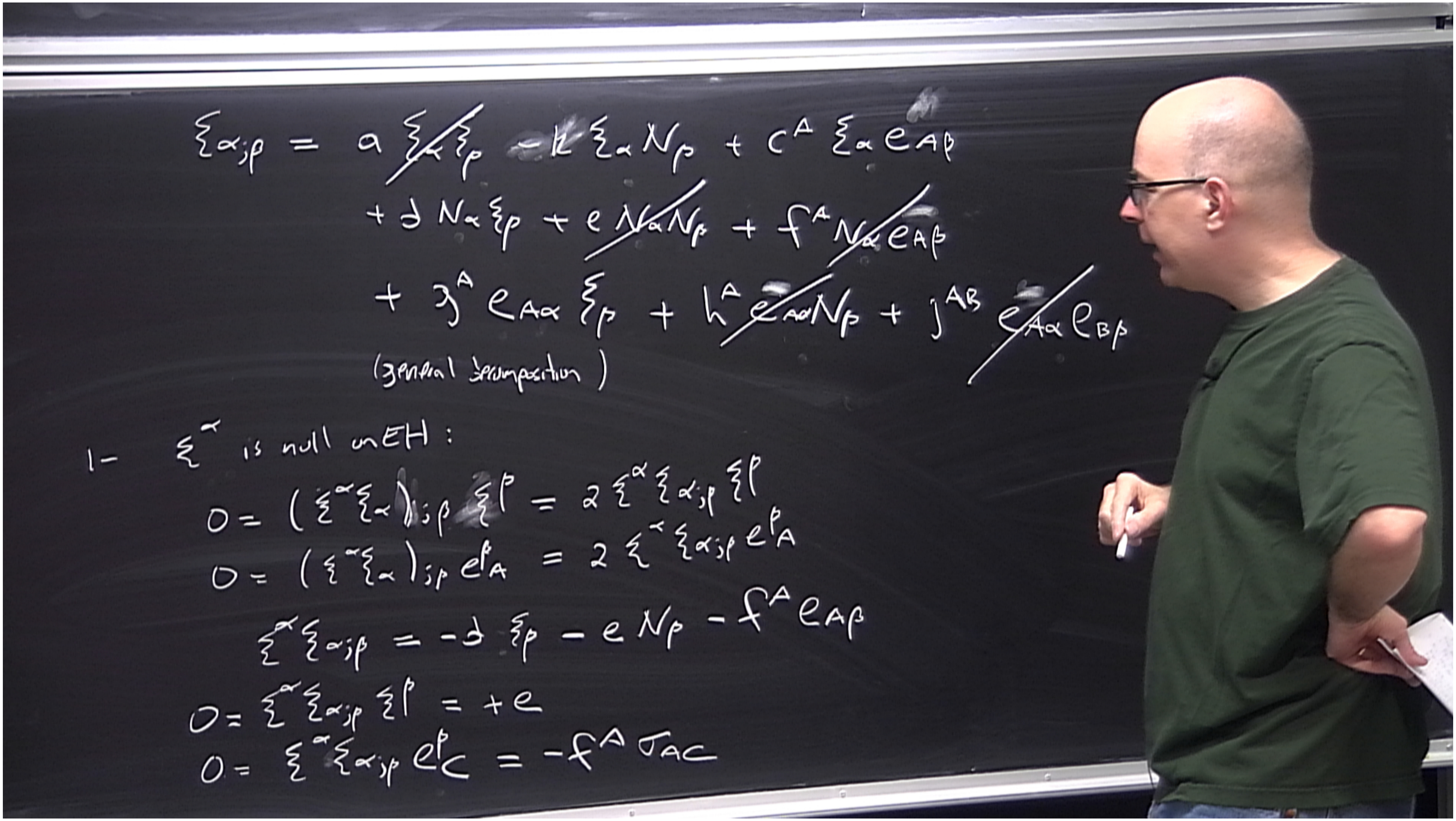
$$= -b\,\xi^\alpha - h^A e^\alpha_A \quad \Rightarrow h^A = 0\,,\; b = -k$$

3- $\theta = \sigma_{\alpha\beta} = \omega_{\alpha\beta} = 0 \quad \Rightarrow \quad \tilde{B}_{\alpha\beta} = h_\alpha{}^\mu h_\beta{}^\nu\,\xi_{\mu;\nu} = 0$

$$= j^{AB}\,e_{A\alpha}\,e_{B\beta} \quad \Rightarrow j^{AB} = 0$$

4- Killing vector

$$\xi_{(\alpha;\beta)} = 0$$


$\xi_{\alpha;\beta} = a\,\cancel{\xi_\alpha \xi_\beta} - k\,\xi_\alpha N_\beta + c^A \xi_\alpha e_{A\beta}$
$+ d\,N_\alpha \xi_\beta + e\,\cancel{N_\alpha N_\beta} + f^A \cancel{N_\alpha} e_{A\beta}$
$+ g^A e_{A\alpha} \xi_\beta + h^A \cancel{e_{A\alpha} N_\beta} + j^{AB} \cancel{e_{A\alpha} e_{B\beta}}$
(general decomposition)
1- ξ^α is null on EH :
$0 = (\xi^\alpha \xi_\alpha)_{;\beta}\, \xi^\beta = 2\, \xi^\alpha \xi_{\alpha;\beta}\, \xi^\beta$
$0 = (\xi^\alpha \xi_\alpha)_{;\beta}\, e^\beta_A = 2\, \xi^\alpha \xi_{\alpha;\beta}\, e^\beta_A$
$\xi^\alpha \xi_{\alpha;\beta} = -d\, \xi_\beta - e\, N_\beta - f^A e_{A\beta}$
$0 = \xi^\alpha \xi_{\alpha;\beta}\, \xi^\beta = +e$
$0 = \xi^\alpha \xi_{\alpha;\beta}\, e^\beta_C = -f^A \sigma_{AC}$

$$\xi_{\alpha;\beta} = a\,\cancel{\xi_\alpha}\xi_\beta - k\,\xi_\alpha N_\beta + c^A\,\xi_\alpha e_{A\beta}$$
$$+ d\,N_\alpha \xi_\beta + \cancel{e\,N_\alpha N_\beta} + \cancel{f^A N_\alpha} e_{A\beta}$$
$$+ g^A e_{A\alpha}\xi_\beta + \cancel{h^A e_{A\alpha} N_\beta} + \cancel{j^{AB} e_{A\alpha} e_{B\beta}}$$

(general decomposition)

1- ξ^α is null on EH :

$$0 = (\xi^\alpha \xi_\alpha)_{;\beta}\,\xi^\beta = 2\,\xi^\alpha \xi_{\alpha;\beta}\,\xi^\beta$$
$$0 = (\xi^\alpha \xi_\alpha)_{;\beta}\,e^\beta_A = 2\,\xi^\alpha \xi_{\alpha;\beta}\,e^\beta_A$$
$$\xi^\alpha \xi_{\alpha;\beta} = -d\,\xi_\beta - e\,N_\beta - f^A e_{A\beta}$$
$$0 = \xi^\alpha \xi_{\alpha;\beta}\,\xi^\beta = +e$$
$$0 = \xi^\alpha \xi_{\alpha;\beta}\,e^\beta_C = -f^A \sigma_{AC}$$

$\xi^\alpha \xi_{\alpha;\beta} = -d\, \xi_\beta - e N_\beta - f^A e_{A\beta}$

$0 = \xi^\alpha \xi_{\alpha;\beta} \xi^\beta = +e$

$0 = \xi^\alpha \xi_{\alpha;\beta} e^\beta_C = -f^A \sigma_{AC}$

$= -b\, \xi^\alpha - h^A e^\alpha_A \quad \Rightarrow h^A = 0, \; b = -k$

3- $\theta = \sigma_{\alpha\beta} = \omega_{\alpha\beta} = 0 \quad \Rightarrow \tilde{B}_{\alpha\beta} = h_\alpha^{\ \mu} h_\beta^{\ \nu} \xi_{\mu;\nu} = 0$

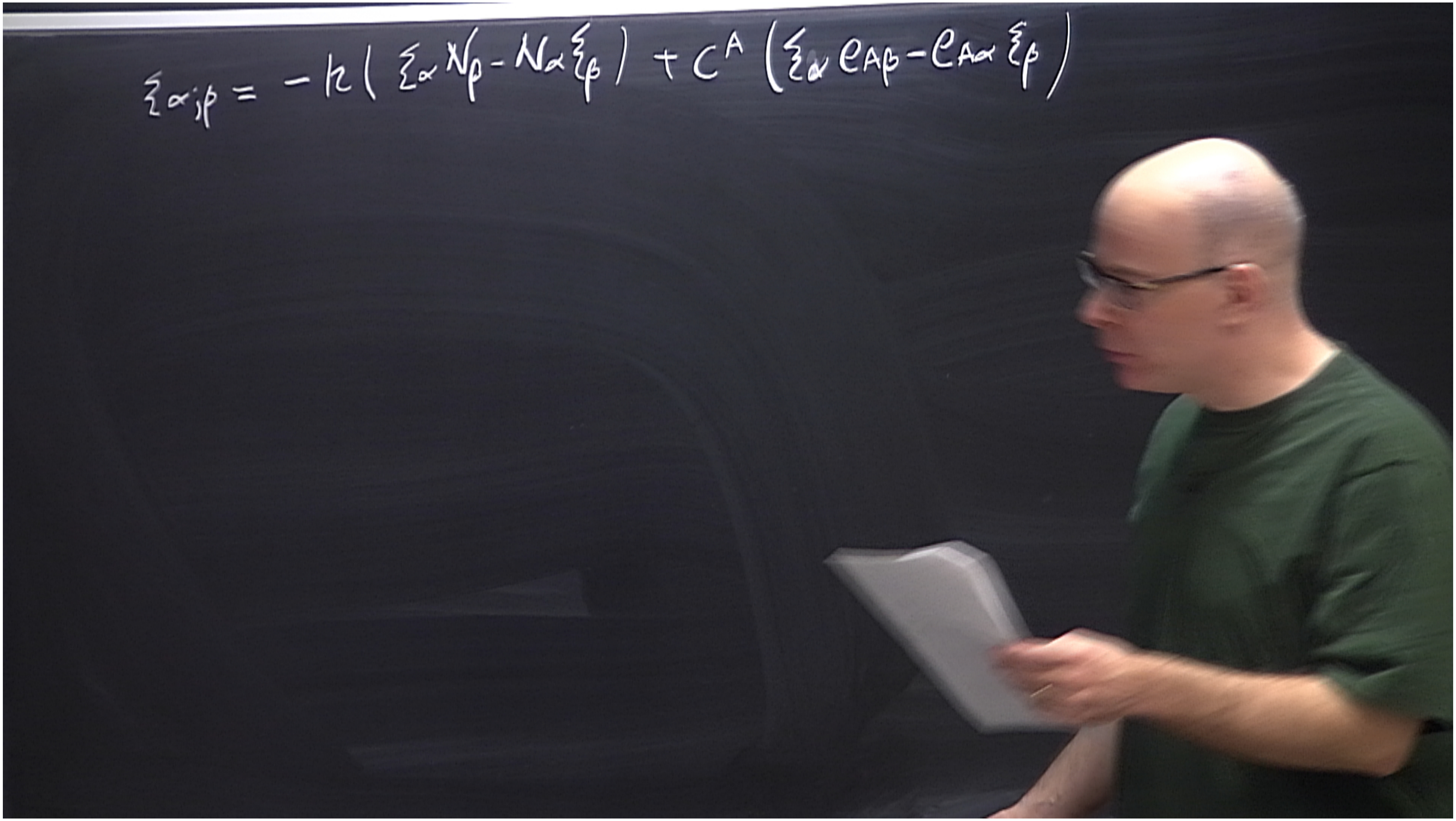
$= j^{AB} e_{A\alpha} e_{B\beta} \quad \Rightarrow j^{AB} = 0$

4- Killing vector

$\xi_{(\alpha;\beta)} = 0$

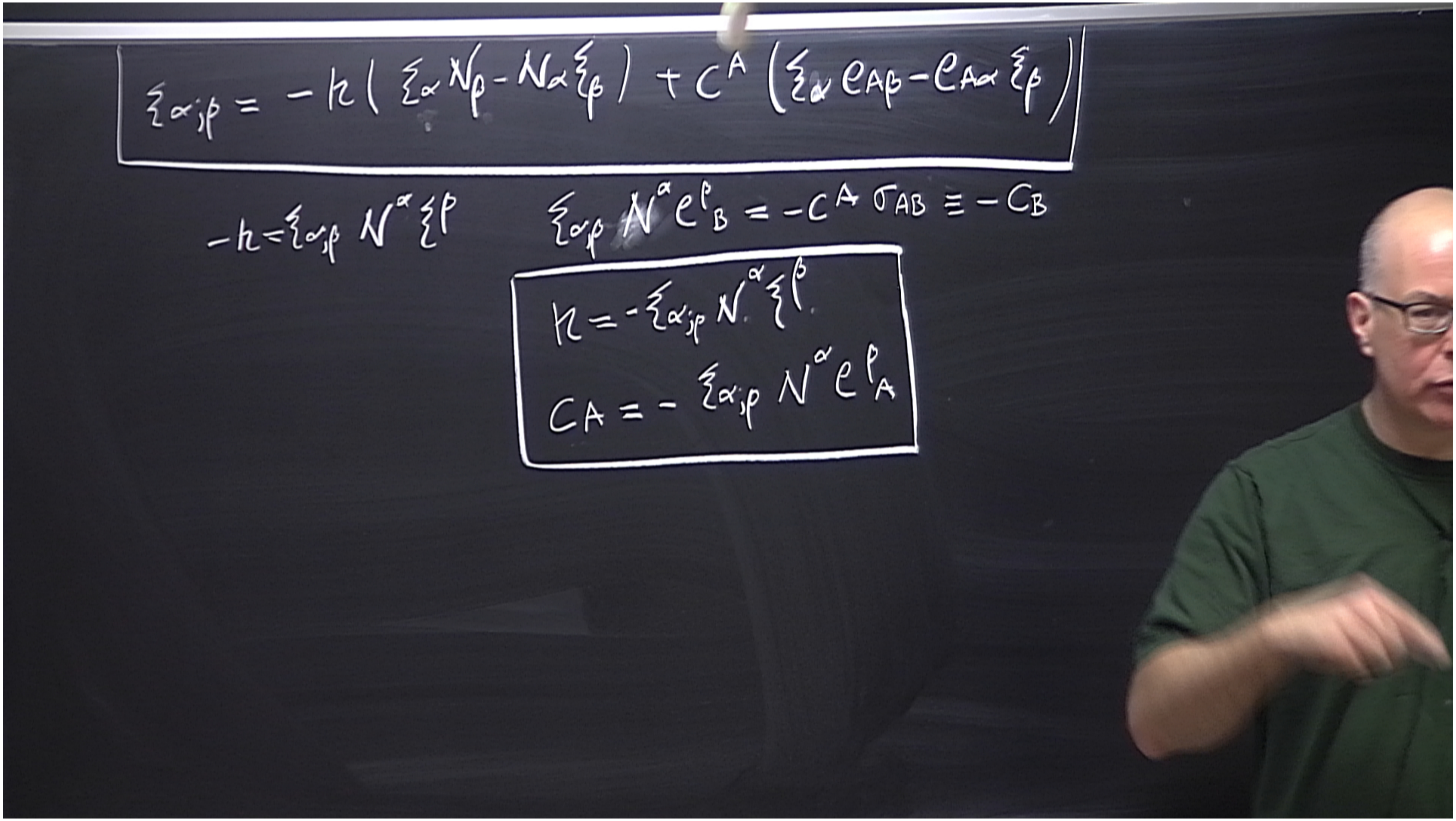
$\Rightarrow d = k$

$g^A = -c^A$


$\xi_{\alpha;\beta} = -k\left(\xi_\alpha N_\beta - N_\alpha \xi_\beta\right) + C^A\left(\xi_\alpha e_{A\beta} - e_{A\alpha}\xi_\beta\right)$

$$\boxed{\xi_{\alpha;\beta} = -k\left(\xi_\alpha N_\beta - N_\alpha \xi_\beta\right) + c^A\left(\xi_\alpha e_{A\beta} - e_{A\alpha}\xi_\beta\right)}$$

$$-k = \xi_{\alpha;\beta} N^\alpha \xi^\beta \qquad \xi_{\alpha;\beta} N^\alpha e^\beta{}_B = -c^A \sigma_{AB}$$


$$\boxed{\xi_{\alpha;\beta} = -k\left(\xi_\alpha N_\beta - N_\alpha \xi_\beta\right) + c^A\left(\xi_\alpha e_{A\beta} - e_{A\alpha}\xi_\beta\right)}$$
$-k = \xi_{\alpha;\beta} N^\alpha \xi^\beta$
$\xi_{\alpha;\beta} N^\alpha e^\beta{}_B = -c^A \sigma_{AB} \equiv -c_B$
$$\boxed{\begin{aligned} k &= -\xi_{\alpha;\beta} N^\alpha \xi^\beta \\ c_A &= -\xi_{\alpha;\beta} N^\alpha e^\beta{}_A \end{aligned}}$$

$$\boxed{\xi_{\alpha;\beta} = -k\left(\xi_\alpha N_\beta - N_\alpha \xi_\beta\right) + C^A\left(\xi_\alpha e_{A\beta} - e_{A\alpha}\xi_\beta\right)}$$

$$-k = \xi_{\alpha;\beta} N^\alpha \xi^\beta \qquad \xi_{\alpha;\beta} N^\alpha e^\beta{}_B = -C^A \sigma_{AB} \equiv -C_B$$

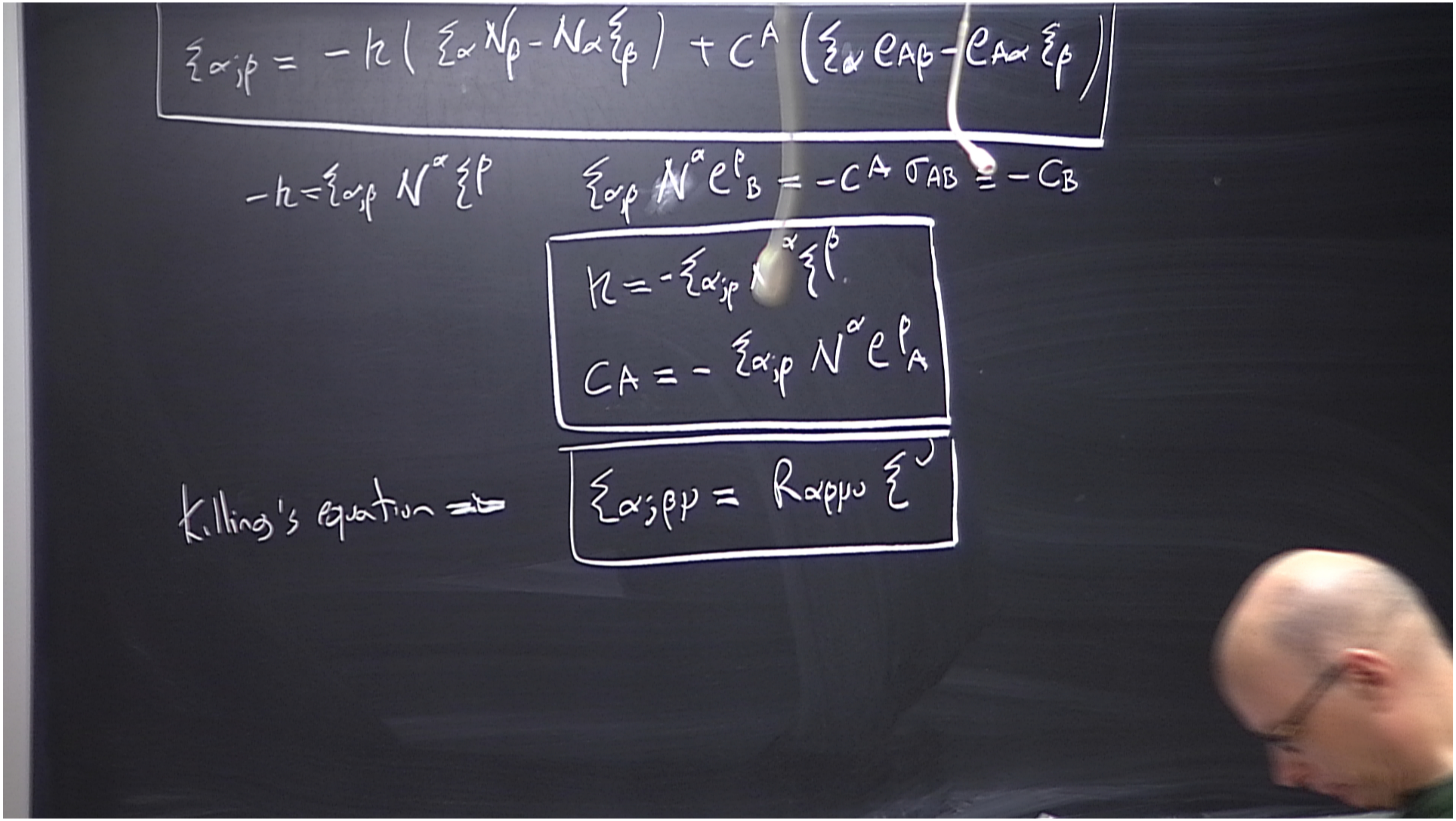
$$\boxed{\begin{aligned} k &= -\xi_{\alpha;\beta} N^\alpha \xi^\beta \\ C_A &= -\xi_{\alpha;\beta} N^\alpha e^\beta{}_A \end{aligned}}$$

$$\boxed{\xi_{\alpha;\beta} = -k\left(\xi_\alpha N_\beta - N_\alpha \xi_\beta\right) + C^A\left(\xi_\alpha e_{A\beta} - e_{A\alpha}\xi_\beta\right)}$$

$$-k = \xi_{\alpha;\beta} N^\alpha \xi^\beta \qquad \xi_{\alpha;\beta} N^\alpha e^\beta{}_B = -C^A \sigma_{AB} \equiv -C_B$$

$$\boxed{\begin{aligned} k &= -\xi_{\alpha;\beta} N^\alpha \xi^\beta \\ C_A &= -\xi_{\alpha;\beta} N^\alpha e^\beta{}_A \end{aligned}}$$

Killing's equation $\Rightarrow$ $\xi_{\alpha;\beta\mu} = R_{\alpha\beta\mu\nu}\xi^\nu$


$\xi_{\alpha;\beta} = -\kappa(\xi_\alpha N_\beta - N_\alpha \xi_\beta) + C^A(\xi_\alpha e_{A\beta} - e_{A\alpha}\xi_\beta)$
$-\kappa = \xi_{\alpha;\beta} N^\alpha \xi^\beta$
$\xi_{\alpha;\beta} N^\alpha e^\beta_B = -C^A \sigma_{AB} = -C_B$
$\kappa = -\xi_{\alpha;\beta} N^\alpha \xi^\beta$
$C_A = -\xi_{\alpha;\beta} N^\alpha e^\beta_A$
Killing's equation $\Rightarrow$
$\xi_{\alpha;\beta\mu} = R_{\alpha\beta\mu\nu} \xi^\nu$

$-\kappa = \xi_{\alpha;\beta} N^\alpha \xi^\beta$

$\xi_{\alpha;\beta} N^\alpha e^\beta{}_B = -C^A \sigma_{AB} \equiv -C_B$

$$\boxed{\begin{aligned} \kappa &= -\xi_{\alpha;\beta} N^\alpha \xi^\beta \\ C_A &= -\xi_{\alpha;\beta} N^\alpha e^\beta{}_A \end{aligned}}$$

Killing's equation $\Rightarrow$ $\boxed{\xi_{\alpha;\beta\mu} = R_{\alpha\beta\mu\nu} \xi^\nu}$

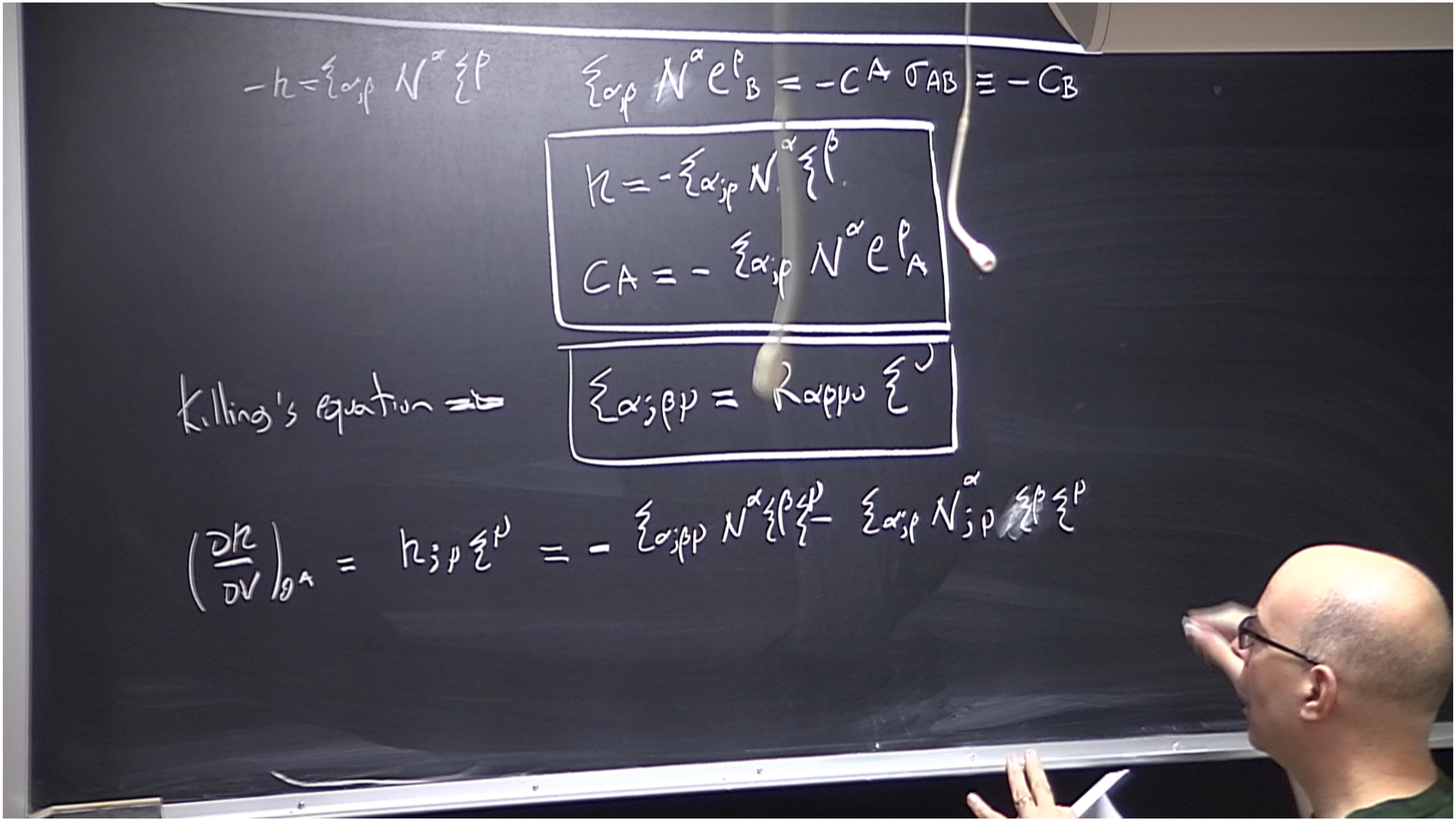
$$\left(\frac{\partial \kappa}{\partial v}\right)_{\theta^A} = \kappa_{;\mu} \xi^\mu$$

$$-\kappa = \xi_{\alpha;\beta} N^\alpha \xi^\beta \qquad \xi_{\alpha;\beta} N^\alpha e^\beta{}_B = -C^A \sigma_{AB} \equiv -C_B$$

$$\boxed{\begin{aligned} \kappa &= -\xi_{\alpha;\beta} N^\alpha \xi^\beta \\ C_A &= -\xi_{\alpha;\beta} N^\alpha e^\beta{}_A \end{aligned}}$$

Killing's equation $\Rightarrow$ $\boxed{\xi_{\alpha;\beta\mu} = R_{\alpha\beta\mu\nu} \xi^\nu}$

$$\left(\frac{\partial \kappa}{\partial v}\right)_{\theta^A} = \kappa_{;\mu} \xi^\mu = -\xi_{\alpha;\beta\mu}$$


$-\kappa = \xi_{\alpha;\beta} N^\alpha \xi^\beta$
$\xi_{\alpha;\beta} N^\alpha e^\beta{}_B = -C^A \sigma_{AB} \equiv -C_B$
$\kappa = -\xi_{\alpha;\beta} N^\alpha \xi^\beta$
$C_A = -\xi_{\alpha;\beta} N^\alpha e^\beta{}_A$
Killing's equation $\Rightarrow$
$\xi_{\alpha;\beta\mu} = R_{\alpha\beta\mu\nu} \xi^\nu$
$\left(\frac{\partial \kappa}{\partial v}\right)_{\theta^A} = \kappa_{;\mu} \xi^\mu = -\xi_{\alpha;\beta\mu} N^\alpha \xi^\beta \xi^\mu - \xi_{\alpha;\beta} N^\alpha{}_{;\mu} \xi^\beta \xi^\mu$

$$-\kappa = \xi_{\alpha;\beta} N^\alpha \xi^\beta \qquad \xi_{\alpha;\beta} N^\alpha e^\beta{}_B = -C^A \sigma_{AB} \equiv -C_B$$

$$\boxed{\begin{aligned} \kappa &= -\xi_{\alpha;\beta} N^\alpha \xi^\beta \\ C_A &= -\xi_{\alpha;\beta} N^\alpha e^\beta{}_A \end{aligned}}$$

Killing's equation $\Rightarrow$ $\boxed{\xi_{\alpha;\beta\mu} = R_{\alpha\beta\mu\nu} \xi^\nu}$

$$\left(\frac{\partial \kappa}{\partial v}\right)_{\theta^A} = \kappa_{;\mu} \xi^\mu = -\xi_{\alpha;\beta\mu} N^\alpha \xi^\beta \xi^\mu - \xi_{\alpha;\beta} N^\alpha{}_{;\mu} \xi^\mu \xi^\beta - \xi_{\alpha;\beta} N^\alpha \xi^\beta{}_{;\mu} \xi^\mu$$

$-k = \xi_{\alpha;\beta} N^\alpha \xi^\beta$ $\qquad \xi_{\alpha;\beta} N^\alpha e^\beta{}_B = -C^A \sigma_{AB} \equiv -C_B$

$$\boxed{\begin{aligned} k &= -\xi_{\alpha;\beta} N^\alpha \xi^\beta \\ C_A &= -\xi_{\alpha;\beta} N^\alpha e^\beta{}_A \end{aligned}}$$

Killing's equation $\Rightarrow$ $\boxed{\xi_{\alpha;\beta\mu} = R_{\alpha\beta\mu\nu} \xi^\nu}$

$$\left(\frac{\partial k}{\partial v}\right)_{\theta^A} = k_{;\mu} \xi^\mu = -\xi_{\alpha;\beta\mu} N^\alpha \xi^\beta \xi^\mu - \xi_{\alpha;\beta} N^\alpha{}_{;\mu} \xi^\beta \xi^\mu - \xi_{\alpha;\beta} N^\alpha \xi^\beta{}_{;\mu} \xi^\mu$$

$$= -R_{\alpha\beta\mu\nu} N^\alpha \xi^\beta \xi^\mu \xi^\nu$$

$$-\kappa = \xi_{\alpha;\beta} N^\alpha \xi^\beta \qquad \xi_{\alpha;\beta} N^\alpha e^\beta{}_B = -C^A \sigma_{AB} \equiv -C_B$$

$$\boxed{\begin{aligned} \kappa &= -\xi_{\alpha;\beta} N^\alpha \xi^\beta \\ C_A &= -\xi_{\alpha;\beta} N^\alpha e^\beta{}_A \end{aligned}}$$

Killing's equation $\Rightarrow$ $\boxed{\xi_{\alpha;\beta\mu} = R_{\alpha\beta\mu\nu}\xi^\nu}$

$$\left(\frac{D\kappa}{Dv}\right)_{\theta^A} = \kappa_{;\mu}\xi^\mu = -\xi_{\alpha;\beta\mu} N^\alpha \xi^\beta \xi^\mu - \xi_{\alpha;\beta} N^\alpha{}_{;\mu} \xi^\mu \xi^\beta - \xi_{\alpha;\beta} N^\alpha \xi^\beta{}_{;\mu} \xi^\mu$$

$$= -R_{\alpha\beta\mu\nu} \cancel{N^\alpha \xi^\beta} \xi^\mu \xi^\nu$$

$$-k = \xi_{\alpha;\beta} N^{\alpha} \xi^{\beta} \qquad \xi_{\alpha;\beta} N^{\alpha} e^{\beta}{}_{B} = -C^{A}\sigma_{AB} \equiv -C_{B}$$

$$\boxed{\begin{aligned} k &= -\xi_{\alpha;\beta} N^{\alpha} \xi^{\beta} \\ C_A &= -\xi_{\alpha;\beta} N^{\alpha} e^{\beta}{}_{A} \end{aligned}}$$

Killing's equation $\Rightarrow$ $\boxed{\xi_{\alpha;\beta\mu} = R_{\alpha\beta\mu\nu}\xi^{\nu}}$

$$\left(\frac{\partial k}{\partial v}\right)_{\theta^A} = k_{;\mu}\xi^{\mu} = -\xi_{\alpha;\beta\mu} N^{\alpha}\xi^{\beta}\xi^{\mu} - \xi_{\alpha;\beta} N^{\alpha}{}_{;\mu}\xi^{\beta}\xi^{\mu} - \xi_{\alpha;\beta} N^{\alpha}\xi^{\beta}{}_{;\mu}\xi^{\mu}$$

$$= \cancel{-R_{\alpha\beta\mu\nu} N^{\alpha}\xi^{\beta}\xi^{\mu}\xi^{\nu}} - k\,\xi_{\alpha} N^{\alpha}{}_{;\mu}\xi^{\mu}$$

$-\kappa = \xi_{\alpha;\beta} N^\alpha \xi^\beta \qquad \xi_{\alpha;\beta} N^\alpha e^\beta_B = -C^A \sigma_{AB} \equiv -C_B$

$$\boxed{\begin{aligned} \kappa &= -\xi_{\alpha;\beta} N^\alpha \xi^\beta \\ C_A &= -\xi_{\alpha;\beta} N^\alpha e^\beta_A \end{aligned}}$$

Killing's equation $\Rightarrow$ $\boxed{\xi_{\alpha;\beta\mu} = R_{\alpha\beta\mu\nu}\, \xi^\nu}$

$$\left(\frac{\partial \kappa}{\partial v}\right)_{\theta^A} = \kappa_{;\mu}\xi^\mu = -\xi_{\alpha;\beta\mu} N^\alpha \xi^\beta \xi^\mu - \xi_{\alpha;\beta} N^\alpha_{;\mu} \xi^\beta \xi^\mu - \xi_{\alpha;\beta} N^\alpha \xi^\beta_{;\mu} \xi^\mu$$

$$= -R_{\alpha\beta\mu\nu} \cancel{N^\alpha \xi^\beta \xi^\mu \xi^\nu} - \kappa\, \xi_\alpha N^\alpha_{;\mu} \xi^\mu - \kappa\, \xi_{\alpha;\beta} N^\alpha \xi^\beta$$

$-k = \xi_{\alpha;\beta} N^\alpha \xi^\beta$

$\xi_{\alpha;\beta} N^\alpha e^\beta_B = -C^A \sigma_{AB} \equiv -C_B$

$$\boxed{\begin{aligned} k &= -\xi_{\alpha;\beta} N^\alpha \xi^\beta \\ C_A &= -\xi_{\alpha;\beta} N^\alpha e^\beta_A \end{aligned}}$$

$N_\alpha \xi^\alpha = -1$

$N_{\alpha;\beta} \xi^\alpha + N_\alpha \xi^\alpha_{;\beta} = 0$

Killing's equation $\Rightarrow$ $\boxed{\xi_{\alpha;\beta\mu} = R_{\alpha\beta\mu\nu} \xi^\nu}$

$$\left(\frac{\partial k}{\partial v}\right)_{\theta^A} = k_{;\mu}\xi^\mu = -\xi_{\alpha;\beta\mu} N^\alpha \xi^\beta \xi^\mu - \xi_{\alpha;\beta} N^\alpha_{;\mu} \xi^\beta \xi^\mu - \xi_{\alpha;\beta} N^\alpha \xi^\beta_{;\mu} \xi^\mu$$

$$= -\cancel{R_{\alpha\beta\mu\nu} N^\alpha \xi^\beta \xi^\mu \xi^\nu} - k \xi_\alpha N^\alpha_{;\mu} \xi^\mu - k \xi_{\alpha;\beta} N^\alpha \xi^\beta$$

$-k = \xi_{\alpha;\beta} N^\alpha \xi^\beta$

$\xi_{\alpha;\beta} N^\alpha e^\beta{}_B = -C^A \sigma_{AB} \equiv -C_B$

$$\boxed{\begin{aligned} k &= -\xi_{\alpha;\beta} N^\alpha \xi^\beta \\ C_A &= -\xi_{\alpha;\beta} N^\alpha e^\beta{}_A \end{aligned}}$$

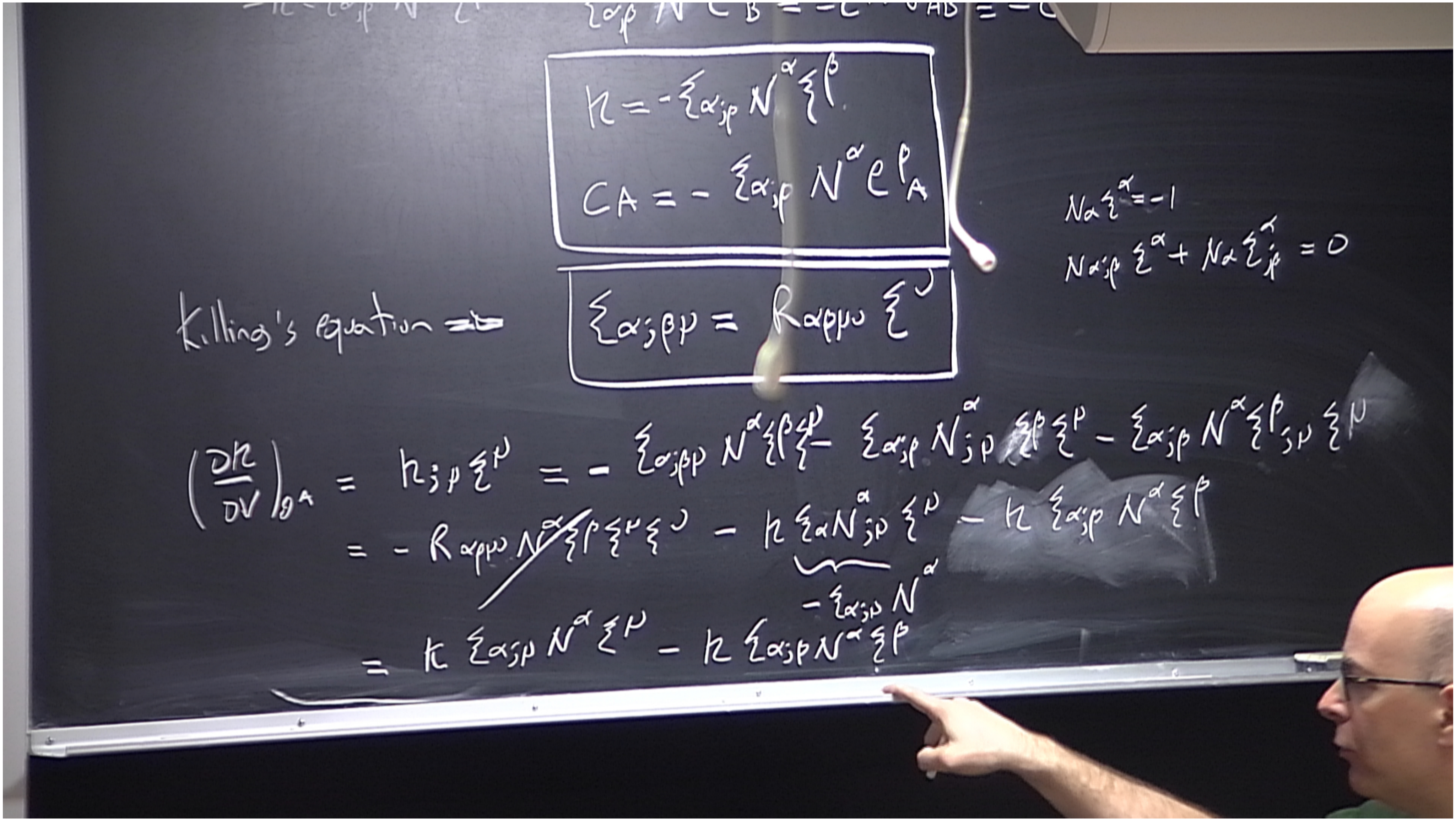
$N_\alpha \xi^\alpha = -1$

$N_{\alpha;\beta} \xi^\alpha + N_\alpha \xi^\alpha{}_{;\beta} = 0$

Killing's equation $\Rightarrow$ $\boxed{\xi_{\alpha;\beta\mu} = R_{\alpha\beta\mu\nu} \xi^\nu}$

$$\left(\frac{\partial k}{\partial v}\right)_{\theta^A} = k_{;\mu} \xi^\mu = -\xi_{\alpha;\beta\mu} N^\alpha \xi^\beta \xi^\mu - \xi_{\alpha;\beta} N^\alpha{}_{;\mu} \xi^\beta \xi^\mu - \xi_{\alpha;\beta} N^\alpha \xi^\beta{}_{;\mu} \xi^\mu$$

$$= -\cancel{R_{\alpha\beta\mu\nu} N^\alpha \xi^\beta \xi^\mu \xi^\nu} - k \underbrace{\xi_\alpha N^\alpha{}_{;\mu}}_{-\xi_{\alpha;\mu} N^\alpha} \xi^\mu - k\, \xi_{\alpha;\beta} N^\alpha \xi^\beta$$



$\boxed{\begin{aligned} \kappa &= -\xi_{\alpha;\beta} N^\alpha \xi^\beta \\ C_A &= -\xi_{\alpha;\beta} N^\alpha e^\beta_A \end{aligned}}$
$N_\alpha \xi^\alpha = -1$
$N_{\alpha;\beta} \xi^\alpha + N_\alpha \xi^\alpha_{;\beta} = 0$
Killing's equation $\Rightarrow$ $\boxed{\xi_{\alpha;\beta\mu} = R_{\alpha\beta\mu\nu} \xi^\nu}$
$\left(\frac{\partial \kappa}{\partial v}\right)_{\theta^A} = \kappa_{;\mu} \xi^\mu = -\xi_{\alpha;\beta\mu} N^\alpha \xi^\beta \xi^\mu - \xi_{\alpha;\beta} N^\alpha_{;\mu} \xi^\beta \xi^\mu - \xi_{\alpha;\beta} N^\alpha \xi^\beta_{;\mu} \xi^\mu$
$= -\cancel{R_{\alpha\beta\mu\nu} N^\alpha \xi^\beta \xi^\mu \xi^\nu} - \kappa \underbrace{\xi_\alpha N^\alpha_{;\mu}}_{-\xi_{\alpha;\mu} N^\alpha} \xi^\mu - \kappa \xi_{\alpha;\beta} N^\alpha \xi^\beta$
$= \kappa \xi_{\alpha;\mu} N^\alpha \xi^\mu - \kappa \xi_{\alpha;\beta} N^\alpha \xi^\beta$

$$\boxed{\begin{aligned} \kappa &= -\xi_{\alpha;\beta} N^\alpha \xi^\beta \\ C_A &= -\xi_{\alpha;\beta} N^\alpha e^\beta_A \end{aligned}}$$

$N_\alpha \xi^\alpha = -1$

$N_{\alpha;\beta}\xi^\alpha + N_\alpha \xi^\alpha_{;\beta} = 0$

Killing's equation $\Rightarrow$ $\boxed{\xi_{\alpha;\beta\mu} = R_{\alpha\beta\mu\nu}\,\xi^\nu}$

$$\left(\frac{\partial \kappa}{\partial v}\right)_{\theta^A} = \kappa_{;\mu}\xi^\mu = -\xi_{\alpha;\beta\mu} N^\alpha \xi^\beta \xi^\mu - \xi_{\alpha;\beta} N^\alpha_{;\mu} \xi^\beta \xi^\mu - \xi_{\alpha;\beta} N^\alpha \xi^\beta_{;\mu} \xi^\mu$$

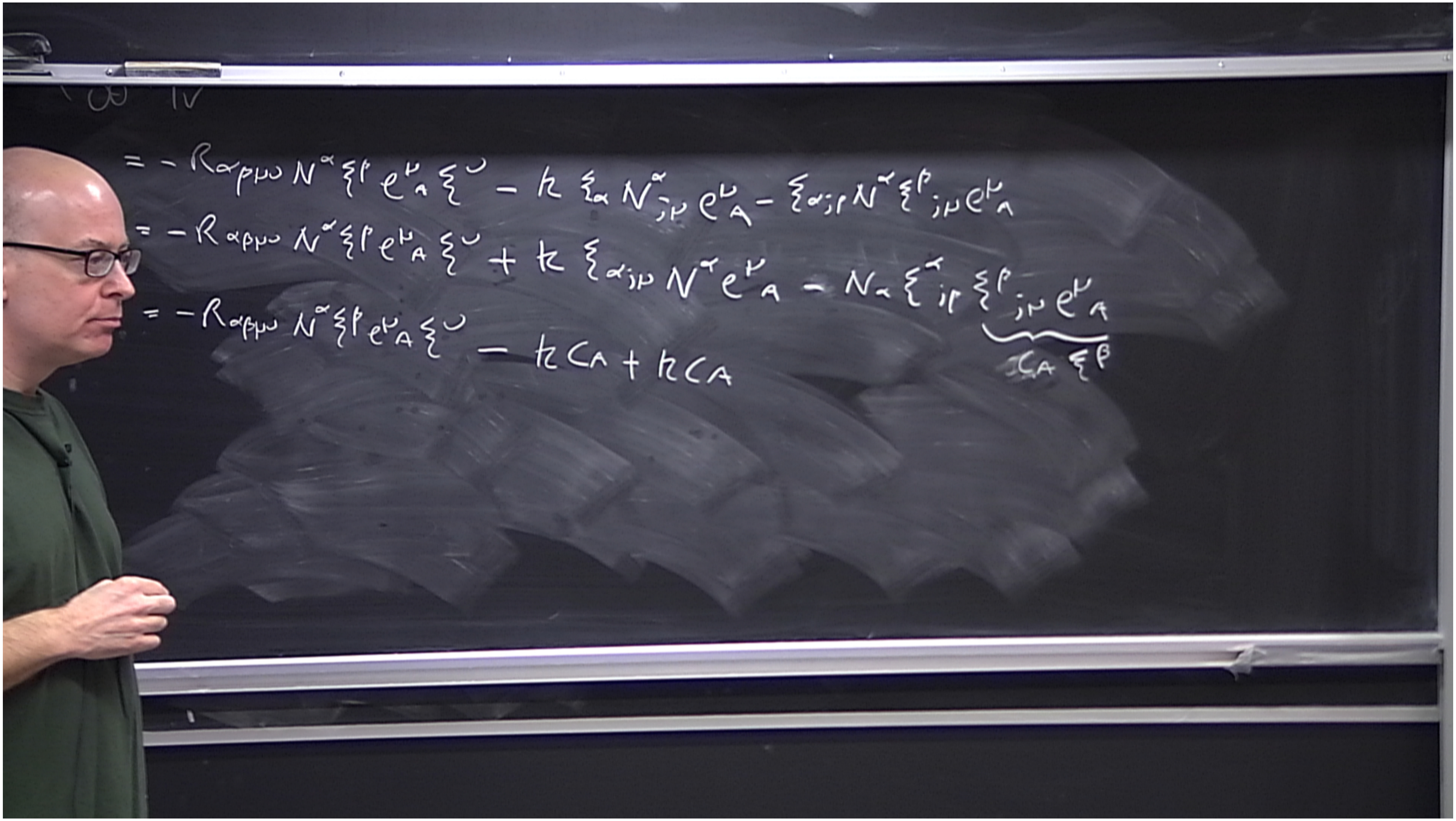
$$= -\cancel{R_{\alpha\beta\mu\nu} N^\alpha \xi^\beta \xi^\mu \xi^\nu} - \kappa \underbrace{\xi_\alpha N^\alpha_{;\mu}}_{-\xi_{\alpha;\mu} N^\alpha} \xi^\mu - \kappa\, \xi_{\alpha;\beta} N^\alpha \xi^\beta$$

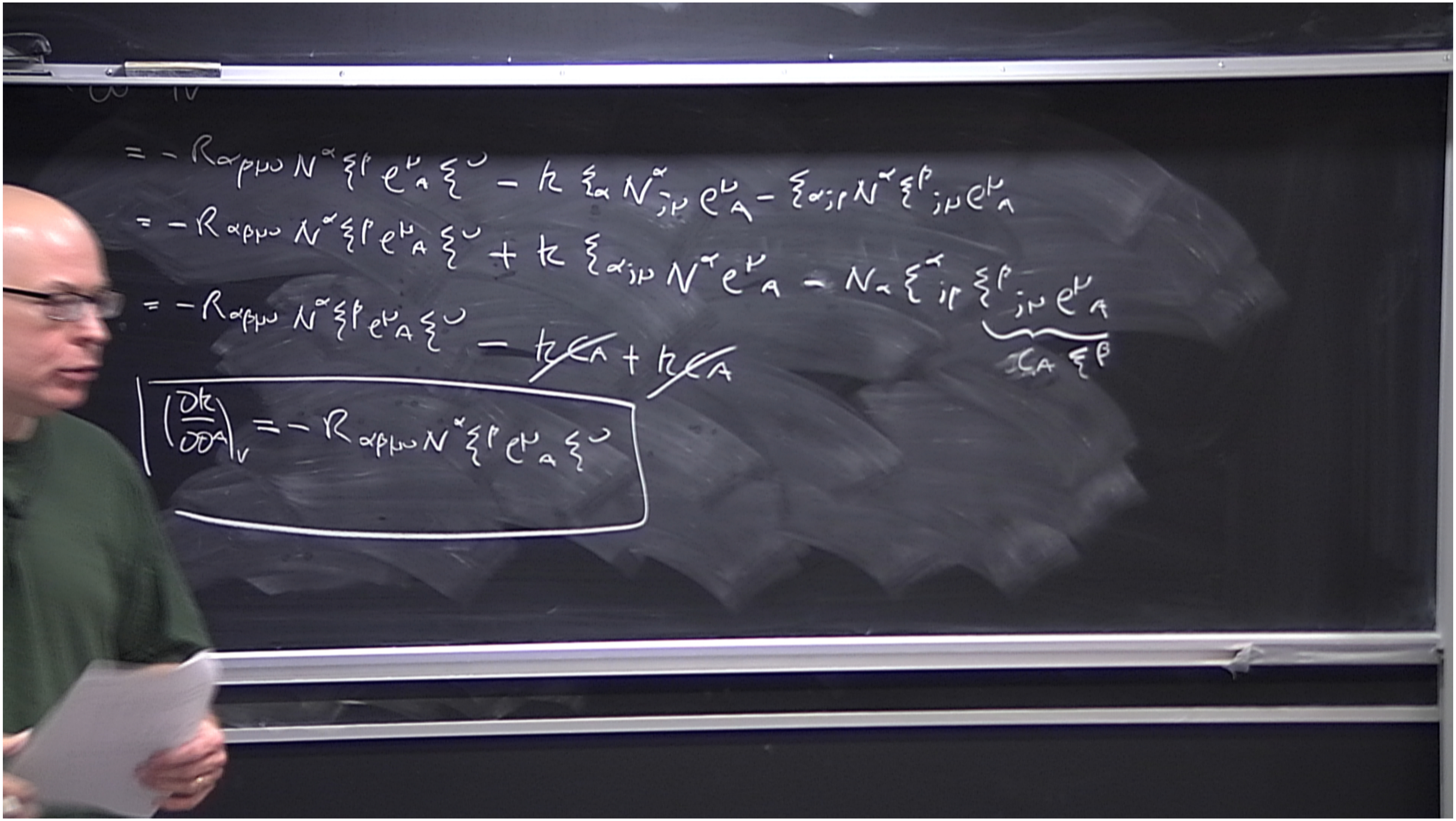
$$= \kappa\, \xi_{\alpha;\mu} N^\alpha \xi^\mu - \kappa\, \xi_{\alpha;\beta} N^\alpha \xi^\beta = 0$$

$$\left(\frac{\partial \kappa}{\partial \theta^A}\right)_V = \kappa_{;\mu} e^\mu_A = -\xi_{\alpha;\rho\mu} N^\alpha \xi^\rho e^\mu_A - \xi_{\alpha;\rho} N^\alpha_{;\mu} \xi^\rho e^\mu_A - \xi_{\alpha;\rho} N^\alpha \xi^\rho_{;\mu} e^\mu_A$$

$$= -R_{\alpha\rho\mu\nu} N^\alpha \xi^\rho e^\mu_A \xi^\nu - \kappa\, \xi_\alpha N^\alpha_{;\mu} e^\mu_A - \xi_{\alpha;\rho} N^\alpha \xi^\rho_{;\mu} e^\mu_A$$

$$= -R_{\alpha\rho\mu\nu} N^\alpha \xi^\rho e^\mu_A \xi^\nu + \kappa\, \xi_{\alpha;\mu} N^\alpha e^\mu_A - N_\alpha \xi^\alpha_{;\rho} \xi^\rho_{;\mu} e^\mu_A$$


$= -R_{\alpha\beta\mu\nu} N^\alpha \xi^\beta e^\mu{}_A \xi^\nu - k\, \xi_\alpha N^\alpha{}_{;\mu} e^\mu{}_A - \xi_{\alpha;\beta} N^\alpha \xi^\beta{}_{;\mu} e^\mu{}_A$
$= -R_{\alpha\beta\mu\nu} N^\alpha \xi^\beta e^\mu{}_A \xi^\nu + k\, \xi_{\alpha;\mu} N^\alpha e^\mu{}_A - N_\alpha \xi^\alpha{}_{;\beta} \underbrace{\xi^\beta{}_{;\mu} e^\mu{}_A}_{C_A \xi^\beta}$
$= -R_{\alpha\beta\mu\nu} N^\alpha \xi^\beta e^\mu{}_A \xi^\nu - k C_A + k C_A$

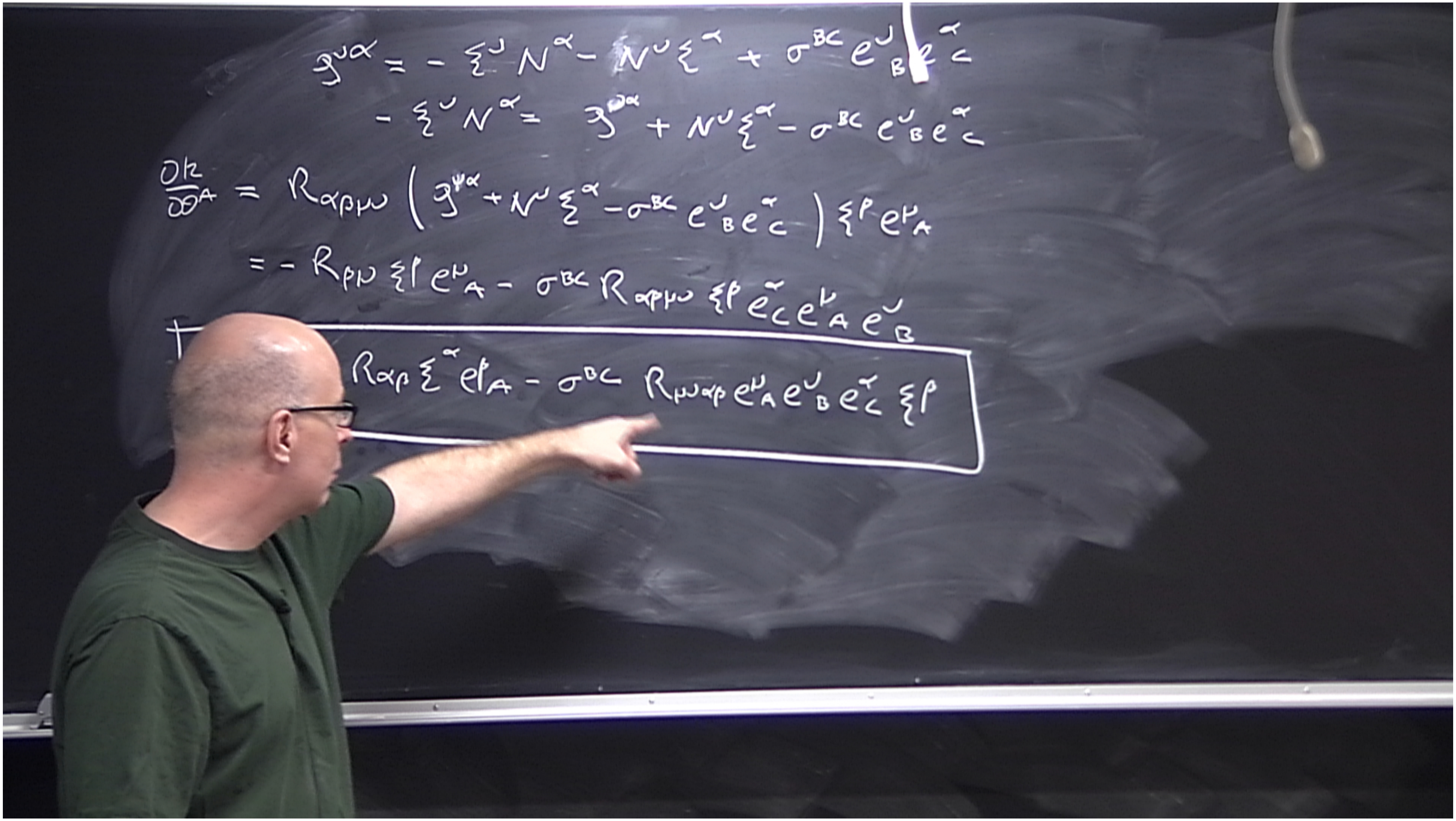

$= -R_{\alpha\beta\mu\nu} N^\alpha \xi^\beta e^\mu{}_A \xi^\nu - k\, \xi_\alpha N^\alpha{}_{;\mu} e^\mu{}_A - \xi_{\alpha;\beta} N^\alpha \xi^\beta{}_{;\mu} e^\mu{}_A$
$= -R_{\alpha\beta\mu\nu} N^\alpha \xi^\beta e^\mu{}_A \xi^\nu + k\, \xi_{\alpha;\mu} N^\alpha e^\mu{}_A - N_\alpha \xi^\alpha{}_{;\beta} \underbrace{\xi^\beta{}_{;\mu} e^\mu{}_A}_{\kappa_A \xi^\beta}$
$= -R_{\alpha\beta\mu\nu} N^\alpha \xi^\beta e^\mu{}_A \xi^\nu - \cancel{k \kappa_A} + \cancel{k \kappa_A}$
$\boxed{\left(\frac{\partial k}{\partial \theta^A}\right)_v = -R_{\alpha\beta\mu\nu} N^\alpha \xi^\beta e^\mu{}_A \xi^\nu}$

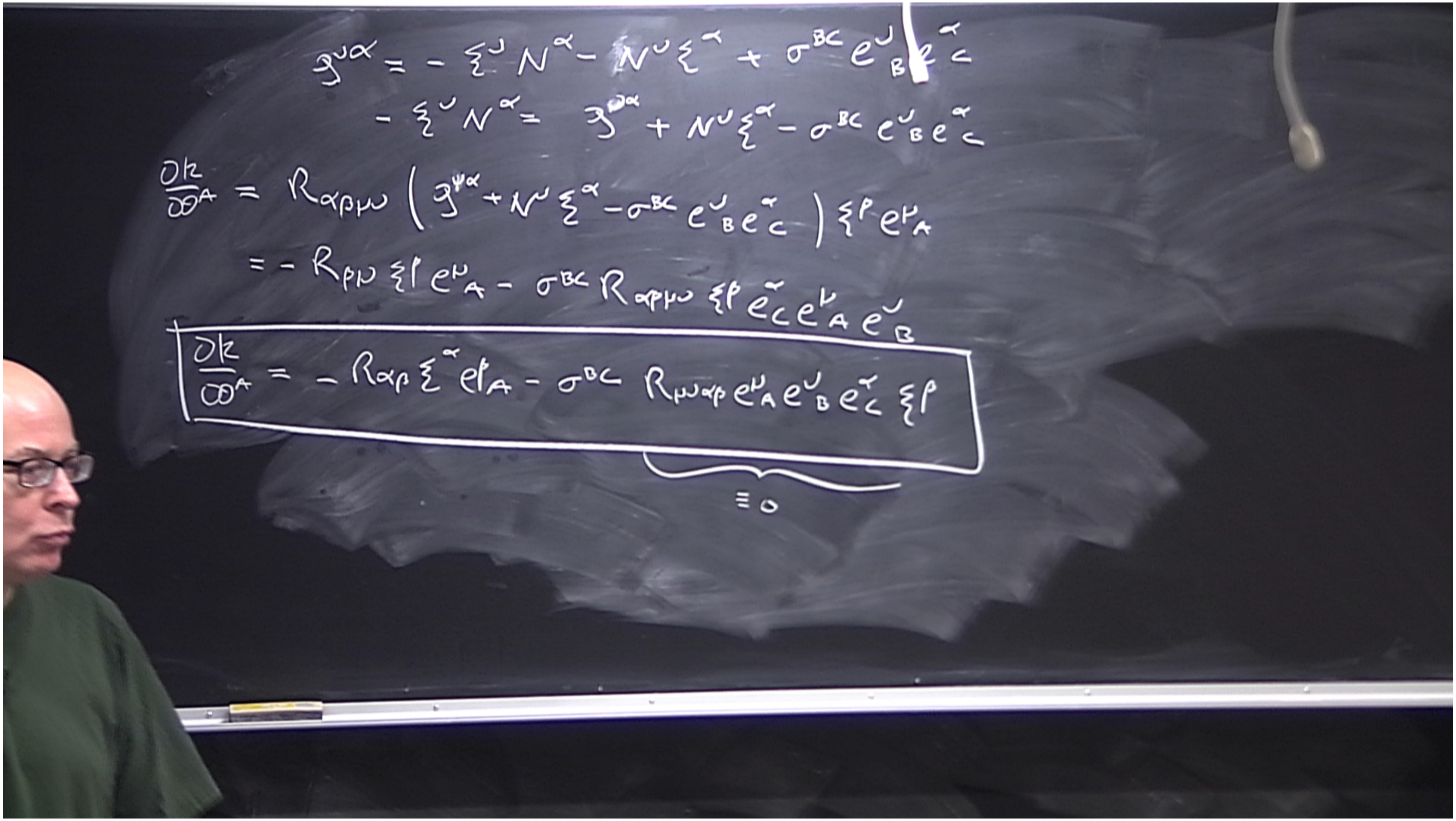
$$= -R_{\alpha\beta\mu\nu} N^\alpha \xi^\beta e^\mu{}_A \xi^\nu - k\, \xi_\alpha N^\alpha{}_{;\mu} e^\mu{}_A - \xi_{\alpha;\beta} N^\alpha \xi^\beta{}_{;\mu} e^\mu{}_A$$

$$= -R_{\alpha\beta\mu\nu} N^\alpha \xi^\beta e^\mu{}_A \xi^\nu + k\, \xi_{\alpha;\mu} N^\alpha e^\mu{}_A - N_\alpha \xi^\alpha{}_{;\beta} \underbrace{\xi^\beta{}_{;\mu} e^\mu{}_A}_{\kappa_A \xi^\beta}$$

$$= -R_{\alpha\beta\mu\nu} N^\alpha \xi^\beta e^\mu{}_A \xi^\nu - \cancel{k \kappa_A} + \cancel{k \kappa_A}$$

$$\boxed{\left(\frac{\partial k}{\partial \theta^A}\right)_v = -R_{\alpha\beta\mu\nu} N^\alpha \xi^\beta e^\mu{}_A \xi^\nu}$$


$g^{\nu\alpha} = -\xi^\nu N^\alpha - N^\nu \xi^\alpha + \sigma^{BC} e^\nu_B e^\alpha_C$
$-\xi^\nu N^\alpha = g^{\nu\alpha} + N^\nu \xi^\alpha - \sigma^{BC} e^\nu_B e^\alpha_C$
$\frac{\partial k}{\partial \theta^A} = R_{\alpha\beta\mu\nu} \left(g^{\nu\alpha} + N^\nu \xi^\alpha - \sigma^{BC} e^\nu_B e^\alpha_C \right) \xi^\beta e^\mu_A$
$= -R_{\beta\mu} \xi^\beta e^\mu_A - \sigma^{BC} R_{\alpha\beta\mu\nu} \xi^\beta e^\alpha_C e^\mu_A e^\nu_B$
$R_{\alpha\beta} \xi^\alpha e^\beta_A - \sigma^{BC} R_{\mu\nu\alpha\beta} e^\mu_A e^\nu_B e^\alpha_C \xi^\beta$


$g^{\nu\alpha} = -\xi^\nu N^\alpha - N^\nu \xi^\alpha + \sigma^{BC} e^\nu_B e^\alpha_C$
$-\xi^\nu N^\alpha = g^{\nu\alpha} + N^\nu \xi^\alpha - \sigma^{BC} e^\nu_B e^\alpha_C$
$\frac{\partial k}{\partial \theta^A} = R_{\alpha\rho\mu\nu}\left(g^{\nu\alpha} + N^\nu \xi^\alpha - \sigma^{BC} e^\nu_B e^\alpha_C \right) \xi^\rho e^\mu_A$
$= -R_{\rho\mu} \xi^\rho e^\mu_A - \sigma^{BC} R_{\alpha\rho\mu\nu} \xi^\rho e^\alpha_C e^\mu_A e^\nu_B$
$\boxed{\frac{\partial k}{\partial \theta^A} = -R_{\alpha\rho} \xi^\alpha e^\rho_A - \underbrace{\sigma^{BC} R_{\mu\nu\alpha\rho} e^\mu_A e^\nu_B e^\alpha_C \xi^\rho}_{\equiv 0}}$

$$g^{\nu\alpha} = -\xi^{\nu} N^{\alpha} - N^{\nu} \xi^{\alpha} + \sigma^{BC} e^{\nu}{}_{B} e^{\alpha}{}_{C}$$

$$-\xi^{\nu} N^{\alpha} = g^{\nu\alpha} + N^{\nu} \xi^{\alpha} - \sigma^{BC} e^{\nu}{}_{B} e^{\alpha}{}_{C}$$

$$\frac{\partial k}{\partial \theta^{A}} = R_{\alpha\beta\mu\nu} \left(g^{\nu\alpha} + N^{\nu} \xi^{\alpha} - \sigma^{BC} e^{\nu}{}_{B} e^{\alpha}{}_{C} \right) \xi^{\beta} e^{\mu}{}_{A}$$

$$= -R_{\beta\nu} \xi^{\beta} e^{\mu}{}_{A} - \sigma^{BC} R_{\alpha\beta\mu\nu} \xi^{\beta} e^{\alpha}{}_{C} e^{\mu}{}_{A} e^{\nu}{}_{B}$$

$$\boxed{\frac{\partial k}{\partial \theta^{A}} = -R_{\alpha\beta} \xi^{\alpha} e^{\beta}{}_{A} - \sigma^{BC} \underbrace{R_{\mu\nu\alpha\beta} e^{\mu}{}_{A} e^{\nu}{}_{B} e^{\alpha}{}_{C}}_{\equiv 0} \xi^{\beta}}$$