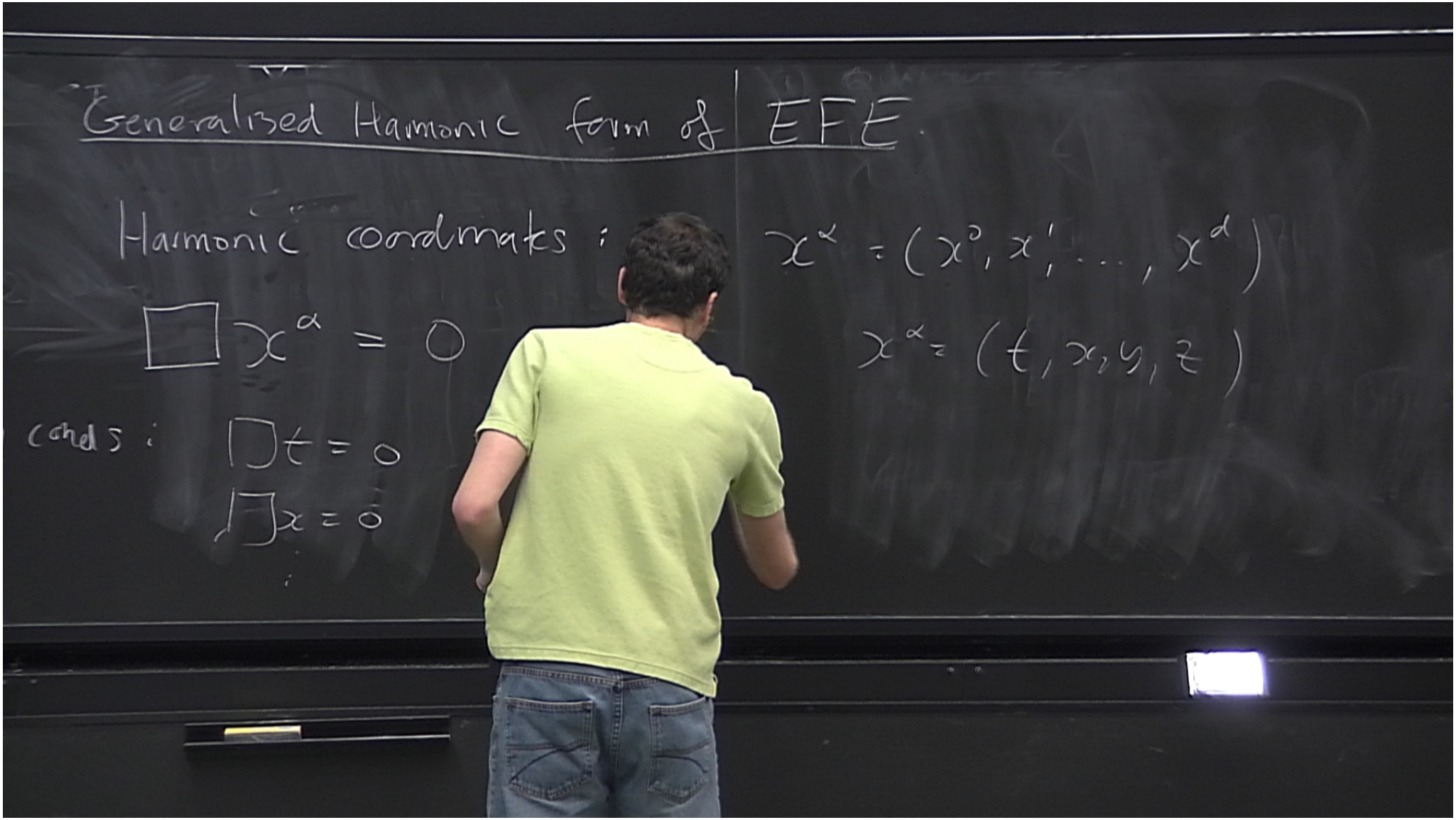
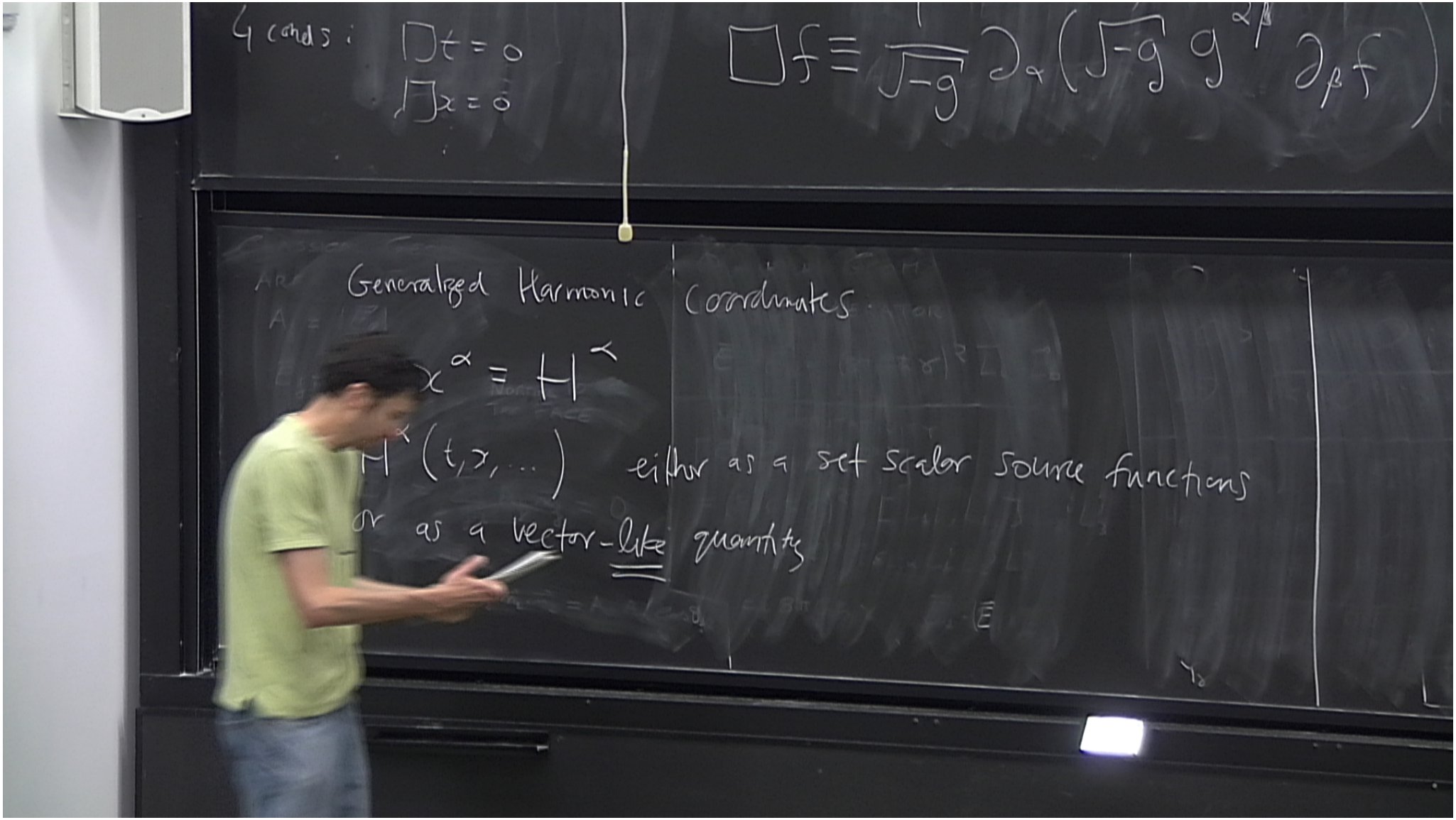
Title: Explorations in Numerical Relativity - Lecture 11

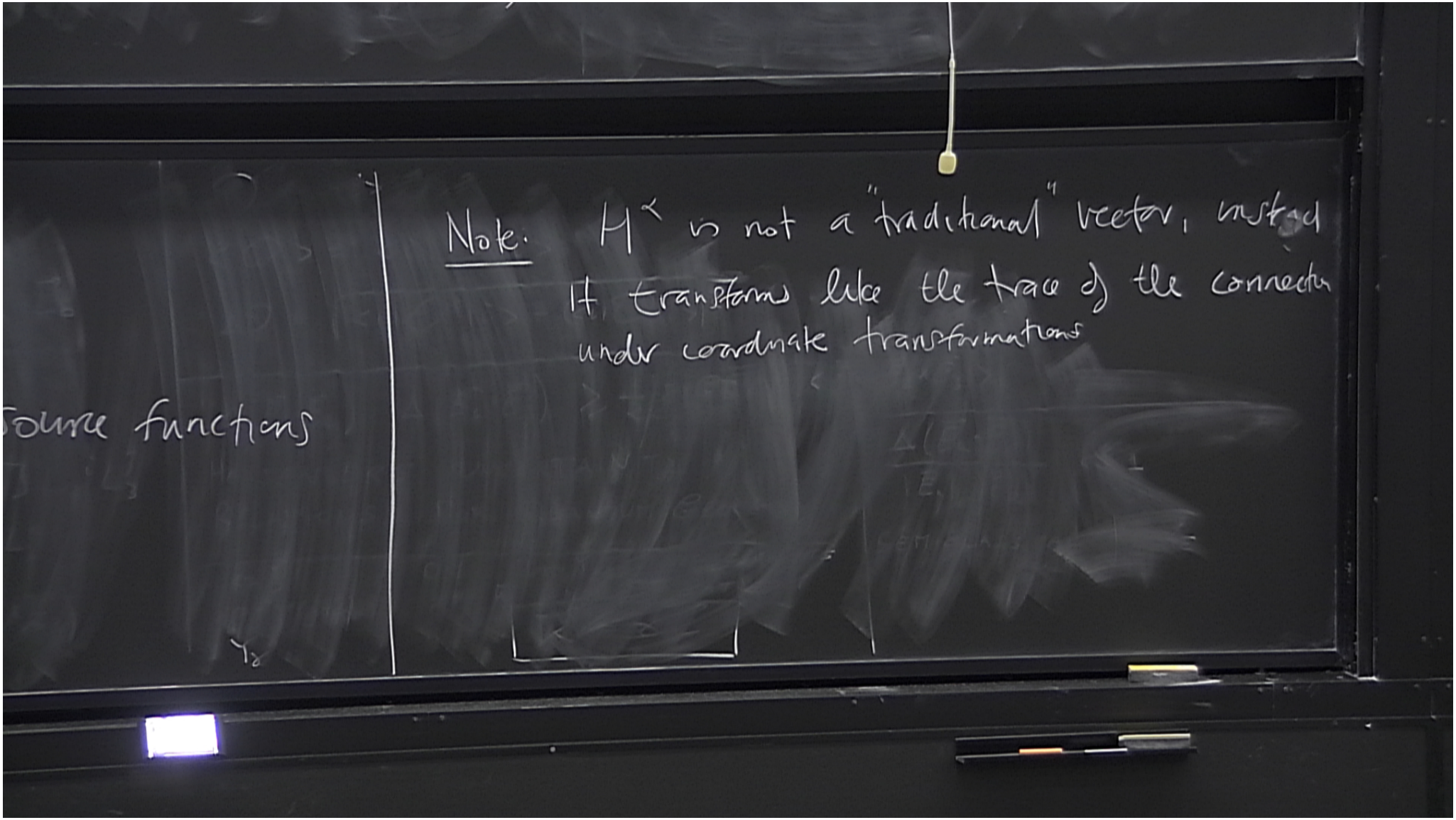
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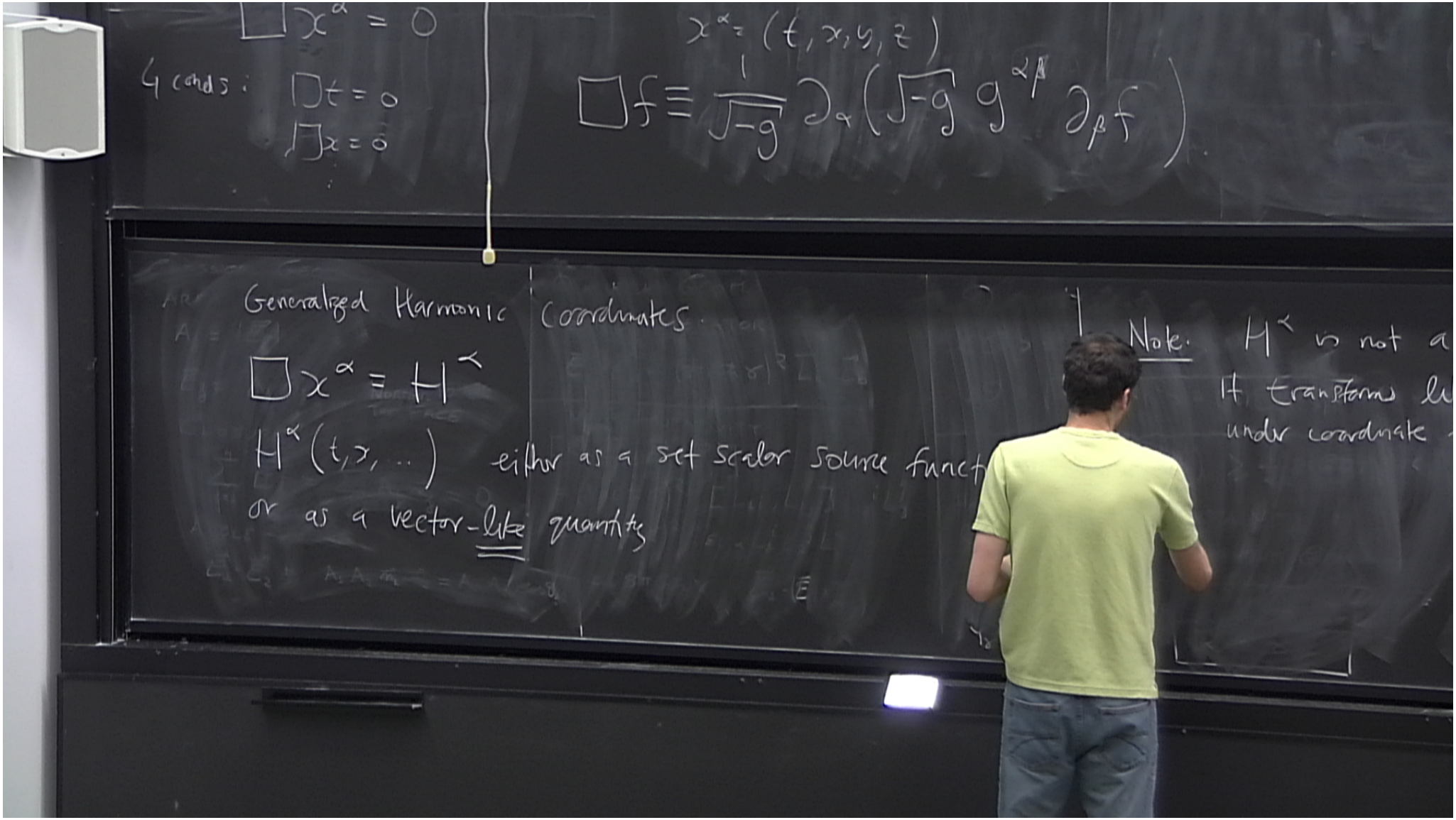
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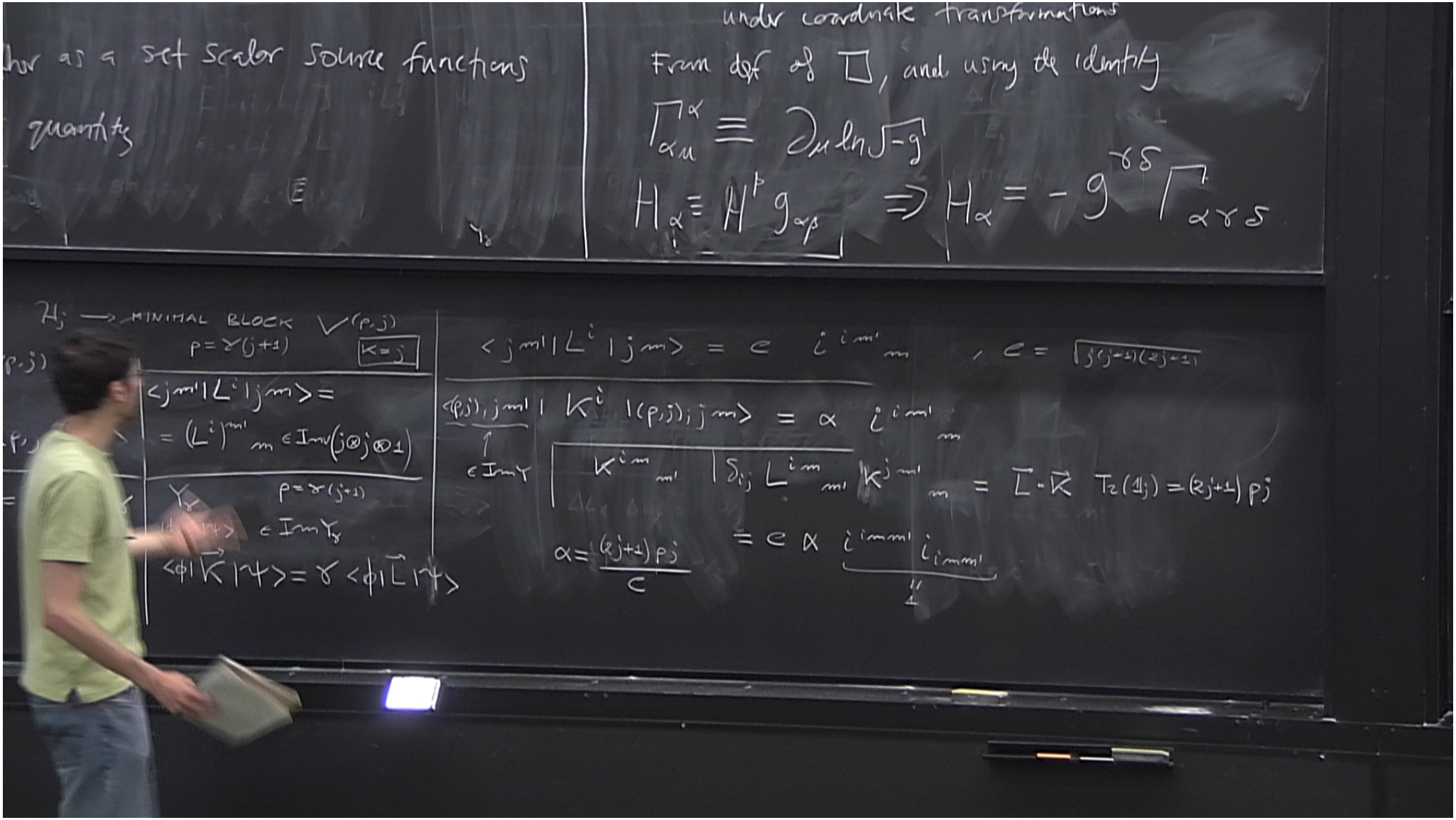
Abstract:

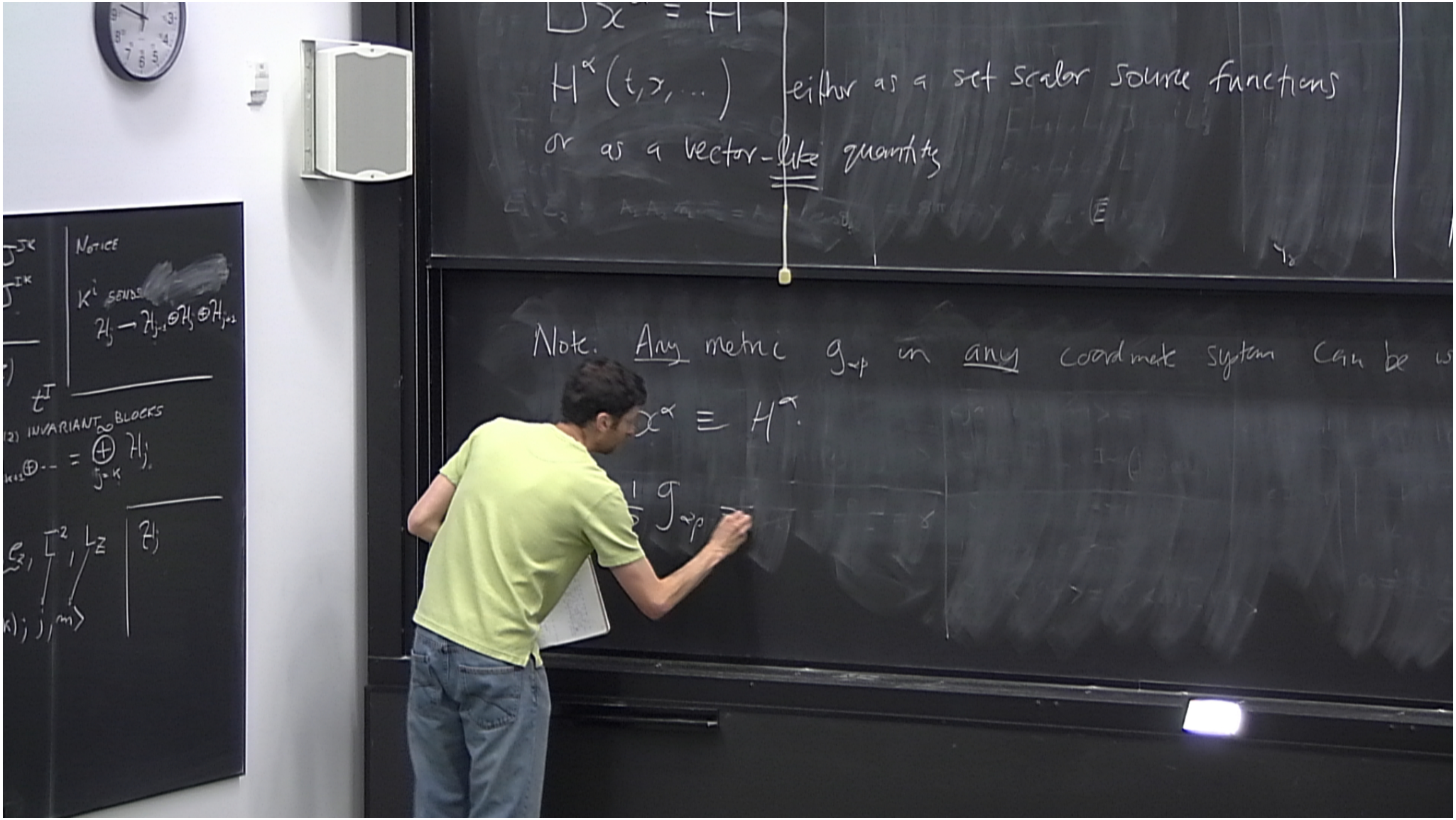

Generalised Harmonic form of EFE.
Harmonic coordinates : $x^\alpha = (x^0, x^1, \ldots, x^d)$
$\Box x^\alpha = 0$
$x^\alpha = (t, x, y, z)$
cords : $\Box t = 0$
$\Box x = 0$


4 conds: $\Box t = 0$
$\Box x = 0$
$\Box f \equiv \frac{1}{\sqrt{-g}} \partial_\alpha \left(\sqrt{-g}\, g^{\alpha\beta} \partial_\beta f \right)$
Generalized Harmonic Coordinates
$x^\alpha = H^\alpha$
$H^\alpha(t, x, \ldots)$ either as a set scalar source functions
or as a vector-like quantity


Note: H^α is not a "traditional" vector, instead
It transforms like the trace of the connection
under coordinate transformations
source functions


$\Box x^\alpha = 0$
4 conds: $\Box t = 0$
$\Box x = 0$
$x^\alpha = (t, x, y, z)$
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or as a vector-like quantity
Note. H^α is not a
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under coordinate


under coordinate transformations
From def of □, and using the identity
$\Gamma^{\alpha}_{\alpha\mu} \equiv \partial_\mu \ln\sqrt{-g}$
$H_\alpha \equiv H^\beta g_{\alpha\beta} \Rightarrow H_\alpha = -g^{\gamma\delta}\Gamma_{\alpha\gamma\delta}$
$\mathcal{H}_j \longrightarrow$ MINIMAL BLOCK $V^{(p,j)}$
$p = \gamma(j+1)$
$\langle jm'|L^i|jm\rangle = \epsilon\ i^{i\,m'}{}_{m}$, $\epsilon = \sqrt{j(j+1)(2j+1)}$
$\langle jm'|L^i|jm\rangle = (L^i)^{m'}{}_{m} \in \mathrm{Inv}(j\otimes j\otimes 1)$
$\langle (p,j), jm'|\ K^i\ |(p,j);jm\rangle = \alpha\ i^{i\,m'}{}_{m}$
$\in \mathrm{Im}\,\Upsilon$
$K^{i\,m}{}_{m'}\ \delta_{ij}\ L^{i\,m}{}_{m'}\ K^{j\,m'}{}_{m} = \vec{L}\cdot\vec{K}$ $T_z(\mathbb{1}_j) = (2j+1)\,p^j$
Υ_γ $p = \gamma(j+1)$
$\in \mathrm{Im}\,\Upsilon_\gamma$
$\langle\phi|\vec{K}|\psi\rangle = \gamma\langle\phi|\vec{L}|\psi\rangle$
$\alpha = \frac{(2j+1)\,p^j}{\epsilon}$ $= \epsilon\,\alpha\ \underbrace{i^{i\,m\,m'}\ i_{i\,m\,m'}}_{1}$

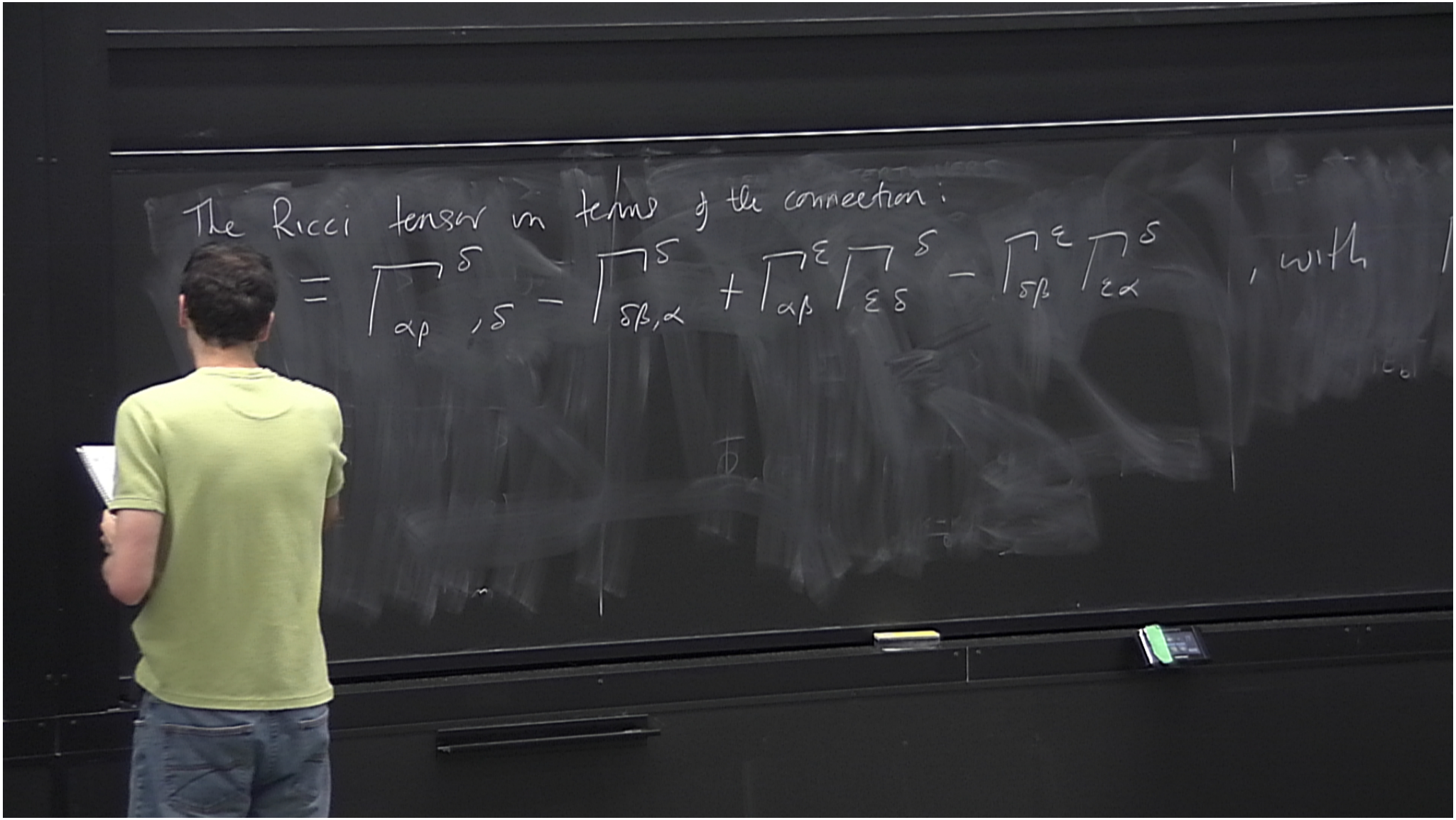

either as a set scalar source functions
or as a vector-like quantity
Note. Any metric $g_{\mu\rho}$ in any coordinate system can be
$x^\alpha \equiv H^\alpha$
NOTICE
INVARIANT BLOCKS

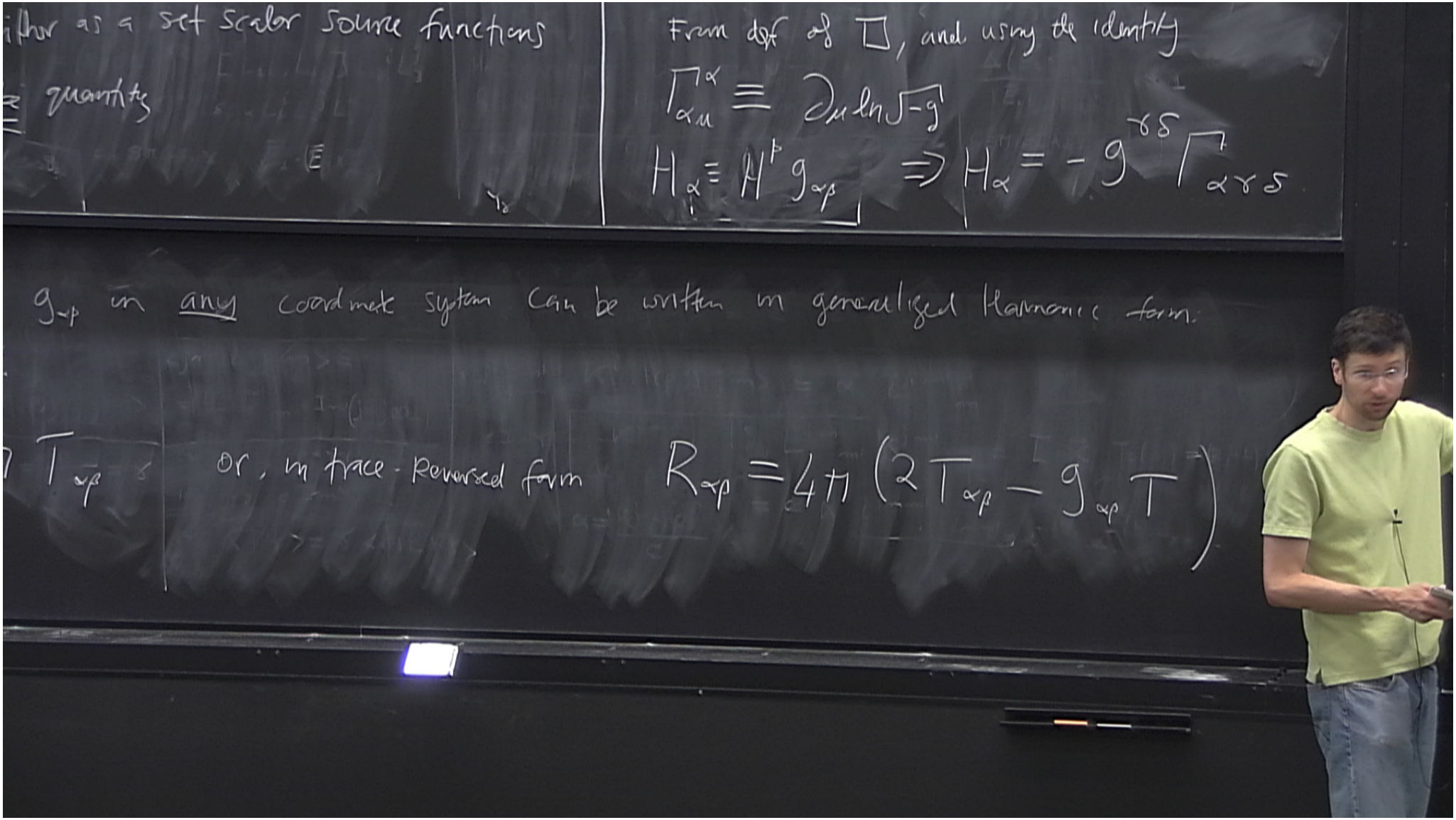
Note: Any metric $g_{\alpha\beta}$ in any coordinate system can be

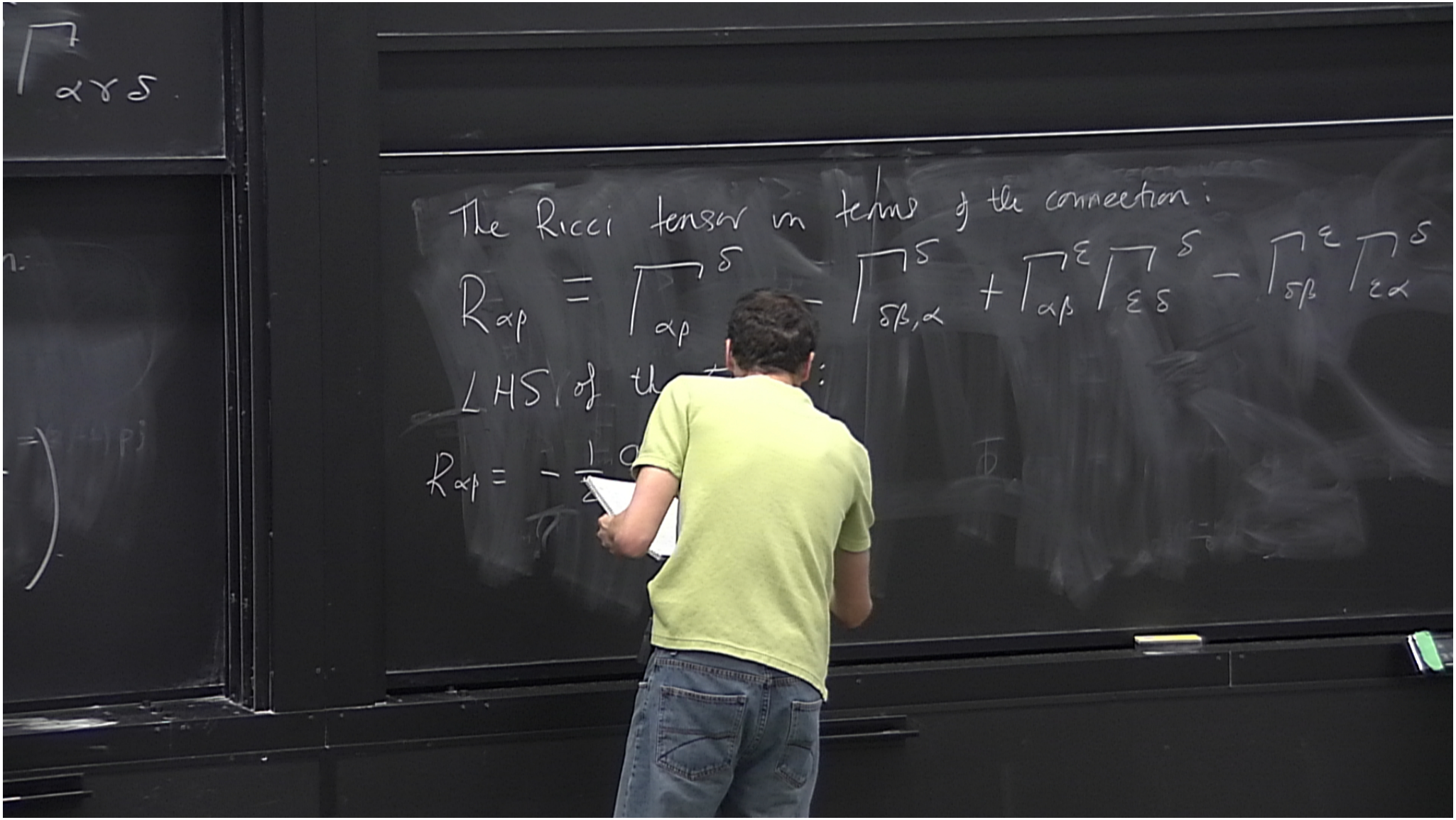
$$\Box x^\alpha \equiv H^\alpha.$$

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = -8\pi T_{\alpha\beta}$$

or, in trace-reversed form


The Ricci tensor in terms of the connection:
$= \Gamma^{\delta}_{\alpha\rho,\delta} - \Gamma^{\delta}_{\delta\beta,\alpha} + \Gamma^{\varepsilon}_{\alpha\rho}\Gamma^{\delta}_{\varepsilon\delta} - \Gamma^{\varepsilon}_{\delta\beta}\Gamma^{\delta}_{\varepsilon\alpha}$, with


ither as a set scalar source functions
quantity
From def of □, and using the identity
$\Gamma^{\alpha}_{\alpha\mu} \equiv \partial_\mu \ln\sqrt{-g}$
$H_\alpha \equiv H^\beta g_{\alpha\beta} \Rightarrow H_\alpha = -g^{\gamma\delta}\Gamma^{\beta}_{\alpha\gamma\delta}$
$g_{\alpha\beta}$ in any coordinate system can be written in generalized Harmonic form.
$T_{\alpha\beta}$
or, in trace-Reversed form $R_{\alpha\beta} = 4\pi\left(2T_{\alpha\beta} - g_{\alpha\beta}T\right)$


The Ricci tensor in terms of the connection:
$R_{\alpha\beta} = \Gamma^{\delta}_{\alpha\beta}$ $- \Gamma^{\delta}_{\delta\beta,\alpha} + \Gamma^{\varepsilon}_{\alpha\beta}\Gamma^{\delta}_{\varepsilon\delta} - \Gamma^{\varepsilon}_{\delta\beta}\Gamma^{\delta}_{\varepsilon\alpha}$
LHS of th
$R_{\alpha\beta} = -\frac{1}{2}$

The Ricci tensor in terms of the connection:

$$R_{\alpha\beta} = \Gamma^{\delta}_{\alpha\beta,\delta} - \Gamma^{\delta}_{\delta\beta,\alpha} + \Gamma^{\varepsilon}_{\alpha\beta}\Gamma^{\delta}_{\varepsilon\delta} - \Gamma^{\varepsilon}_{\delta\beta}\Gamma^{\delta}_{\varepsilon\alpha}$$, with

LHS of the EFE:

$$R_{\alpha\beta} = -\frac{1}{2} g^{\delta\gamma} g_{\alpha\beta,\delta\gamma} + g^{\delta\gamma} g_{\delta(\alpha,\beta)\gamma} - \ln\sqrt{-g}_{,\alpha\beta}$$

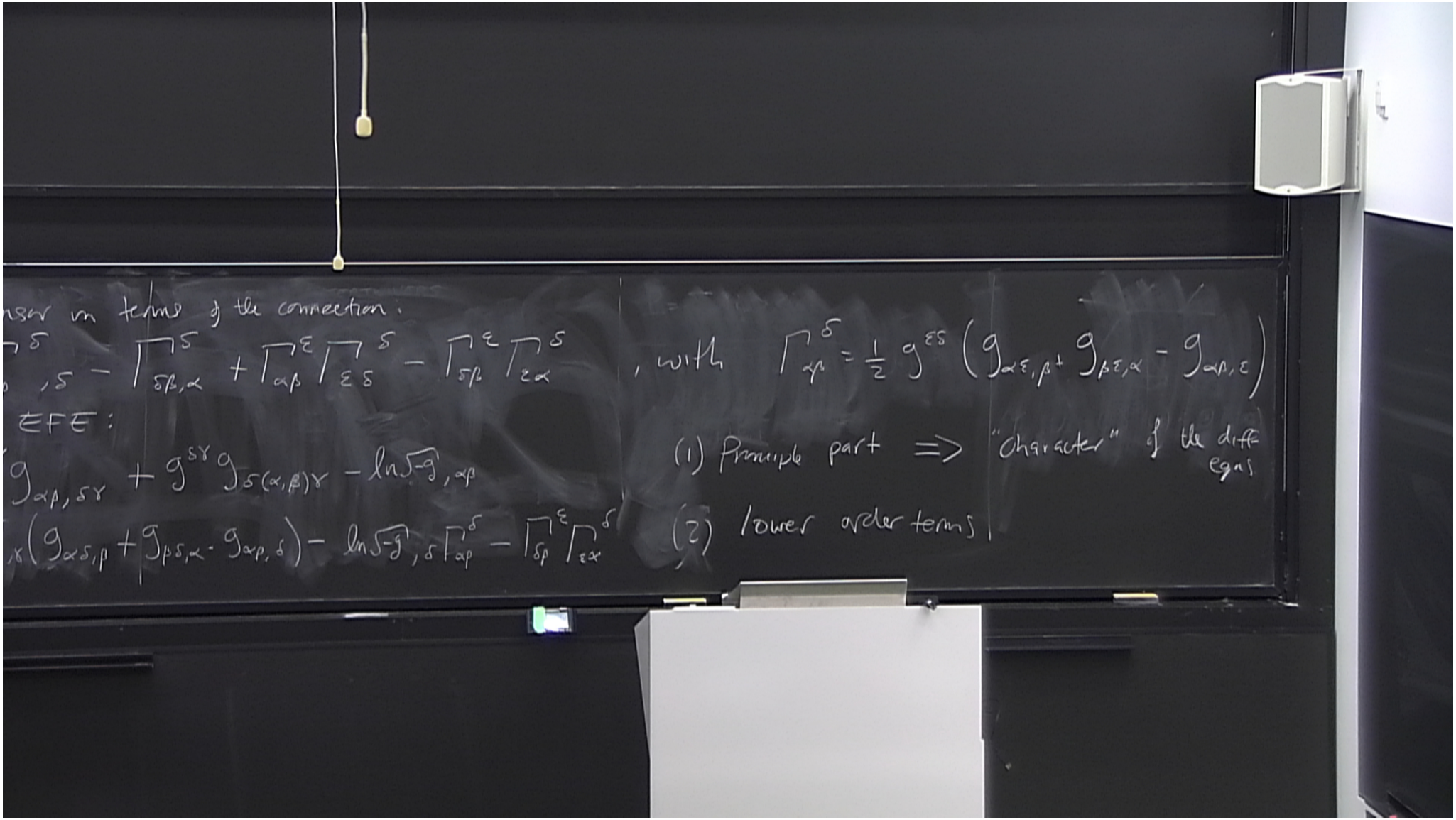
The Ricci tensor in terms of the connection:

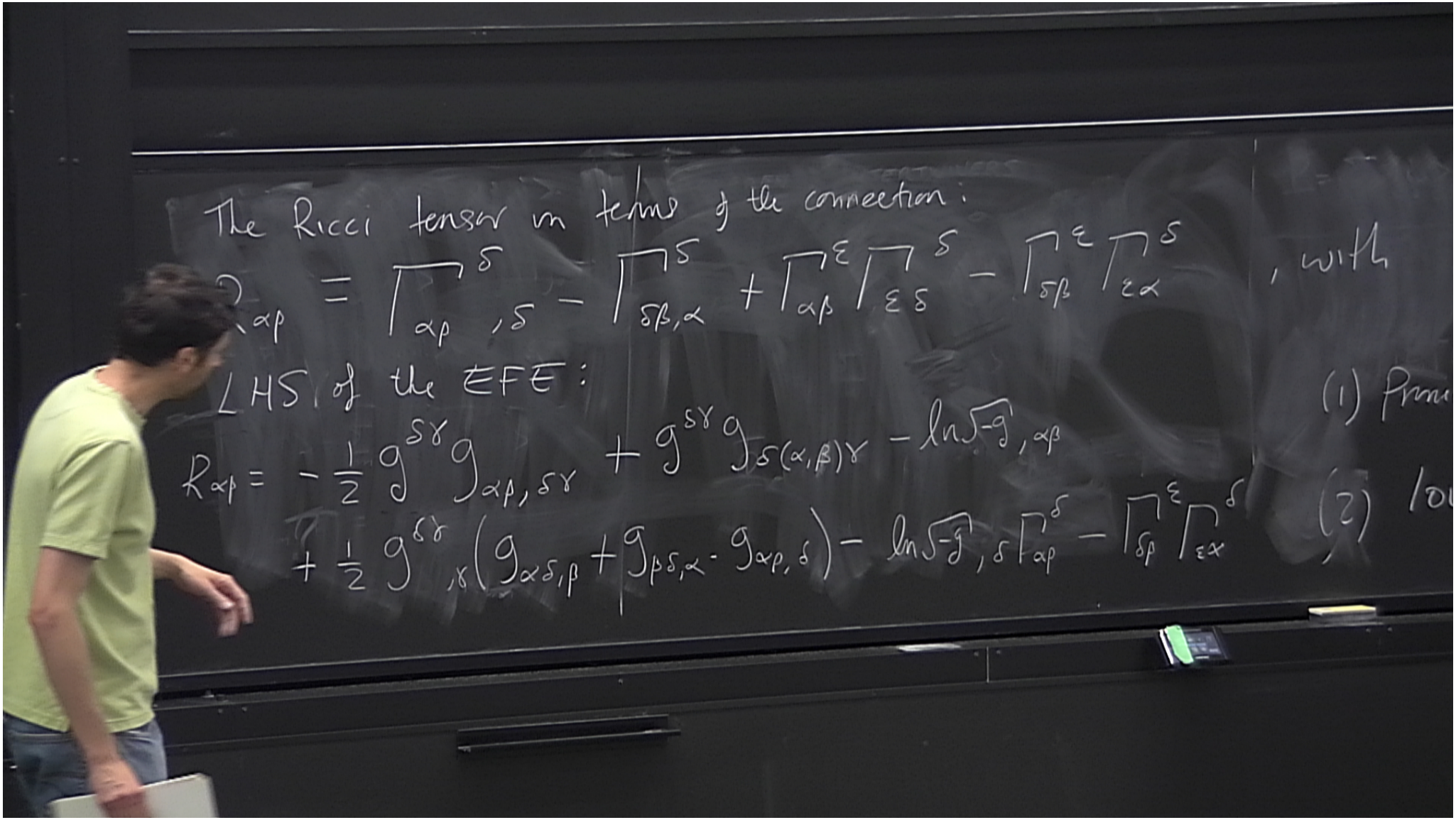
$$R_{\alpha\beta} = \Gamma^{\delta}_{\alpha\beta,\delta} - \Gamma^{\delta}_{\delta\beta,\alpha} + \Gamma^{\varepsilon}_{\alpha\beta}\Gamma^{\delta}_{\varepsilon\delta} - \Gamma^{\varepsilon}_{\delta\beta}\Gamma^{\delta}_{\varepsilon\alpha}$$, with

LHS of the EFE:

$$R_{\alpha\beta} = -\frac{1}{2} g^{\delta\gamma} g_{\alpha\beta,\delta\gamma} + g^{\delta\gamma} g_{\delta(\alpha,\beta)\gamma} - \ln\sqrt{-g}_{,\alpha\beta}$$

$$+ \frac{1}{2} g^{\delta\gamma}{}_{,\gamma}\left(g_{\alpha\delta,\beta} + g_{\beta\delta,\alpha} - g_{\alpha\beta,\delta}\right) - \ln\sqrt{-g}_{,\delta}\Gamma^{\delta}_{\alpha\beta} - \Gamma^{\varepsilon}_{\delta\beta}\Gamma^{\delta}_{\varepsilon\alpha}$$

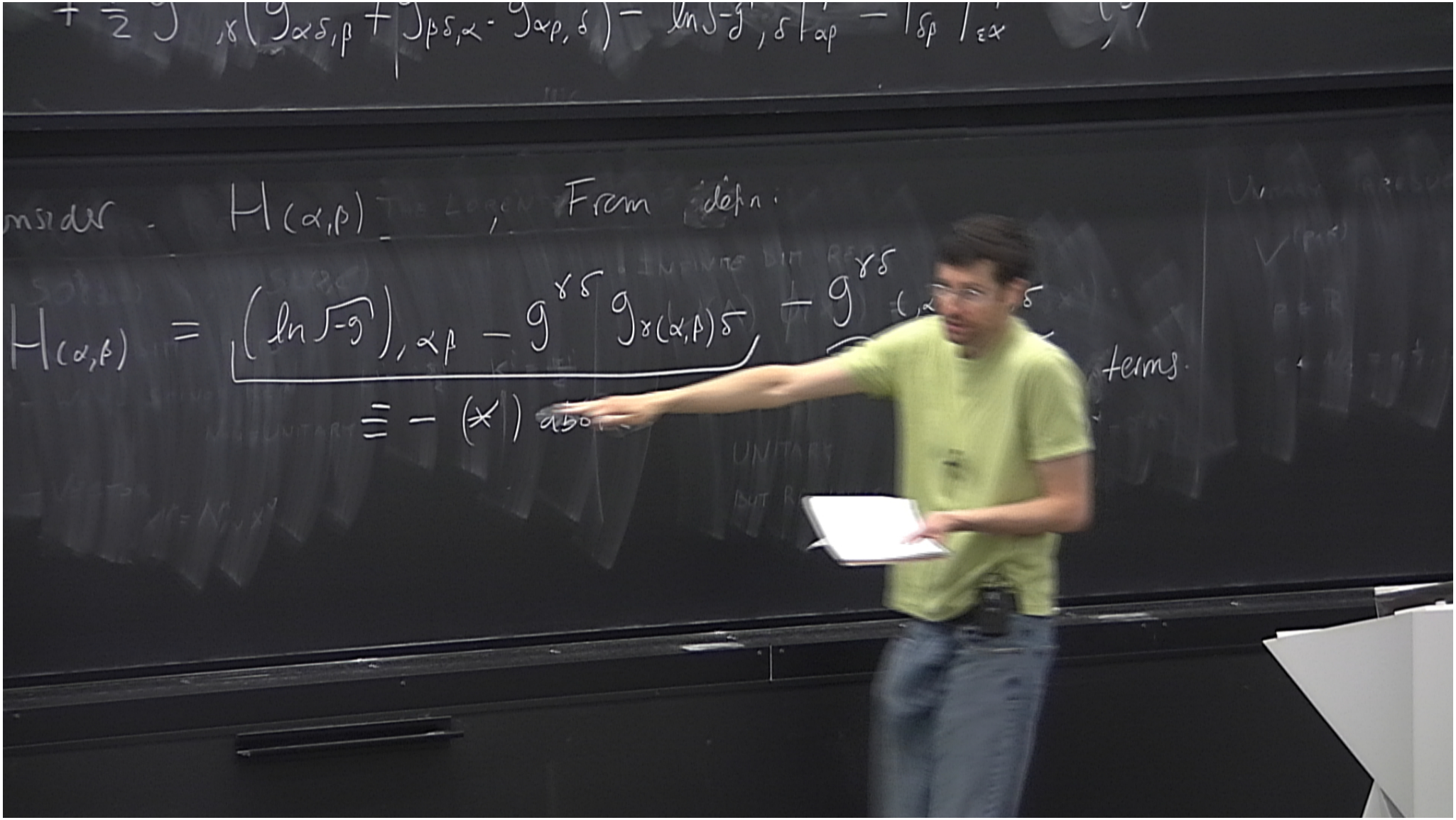

nsor in terms of the connection:
, with
EFE:
(1) Principle part ⇒ "character" of the diff eqns
(2) lower order terms

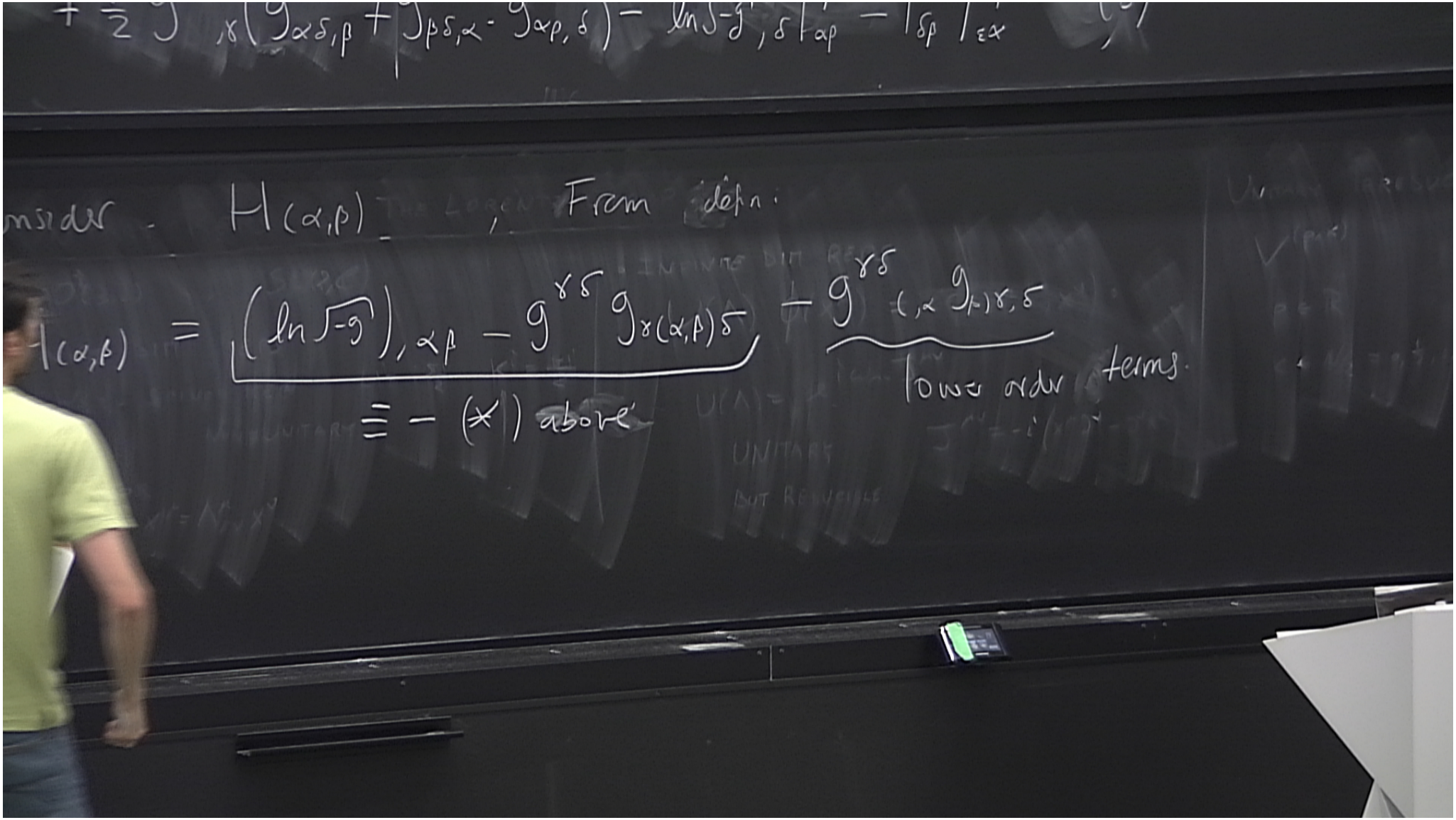

The Ricci tensor in terms of the connection:
$R_{\alpha\rho} = \Gamma^{\delta}_{\alpha\rho,\delta} - \Gamma^{\delta}_{\delta\beta,\alpha} + \Gamma^{\varepsilon}_{\alpha\rho}\Gamma^{\delta}_{\varepsilon\delta} - \Gamma^{\varepsilon}_{\delta\beta}\Gamma^{\delta}_{\varepsilon\alpha}$, with
LHS of the EFE:
$R_{\alpha\rho} = -\frac{1}{2} g^{\delta\gamma} g_{\alpha\rho,\delta\gamma} + g^{\delta\gamma} g_{\delta(\alpha,\beta)\gamma} - \ln\sqrt{-g}_{,\alpha\rho}$
$+ \frac{1}{2} g^{\delta\gamma}{}_{,\gamma}\left(g_{\alpha\delta,\beta} + g_{\beta\delta,\alpha} - g_{\alpha\rho,\delta}\right) - \ln\sqrt{-g}_{,\delta}\Gamma^{\delta}_{\alpha\rho} - \Gamma^{\varepsilon}_{\delta\rho}\Gamma^{\delta}_{\varepsilon\alpha}$
(1) Prin
(2) lo

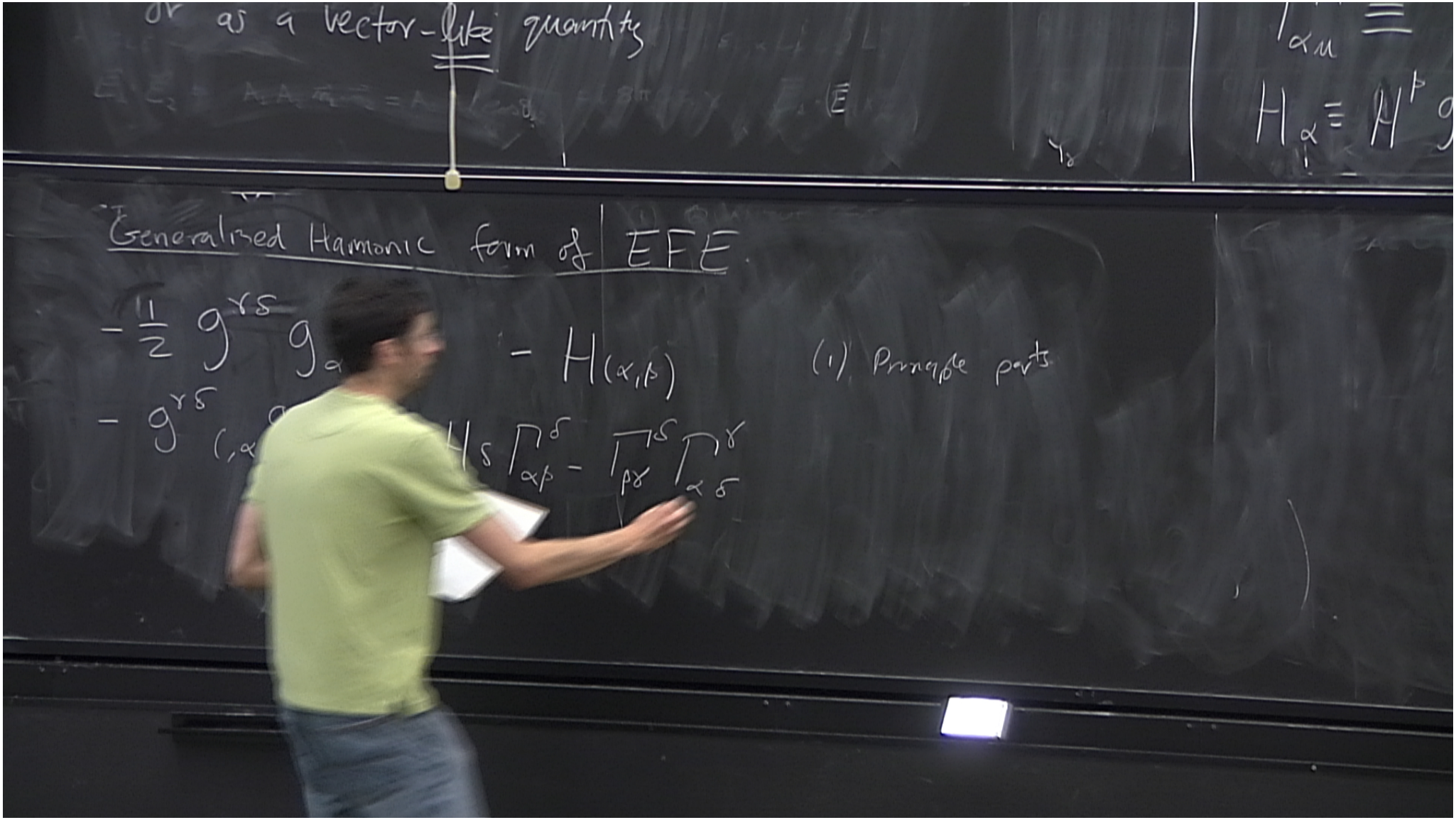
$$+\tfrac{1}{2} g^{\delta\delta}{}_{,\gamma}\left(g_{\alpha\delta,\beta}+g_{\beta\delta,\alpha}-g_{\alpha\beta,\delta}\right)-\left|\ln\sqrt{-g}\right|_{,\delta}\Gamma^{\delta}_{\alpha\beta}-\Gamma^{\varepsilon}_{\delta\beta}\Gamma^{\delta}_{\varepsilon\alpha}$$

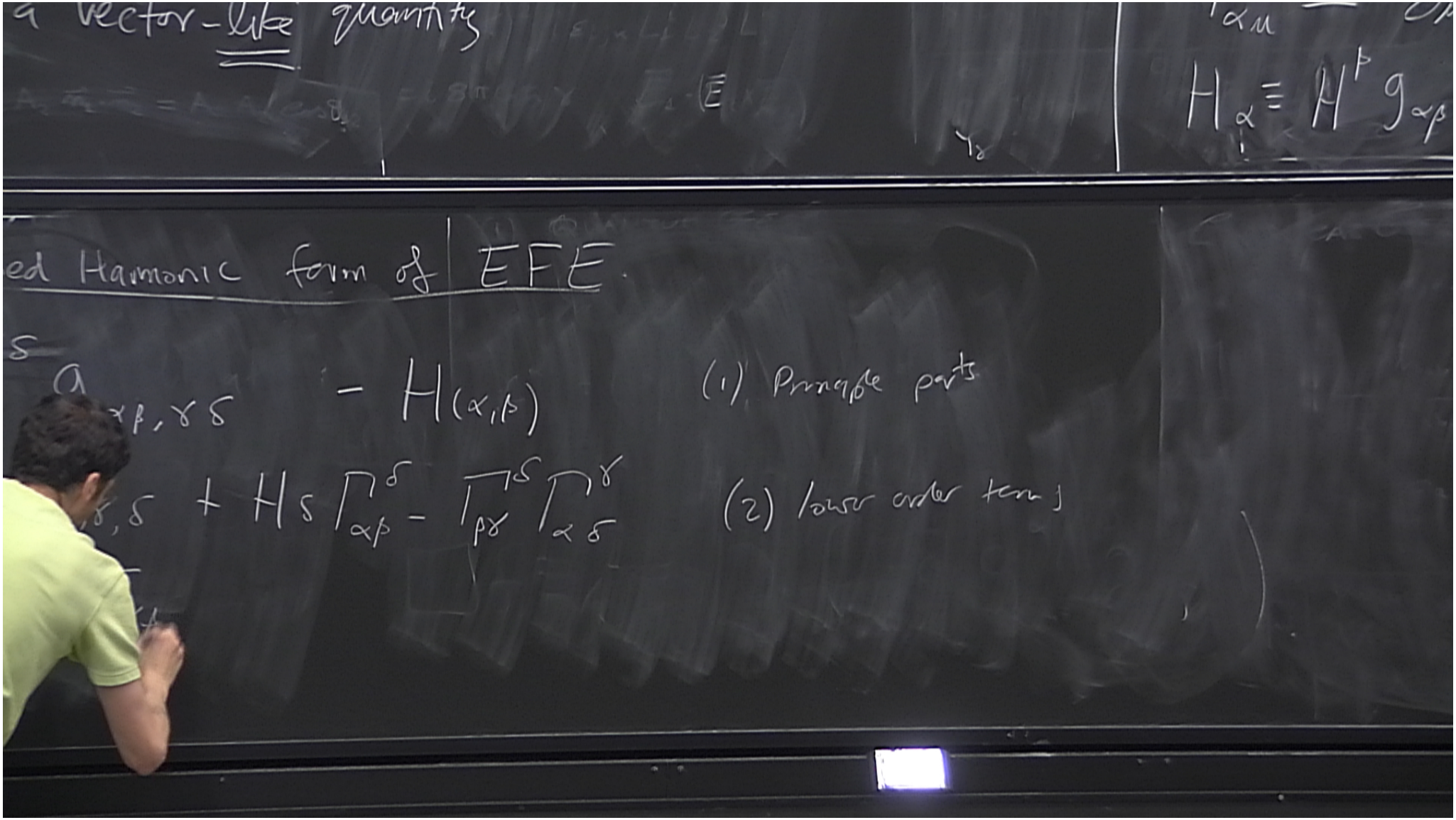
Consider $H(\alpha,\beta)$; From defn:

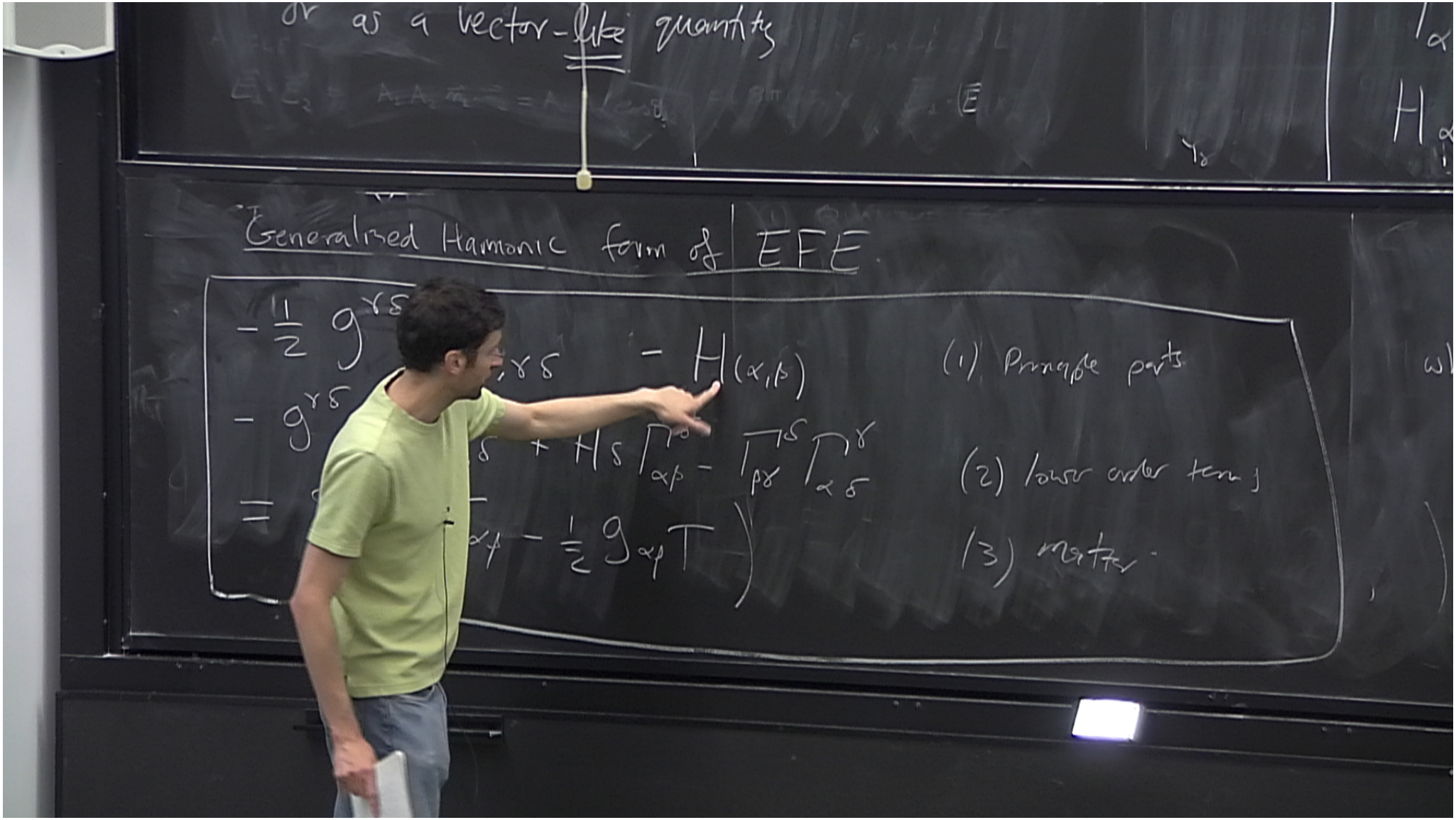
$$H(\alpha,\beta)=\left(\ln\sqrt{-g}\right)_{,\alpha\beta}-g^{\gamma\delta}g_{\gamma(\alpha,\beta)\delta}-g^{\gamma\delta}{}_{(,\alpha}g_{\beta)\gamma,\delta}$$

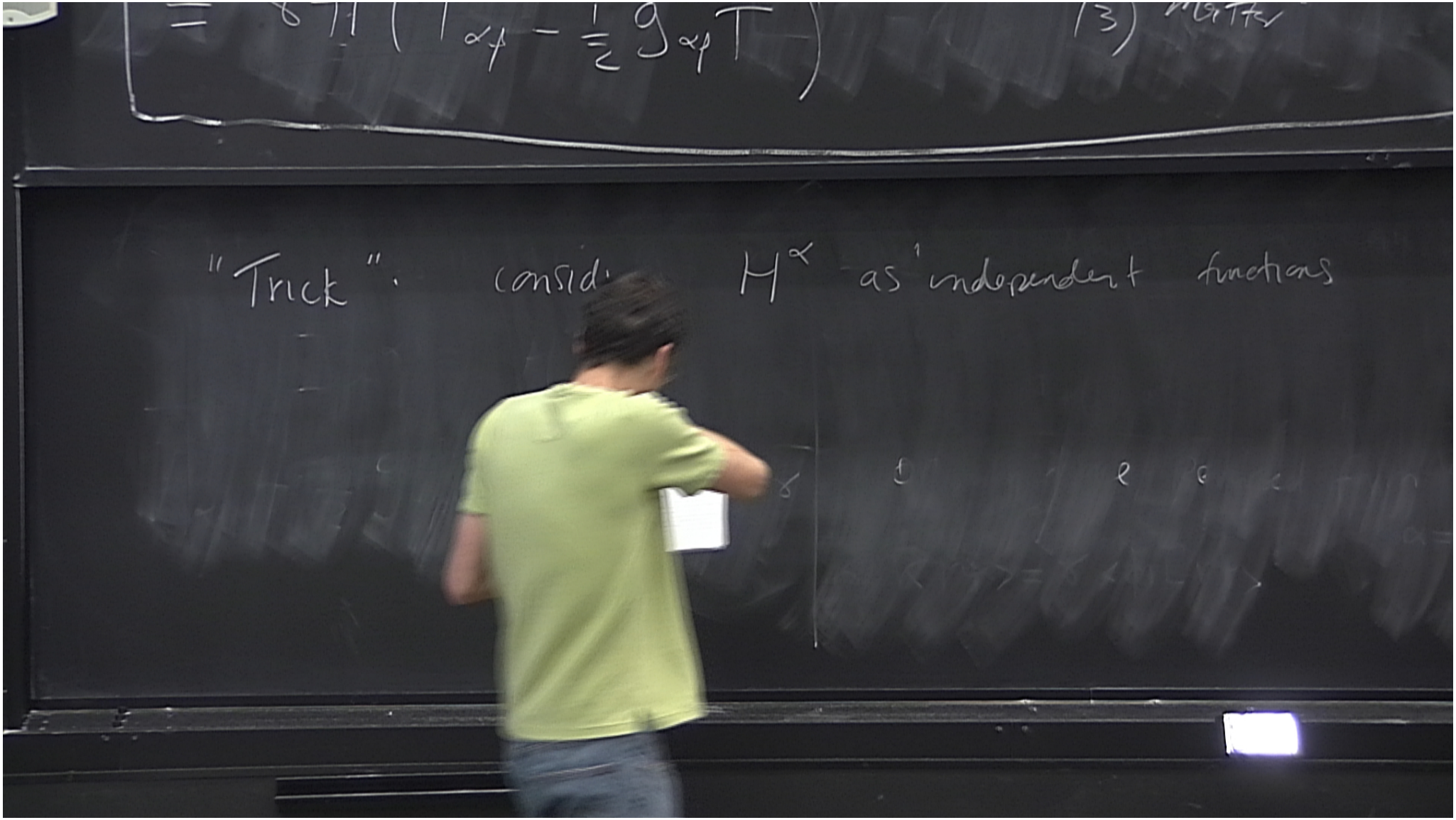

Consider $H(\alpha,\beta)$; From defn:
$H(\alpha,\beta) = (\ln\sqrt{-g})_{,\alpha\beta} - g^{\gamma\delta} g_{\gamma(\alpha,\beta)\delta} + g^{\gamma\delta}$
$\equiv -(*)$ above
terms.

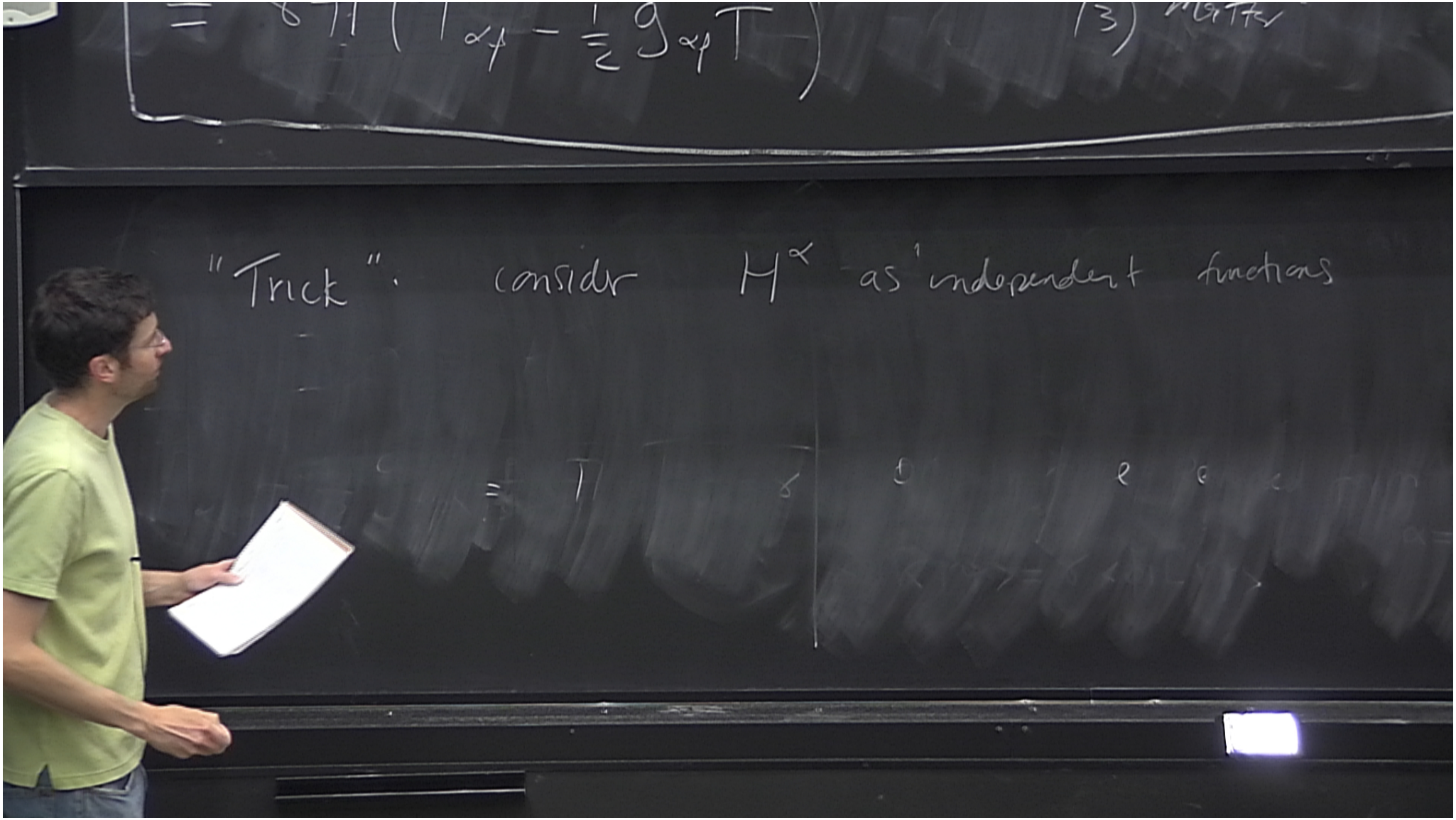

Consider $H(\alpha,\beta)$; From defn:
$H(\alpha,\beta) = \underbrace{(\ln\sqrt{-g})_{,\alpha\beta} - g^{\gamma\delta} g_{\gamma(\alpha,\beta)\delta}}_{\equiv -(*) \text{ above}} + \underbrace{g^{\gamma\delta}{}_{(,\alpha} g_{\beta)\gamma,\delta}}_{\text{lower order terms.}}$


as a vector-like quantity
Generalised Harmonic form of EFE
(1) Principle parts


a vector-like quantity
$H_\alpha \equiv H^\beta g_{\alpha\beta}$
ed Harmonic form of EFE.
$- H(\alpha,\beta)$
(1) Principle parts
$+ H^\delta \Gamma^\delta_{\alpha\beta} - \Gamma^\delta_{\beta\gamma}\Gamma^\gamma_{\alpha\delta}$
(2) lower order terms


or as a vector-like quantity
Generalised Harmonic form of EFE
(1) Principle parts
(2) lower order terms
(3) matter


"Trick": consid. H^α as independent functions


"Trick": consider H^α as independent functions

$= \delta^{\mu}_{\nu}\left(T_{\alpha\mu} - \frac{1}{2} g_{\alpha\mu} T\right)$ (3) Matter

"Trick": consider H^{α} as independent functions

$\Rightarrow$ need additional equations that determine

$= T$

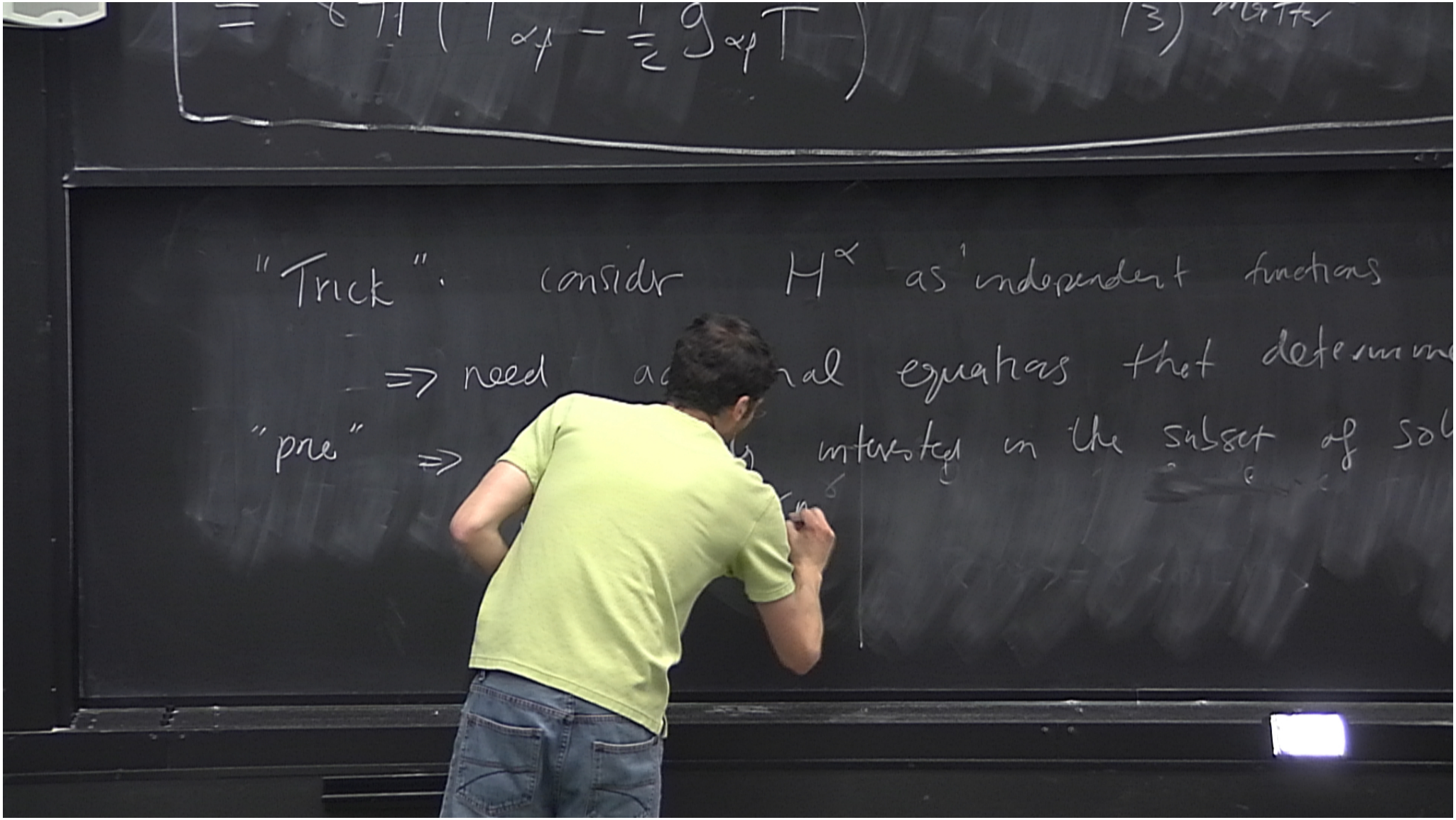
$\Gamma^{\delta}_{\alpha\beta} - \Gamma^{\delta}_{\beta\delta}\Gamma^{\delta}_{\alpha\delta}$ (2) lower order terms

$T_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} T$ (3) matter

consider H^{α} as independent functions

need additional equations that determine H^{α} (eg. $H^{\alpha} = 0$

$\mathcal{L}(H^{\alpha}) = 0$, etc.)


"Trick": consider H^α as independent functions
$\Rightarrow$ need
"pre" $\Rightarrow$
interested in the subset of

$- g_{(\alpha\beta)\gamma,\delta} + H_\delta \Gamma^\delta_{\alpha\beta} - \Gamma^\delta_{\beta\gamma}\Gamma^\gamma_{\alpha\delta}$ (2) lower order terms

$= 8\pi \left(T_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} T \right)$ (3) matter.) (A)

"Trick": consider H^α as independent functions

$\Rightarrow$ need additional equations that determine H^α (eg. $H^\alpha = 0$

"proof" $\Rightarrow$ we're only interested in the subset of solutions to $\mathcal{L}(H$

the system (A) & (B) that satisfy what is now a constraint!

$= 8\pi \left(T_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} T \right)$ (2) lower order terms (3) matter.) (A)

"Trick": consider H^α as independent functions

$\Rightarrow$ need additional equations that determine H^α (eg. $H^\alpha = \partial_\beta \mathcal{L}(H^\alpha)$)

"proof" $\Rightarrow$ we're only interested in the subset of solutions to the system (A) & (B) that satisfy what is now a constraint:

$$C_\alpha \equiv g_{\alpha\beta}\left(H^\beta - \Box x^\beta \right) = 0.$$

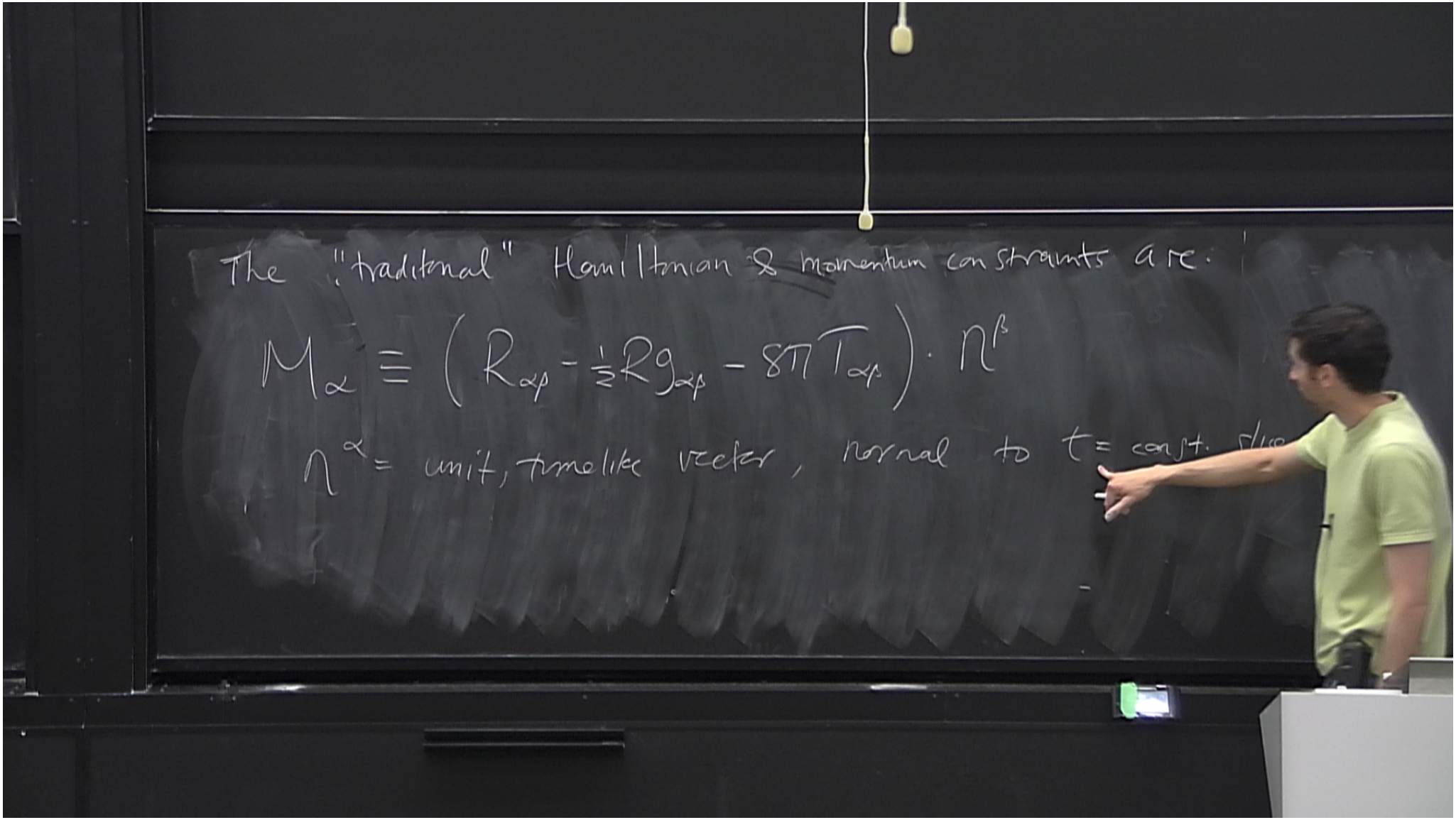
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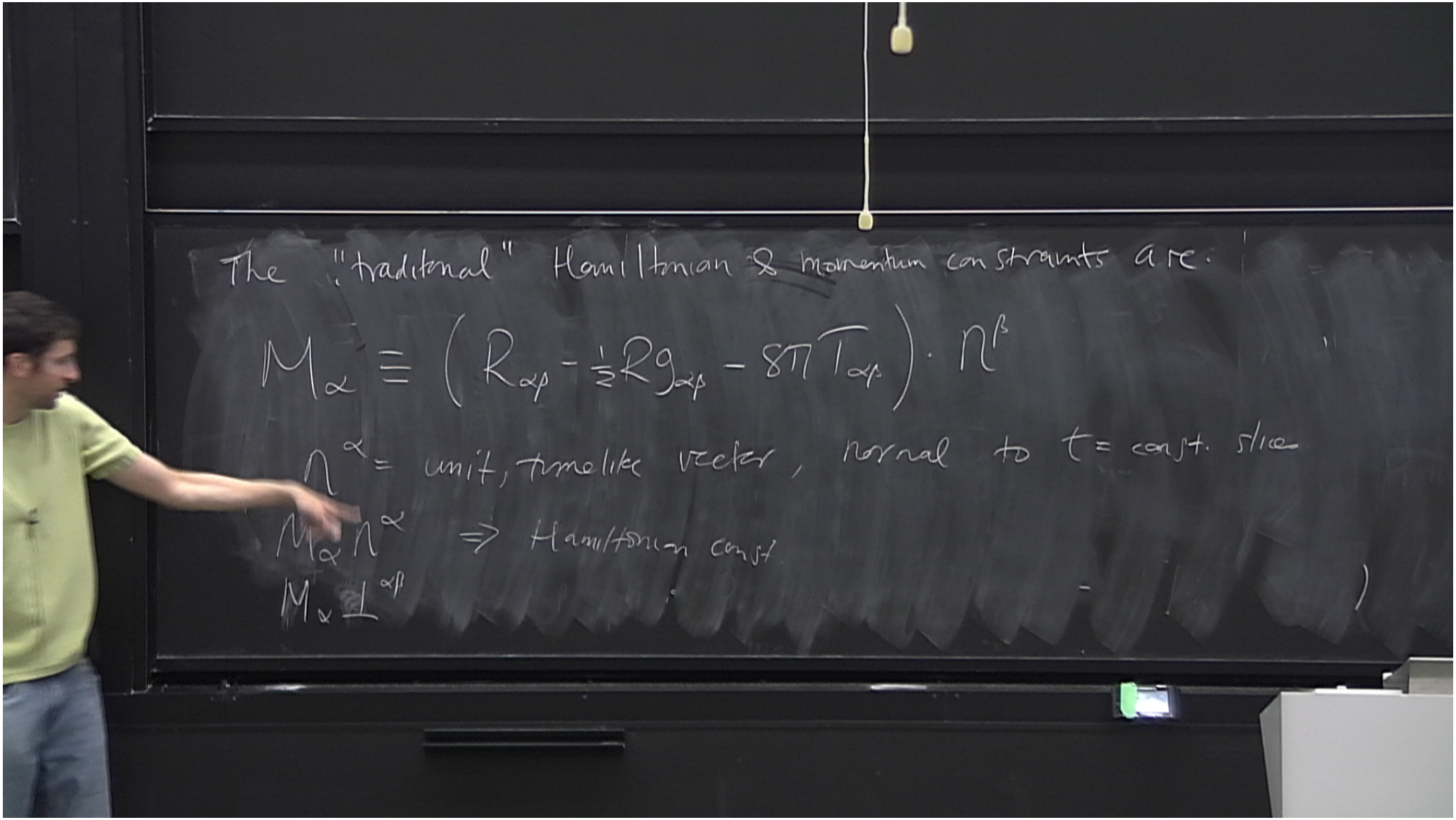
"Trick": consider H^α as independent functions

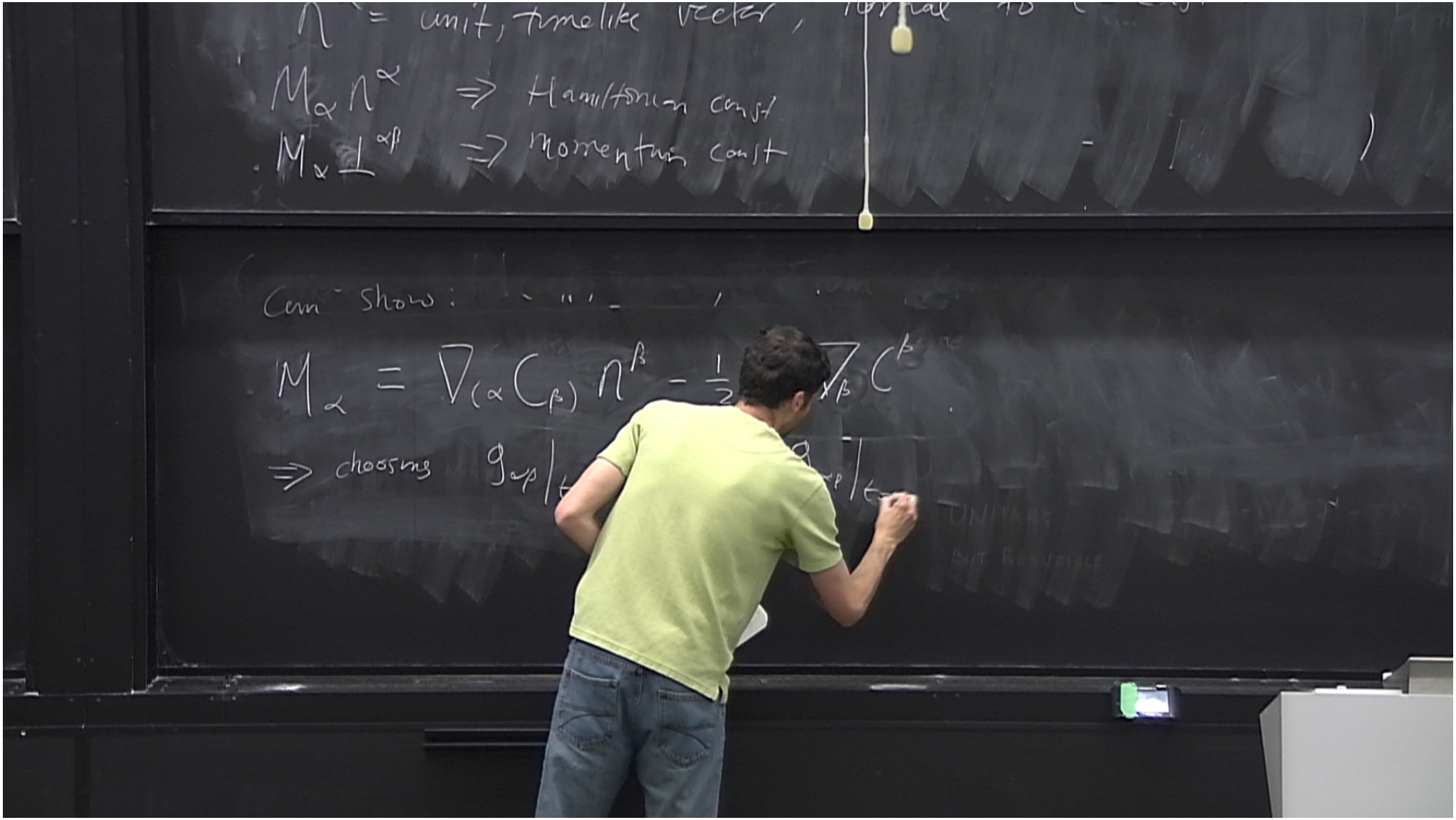
$\Rightarrow$ need additional equations that determine H^α (eg. $H^\alpha = 0$ $\mathcal{L}(H^\alpha)$

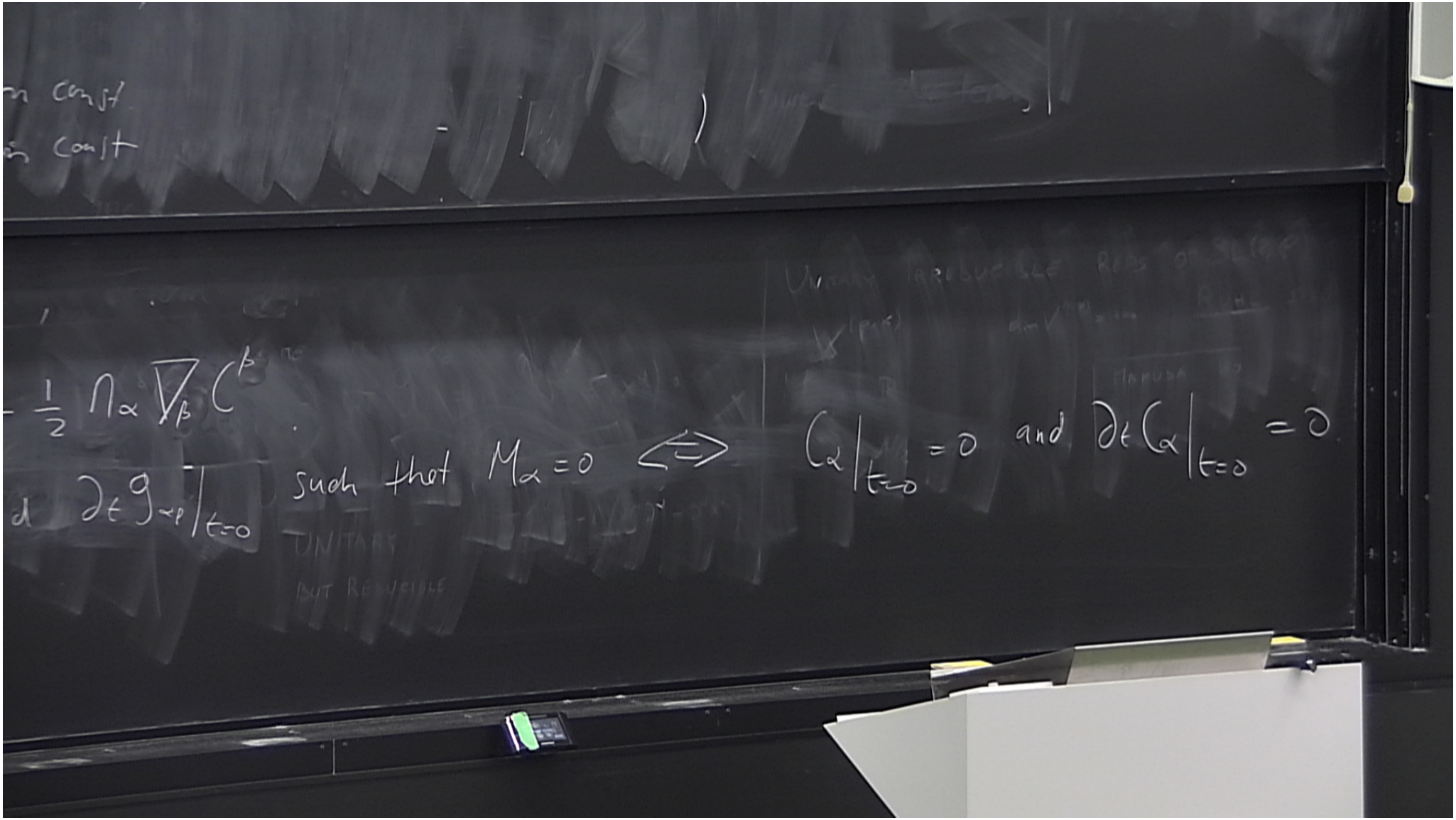
"pres" $\Rightarrow$ we're only interested in the subset of solutions to the system (A) & (B) that satisfy what is now a constraint:

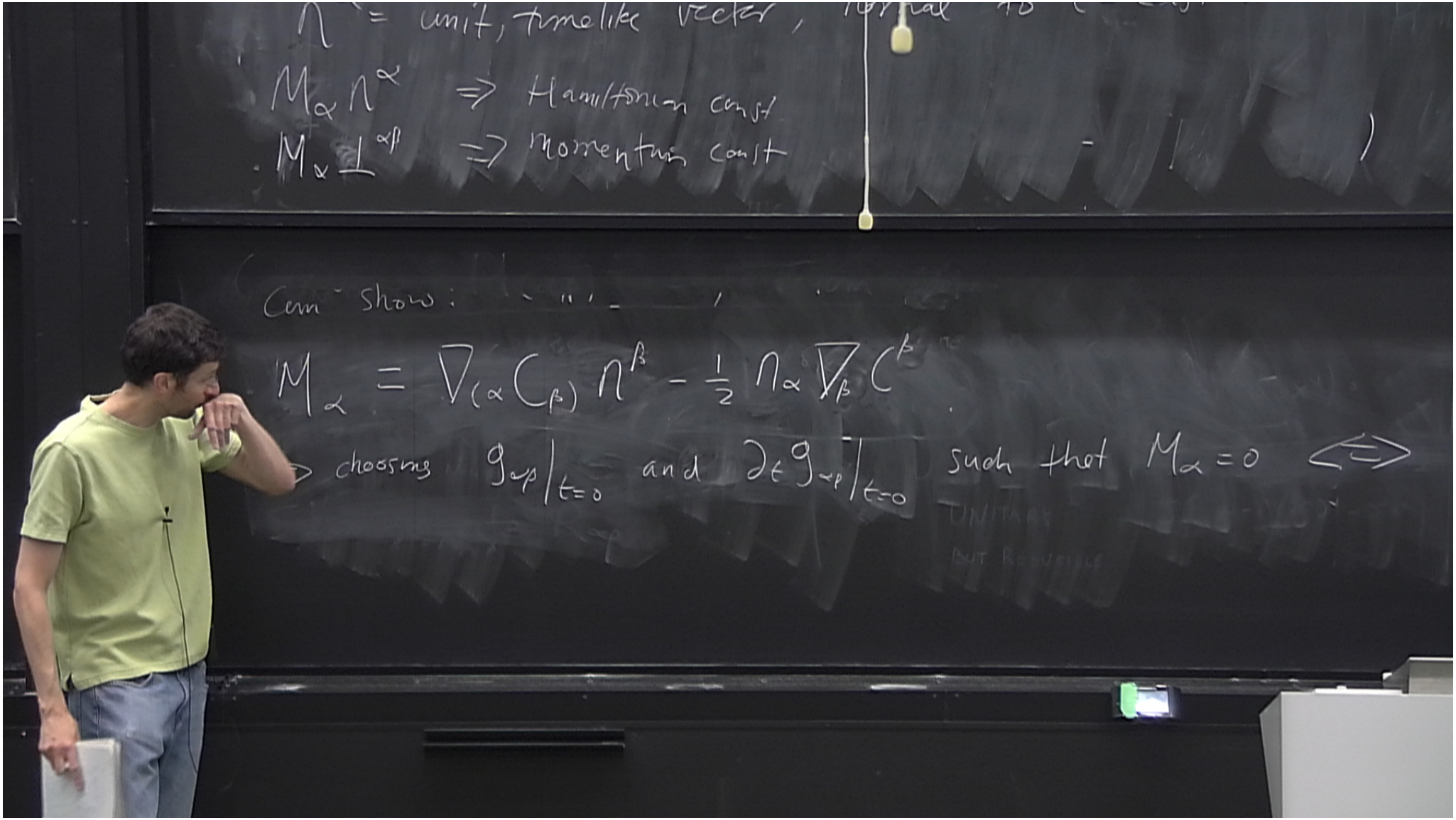
$$C_\alpha \equiv g_{\alpha\beta}\left(H^\beta - \Box x^\beta\right) = 0.$$

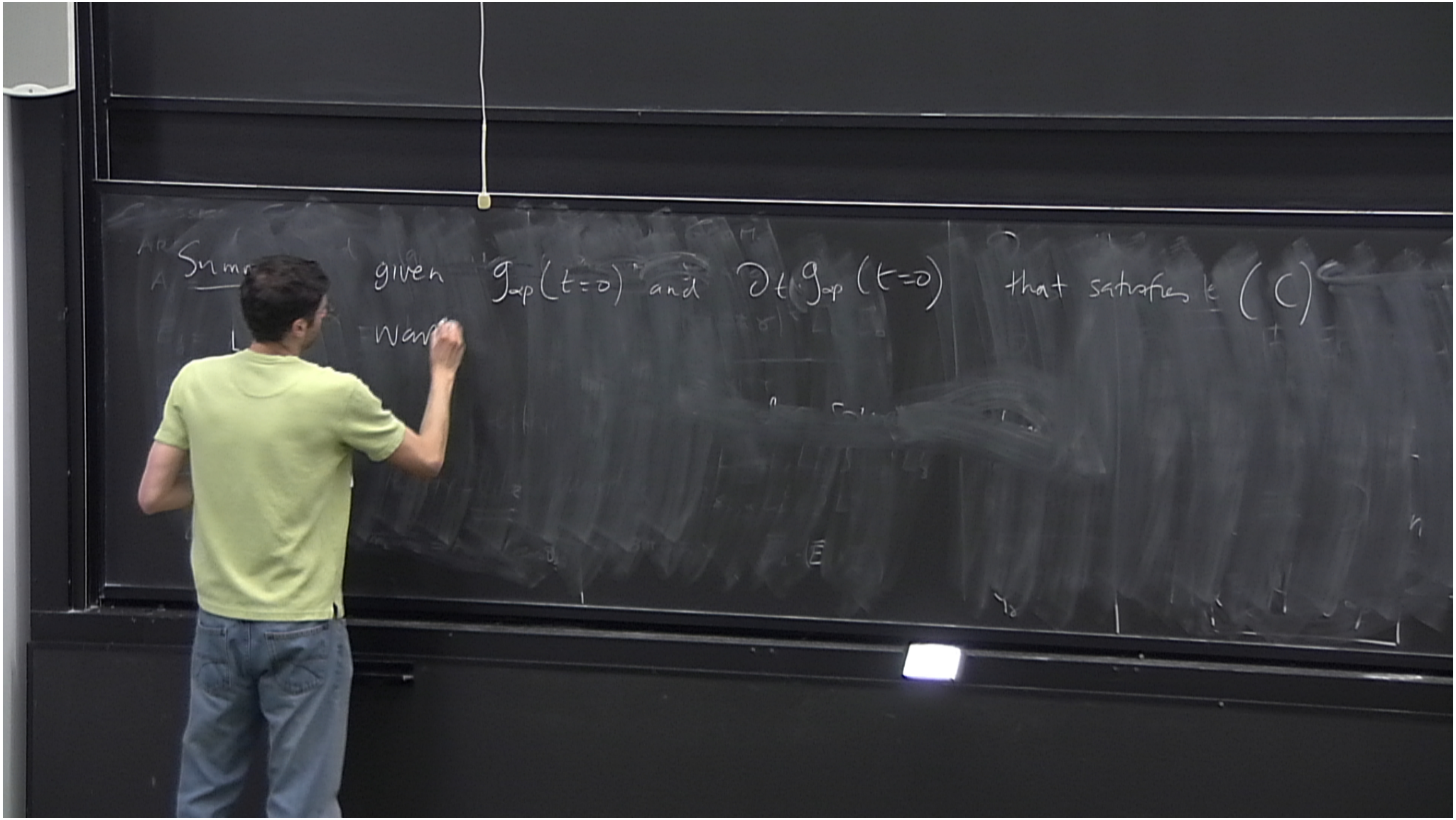

The "traditional" Hamiltonian & momentum constraints are:
$M_\alpha \equiv \left(R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} - 8\pi T_{\alpha\beta} \right) \cdot n^\beta$
n^α = unit, timelike vector, normal to t = const.

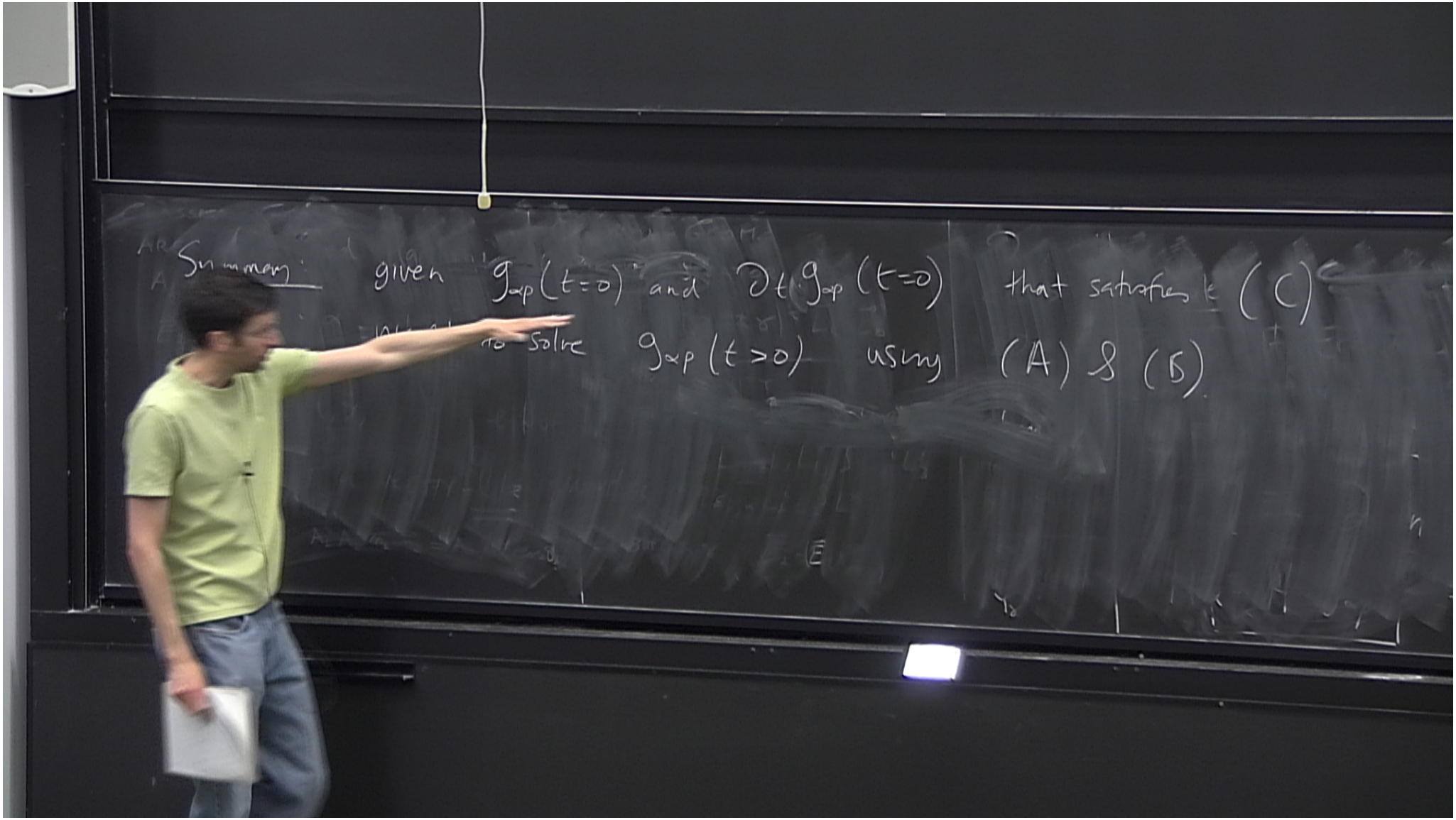

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n^α = unit, timelike vector, normal to t = const. slice
$M_\alpha n^\alpha \Rightarrow$ Hamiltonian const
$M_\alpha \perp^{\alpha\beta}$

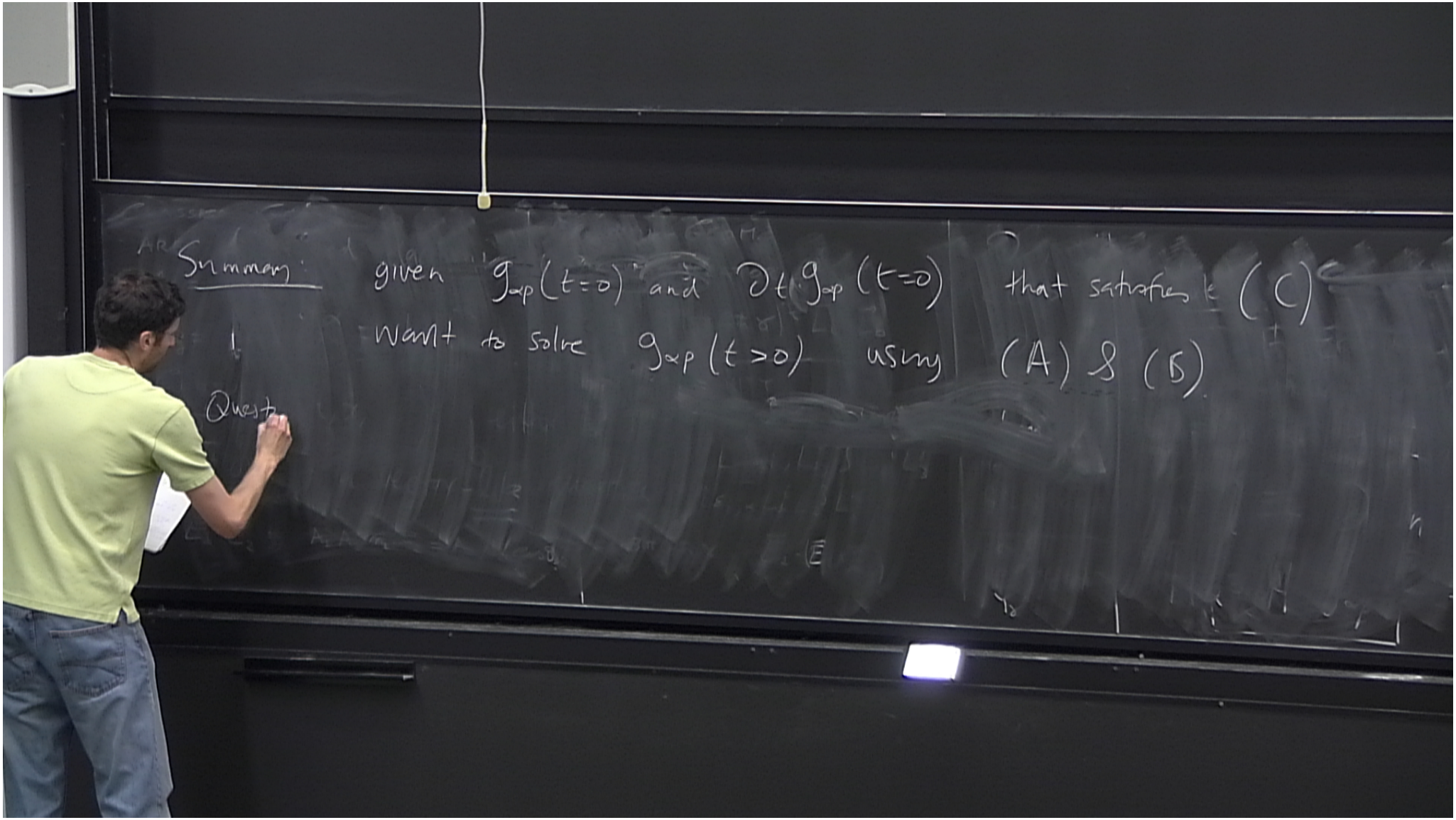

= unit, timelike vector,
$M_\alpha n^\alpha$ ⇒ Hamiltonian const
$M_\alpha \perp^{\alpha\beta}$ ⇒ momentum const
Can show:
$M_\alpha = \nabla_{(\alpha} C_{\beta)} n^\beta - \frac{1}{2}$
⇒ choosing

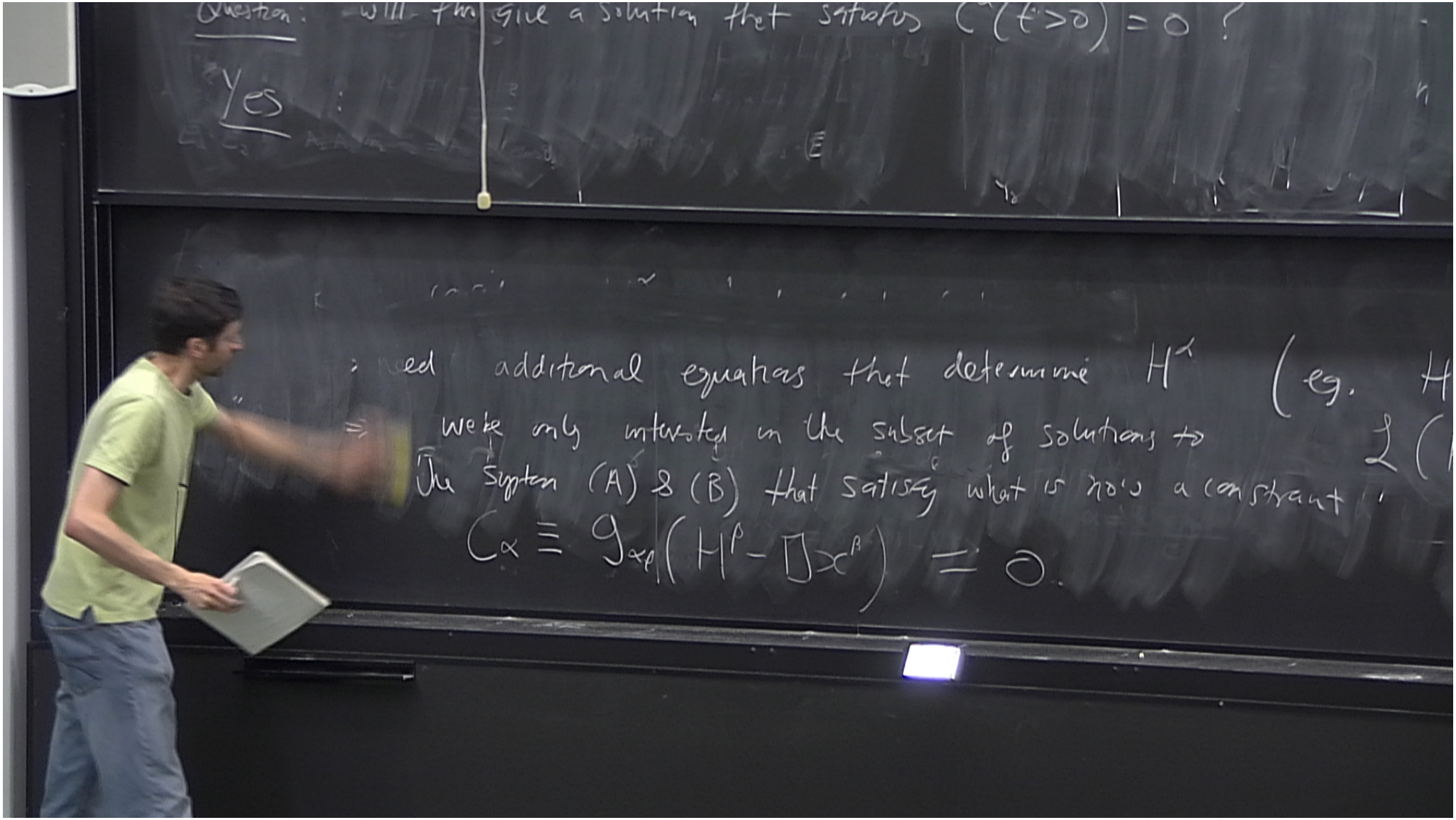

$\frac{1}{2} \Lambda_\alpha \nabla_\beta C^\beta$
such that $M_\alpha = 0 \iff C_\alpha|_{t=0} = 0$ and $\partial_t C_\alpha|_{t=0} = 0$

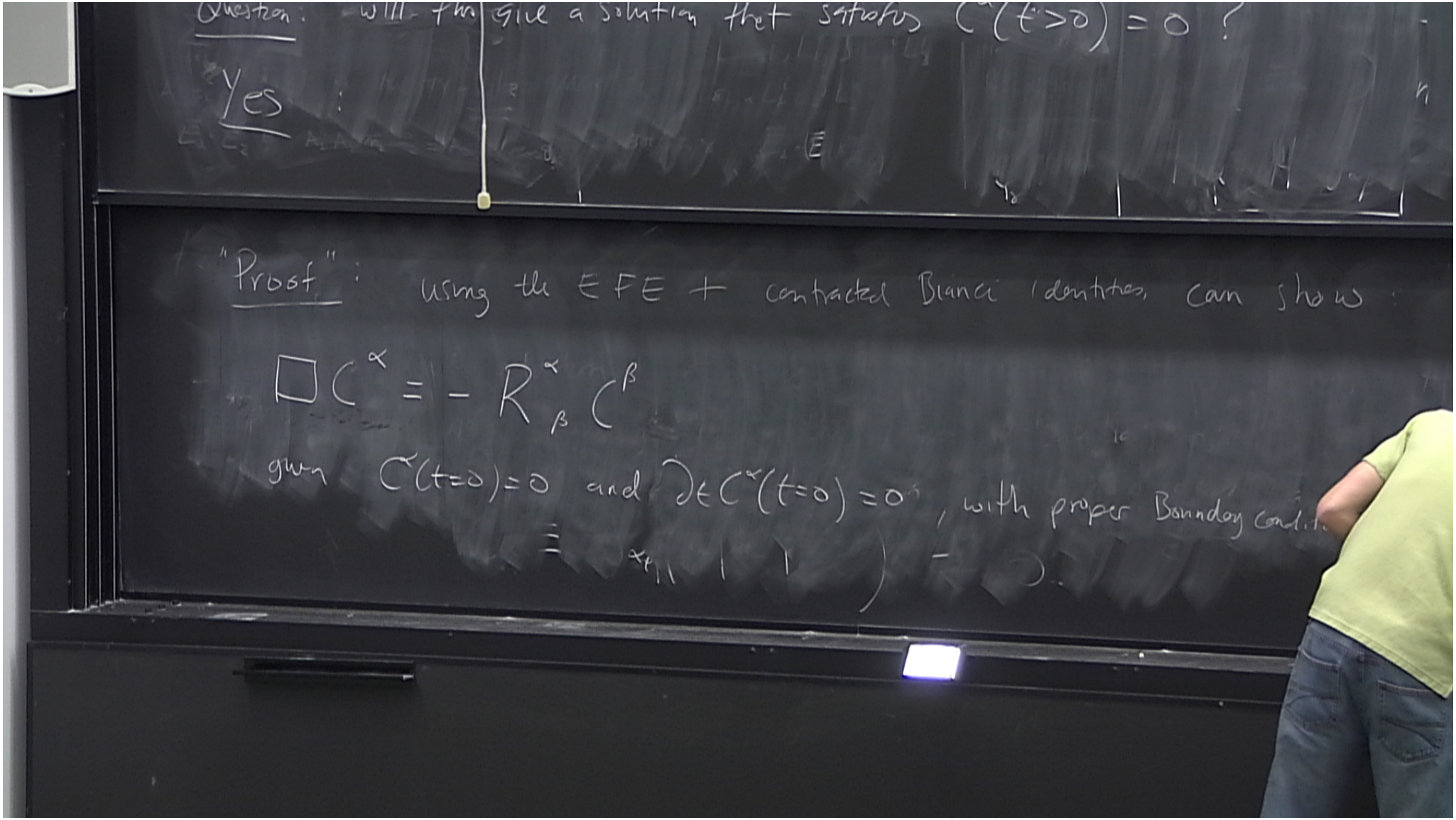

n^α = unit, timelike vector,
$M_\alpha n^\alpha \Rightarrow$ Hamiltonian const
$M_\alpha \perp^\alpha \Rightarrow$ momentum const
Can show:
$M_\alpha = \nabla_{(\alpha} C_{\beta)} n^\beta - \frac{1}{2} n_\alpha \nabla_\beta C^\beta$
$\Rightarrow$ choosing $g_{\alpha\beta}|_{t=0}$ and $\partial_t g_{\alpha\beta}|_{t=0}$ such that $M_\alpha = 0$ $\Leftrightarrow$

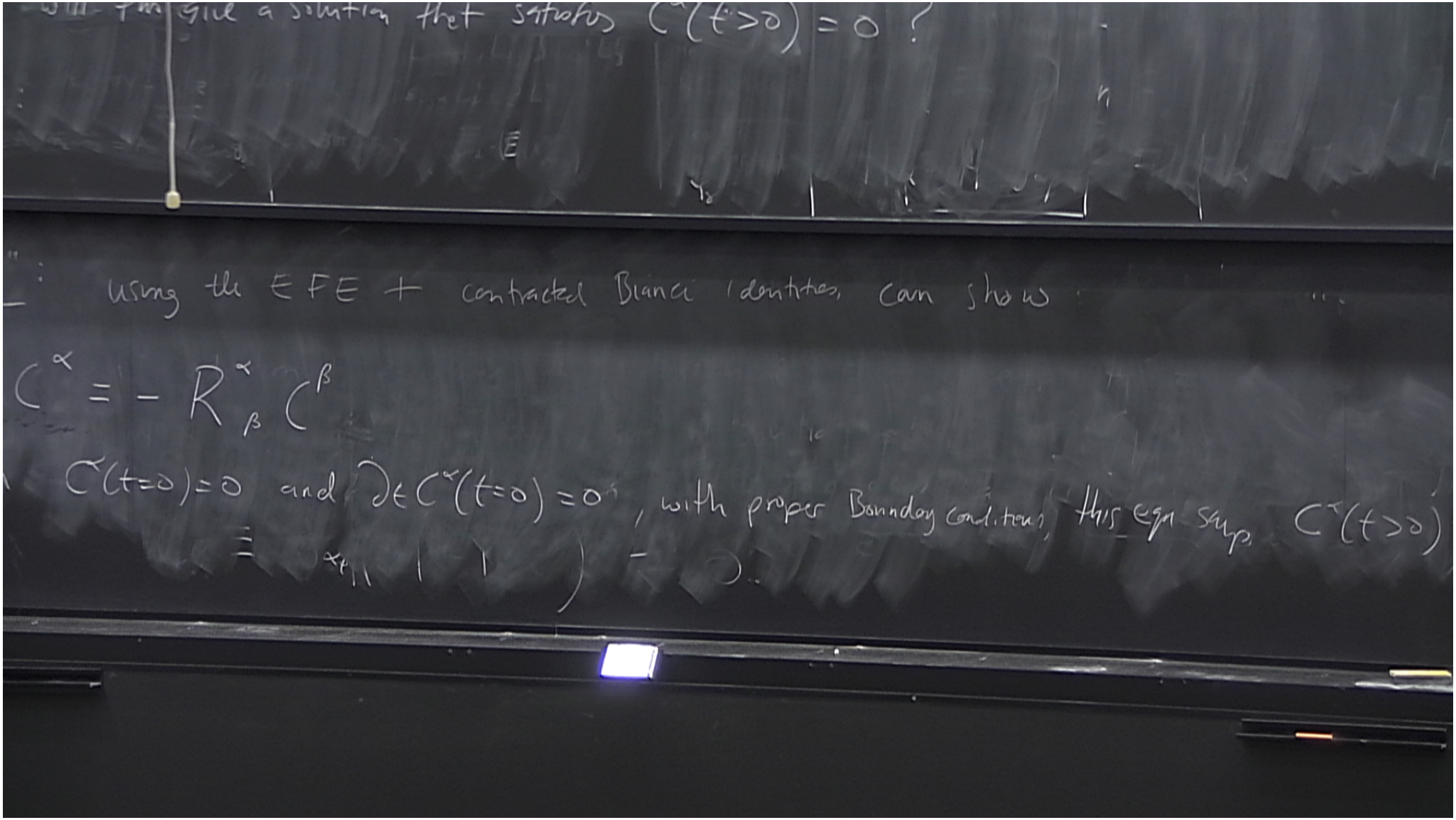

given $g_{ab}(t=0)$ and $\partial_t g_{ab}(t=0)$ that satisfies (C)

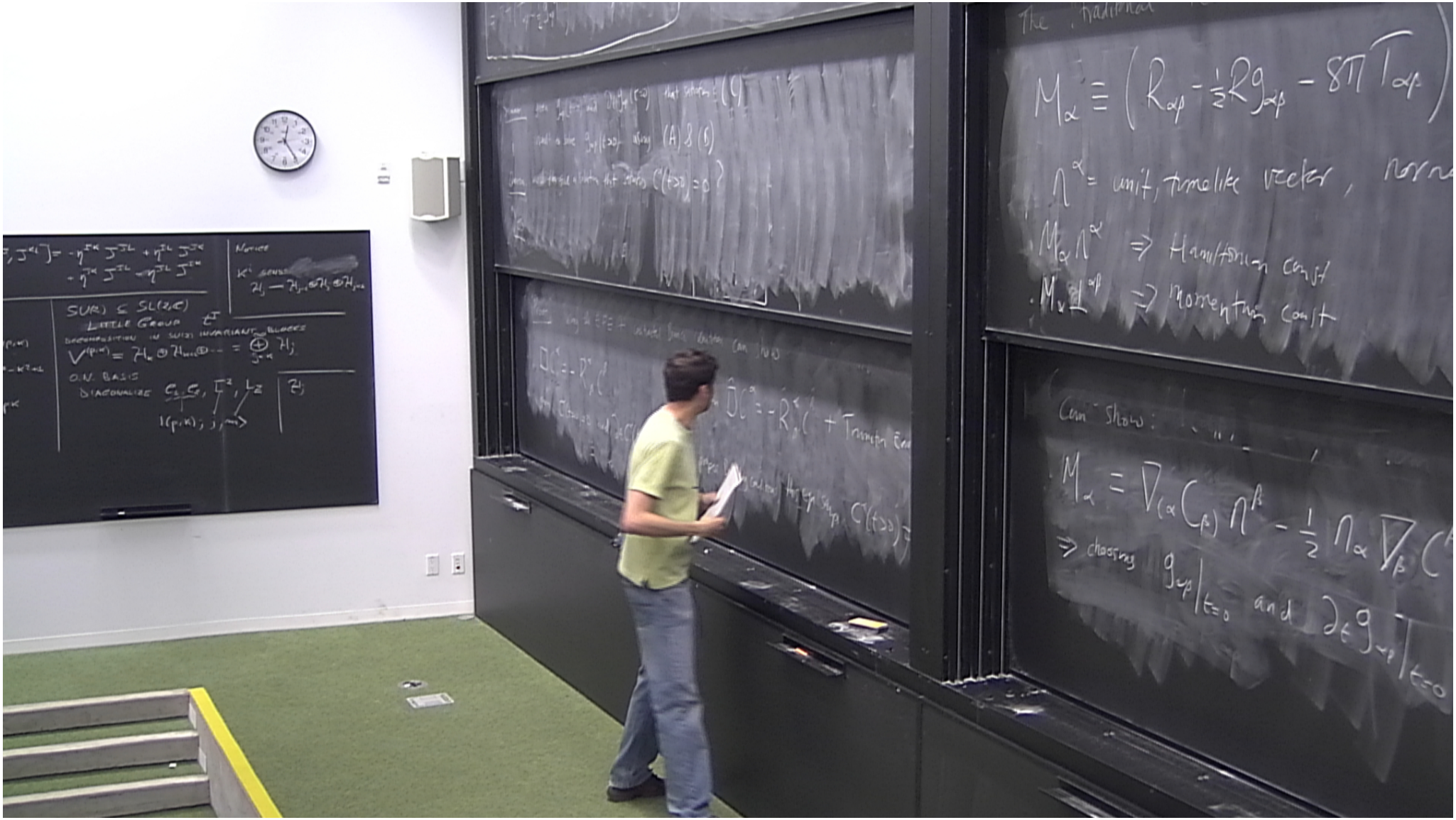

Summary: given $g_{\alpha\beta}(t=0)$ and $\partial_t g_{\alpha\beta}(t=0)$ that satisfies (C)
to solve $g_{\alpha\beta}(t>0)$ using (A) & (B)

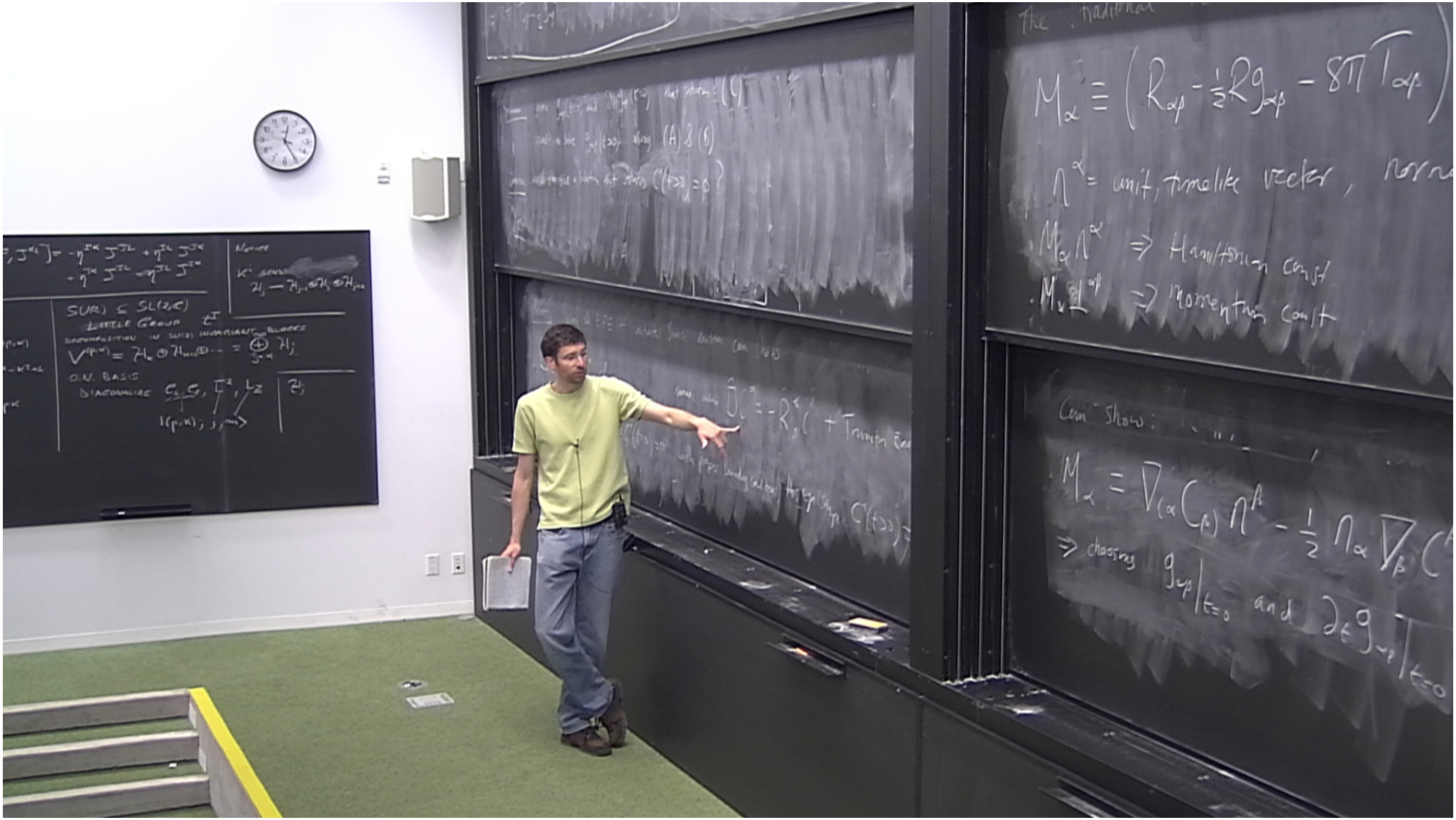

Summary
given $g_{\alpha\beta}(t=0)$ and $\partial_t g_{\alpha\beta}(t=0)$ that satisfies (C)
want to solve $g_{\alpha\beta}(t>0)$ using (A) & (B)


Question: Will this give a solution that satisfies $C_\alpha(t>0) = 0$?
Yes
need additional equations that determine H^α (eg. H
We're only interested in the subset of solutions to
the system (A) & (B) that satisfy what is now a constraint:
$C_\alpha \equiv g_{\alpha\beta}(H^\beta - \Box x^\beta) = 0$.


Question: Will this give a solution that satisfies $C^\alpha(t>0)=0$?
Yes :
"Proof": using the EFE + contracted Bianci identities, can show:
$\Box C^\alpha = -R^\alpha_\beta C^\beta$
given $C^\alpha(t=0)=0$ and $\partial_t C^\alpha(t=0)=0$, with proper Boundary Condi


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Using the EFE + contracted Bianci identities, can show
$C^\alpha = -R^\alpha_\beta C^\beta$
$C^\alpha(t=0) = 0$ and $\partial_t C^\alpha(t=0) = 0$, with proper Boundary conditions, this eqn says $C^\alpha(t>0)$


$M_\alpha \equiv \left(R_{\alpha\beta} - \frac{1}{2}R g_{\alpha\beta} - 8\pi T_{\alpha\beta}\right)$
n^α = unit, timelike vector,
$M_\alpha n^\alpha \Rightarrow$ Hamiltonian constr
$M_\alpha L^{\alpha\beta} \Rightarrow$ momentum constr
$M_\alpha = \nabla_{(\alpha} C_{\beta)} n^\beta - \frac{1}{2} n_\alpha \nabla_\beta C$
$\Rightarrow$ choosing $g_{\alpha\beta}|_{t=0}$ and $\partial_t g_{\alpha\beta}|_{t=0}$


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