Title: Explorations in Numerical Relativity - Lecture 6

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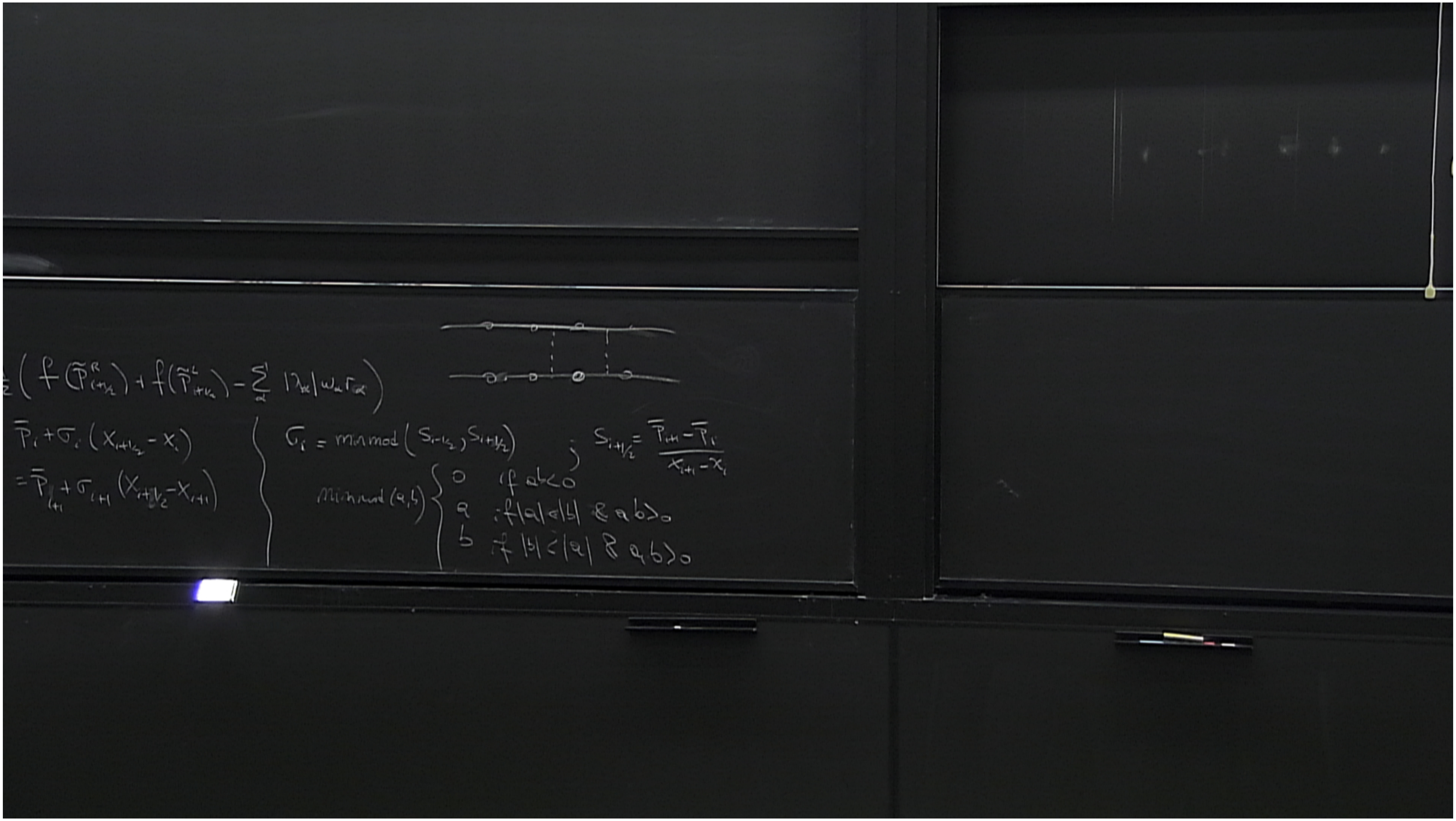
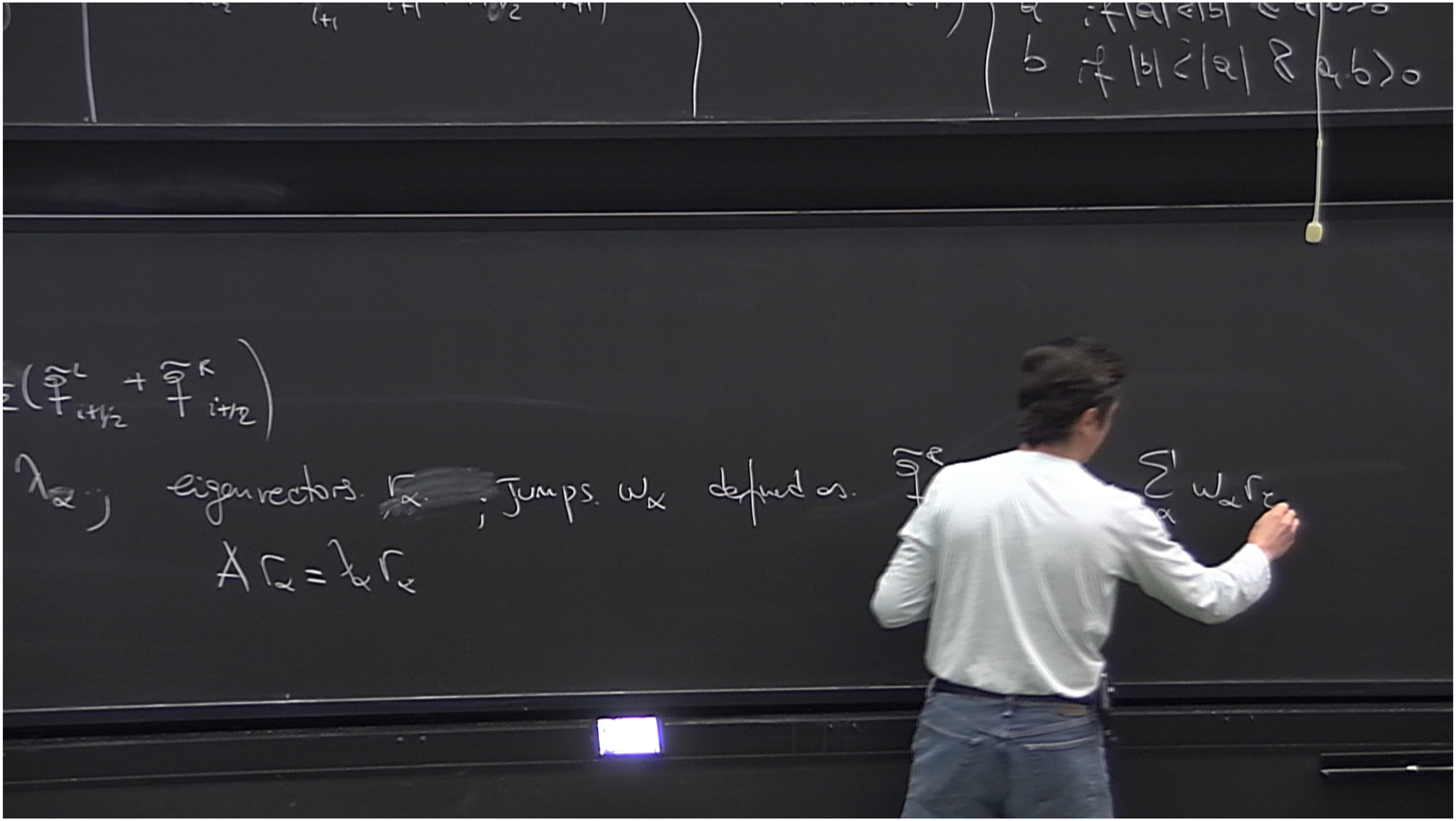
URL: http://pirsa.org/12040046

Abstract:



$\frac{1}{2}\left(f(\tilde{P}^R_{i+1/2}) + f(\tilde{P}^L_{i+1/2}) - \sum_\alpha |\lambda_\alpha| w_\alpha r_\alpha \right)$
$\bar{P}_i + \sigma_i (x_{i+1/2} - x_i)$
$= \bar{P}_{i+1} + \sigma_{i+1} (x_{i+1/2} - x_{i+1})$
$\sigma_i = \mathrm{minmod}(S_{i-1/2}, S_{i+1/2})$; $S_{i+1/2} = \frac{\bar{P}_{i+1} - \bar{P}_i}{x_{i+1} - x_i}$
minmod(a,b)
0 if ab<0
a if |a|<|b| & ab>0
b if |b|<|a| & ab>0


b if |b|<|a| & a·b>0
eigenvectors
jumps w_α defined as
$\sum_\alpha w_\alpha r_\alpha$
$A r_\alpha = \lambda_\alpha r_\alpha$

a if $|a|<|b|$ & $a\cdot b>0$

b if $|b|<|a|$ & $a\cdot b>0$

<u>2-Step method</u>

$$\bar{q}^{n+1/2} = \bar{q}^{n} + \frac{\Delta t}{2} L(\bar{q}^{n})$$

$$\bar{q}^{n+1} = \bar{q}^{n} + \Delta t\, L(\bar{q}^{n+1/2})$$

$\tilde{q}^{R}_{i+1/2})$

eigenvectors r_α ; jumps w_α defined as $\tilde{q}^{R}_{i+1/2} - \tilde{q}^{L}_{i+1/2} = \sum_\alpha w_\alpha r_\alpha$

$$A r_\alpha = \lambda_\alpha r_\alpha$$

a if $|a|<|b|$ & $a\cdot b>0$

b if $|b|<|a|$ & $a\cdot b>0$

$\bar{q}^R_{i+1/2})$

2-Step method

$$\bar{q}^{n+1/2} = \bar{q}^n + \frac{\Delta t}{2} L(\bar{q}^n)$$

$$\bar{q}^{n+1} = \bar{q}^n + \Delta t \, L(\bar{q}^{n+1/2})$$

$$L(\bar{q}^n) = -\left[\frac{F^{Roe}_{i+1/2}(\bar{q}^n) - F^{Roe}_{i-1/2}(\bar{q}^n)}{\Delta x}\right] + \hat{\psi}_i(\bar{q}_n)$$

eigenvectors r_α ; jumps w_α

$$A r_\alpha = \lambda_\alpha r_\alpha \qquad \bar{q}^R_{i+1/2} - \bar{q}^L_{i+1/2} = \sum_\alpha w_\alpha r_\alpha$$

$$\hookrightarrow \frac{\partial q}{\partial t} + \partial_x f^x = 0$$

$$\tilde{p}^L_{i+1/2} = \bar{p}_i + \sigma_i (x_{i+1/2} - x_i)$$

$$\tilde{p}^R_{i+1/2} = \bar{p}_{i+1} + \sigma_{i+1} (x_{i+1/2} - x_{i+1})$$

$$\sigma_i = \text{minmod}(s_{i-1/2}, s_{i+1/2}) \; ; \; s_{i+1/2} = \frac{\;\cdots\;}{x_{i+1} - x_i}$$

$$\text{minmod}(a,b) \begin{cases} 0 & \text{if } a b < 0 \\ a & \text{if } |a| < |b| \;\&\; a \cdot b > 0 \\ b & \text{if } |b| < |a| \;\&\; a \cdot b > 0 \end{cases}$$

$$A|_{i+1/2} \equiv \left.\frac{\partial f}{\partial q}\right|_{q = \frac{1}{2}\left(\tilde{q}^L_{i+1/2} + \tilde{q}^R_{i+1/2}\right)}$$

$\hookrightarrow$ compute eigenvalues λ_α ; eigenvectors r_α ; jumps w_α

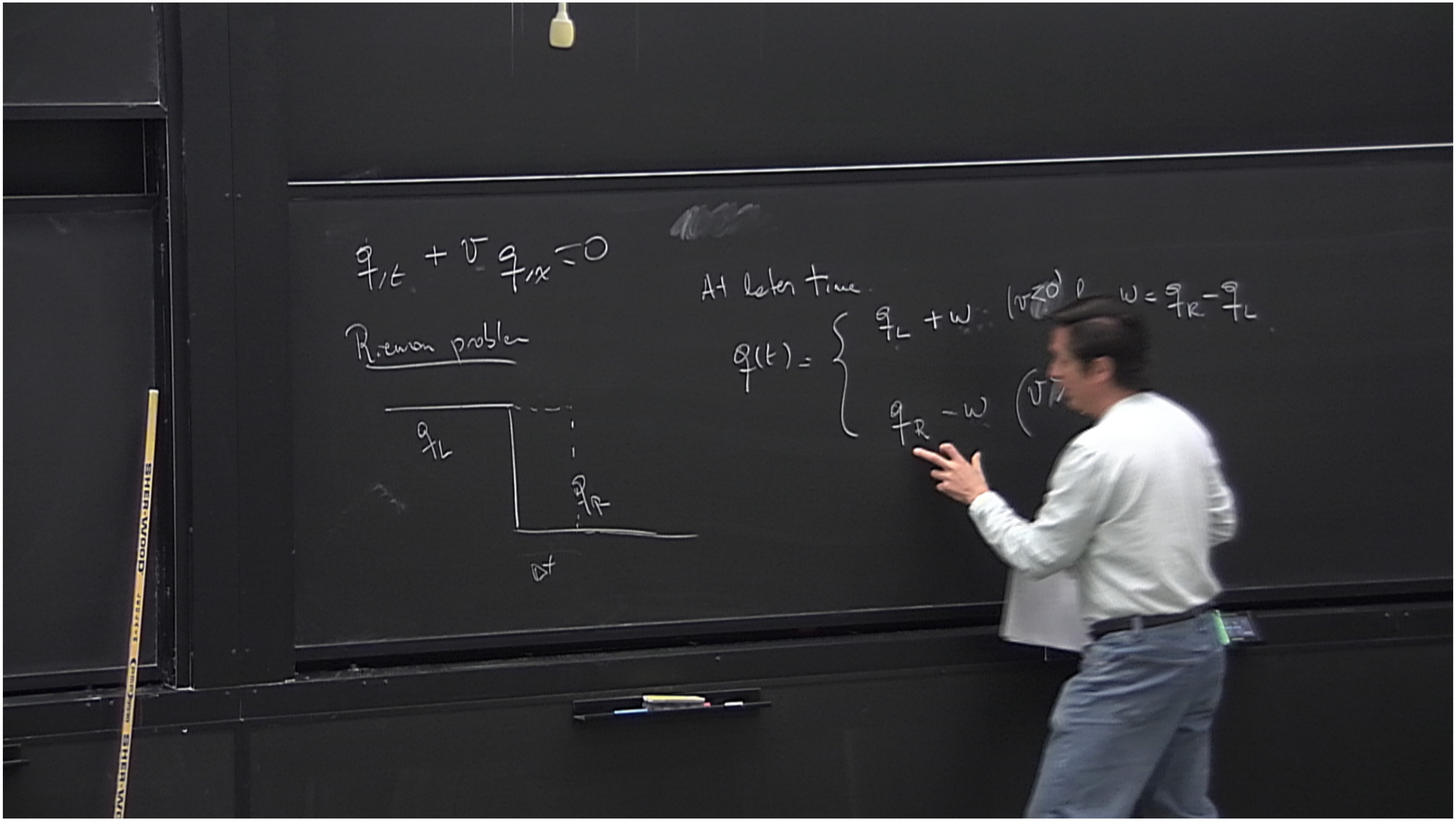
$$A r_\alpha = \lambda_\alpha r_\alpha \qquad \tilde{q}^R_{i+1/2} - \tilde{q}^L_{i+1/2} = \sum_\alpha w_\alpha r_\alpha$$

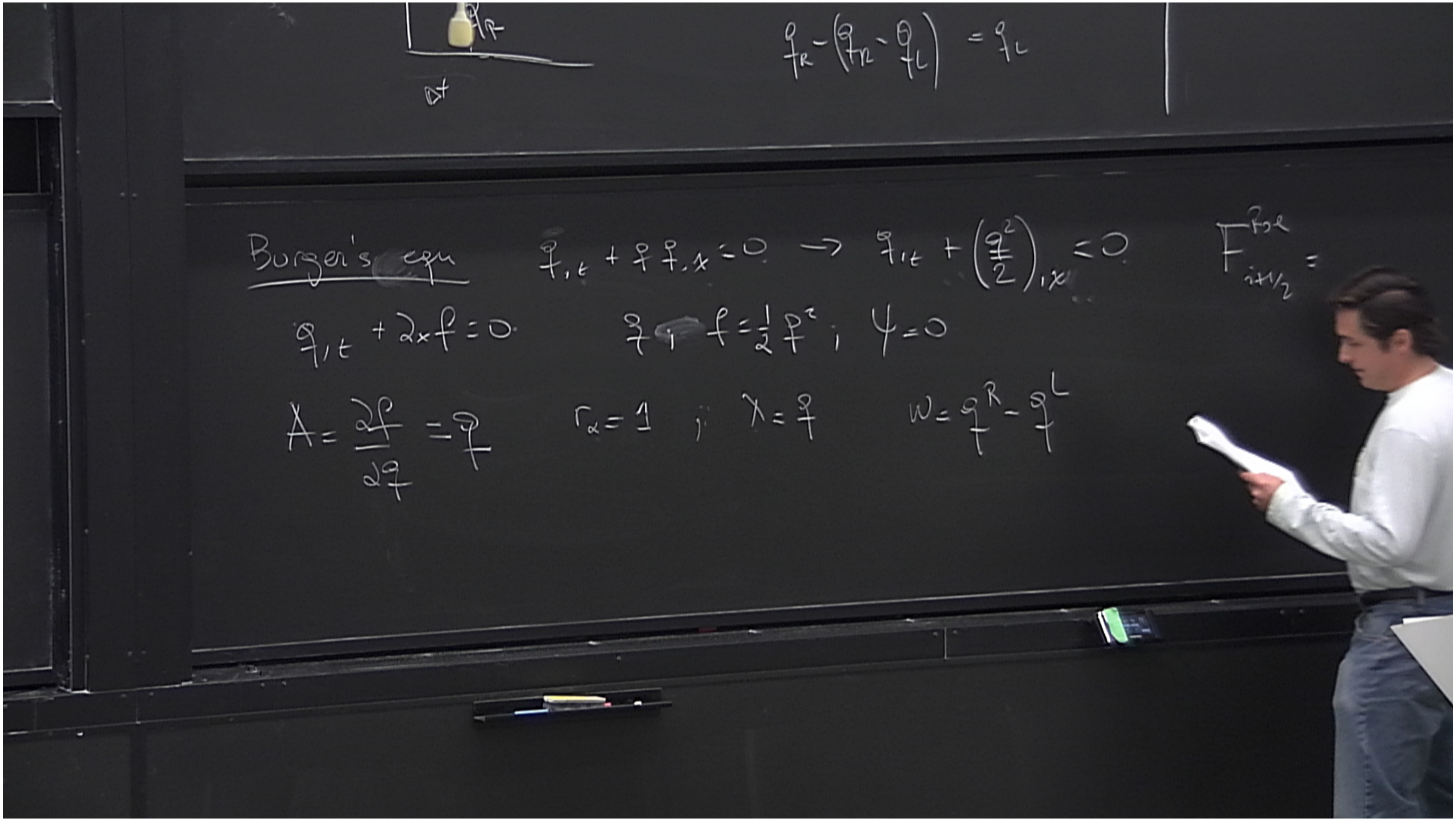
2-step method

$$\bar{q}^{n+1/2} = \bar{q}^n + \frac{\Delta t}{2} L(\bar{q}^n)$$

$$\bar{q}^{n+1} = \bar{q}^n + \Delta t \, L(\bar{q}^{n+1/2})$$

$$L(\bar{q}^n) = -\frac{\left[F^{Roe}_{i+1/2}(\bar{q}^n) - F^{Roe}_{i-1/2}(\bar{q}^n)\right]}{\Delta x} + \hat{\psi}_i(q_n)$$


q,t + v q,x = 0
Riemann problem
qL
qR
At later time
q(t) =
qL + w
qR − w
w = qR − qL



$q_R - (q_R - q_L) = q_L$
Δt
Burger's eqn $q_{,t} + q\,q_{,x} = 0 \rightarrow q_{,t} + \left(\frac{q^2}{2}\right)_{,x} = 0$
$F^{Roe}_{i+1/2} =$
$q_{,t} + \partial_x f = 0$
$q, \; f = \frac{1}{2} q^2 \; ; \; \psi = 0$
$A = \frac{\partial f}{\partial q} = q$
$r_\alpha = 1 \; ; \; \chi = q$
$w = q^R - q^L$

$$q_R - (q_R - q_L) = q_L$$

$$q_{,t} + q\, q_{,x} = 0 \longrightarrow q_{,t} + \left(\frac{q^2}{2}\right)_{,x} = 0$$

$$q \; ; \; f = \frac{1}{2} q^2 \; ; \; \psi = 0$$

$$r_\alpha = 1 \; ; \; \lambda = q \qquad w = q^R - q^L$$

$$F^{Roe}_{i+1/2} = \frac{1}{2}\left[f(q^L) + f(q^R) - |\lambda|\, r\, w \right]_{i+1/2}$$

$$= \frac{1}{2}\left[q^L + q^R - |q|\,(q^R - q^L) \right]$$

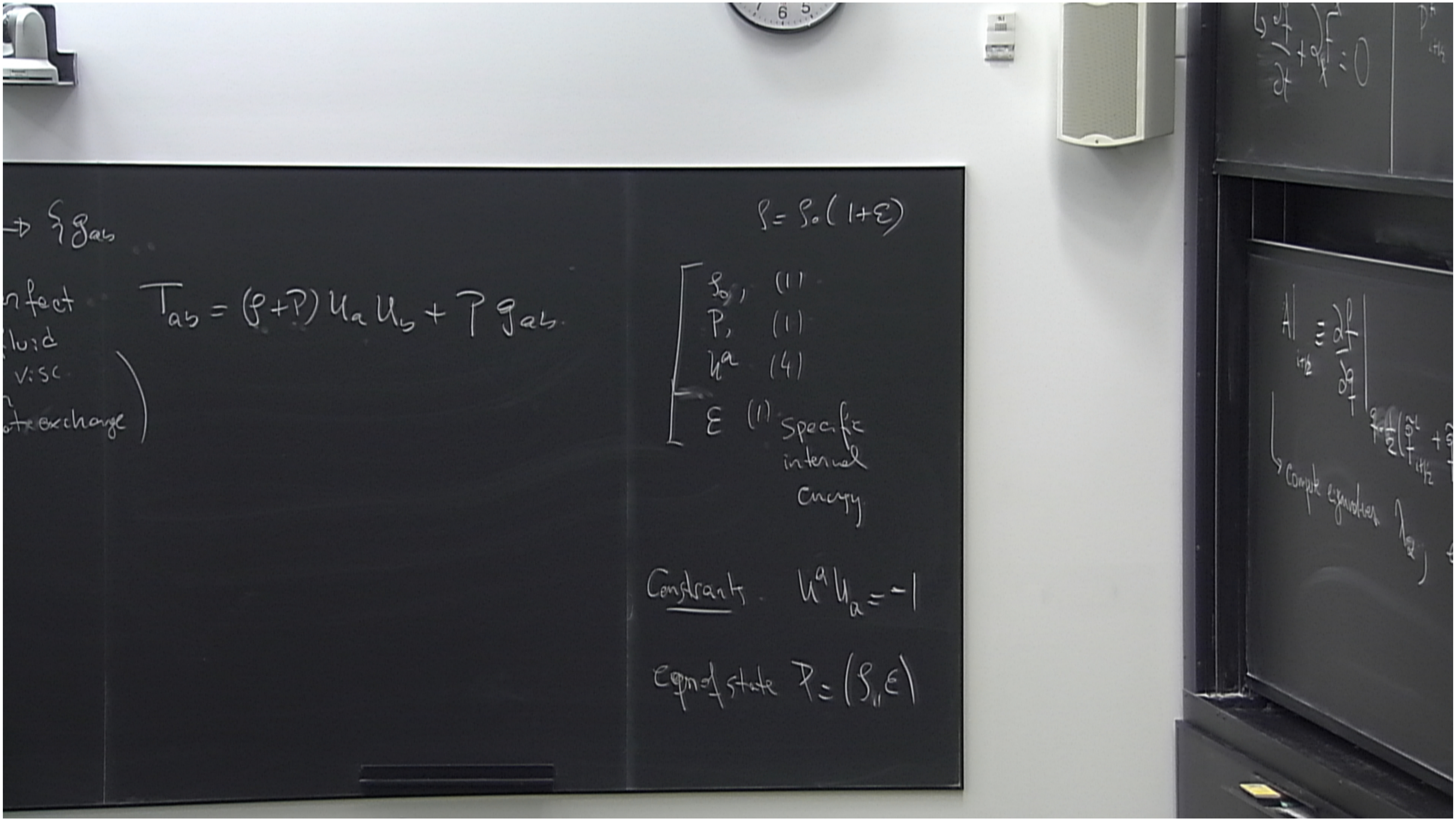
$$|q| \to \frac{1}{2}(q^R + q^L)$$

GR + Hydro.

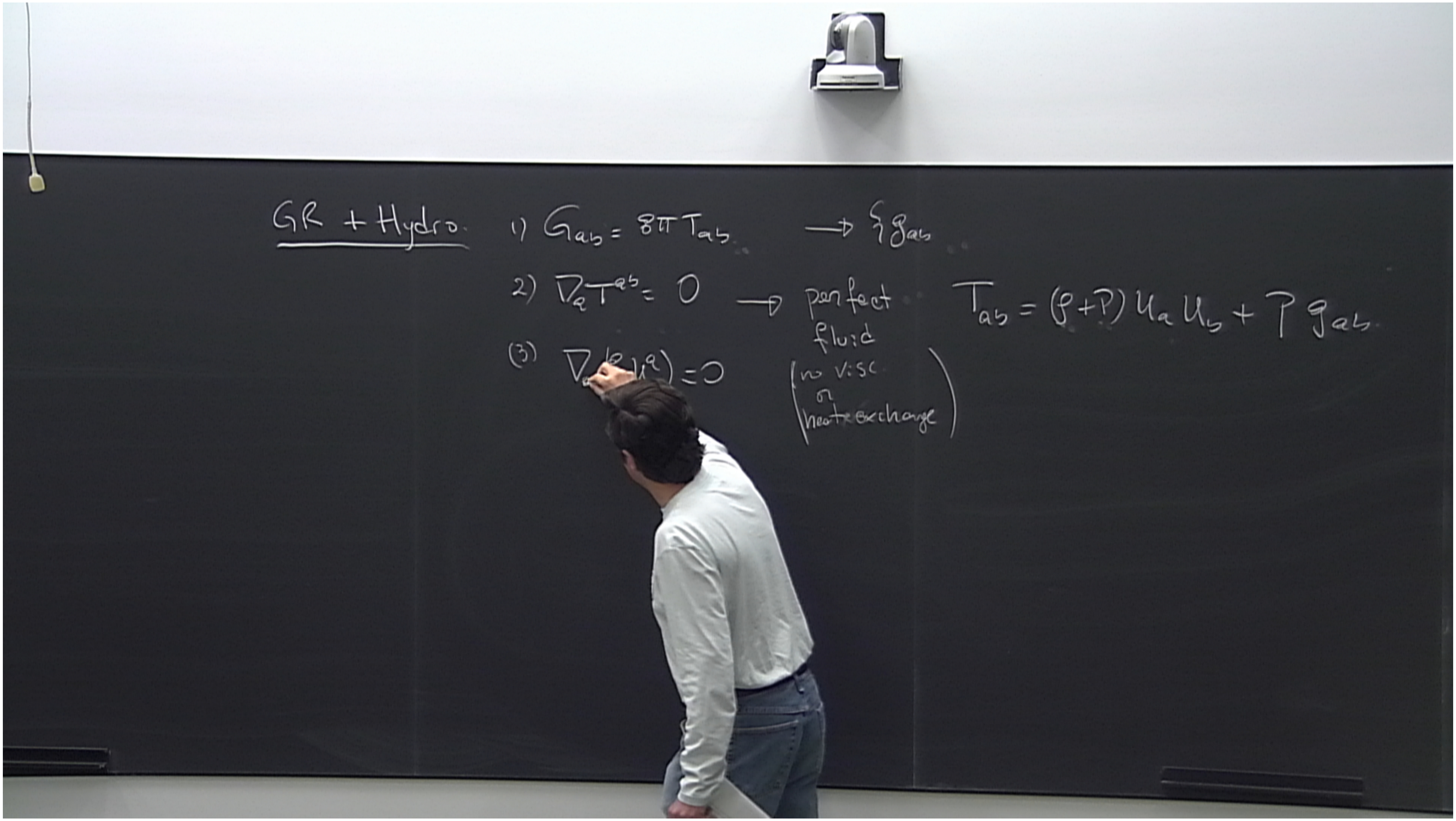
1) $G_{ab} = 8\pi T_{ab} \longrightarrow \{g_{ab}$

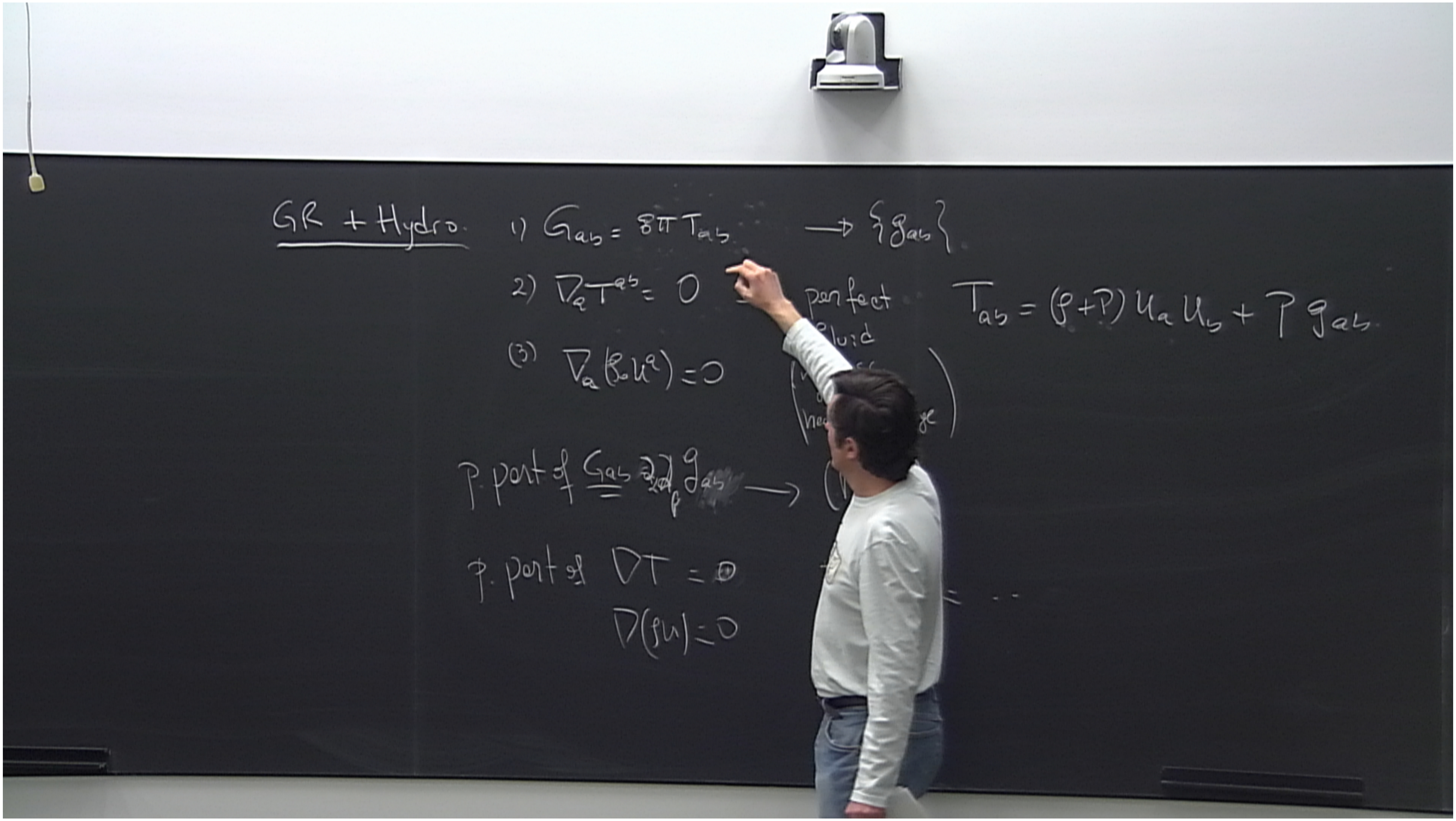
2) $\nabla_a T^{ab} = 0 \longrightarrow$ perfect fluid (no visc. or heat exchange) $T_{ab} = (\rho + P) u_a u_b + P g_{ab}$.

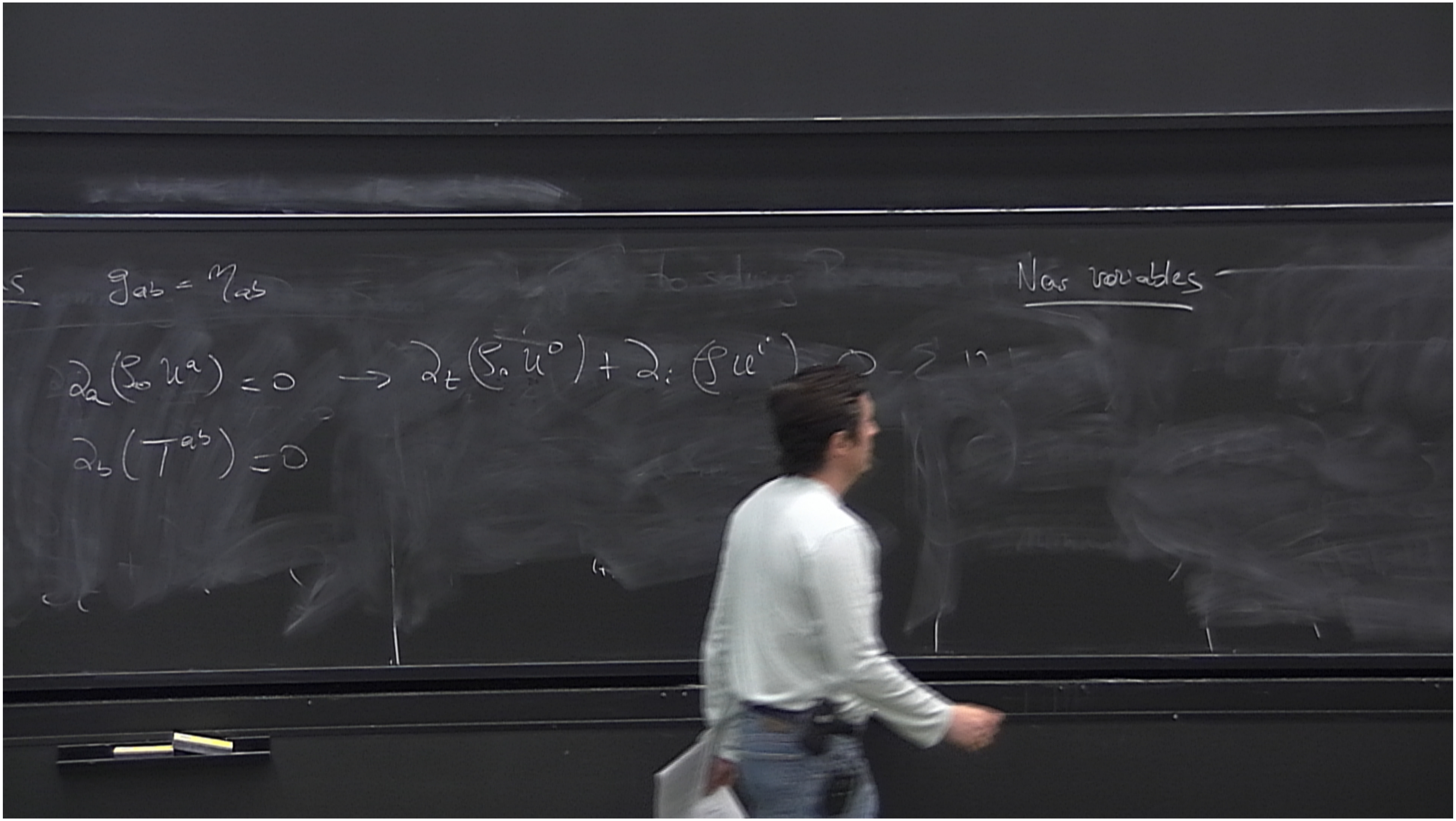
(3)



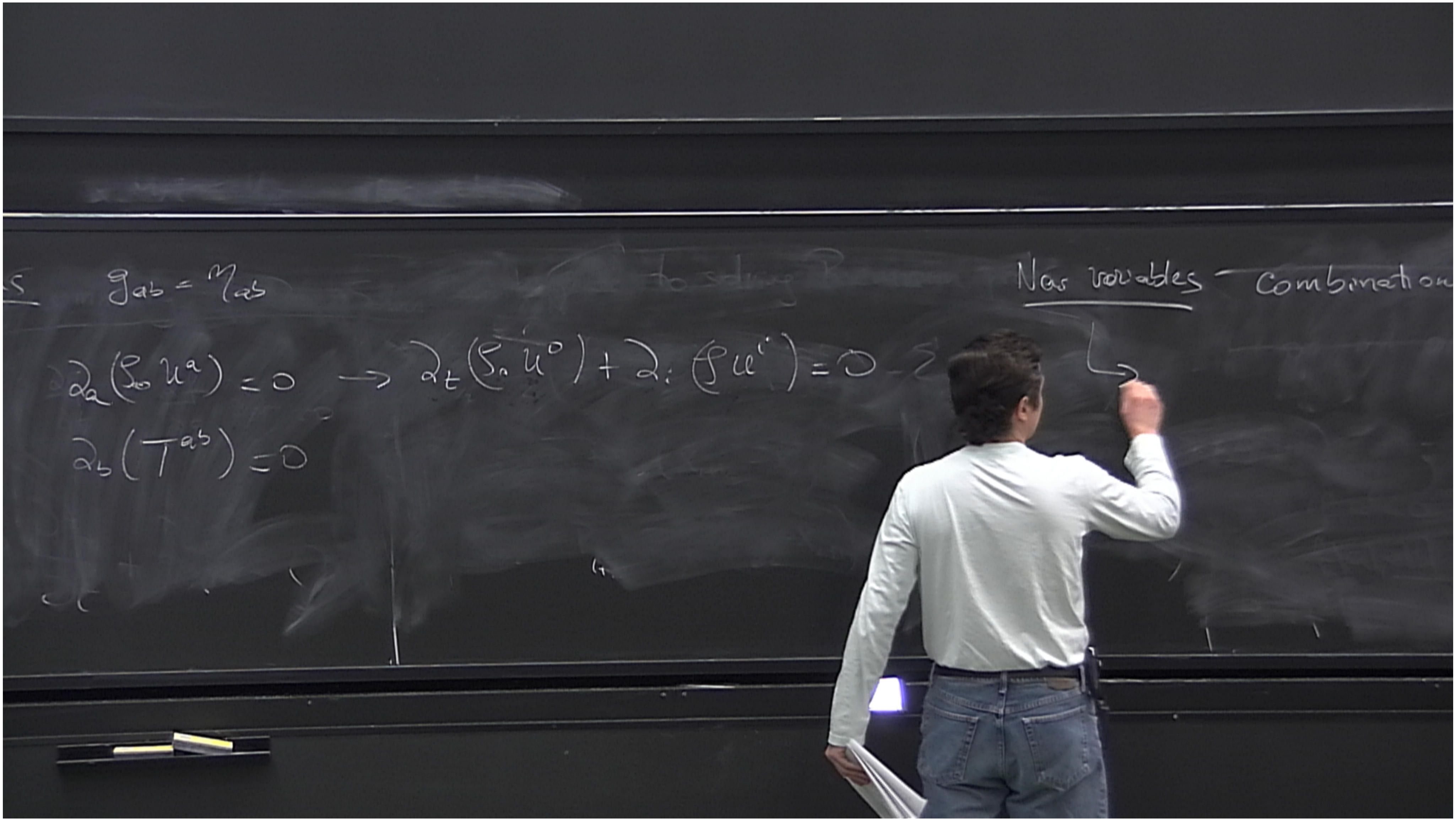
$T_{ab} = (\rho + P) u_a u_b + P g_{ab}$
fluid
visc
exchange
$\rho = \rho_o (1+\varepsilon)$
ρ_o, (1)
P, (1)
u^a (4)
ε (1) specific internal energy
Constraints $u^a u_a = -1$
eqn of state $P = (\rho, \varepsilon)$
$\frac{\partial f}{\partial t} + a \frac{\partial f}{\partial x} = 0$
Compute eigenvalues


GR + Hydro
1) $G_{ab} = 8\pi T_{ab}$ → $\{g_{ab}$
2) $\nabla_a T^{ab} = 0$ → perfect fluid
$T_{ab} = (\rho + P) u_a u_b + P g_{ab}$
(3) $\nabla_a(\rho u^a) = 0$
(no visc or heat exchange)


GR + Hydro.
1) $G_{ab} = 8\pi T_{ab}$ $\longrightarrow \{g_{ab}\}$
2) $\nabla_a T^{ab} = 0$
perfect fluid
$T_{ab} = (\rho + P) u_a u_b + P g_{ab}$
(3) $\nabla_a (\rho_0 u^a) = 0$
p. part of G_{ab} g_{ab} $\longrightarrow$
p. part of $\nabla T = 0$
$\nabla(\rho u) = 0$



$g_{ab} = \eta_{ab}$
New variables
$\partial_a(\rho_0 u^a) = 0 \longrightarrow \partial_t(\rho_0 u^0) + \partial_i(\rho u^i)$
$\partial_b(T^{ab}) = 0$


$g_{ab} = \eta_{ab}$
$\partial_a(\rho_0 u^a) = 0 \rightarrow \partial_t(\rho_0 u^0) + \partial_i(\rho u^i) = 0$
$\partial_b(T^{ab}) = 0$
New variables
Combination

New variables - combinations of "primitive variables"
(ρ_0, ε, u^i)

$\rightarrow \partial_t(\rho_0 u^0) + \partial_i(\rho u^i) = 0$

↳ "conserved/conservation variables"

* Simplify: ultra-relativistic limit: $\rho_0 \varepsilon \gg \rho_0$

GR + Hydro.

1) $G_{ab} = 8\pi T_{ab} \longrightarrow \{g_{ab}\}$

2) $\nabla_a T^{ab} = 0 \longrightarrow$ perfect fluid $\quad T_{ab} = (\rho + P)\, u_a u_b + P\, g_{ab}$

(3) $\nabla_a(\rho_0 u^a) = 0$

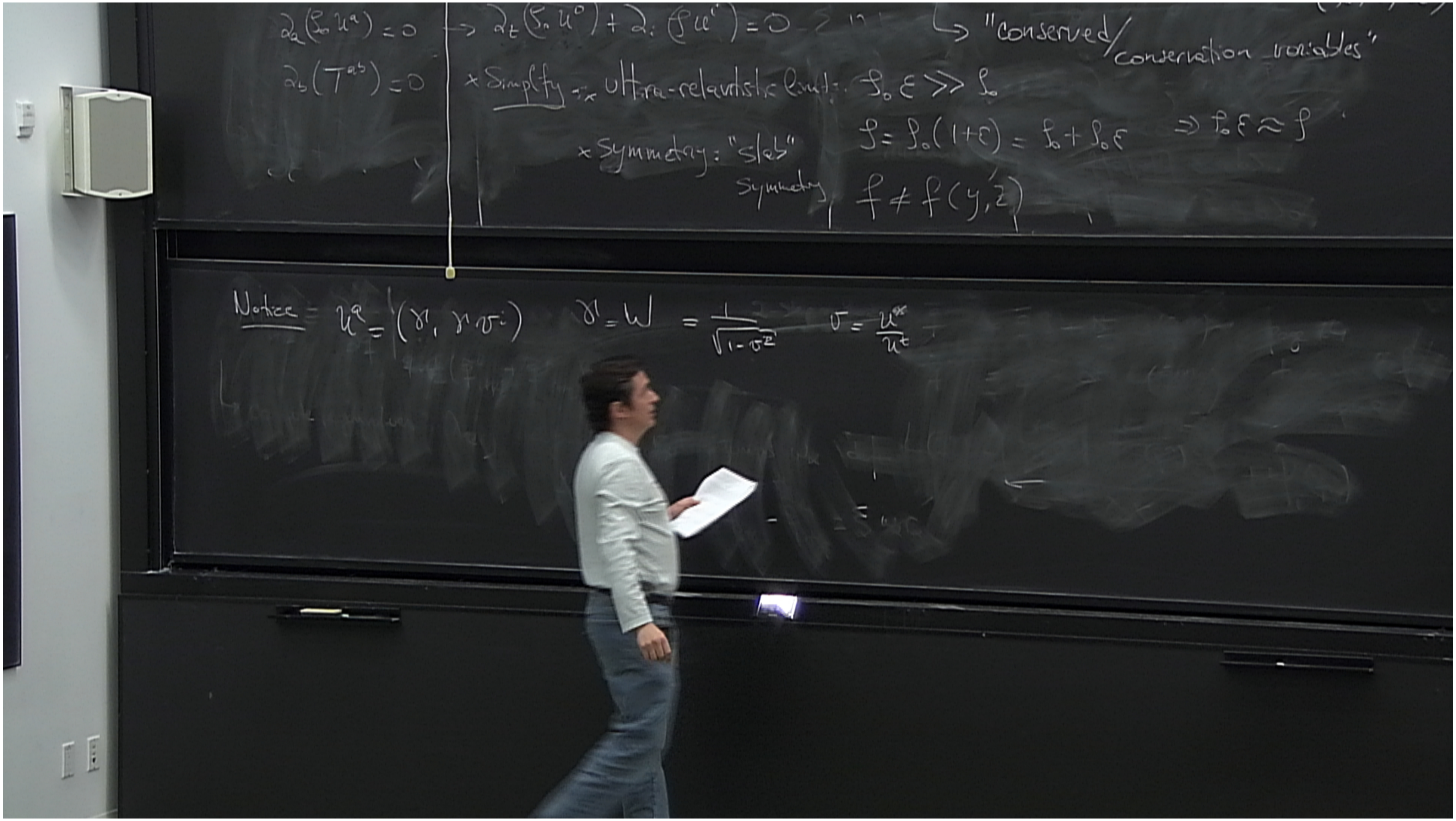
(no visc or heat exchange)

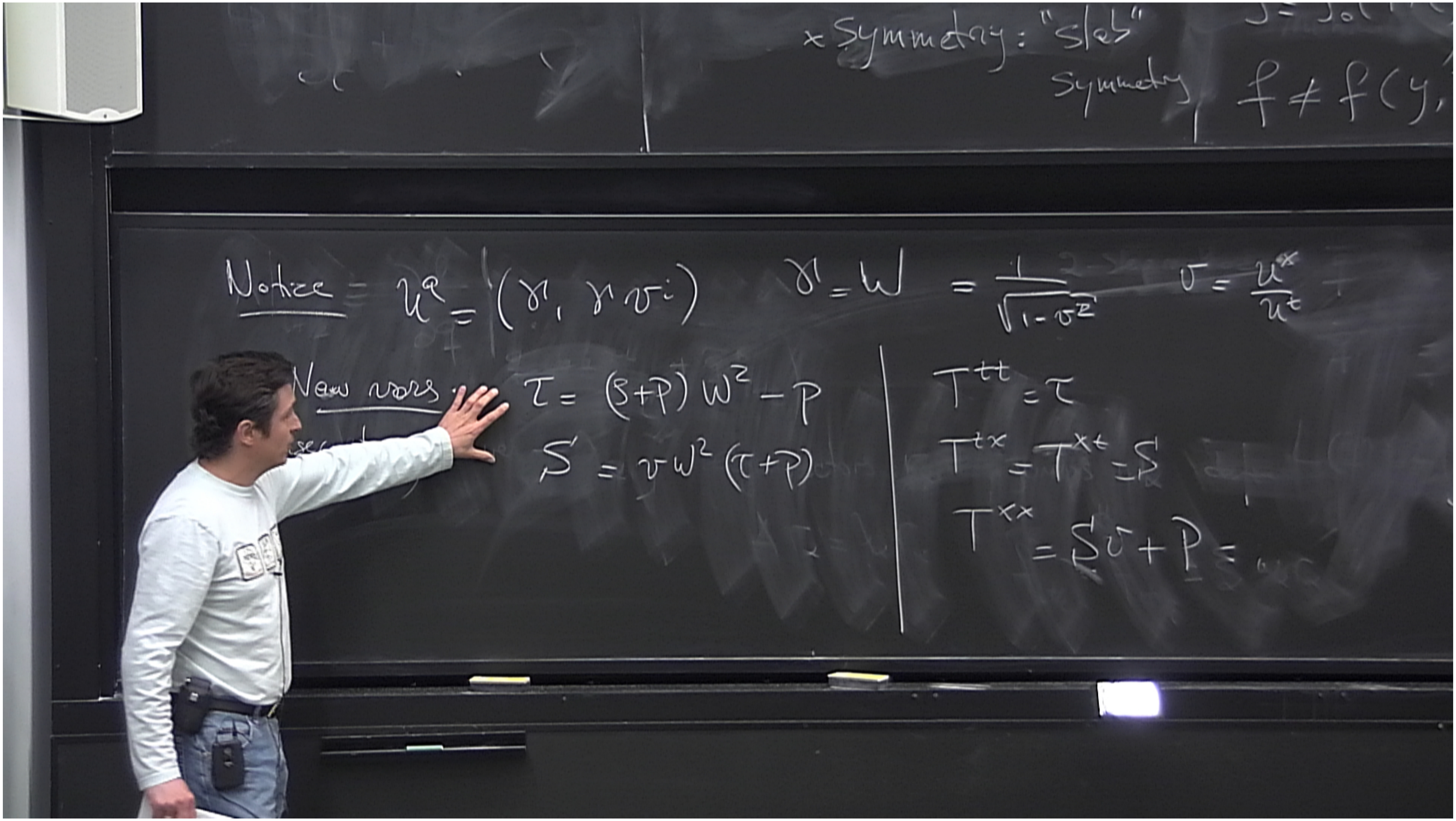
p. part of $\underline{\underline{G_{ab}}}$ $\partial_\alpha \partial_\beta\, g_{ab} \longrightarrow (\Gamma)$

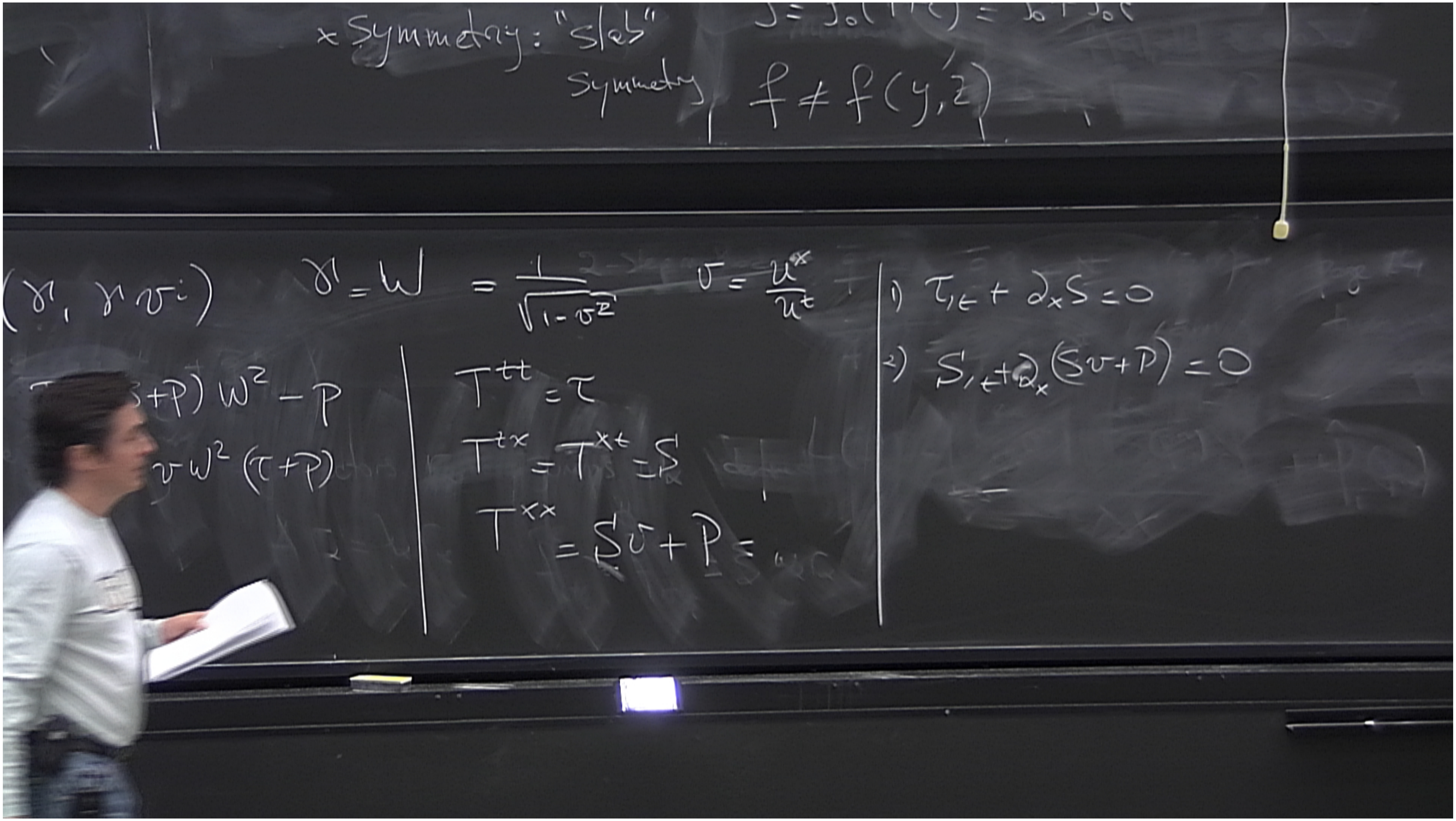
p. part of $\nabla T = 0$

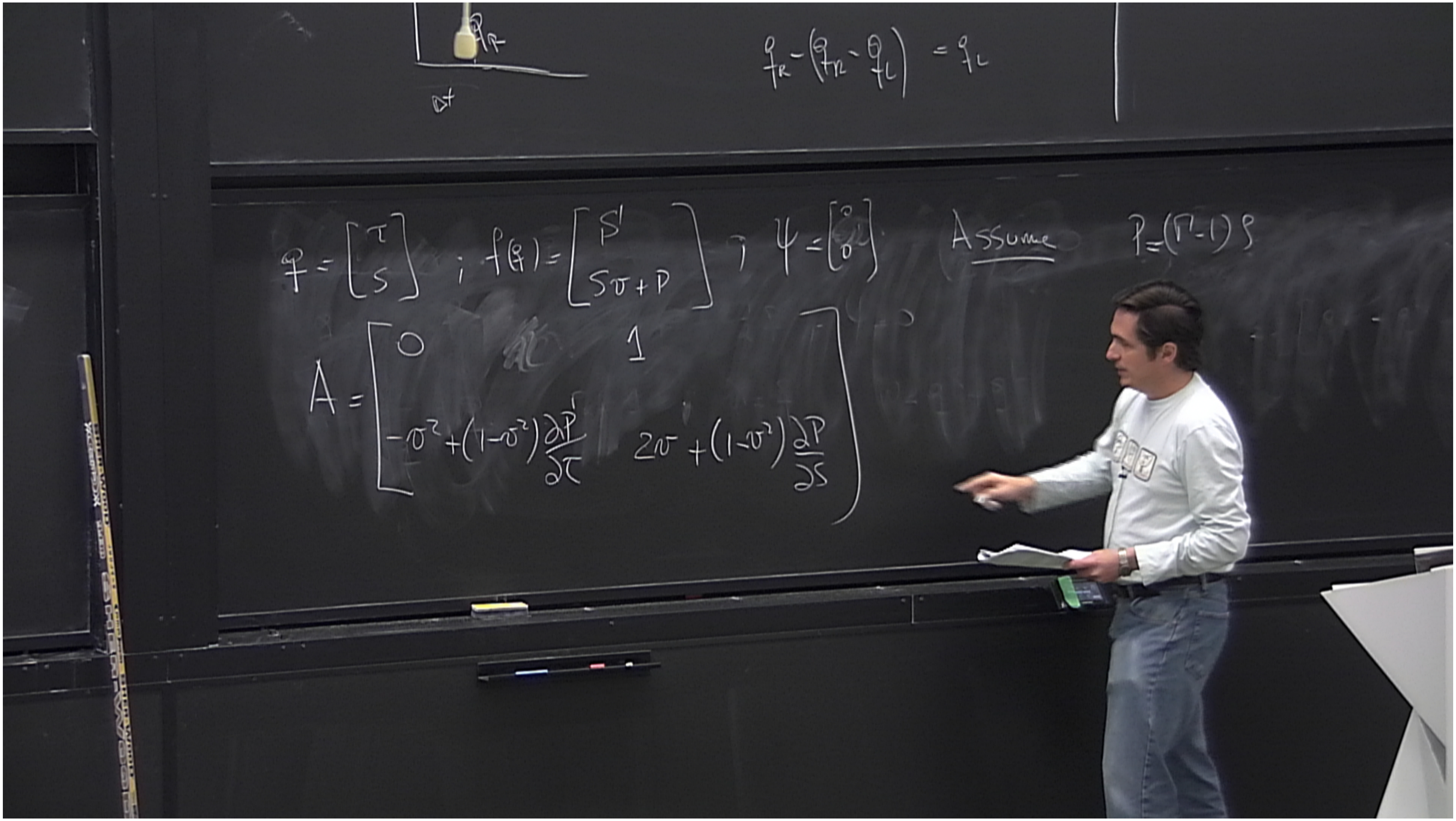
$\nabla(\rho u) = 0$

$\partial(\text{fluid}) = \ldots$


∂a(ρ0 u^a) = 0 → ∂t(ρ0 u^0) + ∂i(ρ u^i) = 0
↳ "conserved/conservation variables"
∂b(T^ab) = 0
× Simplify: ultra-relativistic limit: ρ0 ε >> ρ0
ρ = ρ0(1+ε) = ρ0 + ρ0 ε ⇒ ρ0 ε ≈ ρ
× Symmetry: "slab" symmetry
f ≠ f(y,z)
Notice: u^a = (γ, γ v_i)
γ = W = 1/√(1-v²)
v = u^x/u^t


x Symmetry: "slab"
Symmetry
$f \neq f(y,$
Notice: $u^a = (\gamma, \gamma v^i)$
$\gamma = W = \frac{1}{\sqrt{1-v^2}}$
$v = \frac{u^x}{u^t}$
New vars:
$\tau = (\rho+P)W^2 - P$
$S = vW^2(\tau+P)$
$T^{tt} = \tau$
$T^{tx} = T^{xt} = S$
$T^{xx} = Sv + P =$


x Symmetry: "slab"
Symmetry f ≠ f(y, z)
(γ, γ v^i)
γ = W = 1/√(1 - v²)
v = u^x/u^t
T^tt = τ
T^tx = T^xt = S
T^xx = Sv + P
1) τ,t + ∂x S = 0
2) S,t + ∂x (Sv + P) = 0


Assume