Title: Explorations in Quantum Gravity - Lecture 11

Date: Apr 17, 2012 10:15 AM

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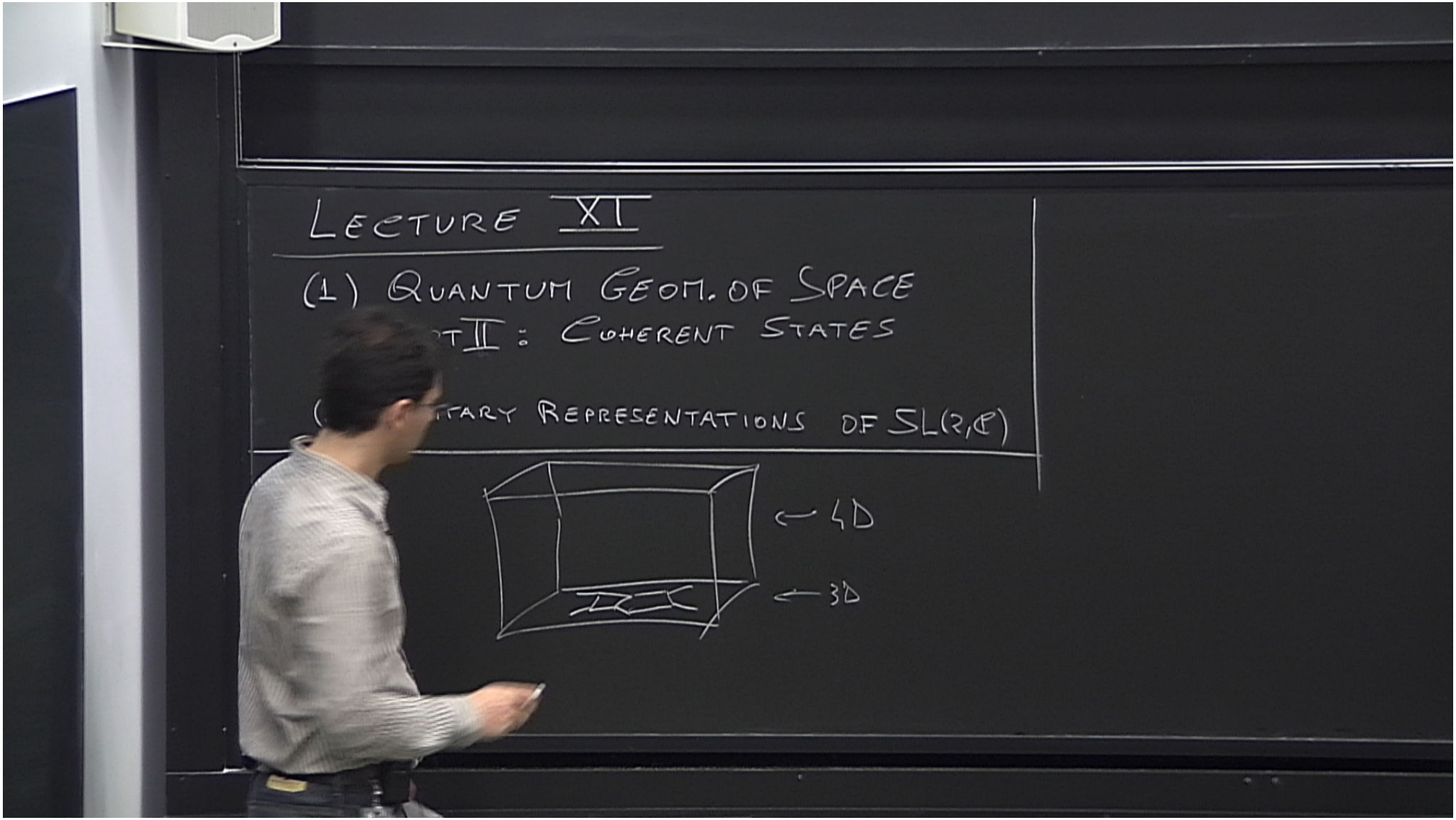
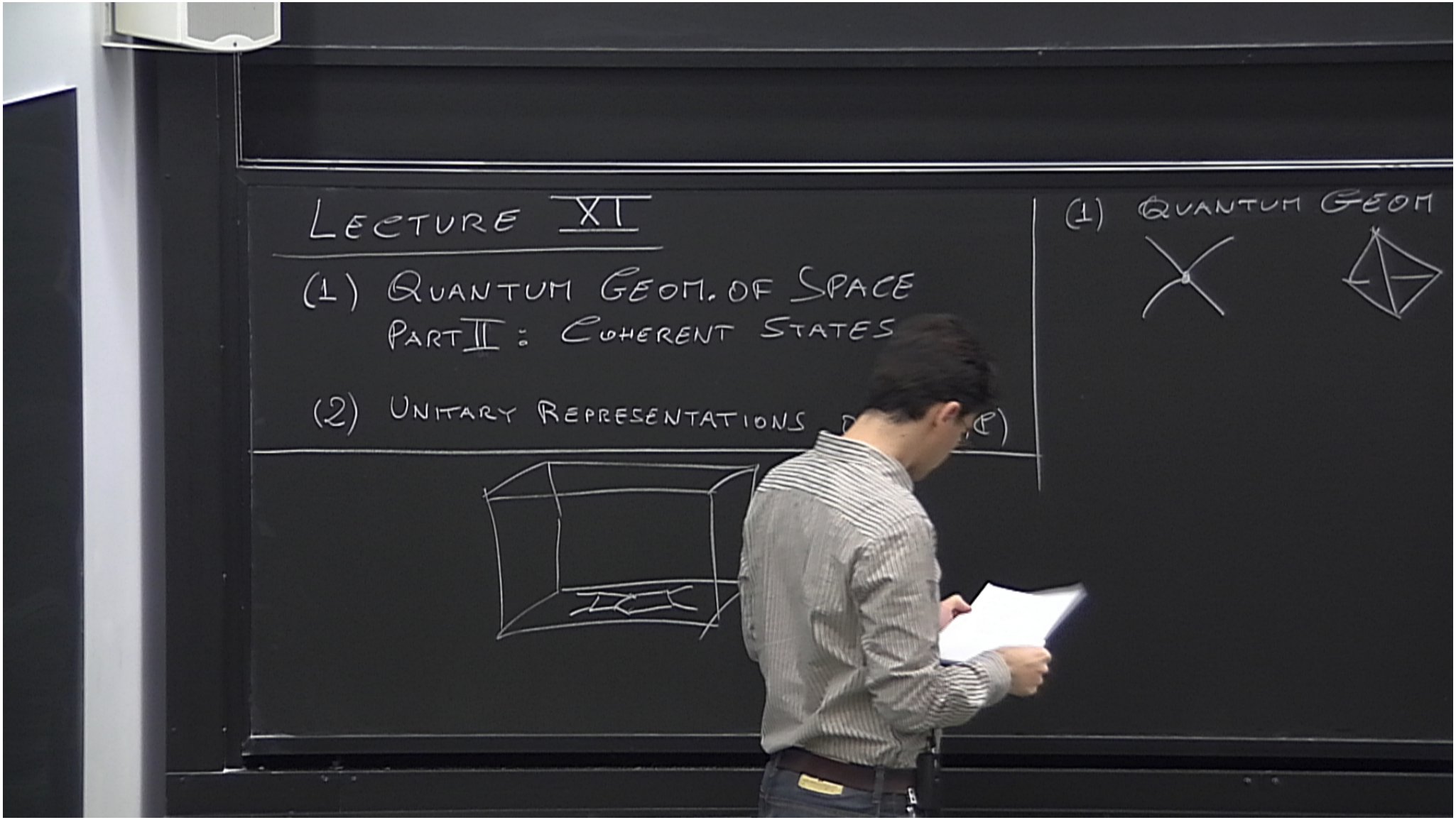
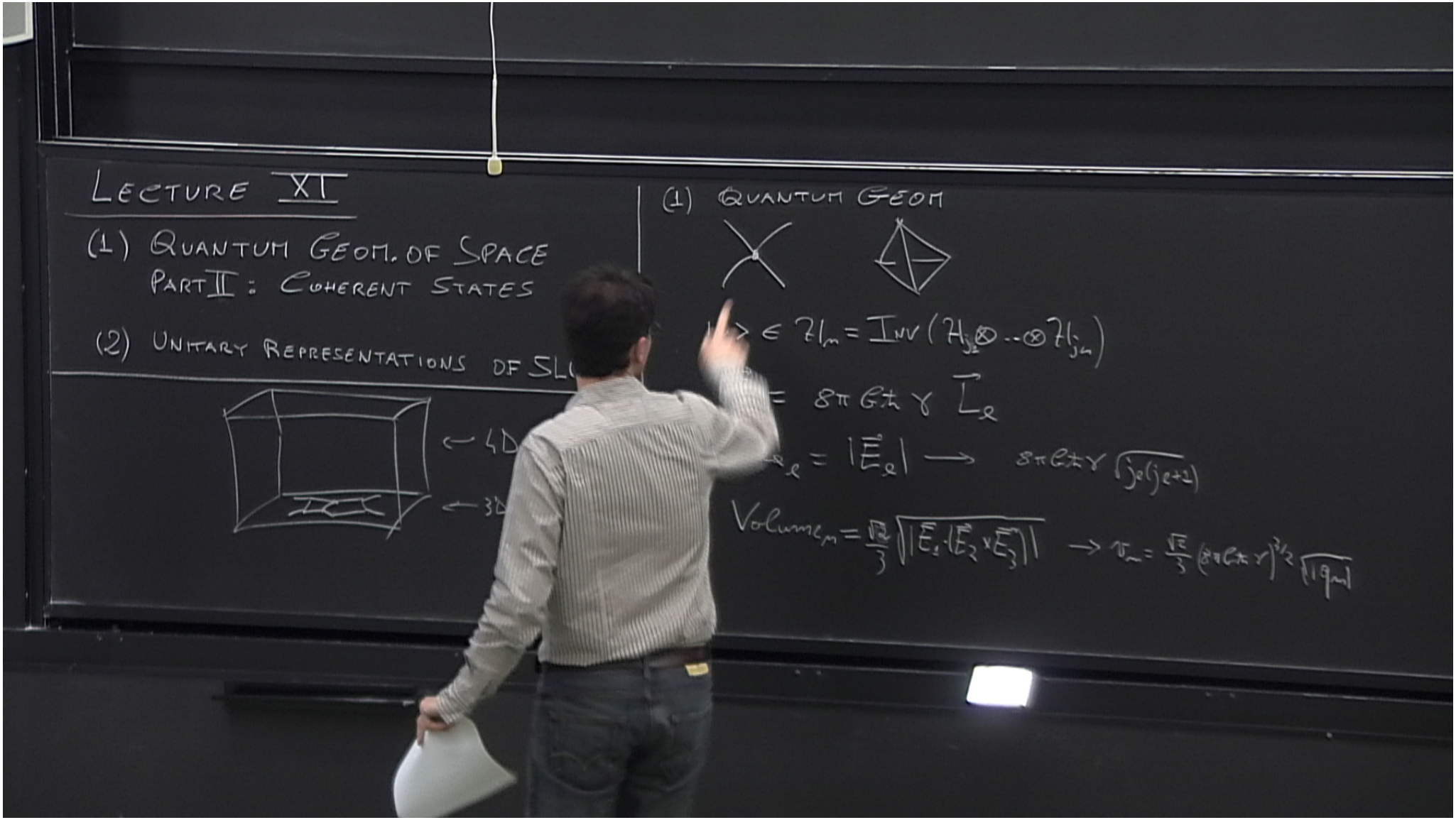
Abstract:


LECTURE XI
(1) QUANTUM GEOM. OF SPACE
PART II: COHERENT STATES
UNITARY REPRESENTATIONS OF SL(2,C)
4D
3D

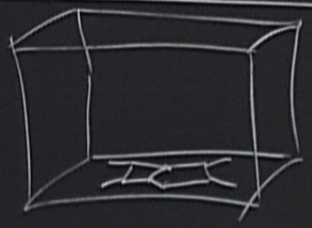

LECTURE XI
(1) QUANTUM GEOM. OF SPACE
PART II : COHERENT STATES
(2) UNITARY REPRESENTATIONS
(1) QUANTUM GEOM


LECTURE XI
(1) QUANTUM GEOM. OF SPACE
PART II : COHERENT STATES
(2) UNITARY REPRESENTATIONS OF SL
(1) QUANTUM GEOM
$|\cdot\rangle \in \mathcal{H}_m = \mathrm{INV}(\mathcal{H}_{j_1} \otimes \cdots \otimes \mathcal{H}_{j_m})$
$= 8\pi G\hbar\gamma \vec{L}_e$
$= |\vec{E}_e| \longrightarrow 8\pi G\hbar\gamma \sqrt{j_e(j_e+2)}$
$\mathrm{Volume}_n = \frac{\sqrt{2}}{3}\sqrt{|\vec{E}_1 \cdot (\vec{E}_2 \times \vec{E}_3)|} \longrightarrow \nu_n = \frac{\sqrt{2}}{3}(8\pi G\hbar\gamma)^{3/2}\sqrt{|q_n|}$

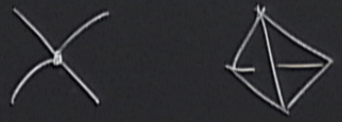
LECTURE XI

(1) QUANTUM GEOM. OF SPACE
PART II : COHERENT STATES

(2) UNITARY REPRESENTATIONS OF $SL(2,\mathbb{C})$



(1) QUANTUM GEOM



$$|i\rangle \in \mathcal{H}_n = \mathrm{INV}(\mathcal{H}_{j_1} \otimes \cdots \otimes \mathcal{H}_{j_N})$$

$$\vec{E}_e = 8\pi G\hbar\gamma\, \vec{L}_e$$

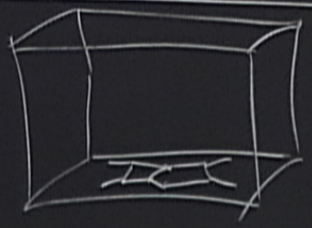
$$\mathrm{Area}_e = |\vec{E}_e| \longrightarrow 8\pi G\hbar\gamma \sqrt{j_e(j_e+1)}$$

$$\mathrm{Volume}_n = \frac{\sqrt{2}}{3}\sqrt{|\vec{E}_1 \cdot (\vec{E}_2 \times \vec{E}_3)|} \longrightarrow v_n = \frac{\sqrt{2}}{3}(8\pi G\hbar\gamma)^{3/2}\sqrt{|q_n|}$$

LECTURE XI

(1) QUANTUM GEOM. OF SPACE
PART II : COHERENT STATES

(2) UNITARY REPRESENTATIONS OF $SL(2,\mathbb{C})$



(1) QUANTUM GEOM

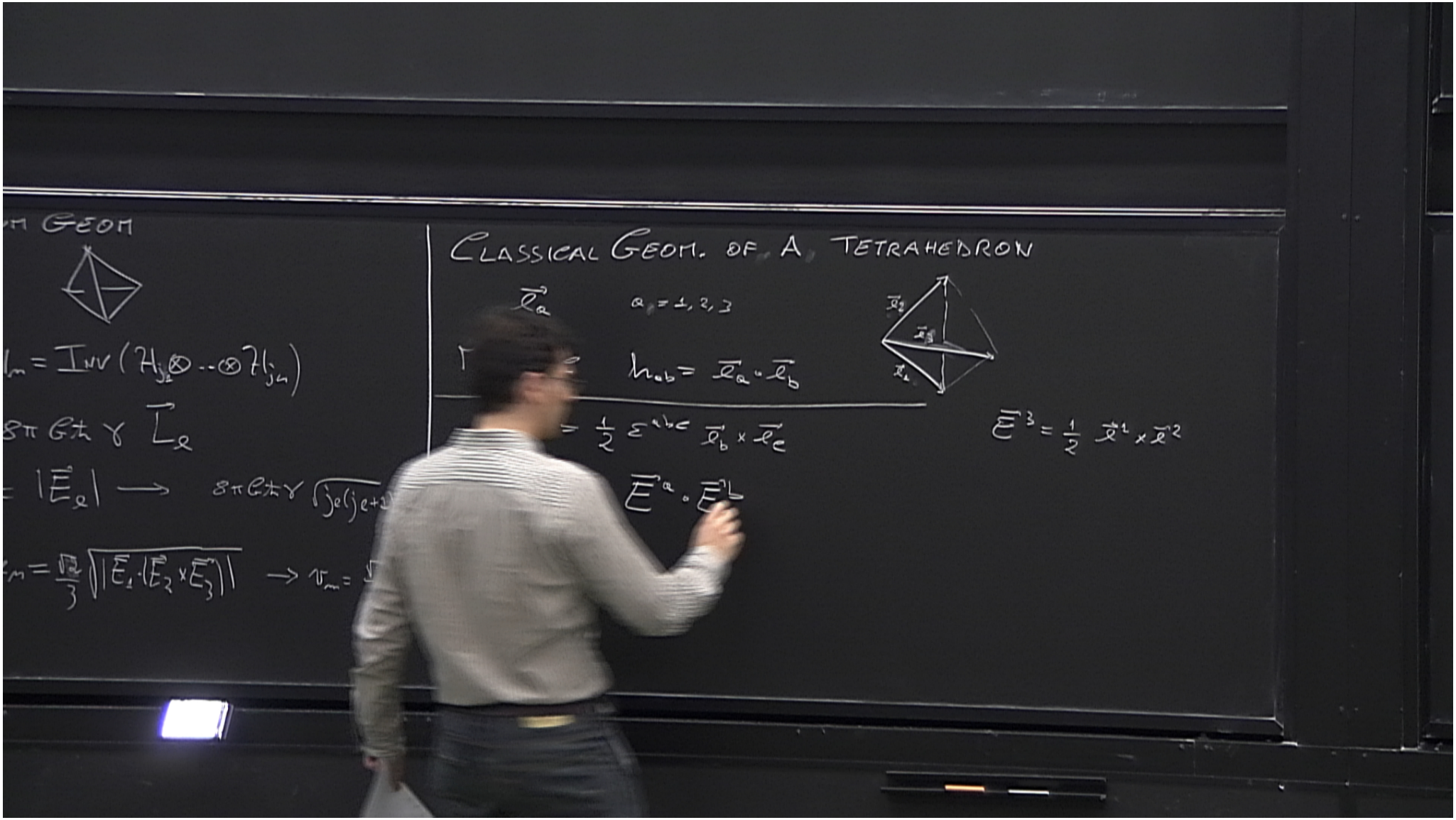


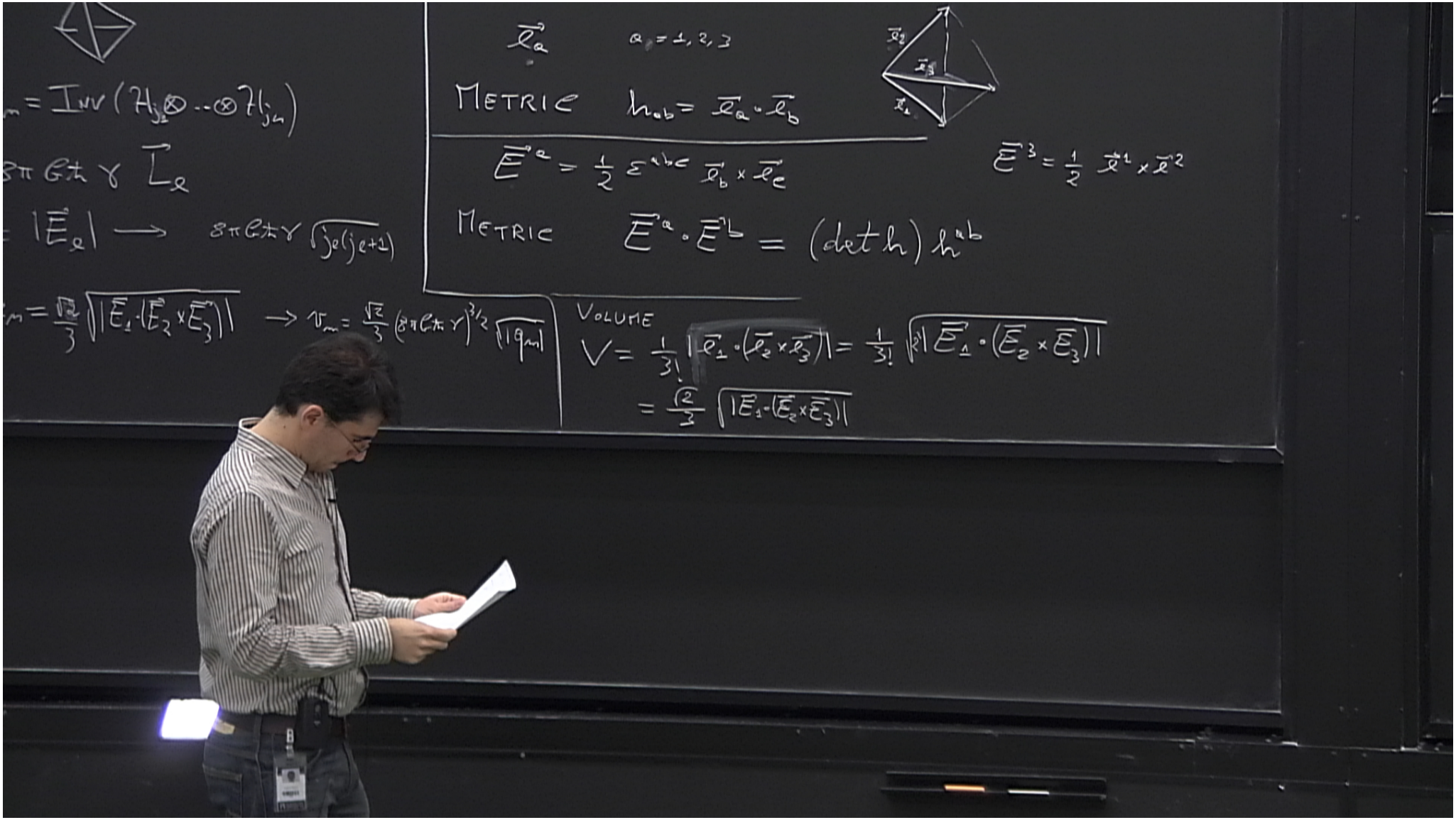
$|i\rangle \in \mathcal{H}_n = \mathrm{INV}(\mathcal{H}_{j_1} \otimes \cdots \otimes \mathcal{H}_{j_N})$

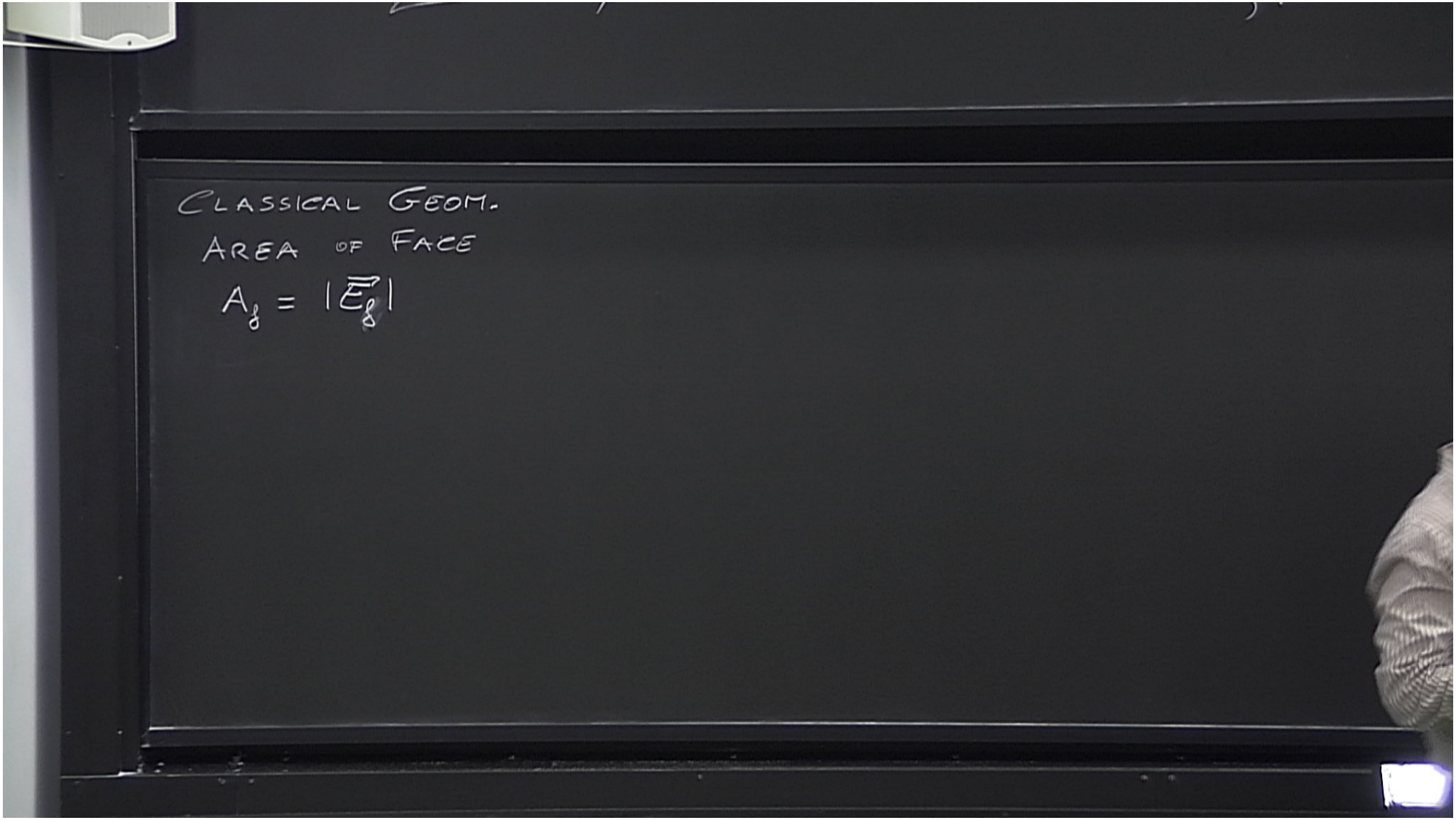
$\vec{E}_e = 8\pi G\hbar\gamma \vec{L}_e$

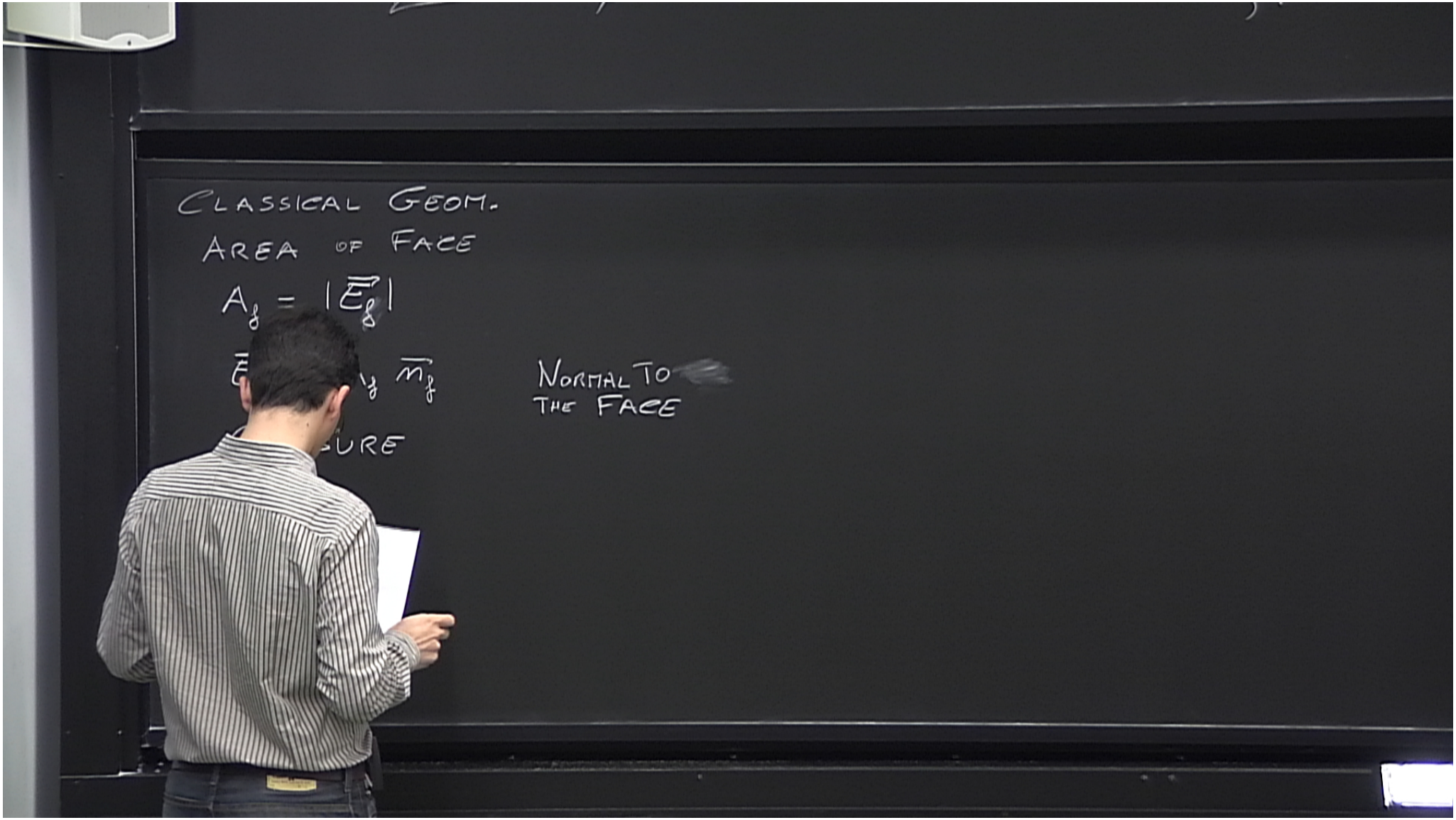
$\mathrm{Area}_e = |\vec{E}_e| \longrightarrow 8\pi G\hbar\gamma\sqrt{j_e(j_e+1)}$

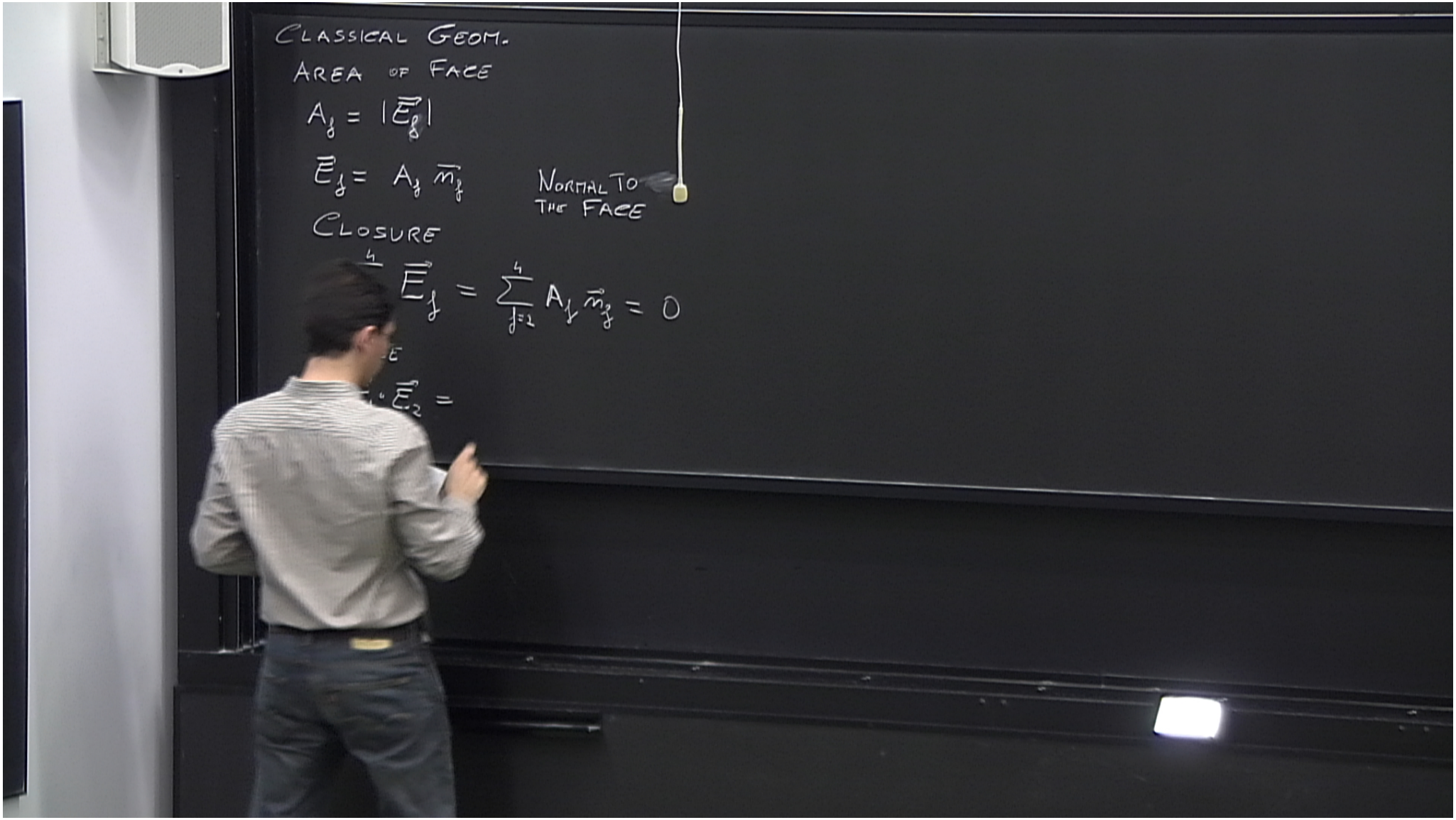
$\mathrm{Volume}_n = \frac{\sqrt{2}}{3}\sqrt{|\vec{E}_1 \cdot (\vec{E}_2 \times \vec{E}_3)|} \longrightarrow V_n = \frac{\sqrt{2}}{3}(8\pi G\hbar\gamma)^{3/2}\sqrt{|q_n|}$

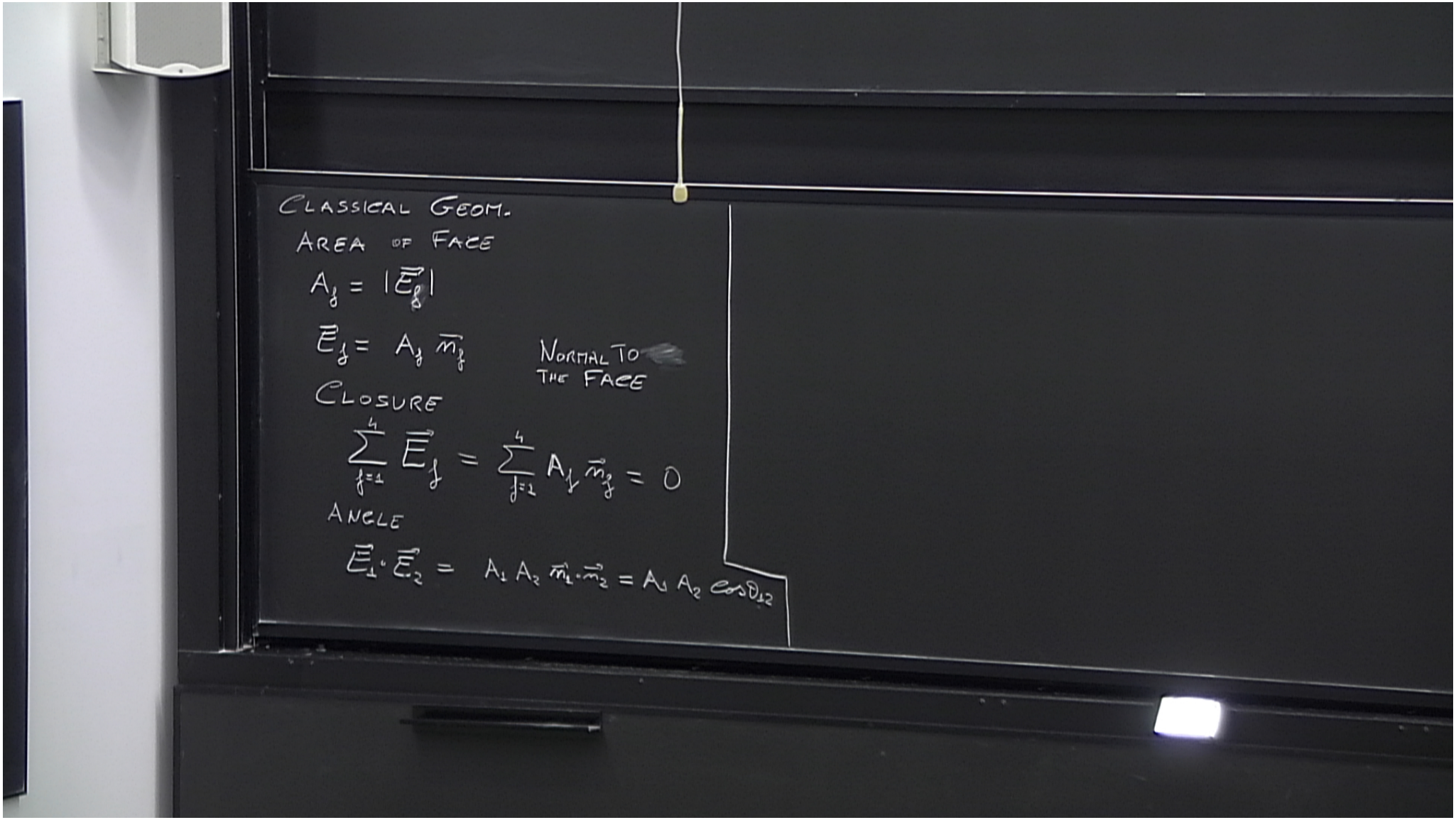

CLASSICAL GEOM. OF A TETRAHEDRON
a = 1, 2, 3
h_ab = e_a · e_b
E^3 = 1/2 e^1 × e^2
E^a · E^b

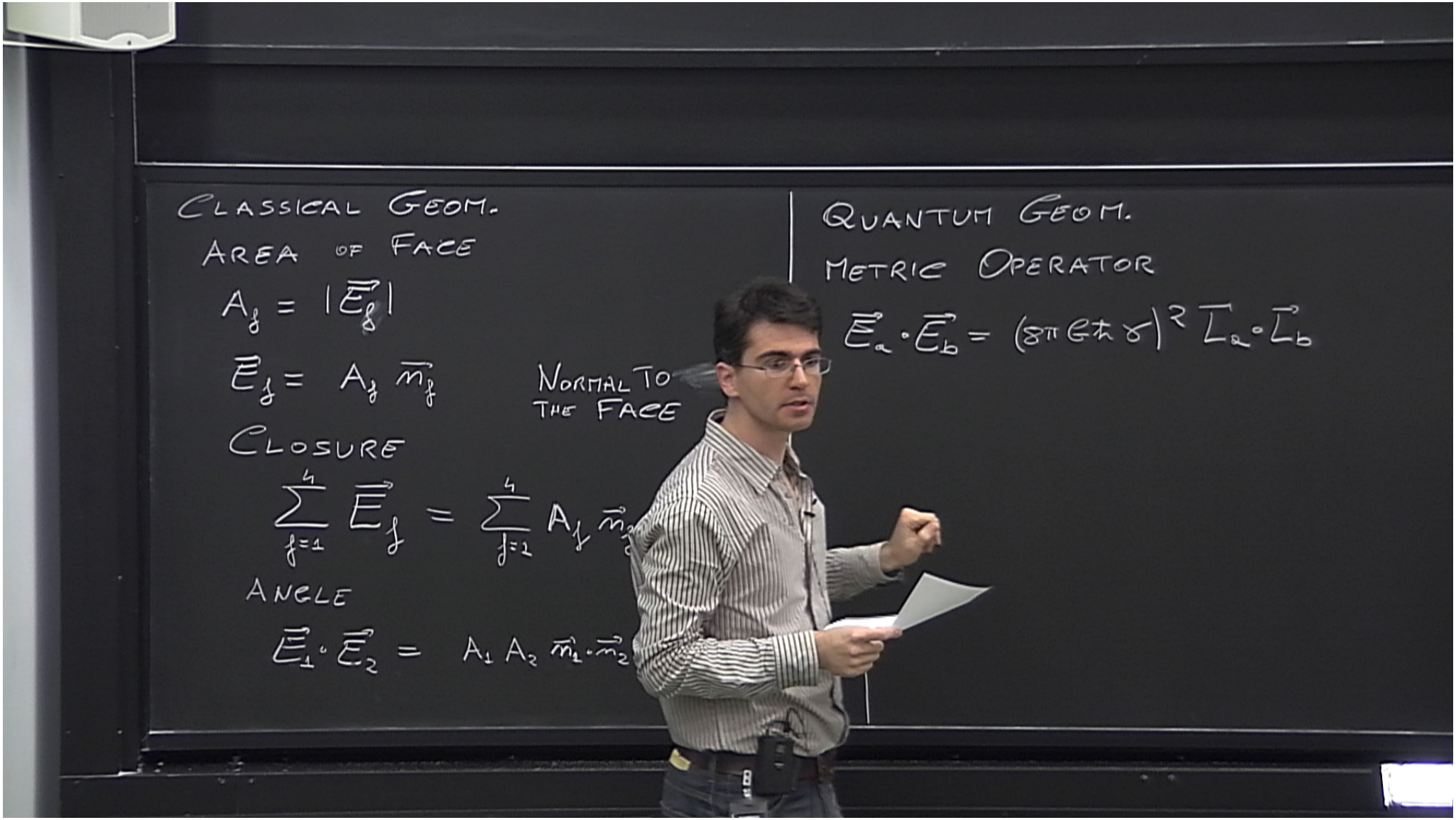

$\vec{e}_a \quad a = 1, 2, 3$
METRIC $h_{ab} = \vec{e}_a \cdot \vec{e}_b$
$\vec{E}^a = \frac{1}{2} \varepsilon^{abc} \vec{e}_b \times \vec{e}_c$
$\vec{E}^3 = \frac{1}{2} \vec{x}^1 \times \vec{x}^2$
METRIC $\vec{E}^a \cdot \vec{E}^b = (\det h) h^{ab}$
VOLUME
$V = \frac{1}{3!} |\vec{e}_1 \cdot (\vec{e}_2 \times \vec{e}_3)| = \frac{1}{3!} \sqrt{|\vec{E}_1 \cdot (\vec{E}_2 \times \vec{E}_3)|}$
$= \frac{\sqrt{2}}{3} \sqrt{|\vec{E}_1 \cdot (\vec{E}_2 \times \vec{E}_3)|}$


CLASSICAL GEOM.
AREA OF FACE
$A_\delta = |\vec{E}_\delta|$


CLASSICAL GEOM.
AREA OF FACE
$A_f = |\vec{E}_f|$
$\vec{m}_f$
NORMAL TO
THE FACE


CLASSICAL GEOM.
AREA OF FACE
$A_f = |\vec{E}_f|$
$\vec{E}_f = A_f \vec{n}_f$
NORMAL TO THE FACE
CLOSURE
$\sum_{f=2}^{4} \vec{E}_f = \sum_{f=2}^{4} A_f \vec{n}_f = 0$


CLASSICAL GEOM.
AREA OF FACE
$A_f = |\vec{E}_f|$
$\vec{E}_f = A_f \vec{n}_f$
NORMAL TO THE FACE
CLOSURE
$\sum_{f=1}^{4} \vec{E}_f = \sum_{f=2}^{4} A_f \vec{n}_f = 0$
ANGLE
$\vec{E}_1 \cdot \vec{E}_2 = A_1 A_2 \vec{n}_1 \cdot \vec{n}_2 = A_1 A_2 \cos\theta_{12}$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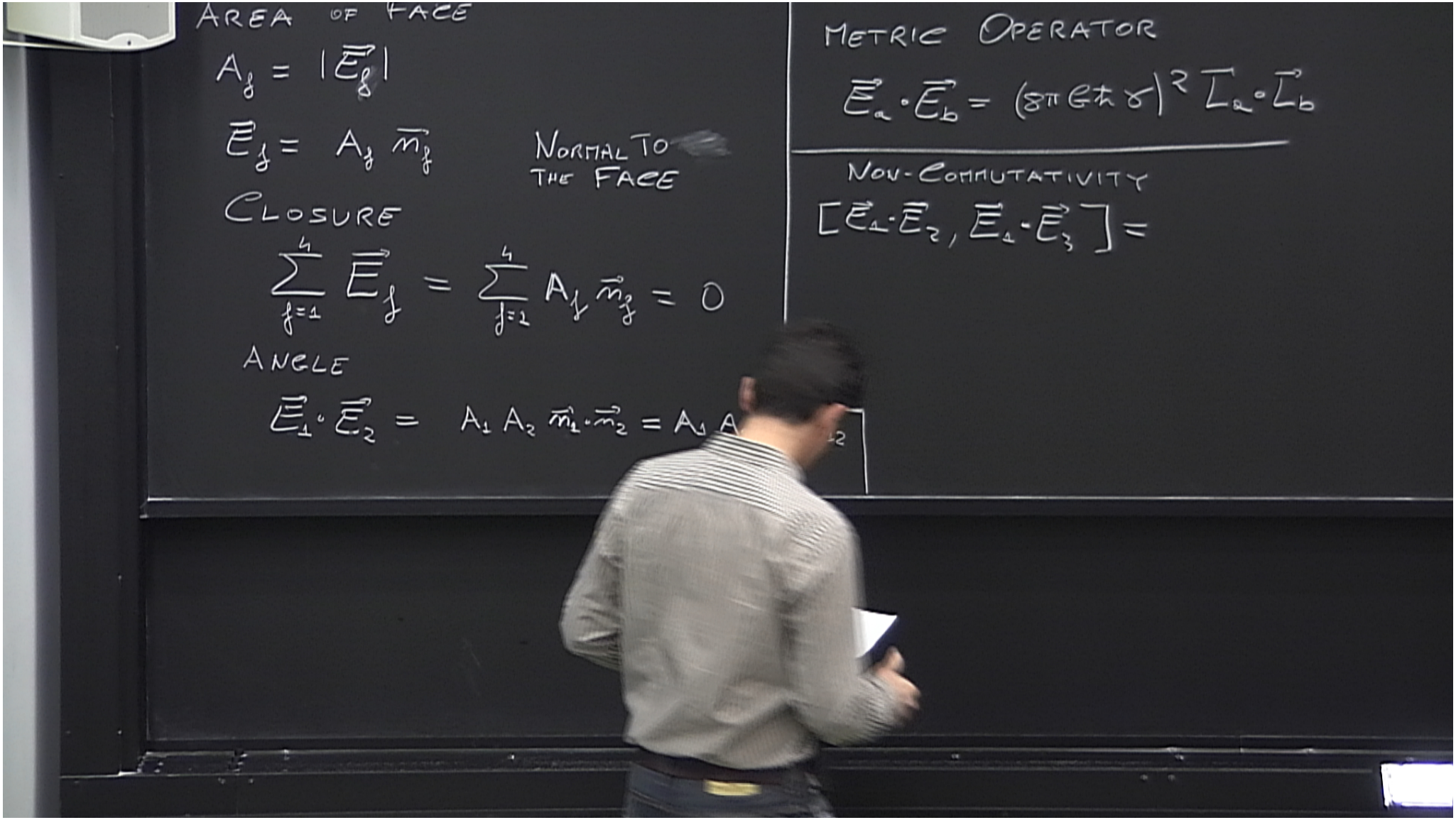
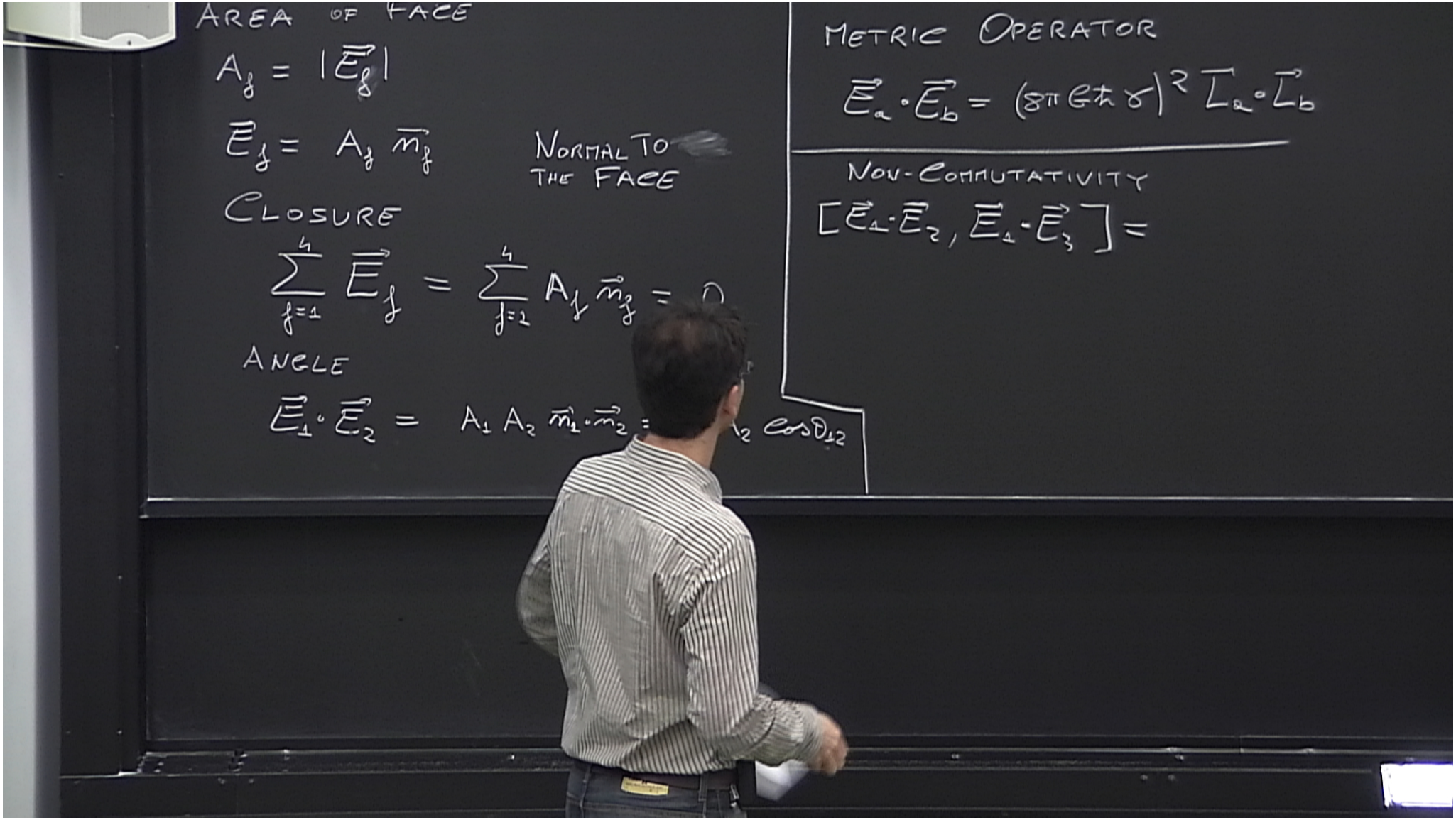
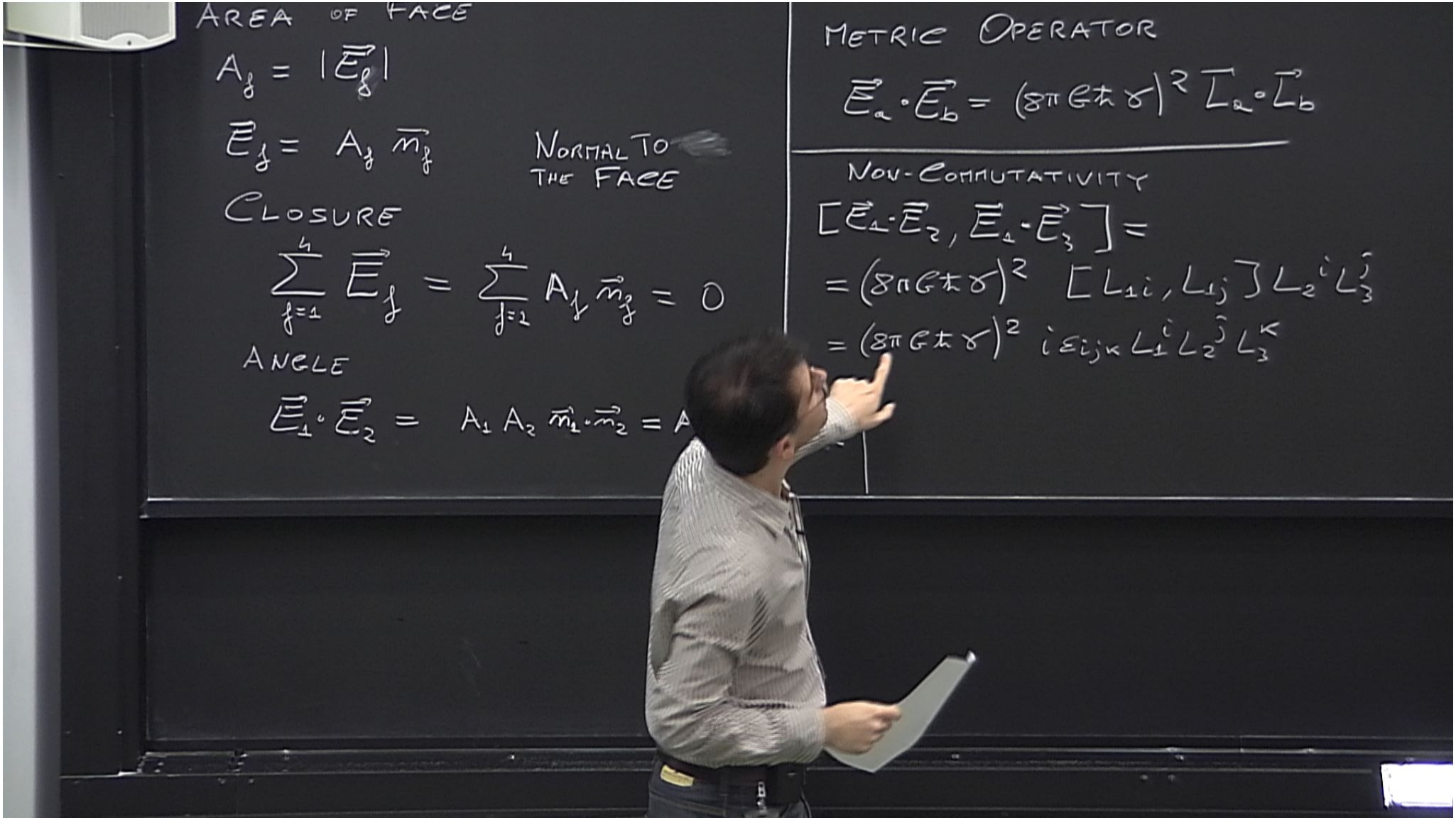
AREA OF FACE
$A_f = |\vec{E}_f|$
$\vec{E}_f = A_f \vec{n}_f$
NORMAL TO THE FACE
CLOSURE
$\sum_{f=1}^{4} \vec{E}_f = \sum_{f=2}^{4} A_f \vec{n}_f$
ANGLE
$\vec{E}_1 \cdot \vec{E}_2 = A_1 A_2 \vec{n}_1 \cdot \vec{n}_2$
QUANTUM GEOM.
METRIC OPERATOR
$\vec{E}_a \cdot \vec{E}_b = (8\pi G\hbar\gamma)^2 \vec{L}_a \cdot \vec{L}_b$


AREA OF FACE
$A_f = |\vec{E}_f|$
$\vec{E}_f = A_f \vec{n}_f$
NORMAL TO THE FACE
CLOSURE
$\sum_{f=1}^{4} \vec{E}_f = \sum_{f=1}^{4} A_f \vec{n}_f = 0$
ANGLE
$\vec{E}_1 \cdot \vec{E}_2 = A_1 A_2 \vec{n}_1 \cdot \vec{n}_2 = A_1 A$
METRIC OPERATOR
$\vec{E}_a \cdot \vec{E}_b = (8\pi G\hbar\gamma)^2 \vec{L}_a \cdot \vec{L}_b$
NON-COMMUTATIVITY
$[\vec{E}_1 \cdot \vec{E}_2, \vec{E}_1 \cdot \vec{E}_3] =$


AREA OF FACE
$A_f = |\vec{E}_f|$
$\vec{E}_f = A_f \vec{n}_f$
NORMAL TO THE FACE
CLOSURE
$\sum_{f=1}^{4} \vec{E}_f = \sum_{f=1}^{4} A_f \vec{n}_f = 0$
ANGLE
$\vec{E}_1 \cdot \vec{E}_2 = A_1 A_2 \vec{n}_1 \cdot \vec{n}_2 =$ $\cos\theta_{12}$
METRIC OPERATOR
$\vec{E}_a \cdot \vec{E}_b = (8\pi G\hbar\gamma)^2 \vec{L}_a \cdot \vec{L}_b$
NON-COMMUTATIVITY
$[\vec{E}_1 \cdot \vec{E}_2, \vec{E}_1 \cdot \vec{E}_3] =$


AREA OF FACE
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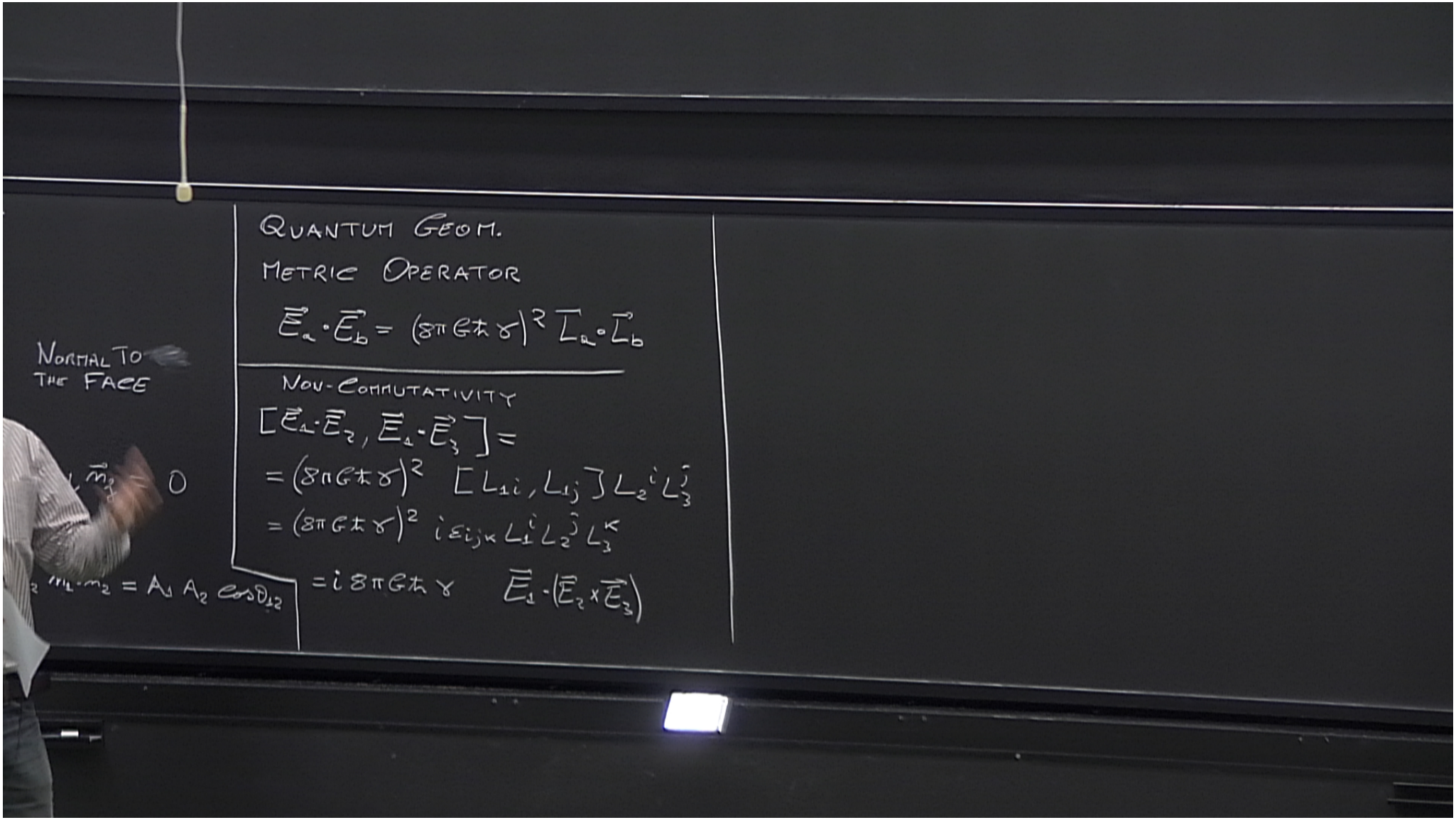
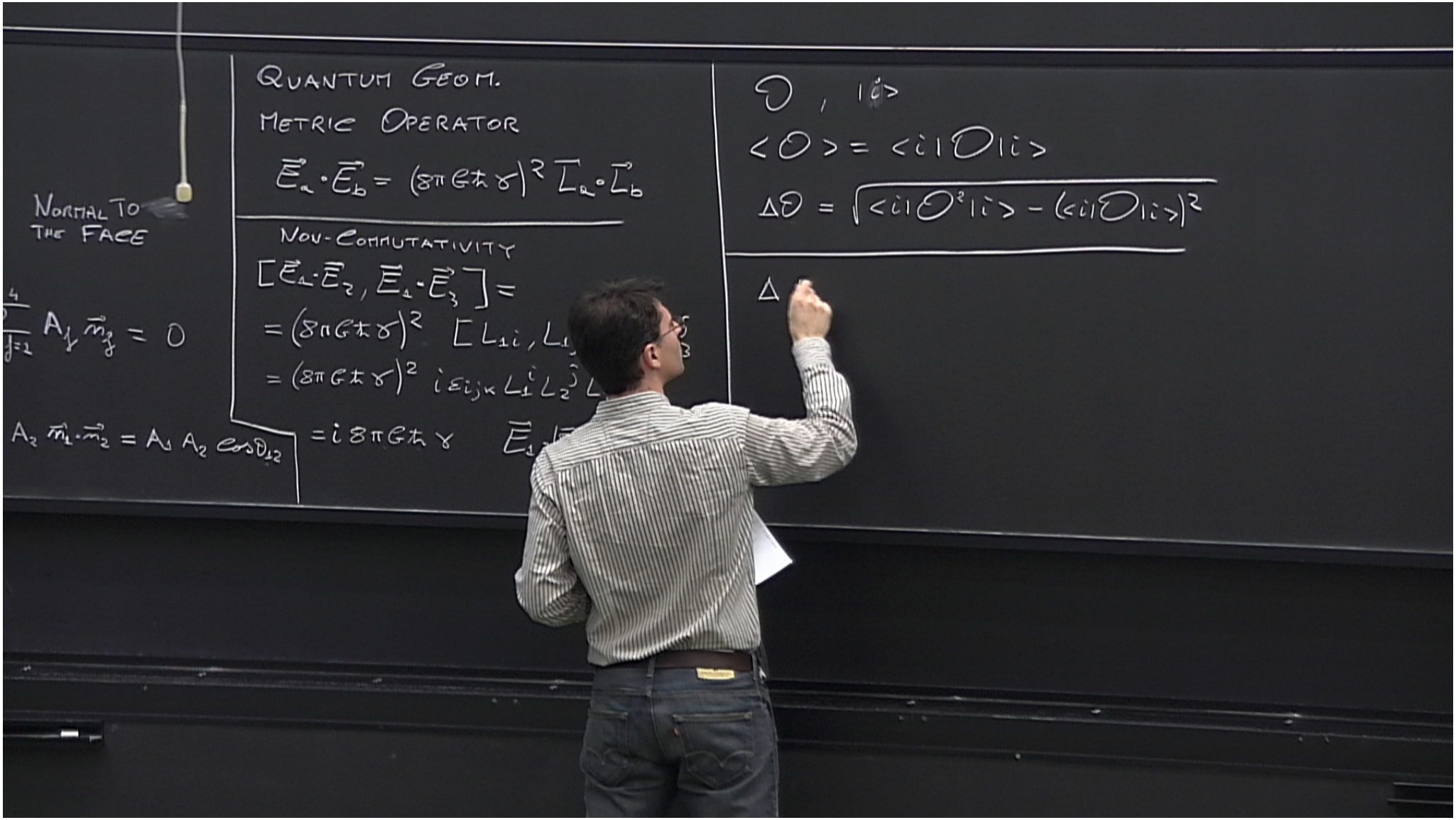
$\vec{E}_f = A_f \vec{n}_f$
NORMAL TO THE FACE
CLOSURE
$\sum_{f=1}^{4} \vec{E}_f = \sum_{f=1}^{4} A_f \vec{n}_f = 0$
ANGLE
$\vec{E}_1 \cdot \vec{E}_2 = A_1 A_2 \vec{n}_1 \cdot \vec{n}_2 = A$
METRIC OPERATOR
$\vec{E}_a \cdot \vec{E}_b = (8\pi G\hbar\gamma)^2 \vec{L}_a \cdot \vec{L}_b$
NON-COMMUTATIVITY
$[\vec{E}_1 \cdot \vec{E}_2, \vec{E}_1 \cdot \vec{E}_3] =$
$= (8\pi G\hbar\gamma)^2 [L_{1i}, L_{1j}] L_2^i L_3^j$
$= (8\pi G\hbar\gamma)^2 i\varepsilon_{ijk} L_1^i L_2^j L_3^k$


QUANTUM GEOM.
METRIC OPERATOR
$\vec{E}_a \cdot \vec{E}_b = (8\pi G\hbar\gamma)^2 \vec{L}_a \cdot \vec{L}_b$
NON-COMMUTATIVITY
$[\vec{E}_1 \cdot \vec{E}_2, \vec{E}_1 \cdot \vec{E}_3] =$
$= (8\pi G\hbar\gamma)^2 [L_{1i}, L_{1j}] L_2^i L_3^j$
$= (8\pi G\hbar\gamma)^2 i\varepsilon_{ijk} L_1^i L_2^j L_3^k$
$= i 8\pi G\hbar\gamma \quad \vec{E}_1 \cdot (\vec{E}_2 \times \vec{E}_3)$
NORMAL TO THE FACE
$\vec{n}_1 \cdot \vec{n}_2 = A_1 A_2 \cos\theta_{12}$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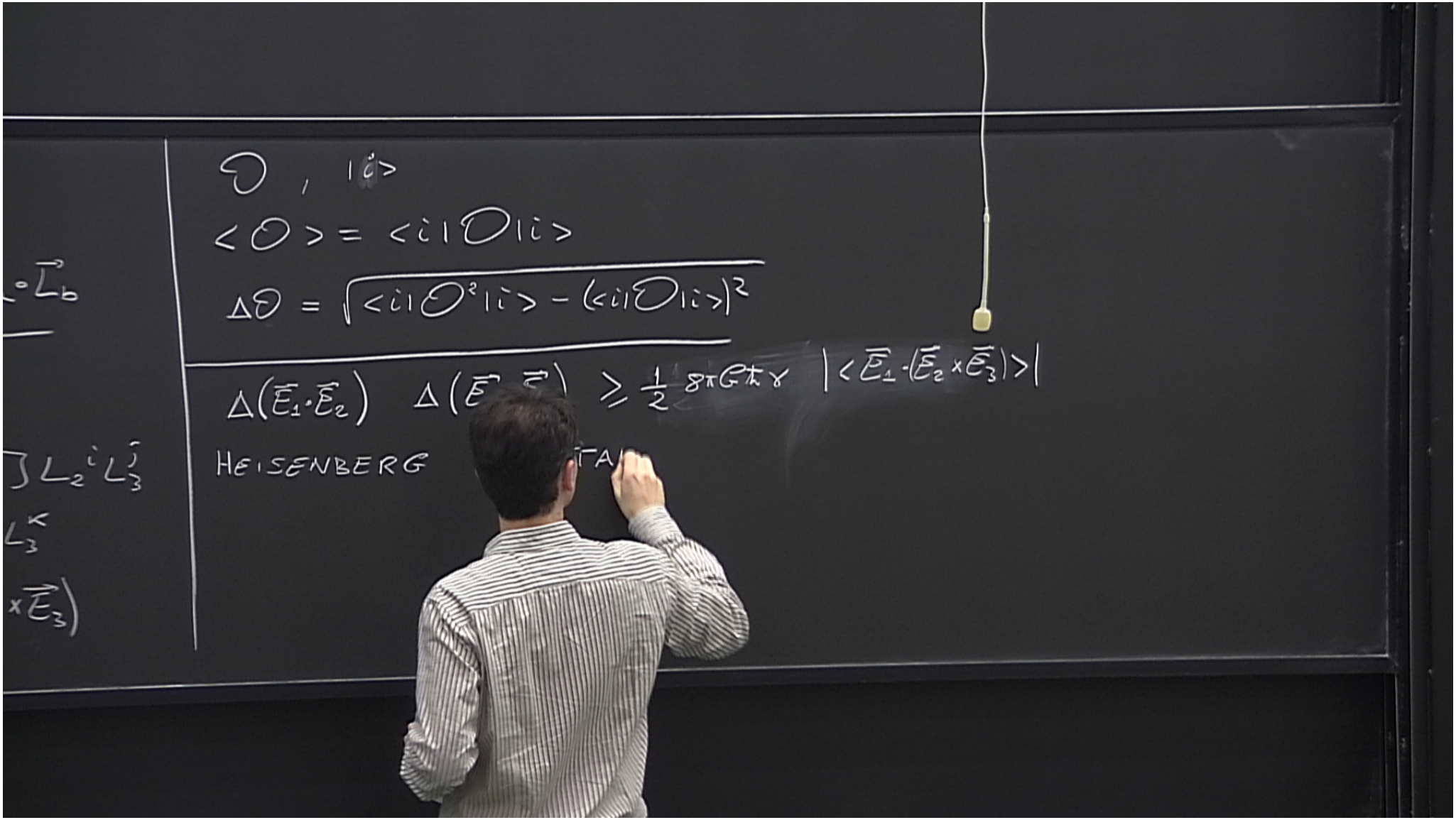
METRIC OPERATOR
$\vec{E}_a \cdot \vec{E}_b = (8\pi G\hbar\gamma)^2 \vec{L}_a \cdot \vec{L}_b$
NORMAL TO THE FACE
NON-COMMUTATIVITY
$[\vec{E}_1 \cdot \vec{E}_2, \vec{E}_1 \cdot \vec{E}_3] =$
$= (8\pi G\hbar\gamma)^2 [L_{1i}, L_1$
$= (8\pi G\hbar\gamma)^2 i\varepsilon_{ijk} L_1^i L_2^j L$
$= i 8\pi G\hbar\gamma$
$\sum_{j=1} A_j \vec{n}_j = 0$
$A_2 \vec{n}_1 \cdot \vec{n}_2 = A_1 A_2 \cos\theta_{12}$
$\mathcal{O}, |i\rangle$
$\langle \mathcal{O} \rangle = \langle i|\mathcal{O}|i\rangle$
$\Delta\mathcal{O} = \sqrt{\langle i|\mathcal{O}^2|i\rangle - (\langle i|\mathcal{O}|i\rangle)^2}$
Δ


$\mathcal{O}$, $|i\rangle$
$\langle \mathcal{O} \rangle = \langle i|\mathcal{O}|i\rangle$
$\Delta\mathcal{O} = \sqrt{\langle i|\mathcal{O}^2|i\rangle - (\langle i|\mathcal{O}|i\rangle)^2}$
$\Delta(\vec{E}_1\cdot\vec{E}_2)$ $\Delta(\vec{E}$ $\geq \frac{1}{2} 8\pi G\hbar\gamma \, |\langle \vec{E}_1\cdot(\vec{E}_2\times\vec{E}_3)\rangle|$
HEISENBERG

$\mathcal{O}$, $|i\rangle$

$\langle \mathcal{O} \rangle = \langle i | \mathcal{O} | i \rangle$

$\Delta \mathcal{O} = \sqrt{\langle i | \mathcal{O}^2 | i \rangle - (\langle i | \mathcal{O} | i \rangle)^2}$

$\Delta(\vec{E}_1 \cdot \vec{E}_2) \quad \Delta(\vec{E}_1 \cdot \vec{E}_3) \geq \frac{1}{2} 8\pi G \hbar \gamma \; |\langle \vec{E}_1 \cdot (\vec{E}_2 \times \vec{E}_3) \rangle|$

HEISENBERG UNCERTAINITY RELATIONS FOR QUANTUM GEOMETRY

$\frac{\Delta(\vec{E}_a \cdot \vec{E}_b)}{|E_a| \, |E_b|} \ll 1$

SEMICLASSICAL STATE

ANGLE

$$\vec{E}_1 \cdot \vec{E}_2 = A_1 A_2 \, \vec{n}_1 \cdot \vec{n}_2 = A_1 A_2 \cos\theta_{12}$$

$$= (8\pi G\hbar\gamma)^2 \, i\varepsilon_{ijk} L_1^i L_2^j L_3^k$$

$$= i\,8\pi G\hbar\gamma \quad \vec{E}_1 \cdot (\vec{E}_2 \times \vec{E}_3)$$

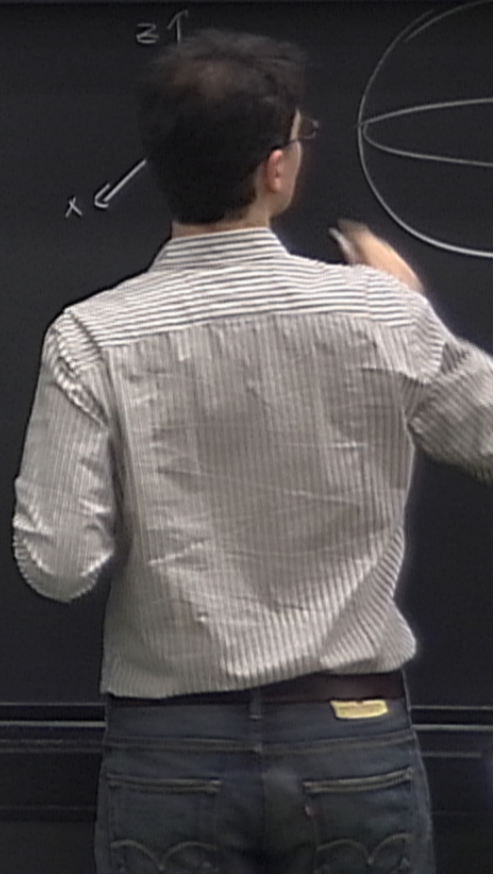
SU(2) COHERENT STATES

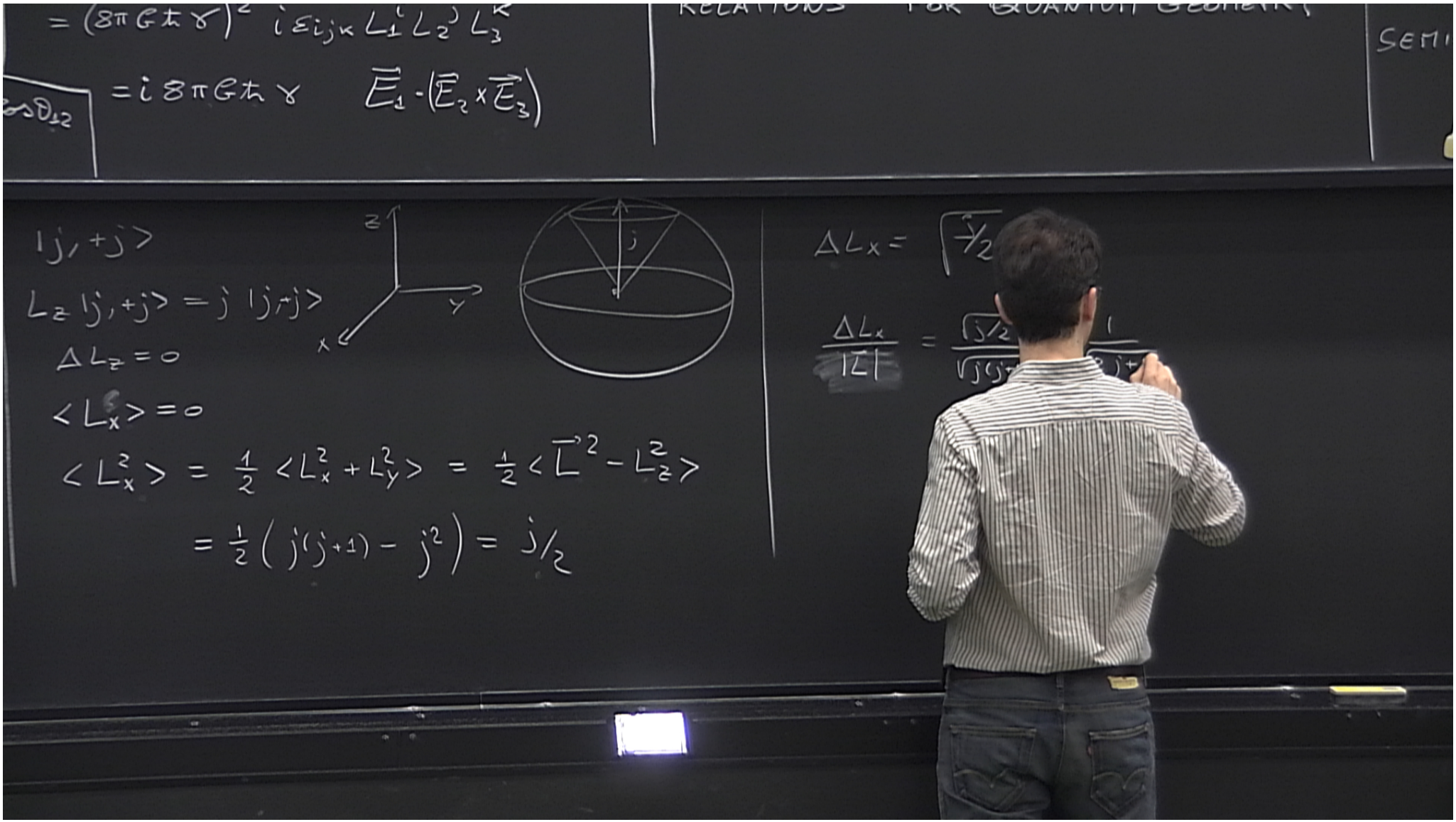
$$|j, m\rangle \in \mathcal{H}_j$$

$$[L_x, L_y] = i L_z$$

$$\Downarrow$$

$$\Delta L_x \, \Delta L_y \geq \frac{1}{2} |\langle L_z \rangle|$$




$= (8\pi G\hbar\gamma)^2 \, i\varepsilon_{ijk} L_1^i L_2^j L_3^k$
$= i\, 8\pi G\hbar\gamma \quad \vec{E}_1 \cdot (\vec{E}_2 \times \vec{E}_3)$
$|j, +j\rangle$
$L_z |j, +j\rangle = j\, |j, +j\rangle$
$\Delta L_z = 0$
$\langle L_x \rangle = 0$
$\langle L_x^2 \rangle = \frac{1}{2} \langle L_x^2 + L_y^2 \rangle = \frac{1}{2} \langle \vec{L}^2 - L_z^2 \rangle$
$= \frac{1}{2} \left(j(j+1) - j^2 \right) = j/2$
$\Delta L_x = \sqrt{j/2}$
$\frac{\Delta L_x}{|\vec{L}|} = \frac{\sqrt{j/2}}{\sqrt{j(j+1)}}$
z
y
x
j

$= (8\pi G\hbar\gamma)^2 \, i\varepsilon_{ijk} L_1^i L_2^j L_3^k$

$= i 8\pi G\hbar\gamma \quad \vec{E}_1 \cdot (\vec{E}_2 \times \vec{E}_3)$

RELATIONS FOR QUANTUM GEOMETRY

SEMI

$\hat{n}_{12}$

$|j, +j\rangle$

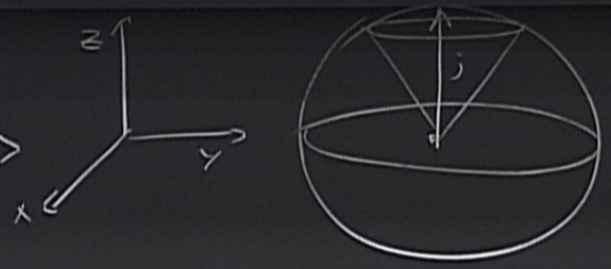
$L_z |j, +j\rangle = j \, |j, +j\rangle$

$\Delta L_z = 0$

$\langle L_x \rangle = 0$

$\langle L_x^2 \rangle = \frac{1}{2} \langle L_x^2 + L_y^2 \rangle = \frac{1}{2} \langle \vec{L}^2 - L_z^2 \rangle$

$= \frac{1}{2} \left(j(j+1) - j^2 \right) = j/2$



$\Delta L_x = \sqrt{j/2}$

$\Delta\theta = \frac{\Delta L_x}{|\vec{L}|} = \frac{\sqrt{j/2}}{\sqrt{j(j+1)}} = \frac{1}{\sqrt{2j+1}} \overset{j>>1}{\approx} \frac{1}{\sqrt{2j}}$

$\Delta L_x \, \Delta L_y = j/2$

$\frac{1}{2} |\langle L_z \rangle| = j/2$

SEMICLASSICAL STATE



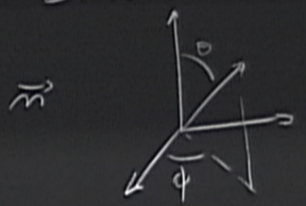
$\Delta L_x = \sqrt{j/2}$

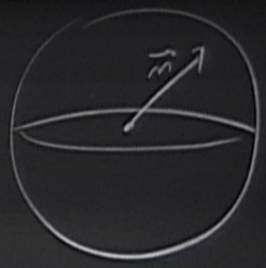
$\Delta\theta = \frac{\Delta L_x}{|\vec{L}|} = \frac{\sqrt{j/2}}{\sqrt{j(j+1)}} = \frac{1}{\sqrt{2j+1}} \overset{j>>1}{\approx} \frac{1}{\sqrt{2j}}$

$\Delta L_x \, \Delta L_y = j/2$

$\frac{1}{2}|\langle L_z \rangle| = j/2$

DIRECTION $\vec{m}$





$R^i_j \, e^j_z = m^i$

$R = e^{-i\phi L_z} e^{-i\theta L_y}$

$|j,\vec{n}\rangle = D(R)\,|j,j\rangle$

$\vec{L}\cdot\vec{n}\,|j,\vec{n}\rangle = j\,|j,\vec{n}\rangle$

$\langle j\vec{n}|\vec{L}\,|j,\vec{n}\rangle = j\vec{n}$

$|j,\vec{n}\rangle = \sum_m \phi_m(\vec{n})\,|j,m\rangle$

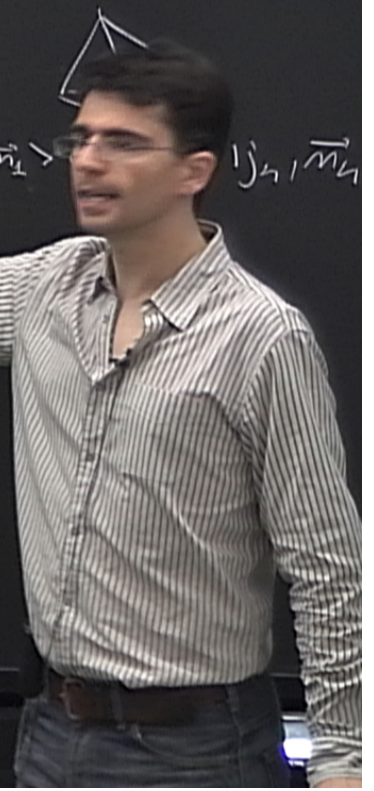
$\phi_m(\vec{n}) = \langle jm|D(R)|j,+j\rangle$

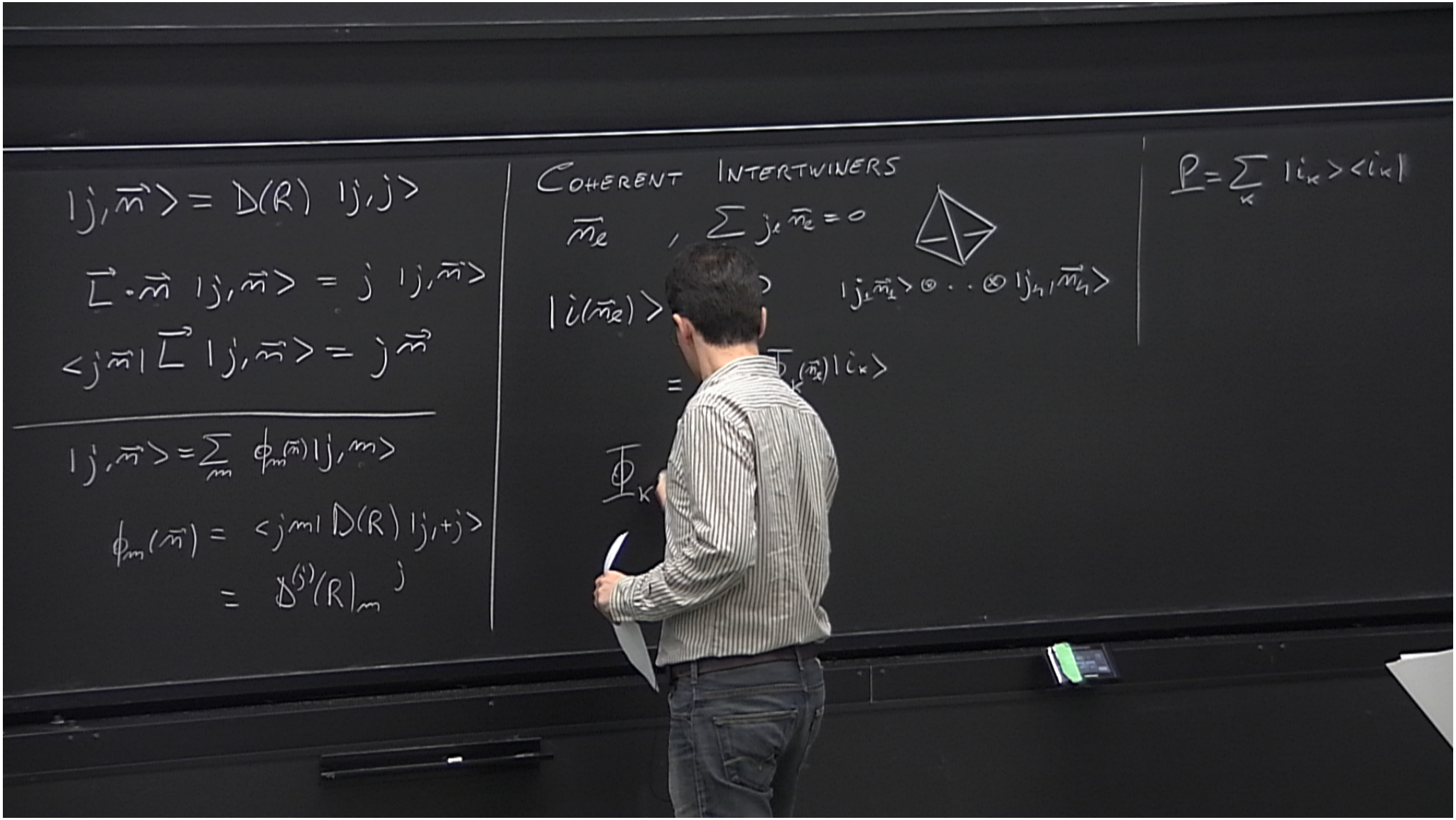
$= \mathcal{D}^{(j)}(R)_m{}^j$

COHERENT INTERTWINERS

$\vec{n}_e \;,\; \sum_e j_e\vec{n}_e = 0$

$|i(\vec{n}_e)\rangle =$ $|j_1,\vec{n}_1\rangle$ $|j_4,\vec{n}_4\rangle$




$|j,\vec{m}\rangle = D(R)\,|j,j\rangle$
$\vec{L}\cdot\vec{n}\,|j,\vec{n}\rangle = j\,|j,\vec{n}\rangle$
$\langle j\vec{m}|\vec{L}\,|j,\vec{n}\rangle = j\vec{m}$
$|j,\vec{m}\rangle = \sum_m \phi_m(\vec{n})\,|j,m\rangle$
$\phi_m(\vec{n}) = \langle jm|D(R)|j,+j\rangle$
$= \mathcal{D}^{(j)}(R)_m{}^j$
COHERENT INTERTWINERS
$\vec{n}_e$, $\sum j_e \vec{n}_e = 0$
$|i(\vec{n}_e)\rangle$
$|j_1\vec{n}_1\rangle \otimes \cdots \otimes |j_4,\vec{n}_4\rangle$
$\hat{P} = \sum_k |i_k\rangle\langle i_k|$

$|j,\vec{n}\rangle = D(R)\,|j,j\rangle$

$\vec{L}\cdot\vec{n}\,|j,\vec{n}\rangle = j\,|j,\vec{n}\rangle$

$\langle j\vec{m}|\vec{L}\,|j,\vec{n}\rangle = j\vec{m}$

$|j,\vec{n}\rangle = \sum_m \phi_m(\vec{n})\,|j,m\rangle$

$\phi_m(\vec{n}) = \langle jm|D(R)|j,+j\rangle$

$= \mathcal{D}^{(j)}(R)_m{}^j$

COHERENT INTERTWINERS

$\vec{n}_e \,,\ \sum_e j_e\vec{n}_e = 0$



$|i(\vec{n}_e)\rangle = \underline{P}\ \ |j_1\vec{n}_1\rangle\otimes\cdots\otimes|j_4,\vec{n}_4\rangle$

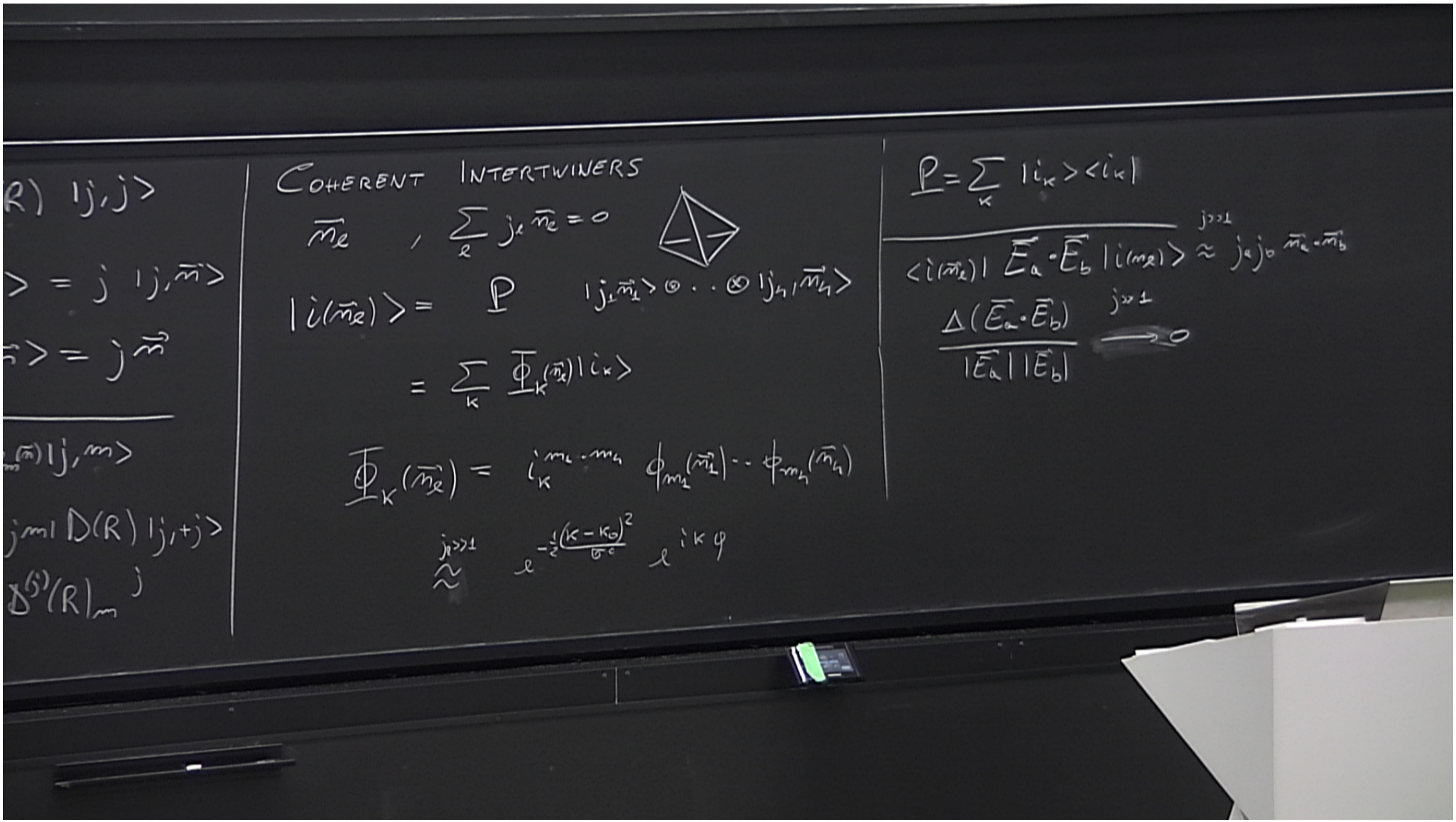
$= \sum_k \underline{\Phi}_k(\vec{n}_e)\,|i_k\rangle$

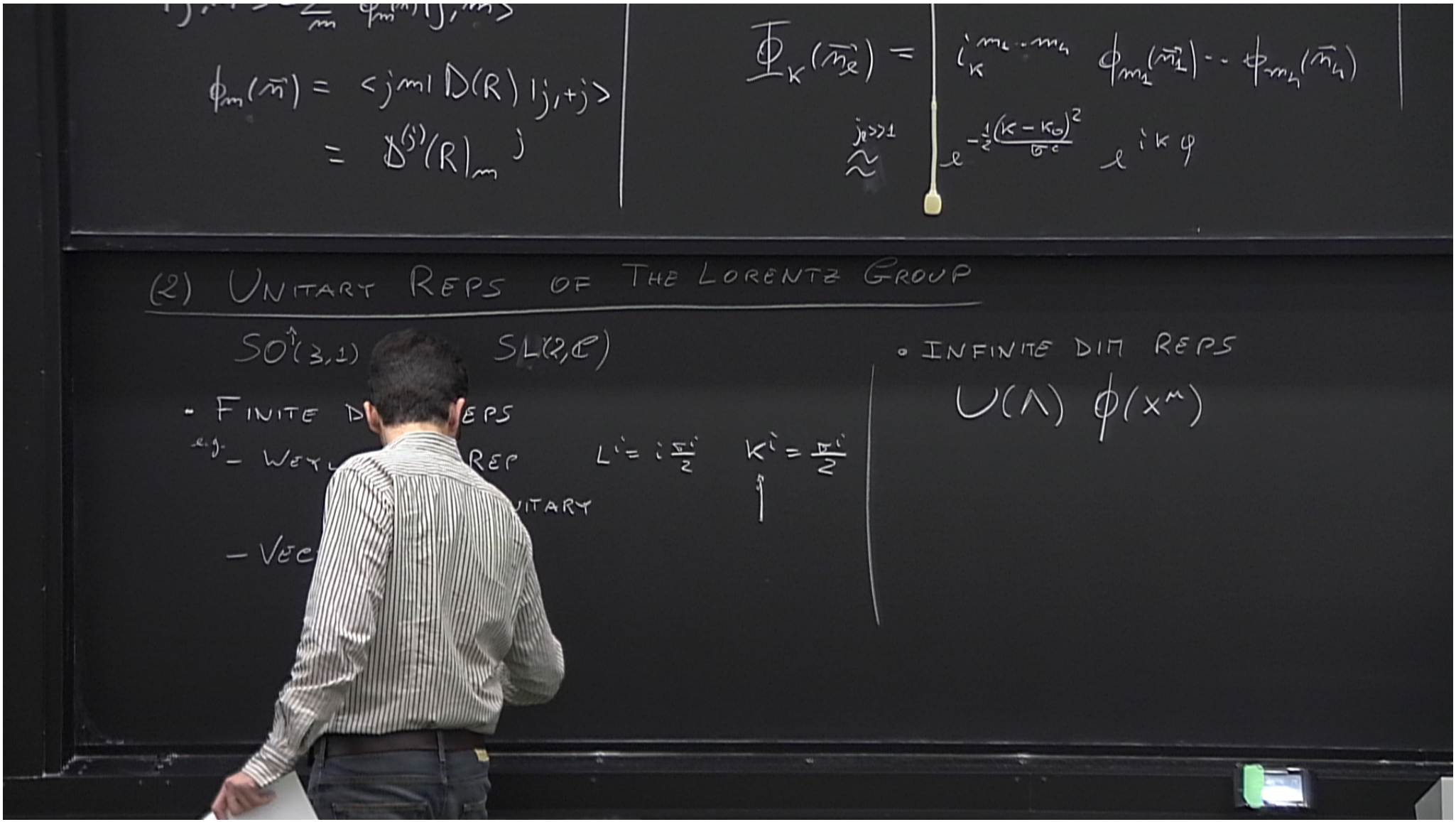
$\underline{\Phi}_k(\vec{n}_e) = i_k^{m_1\cdots m_4}\,\phi_{m_1}(\vec{n}_1)\cdots\phi_{m_4}(\vec{n}_4)$

$\overset{j\gg1}{\approx}\ e^{-\frac{1}{2}\frac{(k-k_0)^2}{\sigma^2}}\ e^{ik\varphi}$

$\underline{P} = \sum_k |i_k\rangle\langle i_k|$

$\langle i(\vec{n}_e)|\ \vec{E}_a\cdot\vec{E}_b\ |i$


COHERENT INTERTWINERS
$\vec{n}_\ell$, $\sum_\ell j_\ell \vec{n}_\ell = 0$
$|i(\vec{n}_\ell)\rangle = P \; |j_1 \vec{n}_1\rangle \otimes \cdots \otimes |j_4, \vec{n}_4\rangle$
$= \sum_k \Phi_k(\vec{n}_\ell) |i_k\rangle$
$\Phi_k(\vec{n}_\ell) = i_k^{m_1 \cdots m_4} \phi_{m_1}(\vec{n}_1) \cdots \phi_{m_4}(\vec{n}_4)$
$\overset{j_\ell \gg 1}{\approx} e^{-\frac{1}{2}\frac{(k-k_0)^2}{\sigma^2}} e^{ik\varphi}$
$P = \sum_k |i_k\rangle\langle i_k|$
$\langle i(\vec{n}_\ell)| \vec{E}_a \cdot \vec{E}_b |i(n_\ell)\rangle \overset{j \gg 1}{\approx} j_a j_b \, \vec{n}_a \cdot \vec{n}_b$
$\frac{\Delta(\vec{E}_a \cdot \vec{E}_b)}{|\vec{E}_a||\vec{E}_b|} \xrightarrow{j \gg 1} 0$


$\phi_m(\vec{n}) = \langle jm | D(R) | j,+j \rangle$
$= D^{(j)}(R)_m{}^j$
$\Phi_K(\vec{n}_\ell) = i_K^{m_1 \cdots m_4} \phi_{m_1}(\vec{n}_1) \cdots \phi_{m_4}(\vec{n}_4)$
$\overset{j_\ell \gg 1}{\approx} e^{-\frac{1}{2}\frac{(K-K_0)^2}{\sigma^2}} e^{iK\varphi}$
(2) UNITARY REPS OF THE LORENTZ GROUP
$SO^{\uparrow}(3,1)$
$SL(2,\mathbb{C})$
• FINITE D
EPS
e.g. – WEY
REP
$L^i = i\frac{\sigma^i}{2}$
$K^i = \frac{\sigma^i}{2}$
ITARY
– VEC
• INFINITE DIM REPS
$U(\Lambda)\ \phi(x^\mu)$

$$\phi_m(\vec{n}) = \langle jm | D(R) | j, +j \rangle$$
$$= D^{(j)}(R)_m{}^j$$

$$\Phi_K(\vec{n}_\ell) = i_K^{m_1 \cdots m_4} \phi_{m_1}(\vec{n}_1) \cdots \phi_{m_4}(\vec{n}_4)$$
$$\overset{j_\ell \gg 1}{\approx} e^{-\frac{1}{2}\frac{(K-K_0)^2}{\sigma^2}} e^{iK\varphi}$$

(2) Unitary Reps of the Lorentz Group

$SO^{\uparrow}(3,1)$, $SL(2,\mathbb{C})$

- Finite dim Reps

e.g. – Weyl Spinor Rep $L^i = i\frac{\sigma^i}{2}$ $K^i = \frac{\sigma^i}{2}$

Non-Unitary

– Vector

$x'^\mu = \Lambda^\mu{}_\nu x^\nu$

- Infinite dim Reps

$$U(\Lambda)\, \phi(x^\mu) = \phi(\Lambda^{-1\,\mu}{}_\nu$$

$$U(\Lambda) = e^{i\omega_{\mu\nu} J^{\mu\nu}}$$

Unitary $J^{\mu\nu} =$

$$\vec{n}_\ell) = i_k^{m_1 \cdots m_4} \phi_{m_1}(\vec{n}_1) \cdots \phi_{m_4}(\vec{n}_4)$$

$$\overset{j_\ell \gg 1}{\approx} e^{-\frac{1}{2}\frac{(k-k_0)^2}{\sigma^2}} e^{ik\varphi}$$

GROUP

- INFINITE DIM REPS

$$U(\Lambda)\,\phi(x^\mu) = \phi(\Lambda^{-1\,\mu}{}_\nu x^\nu)$$

$$U(\Lambda) = e^{i\omega_{\mu\nu} J^{\mu\nu}}$$

$$J^{\mu\nu} = -i\left(x^\mu \partial^\nu - \partial^\nu x^\mu\right)$$

UNITARY

BUT REDUCIBLE

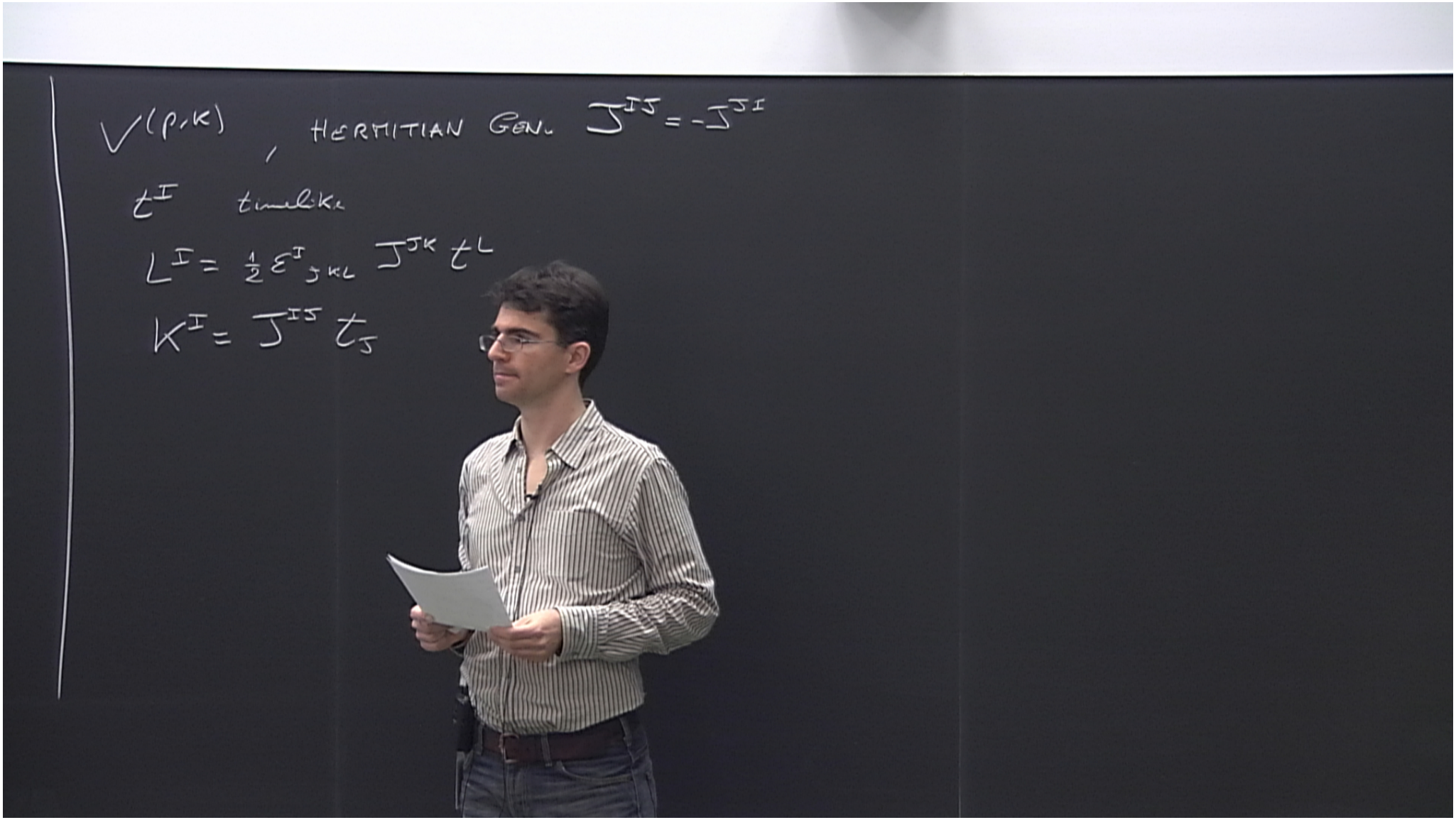
UNITARY IRREDUCIBLE REPS OF $SL(2,\mathbb{C})$

RUHL 1970

$V^{(p,k)}$ $\quad \dim V^{(p,k)} = +\infty$

$p \in \mathbb{R}$

$k \in \mathbb{N}/2 = 0, \frac{1}{2}, 1, \ldots$


V(p,k) , HERMITIAN GEN. $J^{IJ} = -J^{JI}$
t^I timelike
$L^I = \frac{1}{2}\varepsilon^I{}_{JKL} J^{JK} t^L$
$K^I = J^{IJ} t_J$

$V(p,k)$, HERMITIAN GEN. $J^{IJ} = -J^{JI}$

t^I timelike $\qquad t^I = (1,0,0,0)$

$L^I = \frac{1}{2}\varepsilon^I{}_{JKL} J^{JK} t^L = (0, L^i)$

$K^I = J^{IJ} t_J = (0, K^i)$

$[L^i, L^j] = i\,\varepsilon^{ij}{}_k L^k$

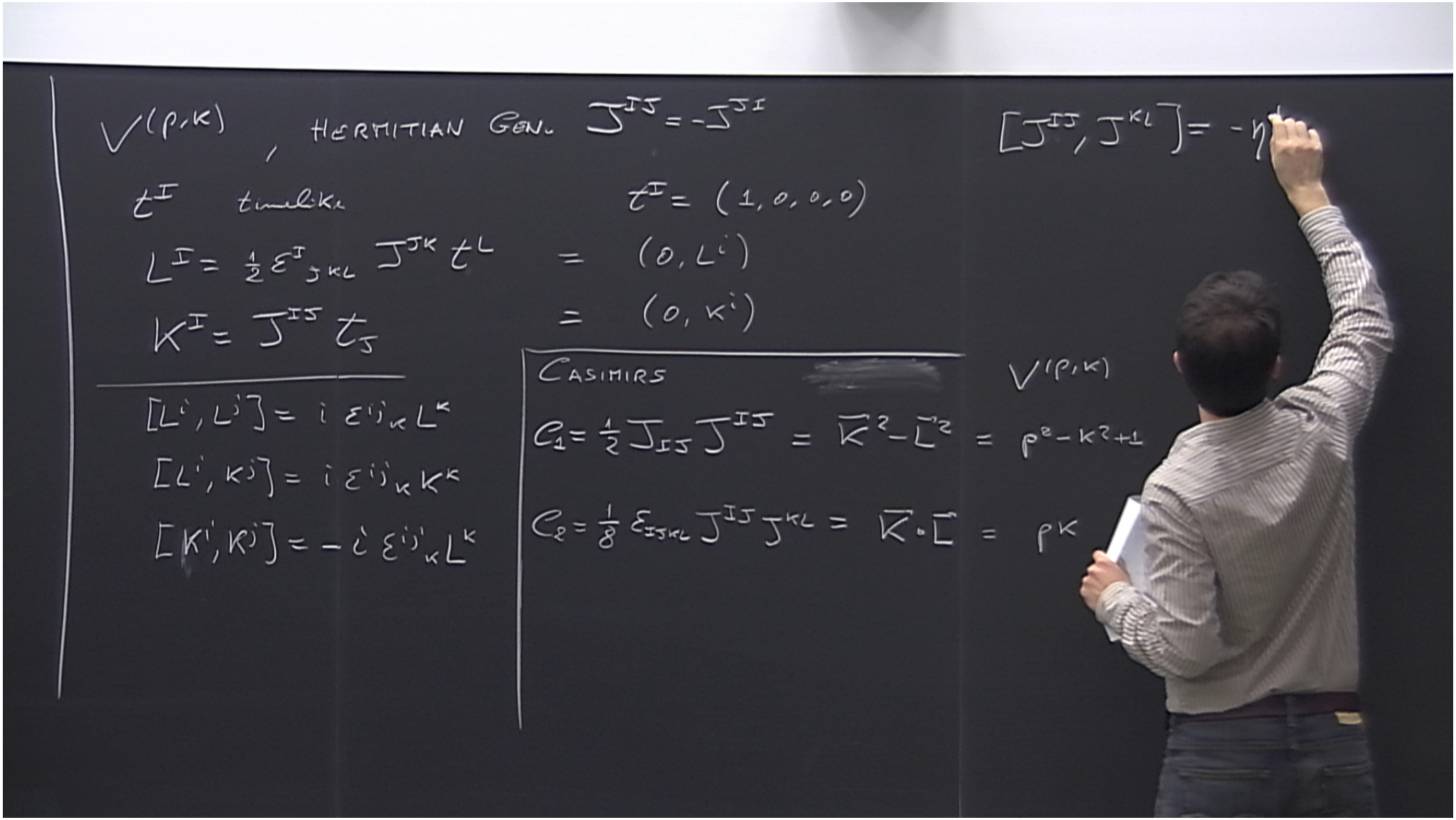
$[L^i, K^j] = i\,\varepsilon^{ij}{}_k K^k$

$[K^i, K^j] = -i\,\varepsilon^{ij}{}_k L^k$

CASIMIRS $\qquad V(p,k)$

$C_1 = \frac{1}{2} J_{IJ} J^{IJ} = \vec{K}^2 - \vec{L}^2 = p^2 - k^2 + 1$

$C_2 = \frac{1}{8} \varepsilon_{IJKL} J^{IJ} J^{KL} = \vec{K} \cdot \vec{L} = pk$


V(P,K), HERMITIAN GEN. $J^{IJ} = -J^{JI}$
$[J^{IJ}, J^{KL}] = -\eta$
t^I timelike
$t^I = (1,0,0,0)$
$L^I = \frac{1}{2}\varepsilon^I{}_{JKL} J^{JK} t^L = (0, L^i)$
$K^I = J^{IJ} t_J = (0, K^i)$
$[L^i, L^j] = i\varepsilon^{ij}{}_k L^k$
$[L^i, K^j] = i\varepsilon^{ij}{}_k K^k$
$[K^i, K^j] = -i\varepsilon^{ij}{}_k L^k$
CASIMIRS
V(P,K)
$C_1 = \frac{1}{2} J_{IJ} J^{IJ} = \vec{K}^2 - \vec{L}^2 = P^2 - K^2 + 1$
$C_2 = \frac{1}{8} \varepsilon_{IJKL} J^{IJ} J^{KL} = \vec{K} \cdot \vec{L} = PK$

MITIAN GEN. $J^{IJ} = -J^{JI}$

$$[J^{IJ}, J^{KL}] = -\eta^{IK} J^{JL} + \eta^{IL} J^{JK} + \eta^{JK} J^{IL} - \eta^{JL} J^{IK}$$

$t^I = (1,0,0,0)$

$J^{jk} t^L = (0, L^i)$

$= (0, K^i)$

$\varepsilon^{ij}{}_k L^k$

$\varepsilon^{ij}{}_k K^k$

$\varepsilon^{ij}{}_k L^k$

CASIMIRS

$C_1 = \frac{1}{2} J_{IJ} J^{IJ} = \vec{K}^2 - \vec{L}^2 = p^2 - k^2 + 1$

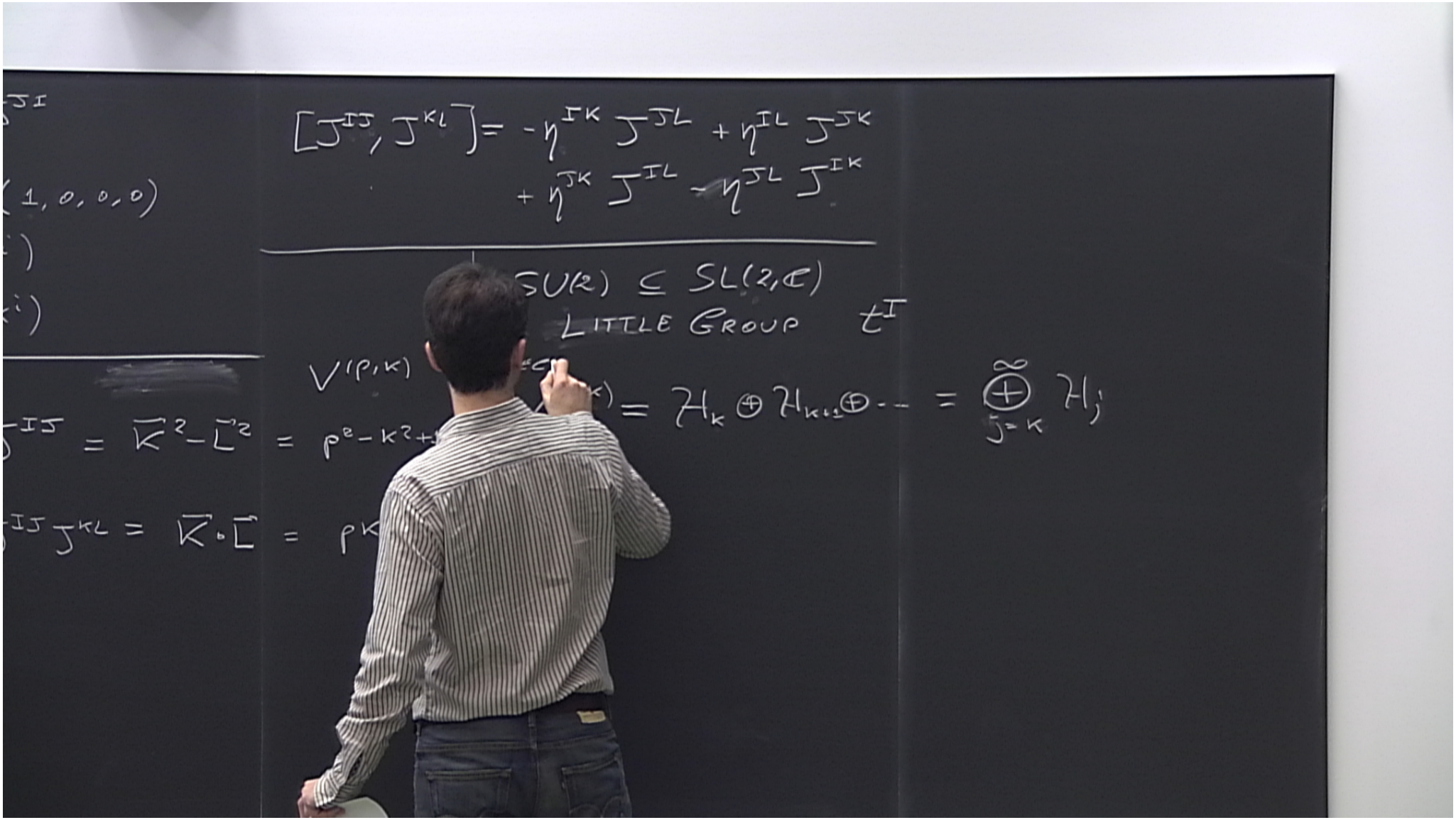
$C_2 = \frac{1}{8} \varepsilon_{IJKL} J^{IJ} J^{KL} = \vec{K} \cdot \vec{L} = pk$

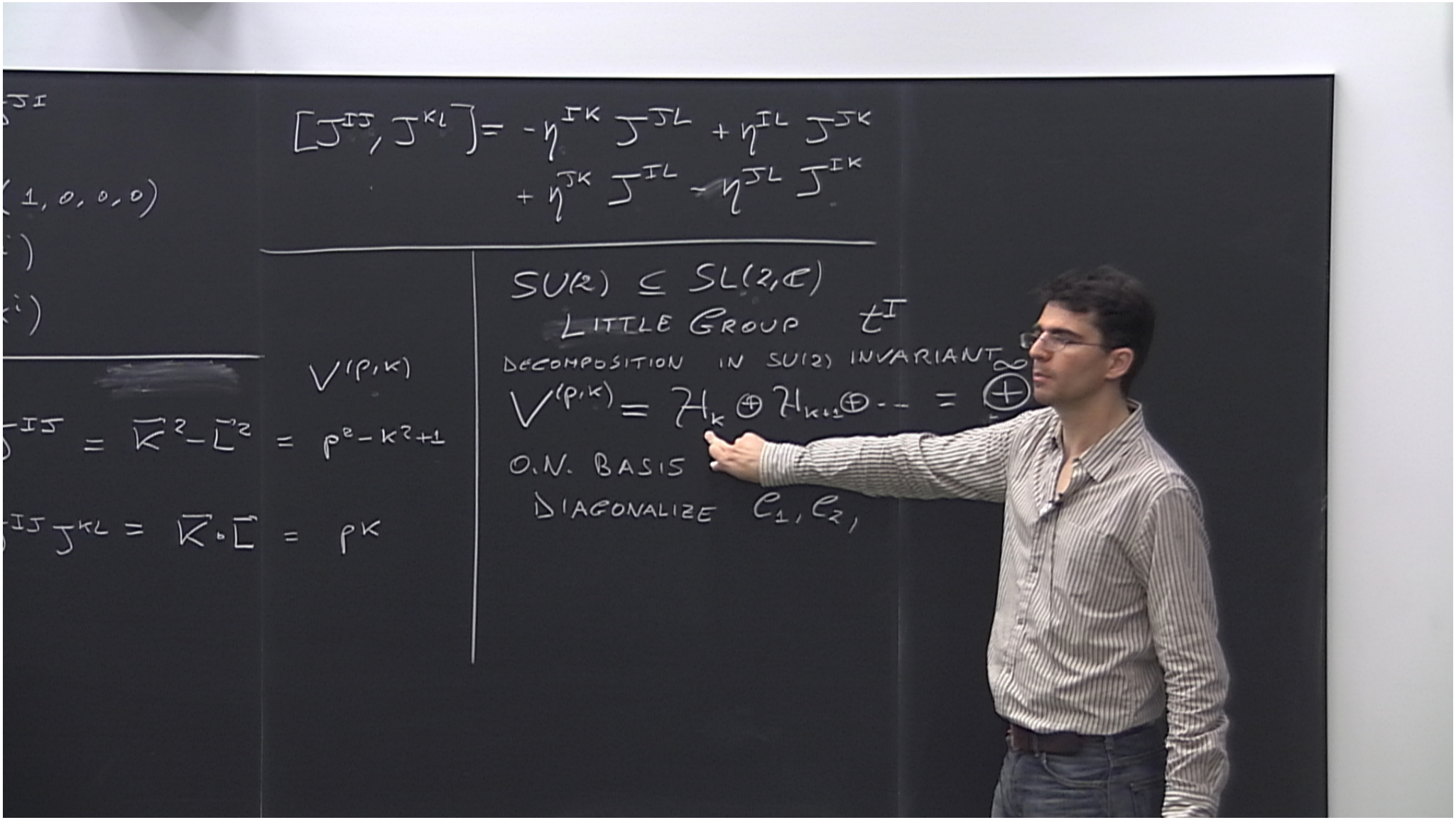
$V^{(p,k)}$

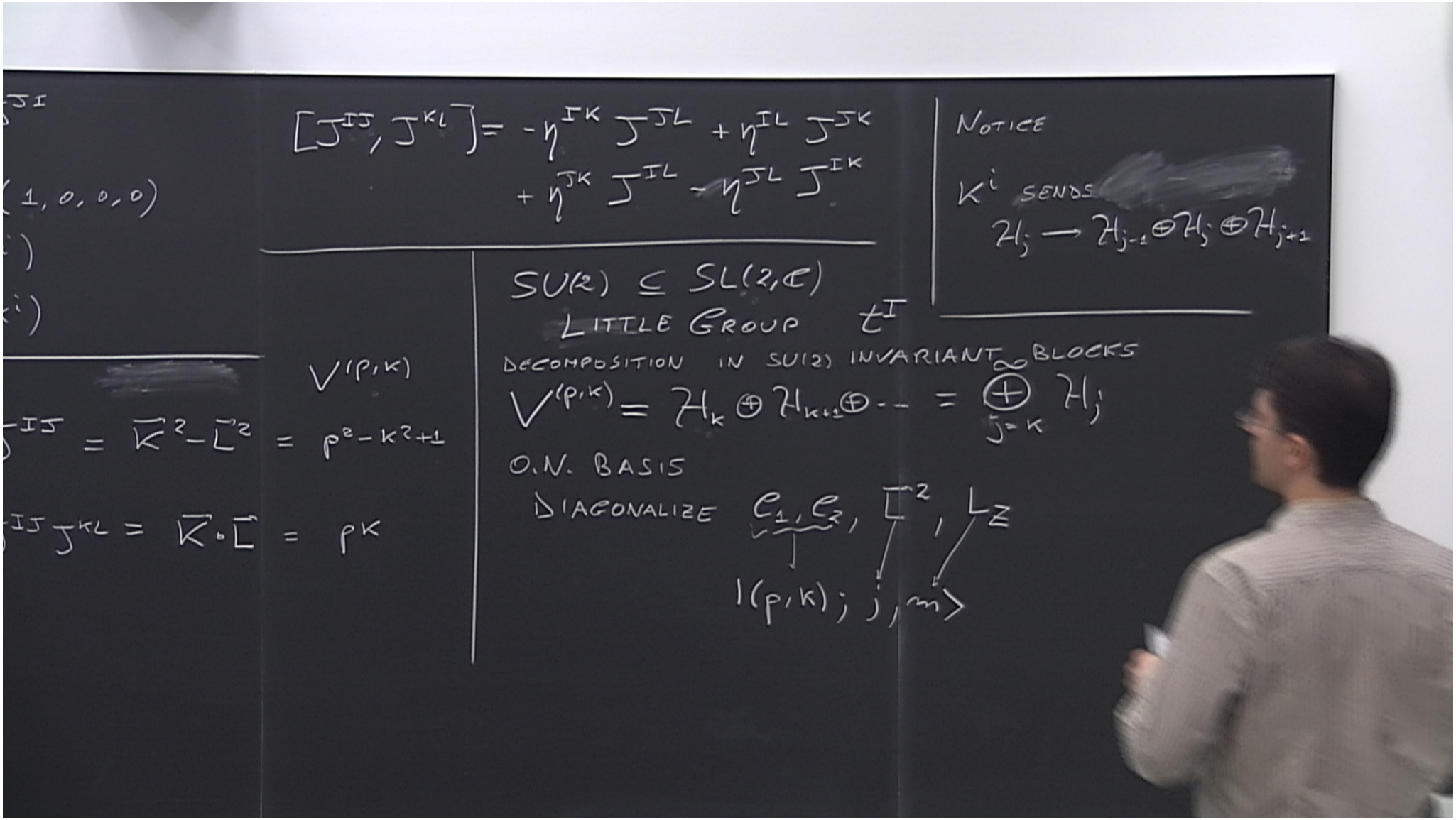
$SU(2) \subset SL(2,\mathbb{C})$

LITT[illegible] GROUP t^I

$V^{(p,k)} =$


$[J^{IJ}, J^{KL}] = -\eta^{IK} J^{JL} + \eta^{IL} J^{JK} + \eta^{JK} J^{IL} - \eta^{JL} J^{IK}$
$SU(2) \subseteq SL(2,\mathbb{C})$
LITTLE GROUP t^I
$V(p,k)$
$\mathcal{H}_k \oplus \mathcal{H}_{k+1} \oplus \cdots = \bigoplus_{j=k}^{\infty} \mathcal{H}_j$
$J^{IJ} = \vec{K}^2 - \vec{L}^2 = p^2 - k^2 +$
$J^{IJ} J^{KL} = \vec{K} \cdot \vec{L} = pK$


$[J^{IJ}, J^{KL}] = -\eta^{IK} J^{JL} + \eta^{IL} J^{JK} + \eta^{JK} J^{IL} - \eta^{JL} J^{IK}$
$V^{(p,k)}$
$J^{IJ} = \vec{K}^2 - \vec{L}^2 = p^2 - k^2 + 1$
$J^{IJ} J^{KL} = \vec{K} \cdot \vec{L} = pk$
$SU(2) \subset SL(2,\mathbb{C})$
LITTLE GROUP t^I
DECOMPOSITION IN SU(2) INVARIANT
$V^{(p,k)} = \mathcal{H}_k \oplus \mathcal{H}_{k+1} \oplus \cdots = \bigoplus_{\infty}$
O.N. BASIS
DIAGONALIZE $C_1, C_2,$


$[J^{IJ}, J^{KL}] = -\eta^{IK} J^{JL} + \eta^{IL} J^{JK} + \eta^{JK} J^{IL} - \eta^{JL} J^{IK}$
NOTICE
K^i SENDS
$\mathcal{H}_j \longrightarrow \mathcal{H}_{j-1} \oplus \mathcal{H}_j \oplus \mathcal{H}_{j+1}$
$SU(2) \subseteq SL(2,\mathbb{C})$
LITTLE GROUP t^I
$V^{(\rho,k)}$
$J^{IJ} = \vec{K}^2 - \vec{L}^2 = \rho^2 - k^2 + 1$
$J^{IJ} J^{KL} = \vec{K} \cdot \vec{L} = \rho k$
DECOMPOSITION IN SU(2) INVARIANT BLOCKS
$V^{(\rho,k)} = \mathcal{H}_k \oplus \mathcal{H}_{k+1} \oplus \cdots = \bigoplus_{j=k}^{\infty} \mathcal{H}_j$
O.N. BASIS
DIAGONALIZE $C_1, C_2, \vec{L}^2, L_z$
$|(\rho,k); j, m\rangle$

J^{JI}

$(1, 0, 0, 0)$

$)$

$^{i})$

$$[J^{IJ}, J^{KL}] = -\eta^{IK} J^{JL} + \eta^{IL} J^{JK} + \eta^{JK} J^{IL} - \eta^{JL} J^{IK}$$

NOTICE

K^i SENDS

$$\mathcal{H}_j \longrightarrow \mathcal{H}_{j-1} \oplus \mathcal{H}_j \oplus \mathcal{H}_{j+1}$$

$V^{(p,k)}$

$$J^{IJ} = \vec{K}^2 - \vec{L}^2 = p^2 - k^2 + 1$$

$$J^{IJ} J^{KL} = \vec{K} \cdot \vec{L} = pk$$

$SU(2) \subseteq SL(2,\mathbb{C})$

LITTLE GROUP t^I

DECOMPOSITION IN SU(2) INVARIANT BLOCKS

$$V^{(p,k)} = \mathcal{H}_k \oplus \mathcal{H}_{k+1} \oplus \cdots = \bigoplus_{j=k}^{\infty} \mathcal{H}_j$$

O.N. BASIS

DIAGONALIZE $\underbrace{C_1, C_2}_{|(p,k);\, j,\, m\rangle}, \vec{L}^2, L_z$

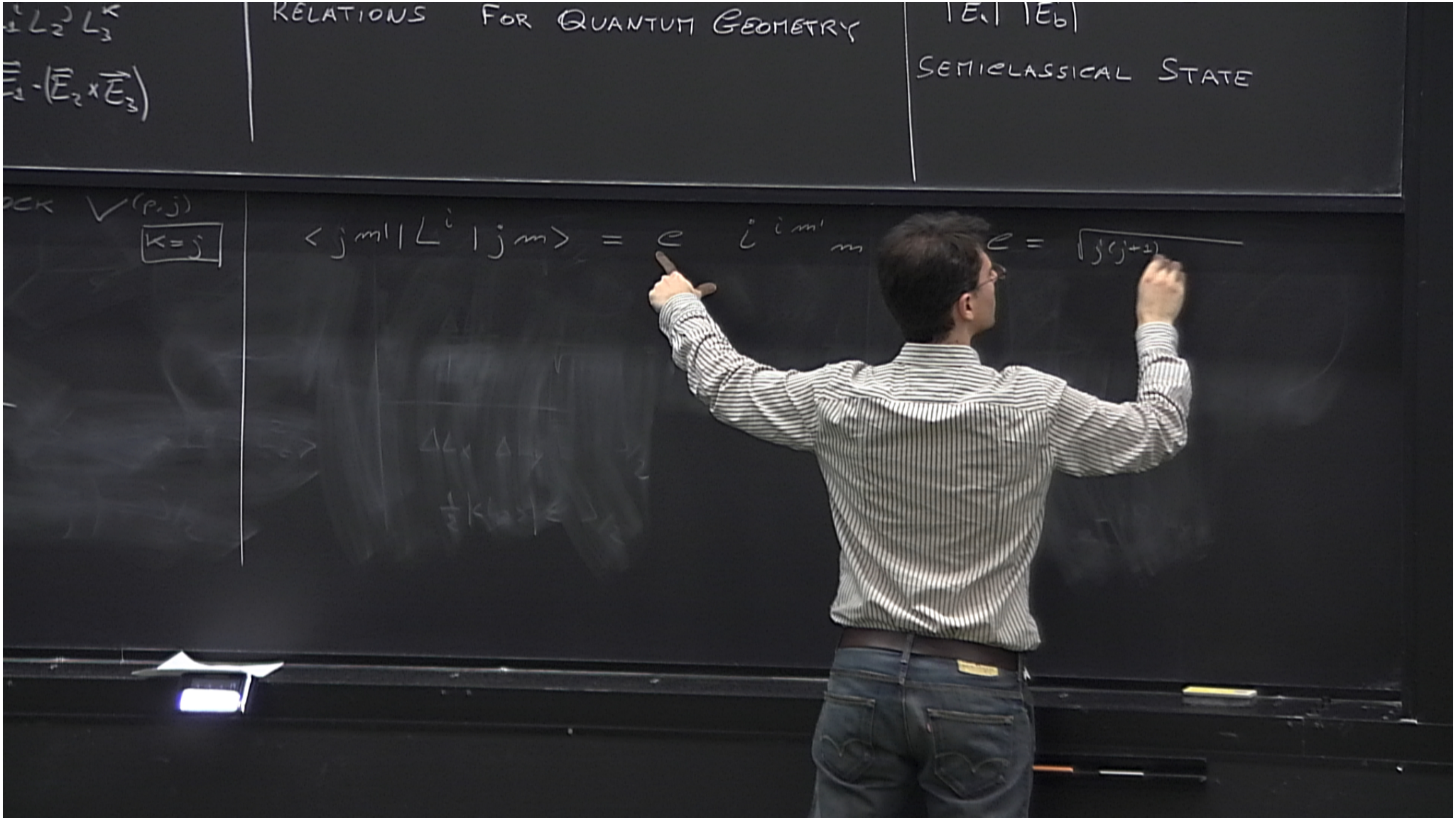
$\mathcal{H}_j$

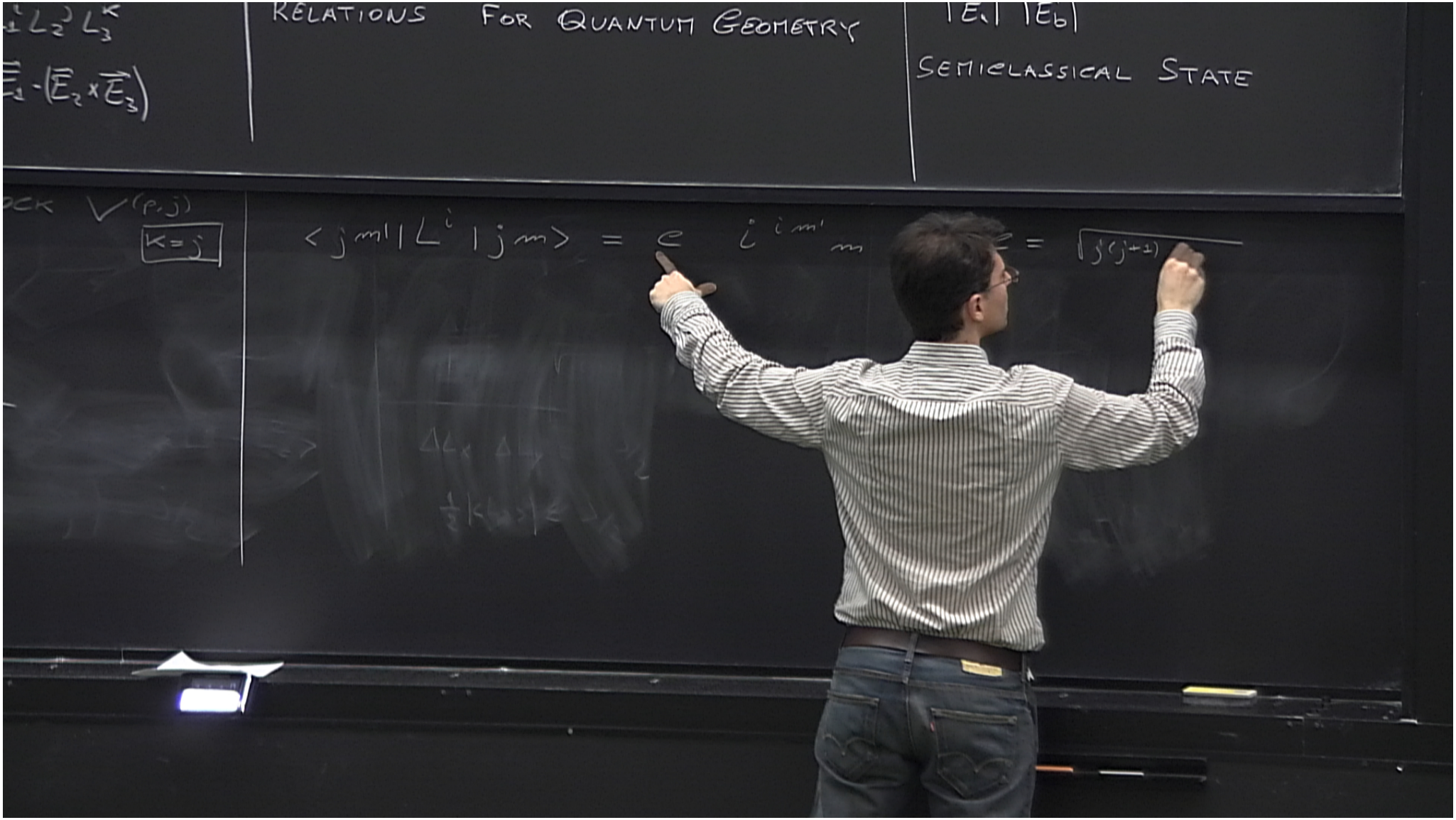
$E_1 \cdot E_2 = A_1 A_2 \vec{n}_1 \cdot \vec{n}_2 = A_1 A_2 \cos\theta_{12}$ $= i 8\pi G \hbar \gamma$ $E_1 \cdot (E_2 \times E_3)$

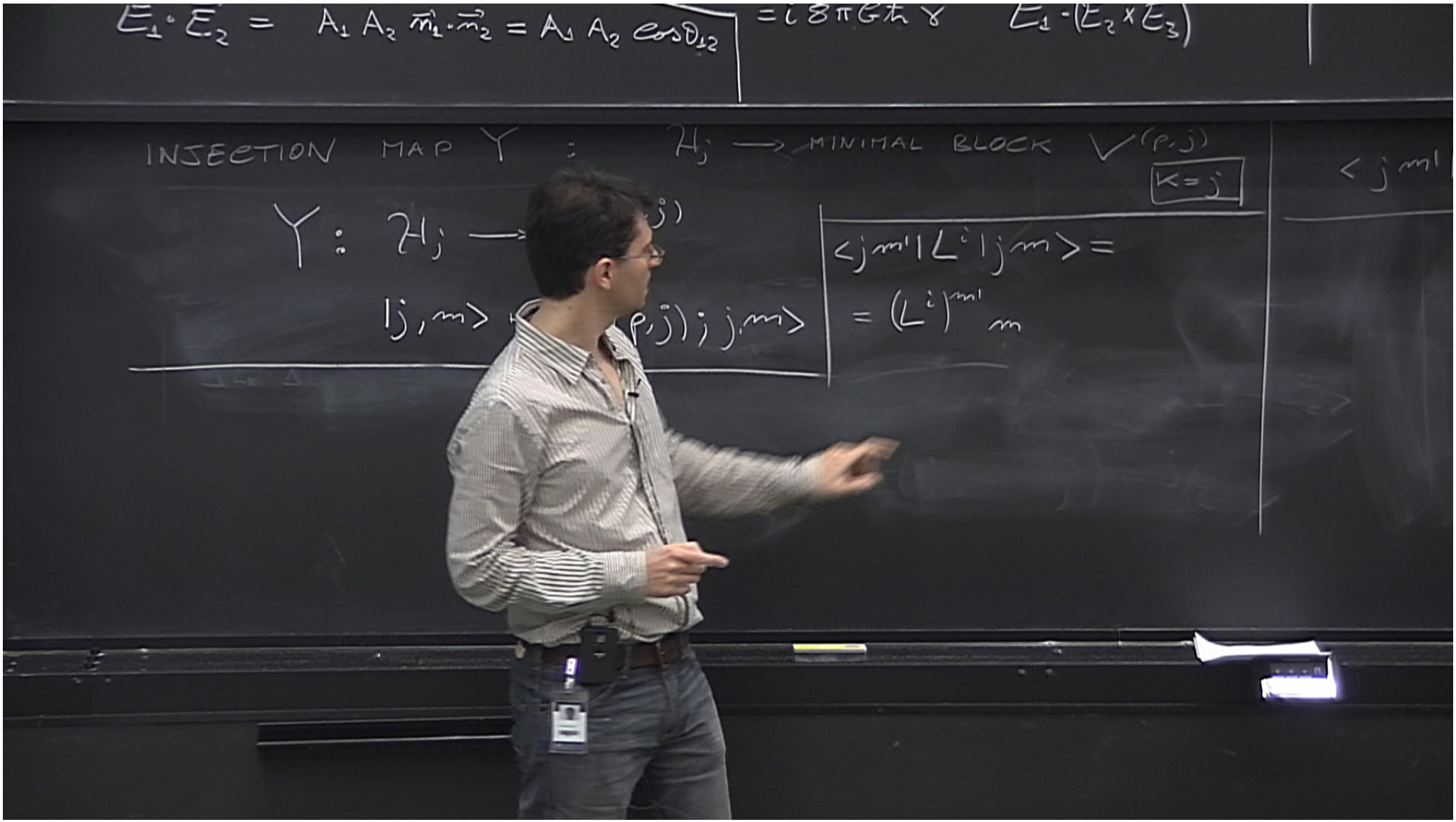
INJECTION MAP Y : $\mathcal{H}_j \longrightarrow$ MINIMAL BLOCK $V^{(\rho,j)}$ $\boxed{k=j}$

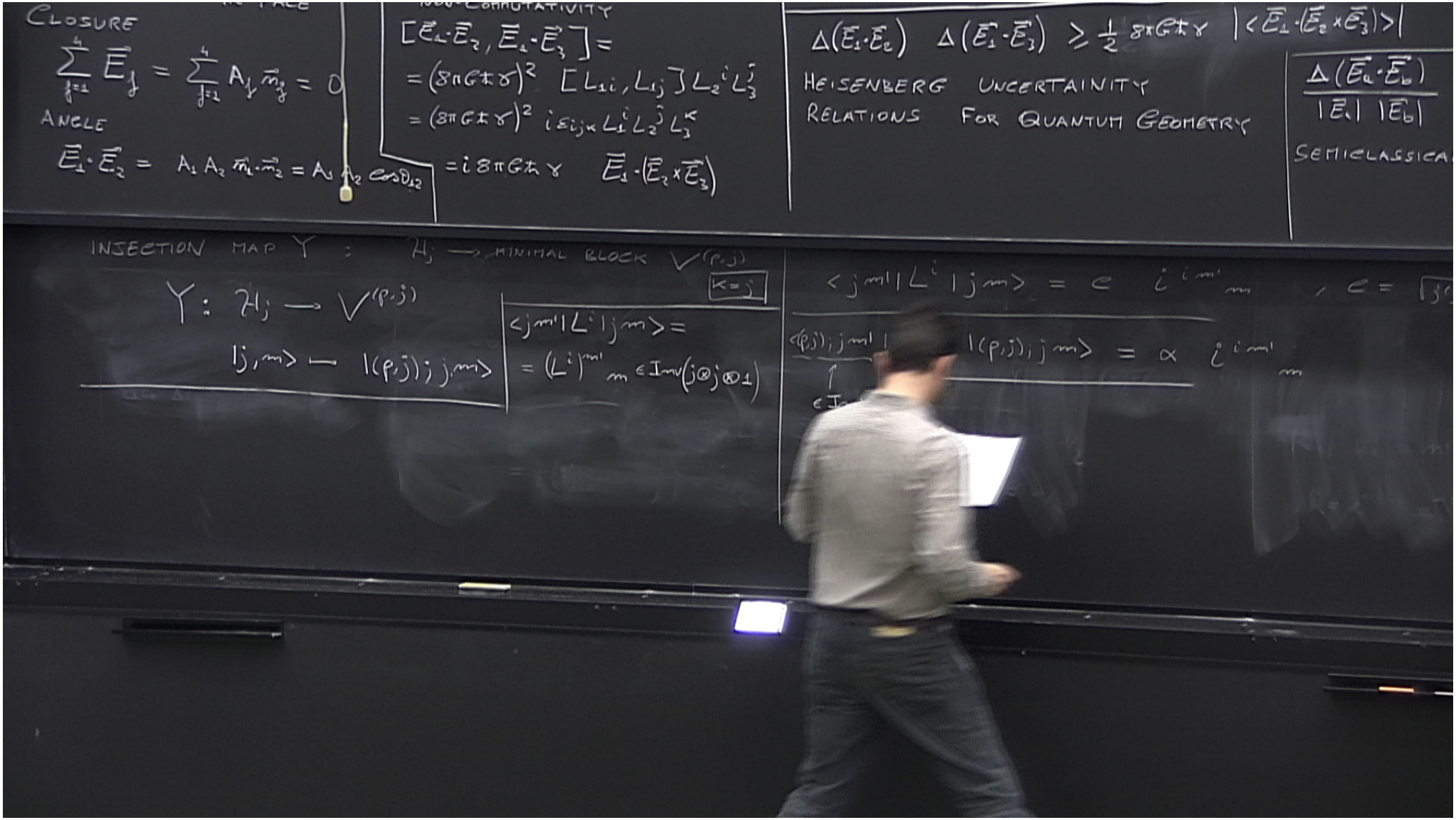
$$Y : \mathcal{H}_j \longrightarrow V^{(\rho,j)}$$

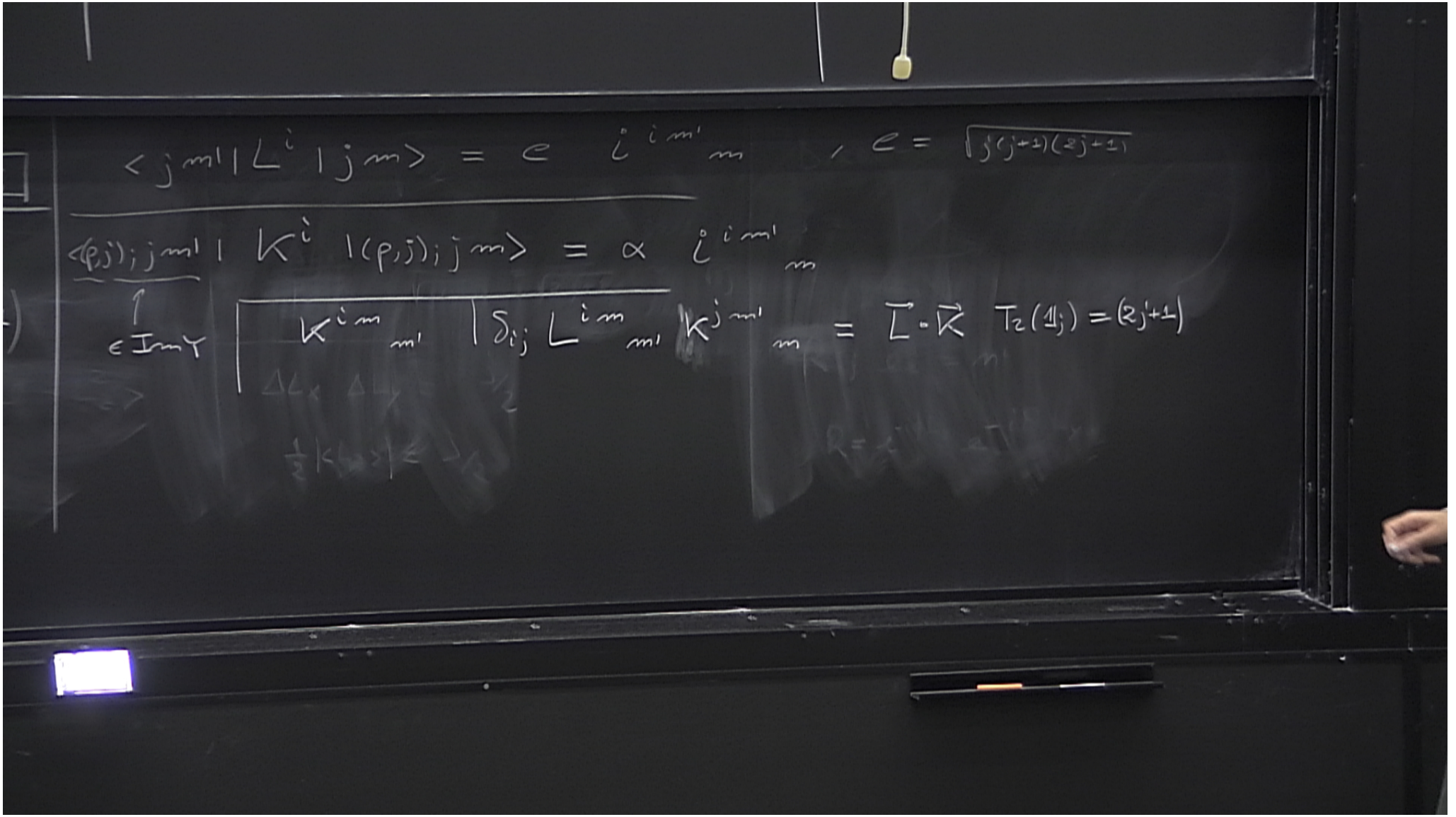
$$|j,m\rangle \longmapsto |(\rho,j); j,m\rangle$$

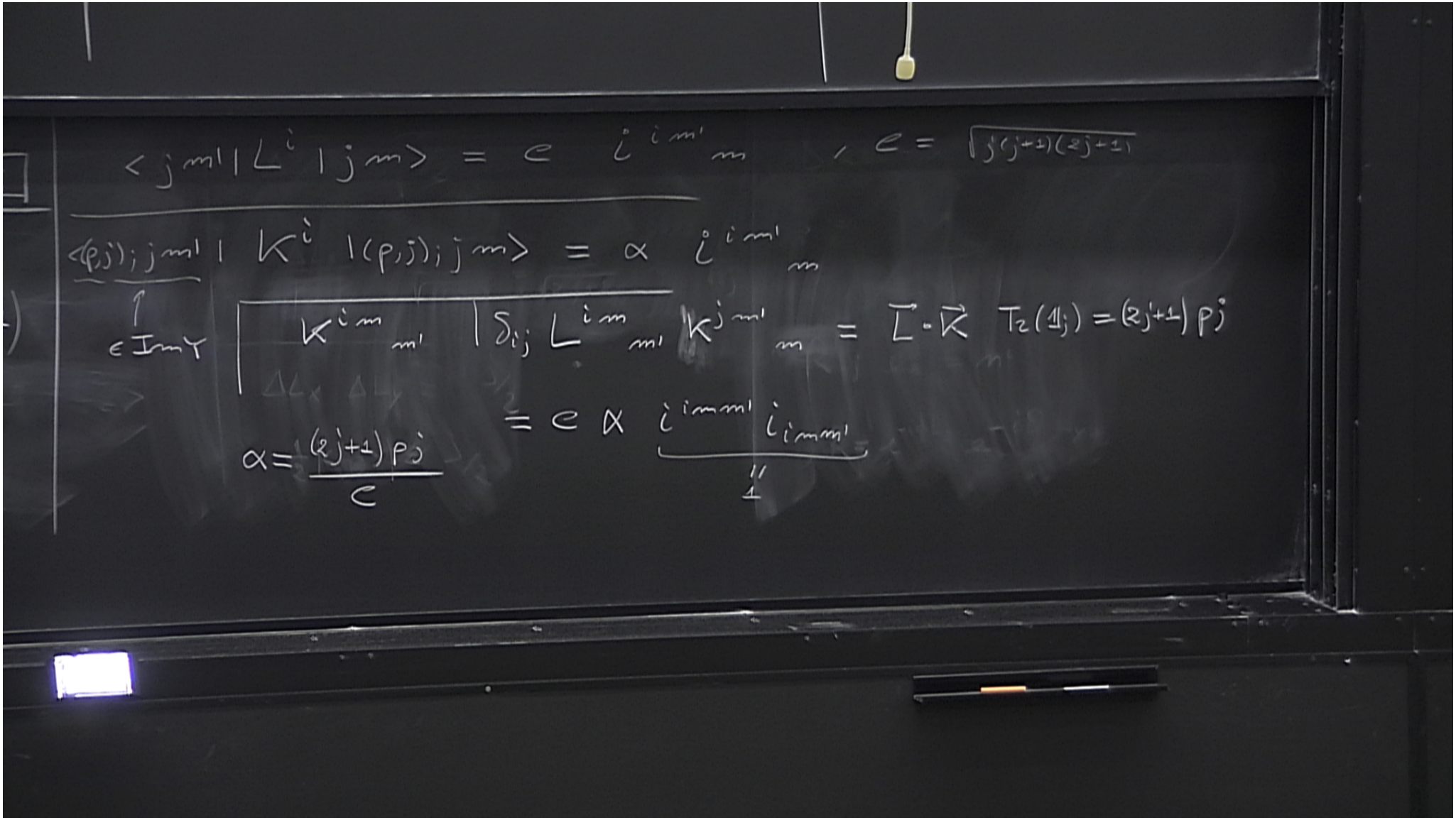

RELATIONS FOR QUANTUM GEOMETRY
SEMICLASSICAL STATE
k=j

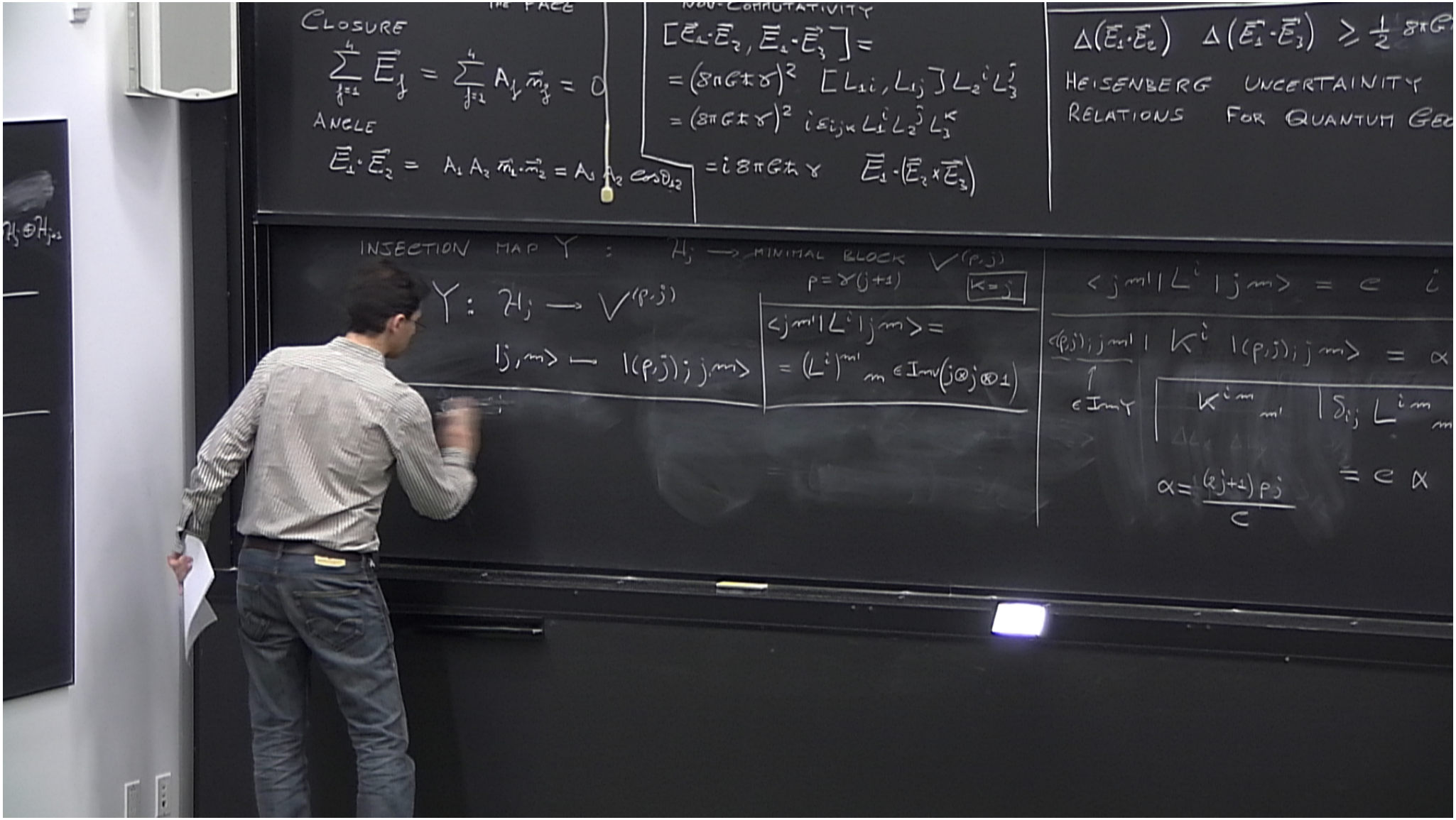

RELATIONS FOR QUANTUM GEOMETRY
SEMICLASSICAL STATE
k=j


E1 · E2 = A1 A2 n1 · n2 = A1 A2 cos θ12
= i 8πGħ γ
E1 · (E2 × E3)
INJECTION MAP Y : Hj → MINIMAL BLOCK V(p,j)
K = j
Y : Hj →
|j, m> ... (p,j) ; j, m>
<jm'|L^i|jm> =
= (L^i)^m' m
<jm'


CLOSURE
$\sum_{j=1}^{4} \vec{E}_j = \sum_{j=1}^{4} A_j \vec{n}_j = 0$
ANGLE
$\vec{E}_1 \cdot \vec{E}_2 = A_1 A_2 \vec{n}_1 \cdot \vec{n}_2 = A_1 A_2 \cos\theta_{12}$
$[\vec{E}_1 \cdot \vec{E}_2, \vec{E}_1 \cdot \vec{E}_3] =$
$= (8\pi G\hbar\gamma)^2 [L_{1i}, L_{1j}] L_2^i L_3^j$
$= (8\pi G\hbar\gamma)^2 i\varepsilon_{ijk} L_1^i L_2^j L_3^k$
$= i 8\pi G\hbar\gamma \; \vec{E}_1 \cdot (\vec{E}_2 \times \vec{E}_3)$
$\Delta(\vec{E}_1 \cdot \vec{E}_2) \; \Delta(\vec{E}_1 \cdot \vec{E}_3) \geq \frac{1}{2} 8\pi G\hbar\gamma \; |\langle \vec{E}_1 \cdot (\vec{E}_2 \times \vec{E}_3) \rangle|$
HEISENBERG UNCERTAINTY RELATIONS FOR QUANTUM GEOMETRY
$\frac{\Delta(\vec{E}_a \cdot \vec{E}_b)}{|\vec{E}_a| \, |\vec{E}_b|}$
SEMICLASSICA
INJECTION MAP Y : $\mathcal{H}_j \longrightarrow$ MINIMAL BLOCK $V^{(p,j)}$
$k = j$
$Y : \mathcal{H}_j \longrightarrow V^{(p,j)}$
$|j, m\rangle \longmapsto |(p,j); j, m\rangle$
$\langle j m' | L^i | j m \rangle =$
$= (L^i)^{m'}{}_m \in \mathrm{Inv}(j \otimes j \otimes 1)$
$\langle j m' | L^i | j m \rangle = c \; i^{i\,m'}{}_m$, $c =$
$\langle (p,j); j m' | \; | (p,j); j m \rangle = \propto i^{i\,m'}{}_m$
$\in$


$\langle j m' | L^i | j m \rangle = c \; i^{i m'}_{\;\; m}$, $c = \sqrt{j(j+1)(2j+1)}$
$\langle (p,j); j m' | K^i | (p,j); j m \rangle = \alpha \; i^{i m'}_{\;\; m}$
$\in \mathrm{Im}\,\gamma$
$K^{i m}_{\;\; m'} \; \delta_{ij} L^{i m}_{\;\; m'} \; K^{j m'}_{\;\; m} = \vec{L} \cdot \vec{K} \quad T_2(\mathbb{1}_j) = (2j+1)$


$\langle j\, m' | L^i | j\, m \rangle = c\ i^{i\,m'}{}_{m}$, $c = \sqrt{j(j+1)(2j+1)}$
$\langle (p,j); j\, m' | K^i | (p,j); j\, m \rangle = \alpha\ i^{i\,m'}{}_{m}$
$\in \mathrm{Im}\,\Upsilon$
$K^{i\,m}{}_{m'} \mid \delta_{ij} L^{i\,m}{}_{m'} K^{j\,m'}{}_{m} = \vec{L} \cdot \vec{K}$ $T_2(\mathbb{1}_j) = (2j+1)\, p_j$
$\alpha = \frac{(2j+1)\, p_j}{c}$
$= c\, \alpha\ \underbrace{i^{i\,m\,m'}\, i_{i\,m\,m'}}_{\| \\ 1}$


CLOSURE
ANGLE
NON-COMMUTATIVITY
HEISENBERG UNCERTAINTY RELATIONS FOR QUANTUM GEO
INJECTION MAP Y : H_j → MINIMAL BLOCK V^(p,j)
p = γ(j+1)
k = j
ϵ Im Y

$E_1 \cdot E_2 = A_1 A_2 \vec{n}_1 \cdot \vec{n}_2 = A_1 A_2 \cos\theta_{12}$

$= i 8\pi G \hbar\gamma \quad E_1 \cdot (E_2 \times E_3)$

INJECTION MAP Y : $\mathcal{H}_j \longrightarrow$ MINIMAL BLOCK $V^{(\rho,j)}$

$\rho = \gamma(j+1)$ $\quad \boxed{K=j}$

$Y : \mathcal{H}_j \longrightarrow V^{(\rho,j)}$

$|j,m\rangle \longmapsto |(\rho,j);j,m\rangle$

$\langle j m' | L^i | j m \rangle =$

$= (L^i)^{m'}{}_m \in Inv(j \otimes j \otimes 1)$

$\frac{\alpha}{c} = \frac{(2j+1)\rho\gamma}{\gamma j(j+1)(2j+1)} = \frac{\rho}{j+1} = \gamma$

$Y_\gamma \qquad \rho = \gamma(j+1)$

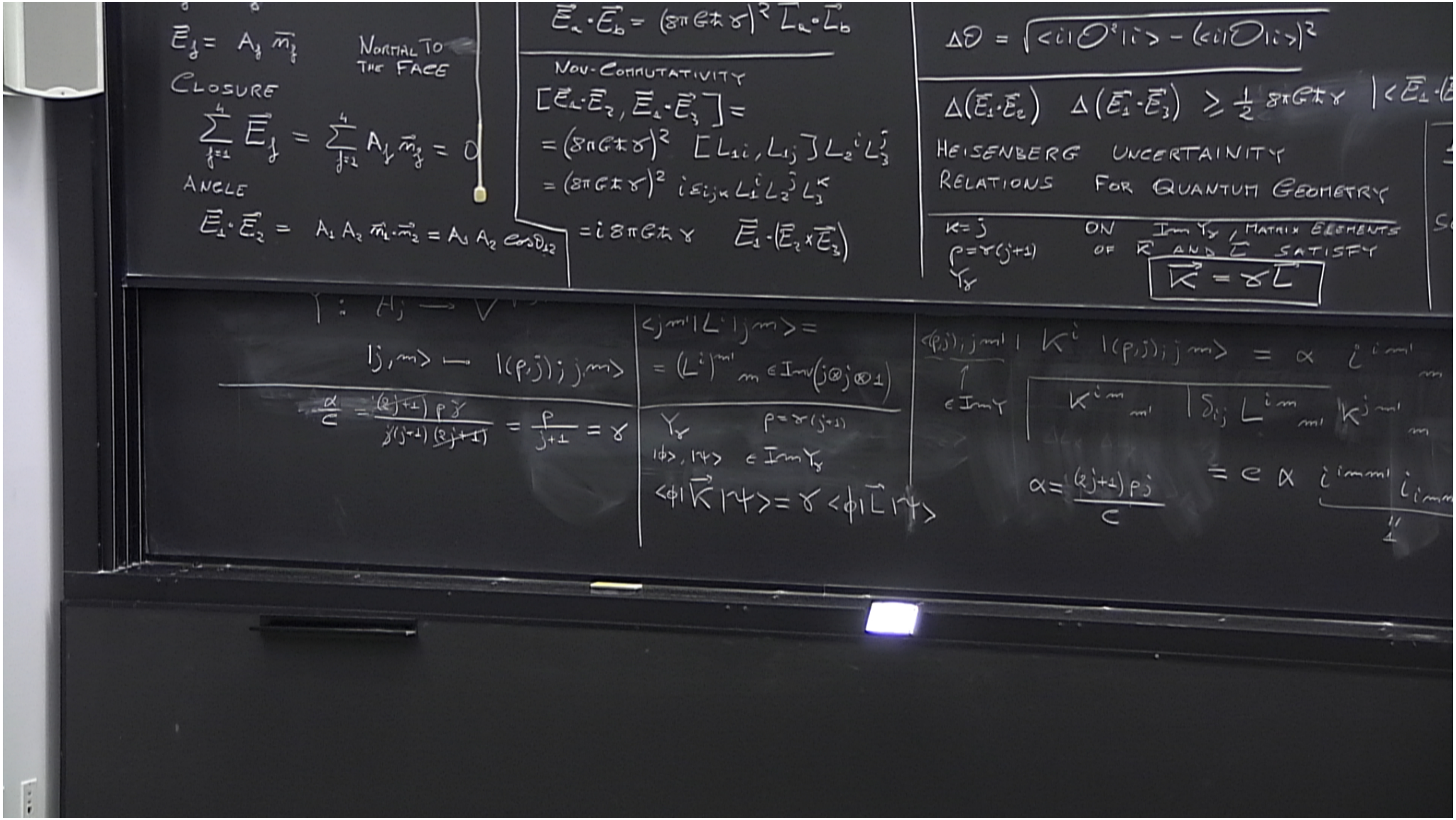
$|\phi\rangle, |\psi\rangle \in Im\, Y_\gamma$

$\langle\phi|\vec{K}|\psi\rangle = \langle\phi|\vec{L}|\psi\rangle$

$\langle (\rho,j); j m' | K^i$

$\in Im\, Y$

$\alpha =$



$\vec{E}_f = A_f \vec{n}_f$ NORMAL TO THE FACE

CLOSURE

$$\sum_{f=1}^{4} \vec{E}_f = \sum_{f=1}^{4} A_f \vec{n}_f = 0$$

ANGLE

$$\vec{E}_1 \cdot \vec{E}_2 = A_1 A_2 \vec{n}_1 \cdot \vec{n}_2 = A_1 A_2 \cos\theta_{12}$$

$$\vec{E}_a \cdot \vec{E}_b = (8\pi G\hbar\gamma)^2 \vec{L}_a \cdot \vec{L}_b$$

NON-COMMUTATIVITY

$$[\vec{E}_1 \cdot \vec{E}_2, \vec{E}_1 \cdot \vec{E}_3] =$$
$$= (8\pi G\hbar\gamma)^2 [L_{1i}, L_{1j}] L_2^i L_3^j$$
$$= (8\pi G\hbar\gamma)^2 i\varepsilon_{ijk} L_1^i L_2^j L_3^k$$
$$= i\, 8\pi G\hbar\gamma \quad \vec{E}_1 \cdot (\vec{E}_2 \times \vec{E}_3)$$

$$\Delta\mathcal{O} = \sqrt{\langle i|\mathcal{O}^2|i\rangle - (\langle i|\mathcal{O}|i\rangle)^2}$$

$$\Delta(\vec{E}_1 \cdot \vec{E}_2) \quad \Delta(\vec{E}_1 \cdot \vec{E}_3) \geq \frac{1}{2} 8\pi G\hbar\gamma \quad |\langle \vec{E}_1 \cdot (\vec{E}$$

HEISENBERG UNCERTAINITY RELATIONS FOR QUANTUM GEOMETRY

$k = j$, $\rho = \gamma(j+1)$, Y_γ ON $\mathrm{Im}\, Y_\gamma$, MATRIX ELEMENTS OF $\vec{K}$ AND $\vec{L}$ SATISFY

$$\boxed{\vec{K} = \gamma \vec{L}}$$

$$Y : H_j \longrightarrow V$$

$$|j, m\rangle \longmapsto |(\rho, j); j, m\rangle$$

$$\frac{\alpha}{c} = \frac{(2j+1)\rho\gamma}{\gamma(j+1)(2j+1)} = \frac{\rho}{j+1} = \gamma$$

$$\langle j m'|L^i|j m\rangle =$$
$$= (L^i)^{m'}{}_m \in \mathrm{Inv}(j \otimes j \otimes 1)$$

Y_γ $\quad \rho = \gamma(j+1)$

$|\phi\rangle, |\psi\rangle \in \mathrm{Im}\, Y_\gamma$

$$\langle\phi|\vec{K}|\psi\rangle = \gamma \langle\phi|\vec{L}|\psi\rangle$$

$$\langle(\rho, j); j m'| K^i |(\rho, j); j m\rangle = \alpha\, i^{i m'}{}_m$$

$\in \mathrm{Im}\, Y$

$$K^{i m}{}_{m'} \quad \delta_{ij} L^{i m}{}_{m'} K^{j m'}{}_m$$

$$\alpha = \frac{(2j+1)\rho j}{c} \qquad = c\,\alpha\, \underbrace{i^{i m m'} i_{i m m}}_{1}$$