Title: Explorations in Quantum Gravity - Lecture 10

Date: Apr 16, 2012 10:15 AM

URL: http://pirsa.org/12040033

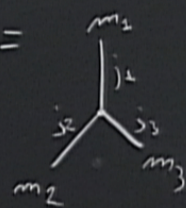
Abstract:

TOOLS LANDAU III ch $\underline{XIV}$

$$\dim \mathrm{Inv}(\mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2} \otimes \mathcal{H}_{j_3}) = \begin{cases} 1 & \text{Triangular} \\ 0 & \end{cases}$$

$$|i_3\rangle = \sum_{m_1 m_2 m_3} i^{m_1 m_2 m_3} |j_1 m_1\rangle \otimes |j_2 m_2\rangle \otimes |j_3 m_3\rangle$$

$$i^{m_1 m_2 m_3} = \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} =$$



$$\eta_{m m'} = (-1)^{j+m} \delta_{m,-m'}$$

$$\langle i_3 | i_3 \rangle = i^{m_1 m_2 m_3} \, i_{m_1 m_2 m_3} = 1$$

$$T^{m_1 m_2 m_3} = c \; i^{m_1 m_2 m_3}$$

$$c = T^{m_1 m_2 m_3} \, i_{m_1 m_2 m_3}$$

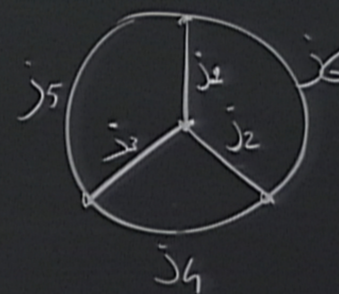
$$\langle j m' | L^i | j m \rangle = (L^i)_m{}^{m'} = c \; i^{i}{}_{m}{}^{m'}$$

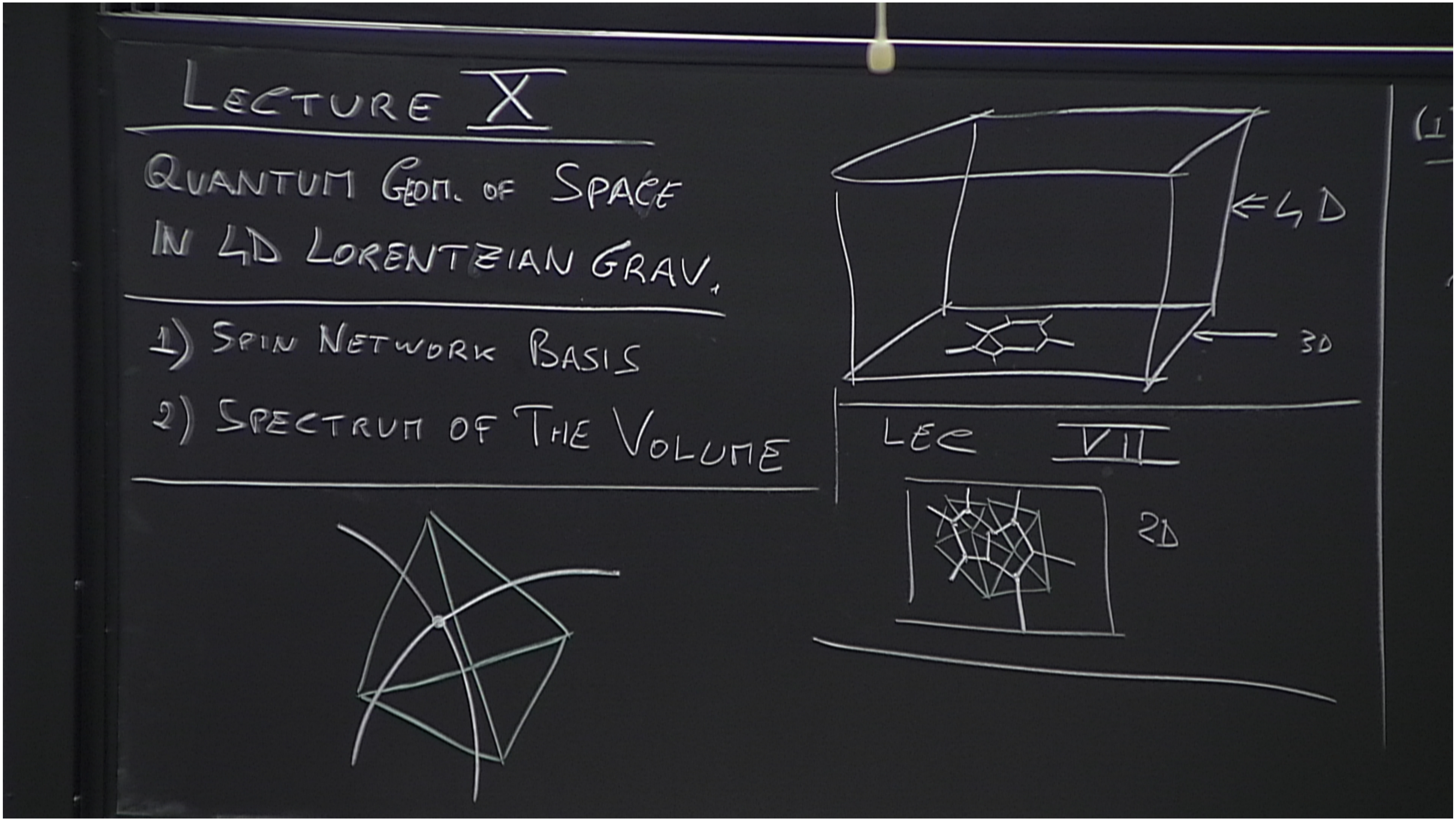
$\underline{ex}$

$$c = \sqrt{j(j+1)(2j+1)}$$

$\{6j\}$

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} =$$




LECTURE X
QUANTUM GEOM. OF SPACE
IN 4D LORENTZIAN GRAV.
1) SPIN NETWORK BASIS
2) SPECTRUM OF THE VOLUME
← 4D
3D
LEC VII
2D

← 4D

3D

(1) SPIN-NETWORK BASIS

$\Gamma = \{l, n\}$ $\qquad L^2(SU(2)^L / SU(2)^N)$

$$\Psi(U_l) = \bigotimes_l D^{(j_l)}(U_l) \cdot \bigotimes_n i_n$$

$$i_n \in \mathcal{H}_n = \mathrm{Inv}(\mathcal{H}_{j_1} \otimes \cdots \otimes \mathcal{H}_{j_4})$$

$$|i\rangle = \sum_{m_1 \cdots m_4} i^{m_1 \cdots m_4} |j_1 m_1\rangle \otimes \cdots \otimes |j_4 m_4\rangle$$

$D(U$

$U =$

$)$

$\otimes\; i_n$

$\mathcal{H}_{j_4})$

$\otimes |j_4 m_4\rangle$

$$D(U)|i\rangle \neq |i\rangle$$

$$U = e^{i\vec{\alpha}\cdot\vec{L}_1} \otimes \cdots \otimes e^{i\vec{\alpha}\cdot\vec{L}_4}$$

$|\vec{\alpha}| \ll 1$

$$D(U)|i\rangle = \left(1 + i\vec{\alpha}\cdot(\vec{L}_1+\vec{L}_2+\vec{L}_3+\vec{L}_4)\right)|i\rangle = |i\rangle$$

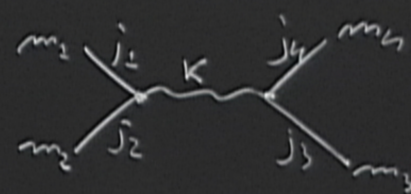
$$\left(\sum_{l=1}^{4}\vec{L}_l\right)|i\rangle = 0$$

example

$$i_K^{m_1 m_2 m_3 m_4} = i^{m_1 m_2 m}\, i_{m m_3 m_4}$$

$$-K \le m \le K$$

$=$



(1) SPIN-NETWORK BASIS

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$$|i\rangle = \sum_{m_1 \ldots m_4} i^{m_1 \ldots m_4} |j_1 m_1\rangle \otimes \cdots \otimes |j_4 m_4\rangle$$

$(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\rangle$

LEC I

$$D(U)|i\rangle = |i\rangle$$

$$U = e^{i\vec{\alpha}\cdot\vec{L}_1} \otimes \cdots \otimes e^{i\vec{\alpha}\cdot\vec{L}_4}$$

$|\alpha| \ll 1$

$$D(U)|i\rangle = \left(1 + i\vec{\alpha}\cdot(\vec{L}_1 + \vec{L}_2 + \vec{L}_3 + \vec{L}_4)\right)|i\rangle = |i\rangle$$

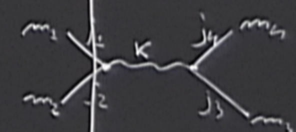
$$\left(\sum_{l=1}^{4} \vec{L}_l\right)|i\rangle = 0$$

example

$$i_K^{m_1 m_2 m_3 m_4} = \sqrt{2K+1}\; i^{m_1 m_2 m}\; i_m^{m_3 m_4}$$

$$-K \le m \le K$$

$= \sqrt{2K+1}$



Observables in $\mathcal{H}_n$, $[O, C] = 0$

$$[\vec{L}_\ell^{\,2}, C] = 0 \qquad C = \sum_{\ell=1}^{4} \vec{L}_\ell$$

$$[(\vec{L}_1 + \vec{L}_2)^2, C] = 0$$

$$(\vec{L}_1 + \vec{L}_2)^2 = (\vec{L}_3 + \vec{L}_4)^2$$

$$[\vec{L}_\ell^{\,2}, (\vec{L}_1 + \vec{L}_2)^2] = 0$$

MAXIMAL COMMUTING SET OF OBSERVABLES

$$\vec{L}_\ell^{\,2} \, |i\rangle = j_\ell(j_\ell + 1) \, |i\rangle$$

$$(\vec{L}_1 + \vec{L}_2)^2 \, |i_k\rangle = k(k+1) \, |i_k\rangle$$

$$|j_1 - j_2| \leq k \leq j_1 + j_2$$

$$|j_3 - j_4| \leq k \leq j_3 + j_4$$

$|i_k\rangle$ o.n. basis

$$\Downarrow \qquad \boxed{\langle i_k | i_{k'}\rangle = \delta_{kk'}}$$

$$d = \dim \mathcal{H}_n = k_{max} - k_{min} + 1$$

$$k_{min} = \mathrm{Max}(|j_1 - j_2|, |j_3 - j_4|)$$

$$k_{max} = \mathrm{Min}(j_1 + j_2, j_3 + j_4)$$

e.g. $j_1 = j_2 = j_3 = j_4 \equiv j$

$$d = 2j + 1$$

$j = 1/2$ $\quad d = 2$

$j = 1$ $\quad d = 3$

$\ldots \rangle = \delta_{kk'}$

$- K_{min} + 1$

$\ldots |, |j_3 - j_4|)$

$\ldots + j_2, j_3 + j_4)$

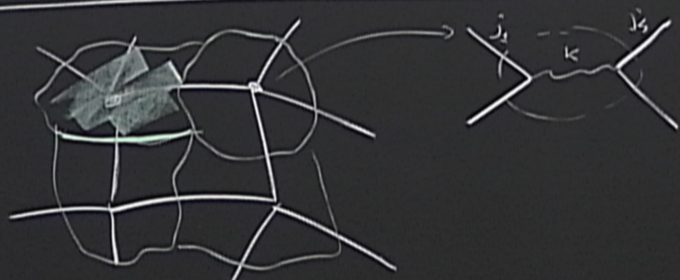
$\ldots_3 = j_4 \equiv j$

$$\psi_{j_\ell, k_n}(U_\ell) = \bigotimes_\ell D^{(j_\ell)}(U_\ell) \cdot_\Gamma \bigotimes_n i_{k_n}$$

$$\| $$

$$\langle U_\ell | j_\ell, k_n \rangle$$

$$\langle j_\ell, k_n | j'_\ell, k'_n \rangle = \delta_{j_\ell j'_\ell}\, \delta_{k_n k'_n}$$



(2) SPECTRUM OF THE VOLUME

$$\hat{V}_n = \frac{\sqrt{2}}{3}(8\pi G \hbar \gamma)^{3/2} \sqrt{|\vec{L}_1 \cdot (\vec{L}_2 \times \vec{L}_3)|}$$

which 3? any $\quad \vec{L}_1 + \vec{L}_2 + \vec{L}_3 + \vec{L}_4 = 0$

$$\hat{V}_n | i_v \rangle = v \, | i_v \rangle$$

$$\vec{L}_1 \cdot (\vec{L}_2 \times \vec{L}_3) | i_q \rangle = q \, | i_q \rangle$$

$$\hat{V}_n | i_q \rangle = \frac{\sqrt{2}}{3}(8\pi G \hbar \gamma)^{3/2} \sqrt{|q|} \; | i_q \rangle$$

$d \times d$

$$\langle i_K | \vec{L}_1 \cdot (\vec{L}_2 \times \vec{L}_3) | i_q \rangle = q \langle i_K | i_q \rangle$$

$$\sum_{K'} Q_K^{\ K'} \langle i_{K'} | i_q \rangle = q \langle i_K | i_q \rangle$$

$$Q_K^{\ K'} = \langle i_K | \vec{L}_1 \cdot (\vec{L}_2 \times \vec{L}_3) | i_{K'} \rangle$$

$$[\vec{L}_1 \cdot \vec{L}_2 , \vec{L}_1 \cdot \vec{L}_3] = [L_{1i}, L_{1j}] L_2^i L_3^j$$

$$= i \epsilon_{ijk} L_1^k L_2^j L_3^k = i \vec{L}_1 \cdot (\vec{L}_2 \times \vec{L}_3)$$

$$= \frac{1}{2} [(\vec{L}_1 + \vec{L}_2)^2 , \vec{L}_1 \cdot \vec{L}_3]$$

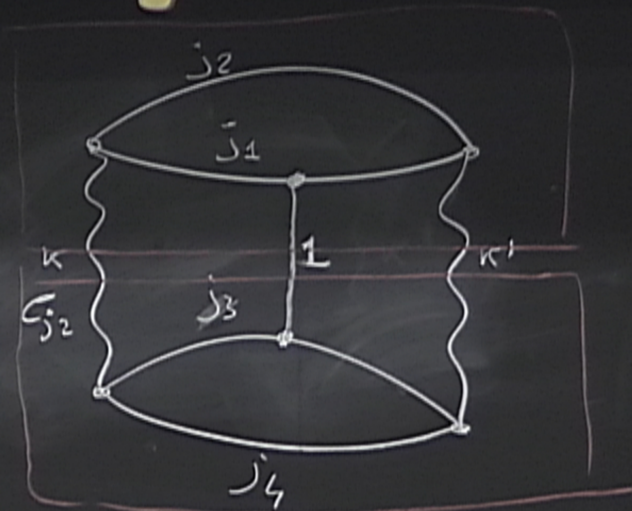
$$Q_k^{\ k'} = \langle k | \vec{L}_1 \cdot (\vec{L}_2 \times \vec{L}_3) | k' \rangle$$

$$= \frac{1}{2} \langle k | \left[(\vec{L}_1 + \vec{L}_2)^2 , \vec{L}_1 \cdot \vec{L}_3 \right] | k' \rangle$$

$$= \frac{1}{2} \Big(k(k+1) - k'(k'+1) \Big) \quad \langle k | \vec{L}_1 \cdot \vec{L}_3 | k' \rangle$$

$\vec{L}_3 | k' \rangle$

$$\langle k | \vec{L}_1 \cdot \vec{L}_3 | k' \rangle =$$



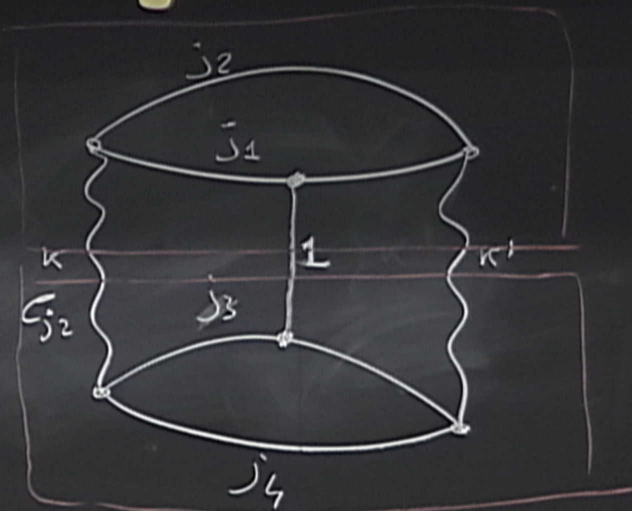
$$= \sqrt{2k+1}\;\sqrt{2k'+1}\;\mathcal{C}_{j_1}\,\mathcal{C}_{j_2}$$

SELECTION RULES

$$k' \neq k \pm 1 \,,\; k$$

$\vec{L}_3 | k' \rangle$.

$$\langle k | \vec{L}_1 \cdot \vec{L}_3 | k' \rangle =$$

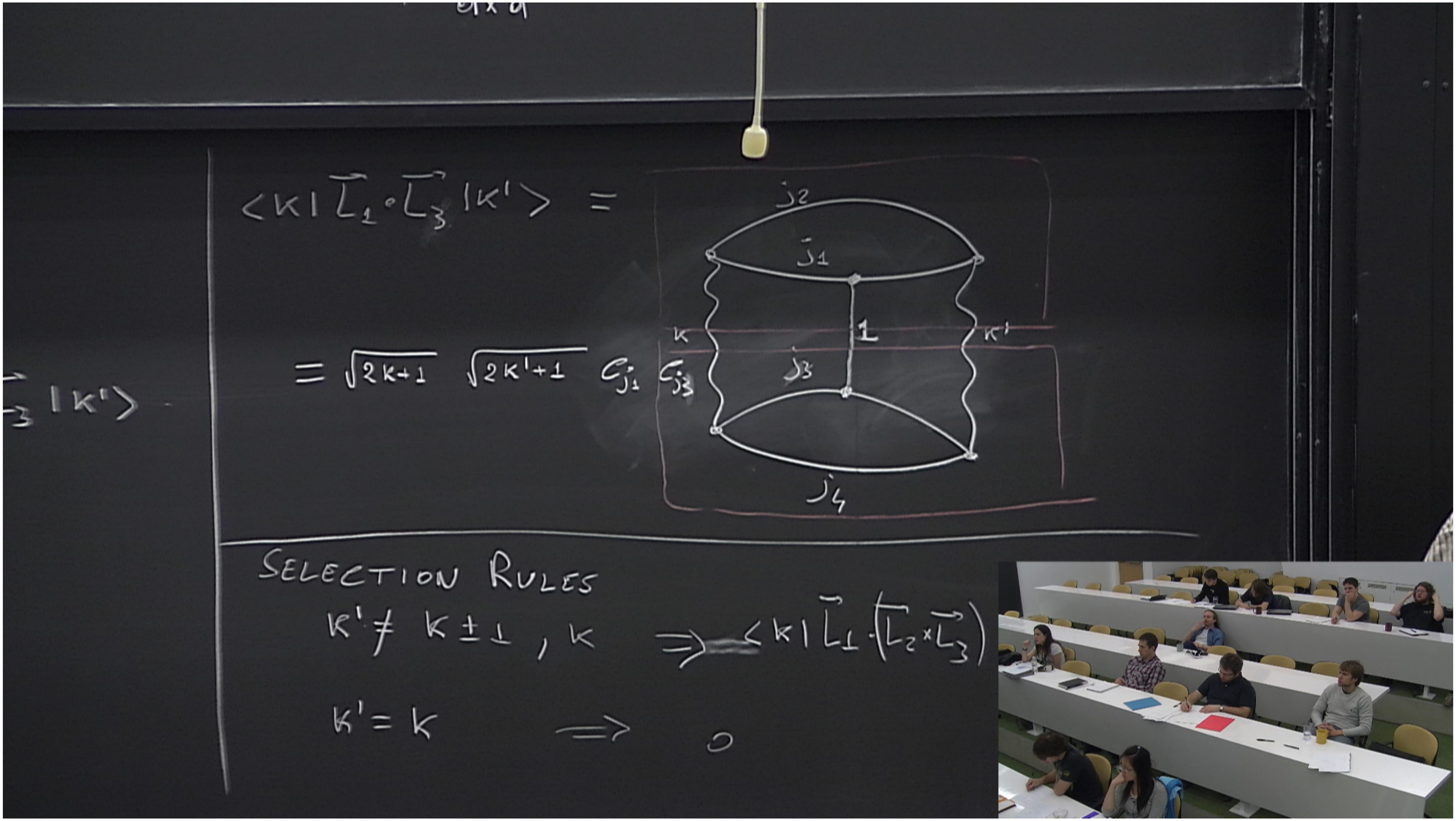


$$= \sqrt{2k+1}\ \sqrt{2k'+1}\ \mathcal{C}_{j_1}\ \mathcal{C}_{j_2}$$

SELECTION RULES

$$k' \neq k \pm 1 , k \quad \Rightarrow \quad \langle k | \vec{L}_1 \cdot (\vec{L}_2 \times \vec{L}_3) | k' \rangle = 0$$

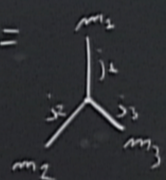
$$k' = k \quad \Rightarrow \quad 0$$


$\langle k | \vec{L}_1 \cdot \vec{L}_3 | k' \rangle =$
$= \sqrt{2k+1} \sqrt{2k'+1}\, C_{j_1} C_{j_3}$
j_2
j_1
1
k
k'
j_3
j_4
$_3 | k' \rangle$
SELECTION RULES
$k' \neq k \pm 1, k \Rightarrow \langle k | \vec{L}_1 \cdot (\vec{L}_2 \times \vec{L}_3)$
$k' = k \Rightarrow 0$

TOOLS   LANDAU III ch XIV

$\dim \mathrm{Inv}(\mathcal{H}_{j_1}\otimes\mathcal{H}_{j_2}\otimes\mathcal{H}_{j_3}) = \begin{cases} 1 \\ 0 \end{cases}$ Triangular

$|i_3\rangle = \sum_{m_1m_2m_3} i^{m_1m_2m_3}\,|j_1 m_1\rangle\otimes|j_2 m_2\rangle\otimes|j_3 m_3\rangle$

$i^{m_1m_2m_3} = \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} =$ 

$\eta_{mm'} = (-1)^{j+m}\,\delta_{m,-m'}$

$\langle i_3|i_3\rangle = i^{m_1m_2m_3}\, i_{m_1m_2m_3} = 1$

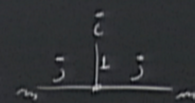
$T^{m_1m_2m_3} = c\; i^{m_1m_2m_3}$

$c = T^{m_1m_2m_3}\, i_{m_1m_2m_3}$

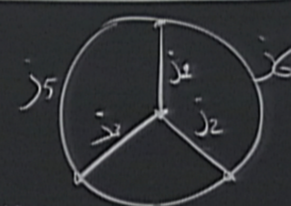
$\langle jm'|L^i|jm\rangle = (L^i)_m{}^{m'} = c\;\, i^i{}_m{}^{m'}$

ex

$c_j = \sqrt{j(j+1)(2j+1)}$

$L^{i\,mm'} = c_j$ 

{6j}

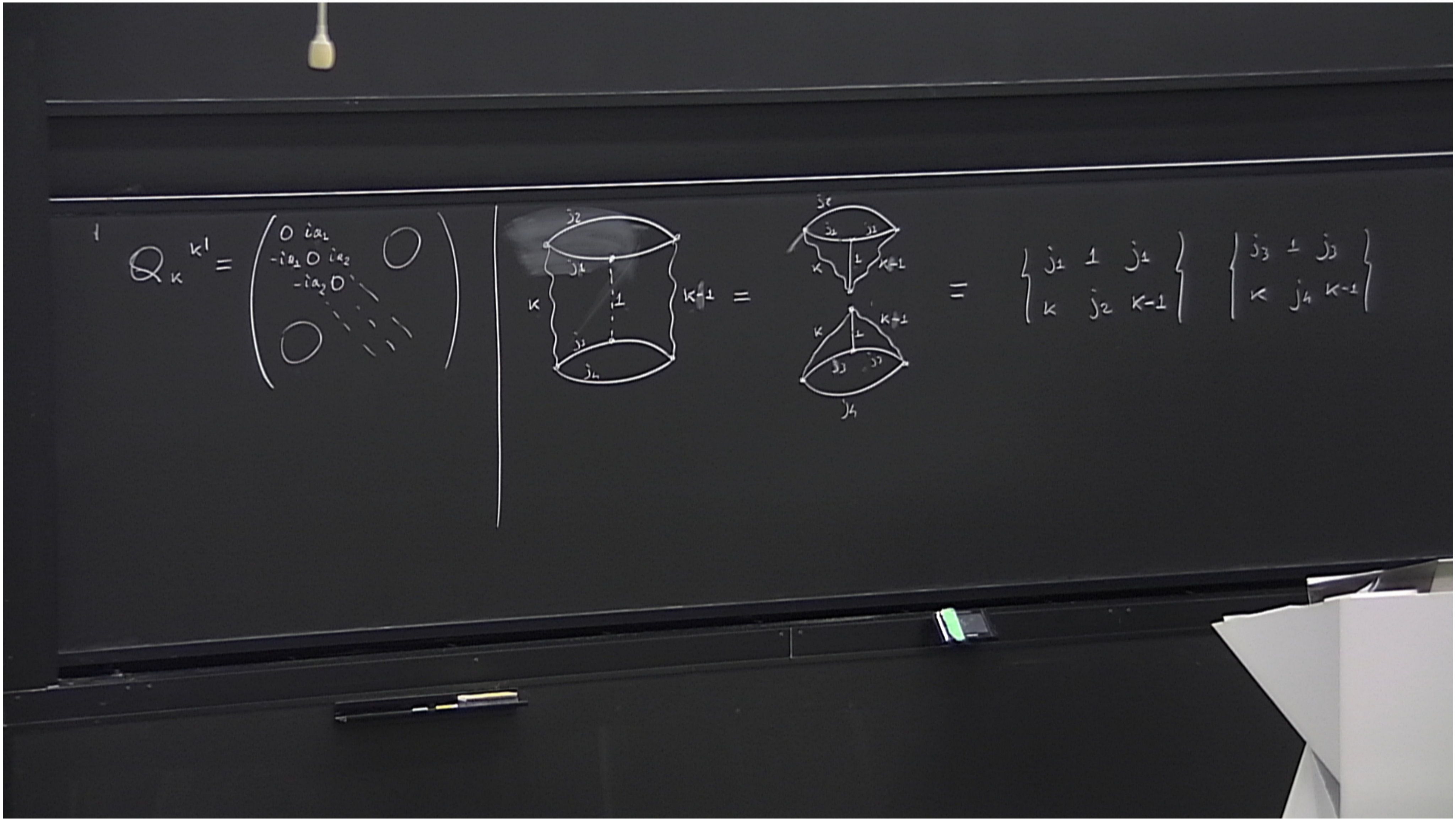
$\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} =$ 

SixJSymbol[{j1, j2, j3}, {j4, j5, j6}]]4

$\begin{Bmatrix} a & 1 & a \\ c & b & c-1 \end{Bmatrix} = (-1)^{a+b+c}\,\dfrac{2\,\Delta(a+\frac{1}{2}, b+\frac{1}{2}, c)}{c_a\,\sqrt{2c+1}\,\sqrt{2c-1}\,\sqrt{c}}$

$\Delta(a,b,c) = \frac{1}{4}\sqrt{(a+b+c)(a+b-c)(a-b+c)(-a+b+c}$

HERON   AREA (a, b, c)


Q K k' =
0 ia1
-ia1 0 ia2
-ia2 0
j2
j1
k
1
k-1
j3
j4
=
=
j1 1 j1
k j2 k-1
j3 1 j3
k j4 k-1

$$= q \langle i_k | i_q \rangle$$

$$= q \langle i_k | i_q \rangle$$

$$\vec{L}_1 \cdot (\vec{L}_2 \times \vec{L}_3) | i_{k'} \rangle$$

$$, L_{1j}] L_2^i L_3^j$$

$$= i \vec{L}_1 \cdot (\vec{L}_2 \times \vec{L}_3)$$

$$Q_k^{k'} = \langle k | \vec{L}_1 \cdot (\vec{L}_2 \times \vec{L}_3) | k' \rangle$$

$$= -\frac{i}{2} \langle k | [(\vec{L}_1 + \vec{L}_2)^2, \vec{L}_1 \cdot \vec{L}_3] | k' \rangle$$

$$= -\frac{i}{2} \left(k(k+1) - k'(k'+1) \right) \langle k | \vec{L}_1 \cdot \vec{L}_3 | k' \rangle$$

$$Q_k^{k-1} = -\frac{i}{2} \, 2k \sqrt{2k+1} \sqrt{2k-1} \, e_{j_1} e_{j_3} \times$$

$$\times \frac{2 \Delta(j_1 + \tfrac{1}{2}, j_2 + \tfrac{1}{2}, k)}{\sqrt{2k+1} \sqrt{2k-1} \sqrt{k} \, e_{j_1}} \quad \frac{2 \Delta(j_3 + \tfrac{1}{2}, j_4 + \tfrac{1}{2}, k)}{\sqrt{2k+1} \sqrt{2k-1} \sqrt{k} \, e_{j_3}}$$

$$= -4i \frac{\Delta(j_1 + 1, j_2 + 1, k) \, \Delta(j_3 + \tfrac{1}{2}, j_4 + \tfrac{1}{2}, k)}{\sqrt{4k^2 - 1}}$$

$$\langle k |$$

$$=$$

SEL

$= q \langle i_k | i_q \rangle$

$= q \langle i_k | i_q \rangle$

$\vec{L}_1 \cdot (\vec{L}_2 \times \vec{L}_3) | i_{k'} \rangle$

$, L_{1j}] L_2^i L_3^j$

$= i \vec{L}_1 \cdot (\vec{L}_2 \times \vec{L}_3)$

$$Q_k^{k'} = \langle k | \vec{L}_1 \cdot (\vec{L}_2 \times \vec{L}_3) | k' \rangle$$

$$= -\frac{i}{2} \langle k | [(\vec{L}_1 + \vec{L}_2)^2, \vec{L}_1 \cdot \vec{L}_3] | k' \rangle$$

$$= -\frac{i}{2} \left(k(k+1) - k'(k'+1) \right) \langle k | \vec{L}_1 \cdot \vec{L}_3 | k' \rangle$$

$$Q_k^{k-1} = -\frac{i}{2} \, 2k \sqrt{2k+1} \sqrt{2k-1} \, c_{j_1} c_{j_3} \times$$

$$= i q_k$$

$$\times \frac{2 \Delta(j_1 + \frac{1}{2}, j_2 + \frac{1}{2}, k)}{\sqrt{2k+1} \sqrt{2k-1} \sqrt{k} \, c_{j_1}} \frac{2 \Delta(j_3 + \frac{1}{2}, j_4 + \frac{1}{2}, k)}{\sqrt{2k+1} \sqrt{2k-1} \sqrt{k} \, c_{j_3}}$$

$$= -4i \frac{\Delta(j_1 + 1, j_2 + 1, k) \, \Delta(j_3 + \frac{1}{2}, j_4 + \frac{1}{2}, k)}{\sqrt{4k^2 - 1}}$$

$\langle k |$

$=$

SELE

$$j_1 = j_2 = j_3 = j_4 \equiv j$$

$$a_k = \frac{1}{4} \frac{(k^2 - (2j+1)^2)\, k^2}{\sqrt{4k^2 - 1}}$$

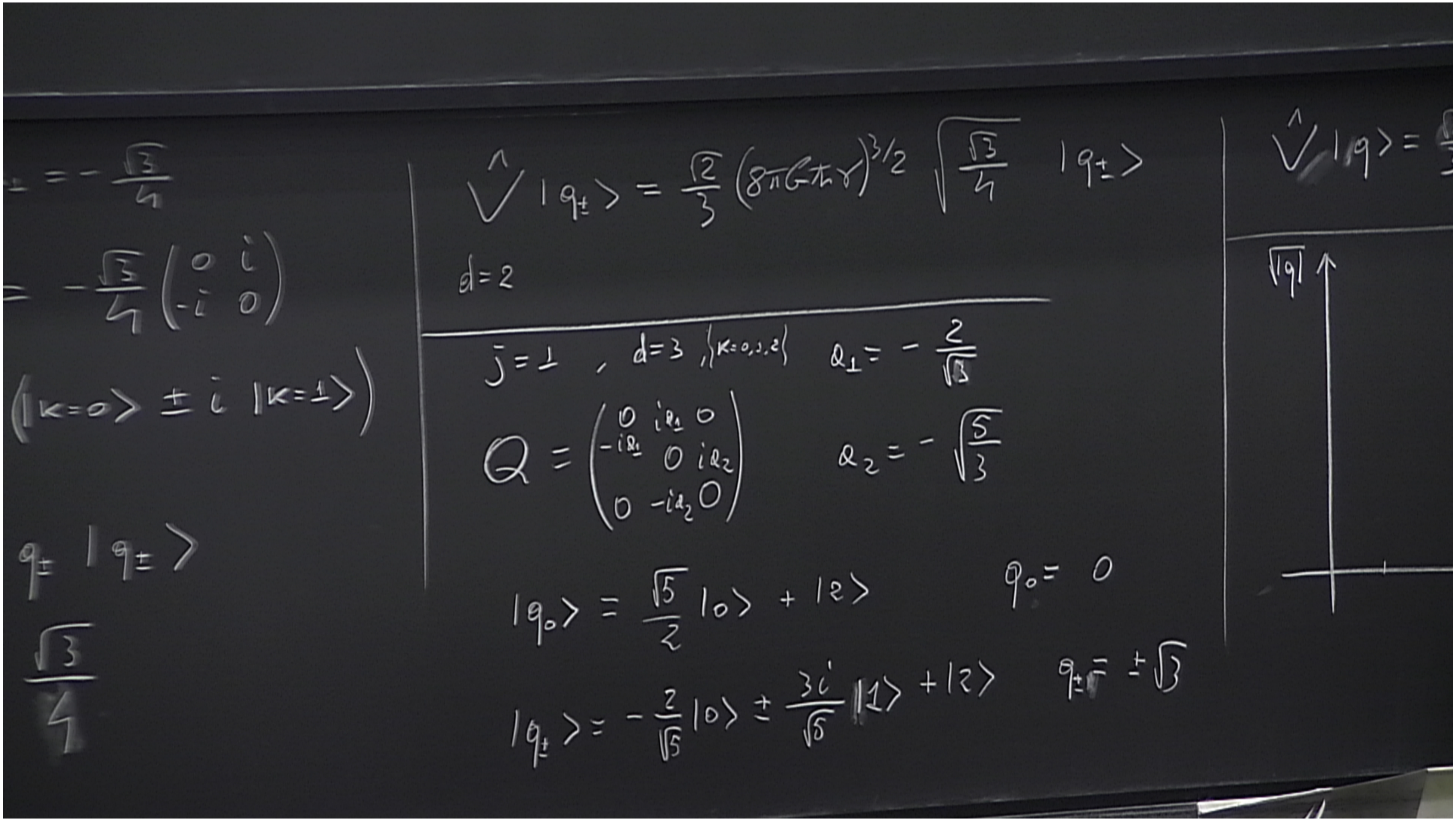
$$j = \frac{1}{2} \quad , \quad a_1 = -\frac{\sqrt{3}}{4}$$

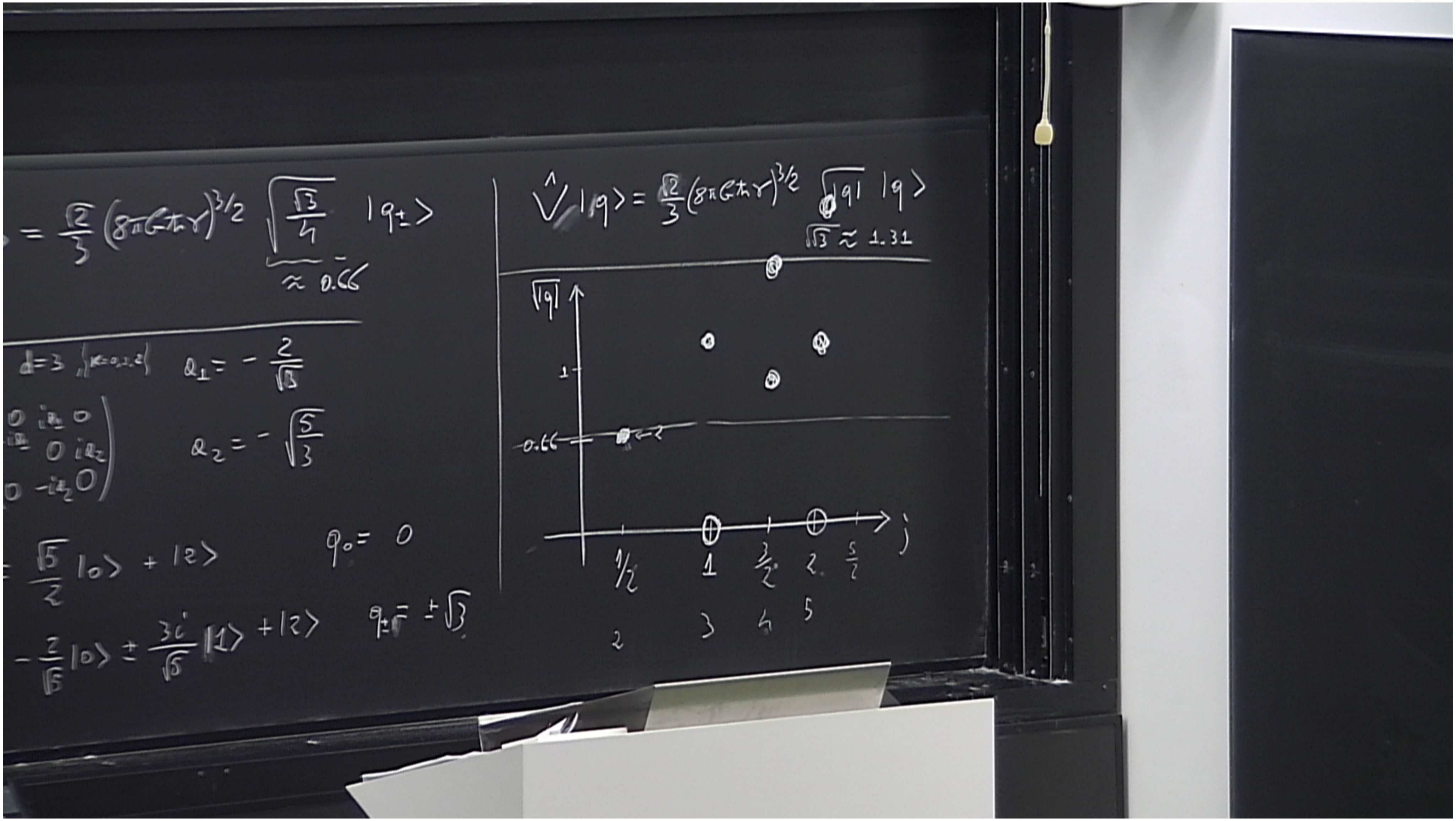
$$Q = \begin{pmatrix} 0 & ia_1 \\ -ia_1 & 0 \end{pmatrix} = -\frac{\sqrt{3}}{4} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

$$|q_\pm\rangle = \frac{1}{\sqrt{2}} \left(|k=0\rangle \pm i\, |k=1\rangle \right)$$

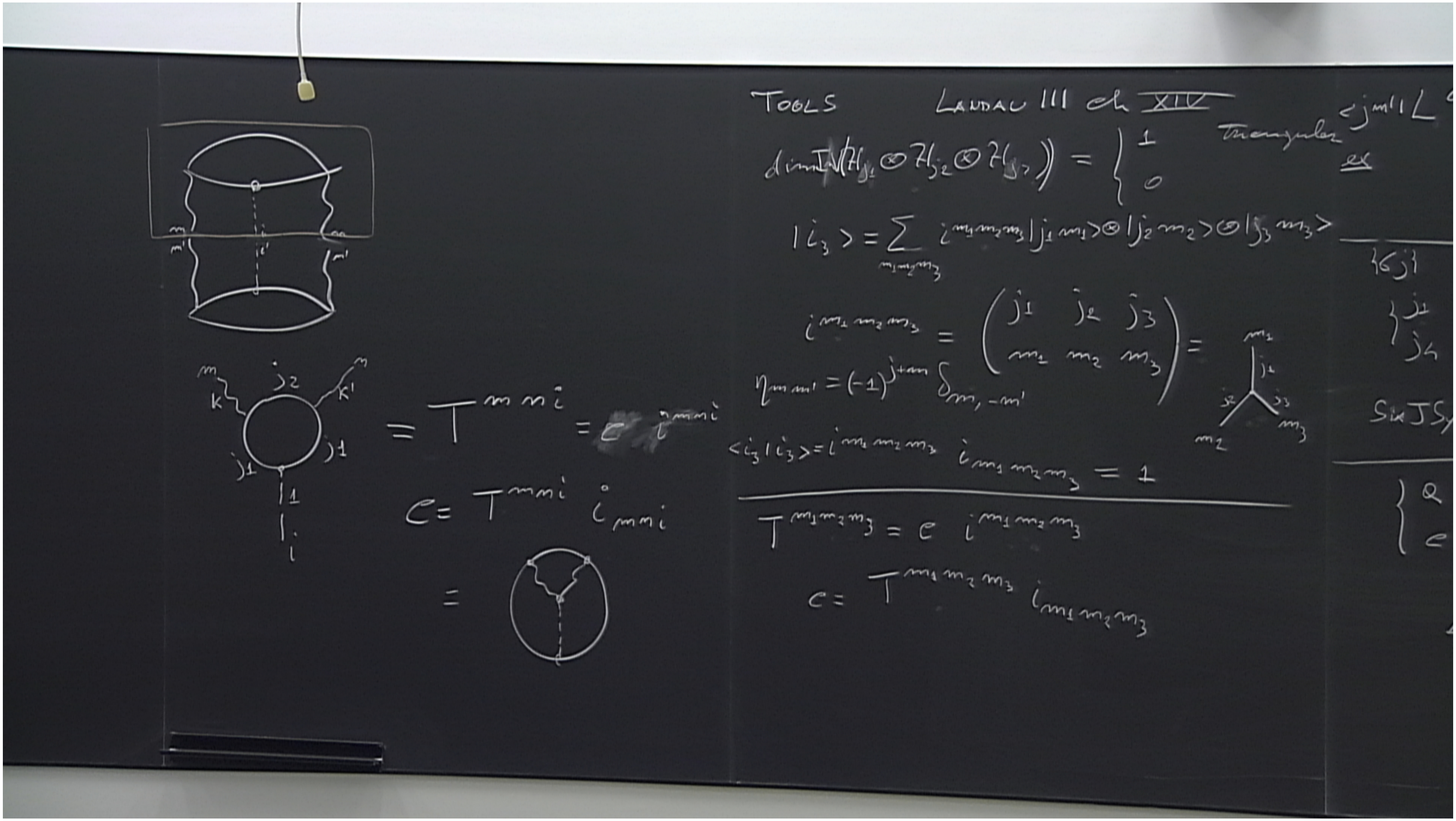
$$\vec{L}_1 \cdot (\vec{L}_2 \times \vec{L}_3) |q_\pm\rangle = q_\pm |q_\pm\rangle$$

$$q_\pm = \pm \frac{\sqrt{3}}{4}$$


$\hat{V}|q_\pm\rangle = \frac{\sqrt{2}}{3}(8\pi G\hbar\gamma)^{3/2}\sqrt{\frac{\sqrt{3}}{4}}\;|q_\pm\rangle$
$d=2$
$j=1\;,\;d=3\;,\{k=0,1,2\}$
$a_1 = -\frac{2}{\sqrt{3}}$
$Q = \begin{pmatrix} 0 & ia_1 & 0 \\ -ia_1 & 0 & ia_2 \\ 0 & -ia_2 & 0 \end{pmatrix}$
$a_2 = -\sqrt{\frac{5}{3}}$
$|q_0\rangle = \frac{\sqrt{5}}{2}|0\rangle + |2\rangle$
$q_0 = 0$
$|q_\pm\rangle = -\frac{2}{\sqrt{5}}|0\rangle \pm \frac{3i}{\sqrt{5}}|1\rangle + |2\rangle$
$q_\pm = \pm\sqrt{3}$
$= -\frac{\sqrt{3}}{4}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$
$(|k=0\rangle \pm i\,|k=1\rangle)$
$q_\pm\,|q_\pm\rangle$
$\frac{\sqrt{3}}{4}$
$\hat{V}|q\rangle =$
$\sqrt{|q|}$



$\hat{V}|q\rangle = \frac{\sqrt{2}}{3}(8\pi G\hbar\gamma)^{3/2}\sqrt{|q|}\,|q\rangle$
$\sqrt{3} \approx 1.31$
$\sqrt{|q|}$
0.66
j
$q_0 = 0$
$q_\pm = \pm\sqrt{3}$
$Q_1 = -\frac{2}{\sqrt{3}}$
$Q_2 = -\sqrt{\frac{5}{3}}$
≈ 0.66


TOOLS
LANDAU III ch XIV
Triangular
ex
SixJS


√3 ≈ 1.31
1/2
1