Title: Sub-Compton Quantum Non-equilibrium and Majorana Systems

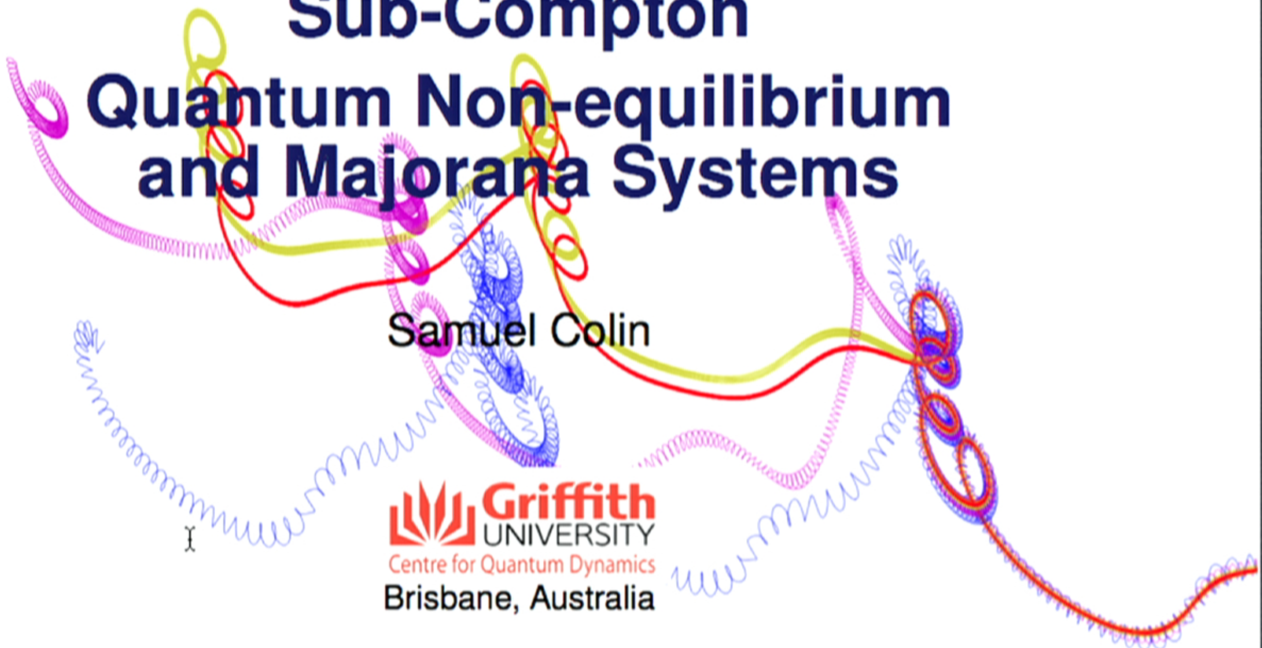
Date: Mar 27, 2012 03:30 PM

URL: http://pirsa.org/12030119

Abstract: In the de Broglie-Bohm pilot-wave theory, an ensemble of fermions is not only described by a spinor, but also by a distribution of position beables. If the distribution of positions is different from the one predicted by the Born rule, the ensemble is said to be in quantum non-equilibrium. Such ensembles, which can lead to an experimental discrimination between the pilot-wave theory and standard quantum mechanics, are thought to quickly relax to quantum equilibrium in most cases.   In this talk, I will look at the Majorana equation from the point of view of the pilot-wave theory and I will show that it predicts peculiar trajectories for the beables; they have to move luminally at all times and they usually undergo complex helical trajectories to give the illusion that their motion is subluminal. The nature of the Majorana trajectory suggests that relaxation to quantum equilibrium could only be partial and that quantum non-equilibrium could still survive at length scales below the Compton wavelength. I investigate this claim, thanks to some numerical simulations of the temporal evolution of non-equilibrium distributions, for three-dimensional confined systems governed by the Dirac and Majorana equations.


WP(1,1,4,6)
r symmetry respects
Batyrev
Hyp.

Sub-Compton Quantum Non-equilibrium and Majorana Systems



Samuel Colin

Griffith UNIVERSITY
Centre for Quantum Dynamics
Brisbane, Australia

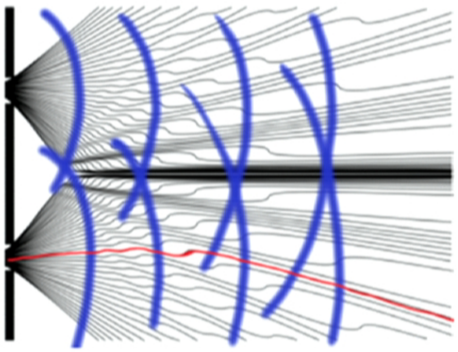
PI QF Seminar, 27 March 2012

The de Broglie-Bohm pilot-wave theory

- Standard quantum mechanics: $\psi(t, X)$
- Element of an ensemble in the dBB PWT: $(\psi(t, X), X(t))$
- Ensemble: $(\psi(t, X), \rho(t, X))$
- Guidance equation for $X(t)$, chosen in such a way that

$$\rho(t, X) = |\psi(t, X)|^2 \quad (\textit{QUANTUM EQUILIBRIUM}) \tag{1}$$

  if that holds for some initial time t_0. $\Rightarrow \dfrac{dX}{dt} = \dfrac{J(t, X)}{|\psi(t, X)|^2}$.
- 2-slit experiment:



Quantum non-equilibrium and relaxation to quantum equilibrium

- QNE and relaxation to quantum equilibrium:

$$\rho(t_0, X) \neq |\psi(t_0, X)|^2 \rightarrow \bar{\rho}(t, X) \approx \overline{|\psi(t, X)|^2} \quad (2)$$

- Standard QM as a special case of the dBB PWT
- New physics of quantum non-equilibrium.

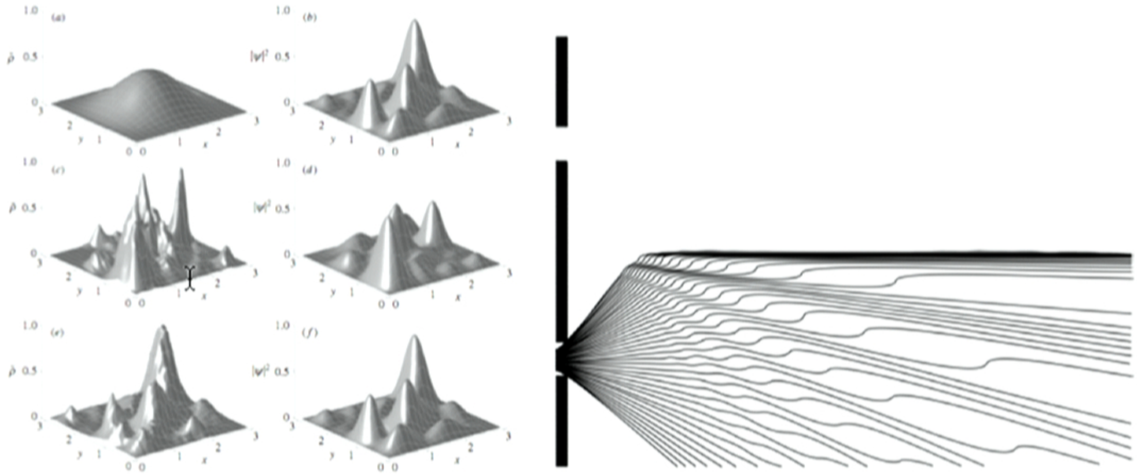


Figure: Left figure, from (Valentini & Westman, *Proc. R. Soc. A*, **461**, 253– (2005).)

The Majorana equation and its relevance today

- Majorana' s work (quoting Peter Woit):

  > *Majorana's most important scientific work appeared in a 1932 Nuovo Cimento paper motivated by the desire to find a replacement for the Dirac equation that would solve the problem of its negative energy states (a problem which disappeared in 1932 with the discovery of the positron). In this paper, Majorana investigated for the first time infinite dimensional representations of the Lorentz group, ones whose role in physics, if any, remains mysterious. As part of this work, he discovered the possibility of a real representation of the Clifford algebra and thus a version of the Dirac equation in which a particle is its own anti-particle.*

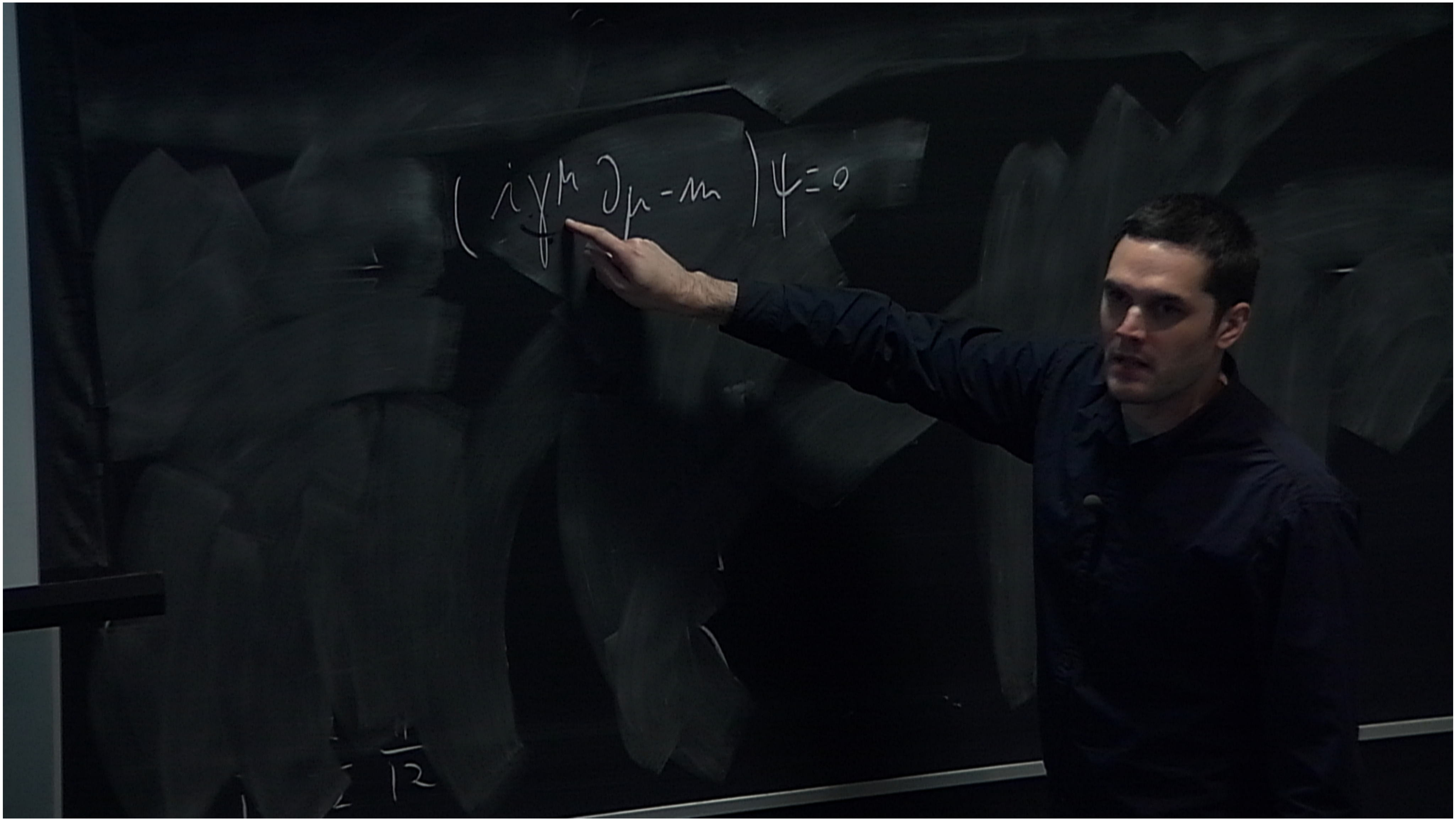
- By-product of Majorana's work, the Majorana equation.

Chi l'ha visto?

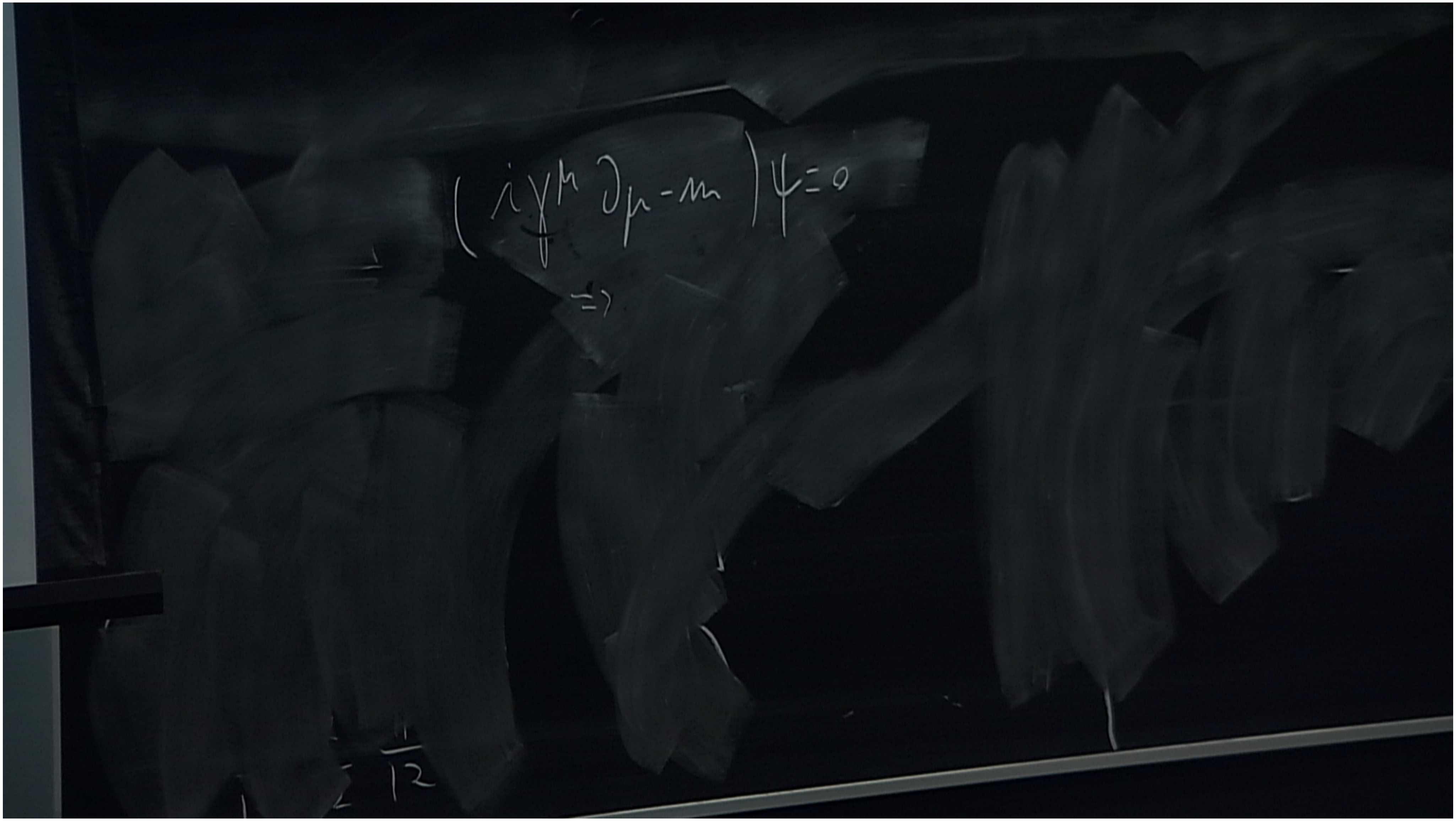
Ettore Majorana, ordinario di fisica teorica all' Università di Napoli, è misteriosamente scomparso dagli ultimi di marzo. Di anni 31, alto metri 1,70, snello, con capelli neri, occhi scuri, una lunga cicatrice sul dorso di una mano. Chi ne sapesse qualcosa è pregato di scrivere al R. P. E. Marianecci, Viale Regina Margherita 66 - Roma.

Ettore Majorana - Questo annuncio della famiglia Majorana apparve sulla «Domenica del Corriere» del 17 luglio 1938.

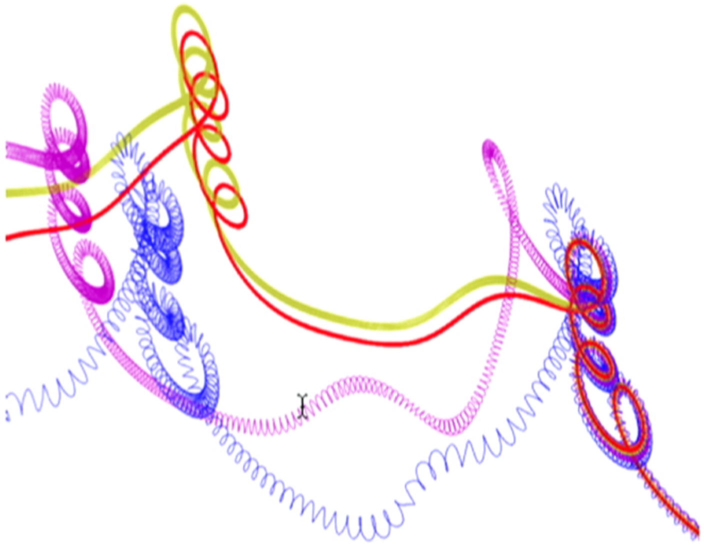
- Relevance today in many domains (neutrino physics, SUSY, condensed matter, quantum information). F.Wilczek, *Nature Physics* **5**, 614 - 618 (2009)

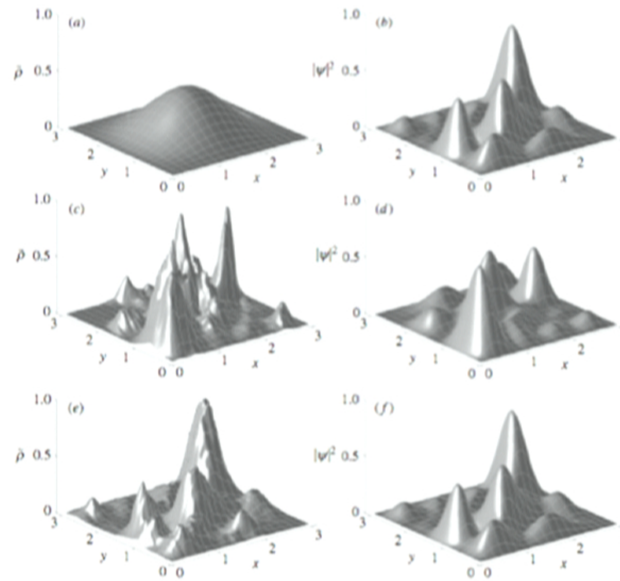


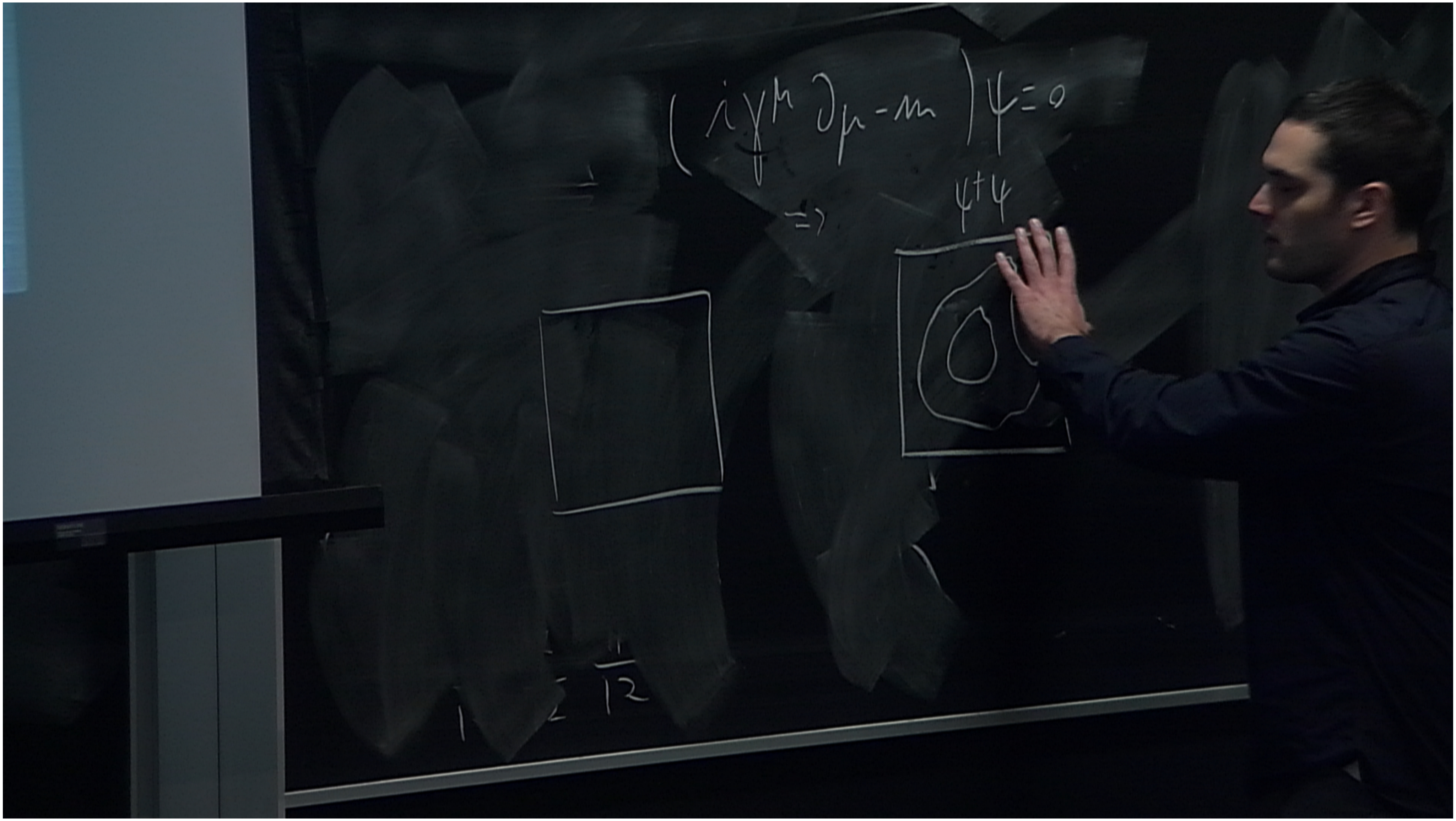
$(i\gamma^\mu \partial_\mu - m)\psi = 0$


$(i\gamma^\mu \partial_\mu - m)\psi = 0$

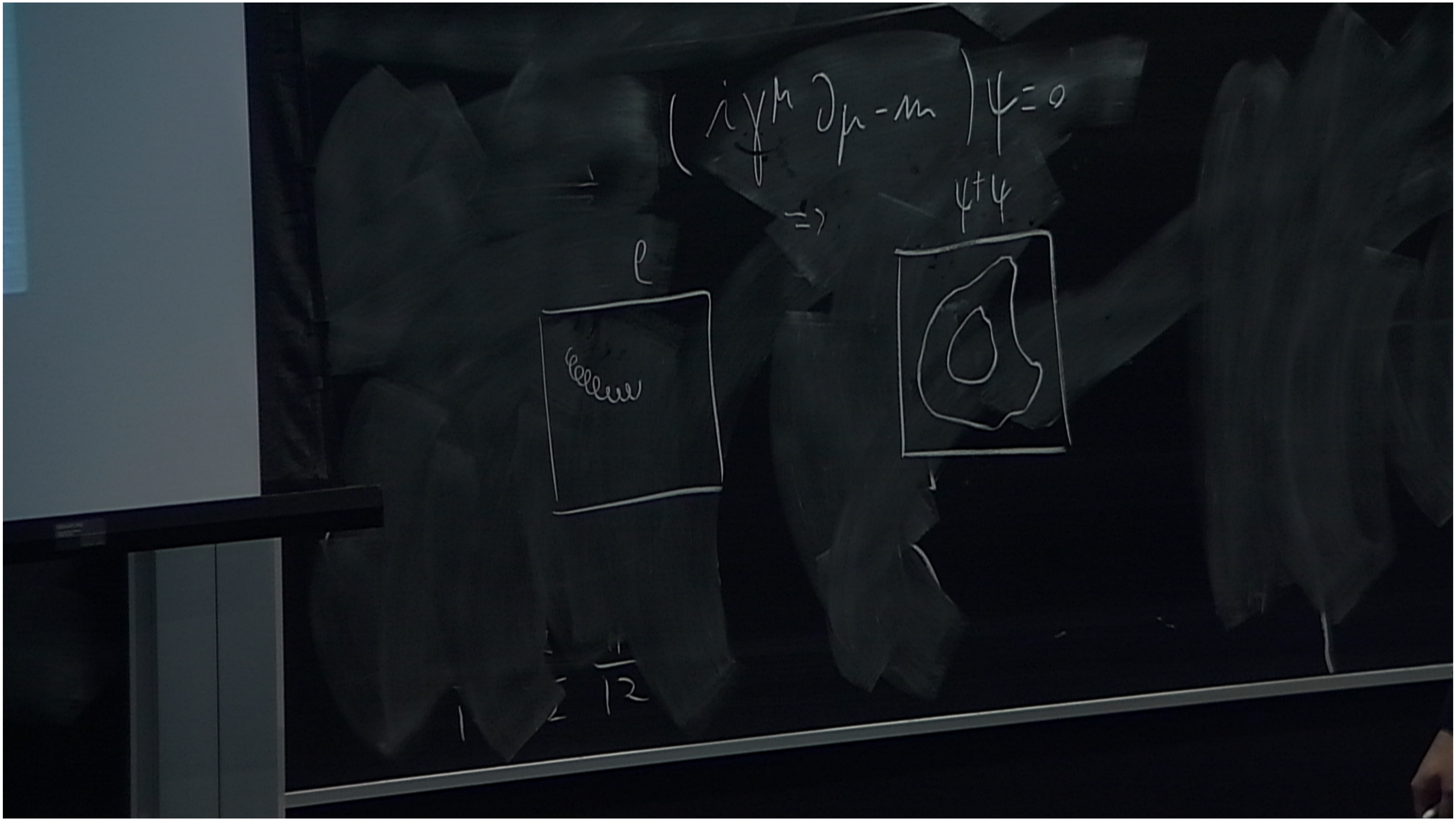
Nature of the Majorana trajectory + relaxation to QE? Today's question.

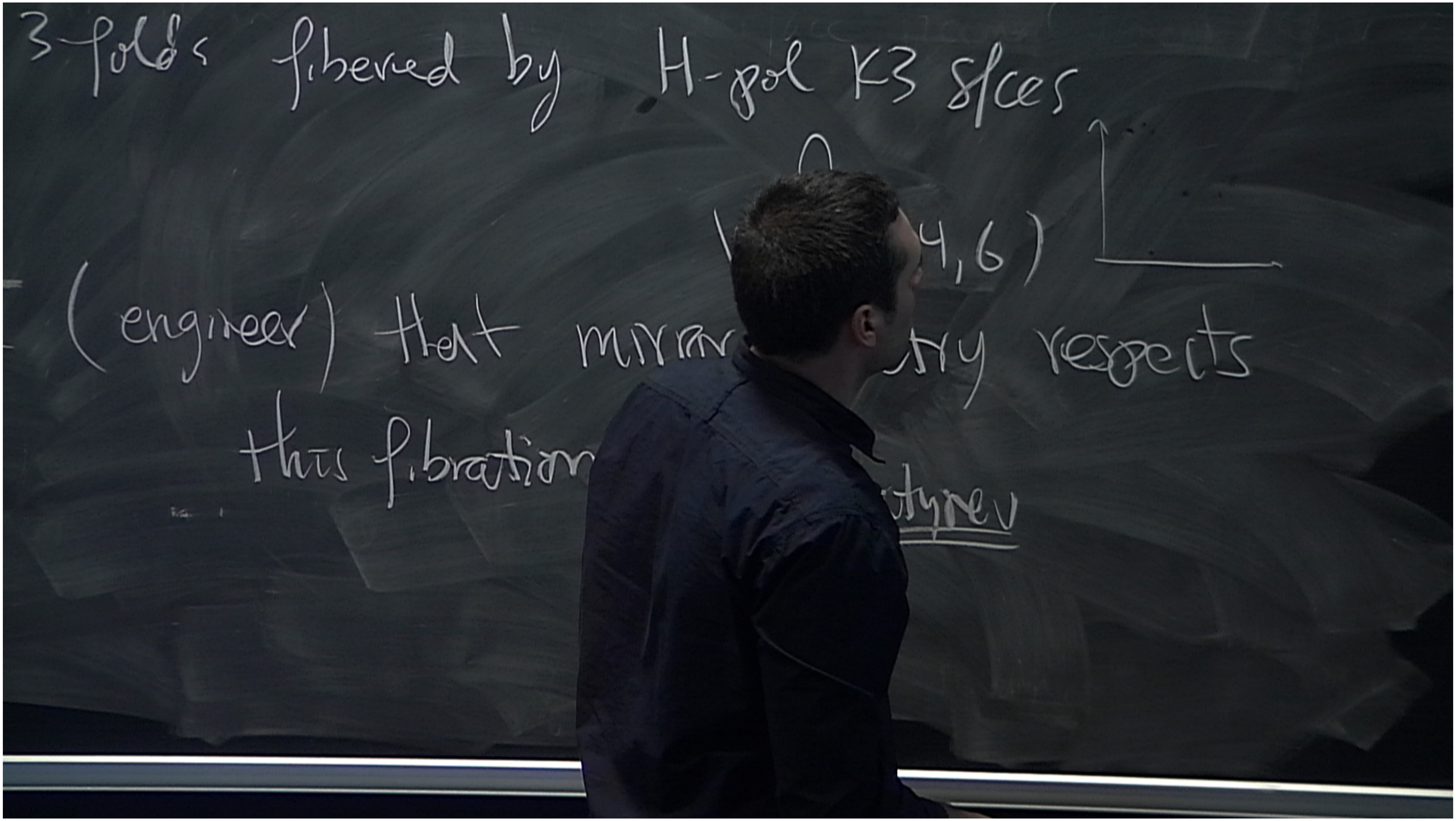






(iγ^μ ∂_μ - m)ψ=0
ψ†ψ


$(i\gamma^\mu \partial_\mu - m)\psi = 0$
$\psi^\dagger \psi$
e


3-folds fibered by H-pol K3 s/ces
4,6)
(engineer) that mirror
try respects
this fibration

Outline

- Relativistic wave equations
- Pilot-wave theory for the Majorana equation
- Relaxation simulations for the Dirac and Majorana equations
- Conclusion

The Dirac equation

The Dirac equation is

$$(i\gamma^\mu \partial_\mu - m)\psi(t, \vec{x}) = 0 \ , \tag{3}$$

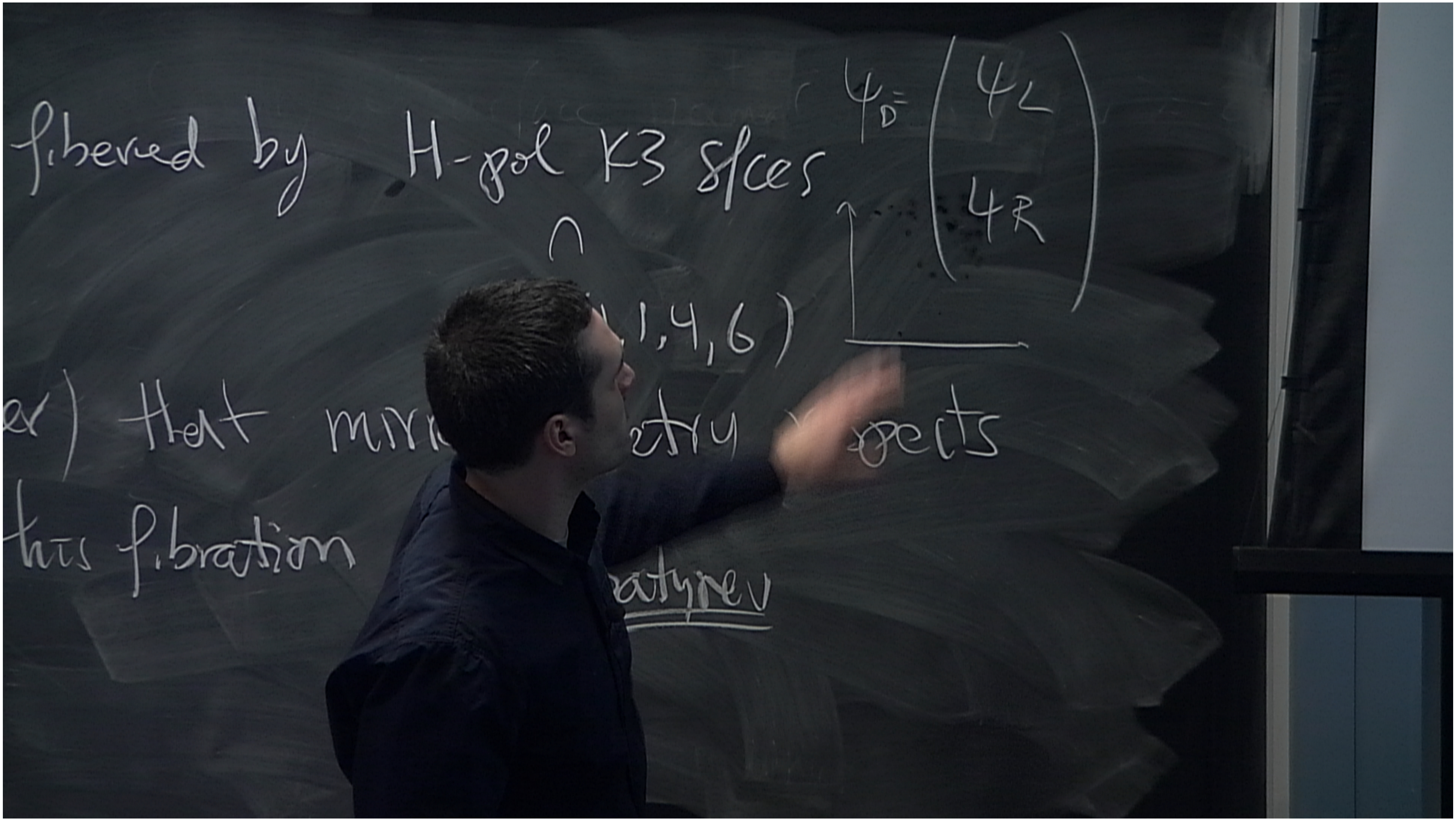
where the γ-matrices satisfy the relations $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$, where $\eta^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$.
The conserved 4-current is given by

$$j^\mu = \bar{\psi}\gamma^\mu\psi \ , \tag{4}$$

where $\bar{\psi} = \psi^\dagger\gamma^0$.
A particular representation of the γ-matrices is the Weyl representation:

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \tilde{\sigma}^\mu & 0 \end{pmatrix} \tag{5}$$

where $\sigma^\mu = (\mathbb{I}, \vec{\sigma})$ and $\tilde{\sigma}^\mu = (\mathbb{I}, -\vec{\sigma})$.


fibered by H-pol K3 sfces
1,4,6)
4L
4R
er) that
this fibration

The Dirac equation: plane-wave solutions

The positive and negative-energy plane-wave solutions are denoted by $u(\vec{p})e^{-iE(\vec{p})t+i\vec{p}\cdot\vec{x}}$ and $v(\vec{p})e^{iE(\vec{p})t+i\vec{p}\cdot\vec{x}}$, where $E(\vec{p}) = \sqrt{|\vec{p}|^2 + m^2}$. In the Weyl representation, if one introduces the right-handed and left-handed eigenstates of helicity $\chi_R(\vec{p})$ and $\chi_L(\vec{p})$, the plane-wave solutions can be given by

$$u_R(\vec{p}) = \begin{pmatrix} \sqrt{\frac{E-p}{2E}}\chi_R(\vec{p}) \\ \sqrt{\frac{E+p}{2E}}\chi_R(\vec{p}) \end{pmatrix} , \tag{6}$$

$$u_L(\vec{p}) = \begin{pmatrix} \sqrt{\frac{E+p}{2E}}\chi_L(\vec{p}) \\ \sqrt{\frac{E-p}{2E}}\chi_L(\vec{p}) \end{pmatrix} , \tag{7}$$

$$v_L(\vec{p}) = \begin{pmatrix} \sqrt{\frac{E-p}{2E}}\chi_L(\vec{p}) \\ \sqrt{\frac{E+p}{2E}}\chi_L(\vec{p}) \end{pmatrix} , \tag{8}$$

$$v_R(\vec{p}) = \begin{pmatrix} \sqrt{\frac{E+p}{2E}}\chi_R(\vec{p}) \\ \sqrt{\frac{E-p}{2E}}\chi_R(\vec{p}) \end{pmatrix} . \tag{9}$$

The Dirac equation: charge conjugation

Dirac equation for an electron with a vector potential:

$$\gamma^\mu(i\partial_\mu + eA_\mu)\psi - m\psi = 0 \,. \tag{10}$$

Dirac equation for the charge conjugate:

$$\gamma^\mu(i\partial_\mu - eA_\mu)\psi_{(c)} - m\psi_{(c)} = 0 \,. \tag{11}$$

Charge conjugation:

$$\psi_{(c)} = i\gamma^2\psi^* \,. \tag{12}$$

The Majorana spinor

- 'Real' solution ψ_M?
- It doesn't change under charge conjugation:

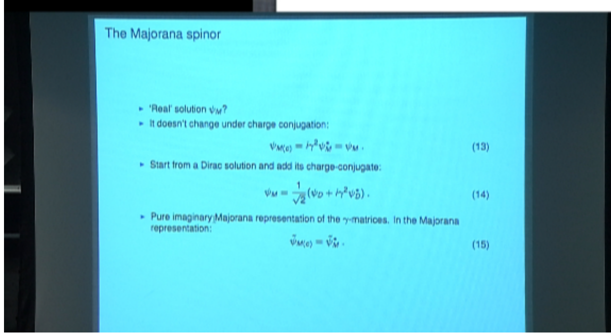
$$\psi_{M(c)} = i\gamma^2 \psi_M^* = \psi_M \, . \tag{13}$$

- Start from a Dirac solution and add its charge-conjugate:

$$\psi_M = \frac{1}{\sqrt{2}} (\psi_D + i\gamma^2 \psi_D^*) \, . \tag{14}$$

- Pure imaginary Majorana representation of the γ-matrices. In the Majorana representation:

$$\tilde{\psi}_{M(c)} = \tilde{\psi}_M^* \, . \tag{15}$$



Pilot-wave theory for the Majorana equation

- Majorana spinor: $\psi_M = \frac{1}{\sqrt{2}}(\psi_D + \psi_{D(c)}) = \frac{1}{\sqrt{2}}(\psi_D + i\gamma^2\psi_D^*)$.
- PWT: Complete description $(\psi_M(t,\vec{x}), \vec{x}_M(t))$
- 4-current:

$$j_M^\mu = \bar{\psi}_M\gamma^\mu\psi_M = \bar{\psi}_D\gamma^\mu\psi_D + \mathfrak{Re}\left(\bar{\psi}_D\gamma^\mu\psi_{D(c)}\right) \ . \qquad (16)$$

- $\psi_M = \begin{pmatrix}\psi_L \\ \psi_R\end{pmatrix}$:

$$j_M^\mu = 2\psi_R^\dagger\sigma^\mu\psi_R \qquad \text{LIGHT-LIKE 4-CURRENT.} \qquad (17)$$

- Guidance equation:

$$\vec{v}_M(t,\vec{x}) = \left.\frac{\vec{j}_M(t,\vec{x})}{j_M^0(t,\vec{x})}\right|_{\vec{x}=\vec{x}(t)} \quad \text{with } |\vec{v}_M| = 1. \qquad (18)$$

- Majorana equation VERSUS Majorana particle:

$$\widehat{\psi}_M(t,\vec{x}) = \frac{1}{\sqrt{(2\pi)^3}}\sum_s\int d^3p\sqrt{\frac{m}{E_{\vec{p}}}}\left(\widehat{c}_s(\vec{p})u_s(\vec{p})e^{-iE_{\vec{p}}t+i\vec{p}\cdot\vec{x}} + \widehat{c}_s^\dagger(\vec{p})v_s(\vec{p})e^{iE_{\vec{p}}t-i\vec{p}\cdot\vec{x}}\right) \qquad (19)$$

Example: plane-wave solution

We consider a right-handed particle moving in the positive z-direction with momentum p_z. Then the Dirac spinor is

$$\psi_D(t, \vec{x}) = \begin{pmatrix} \sqrt{\frac{E-p}{2E}} \\ 0 \\ \sqrt{\frac{E+p}{2E}} \\ 0 \end{pmatrix} e^{-iEt+ip_z z} \tag{20}$$

Therefore the velocity field is

$$j_D^\mu = \bar{\psi}_D \gamma^\mu \psi = (1, 0, 0, \frac{p_z}{E}) \; . \tag{21}$$

Therefore, in the corresponding pilot-wave theory, if the particle is of the Dirac type, the particle moves along a straight line with uniform velocity $(0, 0, \frac{p_z}{E})$.

Example: plane-wave solution

For the corresponding Majorana solution, the spinor is given by

$$\psi_M = \frac{1}{\sqrt{2}}(\psi_D + i\gamma^2\psi_D^*) = \frac{1}{\sqrt{2}}(\psi_D + \psi_{D(c)}) \, . \tag{22}$$

Total current j_M^μ is given by

$$(1, \frac{m}{E}\cos(2Et - 2p_z z), -\frac{m}{E}\sin(2Et - 2p_z z), \frac{p}{E}) \, . \tag{23}$$

Solution (assuming $\vec{x}(0) = (0,0,0)$):

$$\vec{x}(t) = (\frac{1}{2m}\sin(2\frac{m^2}{E}t), \frac{1}{2m}\cos(2\frac{m^2}{E}t) - \frac{1}{2m}, \frac{p}{E}t) \tag{24}$$

Helical trajectory whose radius is the Compton wavelength of the particle

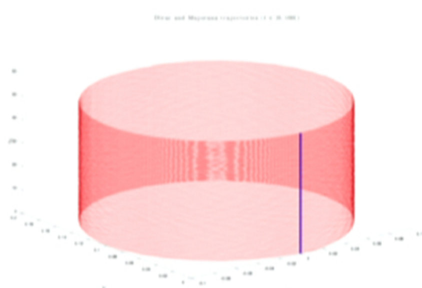


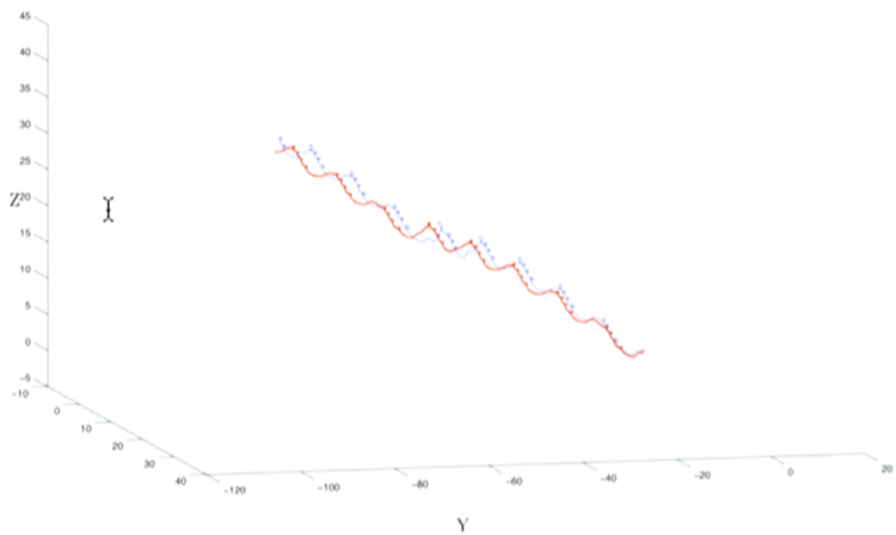
Figure: The mass is $m = 5$ and $p_z = 3$. The beables start from the origin and $t \in [0, 100]$.

Superposition of plane waves for Dirac (blue) and Majorana (red) solutions

State:

$$\psi(t,\vec{x}) = \frac{1}{\sqrt{3}}\left(e^{-iE_1 t}u_R(\vec{p}_1) + e^{i4}e^{-iE_2 t}u_R(\vec{p}_2) + e^{i9}e^{-iE_3 t}u_R(\vec{p}_3)\right) , \quad (25)$$

where $\vec{p}_1 = (1,0,1)$, $\vec{p}_2 = (-1,-2,-1)$ and $\vec{p}_3 = (1,-1,1)$.
Mass $m = 10$. Initial position $(0,0,0)$. $t \in [0,1000]$.

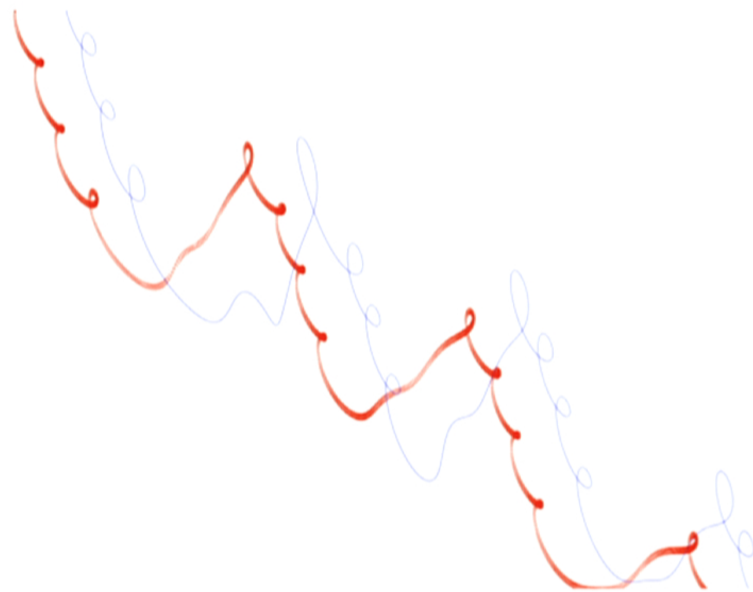


Superposition of plane waves for Dirac (blue) and Majorana (red) solutions ZOOM

State:

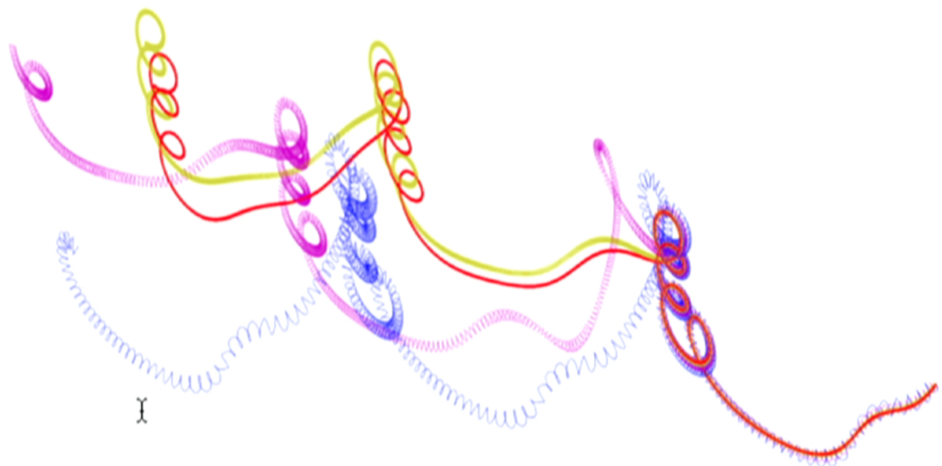
$$\psi(t,\vec{x}) = \frac{1}{\sqrt{3}}\left(e^{-iE_1 t}u_R(\vec{p}_1) + e^{i4}e^{-iE_2 t}u_R(\vec{p}_2) + e^{i9}e^{-iE_3 t}u_R(\vec{p}_3)\right) , \qquad (26)$$

where $\vec{p}_1 = (1,0,1)$, $\vec{p}_2 = (-1,-2,-1)$ and $\vec{p}_3 = (1,-1,1)$.
Mass $m = 10$. Initial position $(0,0,0)$. $t \in [0, 1000]$.



HELICAL TRAJECTORY AGAIN WITH COMPTON WAVELENGTH DIAMETER.

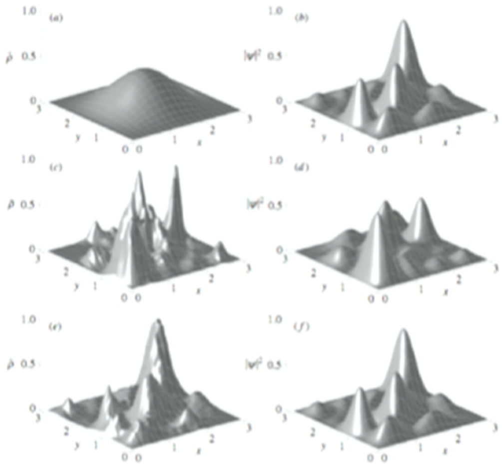
Varying the mass

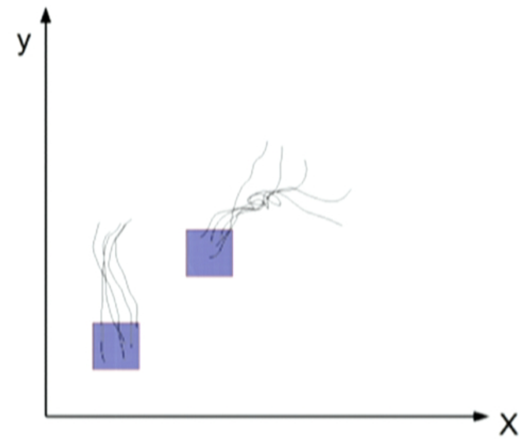


Mass 36, 18, 9, 6 for $\Delta t = 1200, 600, 300, 200$.

Helical trajectory with helix diameter = Compton wavelength.

Relaxation, chaos, node and vorticity

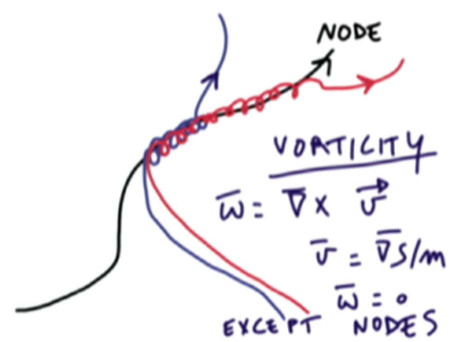




The following quantity

$$\frac{\rho(t,\vec{x})}{|\psi(t,\vec{x})|^2} \qquad (27)$$

is also conserved along a trajectory.
Coarse-graining: $\bar{\rho}(t,\vec{x})$ and $\overline{|\psi(t,\vec{x})|^2}$.



Example: plane-wave solution

For the corresponding Majorana solution, the spinor is given by

$$\psi_M = \frac{1}{\sqrt{2}}(\psi_D + i\gamma^2\psi_D^*) = \frac{1}{\sqrt{2}}(\psi_D + \psi_{D(c)}) \,. \tag{22}$$

Total current j_M^μ is given by

$$(1, \frac{m}{E}\cos(2Et - 2p_z z), -\frac{m}{E}\sin(2Et - 2p_z z), \frac{p}{E}) \,. \tag{23}$$

Solution (assuming $\vec{x}(0) = (0, 0, 0)$):

$$\vec{x}(t) = (\frac{1}{2m}\sin(2\frac{m^2}{E}t), \frac{1}{2m}\cos(2\frac{m^2}{E}t) - \frac{1}{2m}, \frac{p}{E}t) \tag{24}$$

Helical trajectory whose radius is the Compton wavelength of the particle

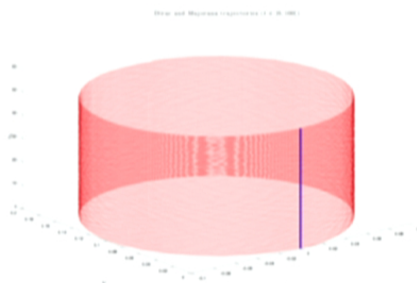
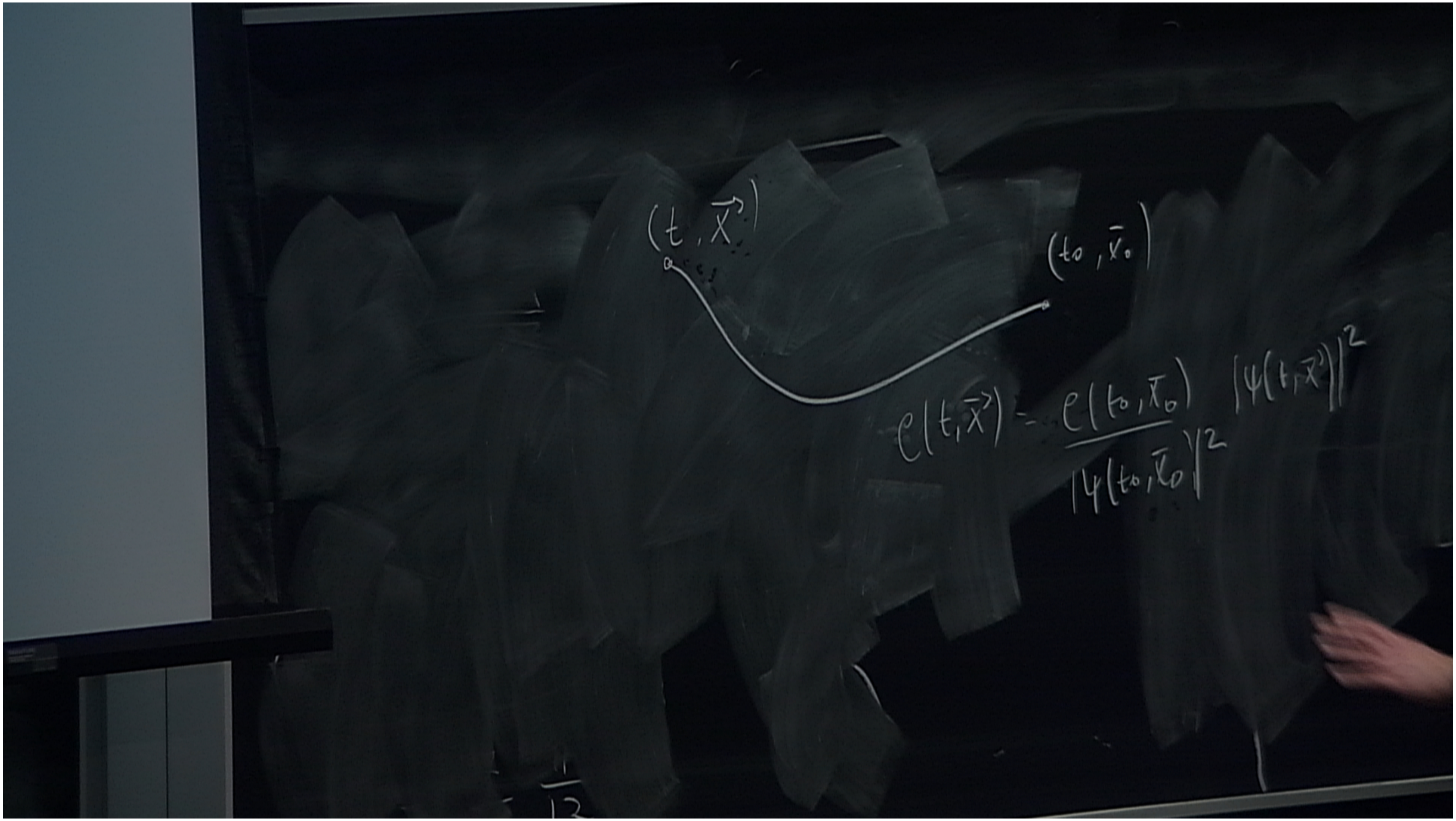
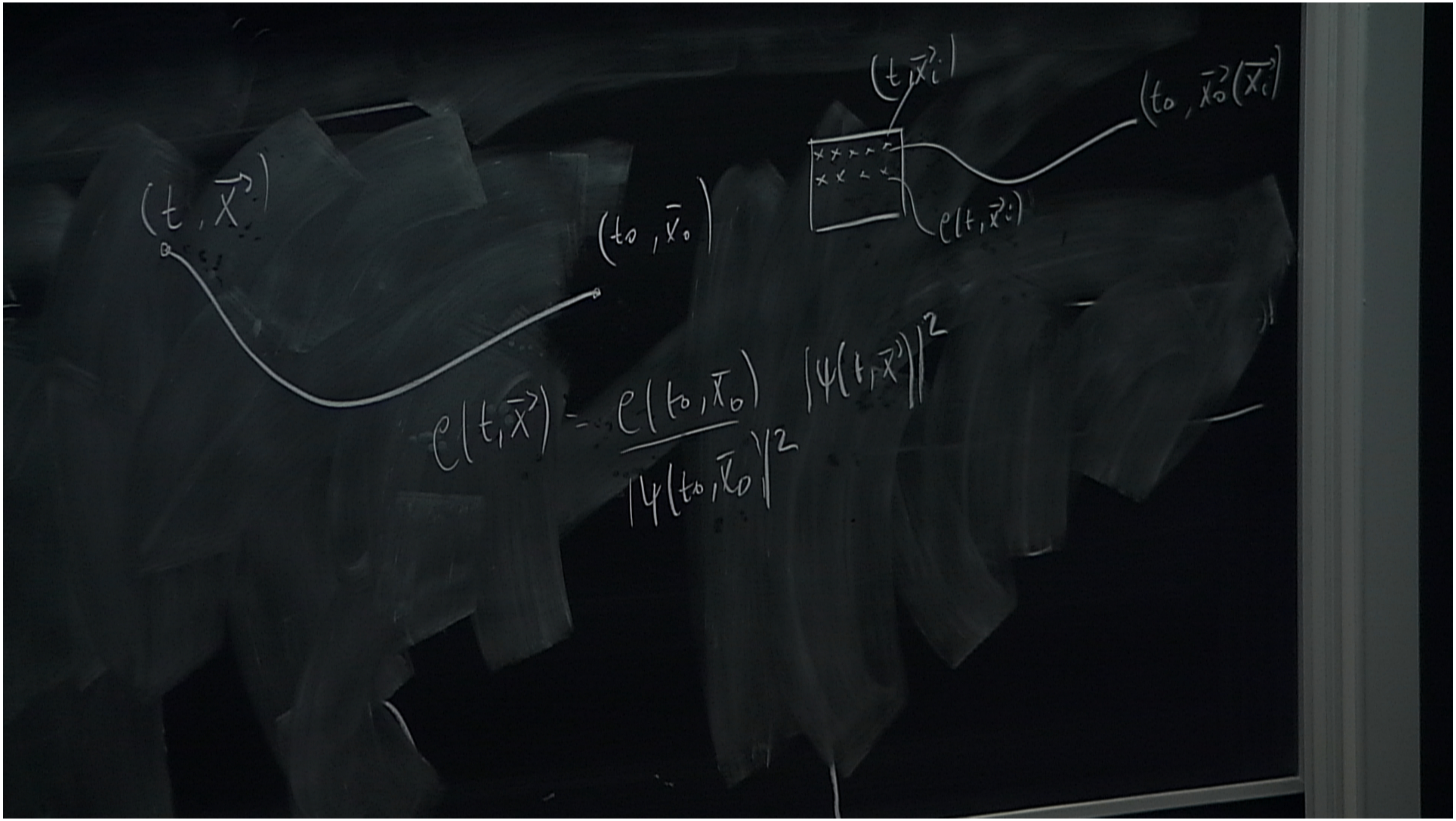
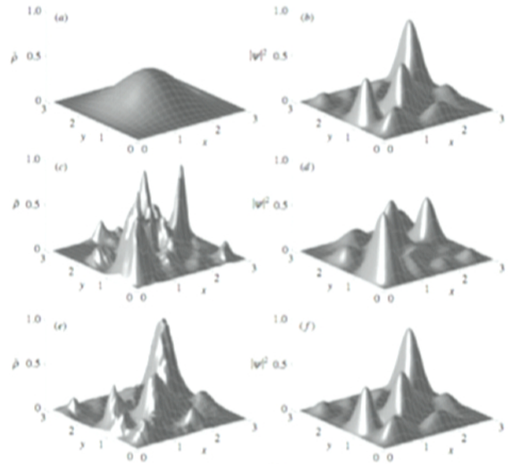


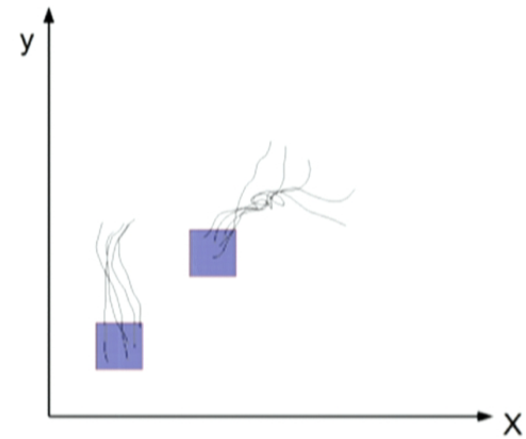
Figure: The mass is $m = 5$ and $p_z = 3$. The beables start from the origin and $t \in [0, 100]$.


(t, x⃗)
(t₀, x̄₀)
ρ(t, x⃗) = ρ(t₀, x̄₀) / |ψ(t₀, x̄₀)|² · |ψ(t, x⃗)|²


(t, x⃗)
(t₀, x̄₀)
(t, x⃗ᵢ)
(t₀, x⃗₀(x⃗ᵢ))
ρ(t, x⃗ᵢ)
ρ(t, x⃗) = ρ(t₀, x̄₀) / |ψ(t₀, x̄₀)|² · |ψ(t, x⃗)|²

Relaxation, chaos, node and vorticity

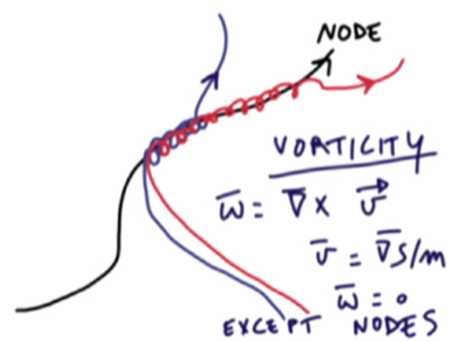




The following quantity

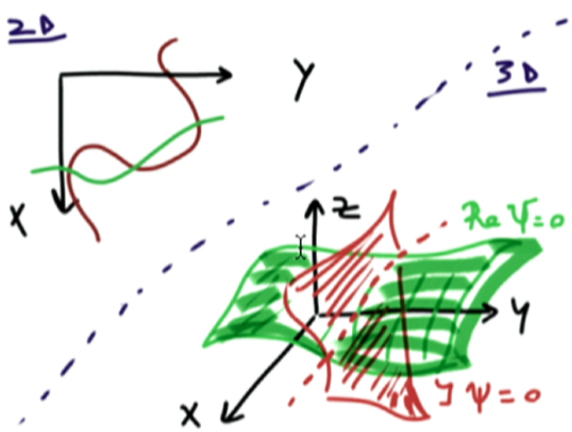
$$\frac{\rho(t,\vec{x})}{|\psi(t,\vec{x})|^2} \qquad (27)$$

is also conserved along a trajectory.
Coarse-graining: $\bar{\rho}(t,\vec{x})$ and $\overline{|\psi(t,\vec{x})|^2}$.



Distribution of nodes (scalar case and Dirac spinor)

NONRELATIVISTIC CASE.



DIRAC SPINOR.

- $\psi_D = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$
- Nodes where

$$\begin{aligned} \mathfrak{Re}(\psi_1) &= \mathfrak{Im}(\psi_1) = \ldots \\ \mathfrak{Re}(\psi_4) &= \mathfrak{Im}(\psi_4) = 0\,. \end{aligned} \tag{28}$$

- 8 conditions!
- Typically no node, but vorticity even in the absence of nodes because

$$\vec{v} = \frac{\psi^\dagger \vec{\alpha} \psi}{\psi^\dagger \psi}\,. \tag{29}$$

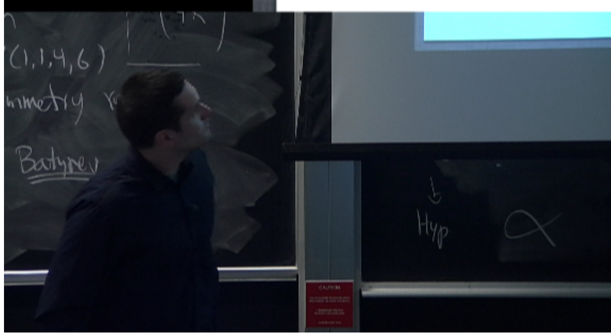
S. Colin, *Proc. R. Soc. A* **468**, 1116–1135 (2012).

Relaxation simulations for spinors

- Interesting to do a relaxation simulation for any spinor (Dirac, Weyl, Majorana).
- Majorana has additional constraints with respect to Dirac.
  More symmetry = more regularity in the trajectory.
  Superposition of states with $m = 0$ in the H-atom.
- Majorana is highly constrained on the Compton wavelength scale. If there is relaxation, there might still be residual non-equilibrium at the Compton wavelength scale.

Which system shall we simulate?

- Simulations for ψ_D and $\psi_M = \frac{(\psi_D + i\gamma^2\psi_D^*)}{\sqrt{2}}$ and comparison.
- 3D simulations (because Majorana solutions can have nodes in 2D with a 2D representation [1]).
- Confined system.
- Invariance under charge conjugation which excludes a spherical step potential.
- Dirac particle with a position-dependent mass (m, M) plus spherical coordinates.
- Keep helical nature of the trajectory.
- Coarse-graining length smaller to the helix diameter.



…n still use a 4D representation in a $2 + 1$ space-time.

System under consideration

- Dirac equation with position-dependent mass:

$$i\partial_t\psi(t,\vec{x}) = (-i\vec{\alpha}\cdot\vec{\nabla} + m(|\vec{x}|)\beta)\psi(t,\vec{x}) \quad (30)$$

  with $m(r) = m$ if $r \leq R$ and $m(r) = M$ otherwise.
- Observables that commute with H: K, J and J_3 (eigenvalues E, $-\kappa\hbar$, $j(j+1)\hbar^2$ and $j_3\hbar$). κ is non-zero and κ and j are related by $\kappa = \pm(j+\frac{1}{2})$.
- If we look for eigenstates E such that $E^2 - m^2 > 0$ and $M^2 - E^2 > 0$, the energy eigenstates can be expressed in terms of spherical Bessel functions of the first kind and modified spherical Bessel functions of the second kind of integer order:

$$\psi_{in} = Ae^{-iEt}\begin{pmatrix} j_{l(\kappa)}(p_{in}r)\mathcal{Y}^{j_3}_{jl_A}(\theta,\phi) \\ i\frac{\kappa}{|\kappa|}\frac{p_{in}}{E+m}j_{lm(\kappa)}(p_{in}r)\mathcal{Y}^{j_3}_{jl_B}(\theta,\phi) \end{pmatrix} \quad (31)$$

  and

$$\psi_{ext} = Be^{-iEt}\begin{pmatrix} j_{l(\kappa)}(p_{ext}r)\mathcal{Y}^{j_3}_{jl_A}(\theta,\phi) \\ -i\frac{p_{ext}}{E+M}j_{lm(\kappa)}(p_{ext}r)\mathcal{Y}^{j_3}_{jl_B}(\theta,\phi) \end{pmatrix} \quad (32)$$

- The energy eigenvalues have to be found numerically.

System under consideration

- Dirac equation with position-dependent mass:

$$i\partial_t \psi(t,\vec{x}) = (-i\vec{\alpha}\cdot\vec{\nabla} + m(|\vec{x}|)\beta)\psi(t,\vec{x}) \tag{30}$$

  with $m(r) = m$ if $r \leq R$ and $m(r) = M$ otherwise.
- Observables that commute with H: K, J and J_3 (eigenvalues E, $-\kappa\hbar$, $j(j+1)\hbar^2$ and $j_3\hbar$). κ is non-zero and κ and j are related by $\kappa = \pm(j+\frac{1}{2})$.
- If we look for eigenstates E such that $E^2 - m^2 > 0$ and $M^2 - E^2 > 0$, the energy eigenstates can be expressed in terms of spherical Bessel functions of the first kind and modified spherical Bessel functions of the second kind of integer order:

$$\psi_{in} = Ae^{-iEt}\begin{pmatrix} j_{l(\kappa)}(p_{in}r)\mathcal{Y}^{j_3}_{jl_A}(\theta,\phi) \\ i\frac{\kappa}{|\kappa|}\frac{p_{in}}{E+m} j_{lm(\kappa)}(p_{in}r)\mathcal{Y}^{j_3}_{jl_B}(\theta,\phi) \end{pmatrix} \tag{31}$$

  and

$$\psi_{ext} = Be^{-iEt}\begin{pmatrix} j_{l(\kappa)}(p_{ext}r)\mathcal{Y}^{j_3}_{jl_A}(\theta,\phi) \\ -i\frac{p_{ext}}{E+M} j_{lm(\kappa)}(p_{ext}r)\mathcal{Y}^{j_3}_{jl_B}(\theta,\phi) \end{pmatrix} \tag{32}$$

- The energy eigenvalues have to be found numerically.

System under consideration

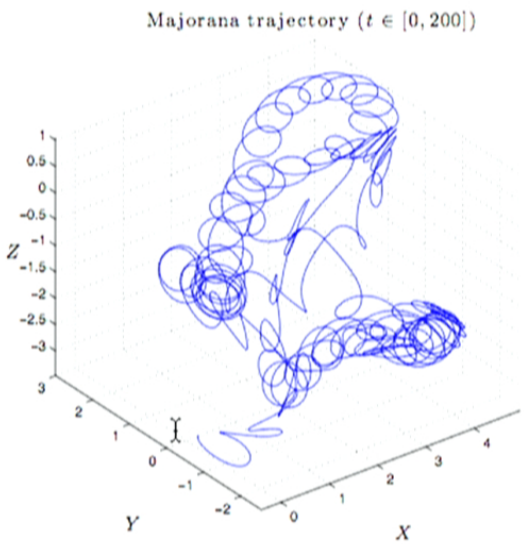
Superposition of 8 modes. $R = 5$.

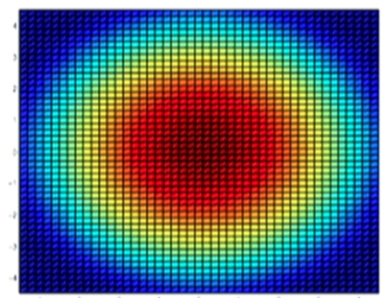
simulation	m	M	helix radius?	time
1	1	3	[1/3, 1]	300
2	1	1.5	[2/3, 1]	200
3	1	1.5	[2/3, 1]	300

mode	κ	j	j_3	phase
1	1	1/2	1/2	$e^{i5.11905989575681}$
2	1	1/2	$-1/2$	$e^{i5.69125859039527}$
3	-1	1/2	1/2	$e^{i0.79788169834087}$
4	-1	1/2	$-1/2$	$e^{i5.73890975922526}$
5	2	3/2	$-1/2$	$e^{i3.97323032474265}$
6	2	3/2	3/2	$e^{i0.61286443954863}$
7	-2	3/2	$-3/2$	$e^{i1.74985591686112}$
8	-2	3/2	1/2	$e^{i3.43615792623681}$

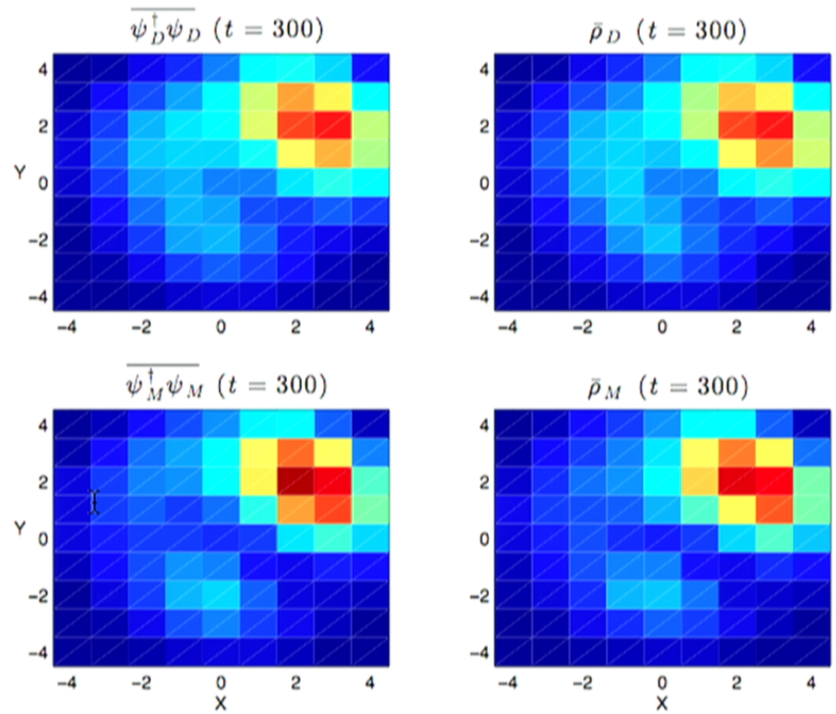
Equal weight in the superposition.

System under consideration

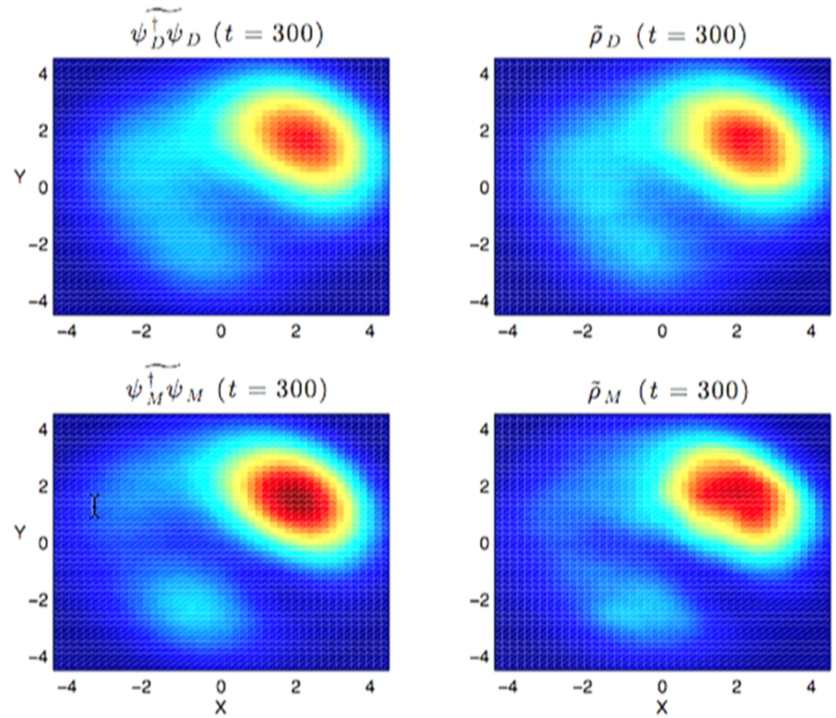




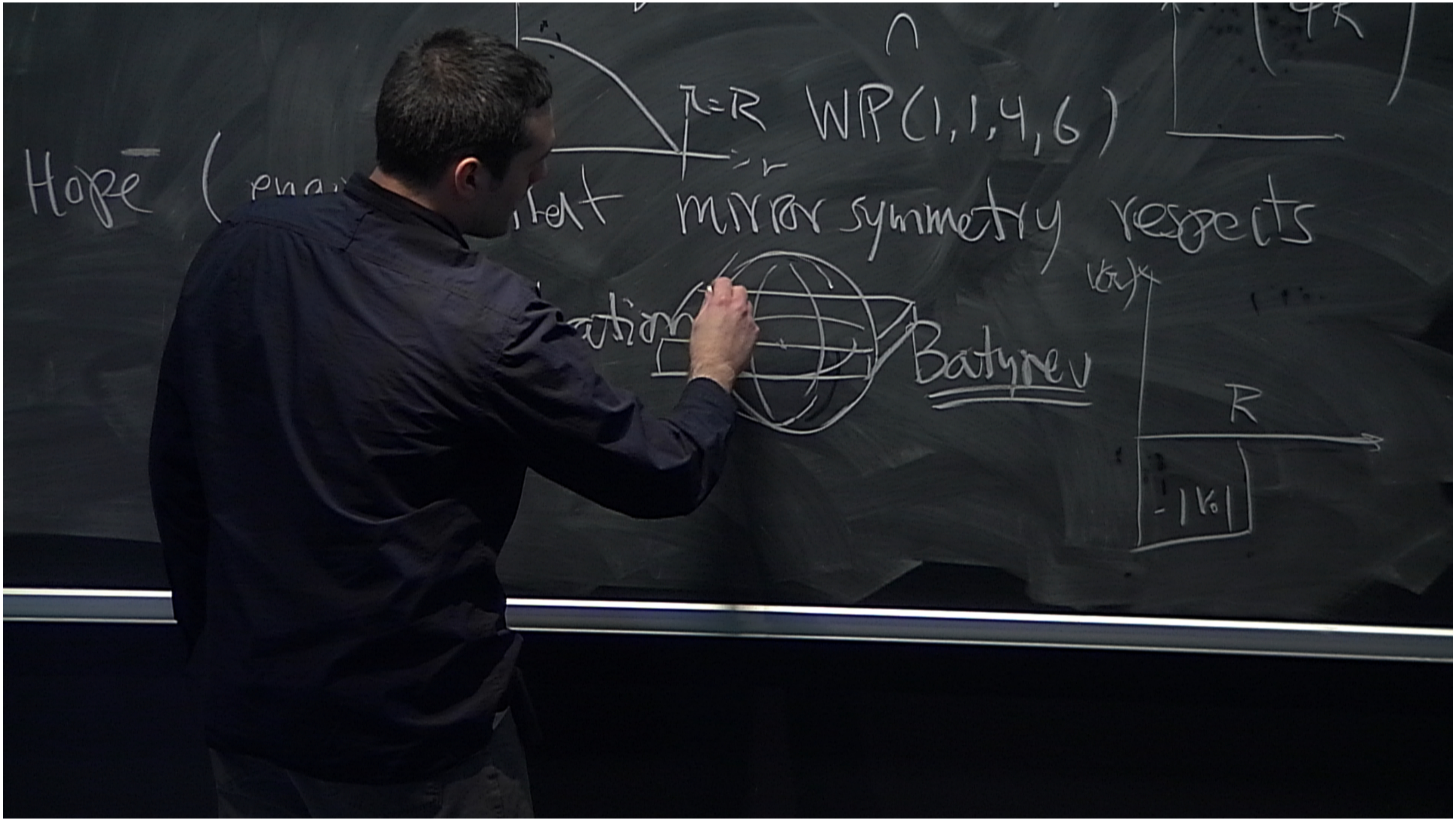
Simulation 1

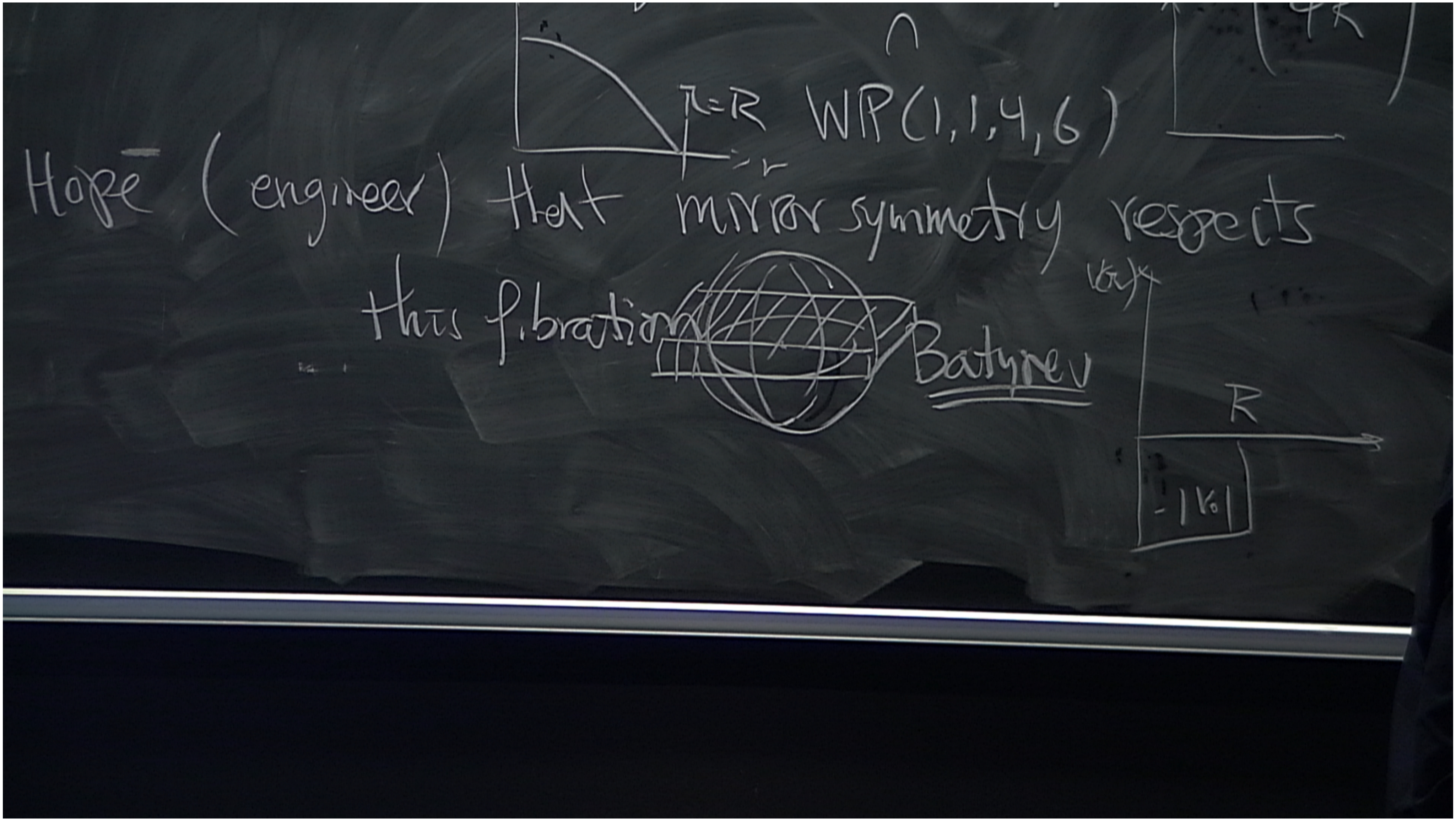


Simulation 1



Page 25 of 35


r=R WP(1,1,4,6)
Hope (
that mirror symmetry respects
Batyrev
R


r=R
WP(1,1,4,6)
Hope (engineer) that mirror symmetry respects
this fibration
Batyrev
V(R)
R
-|V0|

Simulation 1

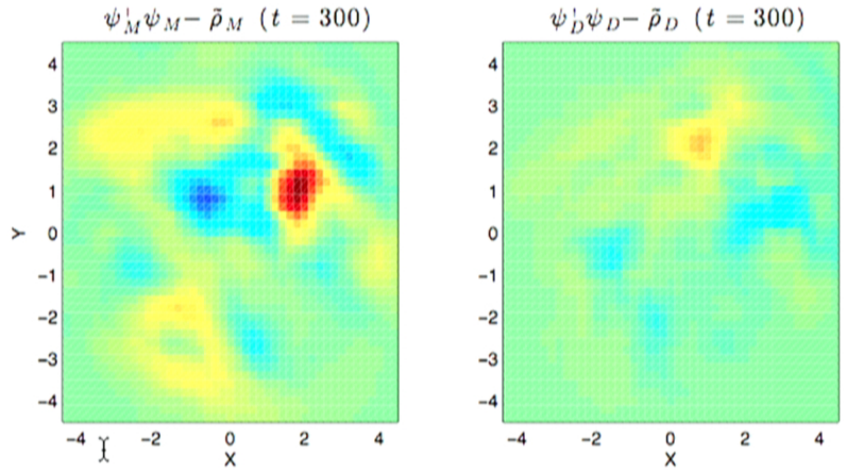
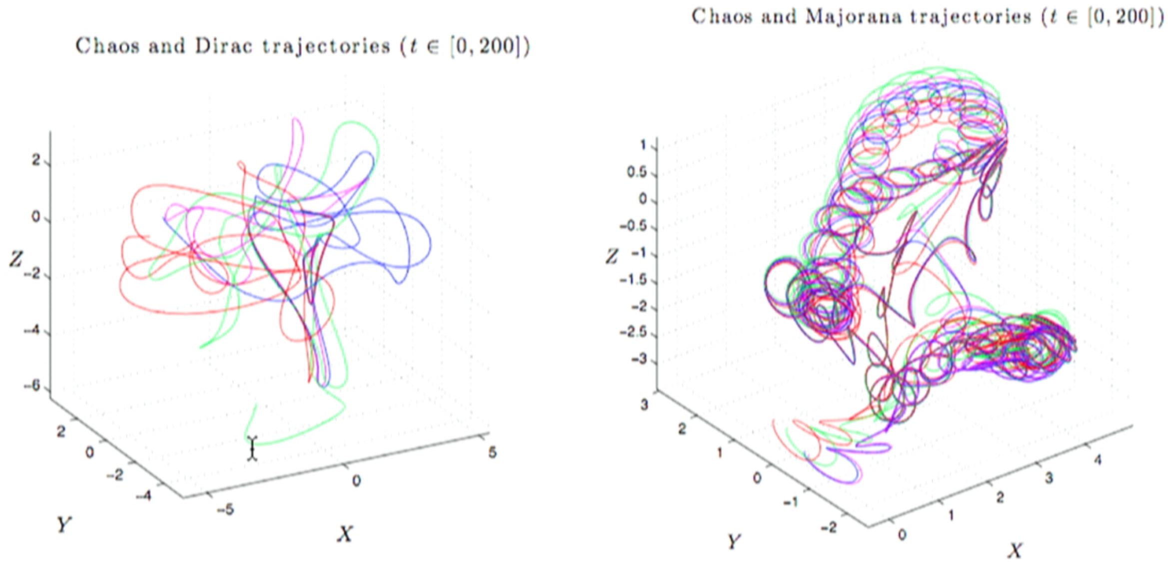


Figure: $max_M = 0.0098$, $mean_M = 0.0018$, $max_D = 0.0085$, $mean_D = 0.0020$ and $diff \in [-0.0014, 0.0014]$

Chaos for Dirac and Majorana trajectories



Larger percentage of good trajectories for the Majorana systems.

Conclusion

- Illustration of the peculiar nature of the Majorana trajectory (luminal, helical for the right set of parameters).
- Safe to say that the Majorana non-equilibrium distributions relax more slowly that the ones guided by a Dirac spinor (related to chaos).
- Claim that non-equilibrium is preserved below the Compton wavelength is unwarranted at this stage.
- A part of this work shows that non-equilibrium distributions guided by a Dirac spinor relax efficiently in 3D.
- This is first-quantization! In order to study Majorana neutrinos, one has to go to second quantization.