Title: The MSSM Spectrum from Deforming the Heterotic Standard Embedding

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Abstract: The heterotic standard embedding on a Calabi-Yau threefold yields an E_6 grand unified theory in four dimensions. The gauge symmetry can be partially broken by turning on discrete Wilson lines, but such models necessarily feature an extended gauge group and exotic light matter fields.
I will describe the recent construction of compactifications on a new three-generation manifold which solve both these problems via supersymmetric deformations of the gauge bundle. Such deformations can be interpreted as the supersymmetric Higgs mechanism in four dimensions, but our top-down construction allows us to keep control of the theory, and perform a reliable computation of the resulting spectrum. The moduli space breaks up into a number of branches depending on the initial choice of Wilson lines, and on eight of these branches we find models whose light charged spectrum is exactly that of the minimal supersymmetric standard model: unbroken gauge group SU(3)xSU(2)xU(1), three generations of quarks and leptons, and one pair of Higgs doublets.

Heterotic compactification — a crash course

Geometric compactification of the $E_8 \times E_8$ heterotic string to 4D with unbroken $\mathcal{N} = 1$ supersymmetry requires a triple $(X; V_1, V_2)$, where

- X is a smooth Calabi–Yau threefold.
- V_i is a vector bundle with holonomy group $G_i \subset E_8$, satisfying

$$F_{ij} = F_{\bar{\imath}\bar{\jmath}} = 0 \ , \ \ g^{i\bar{\jmath}} F_{i\bar{\jmath}} = 0 \ ,$$

  i.e., V_i is holomorphic and stable (by Donaldson-Uhlenbeck-Yau).

Green-Schwarz anomaly cancellation implies a topological condition:

$$c_2(TX) - c_2(V_1) - c_2(V_2) = [C] \quad \text{assuming } c_1(V_i) = 0 \ .$$

where C is the (co)homology class of some holomorphic curve in X, which must be wrapped by a 5-brane (or 5-branes, if it has multiple components).

The geometry of $(X; V_1, V_2)$ determines the four-dimensional theory. Focus on one E_8, i.e., one bundle V, with holonomy group $G \subset E_8$.

- The unbroken gauge group is the centraliser of G in E_8.
- $G = SU(5) \times U(1)_Y$ gives $G_{\text{SM}} \cong SU(3) \times SU(2) \times U(1)_Y$.
  However, the hypercharge gauge boson is massive in this case.

Massless matter is determined by cohomology groups of V and associated bundles:

- Let $G = SU(4)$.

$$
\begin{aligned}
E_8 &\supset \big(SU(4) \times \text{Spin}(10)\big)/\mathbb{Z}_4 \\
\mathbf{248} &= (\mathbf{1}, \mathbf{45}) \oplus (\mathbf{15}, \mathbf{1}) \oplus (\mathbf{4}, \mathbf{16}) \oplus (\overline{\mathbf{4}}, \overline{\mathbf{16}}) \oplus (\mathbf{6}, \mathbf{10})
\end{aligned}
$$

- Massless chiral multiplets in the **16** of Spin(10) $\leftrightarrow$ $H^1(X, V)$.

The standard embedding

For any Calabi–Yau threefold X, there is a canonical solution:

$V_1 = TX$ (the tangent bundle of X), V_2 trivial.

Notice that anomaly cancellation is satisfied with $C = 0$, i.e., no five-branes in the vacuum.

This is the 'standard embedding' (Candelas et al., 1985). Features:

- $\mathrm{Hol}(TX) \cong SU(3)$ (Calabi-Yau), so gauge group E_6. Decomposition:

$$\begin{aligned} E_8 &\supset (SU(3) \times E_6)/\mathbb{Z}_3 \\ \mathbf{248} &= (\mathbf{1}, \mathbf{78}) \oplus (\mathbf{8}, \mathbf{1}) \oplus (\mathbf{3}, \mathbf{27}) \oplus (\overline{\mathbf{3}}, \overline{\mathbf{27}}) \end{aligned}$$

- There are $h^1(X, TX) = h^2(X, TX^*) = h^{2,1}(X)$ chiral $\mathbf{27}$s and $h^1(X, TX^*) = h^{1,1}(X)$ chiral $\overline{\mathbf{27}}$s.

So 'three-generation' manifolds are those with $\chi = 2(h^{1,1} - h^{2,1}) = \pm 6$.

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The new models

- We construct a Calabi–Yau threefold $X = \widetilde{X}/\mathbb{Z}_{12}$.

$$(h^{1,1}, h^{2,1})(\widetilde{X}) = (8, 44) \ \ , \ (h^{1,1}, h^{2,1})(X) = (1, 4) \ .$$

- Upstairs, we deform $T\widetilde{X} \oplus \mathcal{O}_{\widetilde{X}} \oplus \mathcal{O}_{\widetilde{X}}$ to an irreducible supersymmetric $SU(5)$ bundle, equivariant under $\mathbb{Z}_{12}$.
- Downstairs, $SU(5)$ is broken to G_{SM} by Wilson lines.
- Discrete parameters in the above construction lead to several hundred branches. Eight of these give the light spectrum of the MSSM.

(The construction can also be viewed as follows:
The standard embedding on X, augmented with Wilson lines, gives gauge groups like $G_{\mathrm{SM}} \times U(1) \times U(1)$ or $SU(4) \times SU(2) \times U(1) \times U(1)$. These are then Higgsed to G_{SM}. I can explain the equivalence later.)

Several heterotic string compactifications with the MSSM spectrum have been found already. Distinguishing features of our models:

- A minimal hidden sector: no five-branes, and no hidden matter.
- Only five geometric moduli, and a similar number of vector bundle moduli.

Outline

$\widetilde{X}$ as a CICY

There are various ways to represent $\widetilde{X}$ as a complete intersection in a product of projective spaces. A convenient one is

$$\widetilde{X} \in \begin{matrix} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \\ \end{matrix} \begin{matrix} \left[\begin{matrix} \mathbf{1} & 0 & \mathbf{1} \\ \mathbf{1} & 0 & \mathbf{1} \\ \mathbf{1} & 0 & \mathbf{1} \\ 0 & \mathbf{1} & \mathbf{1} \\ 0 & \mathbf{1} & \mathbf{1} \\ 0 & \mathbf{1} & \mathbf{1} \end{matrix}\right] \\ \begin{matrix} p_1 & p_2 & r \end{matrix} \end{matrix}$$

$p_i = 0$ defines a copy of dP_6, the del Pezzo surface of degree 6, as a hypersurface in $\left(\mathbb{P}^1\right)^3$. r is then a section of the anti-canonical bundle of $\mathrm{dP}_6 \times \mathrm{dP}_6$, so its vanishing defines a Calabi-Yau threefold.

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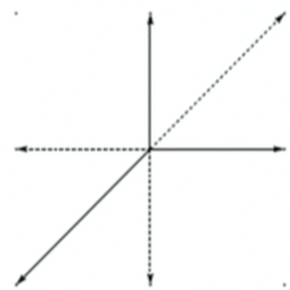
$\widetilde{X}$ as a toric hypersurface

The surface dP_6 is actually toric; its fan has the symmetry of a hexagon:



Removing the dashed rays gives the projection to $\mathbb{P}^2$.

The fourfold $Z \equiv \mathrm{dP}_6 \times \mathrm{dP}_6$ is therefore also toric.

$H^{1,1}(Z) \cong \mathbb{C}^8$, and this is generated (with redundancy) by the twelve toric divisors. The hypersurface $\widetilde{X}$ inherits all its harmonic $(1,1)$ forms from Z, so $h^{1,1}(\widetilde{X}) = 8$. We can also calculate $\chi(\widetilde{X}) = -72$; hence $h^{2,1}(\widetilde{X}) = 44$.

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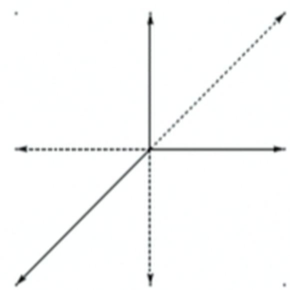

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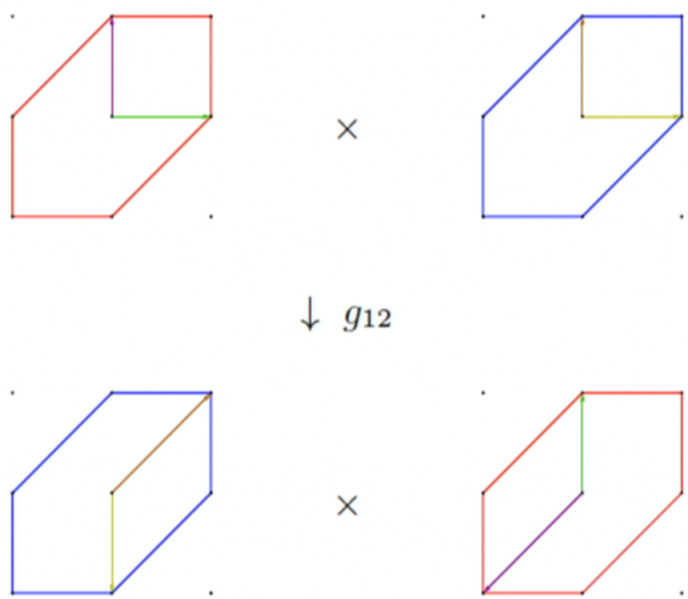
The symmetry group of the fan of Z is $(D_6 \times D_6) \rtimes \mathbb{Z}_2$, which therefore acts on Z. We pick an order twelve element defined geometrically as follows:



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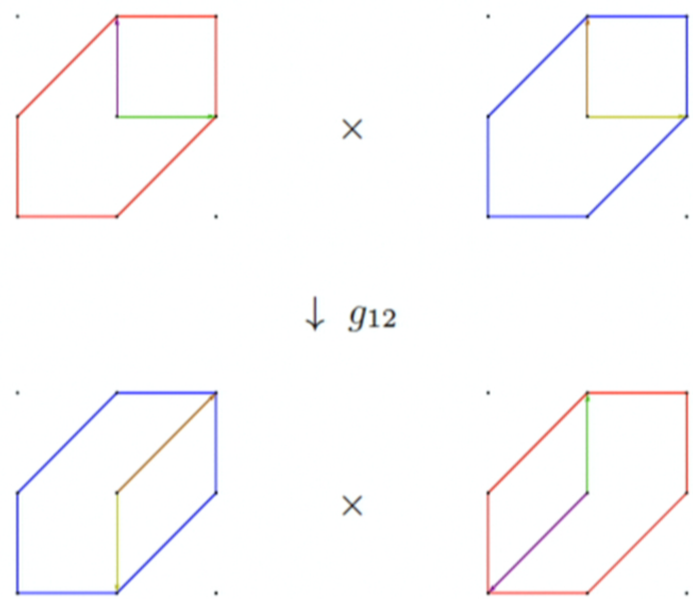

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The quotient manifold

Consider those $\widetilde{X}$ invariant under g_{12}, and define $X = \widetilde{X}/\mathbb{Z}_{12}$. The following can be checked:

- The generic symmetric $\widetilde{X}$ is smooth.
- $\mathbb{Z}_{12}$ acts without fixed points on the generic symmetric $\widetilde{X}$.
- The family of symmetric $\widetilde{X}$ has four parameters.

Hodge numbers of X:

- The $\mathbb{Z}_{12}$ action on $H^{1,1}(\widetilde{X})$ corresponds to the representation

  $$\mathbf{0} \oplus \mathbf{2} \oplus \mathbf{3} \oplus \mathbf{4} \oplus \mathbf{6} \oplus \mathbf{8} \oplus \mathbf{9} \oplus \mathbf{10} \; .$$

  The single invariant implies $h^{1,1}(X) = 1$.
- χ divides by the order of a freely-acting group, so $\chi(X) = -6$.
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The classical Euler sequence

We must first understand the tangent bundle. Projective space is simple:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^N} \longrightarrow \mathcal{O}_{\mathbb{P}^N}(1)^{\oplus N+1} \longrightarrow T\mathbb{P}^N \longrightarrow 0$$

This is intuitive:

- $\mathbb{P}^N$ is parametrised by $(z_0, \ldots, z_N) \sim (\lambda\, z_0, \ldots, \lambda\, z_N)$, $\lambda \in \mathbb{C}^*$.
- Vector fields must therefore be invariant under this: $v = \sum_i v_i(z) \frac{\partial}{\partial z_i}$, where each v_i is linear, i.e., a section of $\mathcal{O}_{\mathbb{P}^N}(1)$.
- But vectors pointing along equivalence classes project to zero, so $\sum_i z_i \frac{\partial}{\partial z_i} \sim 0$.

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The tangent bundle of a hypersurface

As for any submanifold, there is a short exact sequence on $\widetilde{X}$:

$$0 \longrightarrow T\widetilde{X} \longrightarrow TZ\big|_{\widetilde{X}} \xrightarrow{df} N_{\widetilde{X}|Z} \longrightarrow 0 \; .$$

Since $\widetilde{X}$ is an anti-canonical hypersurface, we have

$$N_{\widetilde{X}|Z} \;\cong\; \mathcal{O}_{\widetilde{X}}(-K_Z\big|_{\widetilde{X}}) \;\cong\; \mathcal{O}_{\widetilde{X}}(\sum_{i=1}^{12} D_i) \; .$$

Combining this with the Euler sequence, we get $T\widetilde{X}$ as the cohomology of the following complex:

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In other words, $T\widetilde{X} = \ker df / \operatorname{im} E$.

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The toric Euler sequence

The Euler sequence generalises in an obvious way to other toric varieties.

$Z \cong \mathrm{dP}_6 \times \mathrm{dP}_6$ has eight independent scalings, i.e.,

$$(z_1, \ldots, z_{12}) \sim (\lambda^{Q_{\alpha,1}} z_1, \ldots, \lambda^{Q_{\alpha,12}} z_{12}), \; \lambda \in \mathbb{C}^*, \; \alpha = 1, \ldots, 8$$

We therefore get eight independent Euler vector fields: $E_\alpha = \sum_i Q_{\alpha,i} z_i \frac{\partial}{\partial z_i}$

$$0 \longrightarrow 8\mathcal{O}_Z \xrightarrow{E} \bigoplus_{i=1}^{12} \mathcal{O}_Z(D_i) \longrightarrow TZ \longrightarrow 0$$

The rank-five bundles

To obtain a rank-five bundle, divide by only 6 of the Euler vectors

$$6\mathcal{O}_{\widetilde{X}} \xrightarrow{E'} \bigoplus_{i=1}^{12} \mathcal{O}_{\widetilde{X}}(D_i) \xrightarrow{df} \mathcal{O}_{\widetilde{X}}(\sum_{i=1}^{12} D_i)$$

The resulting bundle $\widetilde{V}_e$ is given by the exact sequence

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Slope stability

The slope of a bundle $\widetilde{V}$ on $\widetilde{X}$ is

$$\mu(\widetilde{V}) = \frac{1}{\mathrm{rk}(\widetilde{V})} \int_{\widetilde{X}} c_1(\widetilde{V}) \wedge \omega \wedge \omega \ ,$$

where ω is the Kähler form. $\widetilde{V}$ is stable if for every coherent sub-sheaf $\mathcal{F} \subset \widetilde{V}$, $\mu(\mathcal{F}) < \mu(\widetilde{V})$.

- Obviously $c_1(\widetilde{V}) = 0 \ \Rightarrow \ \mu(\widetilde{V}) = 0$.
- We see now that $\widetilde{V}_e$ is unstable, since $\mathcal{O}_{\widetilde{X}} \subset \widetilde{V}_e$, and $\mu(\mathcal{O}_{\widetilde{X}}) = 0 = \mu(\widetilde{V}_e)$.
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The stable bundles

- Instability comes from $H^0(\widetilde{X}, \ker df) \cong \mathbb{C}^8$; we only mod out a 6D subspace.
- We must deform df to a map Φ which annihilates only six Euler vectors:

$$\bigoplus_{i=1}^{12} \mathcal{O}_{\widetilde{X}}(D_i) \xrightarrow{\Phi} \mathcal{O}_{\widetilde{X}}(\sum_{i=1}^{12} D_i)$$

- Φ is a twelve-vector of homogeneous polynomials: sections of $\mathcal{O}_{\widetilde{X}}(\sum_{i \neq j} D_i)$.
- The space of possible Φ is 420-dimensional; we can choose Φ to annihilate any six Euler vectors we like.
- Our vector bundles $\widetilde{V}$ upstairs are the cohomology of

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Proof of stability

$H^0(\widetilde{X}, \widetilde{V}) = 0$ is necessary, but not sufficient, for stability.

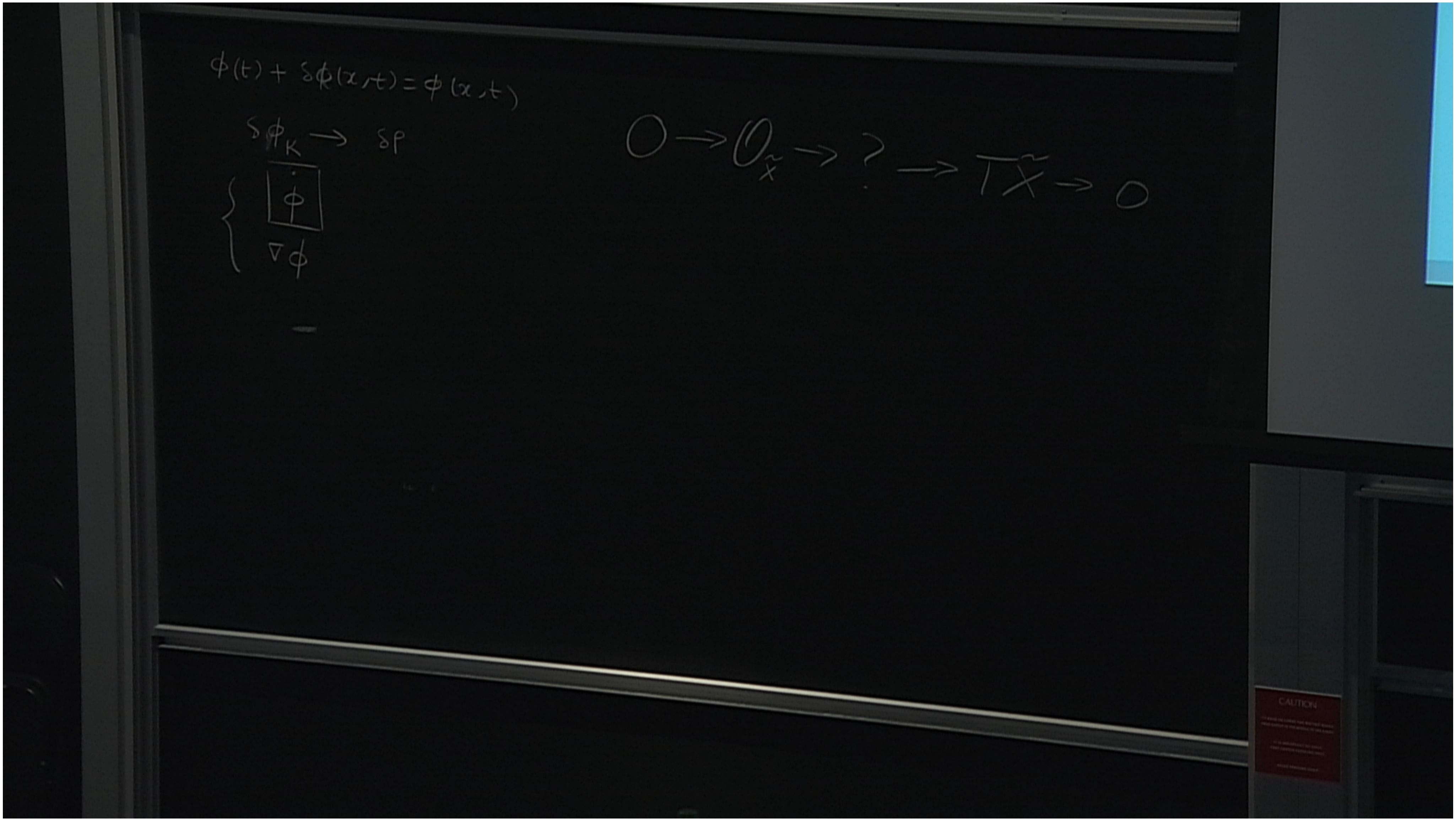
We will shortly quotient $\widetilde{V}$ by $\mathbb{Z}_{12}$ to obtain a bundle V on X, and $h^{1,1}(X) = 1$.

This allows us to apply Hoppe's criterion:

> Let V be a holomorphic bundle with $c_1(V) = 0$ on a Calabi-Yau manifold X with $h^{1,1}(X) = 1$. Then V is stable if $H^0(X, \wedge^p V) = 0$ for $p = 1, \ldots, \mathrm{rk}(V) - 1$.

We can check that in fact $H^0(\widetilde{X}, \wedge^p \widetilde{V}) = 0$, and conclude that V is stable.

Stability of V is all we need, but this also implies that $\widetilde{V}$ is stable, because the Hermitian-Yang-Mills metric on V pulls back to one on $\widetilde{V}$.


φ(t) + δφ(x,t) = φ(x,t)
δφ_K → δρ
φ̇
∇φ
O → O_x̃ → ? → TX̃ → O

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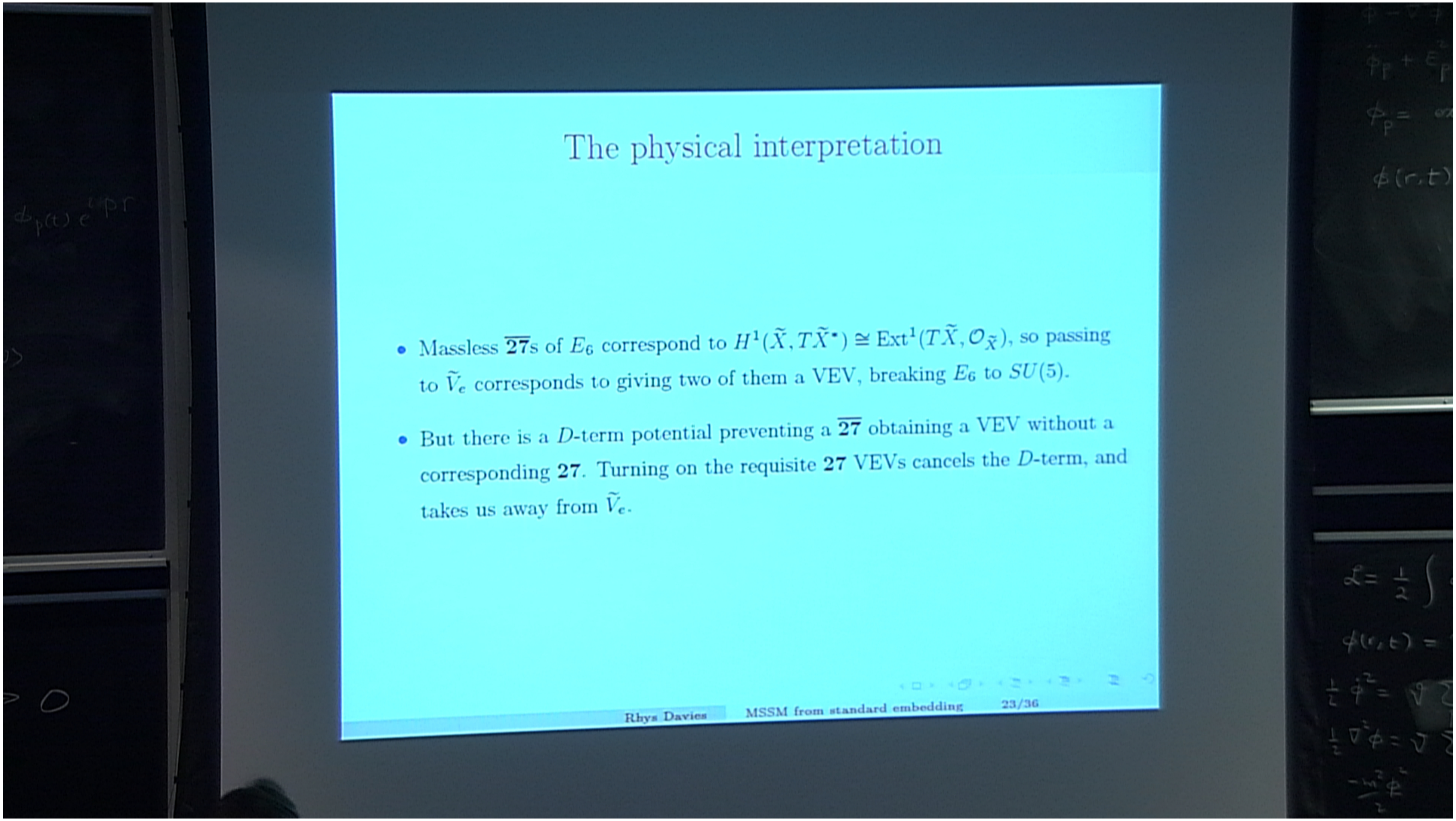
$$0 \longrightarrow 2\mathcal{O}_{\widetilde{X}} \longrightarrow \widetilde{V}_e \longrightarrow T\widetilde{X} \longrightarrow 0\ .$$

Is this okay? No, but we need to discuss stability to see why.


The physical interpretation
Massless $\overline{\mathbf{27}}$s of E_6 correspond to $H^1(\widetilde{X}, T\widetilde{X}^*) \cong \mathrm{Ext}^1(T\widetilde{X}, \mathcal{O}_{\widetilde{X}})$, so passing to $\widetilde{V}_e$ corresponds to giving two of them a VEV, breaking E_6 to $SU(5)$.
But there is a D-term potential preventing a $\overline{\mathbf{27}}$ obtaining a VEV without a corresponding $\mathbf{27}$. Turning on the requisite $\mathbf{27}$ VEVs cancels the D-term, and takes us away from $\widetilde{V}_e$.
Rhys Davies
MSSM from standard embedding
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Outline

Equivalence of $\widetilde{V}$

- Recall: $\widetilde{V} = \ker \Phi / \operatorname{im} E'$.
- The $\mathbb{Z}_{12}$ action simply permutes the toric divisors, and hence lifts to an action on $\bigoplus_i \mathcal{O}_{\widetilde{X}}(D_i)$ and $\mathcal{O}_{\widetilde{X}}(\sum_{i=1}^{12} D_i)$.
- We want $\ker \Phi$ to be equivariant; it will be if the following commutes:

$$\begin{array}{ccc} \bigoplus_i \mathcal{O}_{\widetilde{X}}(D_i) & \xrightarrow{g_{12}} & \bigoplus_i \mathcal{O}_{\widetilde{X}}(D_i) \\ \Phi \downarrow & & \downarrow \Phi \\ \mathcal{O}_{\widetilde{X}}(\sum_{i=1}^{12} D_i) & \xrightarrow{g_{12}} & \mathcal{O}_{\widetilde{X}}(\sum_{i=1}^{12} D_i) \end{array}$$

- Actually easy: Φ is a twelve-vector of homogeneous polynomials, and equivariance means its components are related by the g_{12} action.
- The space of equivariant Φ is therefore $420/12 = 35$-dimensional.

- Now for $\mathrm{im}\, E'$. Diagonalising the action on the Euler vectors, we find

$$H^0(\widetilde{X}, 8\mathcal{O}_{\widetilde{X}}) \sim \mathbf{0} \oplus \mathbf{2} \oplus \mathbf{3} \oplus \mathbf{4} \oplus \mathbf{6} \oplus \mathbf{8} \oplus \mathbf{9} \oplus \mathbf{10} \ .$$

We may choose any six-dimensional sub-representation for $\mathrm{im}\, E'$.

Hence there are $\binom{8}{6} = 28$ distinct branches of the moduli space.

- Recall the (unstable) extension bundle $\widetilde{V}_e$:

$$0 \longrightarrow 2\mathcal{O}_{\widetilde{X}} \longrightarrow \widetilde{V}_e \longrightarrow T\widetilde{X} \longrightarrow 0 \ .$$

Its two global sections carry the two $\mathbb{Z}_{12}$ charges *not* in $\mathrm{im}\, E'$.

- Non-trivial equivariant structure on $\mathcal{O}_{\widetilde{X}}$ $\leftrightarrow$ Non-trivial flat line bundle on X $\leftrightarrow$ non-zero Wilson line. Notation: rep. $\mathbf{n} \leftrightarrow \mathcal{L}_n$.
- So downstairs, we actually get a deformation of $TX \oplus \mathcal{L}_{n_1} \oplus \mathcal{L}_{n_2}$.

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Equivariance of V — a subtlety

- We require $\mathrm{Hol}(V) \subset SU(5)$, but for $U \in SU(3)$, we have

$$\det \operatorname{diag}(U, e^{\frac{\pi i}{6} n_1}, e^{\frac{\pi i}{6} n_2}) = e^{\frac{\pi i}{6}(n_1+n_2)}$$

- We must add an overall phase to the equivariant structure on $\widetilde{V}$, so that V is a deformation of

$$\mathcal{L}_{\tilde{n}} \otimes \left(TX \oplus \mathcal{L}_{n_1} \oplus \mathcal{L}_{n_2}\right) ,$$

where $5\tilde{n} + n_1 + n_2 \equiv 0 \bmod 12$. Solve this: $\tilde{n} = 7(n_1 + n_2)$.

- It is important to remember this extra phase when calculating the $\mathbb{Z}_{12}$ action on cohomology.

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Wilson line breaking of $SU(5)$

There are 11 different choices of homomorphism $\mathbb{Z}_{12} \to SU(5)$ which correspond to Wilson line breaking to $SU(3)\times SU(2)\times U(1)_Y$:

$$g_{12} \mapsto \mathrm{diag}(e^{-\frac{\pi i}{6}k}, e^{-\frac{\pi i}{6}k}, e^{-\frac{\pi i}{6}k}, e^{\frac{\pi i}{4}k}, e^{\frac{\pi i}{4}k}) \in SU(5) \quad , \; k = 1, \ldots, 11$$

Field	u^c	Q	e^c	d^c	L, H_d
$SU(5)$ provenance	$\mathbf{10}$	$\mathbf{10}$	$\mathbf{10}$	$\overline{\mathbf{5}}$	$\overline{\mathbf{5}}$
G_{SM} rep.	$(\overline{\mathbf{3}}, \mathbf{1})_{-4}$	$(\mathbf{3}, \mathbf{2})_1$	$(\mathbf{1}, \mathbf{1})_6$	$(\overline{\mathbf{3}}, \mathbf{1})_2$	$(\mathbf{1}, \mathbf{2})_{-3}$
$\mathbb{Z}_{12}$ charge	$8k$	k	$6k$	$2k$	$9k$

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Outline

Introduction

The manifold

Constructing the bundles

Equivariance and Wilson lines

Calculating the massless spectrum

Conclusions

The massless spectrum

Upstairs, we have a rank-five bundle $\widetilde{V}$ on $\widetilde{X}$, and the massless chiral multiplets are counted by certain cohomology groups:

$$n_{\mathbf{10}} = h^1(\widetilde{X}, \widetilde{V}) \ , \quad n_{\overline{\mathbf{5}}} = h^1(\widetilde{X}, \wedge^2 \widetilde{V})$$
$$n_{\overline{\mathbf{10}}} = h^1(\widetilde{X}, \widetilde{V}^*) \ , \quad n_{\mathbf{5}} = h^1(\widetilde{X}, \wedge^2 \widetilde{V}^*) \ ,$$

We now take the quotient by $\mathbb{Z}_{12}$:

- $\widetilde{V}$ is equivariant under $\mathbb{Z}_{12} \implies \mathbb{Z}_{12}$ acts on $H^*(\widetilde{X}, \widetilde{V})$ et cetera.
- We also choose non-zero Wilson lines, which act on the fields according to their hypercharge.
- The spectrum on X consists of fields invariant under the combined transformation; these need not fill out complete $SU(5)$ multiplets.

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Projecting out the $\overline{\mathbf{10}}$ states

The massless $\overline{\mathbf{10}}$ states come from $H^1(\widetilde{X}, \widetilde{V}^*)$; we must calculate this group and the $\mathbb{Z}_{12}$ action on it. Introduce $\mathcal{F} = \ker \Phi$; our bundle $\widetilde{V}$ fits into an exact sequence

$$0 \longrightarrow 6\mathcal{O}_{\widetilde{X}} \longrightarrow \mathcal{F} \longrightarrow \widetilde{V} \longrightarrow 0 \ ,$$

and $\mathcal{F}$ is in turn determined by another short exact sequence:

$$0 \longrightarrow \mathcal{F} \longrightarrow \bigoplus_i \mathcal{O}_{\widetilde{X}}(D_i) \xrightarrow{\Phi} \bigoplus_i \mathcal{O}_{\widetilde{X}}(D_i) \longrightarrow 0 \ .$$

Dualising gives

$$0 \longrightarrow \widetilde{V}^* \longrightarrow \mathcal{F}^* \longrightarrow 6\mathcal{O}^*_{\widetilde{X}} \longrightarrow 0 \ .$$

We can calculate that $H^i(\widetilde{X}, \mathcal{F}^*) = 0$ for $i = 0, 1$, so the long exact cohomology sequence contains

$$0 \longrightarrow H^0(\widetilde{X}, 6\mathcal{O}^*_{\widetilde{X}}) \longrightarrow H^1(\widetilde{X}, \widetilde{V}^*) \longrightarrow 0 \ .$$

We conclude that $H^1(\widetilde{X}, \widetilde{V}^*) \cong \tilde{\mathbf{n}}^* \otimes H^0(\widetilde{X}, \ker \Phi)^*$

(the other two are eaten by the gauge bosons of broken generators).

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Extra states from the $\mathbf{10}$?

The massless $\mathbf{10}$ fields come from $H^1(\widetilde{X}, \widetilde{V})$, but we do not need to consider these separately. In the case of a freely-acting group $\mathcal{G}$ and an equivariant bundle $\widetilde{V}$, the holomorphic Atiyah-Bott fixed-point formula reduces to

$$\sum_p (-1)^p n^p(\mathbf{r}) = \chi(\widetilde{V}) \frac{\dim \mathbf{r}}{\mathcal{G}} \ .$$

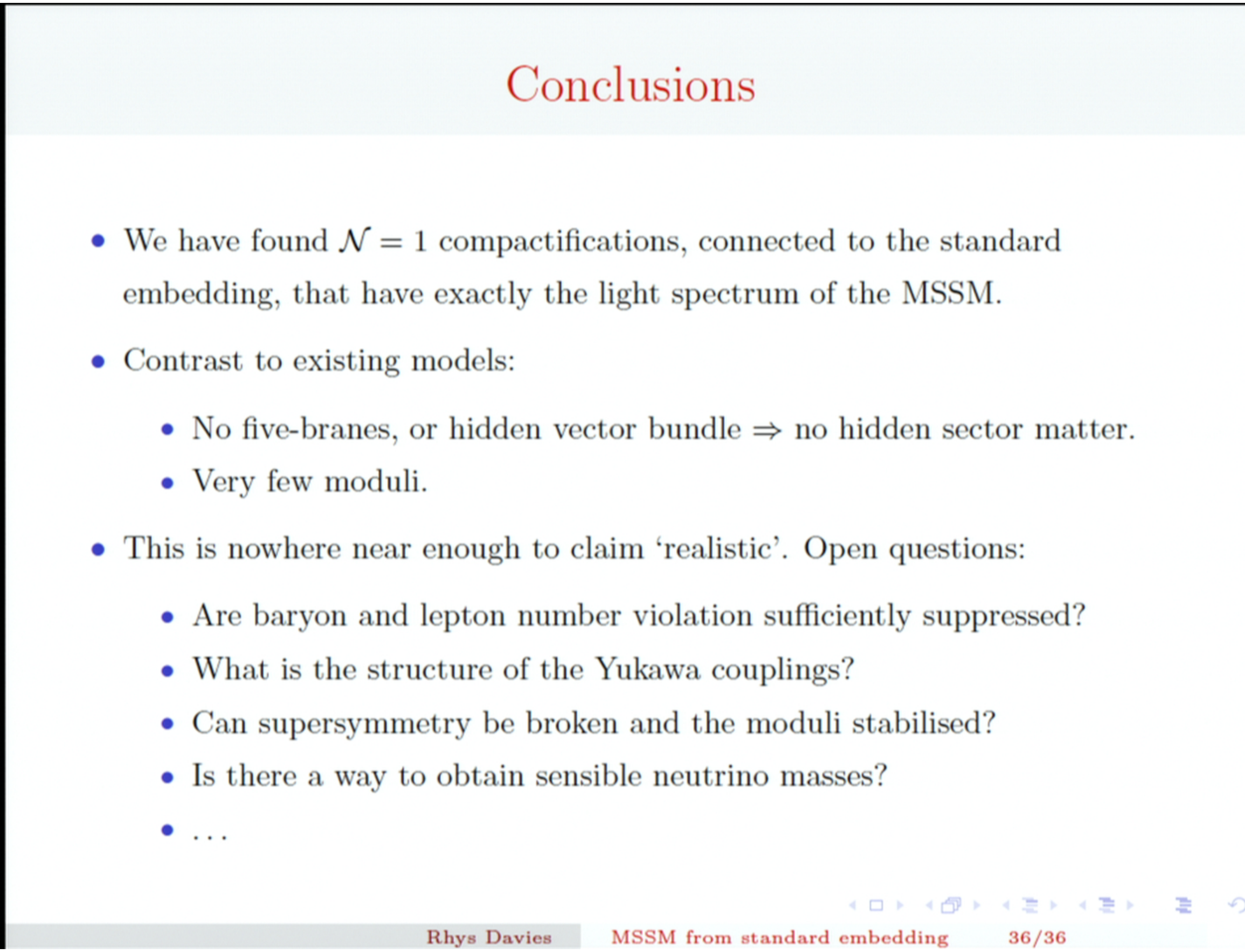
Combined with Serre duality, $H^2(\widetilde{X}, \widetilde{V}) \cong H^1(\widetilde{X}, \widetilde{V}^*)^*$, this guarantees that we get exactly three copies of the $\mathbf{10}$, as long as there are no massless $\overline{\mathbf{10}}$ states.

Doublet-triplet splitting

The Higgs fields of the MSSM are vector-like, and originate in the $\mathbf{5} \oplus \overline{\mathbf{5}}$ of $SU(5)$. We want to retain a single pair of Higgs doublets, but remove their associated colour triplets.

This is analogous to the $\mathbf{10} \oplus \overline{\mathbf{10}}$ analysis, but much harder. In the end, we find eight models with the MSSM spectrum:

(n_1, n_2)	k
$(3, 4)$	$4, 8$
$(3, 8)$	$4, 8$
$(4, 9)$	$4, 8$
$(8, 9)$	$4, 8$



Conclusions
We have found $\mathcal{N} = 1$ compactifications, connected to the standard embedding, that have exactly the light spectrum of the MSSM.
Contrast to existing models:
No five-branes, or hidden vector bundle $\Rightarrow$ no hidden sector matter.
Very few moduli.
This is nowhere near enough to claim 'realistic'. Open questions:
Are baryon and lepton number violation sufficiently suppressed?
What is the structure of the Yukawa couplings?
Can supersymmetry be broken and the moduli stabilised?
Is there a way to obtain sensible neutrino masses?
...
Rhys Davies
MSSM from standard embedding
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