Title: A Toy Model to Study the Imposition of the Spin Foam Simplicity Constraints

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Abstract: We introduce an exactly solvable model to test various proposals for the imposition of the spin foam simplicity constraints. This model is a three-dimensional Holst-Plebanski action for the gauge group SO(4), in which the simplicity constraints mimic the situation of the four-dimensional theory. In particular, the canonical analysis reveals the presence of secondary second class constraints conjugated to the primary ones. We perform the spin foam quantization of the theory in the spirit of the BC and EPRL models, and give arguments for modifying the measure over the holonomies in order to account for the presence of the secondary second class constraints.

A toy model to test the imposition of the spin foam simplicity constraints

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Based on 1112.1965 [gr-qc] with K. Noui

Introduction and motivations

Statement of intent:

$$\langle \Sigma_1, q_1 | \Sigma_2, q_2 \rangle_{\text{phys}} = \int_{g|_\Sigma = q} [\mathcal{D}g] \exp(iS).$$

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The spin foam approach takes as a starting point the (Holst-)Plebanski action

$$S_{\text{Pl}}[B, \omega, \phi] = \int_{\mathcal{M}_4} \left[\left(1 + \frac{\star}{\gamma} \right) B^{IJ} \wedge F_{IJ} + \phi_{IJKL} B^{IJ} \wedge B^{KL} \right],$$

where the simplicity constraints imposed by ϕ ensure that B comes from a tetrad.

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- BC model: simplicity constraints imposed too strongly,
- EPRL and FK: weak imposition through the linear simplicity constraints,
- Many other models: Han-Thiemann, Baratin-Oriti, Conrady-Hnybida, ...

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→ Semi-classical limit, physical predictions, mathematical (internal) consistency.

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In the canonical theory, the primary simplicity constraints ϕ are second class because they are conjugated to secondary constraints ψ.

The usual point of view is that the phase space path integral

$$\mathcal{Z} = \int \mathcal{D}p\,\mathcal{D}q\sqrt{|\det\{\phi,\psi\}|}\delta(\phi)\delta(\psi)\exp\left(i\int \mathrm{d}t\,\left(p_a\dot{q}^a - H_0\right)\right)$$

can be cleared of the secondary constraints (Henneaux-Slavnov) to give

$$\mathcal{Z} = \int \mathcal{D}p\,\mathcal{D}q\,\mu(p,q)\mathcal{D}\lambda\exp\left(i\int \mathrm{d}t\,\left(p_a\dot{q}^a - H_0 - \lambda\phi\right)\right).$$

It has been shown (Buffenoir-Henneaux-Noui-Roche and Engle-Han-Thiemann) that

$$\mu_{\text{Plebanski}} = \mathcal{V}^9 V, \qquad \text{and} \qquad \mu_{\text{Holst}} = \mathcal{V}^3 V.$$

- ? Can we always get rid of the secondary second class constraints?
- ? What is the influence of the measure on the face and edge amplitudes?

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Introduction and motivations

- ⋆ How should we impose the primary simplicity constraints?
- ⋆ Is it legitimate to forget about the secondary second class constraints?

One can try to look for models of the form

$$S_1 = S_2 + \mathcal{C}$$

where the theories given by S_1 and S_2 have known spin foam quantizations, and $\mathcal{C}$ are simplicity constraints.

Such models exist in 3d (this talk) and in 4d (Alexandrov 1202.5039).

Outline

1. Classical theory
2. The strategy
3. Quantum theory, take I
4. Quantum theory, take II
5. Conclusion and future directions

Classical theory

Let us consider the 3-dimensional SO(4) Plebanski action

$$S_{\text{Pl}}[B,\omega,\phi]=\frac{1}{2}\int_{\mathcal{M}}\mathrm{d}^3x\Big(\varepsilon^{\mu\nu\rho}\mathrm{Tr}(B_\mu F_{\nu\rho})+\phi^{\mu\nu}\mathrm{Tr}(\star B_\mu B_\nu)\Big).$$

Just like in four dimensions, the simplicity constraints

$$\mathcal{C}_{\mu\nu}=\frac{1}{2}\varepsilon_{IJKL}B^{IJ}_\mu B^{KL}_\nu\approx 0$$

have three sectors of solutions:

- gravitational sector: $B^{IJ}_\mu=\varepsilon^{IJ}{}_{KL}x^K e^L_\mu,$
- topological sector: $B^{IJ}_\mu=x^I e^J_\mu-x^J e^I_\mu,$
- degenerate sector,

with $x^I\in\mathbb{R}^4$ and e^I_μ a one-form.

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with $x^I\in\mathbb{R}^4$ and e^I_μ a one-form. The gravitational sector in the time gauge becomes

$$S_{\mathrm{Pl}}[B,\omega]\longrightarrow S_{\text{gravity}}[B(e),\omega]=\frac{1}{2}\int_{\mathcal{M}}\mathrm{d}^3x\,\varepsilon^{\mu\nu\rho}\varepsilon_{ijk}e^i_\mu F^{jk}_{\nu\rho}.$$

Notice that we have the 4 symmetries

$$\left(x^I\longrightarrow\alpha x^I,e^I_\mu\longrightarrow\alpha^{-1}e^I_\mu\right),\qquad\text{and}\qquad\left(e^i_\mu\longrightarrow e^i_\mu+\beta_\mu x^i,e^0_\mu\longrightarrow e^0_\mu+\beta_\mu x^0\right),$$

so the simple B field has indeed $18-6=16-4=12$ components.

Classical theory

If we write the SO(4) BF part of the action as

$$S[B,\omega] = \frac{1}{2}\int_{\mathcal{M}} \mathrm{d}^3x\, \varepsilon^{\mu\nu\rho}\Big(\mathrm{Tr}\big({}^{(+)}B_\mu\, {}^{(+)}F_{\nu\rho}\big) + \mathrm{Tr}\big({}^{(-)}B_\mu\, {}^{(-)}F_{\nu\rho}\big)\Big),$$

the torsion-free condition can be solved to obtain the second order action

$$S[g_{\mu\nu}] = \frac{1}{2}s^+ \int_{\mathcal{M}} \mathrm{d}^3x \sqrt{|{}^{(+)}g|}\,\mathcal{R}\big[{}^{(+)}g_{\mu\nu}\big] + \frac{1}{2}s^- \int_{\mathcal{M}} \mathrm{d}^3x \sqrt{|{}^{(-)}g|}\,\mathcal{R}\big[{}^{(-)}g_{\mu\nu}\big],$$

where ${}^{(\pm)}g_{\mu\nu} = {}^{(\pm)}B_\mu \cdot {}^{(\pm)}B_\nu$ and $s^\pm = \mathrm{sign}\,\big(\det\,\big({}^{(\pm)}B^i_\mu\big)\big)$.

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One can show that

- gravitational sector: ${}^{(+)}g_{\mu\nu} = {}^{(-)}g_{\mu\nu}$, $s^+ = s^-$ $\longrightarrow$ gravity,
- topological sector: ${}^{(+)}g_{\mu\nu} = {}^{(-)}g_{\mu\nu}$, $s^+ = -s^-$ $\longrightarrow$ nothing.

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Classical theory - Canonical analysis

The canonical analysis is similar to that of the 4-dimensional Plebanski theory (Buffenoir-Henneaux-Noui-Roche and Alexandrov-Krasnov).

The 36 phase space variables ω_μ^{IJ} and B_μ^{IJ} are subjected to

- 12 first class constraints: SO(4) gauge, diffeos, vanishing of π_N and π_{N^a},
- 12 second class constraints, including in particular:

$$\begin{array}{lll} \text{primary} & \mathcal{C}_{ab} & \approx 0, \\ \text{secondary} & \dot{\mathcal{C}}_{ab} & \approx 0 \quad \Longrightarrow \quad \psi_{ab} \equiv \mathrm{Tr}(D_a B_0 \star B_b) + \mathrm{Tr}(D_b B_0 \star B_a) \approx 0. \end{array}$$

In fact the secondary second class constraint ψ_{ab} can be combined with the linear simplicity constraint to give

$$\mathcal{P}^{IJ}_{KL}\omega_a^{KL} = x^{[I}\partial_a x^{J]}.$$

The same property is satisfied in the four-dimensional theory by the Lorentz-covariant extension of the Ashtekar-Barbero connection, where $\mathcal{P}$ projects onto the boost part of $\mathfrak{sl}(2,\mathbb{C})$.

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The strategy

With this classical theory at hand, we can study the commutativity of the diagram

$$\begin{tikzcd} S_{\mathrm{BF}} \arrow[r, "\mathcal{C}=0"] \arrow[d] & S_{\mathrm{gravity}} \arrow[d] \\ \mathcal{Z}_{\mathrm{BF}} \arrow[r, "??"] & \mathcal{Z}_{\mathrm{gravity}} \end{tikzcd}$$

We know that

$$\mathcal{Z}_{\mathrm{gravity}} = \sum_{j\to f} \prod_{f\in\Delta^*} (2j_f+1) \prod_{v\in\Delta^*} \{6j\},$$

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Quantum theory, take I

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3. Quantum theory, take I

4. Quantum theory, take II

5. Conclusion and future directions

Quantum theory, take I

The one-forms B_μ^{IJ} are integrated along the links $\ell \in \Delta$ dual to $f \in \Delta^*$ to give $B_f^{IJ} \in \mathfrak{so}(4)$. The discrete simplicity constraints are

- diagonal simplicity: $$\varepsilon_{IJKL} B_f^{IJ} B_f^{KL} \approx 0 \qquad \forall\, f \in \partial t, \tag{1}$$
- cross simplicity: $$\varepsilon_{IJKL} B_f^{IJ} B_{f'}^{KL} \approx 0 \qquad \forall\, f, f' \in \partial t. \tag{2}$$

By fixing the normal x^I, we can write the linear simplicity constraint

$$x_J B_\mu^{IJ}, \tag{3}$$

which selects the gravitational sector, and breaks SO(4) to SU(2) stabilizing x^I.

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Different ways to impose these (primary second class) constraints:

- ⋆ BC (Barrett-Crane) model: (1) and (2) strongly,
- ⋆ EPR(L $\gamma \neq 0$) (Engle-Pereira-Rovelli-Livine) model: (1) strongly and (3) weakly.

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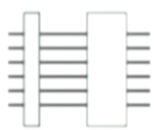
(1) implies the restriction $j_f^+ = j_f^- \equiv j_f$ to simple SO(4) representations.

Then we have the decomposition

$$\mathcal{H}^{(j_f, j_f)}_{\mathrm{SO}(4)} = \bigoplus_{j_f=0}^{2j_f} \mathcal{H}^{(j_f)}_{\mathrm{SU}(2)},$$

in which BC selects the subspace $j_f = 0$, and EPRL (via master constraint) selects the highest spin $j_f^+ + j_f^- = 2j_f$.

We can represent this graphically as



$$\equiv \int_{[\mathrm{SO}(4)]^3} \mathrm{d}g\, \delta(\hat{\mathcal{C}}).$$

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By fixing the normal x^I, we can write the linear simplicity constraint

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which selects the gravitational sector, and breaks SO(4) to SU(2) stabilizing x^I.

Different ways to impose these (primary second class) constraints:

- ⋆ BC (Barrett-Crane) model: (1) and (2) strongly,
- ⋆ EPR(L $\gamma \neq 0$) (Engle-Pereira-Rovelli-Livine) model: (1) strongly and (3) weakly.

Quantum theory, take I

(1) implies the restriction $j_f^+ = j_f^- \equiv j_f$ to simple SO(4) representations.

Then we have the decomposition

$$\mathcal{H}_{\mathrm{SO}(4)}^{(j_f, j_f)} = \bigoplus_{j_f=0}^{2j_f} \mathcal{H}_{\mathrm{SU}(2)}^{(j_f)},$$

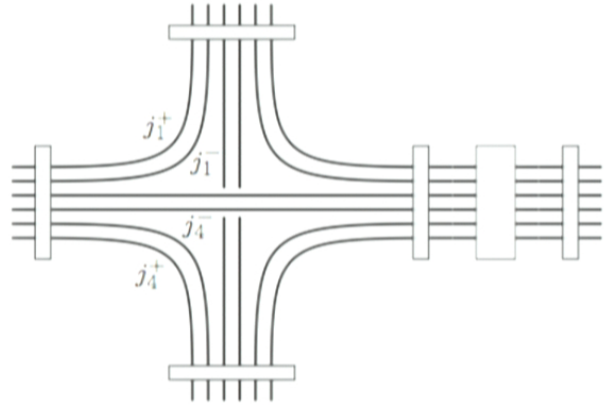
in which BC selects the subspace $j_f = 0$, and EPRL (via master constraint) selects the highest spin $j_f^+ + j_f^- = 2j_f$.

We can represent this graphically as

 $\equiv \displaystyle\int_{[\mathrm{SO}(4)]^3} \mathrm{d}g\, \delta(\hat{\mathcal{C}}).$

Quantum theory, take I - BC

$$\mathcal{Z}_{\mathrm{BC}} = \sum_{\{j^+,j^-\}\to\{f\}} \prod_{f\in\Delta^*} (2j_f^+ + 1)(2j_f^- + 1)$$



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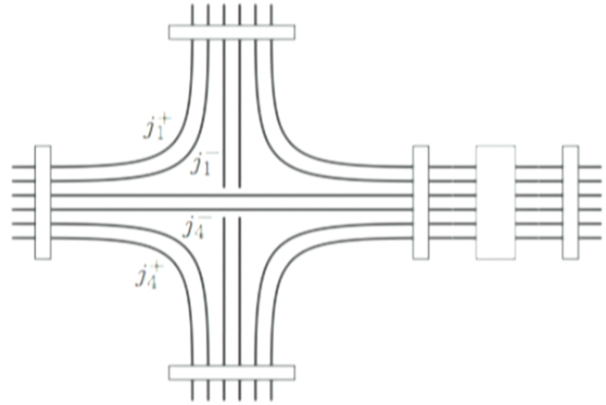
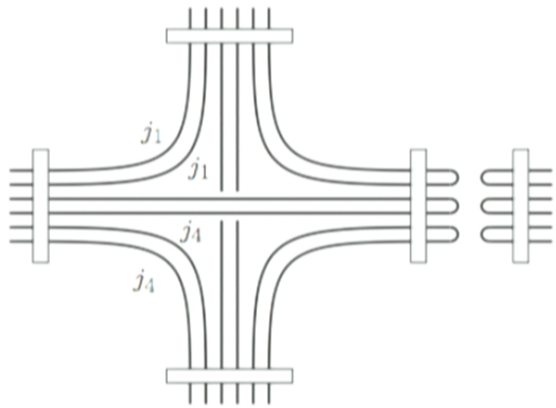


$$= \sum_{\{j\}\to\{f\}} \prod_{f\in\Delta^*} (2j_f + 1)^2$$



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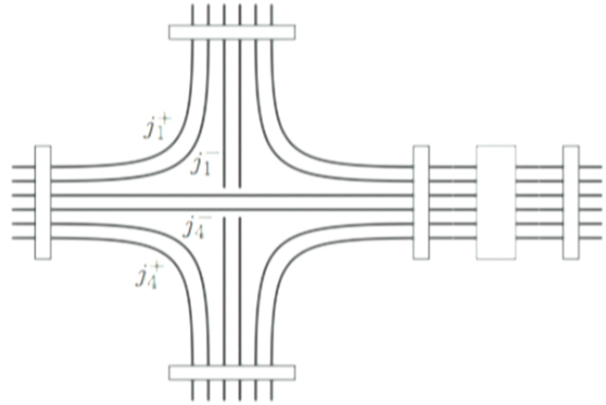

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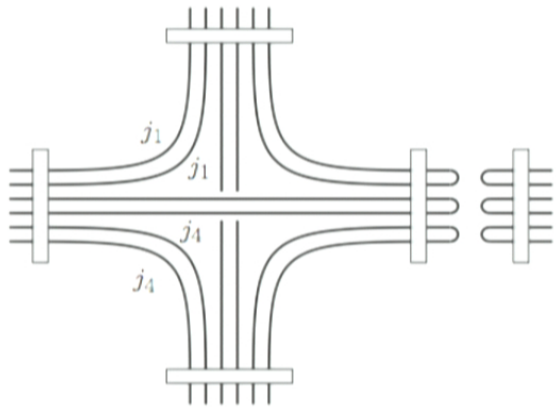

$$= \sum_{\{j\}\to\{f\}} \prod_{f\in\Delta^*} (2j_f + 1)^2 \prod_{v\in\Delta^*} \{6j\}^2$$

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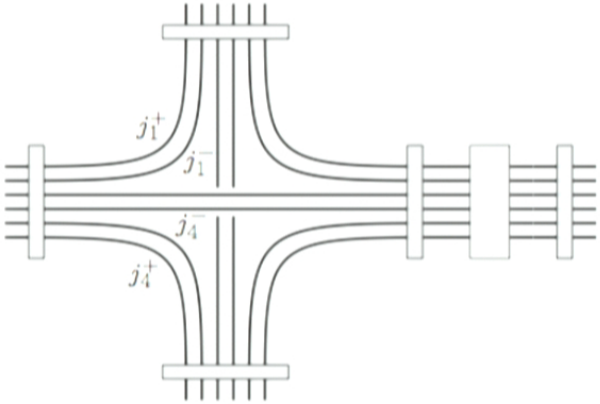

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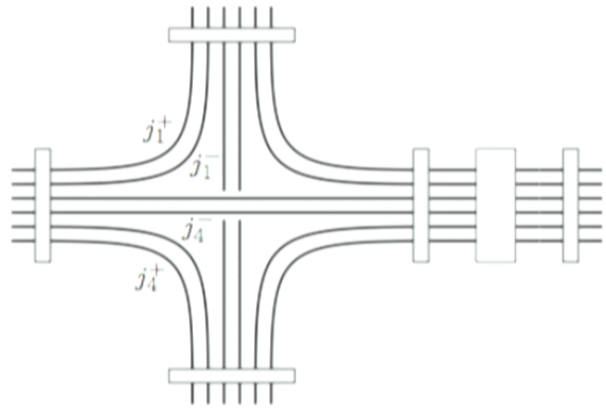

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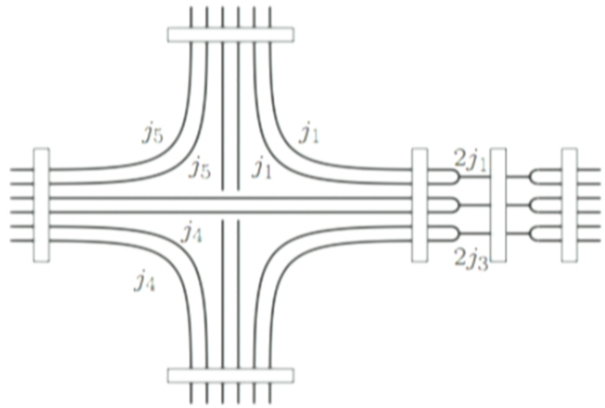

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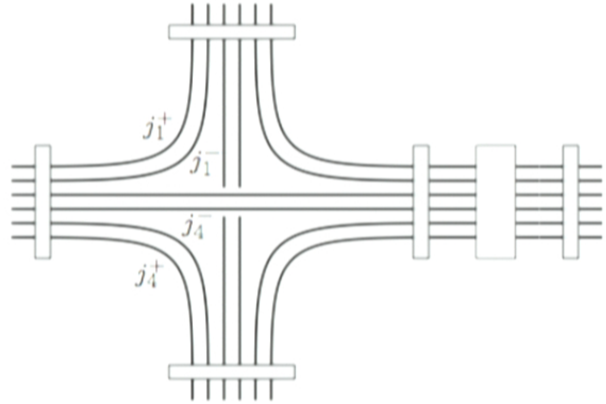

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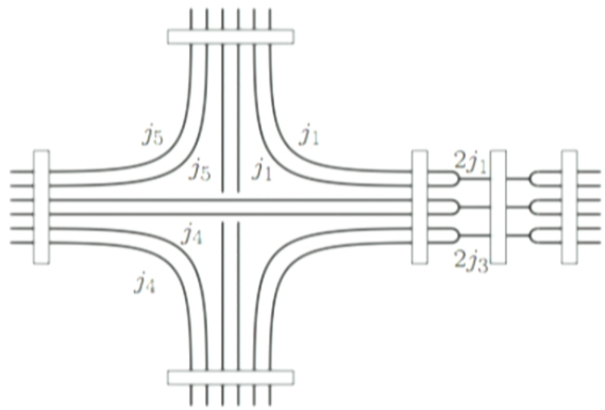

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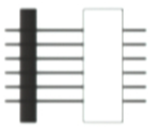
Let us now take into account the secondary second class constraints ψ_{ab} coming from the preservation of the primary simplicity constraints $\mathcal{C}_{ab}$.

In the time gauge $x^I = (1,0,0,0)$, they just say that ${}^{(+)}\omega_a = {}^{(-)}\omega_a$, so let us use



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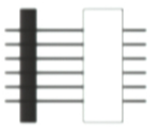

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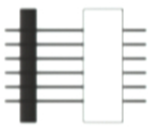

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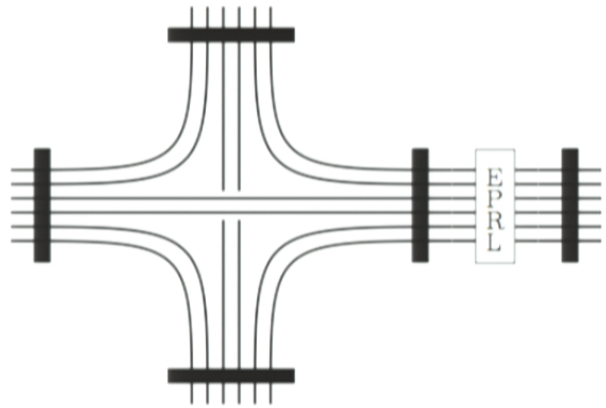

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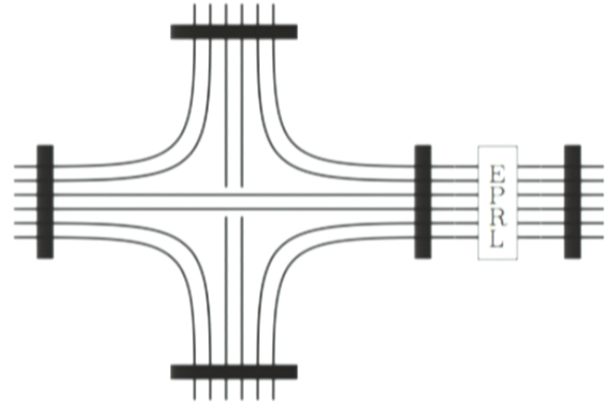

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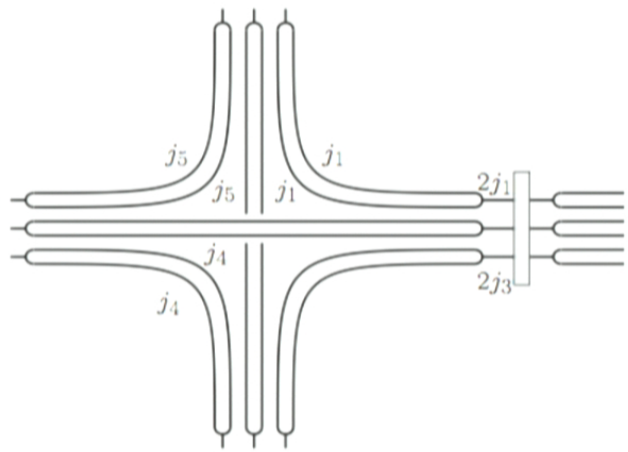

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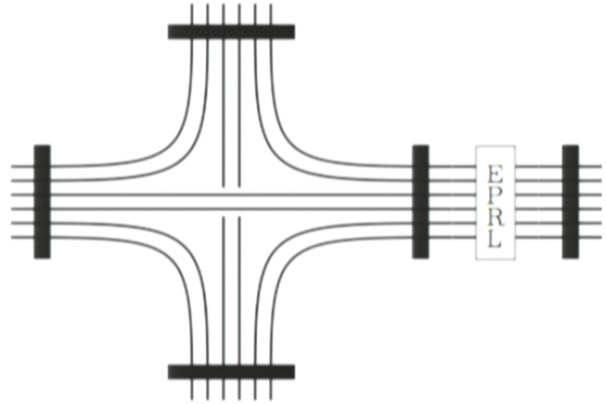

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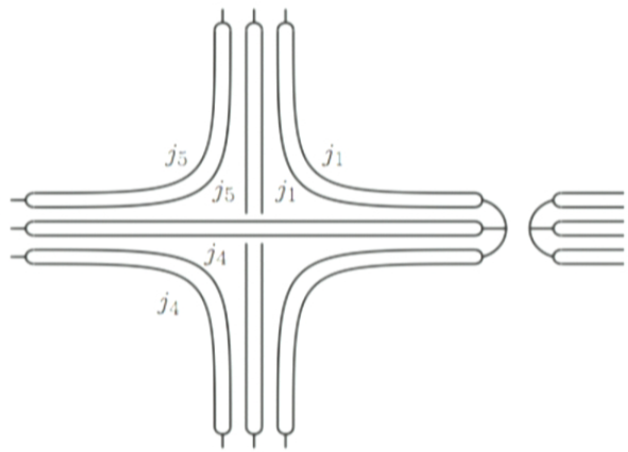

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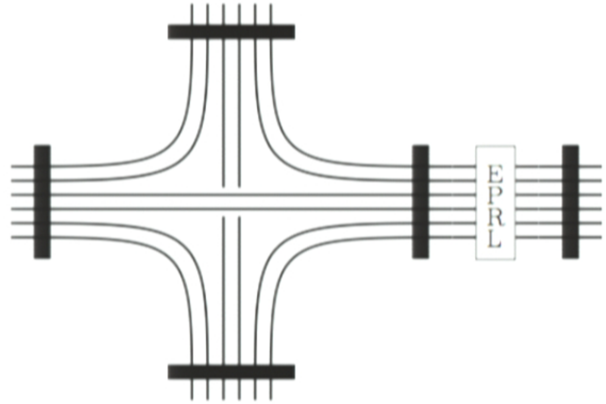

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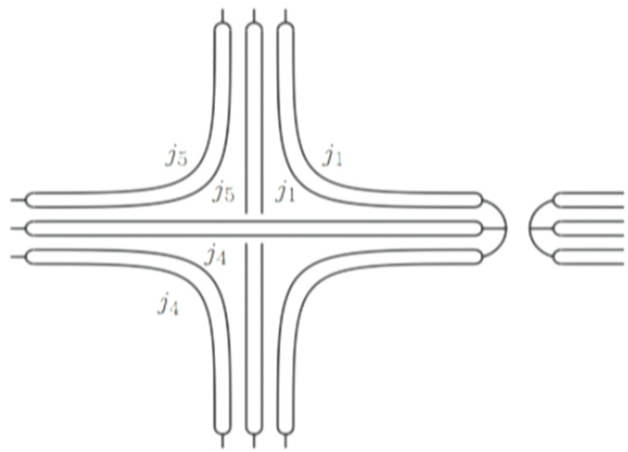

$$= \sum_{\{j\}\to\{f\}} \prod_{f\in\Delta^*} (2j_f+1)^2 \prod_{v\in\Delta^*} \frac{1}{d_{j_1}\dots d_{j_6}} \left\{ \begin{matrix} 2j_1 & 2j_2 & 2j_3 \\ 2j_4 & 2j_5 & 2j_6 \end{matrix} \right\}.$$

Quantum theory, take II

Can we have a more generic and constructive point of view? Write the total simplicial path integral as

$$\mathcal{Z} = \int \prod_f \mathcal{D}^{(x)}[B_f] \left(\bigstar_v A_v[B_f] \right),$$

where the vertex amplitude is

$$A_v[B_f] = \int \prod_e \mathcal{D}^{(B,x_e)}[g_e] \prod_f \exp\left(i\mathrm{Tr}\left[B_f g_{u(f)}^{-1} g_{d(f)} \right] \right),$$

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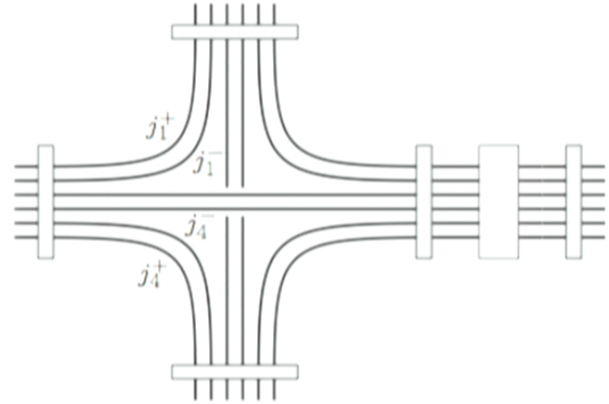

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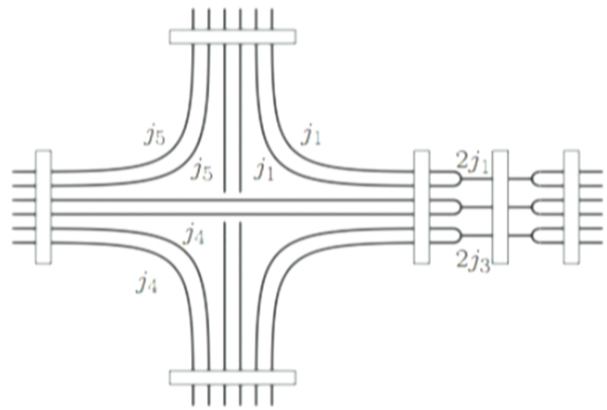

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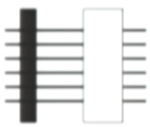

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Conclusion and future directions

- ! Restrictions of this model: no non-trivial diagonal simplicity constraints, 3d, ...
- ⋆ There is an analogue four-dimensional model describing the degenerate sector of Plebanski theory.
- ⋆ Use this exemple to test other proposals for building spin foam models.
- ? What about the secondary second class constraints in the four dimensions?
- ? Maybe clarify the role of the Barbero-Immirzi parameter?

The three-dimensional SO(4) Plebanski action can be generalized to

$$S_{\mathrm{Pl}\gamma}[B,\omega,\phi]=\frac{1}{2}\int_{\mathcal{M}}\mathrm{d}^3x\left[\varepsilon^{\mu\nu\rho}\left(\mathrm{Tr}(B_\mu F_{\nu\rho})+\frac{1}{\gamma}\mathrm{Tr}(\star B_\mu F_{\nu\rho})\right)+\phi^{\mu\nu}\,\mathrm{Tr}(\star B_\mu B_\nu)\right],$$

where γ is again irrelevant at the classical level.

Merci

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