


Title: Renormalization of Tensor-network States/Models

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Abstract: One of the biggest challenges in physics is to develop accurate and efficient methods that can solve many currently intractable problems in correlated quantum or statistical systems. Tensor-network model/state is drawing more and more attention since it captures the feature of the area law and is absent from the sign problem. The evaluation of the expectation value of the observables can be reduced to the contraction of a tensor-network, which can be done by means of renormalization group method, and this is exactly what tensor renormalization group (TRG) method has done. In the light of comparison between numerical renormalization group (NRG) and density matrix renormalization group (DMRG), having considered the renormalization effect of the environment, second renormalization group (SRG) method has been proposed to improve the performance of TRG. Although TRG and SRG have achieved great success in 2D lattice, application to 3D is not easy. The talk will give a review of TRG and SRG, and talk about the HOTRG, i.e., TRG based on the higher-order singular value decomposition, and its SRG version, HOSRG. They can be easily applied to 3D lattice with relatively low memory and computation cost. The well-known 3D Ising model on simple cubic lattice has been tested. As far as our group know, HOSRG have achieved by far the most accurate renormalization group results for the 3D Ising model. Some analysis and possible applications will be also discussed.



Renormalization *of* Tensor-Network States/Models

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Prof. Zhengyu Weng

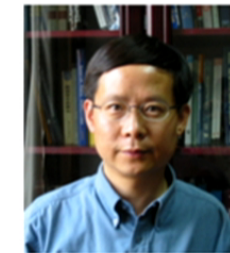
Dr. Hongchen Jiang (*KITP, UCSB*)

Academy of Mathematics and Systems Science, CAS

Dr. Jinwei Zhu

Beijing Computational Science Research Center

Dr. Liping Yang





- *References:*

- Z. Y. Xie, H. C. Jiang, Q. N. Chen, Z. Y. Weng, and T. Xiang, Phys. Rev. Lett. 103, 160601 (2009).
- H. H. Zhao, Z. Y. Xie, Q. N. Chen, Z. C. Wei, J. W. Cai, and T. Xiang, Phys. Rev. B 81, 174411 (2010)
- Z. Y. Xie, J. Chen, J. W. Zhu, L. P. Yang, and T. Xiang, arXiv:1201.1144.

Challenge in the study of strongly correlated systems

- **Essentially non-perturbative:** no obvious small parameters:
Conventional methods of quantum field theory are not applicable
- **High cost:** the total degree of freedom grows **exponentially** with system size

To face the challenges:

Weak Coupling Approach:

- Convert a many-body problem into one-body
- mean field theory, density functional theory

Strong Coupling Approach:

- use only a finite set of many-body basis states, throw rest
- Configuration interactions(CI), QMC, Numerical RG.

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Two widely-used strong-coupling methods

✓ **Quantum Monte Carlo**

- No dimension consideration
- Suffer from the “minus-sign” problem for fermion and frustrated spin system

✓ **Density Matrix Renormalization Group**

- Best method for 1D quantum model
- Finite size for 2D

New method?

A possible way out:

Tensor-network states/models! ^[1]

[1]. H. Niggemann, A. Klumper, and J. Zittartz, Z. Phys. B 104, 103 (1997).

What is a tensor-network state/models?

➤ Classical model of statistical physics:

- all statistical models with local interactions can be represented as *tensor-network models*, defined on *real lattice or dual lattice*.
- partition function and expectation value of a local operator.

$$Z = \text{Tr} \left[\prod_i T^i \right]$$
$$\langle O \rangle = \frac{\text{Tr} \left[S_o^j \prod_{i \neq j} T^i \right]}{Z}$$

1D case: partition function is a matrix (order-2 tensor) product

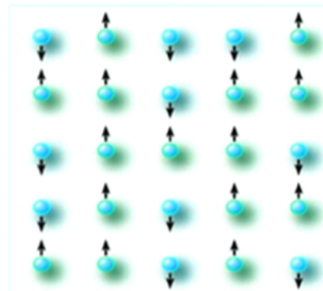
➤ Example: Ising model



Ernst Ising

$$H = -J \sum_{\langle ij \rangle} S_i^z S_j^z$$

$$S_i^z = -1, 1$$



$$\begin{aligned} Z &= \sum_{S_1 \dots S_N} \exp\left(\beta \sum_i S_i S_{i+1}\right) \\ &= \text{Tr}(A \cdots A) \\ &= \lambda_{\max}^N \quad N \rightarrow \infty \end{aligned}$$

$$A = \begin{pmatrix} e^{\beta} & e^{-\beta} \\ e^{-\beta} & e^{\beta} \end{pmatrix}$$

Tensor now is a matrix!

1D case: partition function is a matrix (order-2 tensor) product

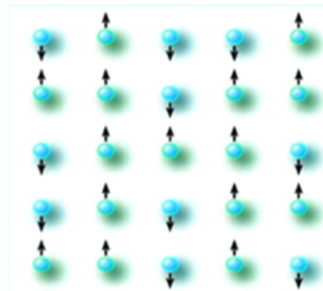
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What is a tensor-network state/models?

➤ Quantum lattice models:

- *Tensor network state* is a faithful representation^[1] of the ground state wavefunction of quantum lattice model that satisfies the area law of entanglement.
- wave function and expectation value of a local operator

$$|\Psi\rangle = \text{Tr} \prod_i T^i[m_i] |m_i\rangle$$

$$\langle O \rangle = \frac{\langle \Psi | O | \Psi \rangle}{\langle \Psi | \Psi \rangle} = \frac{\text{Tr} \left[\prod_i A^i \right]}{\text{Tr} \left[\prod_i B^i \right]}$$

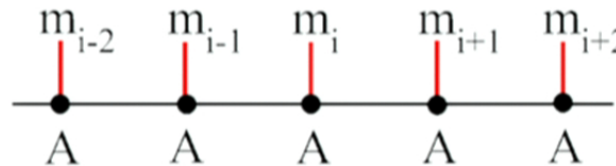
[1]. F. Verstraete, J. I. Cirac, arXiv:0407066

1D case: matrix-product state(MPS), DMRG's fix-point

Example: S=1 AKLT chain

$$H = \sum_i \left(S_i \cdot S_{i+1} + \frac{1}{3} (S_i \cdot S_{i+1})^2 \right)$$

Its ground state can be represented as valence bond solid(VBS) state:



$$|\Psi\rangle = \sum_{m_1 \cdots m_L} \text{Tr}(\cdots A[m_1] \cdots A[m_L] \cdots) |\cdots m_1 \cdots m_L \cdots\rangle$$

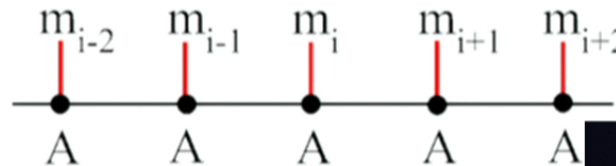
$$A[\pm 1, 0] = \{ \sqrt{2}\sigma_+, -\sqrt{2}\sigma_-, \sigma_z \}$$

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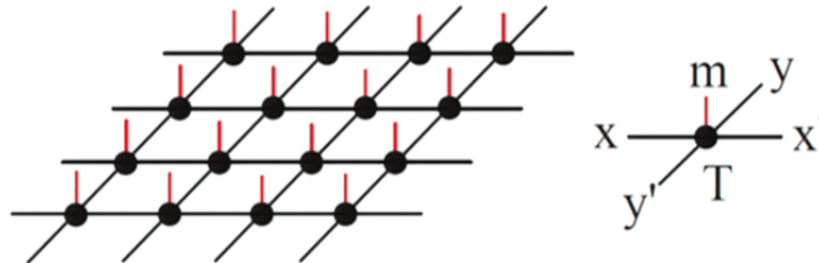


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2D extension:

projected-entangled-paired-state(PEPS), or tensor-product state(TPS)

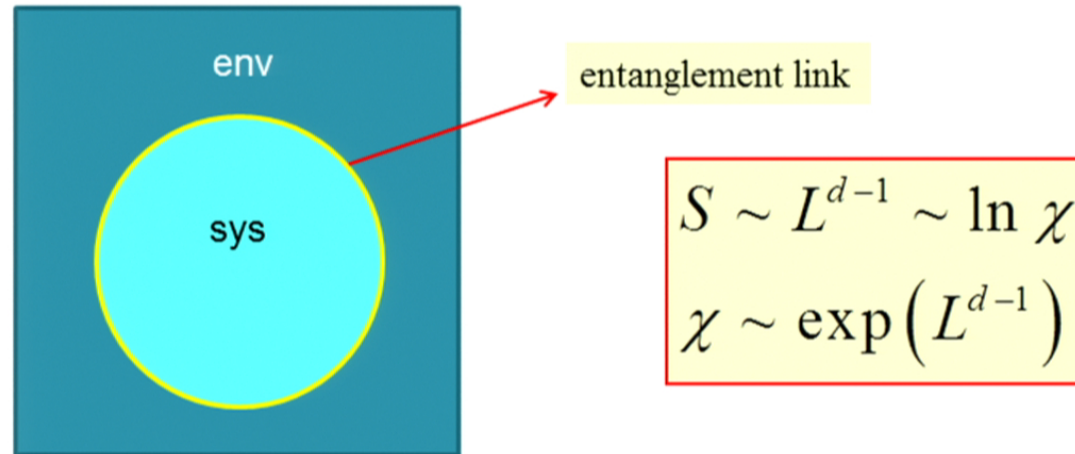


$$|\Psi\rangle = \text{Tr} \prod T_{x_i x'_i y_i y'_i} [m_i] |m_i\rangle \quad \langle \hat{O} \rangle = \frac{\langle \Psi | \hat{O} | \Psi \rangle}{\langle \Psi | \Psi \rangle}$$

- [1]: H. C. Jiang, Z. Y. Weng, and T. Xiang, Phys. Rev. Lett. 101, 090603 (2008).
[2]: G. Vidal, Phys. Rev. Lett. 98, 070201 (2007).

Natural advantage of tensor-network states/models:

→ Area Law in quantum information: for a gapped system



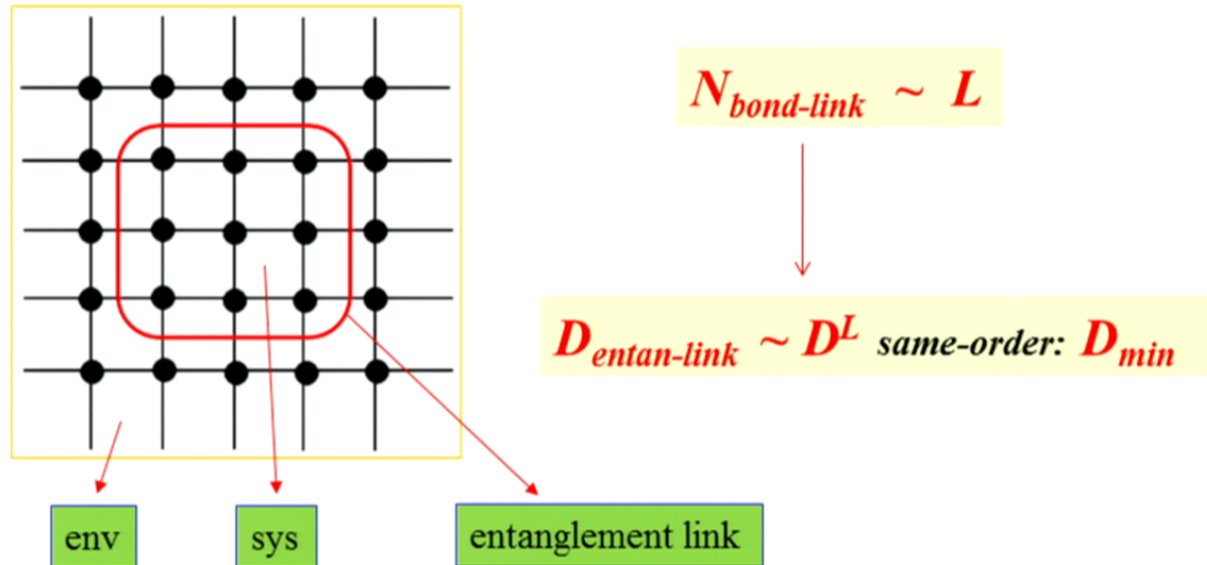
S : entanglement between sys and env

d : spatial dimension L : length scale of sys

χ : min number of basis states needed to describe the ground state, or

D_{\min} of the entanglement link to describe the entanglement.

- $d = 1$: $D_{min} \sim const$, MPS/DMRG can cover
- $d = 2$: $D_{min} \sim e^L$, DMRG can not cover easily



- **Tensor network state/model** can naturally describe the entanglement in **higher dimension**, and is absent from the *locality broken* as appeared in the application of DMRG to 2D.

How to evaluate the tensor network?

- **difficulty:** high cost of memory and computation
- **Some possible answers:**
 1. **tensor renormalization group(TRG)** algorithms:
contract the network by coarse-graining process.
 2. **infinite time-evolved block decimation algorithm(iTEBD)^[1], PEPS^[2]/iPEPS^[3] algorithm, corner transfer matrix(CTM)**
and related: based on the idea of transfer-matrix.
 3. **Entanglement renormalization(MERA)^[4]:** a different real-space renormalization scheme deal with physical degree of freedom.

[1]. G. Vidal, Phys. Rev. Lett 98, 070201 (2007), R. Orus, G. Vidal, Phys. Rev.B 78, 155117(2008).

[2]. V. Murg, F. Verstraete, J. I. Cirac, Phys. Rev. A 75, 033605 (2007).

[3]. J. Jordan, R. Orus, G. Vidal, F. Verstraete, J. I. Cirac, Phys.Rev. Lett 101, 250602 (2008)

[4]. G. Vidal, Phys. Rev. Lett **99**, 220405 (2007)

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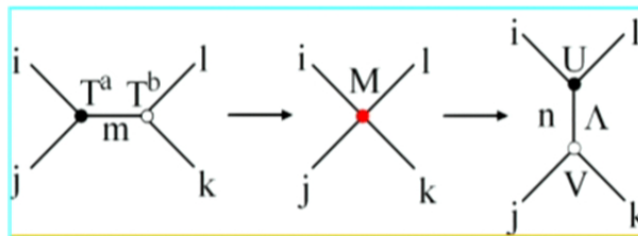
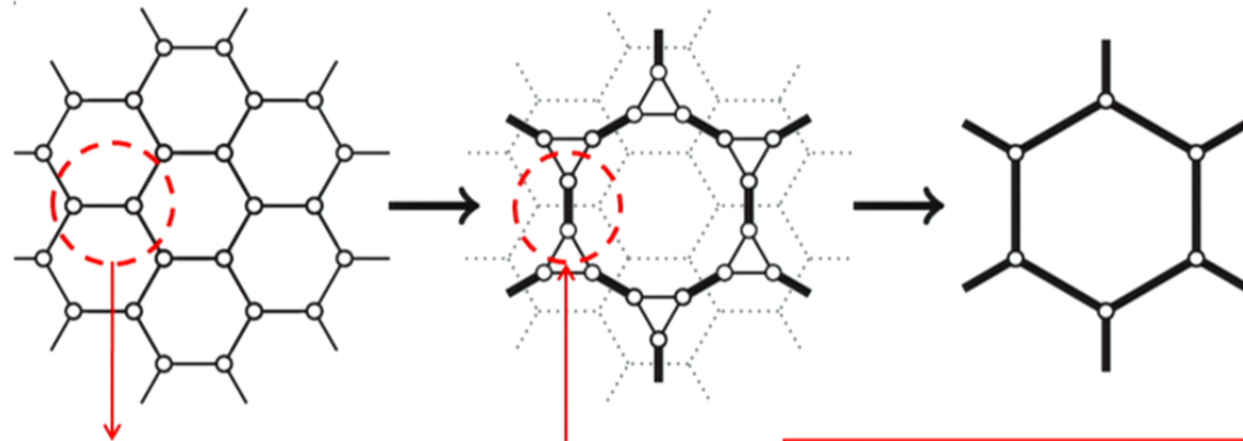
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To speak in detail:

Step I: Rewiring

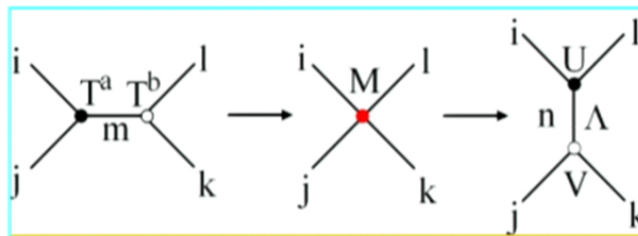
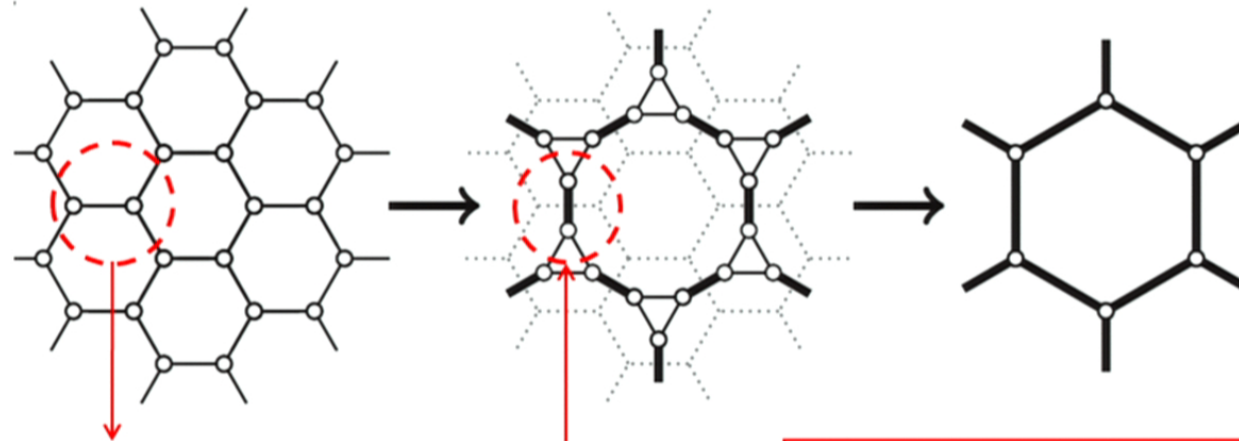


$$\begin{aligned}
 M_{kj,il} &= \sum_m T_{mji} T_{mlk} \\
 &= \sum_{n=1}^D U_{kj,n} \Lambda_n V_{il,n}
 \end{aligned}$$

SVD: best scheme to truncate a matrix

To speak in detail:

Step I: Rewiring



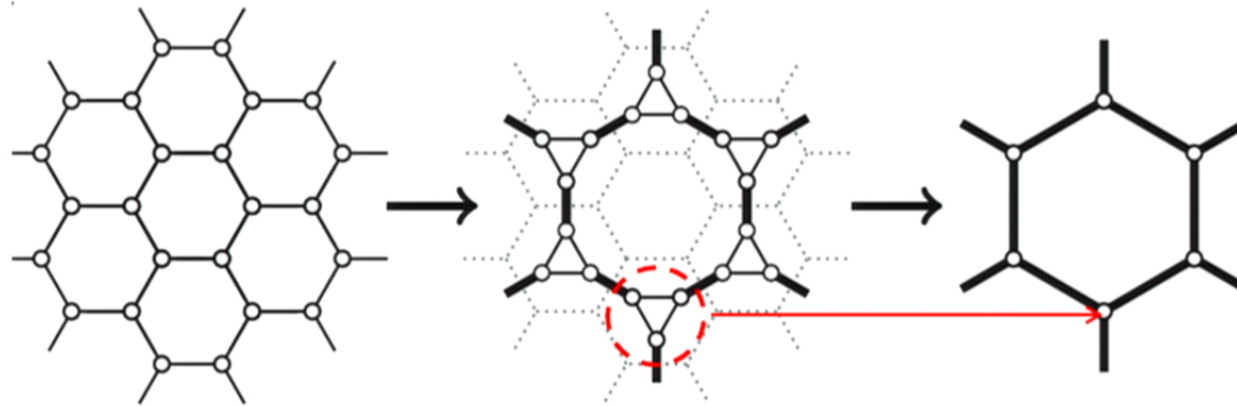
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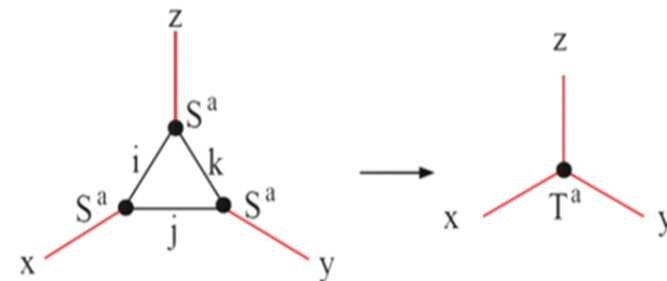
SVD: best scheme to truncate a matrix

Core step of TRG: Singular value decomposition (SVD)

Step II: decimation

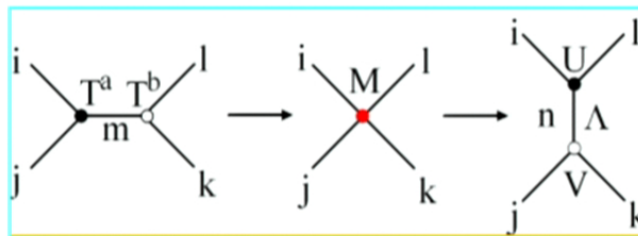
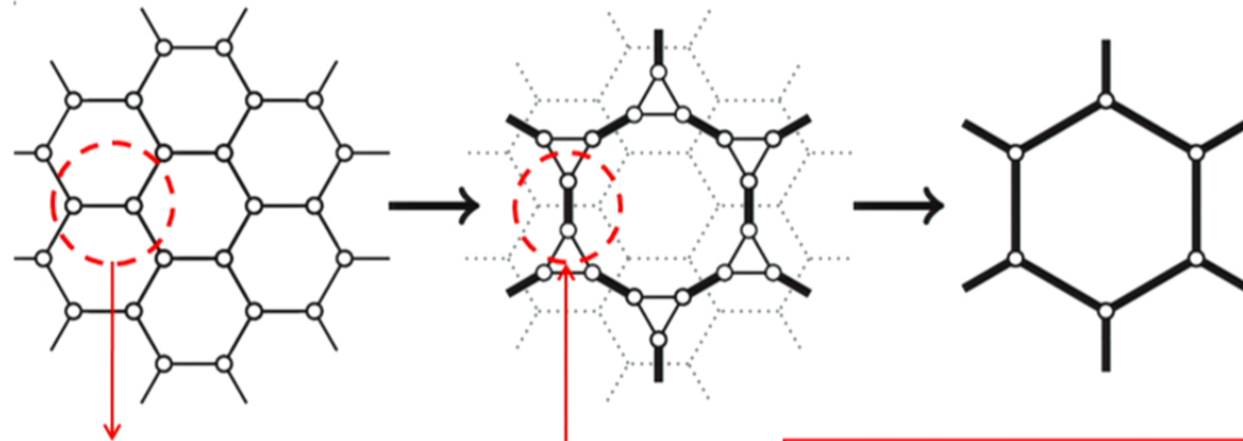


$$T_{xyz} = \sum_{ijk} S_{xik} S_{yji} S_{zjk}$$



To speak in detail:

Step I: Rewiring



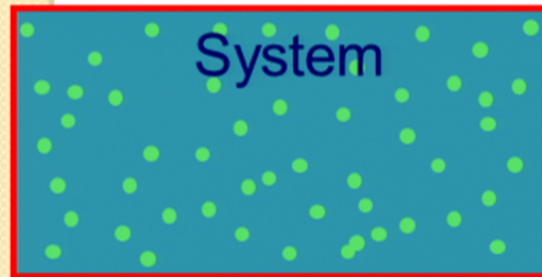
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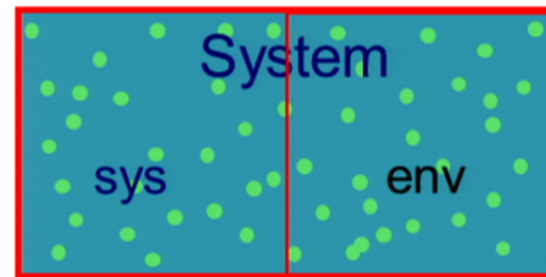
Let's get some idea from comparison between NRG and DMRG

1974 Wilson NRG



States are weighted according to the **spectra** of the system

1992 White's DMRG

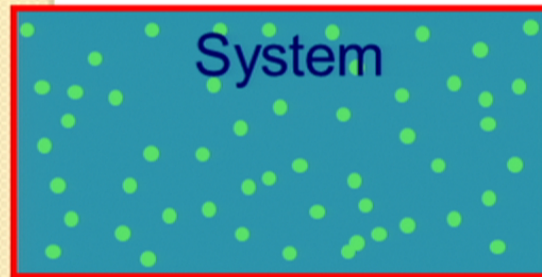


States are weighted according to the **entanglement** between a sub-system and its env

Better performance

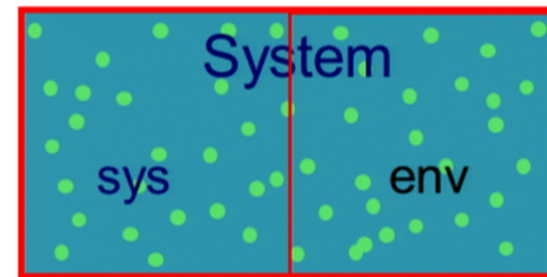
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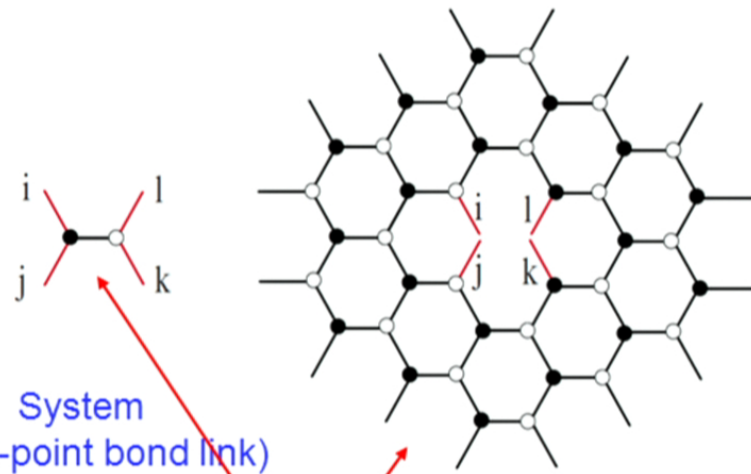
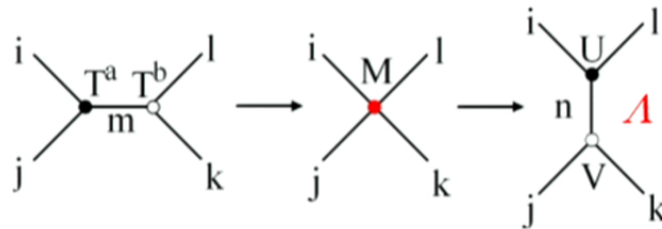
1992 White's DMRG



States are weighted according to the **entanglement** between a sub-system and its env

Better performance

Second renormalization of tensor-network state (**SRG**)



System
(A two-point bond link)

environment

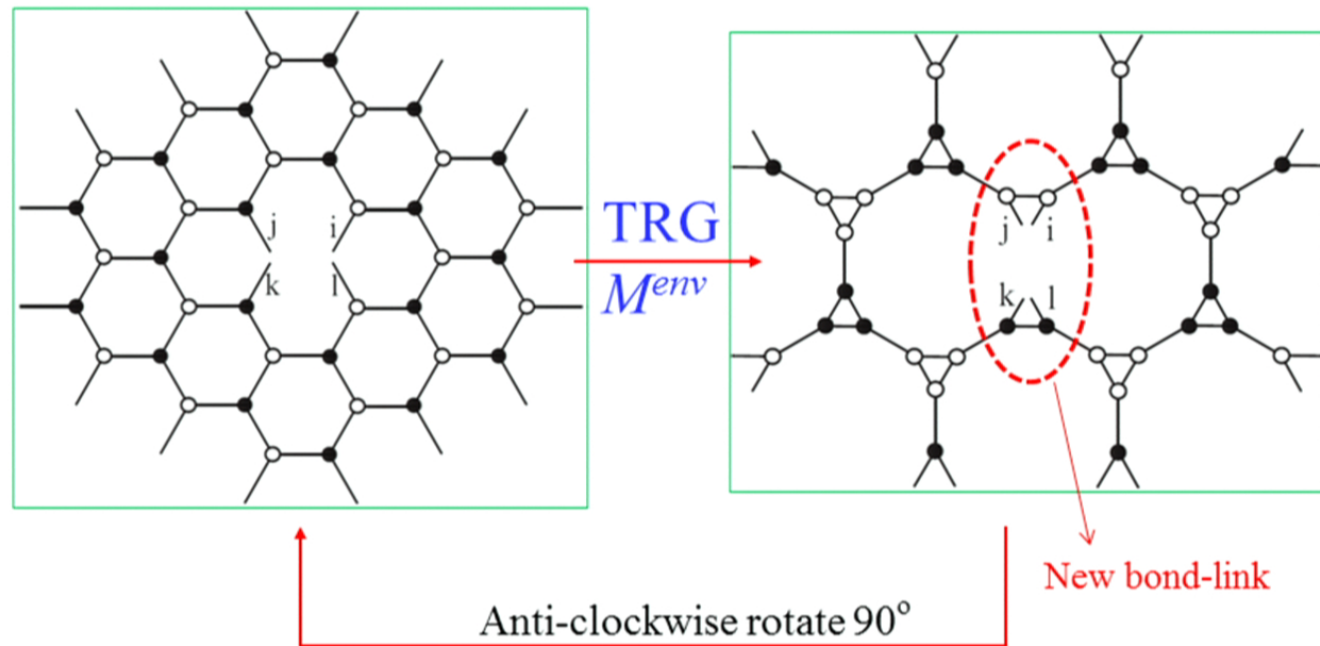
$$Z = \text{Tr} (MM^{env})$$

➤ **TRG**: optimize $M(\text{system})$ by its SVD

➤ **SRG**: optimize full Z by M and M^{env}

➤ ***Z not M!***

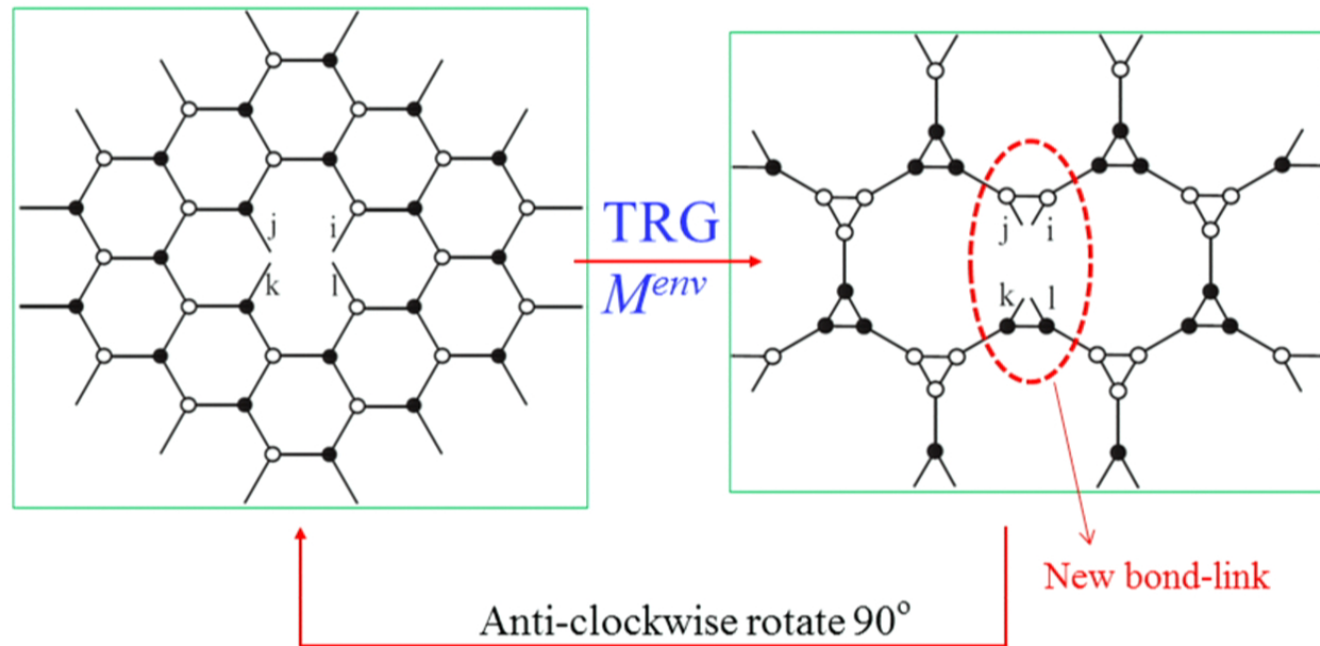
How to get the environment? TRG!



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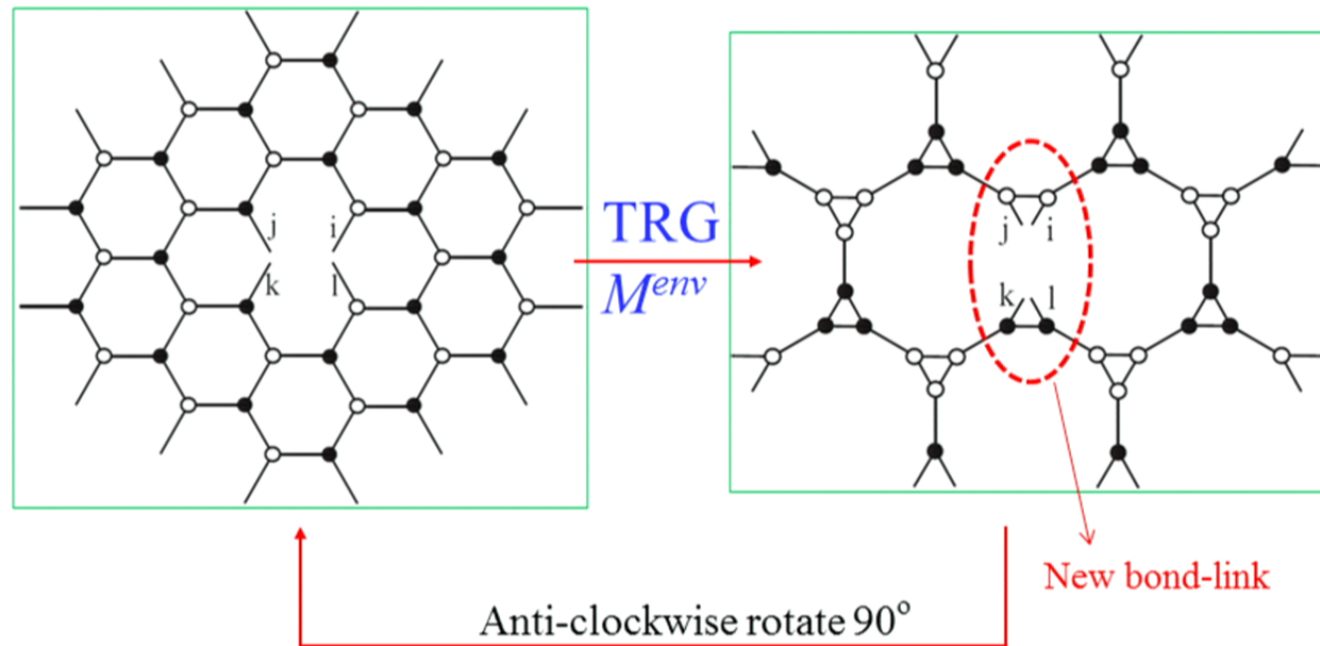
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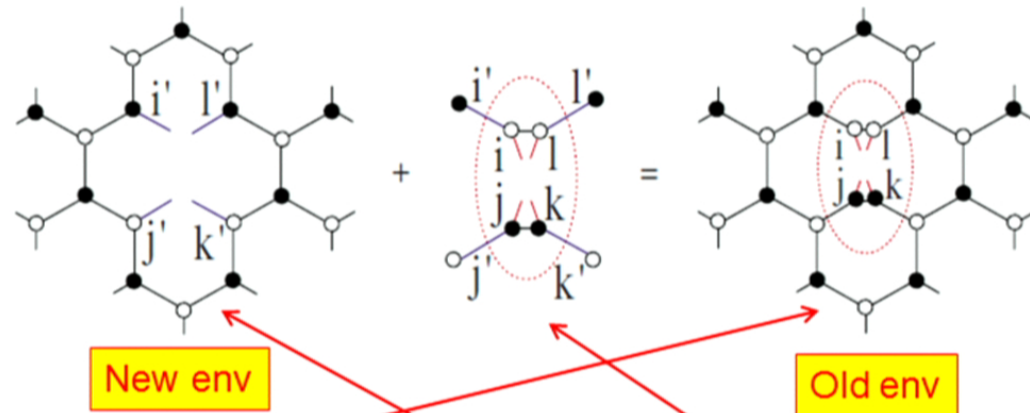
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Between every two neighboring RG steps



$$M_{ijkl}^{(n-1)} = \sum_{i'j'k'l'} M_{i'j'k'l'}^{(n)} \sum_{pq} S_{k'jp} S_{j'pi} S_{i'lq} S_{l'qk}$$

1. **Forward iteration:**

TRG to get all S and the final $M^{(N)}$.

2. **Backward iteration:**

Successively get $M^{(N-1)}$, $M^{(N-2)}$, ..., and at last $M^{(0)} = M^{\text{env}}$

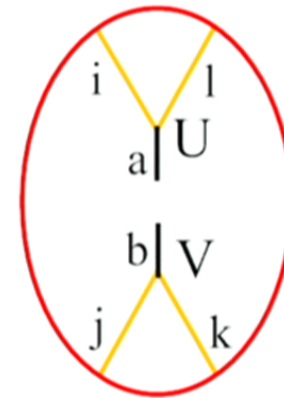
Modify the local decomposition by M^{env}

➤ SVD targeting Z!

$$\begin{aligned} Z &= \text{Tr}(MM^e) \\ &= \text{Tr}(\sqrt{\Lambda_e} V_e^+ M U_e \sqrt{\Lambda_e}) \\ &= \text{Tr}(U \Lambda V^+) \end{aligned}$$

➤ Cut at Λ

➤ Bond Density Matrix

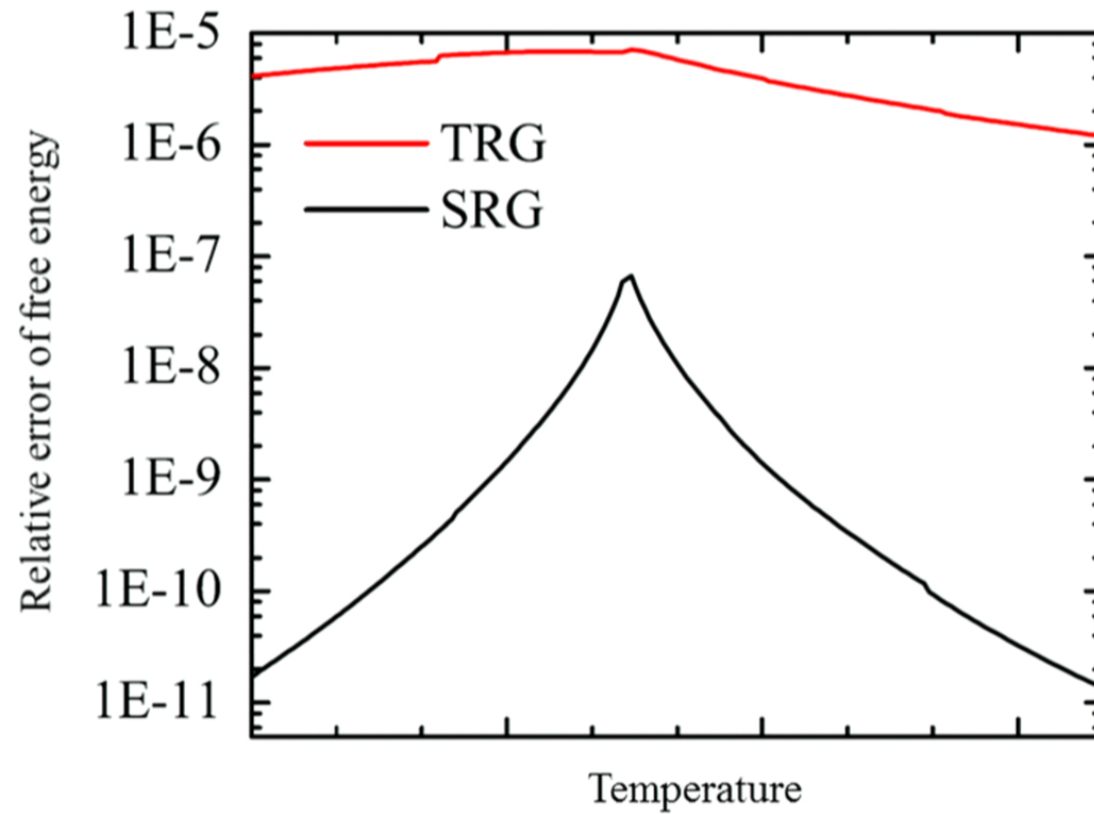


$$Z = \text{Tr} \rho_{ab} = \text{Tr}(P \Lambda P^{-1})$$

➤ Insert PP^{-1} and cut at Λ

Performance: Ising model on a triangular lattice

$D = 24$



Difficulty to extend to 3D

- **PEPS/iPEPS/CTM:** two problems need to solve
no well-controlled truncation scheme of PEPS
heavy cost of expectation value
- **TRG(TERG)/SRG:** still two
change of lattice configuration
heavy cost of coarse-graining process

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Prof. T.Nishino's Group:

- Corner **Transfer Tensor** Renormalization Group method in 3D^[1]
performance: **10 percent**, $D = 2$. (with respect to MC)
- Variational wavefunction method, such as
KWA(wave)+ CTMRG(env)^[2], vertical density matrix algorithm(VDMA)^[3],
vertex-type TPS(wave)+CTMRG(env)^[4],
tensor-product variation ansatz(TPVA) with 64 parameters^[5].

performance: $D: 2\sim 5$,

benchmark^[4]: **0.6 percent deviation**

[1]. *T. Nishino and K. Okunishi, J. Phys. Soc. Jpn. 67, 3066 (1998)*

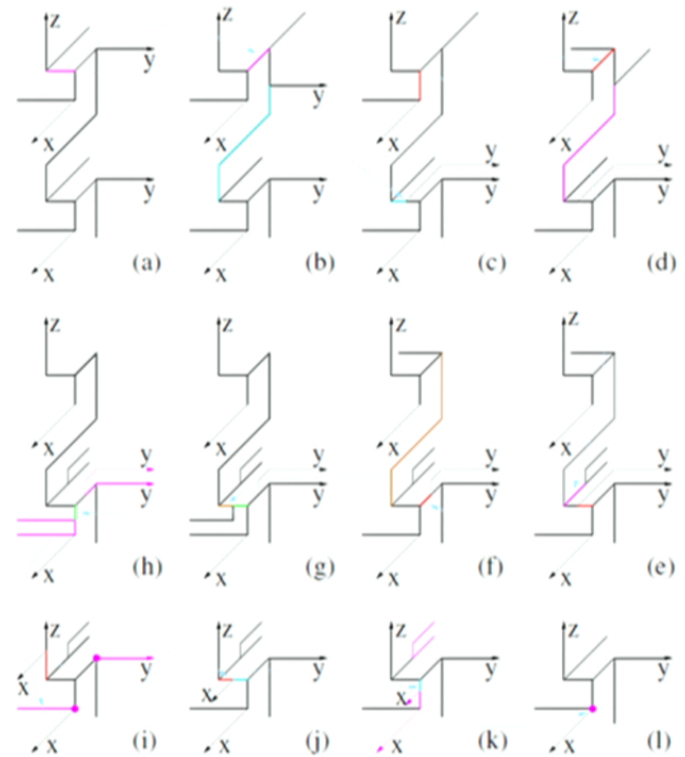
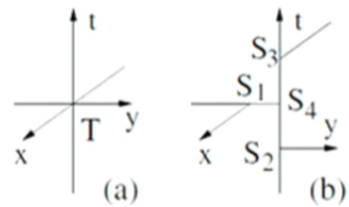
[2]. *K. Okunishi, T. Nishino, Prog. Theor. Phys, 103, 541 (2000);*

[3]. *N. Maeshima, et.al, Phys. Rev. E 64, 016705 (2001);*

[4]. *T. Nishino, et.al, Prog. Theo. Phys. 105, 409 (2001).*

[5]. *A. Gendiar and T. Nishino, Phys. Rev. B 71, 024404 (2005)*

Prof. X.G.Wen's Group:

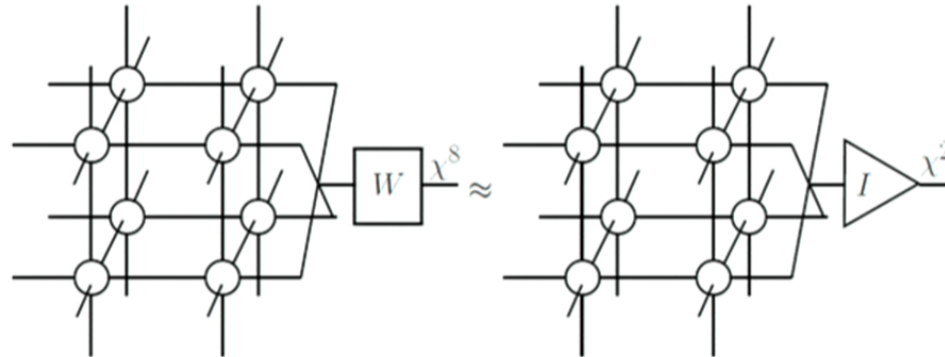


- * 11 moves in 1 RG step
- * 1 percent deflection
- * Critical behavior

Ref: Z. C. Gu, M. Levin, and X. G. Wen, unpublished

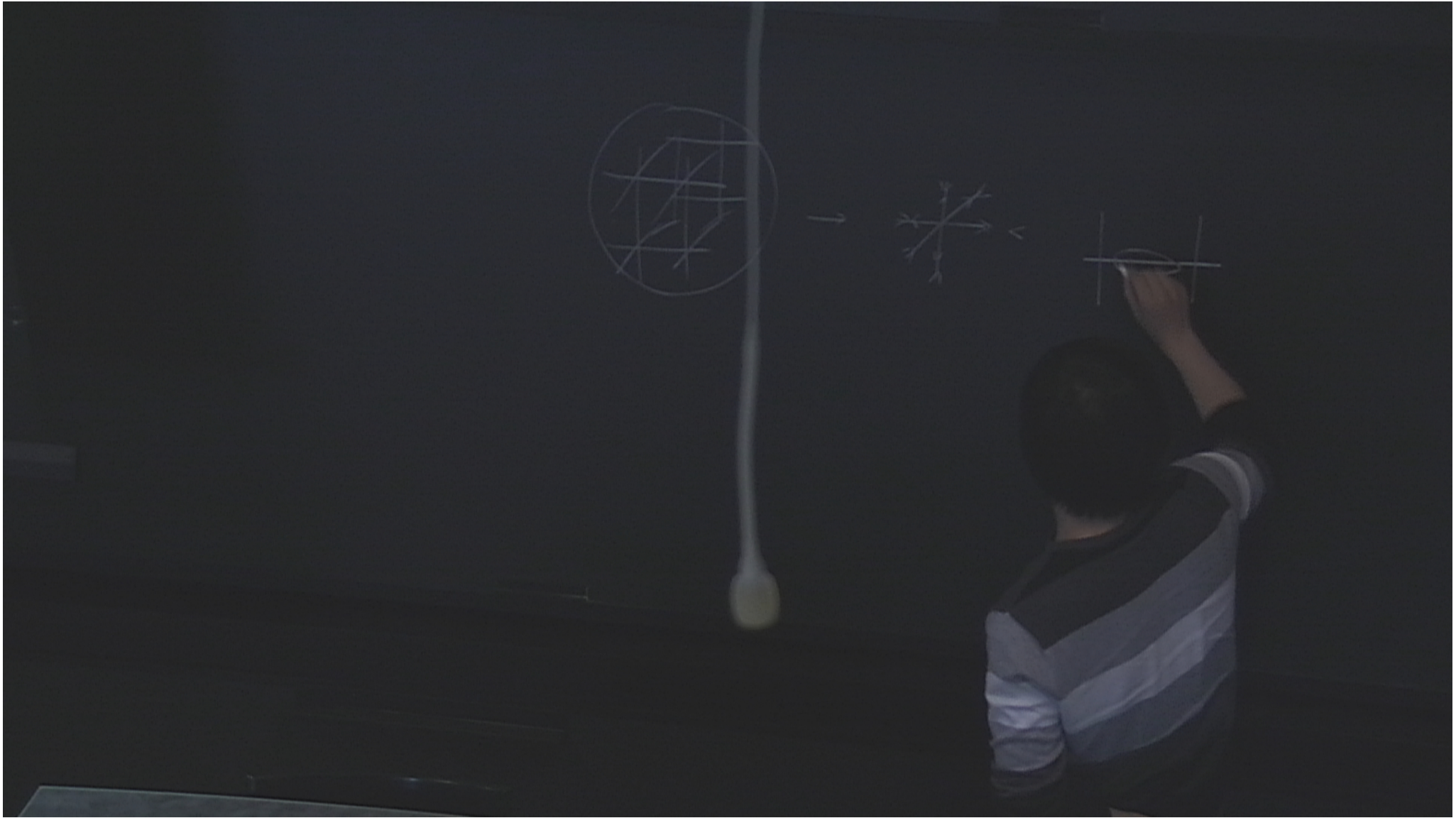
When we are preparing paper:

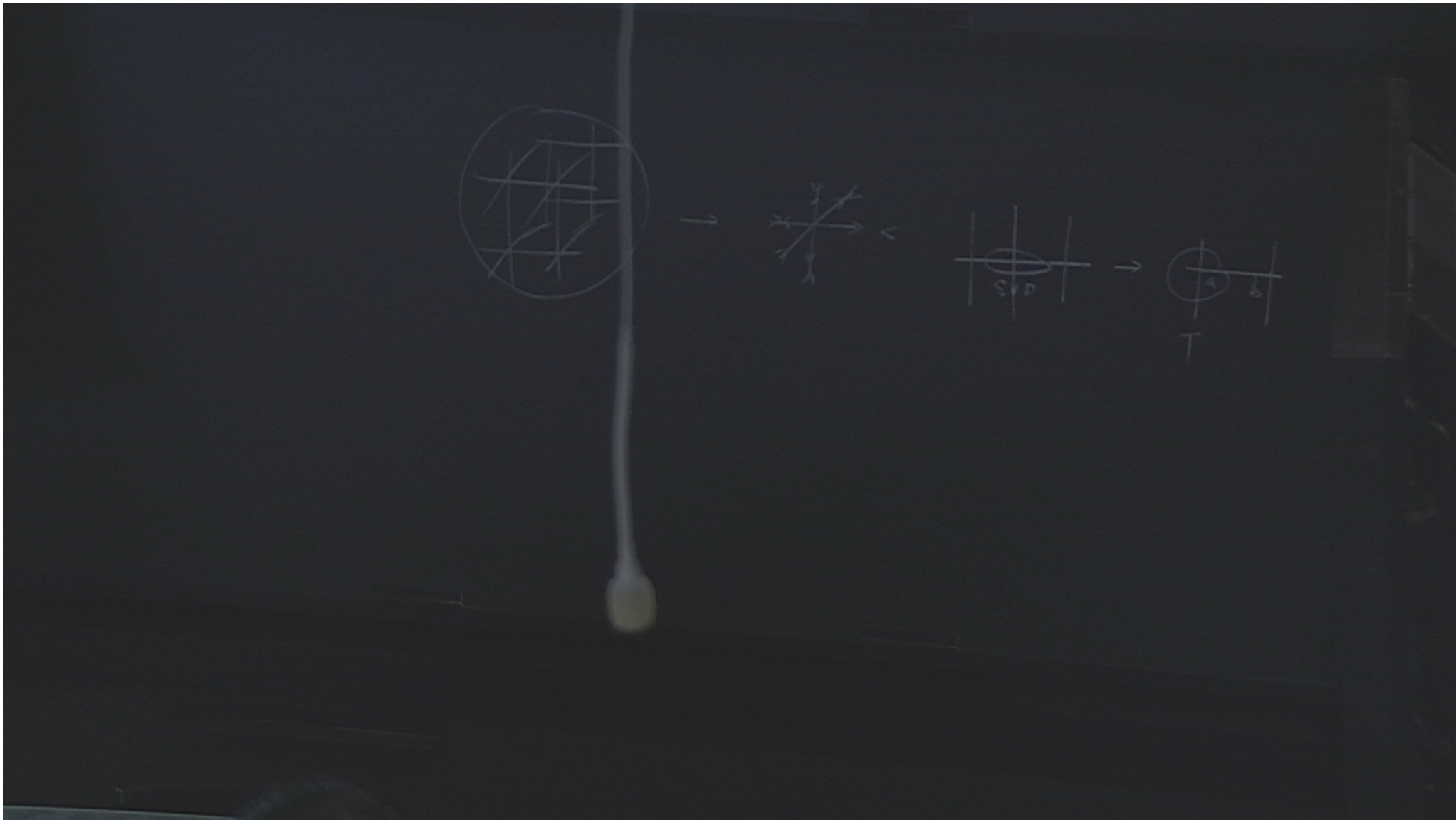
Prof. J.I.Latorre's Group:



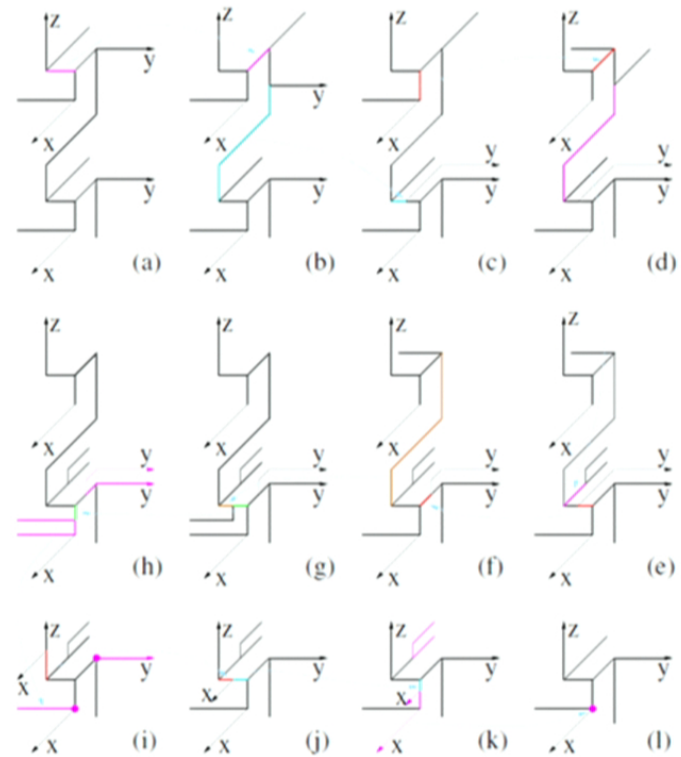
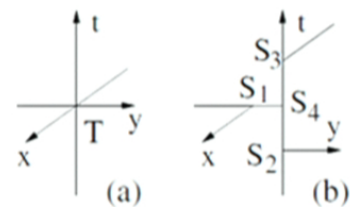
- * Successive SVD targeting bond, not point tensor
- * Very high cost, $D_{\max} \sim 5$
- * *Benchmark:* **2.9 percent** deflection (3D TIF).

Ref: A. Garcia-Saez, and J. I. Latorre, arXiv:1112.1412.





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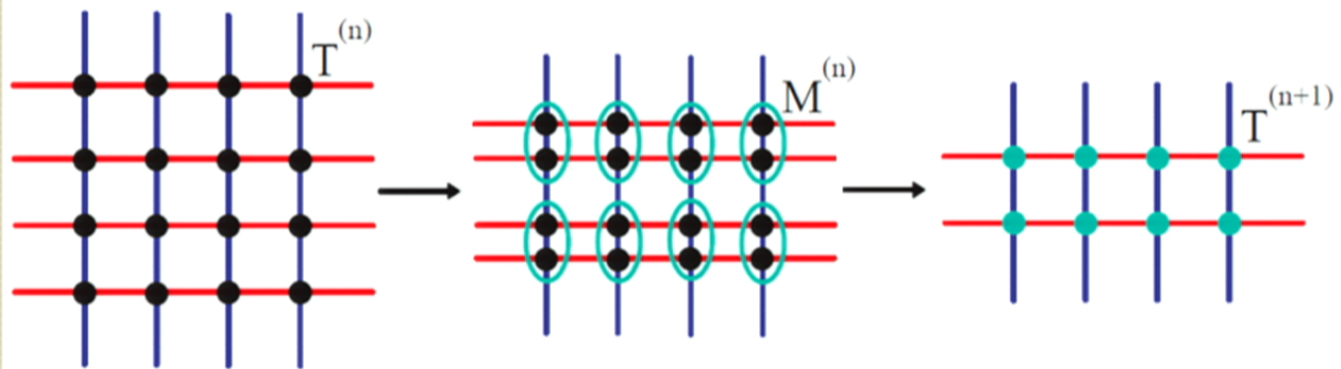


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TRG by Higher-Order SVD (HOTRG)

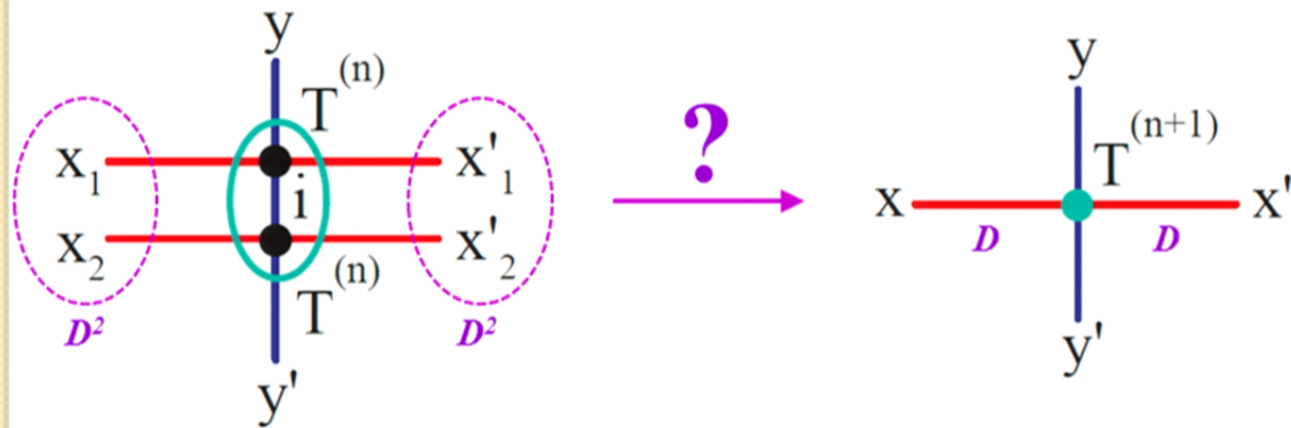
- Coarse-graining process^[1] in 2D



y, x successively

[1]. Z. Y. Xie, J. Chen, J. W. Zhu, L. P. Yang, and T. Xiang, arXiv:1201.1144

Key: How to truncate the tensor?



2 cutoff simultaneously! Open problem^[1]!

[1]. V. de Silva and L. H. Lim, SIAM. J. Matrix Anal. & Appl. 30, 1084 (2008)

Good answer: Higher-order SVD (HOSVD)!

- Definition: (2000, Lieven)

$$M_{xx'yy'}^{(n)} = \sum_{ijkl} S_{ijkl} U_{xi}^L U_{x'j}^R U_{yk}^U U_{y'l}^D$$

- all-orthogonal:

$$\langle S_{:,j,::} | S_{:,j',::} \rangle = 0, \quad \text{if } j \neq j'$$

- Pseudo-diagonal / ordering:

$$|S_{:,j,::}| \geq |S_{:,j',::}|, \quad \text{if } j < j'$$

- Nearly optimal, at least good, low-rank approximation.

Ref: L. de Lathauwer, B. de Moor, and J. Vandewalle, SIAM, J. Matrix Anal. Appl., 21, 1253 (2000).

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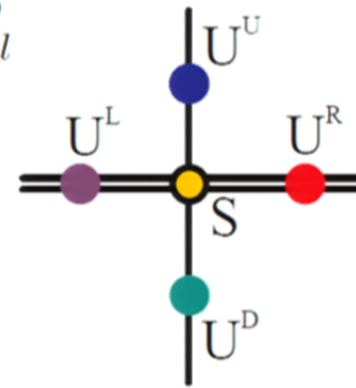
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In our case:

- Express M in terms of S

$$M_{xx'yy'}^{(n)} = \sum_{ijkl} S_{ijkl} U_{xi}^L U_{x'j}^R U_{yk}^U U_{y'l}^D$$

- Regroup the elements:
absorb U^U and U^D in S , cut left or right:

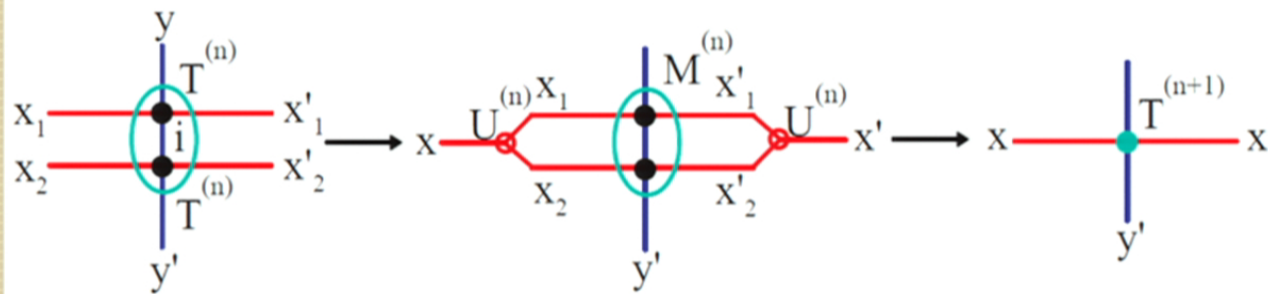


- if $\varepsilon_1 < \varepsilon_2$, cut left, $U^{(n)} = U^L$
if $\varepsilon_1 > \varepsilon_2$, cut right, $U^{(n)} = U^R$

$$\varepsilon_1 = \sum_{i>D} |S(i, :, :, :)|^2$$

$$\varepsilon_2 = \sum_{j>D} |S(:, j, :, :)|^2$$

- Simply amounts to acting two unitaries on a fat tensor, i.e.



$$T_{xx'yy'}^{(n+1)} = \sum_{ij} U_{ix}^{(n)} M_{ijyy'}^{(n)} U_{jx'}^{(n)}$$

- **U**: information of **local** cluster tensor.

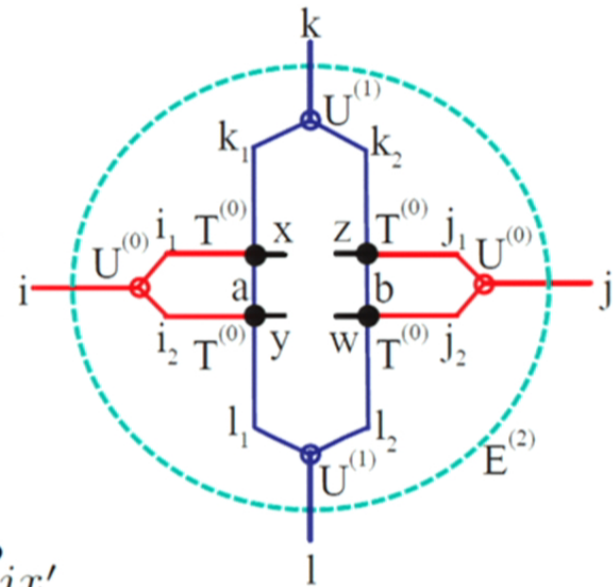
Renormalization effect of Environment

- Bond density matrix.

$$\rho_{zw,xy} = \sum E_{ijkl}^{(2)} U_{i_1 i_2 i}^{(0)} U_{j_1 j_2 j}^{(0)} U_{k_1 k_2 k}^{(1)} U_{l_1 l_2 l}^{(1)} T_{i_1 x k_1 a}^{(0)} T_{i_2 y a l_1}^{(0)} T_{z j_1 k_2 b}^{(0)} T_{w j_2 b l_2}^{(0)}$$

$$Z = \text{Tr} \rho = \text{Tr} (P \Lambda P^{-1})$$

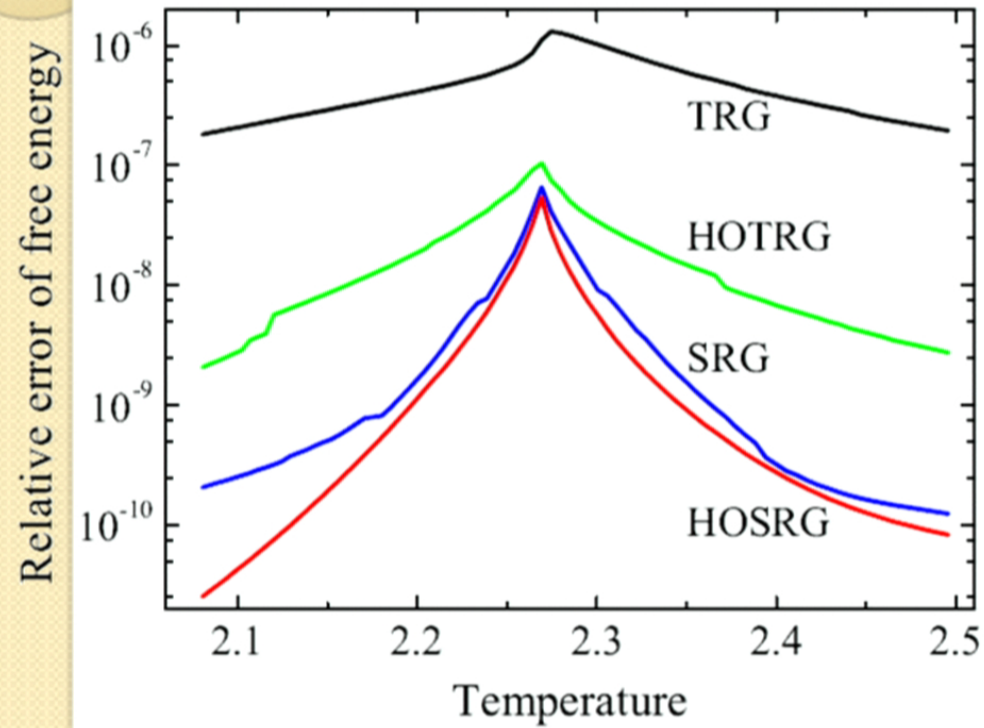
$$T_{xx'yy'} = \sum_{ij} P_{xi}^{-1} M_{ijyy'}^{(0)} P_{jx'}$$



P: approximate information of entire system.

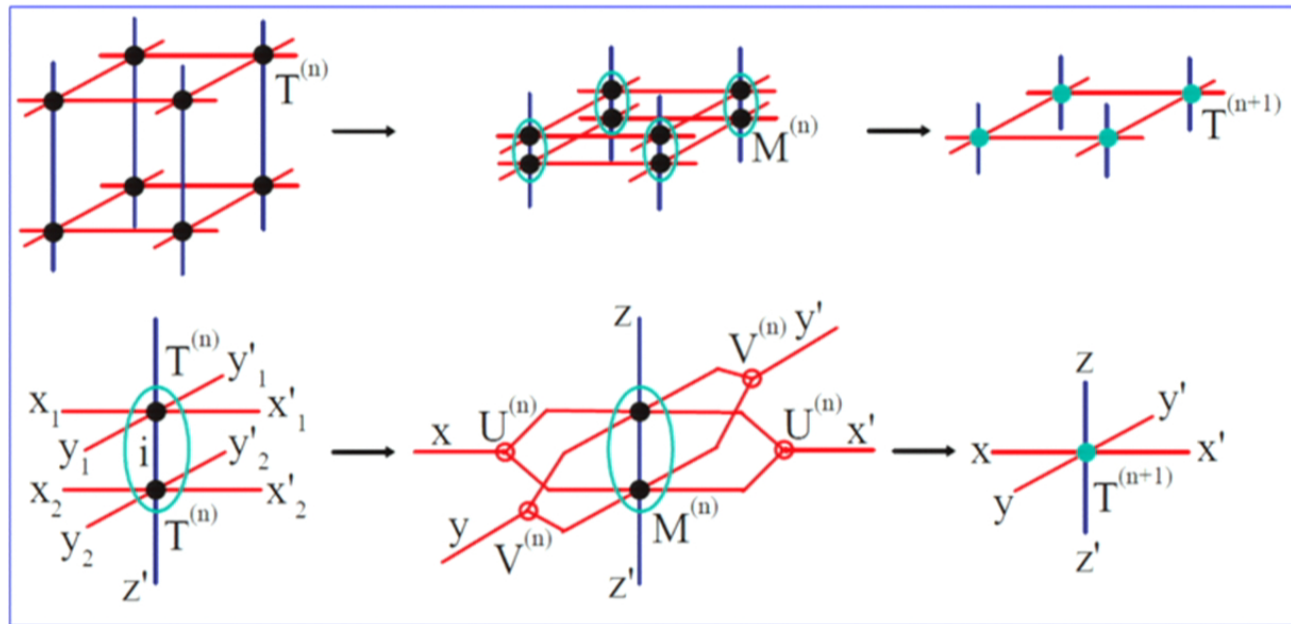
Performance of HOSRG:

D = 24



1. T vs S
2. HOTRG near T_c

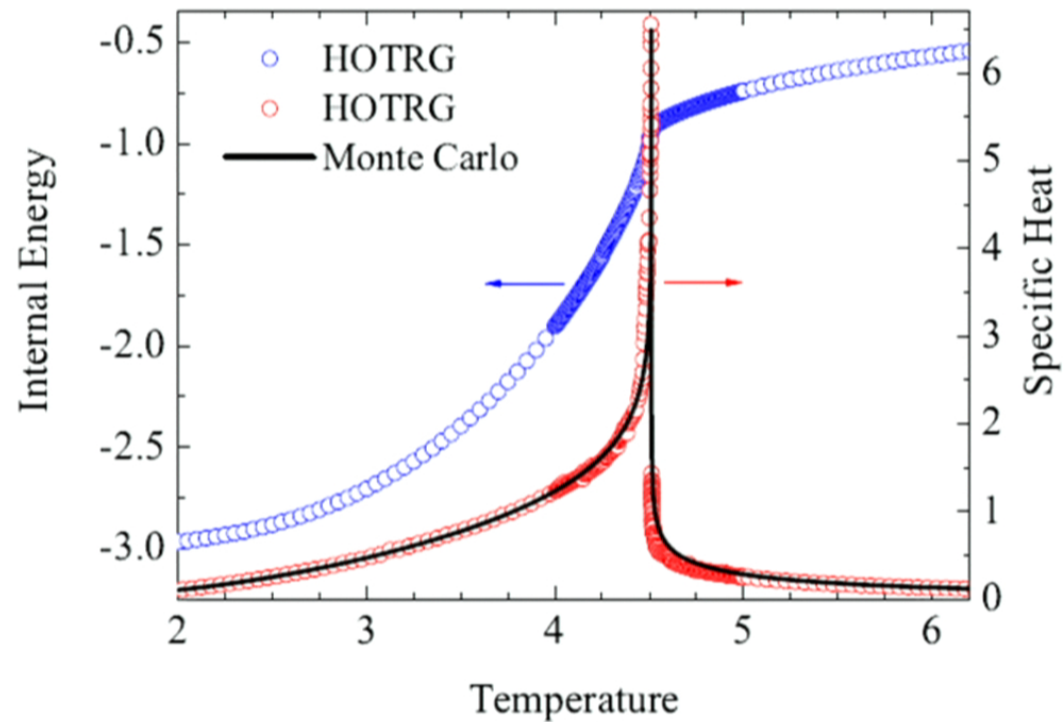
Most important: HOTRG at 3D stage



- * HOSVD of M , order-6, with 4 cuts for core tensor
- * x, y, z axis direction

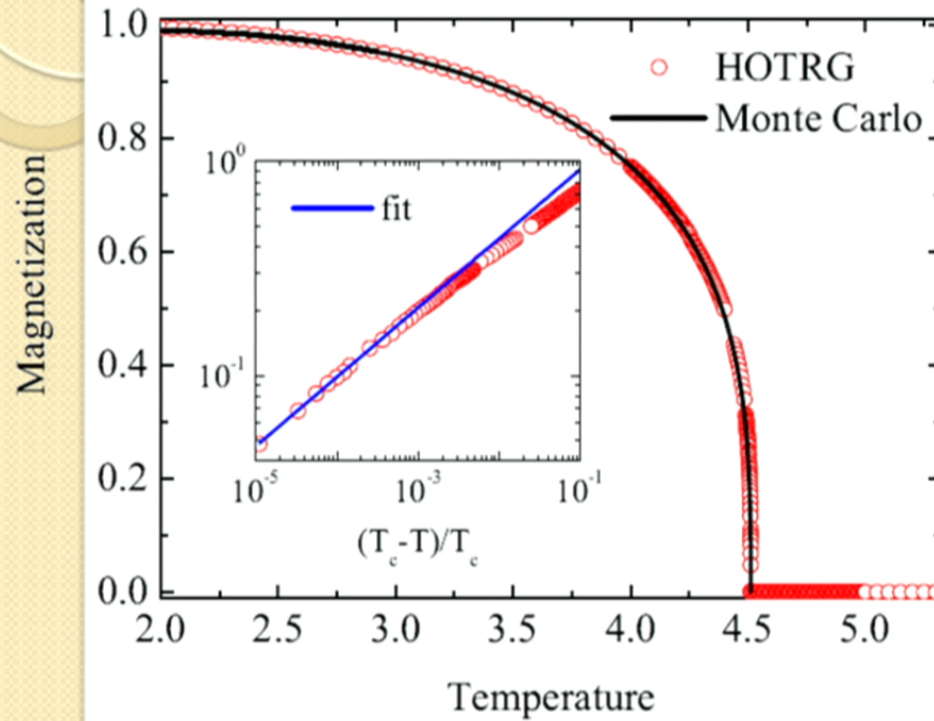
Ising model on simple cubic: HOTRG

$D = 14$



MC data: X. M. Feng, and H. W. J. Blote, Phys. Rev. E 81, 031103 (2010)

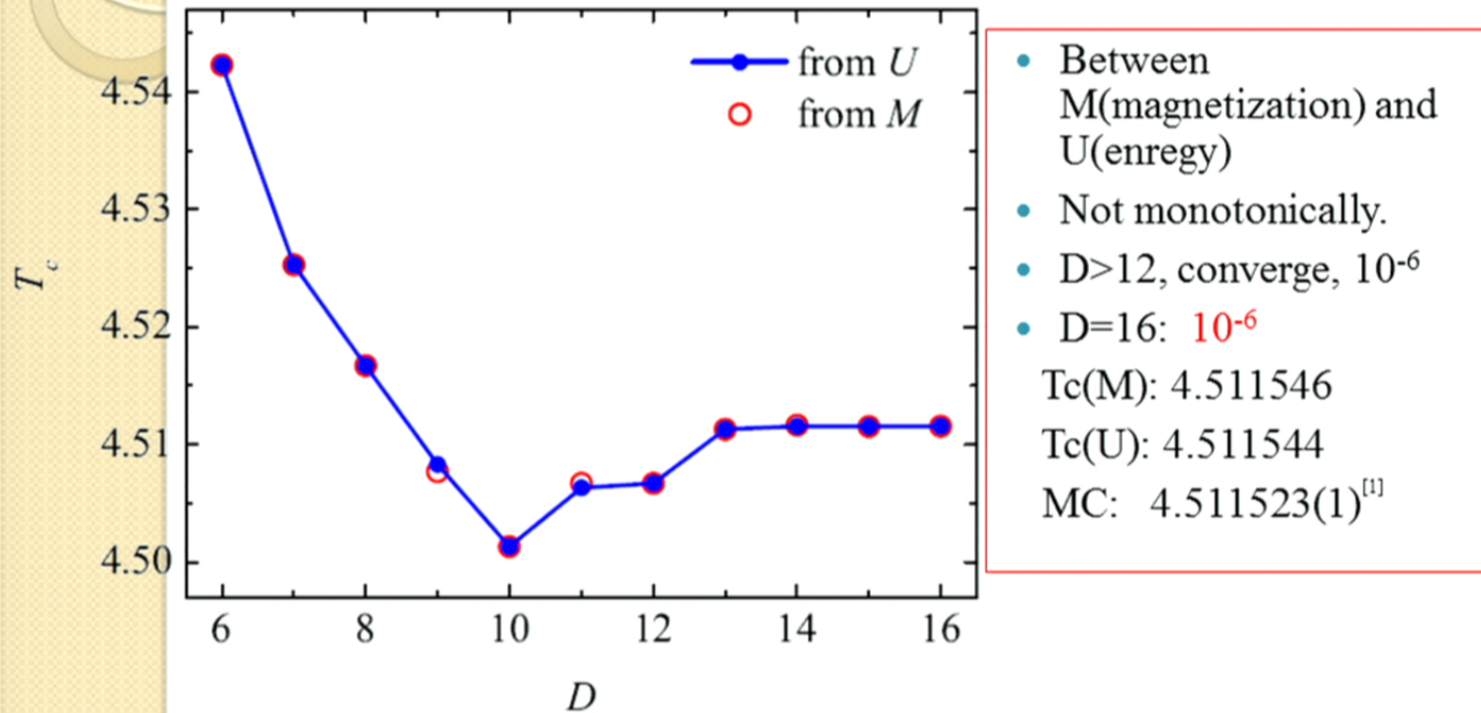
$D = 14$



$\beta:$	0.3237
MC:	0.3262
HTSE:	0.3265

MC data: A. L. Talapov, H. W. J. Blote, J. Phys. A: Math. Gen. 29, 5727 (1996).

Bond dimension (D) analysis



[1]. M. Hasenbusch, Phys. Rev. B 82, 174433 (2010).

Conclusion and Discussion

- HOTRG provides an accurate and efficient numerical tool for **2D** and **3D lattice**.
 - relatively low cost: 2D(**D^4** - **D^7**), 3D(**D^6** - **D^{11}**)
 - $D_{\max} = 16$, accuracy: 10^{-6} , most accurate RG result in 3D
 - Other method on 3D: $D: 2\sim 5$, 0.6%
- HOTRG performs better than TRG in 2D.
 - *cluster size vs efficiency*
 - Essential importance: easy application in 3D
- Renormalization effect of environment
 - HOSRG/SRG performs better than HOTRG/TRG
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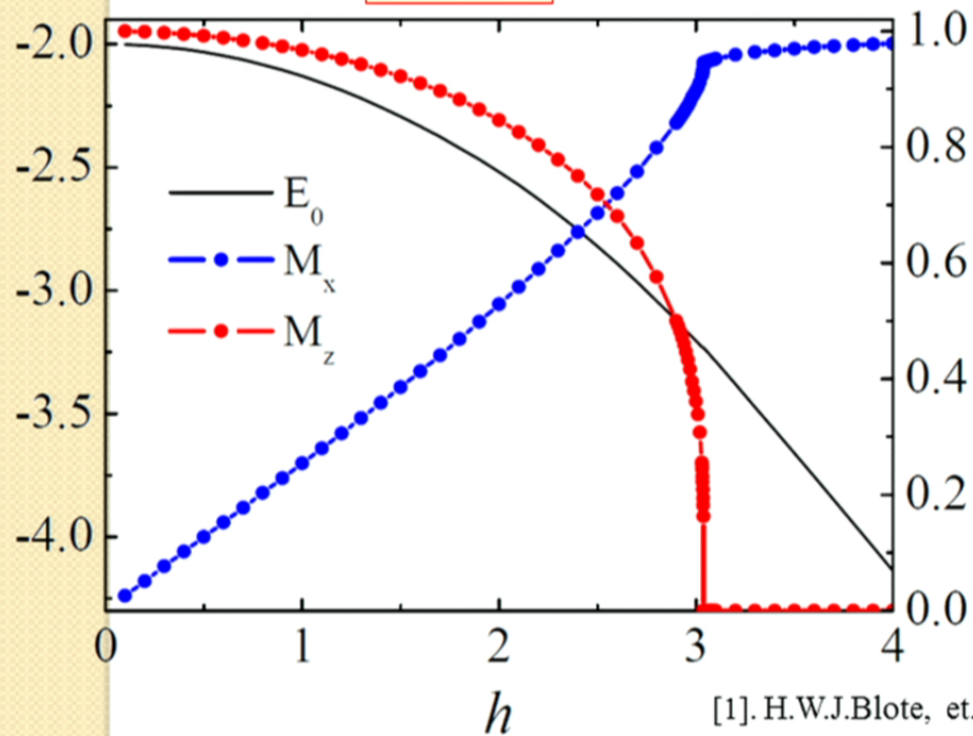
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Quantum model:

- projection to get wave function
- map to (d+1) classical model / finite T

$$H = - \sum_{\langle i,j \rangle} S_z^i S_z^j - h \sum_i S_x^i$$

$D = 12$



Larger D in progress

$h_c = 3.039$

MC: 3.044^[1]

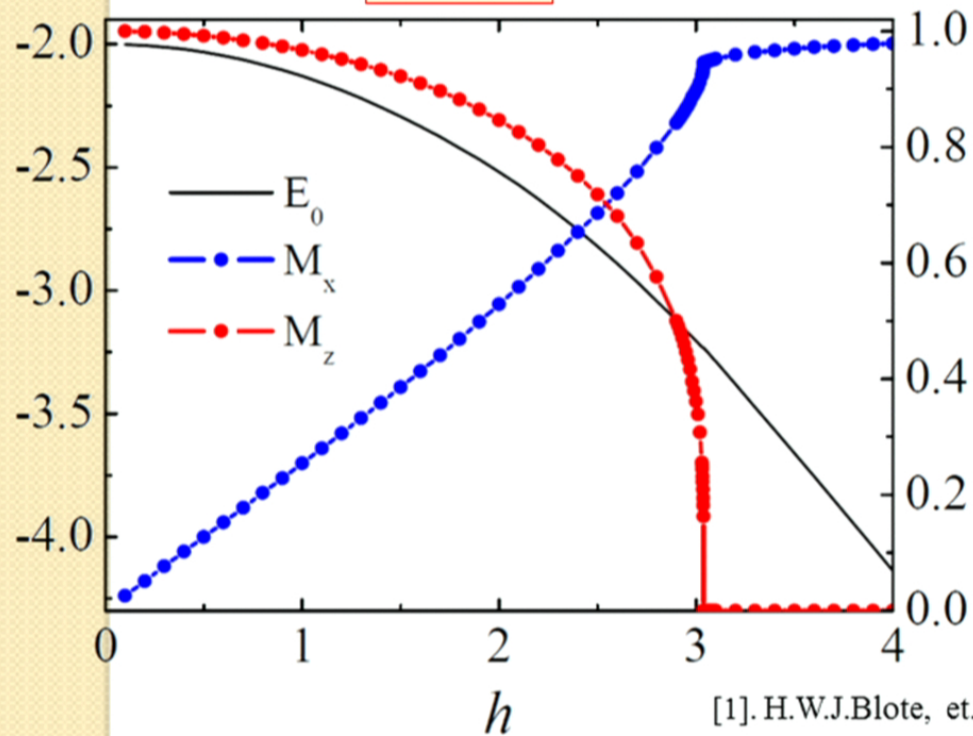
[1]. H.W.J.Blote, et.al, Phys.Rev.E 66, 66110(2002)

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[1]. H.W.J.Blote, et.al, Phys.Rev.E 66, 66110(2002)

Other lattices

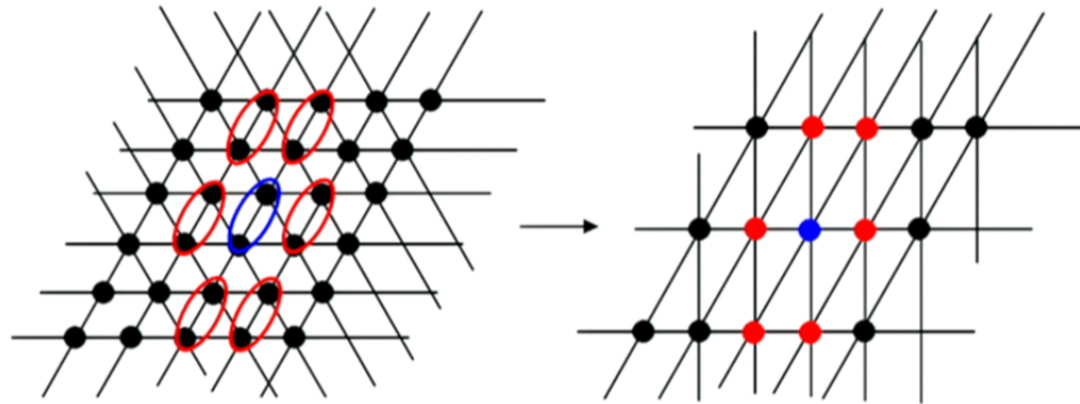
- Simplest way is to convert to square!

honeycomb: combine 1 direction

triangular: dual, then combine

kagome: decompose, then combine

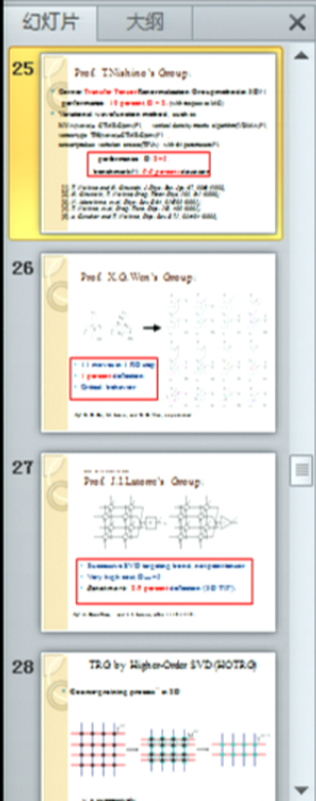
- Or contract along the lattice direction:





➤ Symmetry or quantum number:

- Reflection symmetry
- Initial T:
 - super-symmetric
 - $T(i,j,k,l,m,n) = 0$, if number of 1(2) of index is odd!
- This can be used to reduce the memory and computation cost.
- Ground state wave function symmetry



Prof. T.Nishino's Group:

- **Corner Transfer Tensor Renormalization Group method in 3D^[1]**
performance: **10 percent, $D = 2$** . (with respect to MC)
- **Variational wavefunction method**, such as
KWA(wave)+CTMRG(env)^[2], vertical density matrix algorithm(VDMA)^[3],
vertex-type TPS(wave)+CTMRG(env)^[4],
tensor-product variation ansatz(TPVA) with 64 parameters^[5].

performance: $D: 2 \sim 5$,
benchmark^[4]: **0.6 percent deviation**

[1]. [T. Nishino and K. Okunishi, J. Phys. Soc. Jpn. 67, 3066 \(1998\)](#)
 [2]. [K. Okunishi, T. Nishino, Prog. Theor. Phys. 103, 541 \(2000\);](#)
 [3]. [N. Maeshima, et.al, Phys. Rev. E 64, 016705 \(2001\);](#)
 [4]. [T. Nishino, et.al, Prog. Theo. Phys. 105, 409 \(2001\).](#)
 [5]. [A. Gendiar and T. Nishino, Phys. Rev. B 71, 024404 \(2005\)](#)

单击此处添加备注





