Title: K3 Modular Parametrization and Calabi-Yau Threefold Moduli

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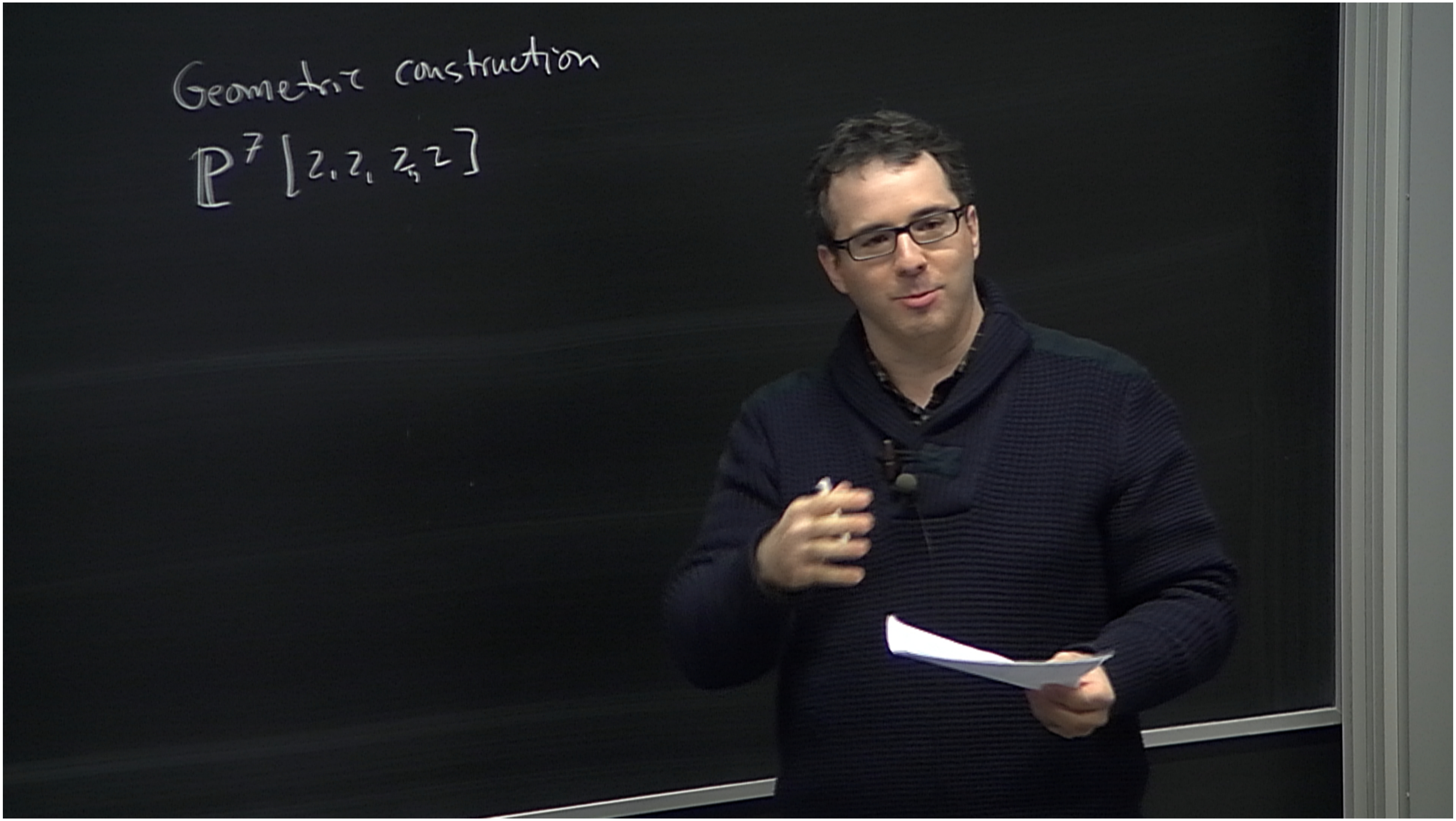
Abstract: A series of generalizations of the Weierstrass normal form for elliptic curves to the case of K3 surfaces will be presented. These have already been applied to better understand F-theory/Heterotic string duality.
We will see how they also resolve a long-standing question of which "mirror-compatible" variations of Hodge structure over the thrice-punctured sphere can arise from families of Calabi-Yau threefolds.


Geometric construction
$\mathbb{P}^7[2,2,2,2]$

K3 Modular Parametrization and Calabi-Yau Threefold Moduli

elliptic curves $\rightsquigarrow$ ell. K3 s/ces

$$\begin{array}{ccc} E & \to & K3 \\ & & \downarrow \\ & & \mathbb{P}^1 \end{array}$$

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$$\begin{tikzcd} E \arrow[r] & K3 \arrow[d] \\ & \mathbb{P}^1 \arrow[u, bend right] \end{tikzcd}$$

1. To build K3 generalizations of Weierstrass normal forms for elliptic curves
2. To apply this to CY 3-folds

elliptic curves $\rightsquigarrow$ ell. K3 s/ces $\quad$ $E \to K3 \to \mathbb{P}^1$

1. To build K3 generalizations of Weierstrass normal forms for elliptic curves
2. To apply this to CY 3-folds as "relative K3 s/ces" to answer questions about CY 3-fold moduli

W. (n. form for ell. cves

$$E: \quad y^2 z - 4x^3 + g_2 x z^2 + g_3 z^3 = 0 \quad , (g_2, g_3) \in \mathbb{C}^2$$

$$\Delta := g_2^3 - 27 g_3^2 \neq 0$$

$[g_2, g_3] \in \mathbb{WP}(2,3)$

classifies E up to isomorphism.

$$\mathcal{M} = \{ [g_2, g_3] \in \mathbb{WP}(2,3) \,|$$

Classifying space for periods for E : $\Gamma \backslash \mathfrak{h}$

$\Gamma = PSL(2, \mathbb{Z})$

$$\mathrm{per} : M \longrightarrow \Gamma \backslash \mathfrak{h}$$

$$\mathrm{per}^{-1} = [60 \cdot E_4,\ 140 \cdot E_6]$$

E_4, E_6 Eisenstein series of wts 4, 6 resp.

K3 analogues of W. n. form ?

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

X alg. K3 sfce / $\mathbb{C}$

alg curves generate Néron-Severi lattice.

$NS(K3) \subset$

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$$NS(K3) \subset H \oplus H \oplus H \oplus E_8 \oplus E_8 \qquad H^{1,1}_{\mathbb{Z}}$$

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M even lattice of signature $(1, \ell-1)$, $\ell > 0$

An M-polarization on X is a primitive lattice embedding

$$i : M \hookrightarrow NS(X)$$

$\langle 4 \rangle$-polarized K3 : $M = \langle 4 \rangle$ smooth quartic hypsfces in $\mathbb{P}^3$

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$\mathcal{M}_{M\text{-pol}}$: coarse mod spce for equiv classes of pairs (M, i)

How do M-polarizations arise for, say, quartics?

- Symmetry ⟵ Fermat quartic K3

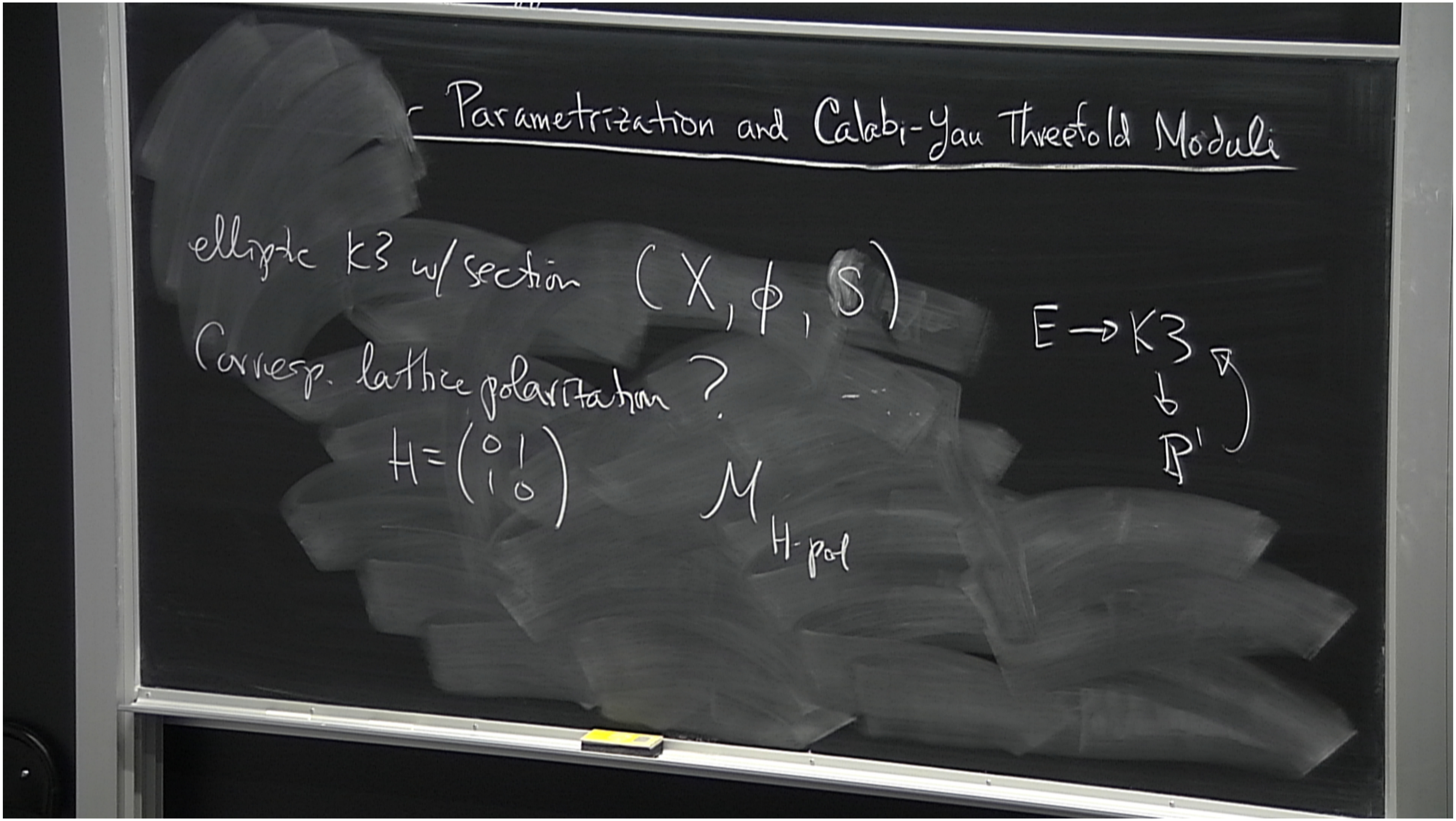
$$x^4+y^4+z^4+w^4=0$$

$(\mathbb{Z}/4\mathbb{Z})^2$ acts. $\begin{cases} x \mapsto \lambda x \\ y \mapsto \mu y \\ z \mapsto \lambda^{-1}\mu^{-1} z \end{cases}$

get $\mathrm{rk}\, NS = \underline{\underline{20}}$

- singularities

ADE singularities, upon minimal resoln yield contributions to $NS(X)$


Parametrization and Calabi-Yau Threefold Moduli
elliptic K3 w/ section (X, ϕ, S)
Corresp. lattice polarization ?
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$E \to K3$
$\downarrow$
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Corresp. lattice polarization?

$$H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\mathcal{M}_{H\text{-pol}}$$

$$\dim_{\mathbb{C}} = 18$$

$$24 \times I_1$$

$$NS(X) = H \oplus W_X, \quad W_X^{root} = \{ r \in W_X \mid \langle r, r \rangle = -2 \}$$

$$W_X^{root} = \bigoplus \text{ADE type lattices.}$$

$$M := \underline{H} \oplus E_8 \oplus E_8$$

$NS(X) = H \oplus W_X$, $W_X^{root} = \{ v \in W_X \mid \langle v, v \rangle = -2 \}$

$\parallel$

$\oplus$ ADE type lattices.

$M := H \oplus E_8 \oplus E_8$

$\mathcal{M}_{M\text{-pol}}$ has $\dim_{\mathbb{C}} = 2$ (X, i) is M-pol K3

Then there is a triple $(a, b, d) \in \mathbb{C}^3$ st X is isomorphic to the minimal resol'n of the ADE-singular quartic :

$$Q_M(a,b,d) := y^2 z w - 4x^3 z + 3a\,x z w^2 + b\,z w^3 - \tfrac{1}{2}(d\,z^2 w^2 + w^4) = 0$$

Two such quartics $Q_M(a_1,b_1,d_1)$ and $Q_M(a_2,b_2,d_2)$

$\longrightarrow$ isomorphic K3 surfaces iff $(a_2,b_2,d_2) = (\lambda^2 a_1, \lambda^3 b_1, \lambda^6 d_1)$ for some $\lambda \in \mathbb{C}^*$.

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$$\mathcal{M}_{M\text{-pol}} = \{ [a,b,d] \in \mathbb{WP}(2,3,6) \mid d \neq 0 \}$$

On $Q_M(a,b,d) \subset \mathbb{P}^3$, there are two distinct lines

$\{x=w=0\}$, $\{z=w=0\}$,

and the pts $P_1 := [0,1,0,0]$ and $P_2 := [0,0,1,0]$

A_{11} E_6

– "standard" fibration: proj to $[z,w]$ $W_X = E_8 \oplus E_8$

– "alternate" fibration: proj to $[x,w]$ $W_X = D_{16}^+$

Toric anticanonical hypersurface normal form.

s' 2^{nd} section, $\sigma' \in MW(\phi, s)$

$\widetilde{X}/\sigma'$

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$$\widetilde{X/\sigma'} \cong \mathrm{Kum}(E_{\tau_1} \times E_{\tau_2})$$

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$$\mathcal{M}_{M\text{-pol}} \longrightarrow \left(PSL(2,\mathbb{Z}) \times PSL(2,\mathbb{Z}) \rtimes \mathbb{Z}/2\mathbb{Z}\right) \backslash \mathfrak{h} \times \mathfrak{h}$$

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E_4, E_6 Eisenstein series of wts 4, 6 resp.

$$j(\tau) = g_2(\tau) / \Delta(\tau)$$

$$Q_M(a,b,d) = y^2 z w - 4x^3 z + 3a\, x z w^2 + b\, z w^3 - \tfrac{1}{2}\left(d\, z^2 w^2 + w^4\right) = 0$$
$$+\, c\, x z^2 w$$

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$$\mathcal{M}_{N\text{-pol}} = \left\{ [a,b,c,d] \in W\mathbb{P}(2,3,5,6) \;\middle|\; \begin{array}{c} c \neq 0 \\ \text{or} \\ d \neq 0 \end{array} \right\}$$

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$$[a,b,c,d] = [\mathcal{E}_4, \mathcal{E}_6, 2^{12}\cdot 3^5 \cdot \mathcal{C}_{10}, 2^{12}\cdot 3^6 \cdot \mathcal{C}_{12}]$$

$$\frac{a^3}{d} = \pi \quad , \quad \frac{b^2}{d} = \pi - \sigma + 1$$

$$N := H \oplus E_8 \oplus E_7$$

$$H \oplus E_7 \oplus E_7$$

$$\widetilde{X/_{\sigma'}} = \mathrm{Kum}(\mathrm{Jac}(C_2))$$

Toric anticanonical hypersurface normal form.

$\Delta \rightsquigarrow \mathbb{P}_\Delta \supset X$

3-D Refl. polytope

toric Fano 3-fold

X $M = H \oplus E_8 \oplus E_8$ - pol. K3 has mirror

an

Toric anticanonical hypersurface normal form

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$$\mathbb{WP}(1,1,4,6) \cup$$

H-pol. K3 s'face.

PF eqn: $\left[\Theta^4 - z(\Theta + a_1)(\Theta + a_2)(\Theta + a_3)(\Theta + a_4)\right] \varpi(z)$, $\Theta = z \frac{d}{dz}$

a_1	a_2	a_3	a_4	Geometric construction	
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\mathbb{P}^7[2,2,2,2]$	#1
$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{2}{3}$	$\mathbb{P}^6[2,2,3]$	#2
	$\vdots$			$\vdots$	
$\frac{1}{6}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{5}{6}$	$\mathbb{WP}(1,1,1,1,2)[6]$	#8
	$\vdots$			$\vdots$	
$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	$\mathbb{P}^4[5]$	#11
	$\vdots$			$\vdots$	
$\frac{1}{12}$	$\frac{5}{12}$	$\frac{7}{12}$	$\frac{11}{12}$	??	#14

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$$\varpi_0(z) = \sum_{m=0}^{\infty} \frac{(2m)!\,(12m)!}{(m!)^4\,(4m)!\,(6m)!} z^m$$

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$$h^{2,1} = 3$$

CY 3-folds fibered by H-pol K3 s/ces

$\cap$

$\mathbb{WP}(1,1,4,6)$

Hope (engineer) that mirror symmetry respects this fibration

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<u>Batyrev</u>

$\mathbb{WP}(1,1,2,8,12)[24]$

$h^{2,1}$ of mirror CY's $= \underline{\underline{3}}$

CI $\quad$ K3 surface fibers $\underline{\underline{\text{as}}}$ M-pol. K3's.

$\downarrow$

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$\downarrow$

Hyp  $\mathrm{Kum}(E_1 \times E_2)$-fibered CY 3fold w/ <u>1 node</u>