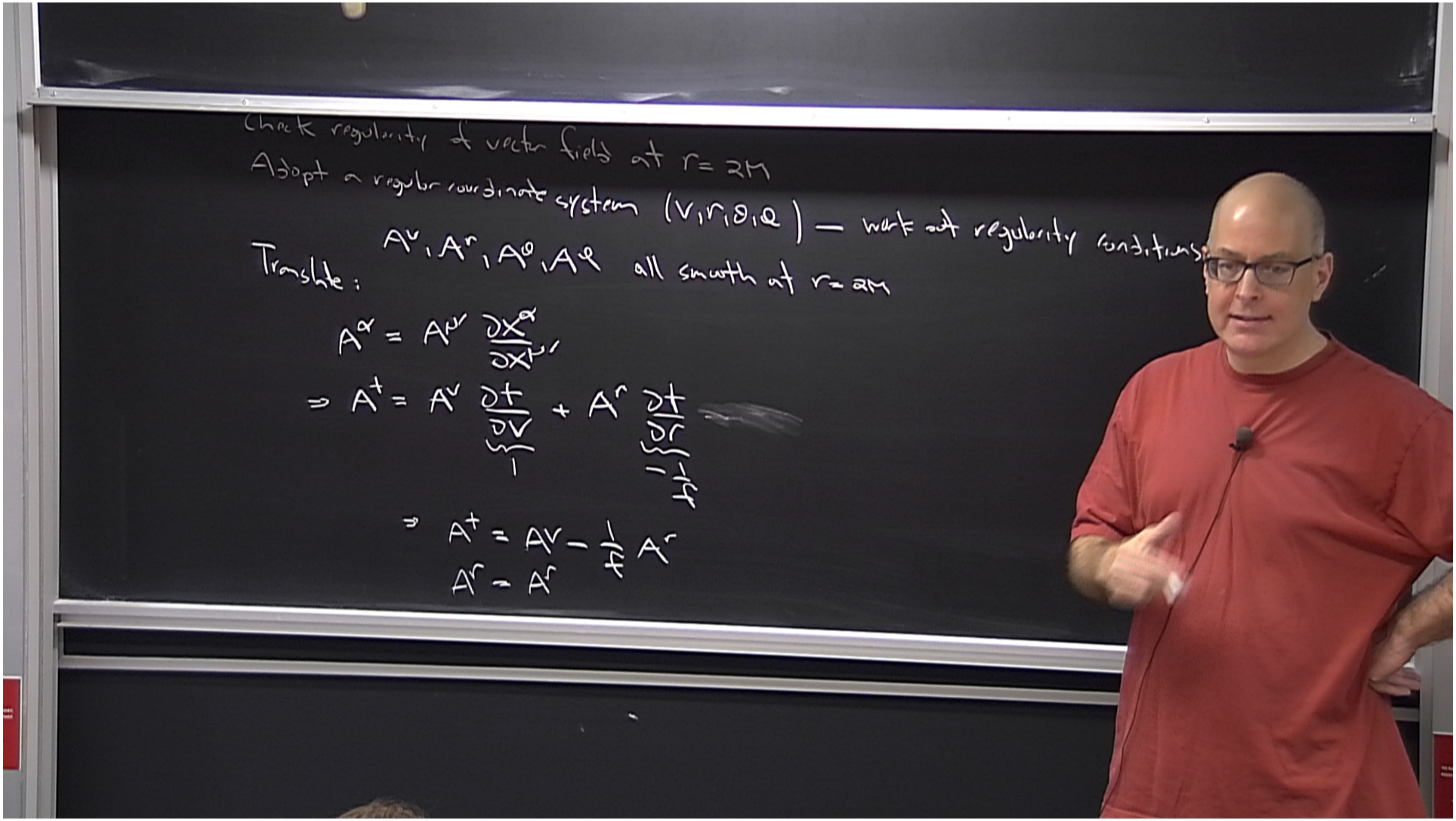
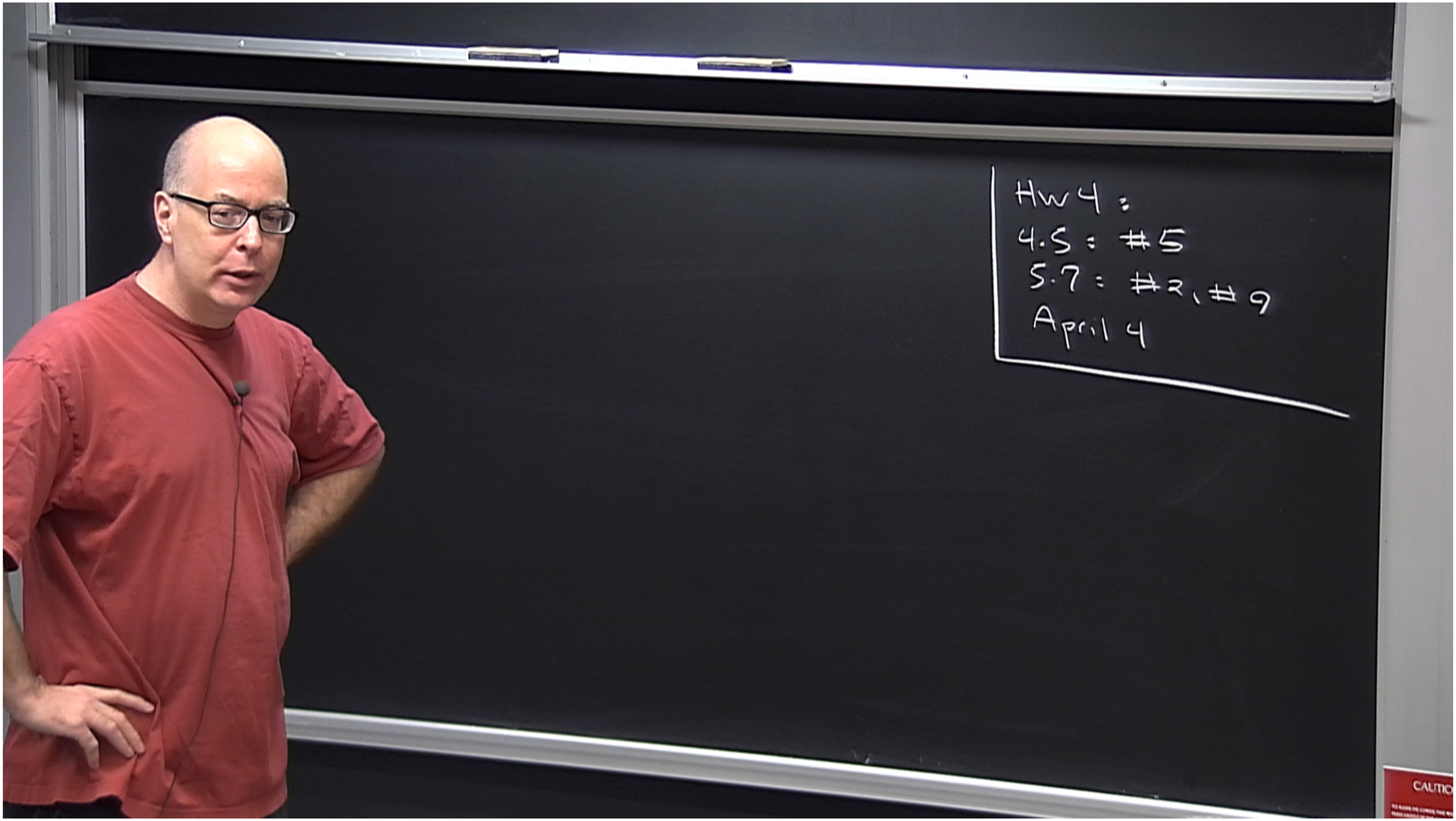
Title: Advanced General Relativity - Lecture 19

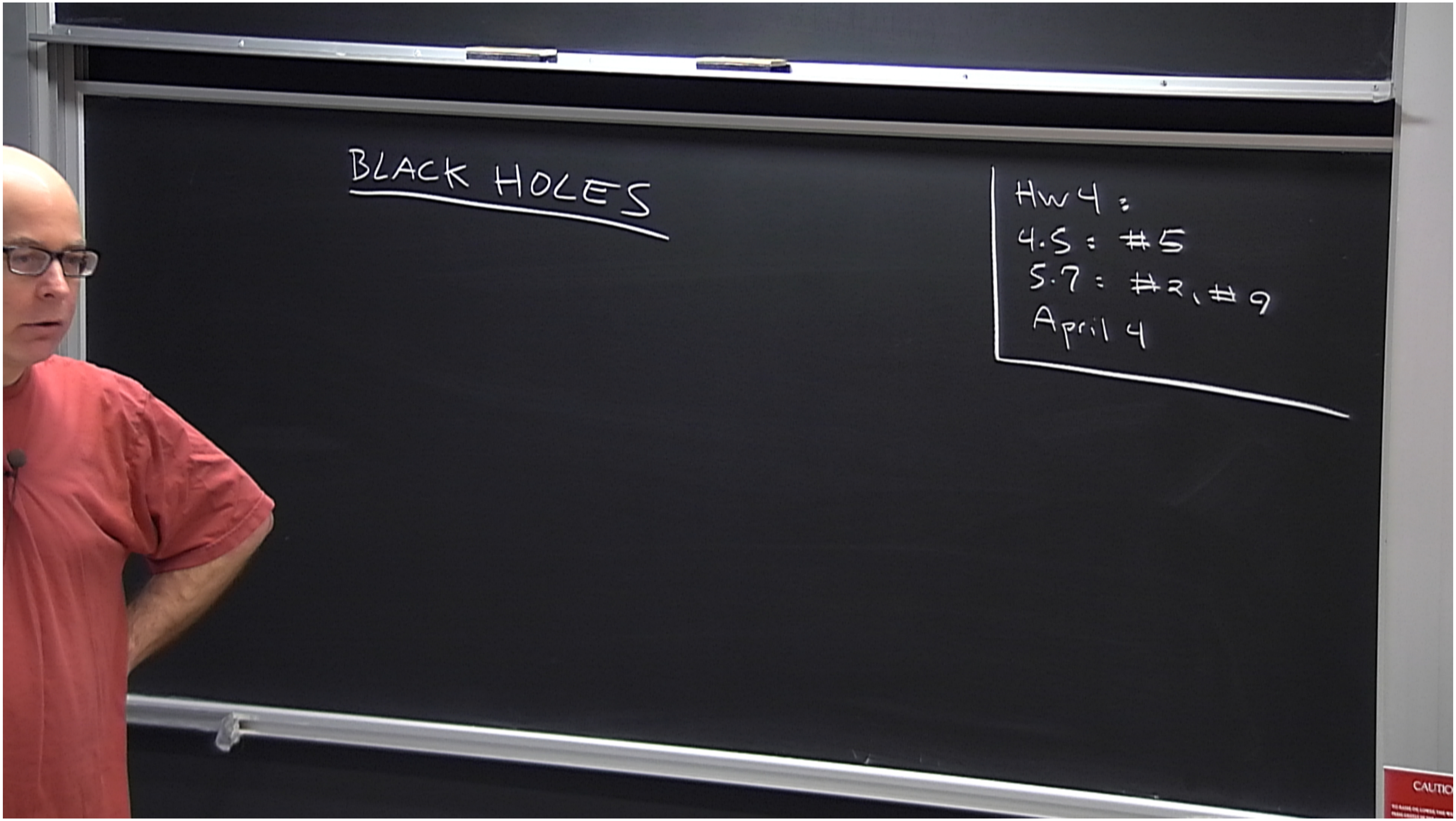
Date: Mar 21, 2012 10:00 AM

URL: http://pirsa.org/12030070

Abstract:


Check regularity of vector fields at r = 2M
Adopt a regular coordinate system (v, r, θ, φ) — work out regularity conditions
Translate: A^v, A^r, A^θ, A^φ all smooth at r = 2M
A^α = A^μ' ∂x^α/∂x^μ'
⇒ A^t = A^v ∂t/∂v + A^r ∂t/∂r
1
−1/f
⇒ A^t = A^v − (1/f) A^r
A^r = A^r


Hw 4 :
4.5 : #5
5.7 : #2, #9
April 4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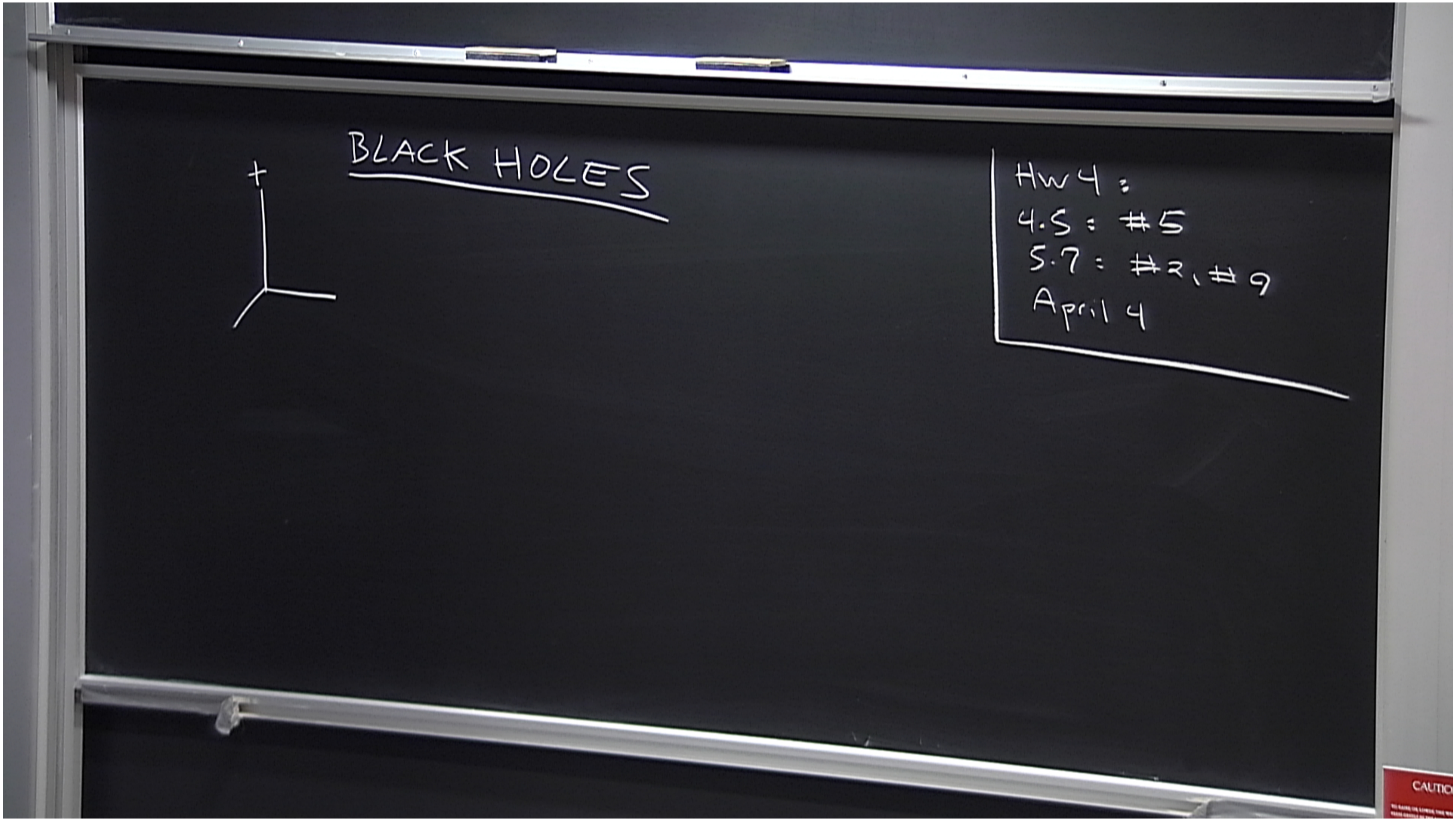
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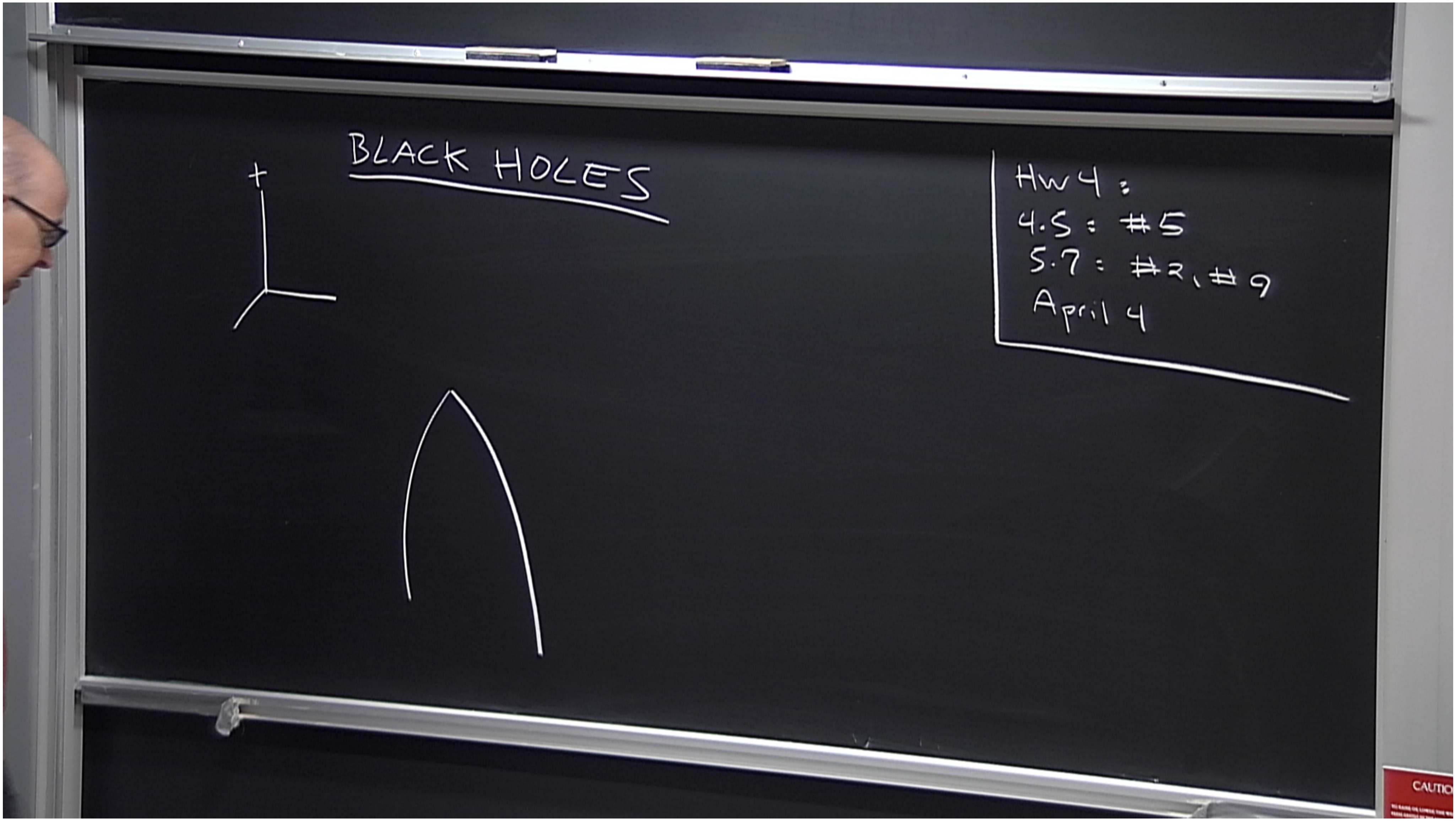
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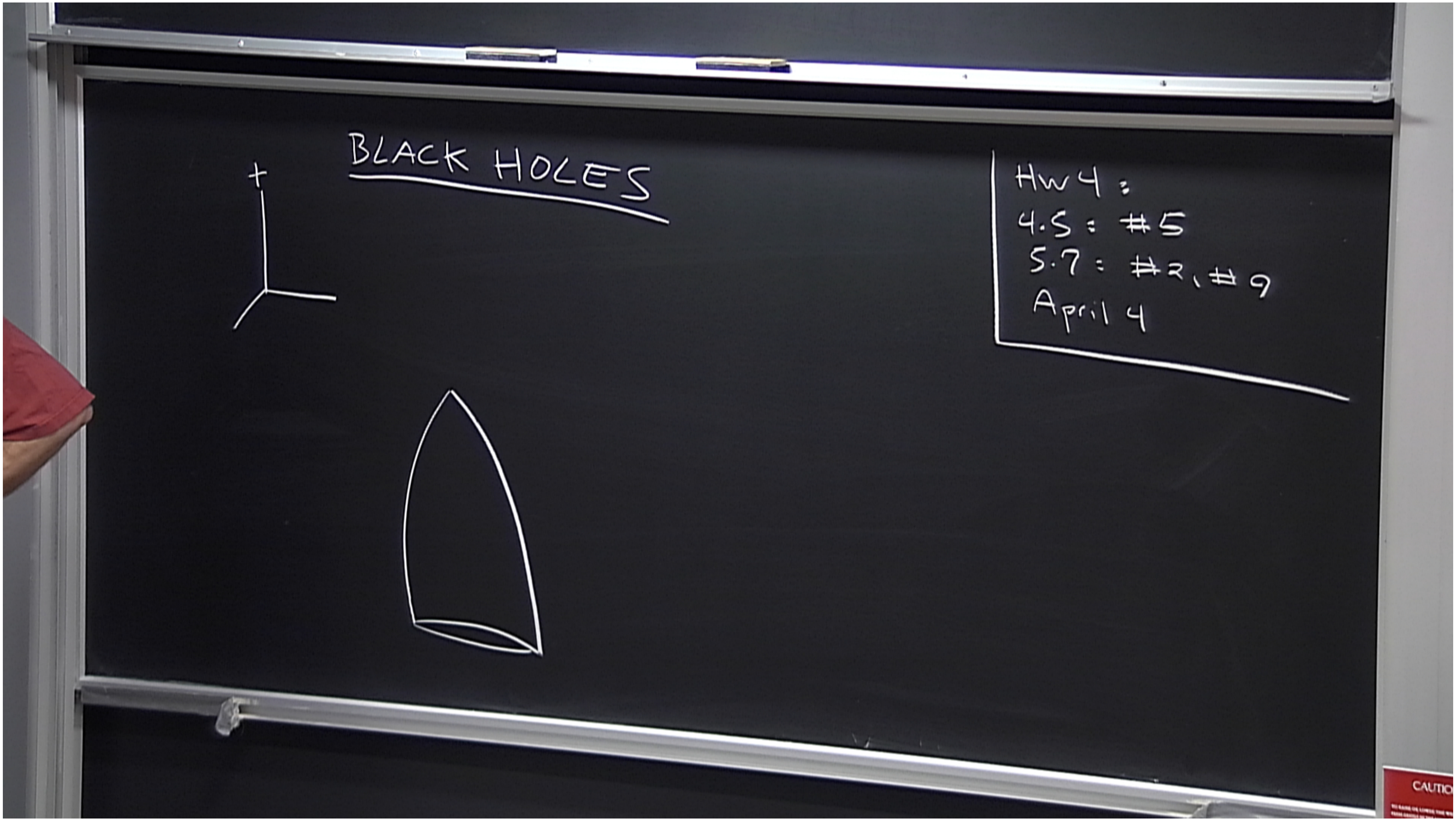

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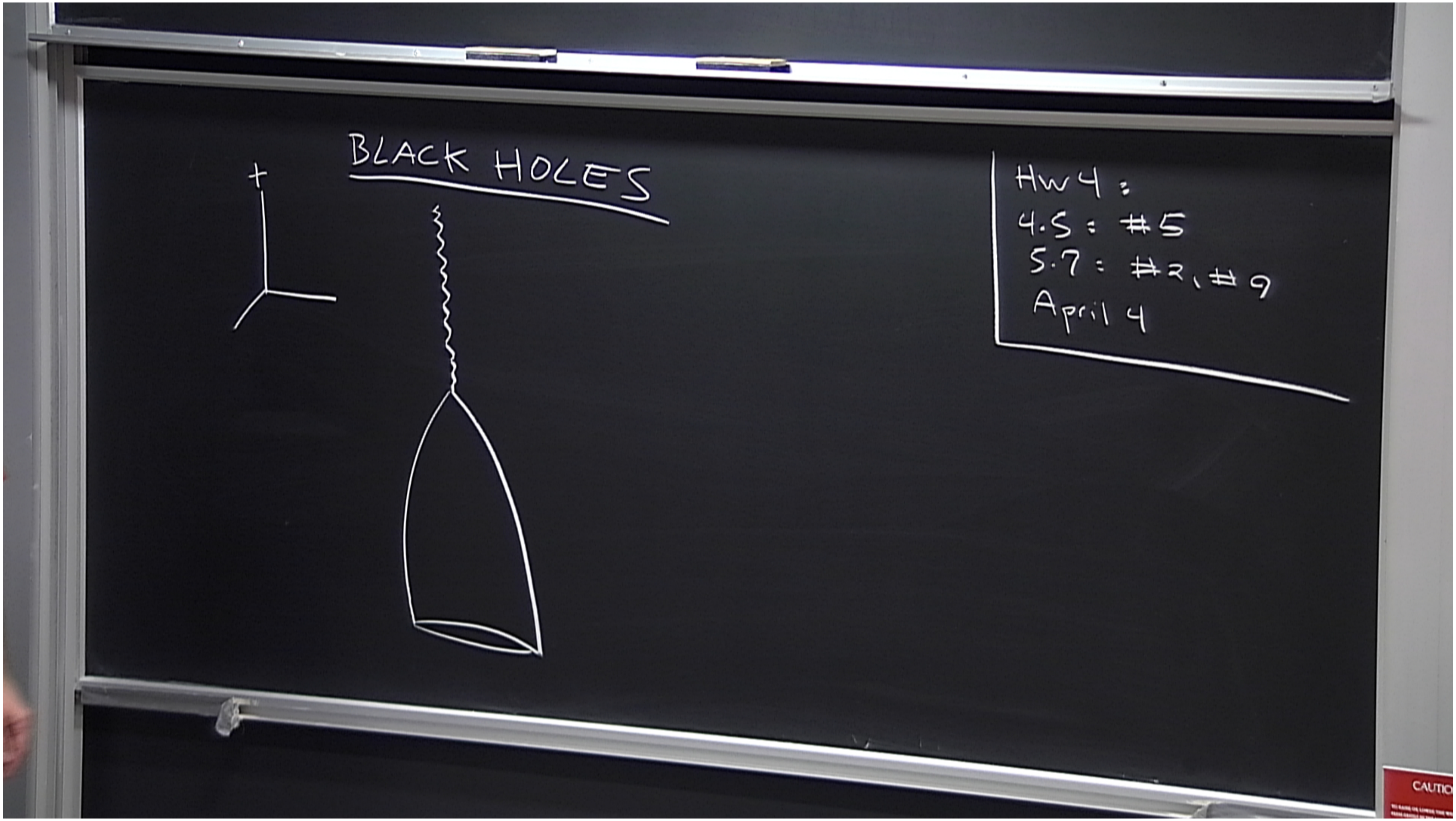
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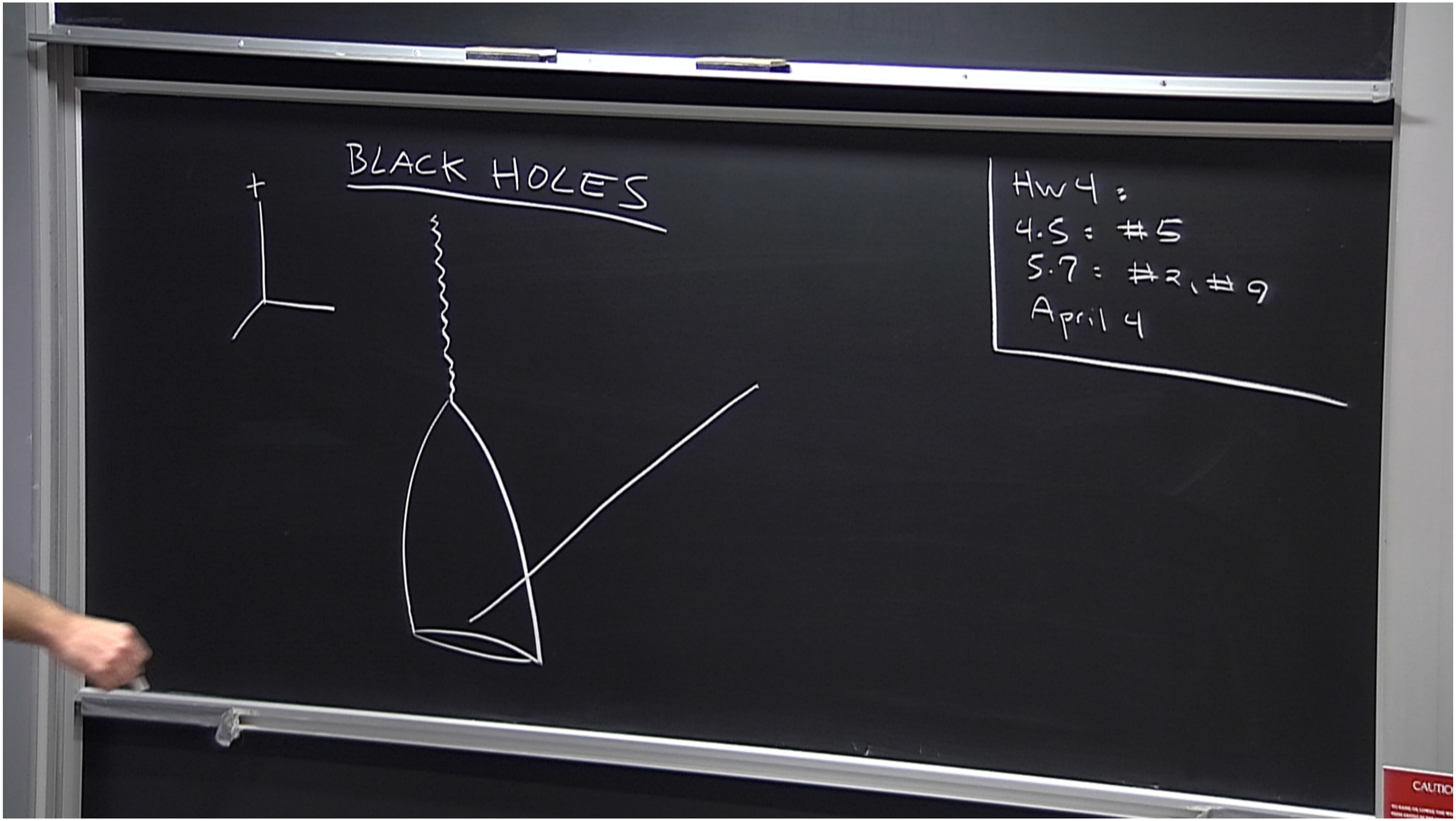
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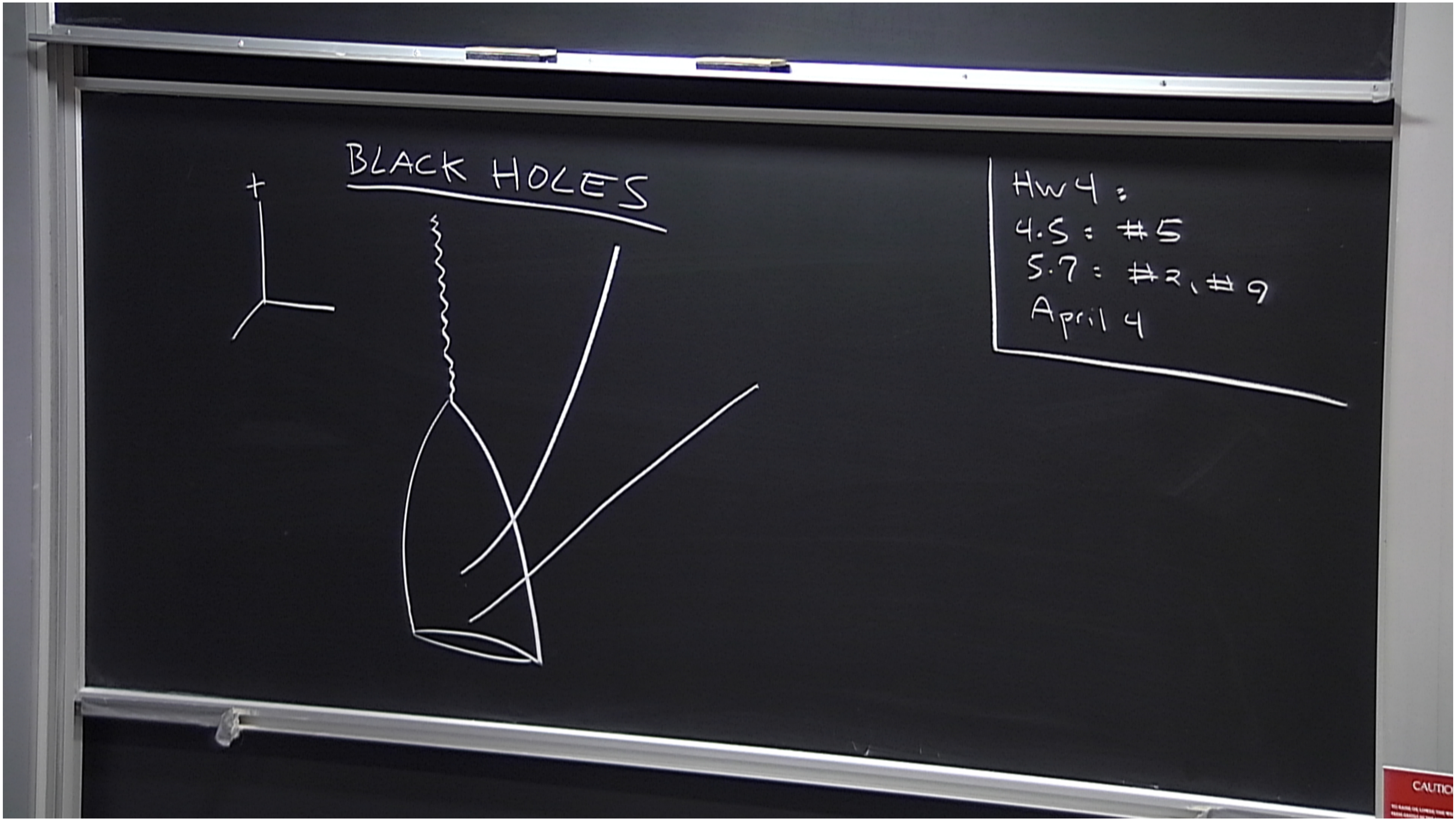
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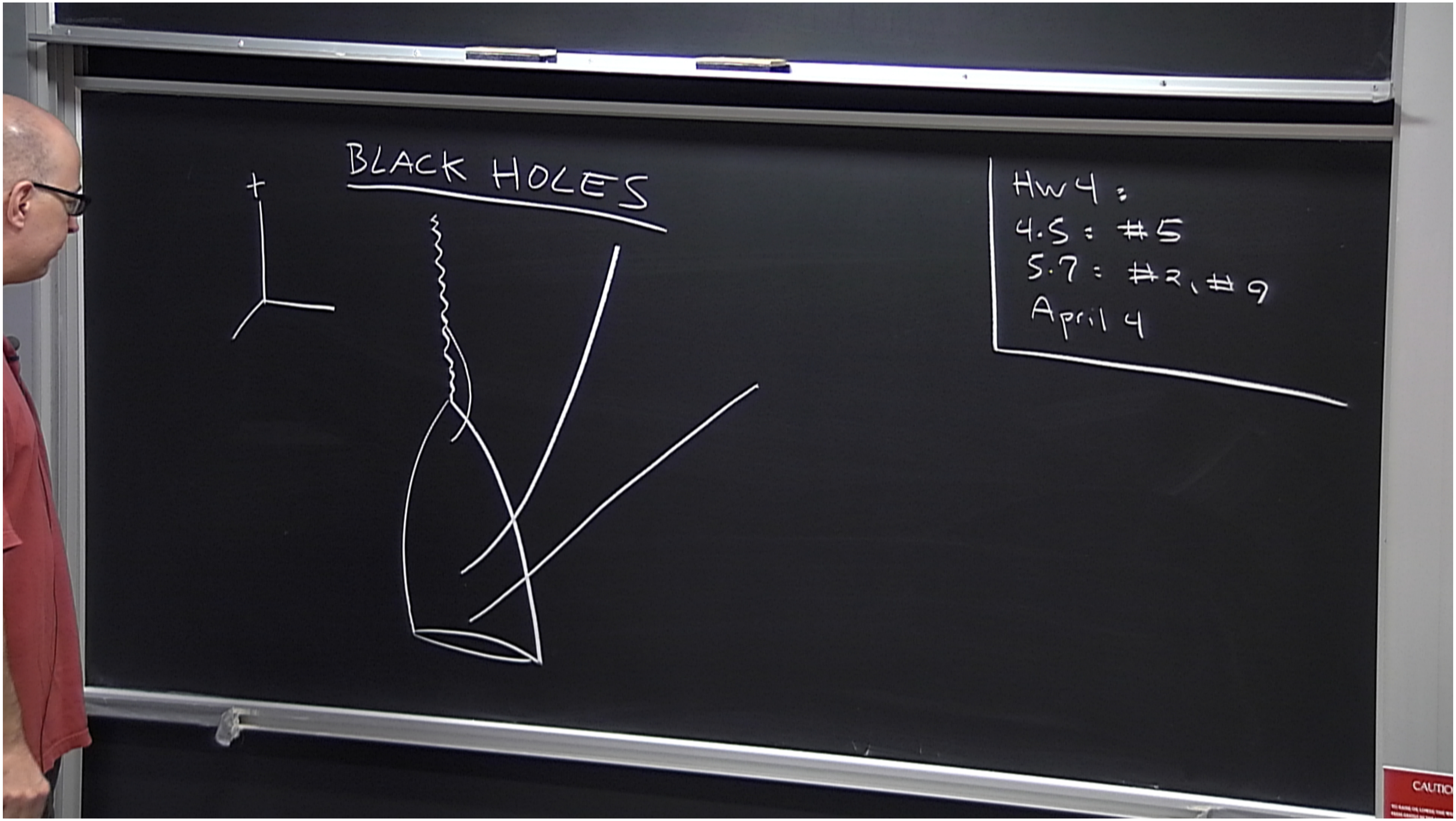
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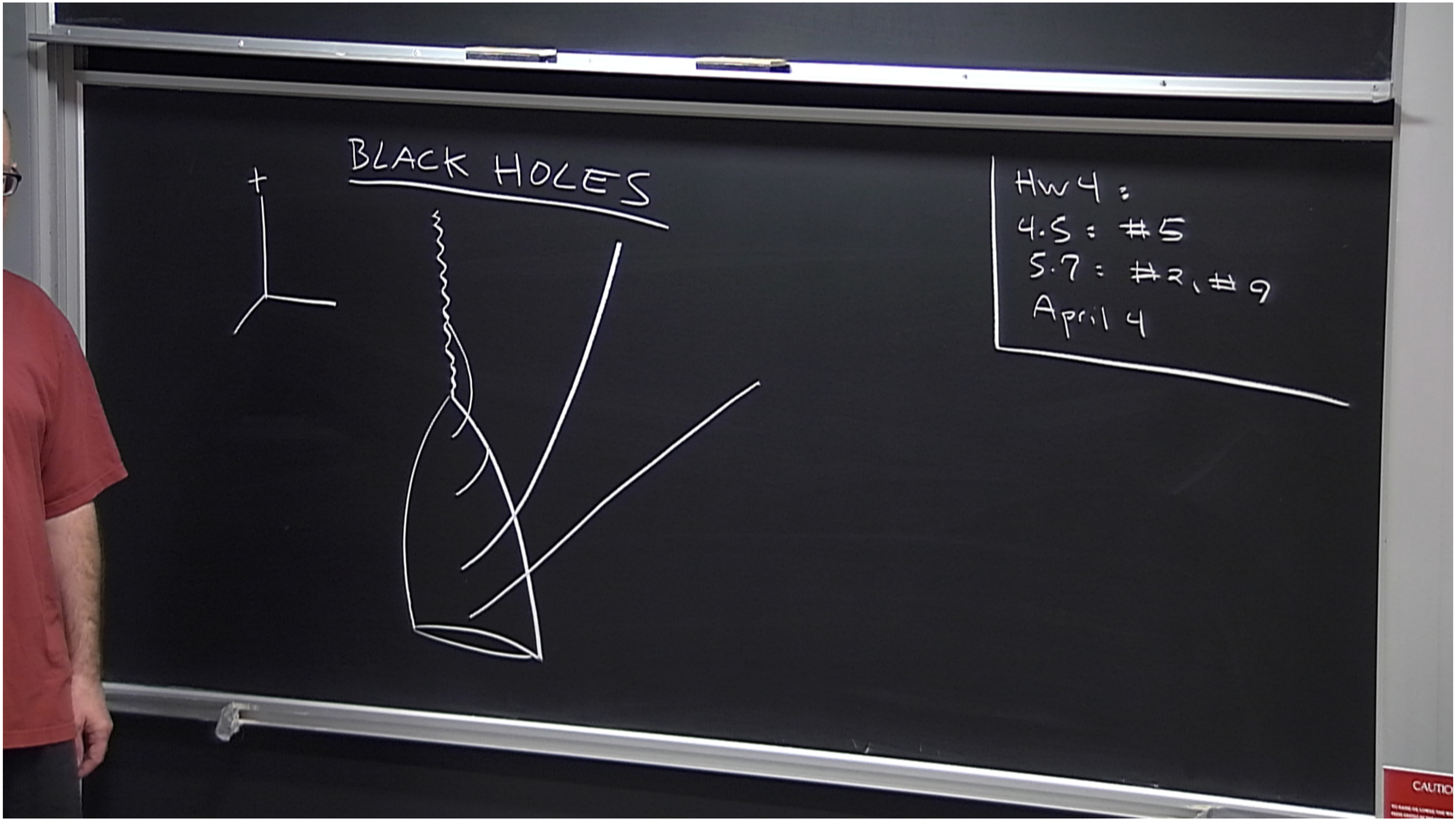
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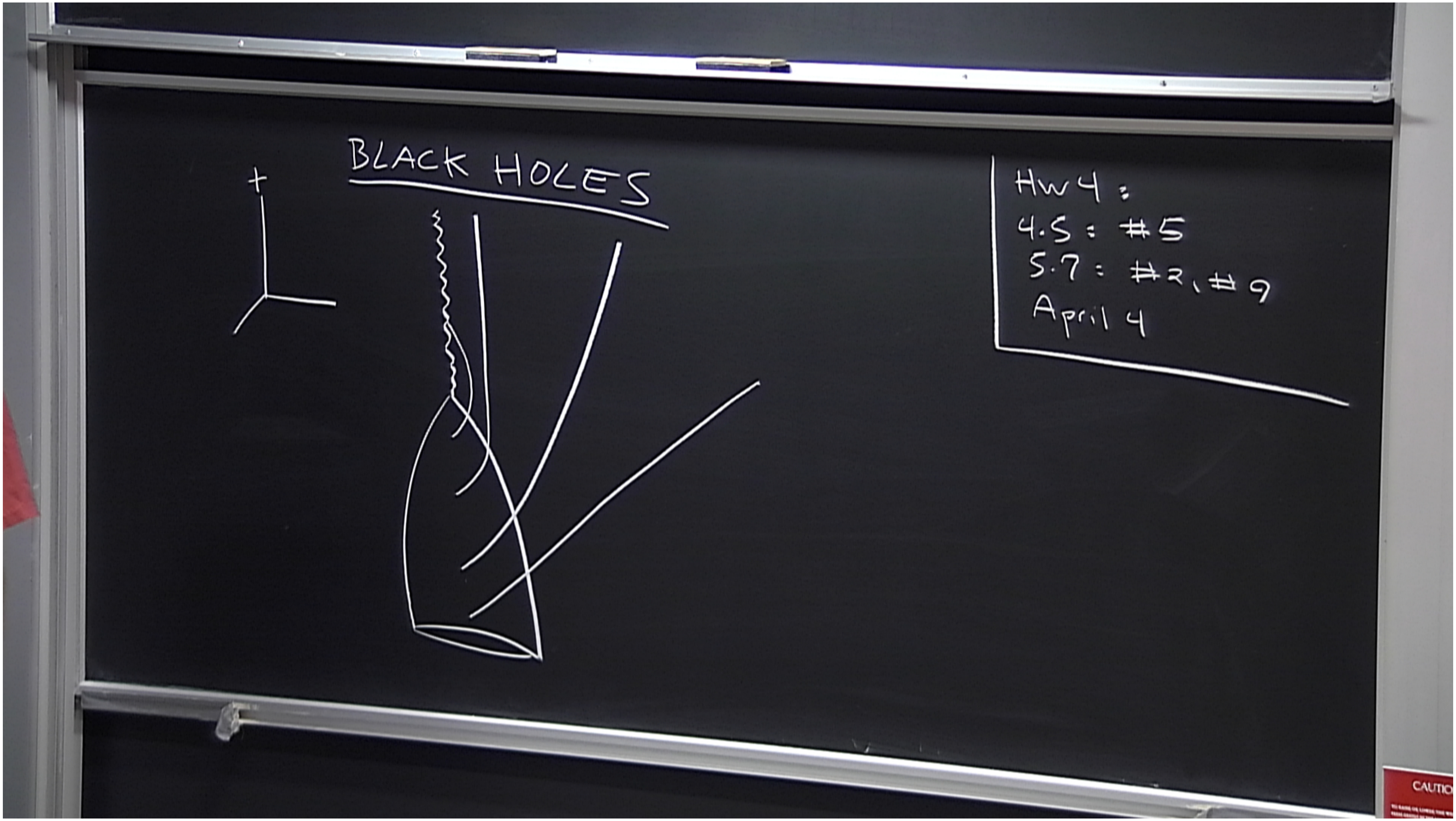
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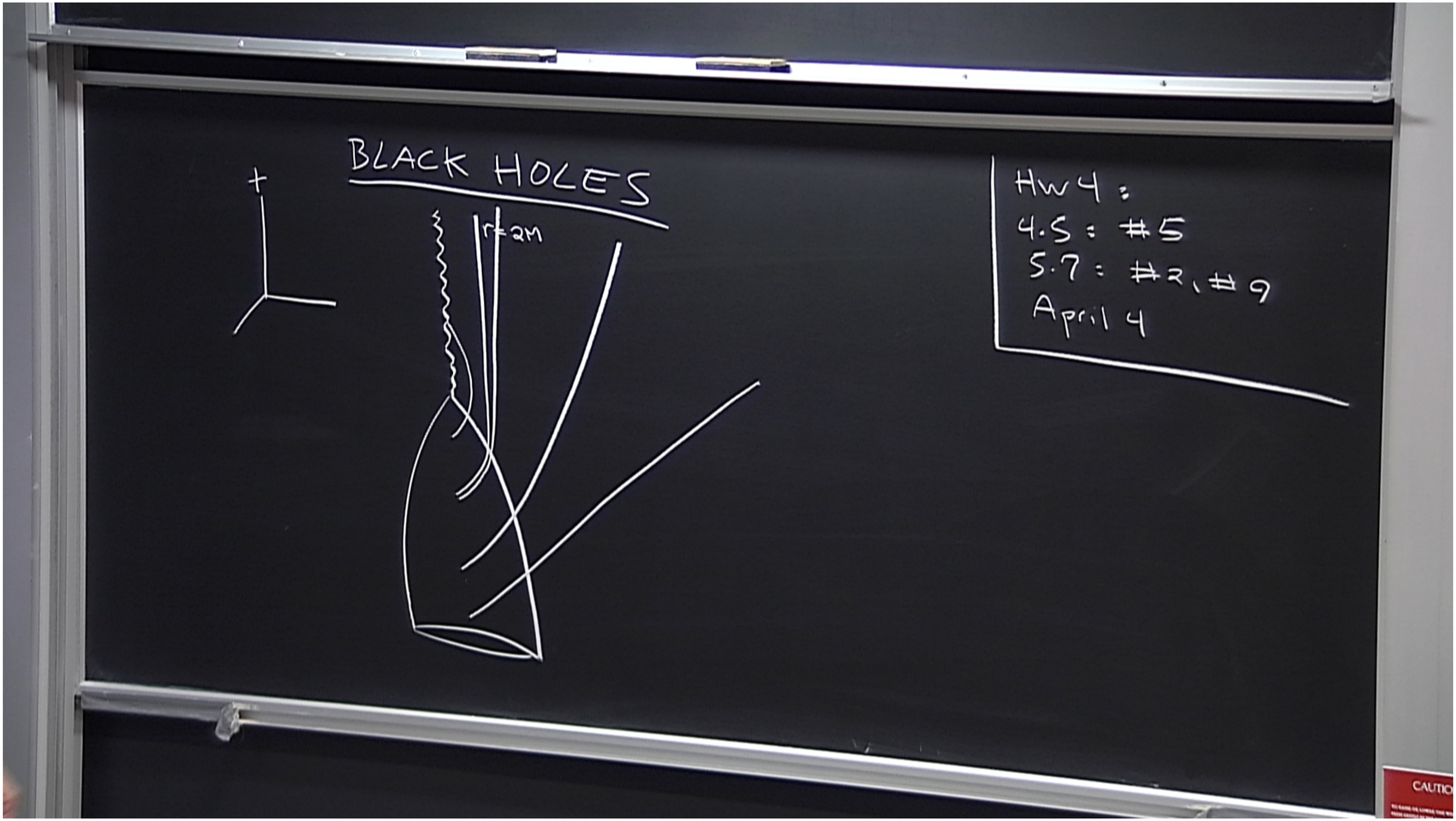
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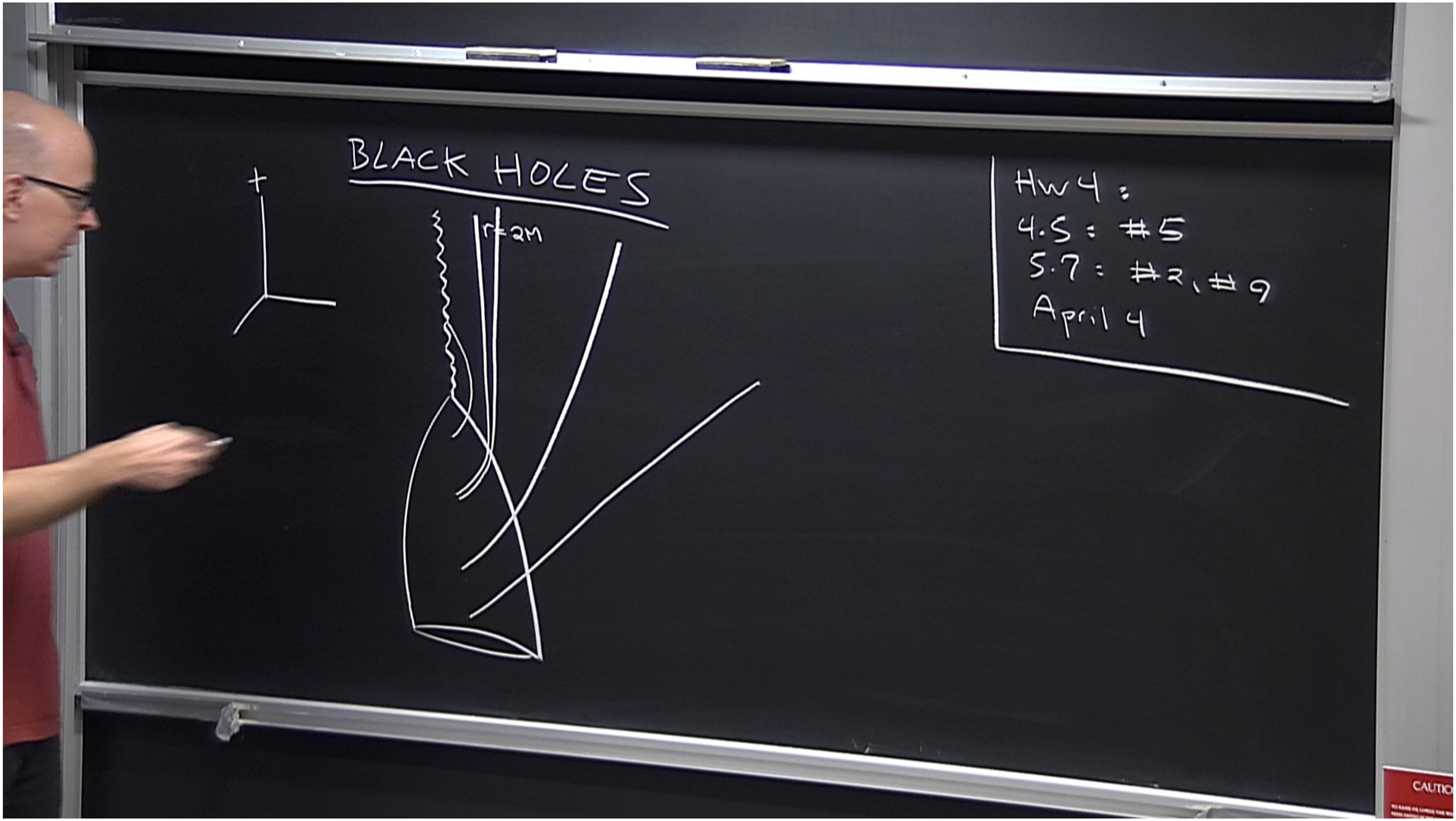
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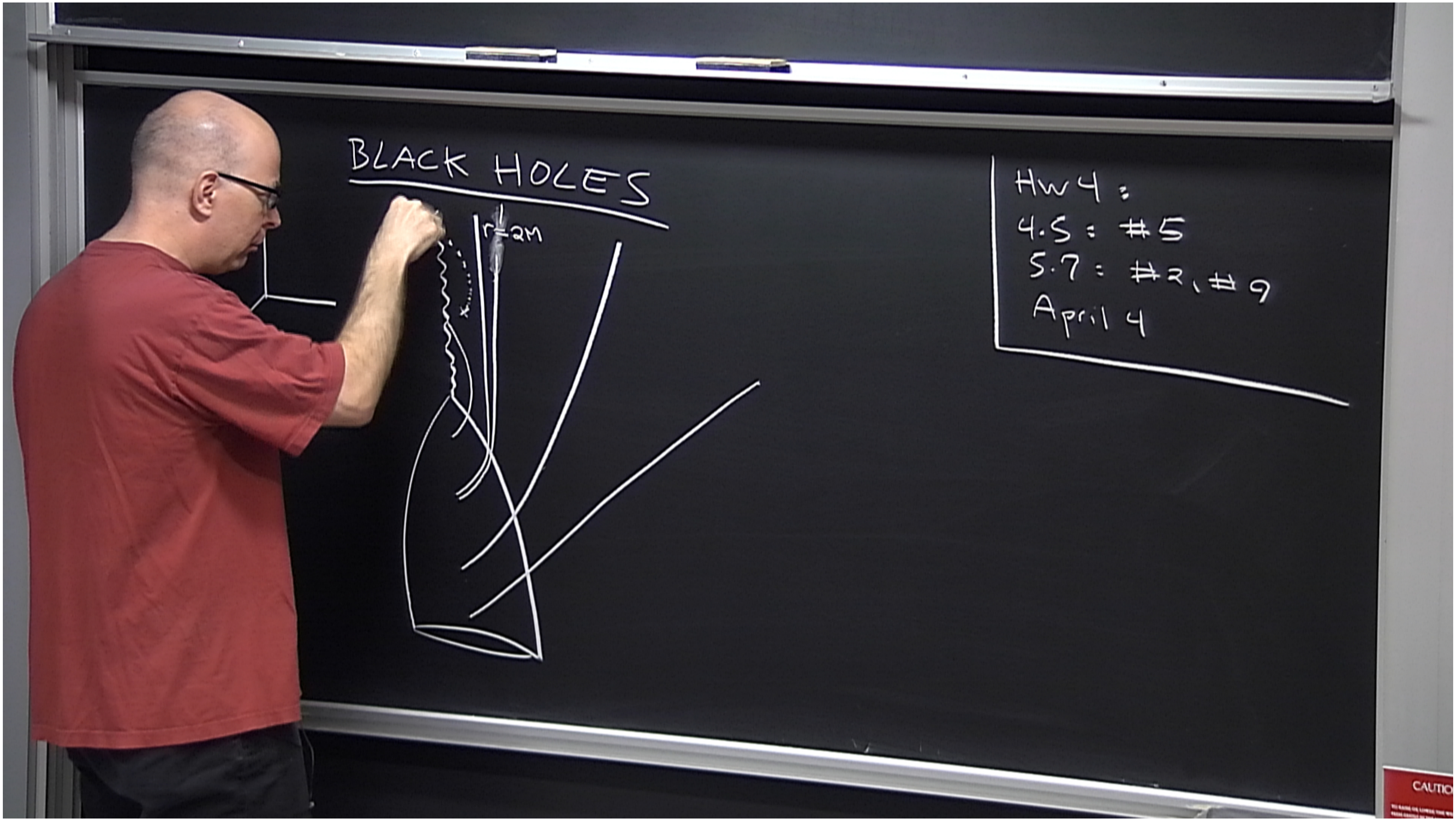
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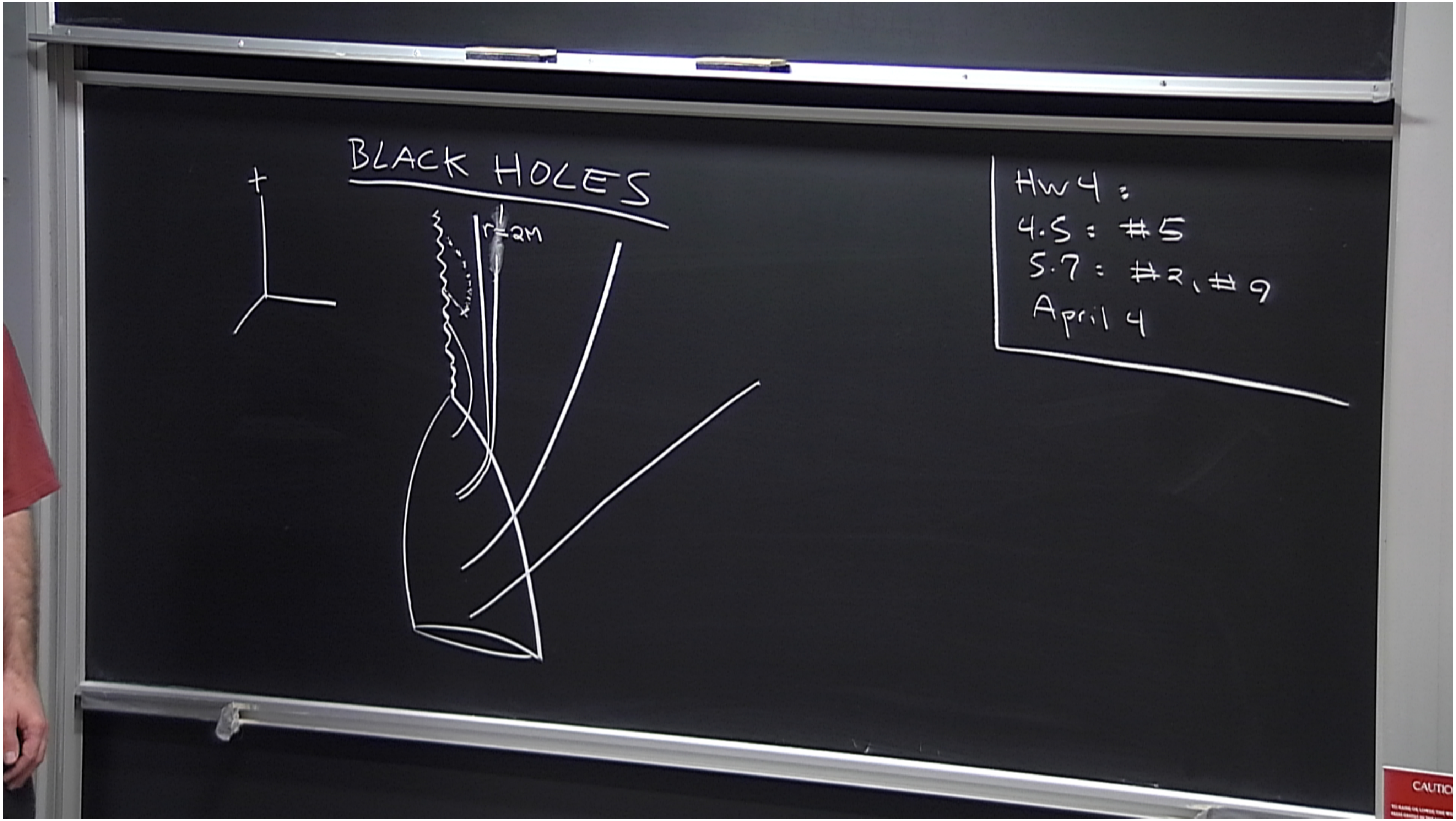
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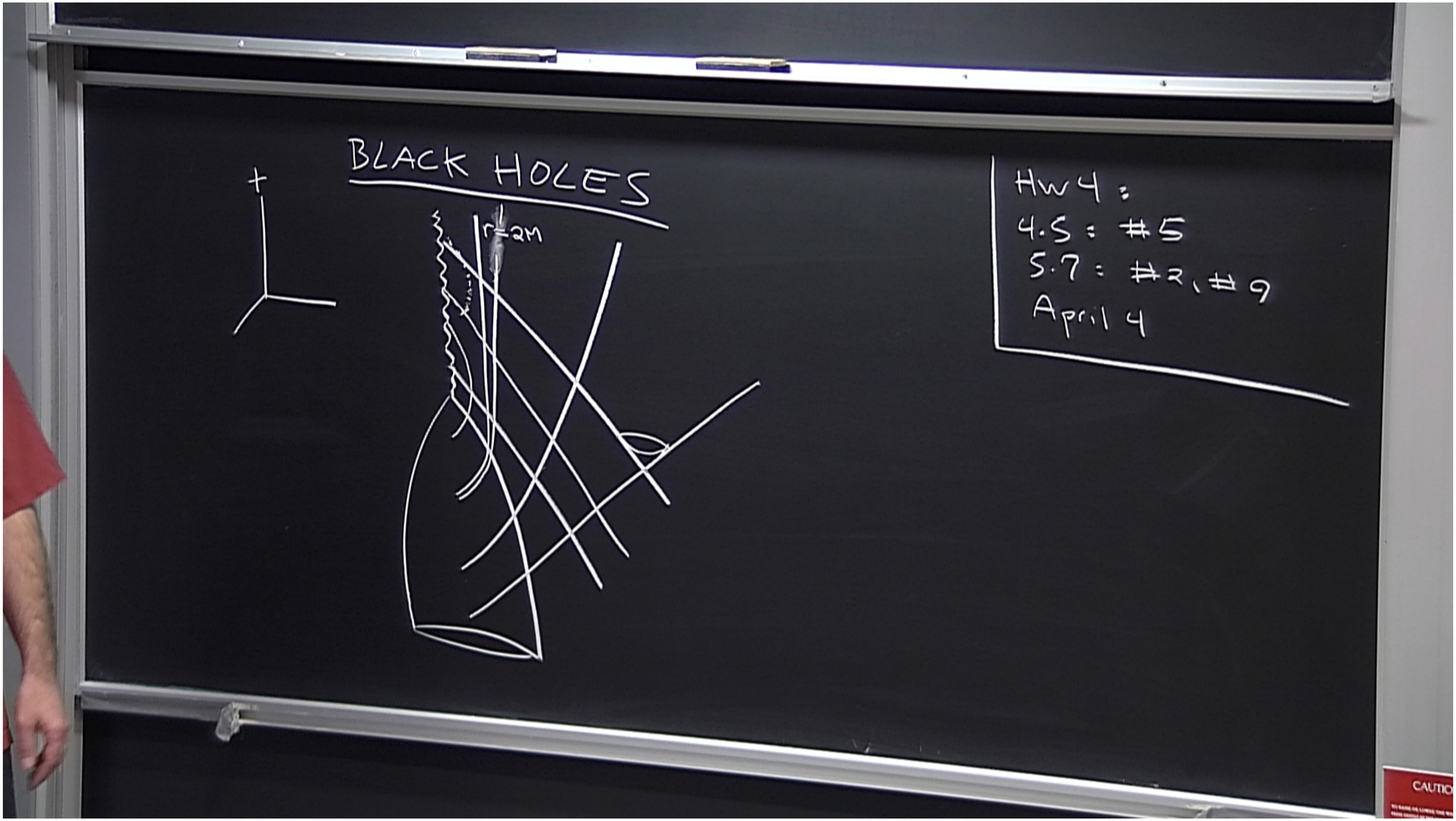
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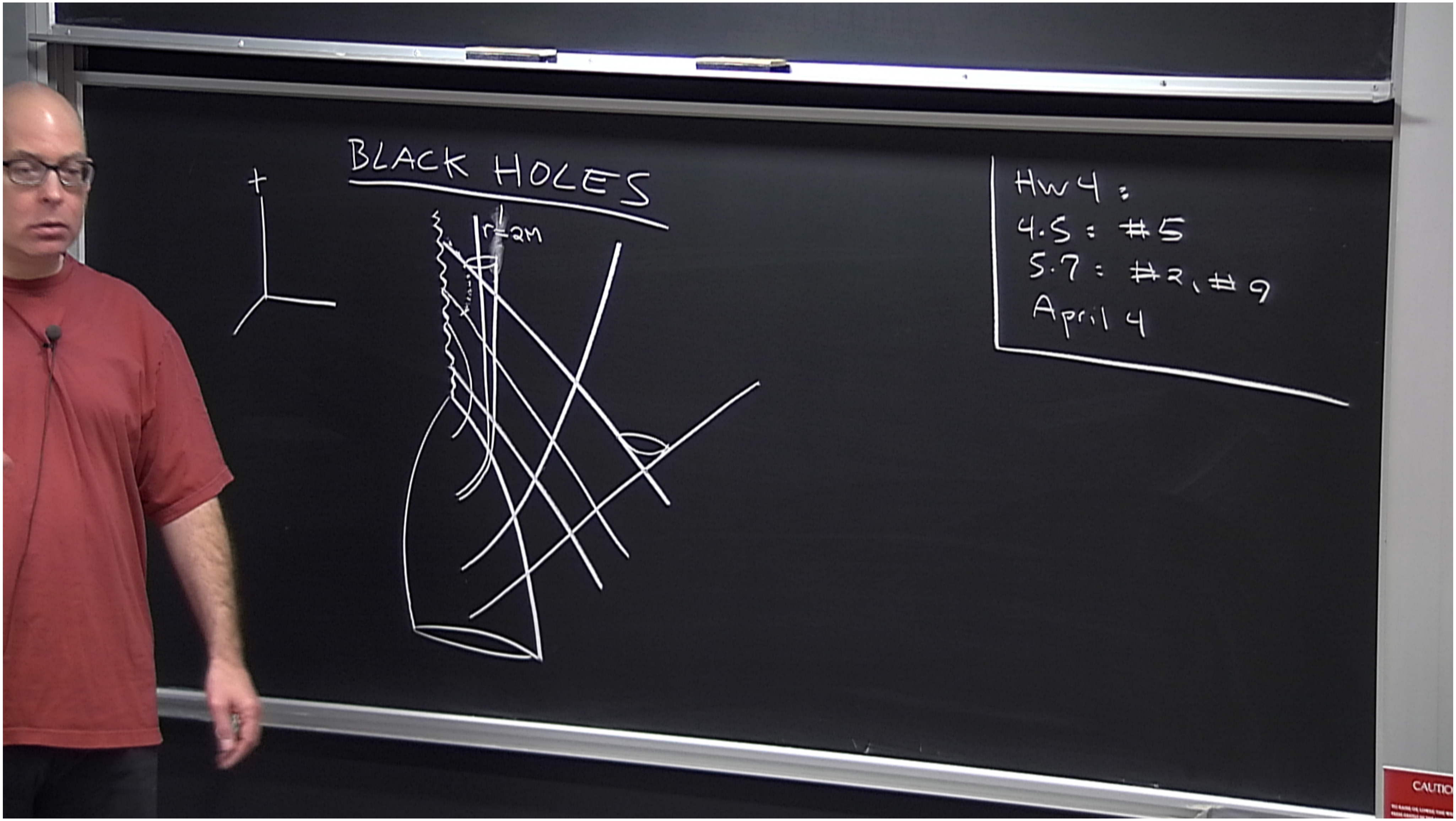
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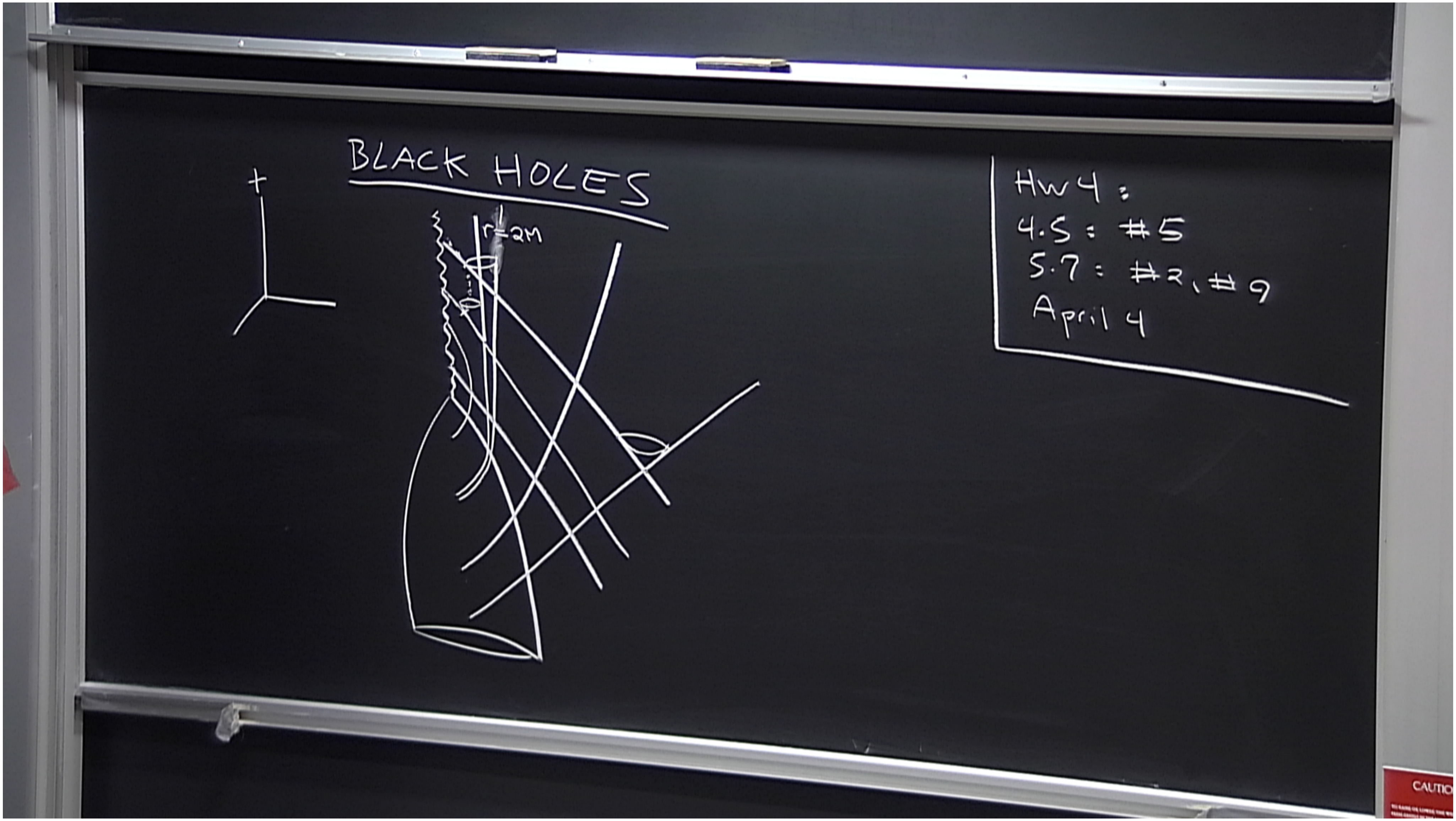
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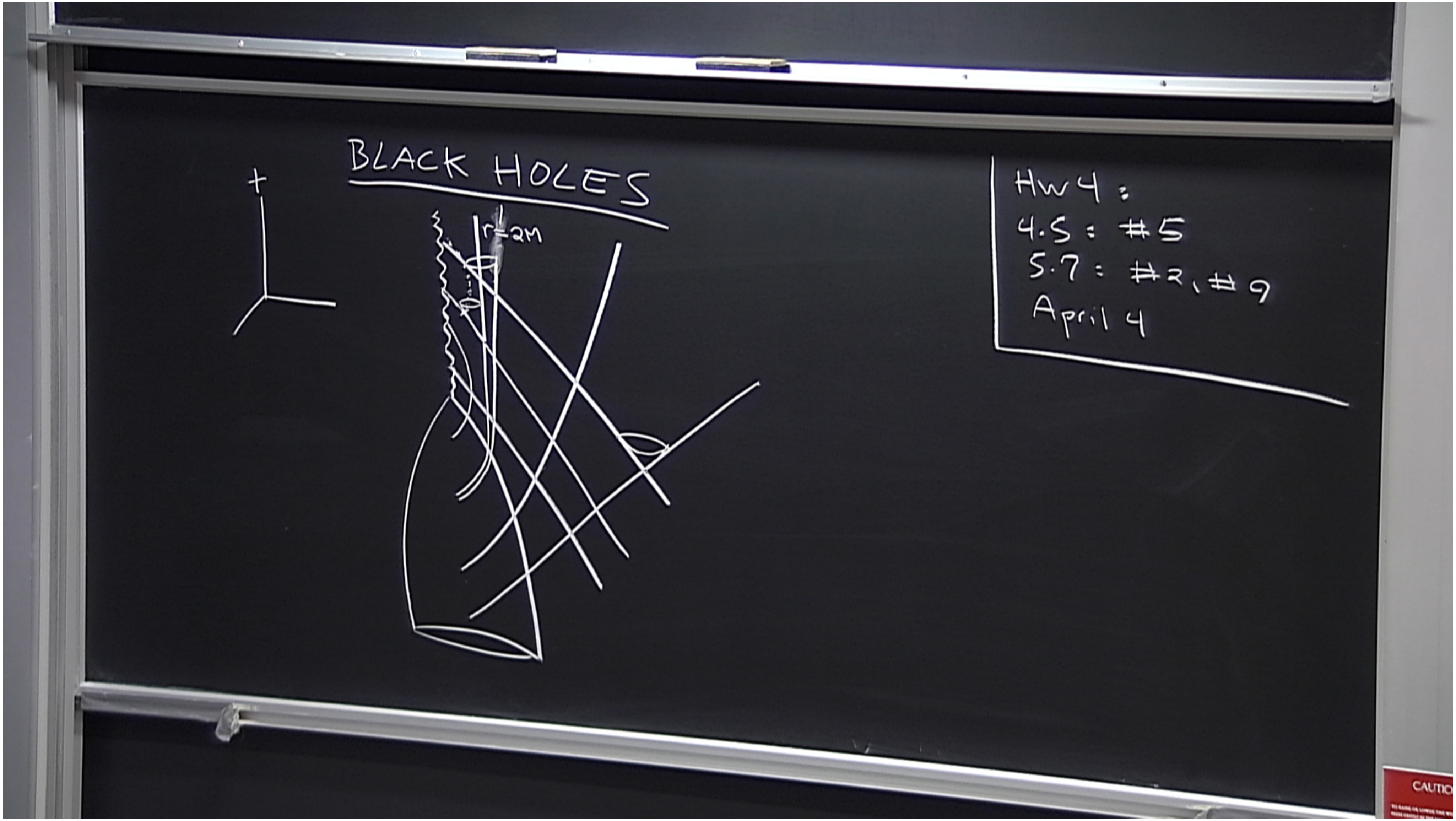
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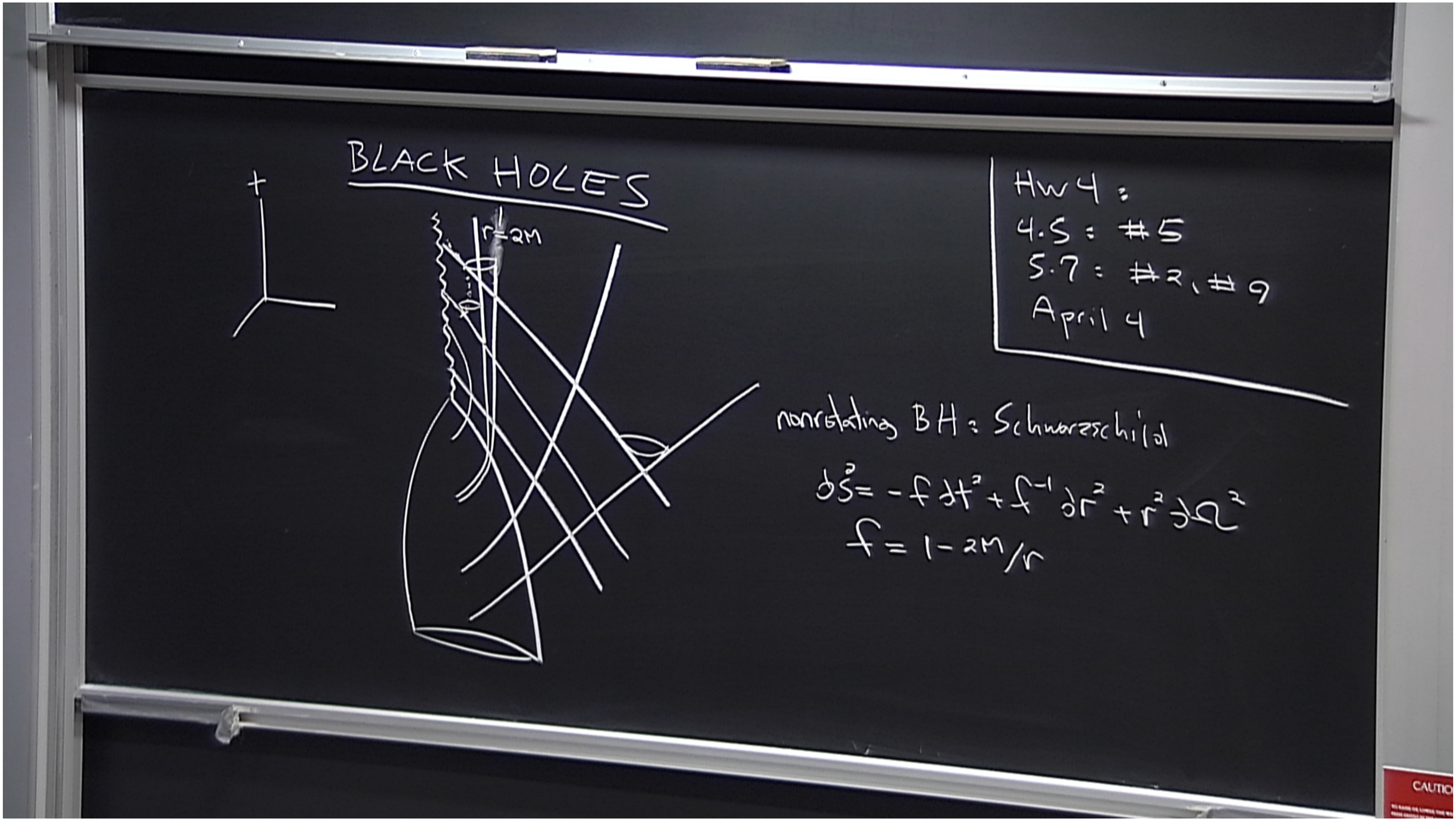
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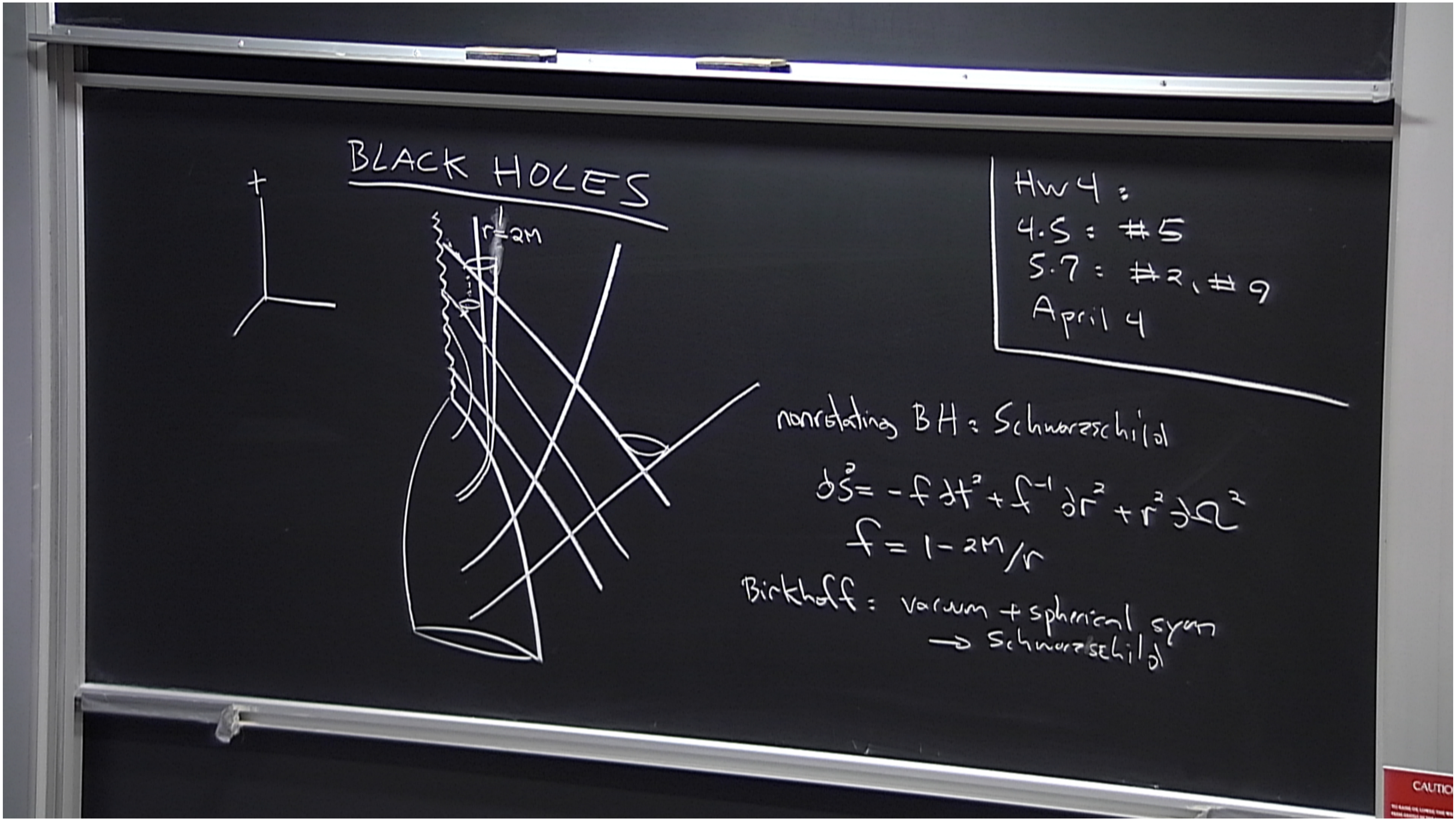
nonrotating BH = Schwarzschild
$ds^2 = -f\,dt^2 + f^{-1}dr^2 + r^2 d\Omega^2$
$f = 1 - 2M/r$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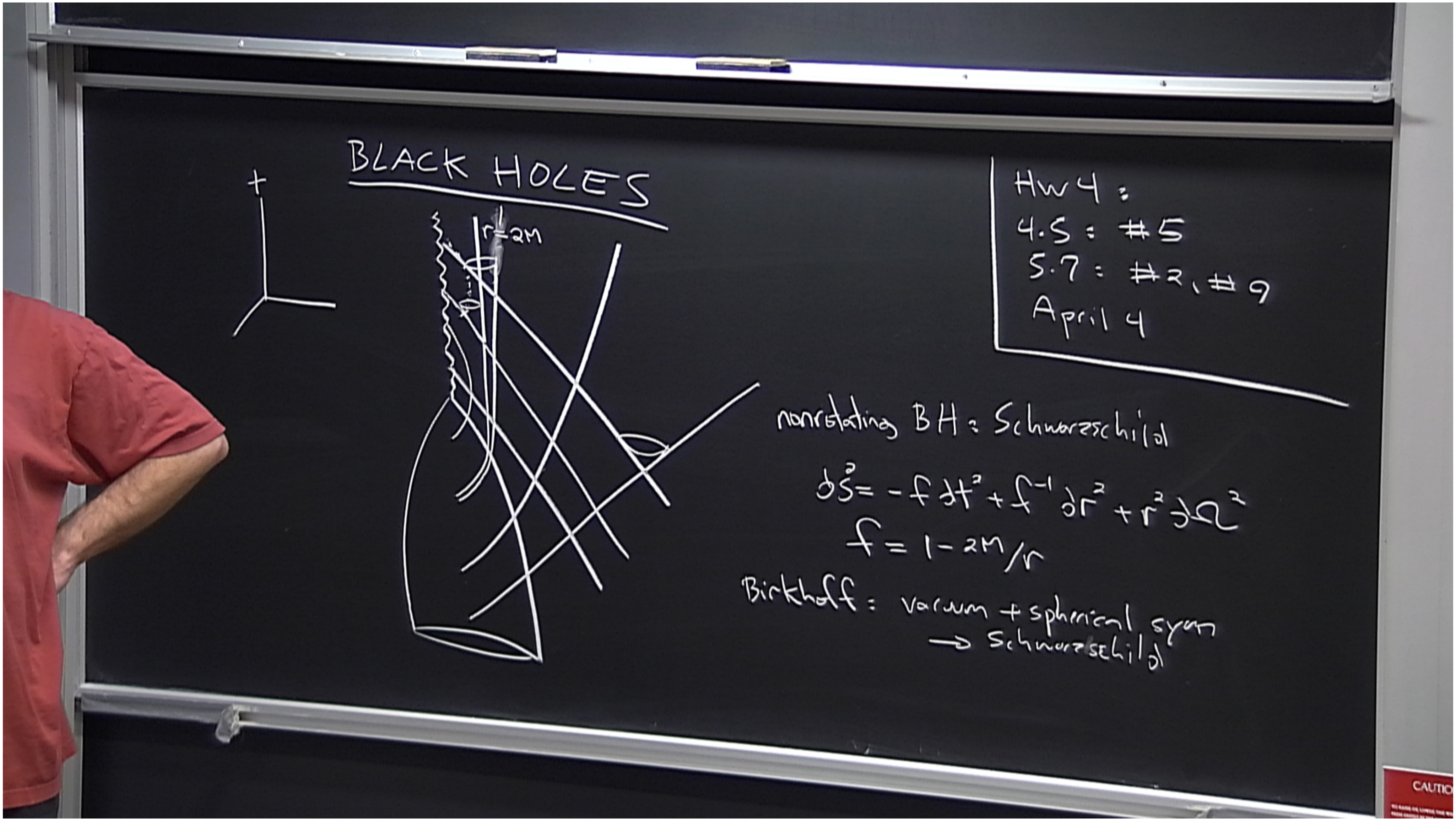
Birkhoff: vacuum + spherical sym
→ Schwarzschild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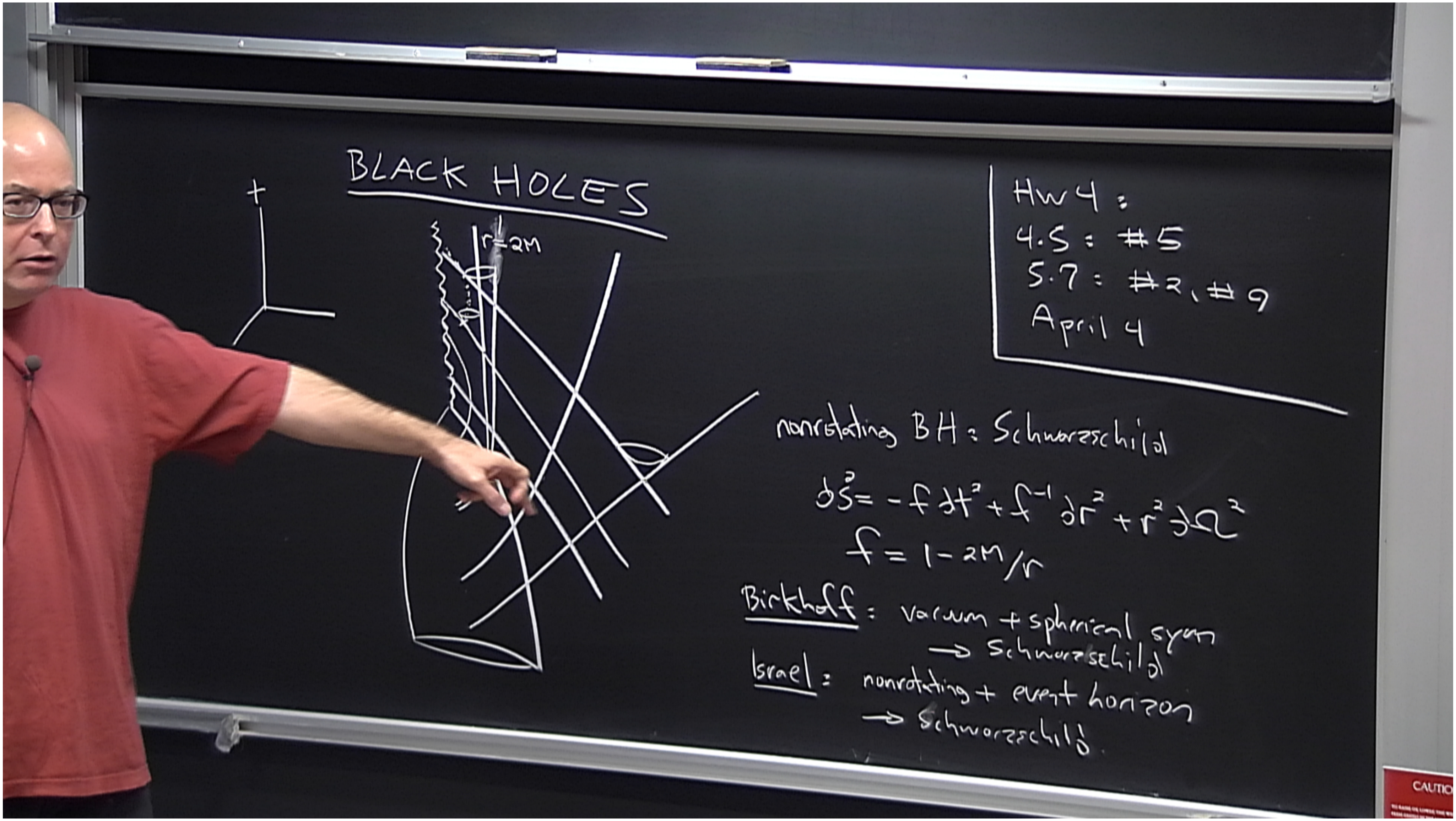
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Israel: nonrotating + event horizon
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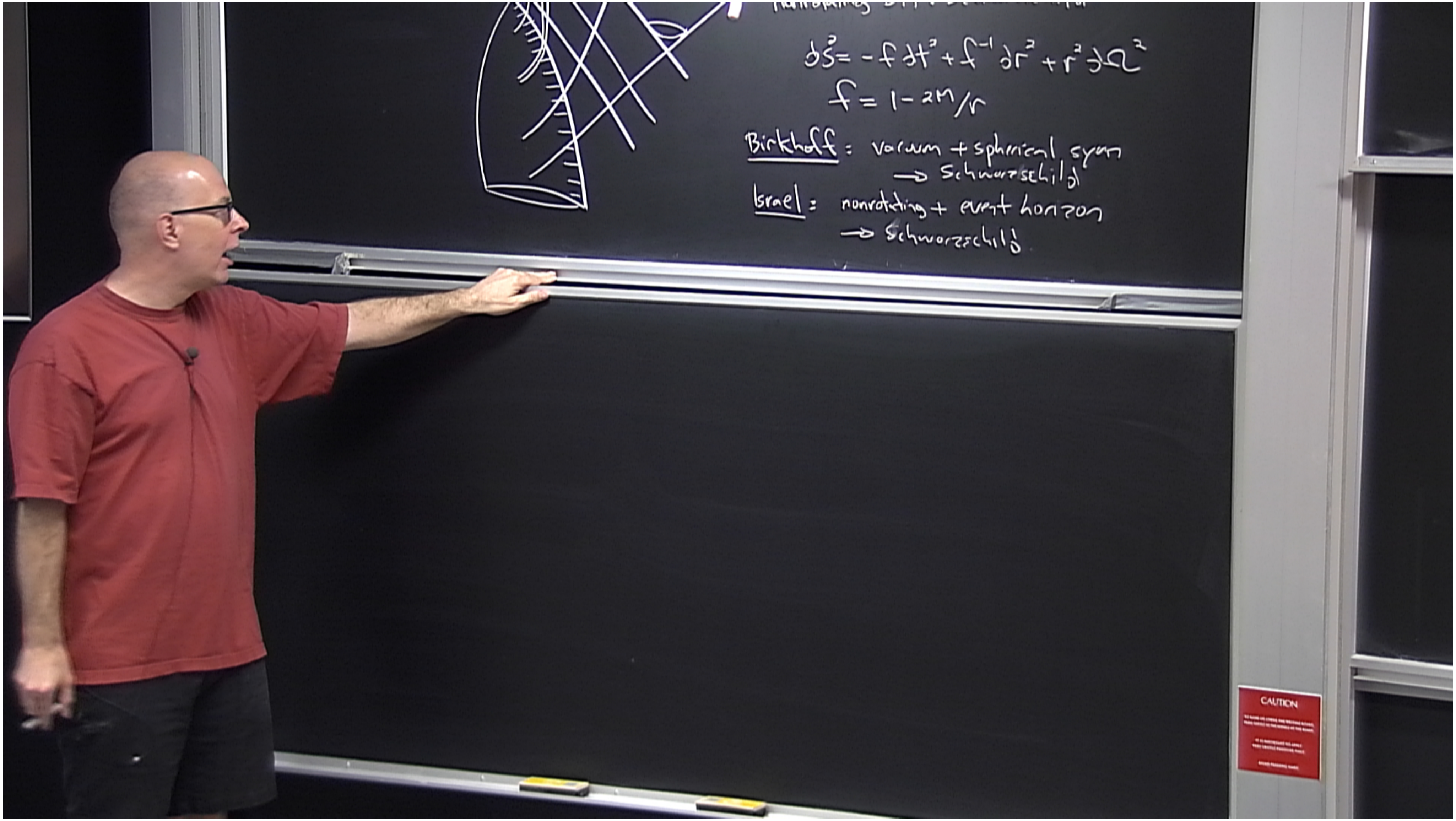

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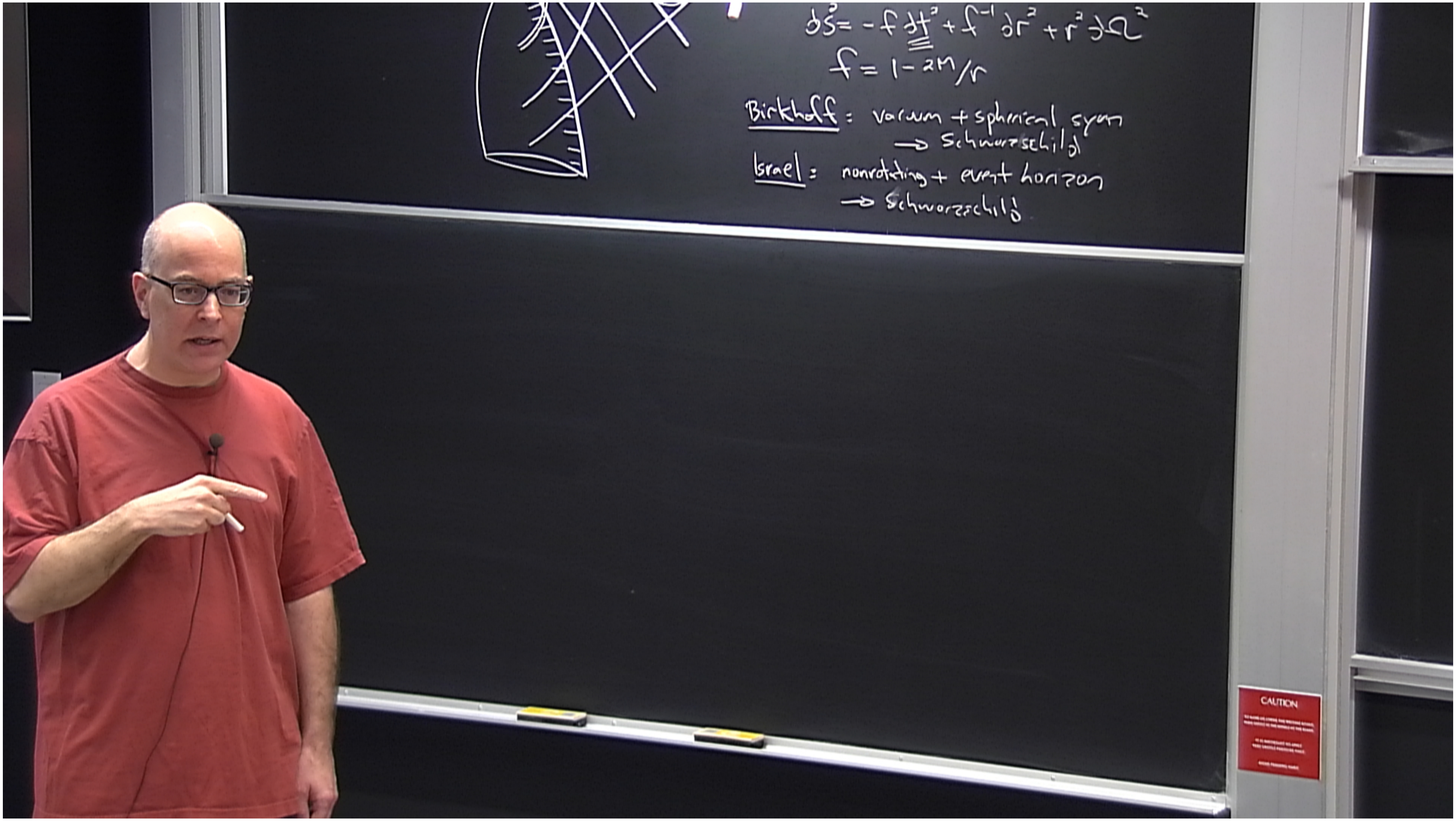
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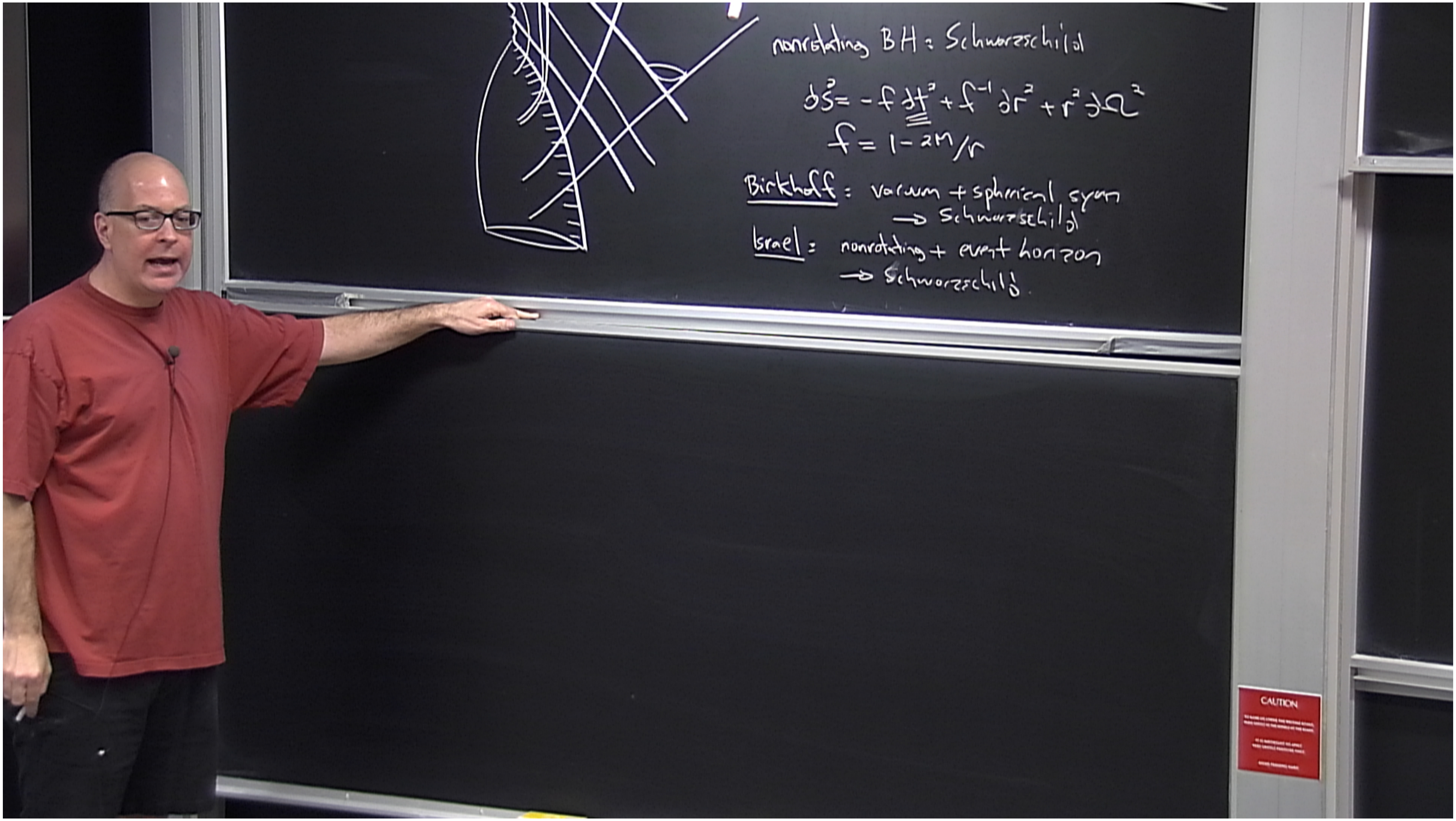
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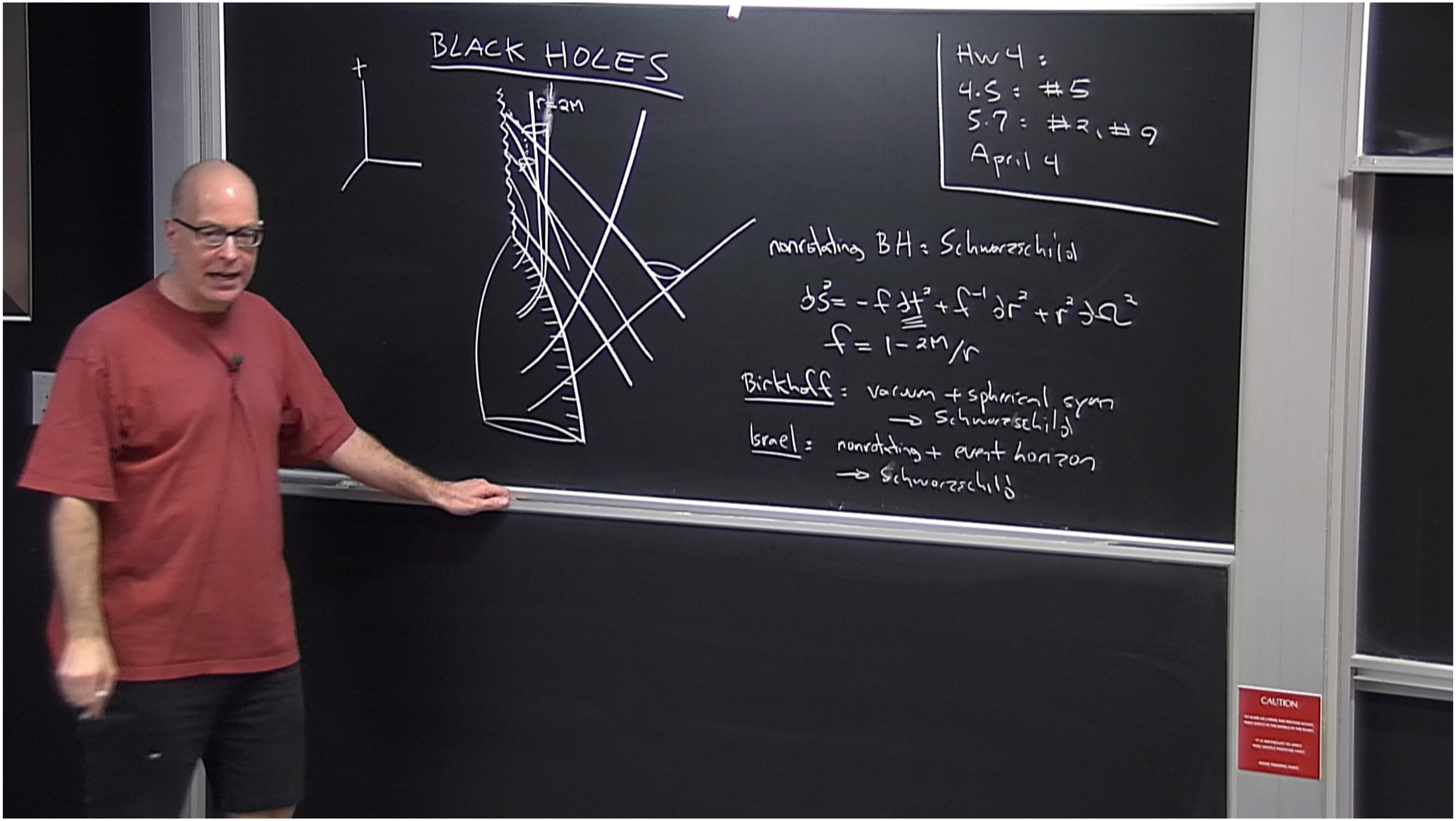
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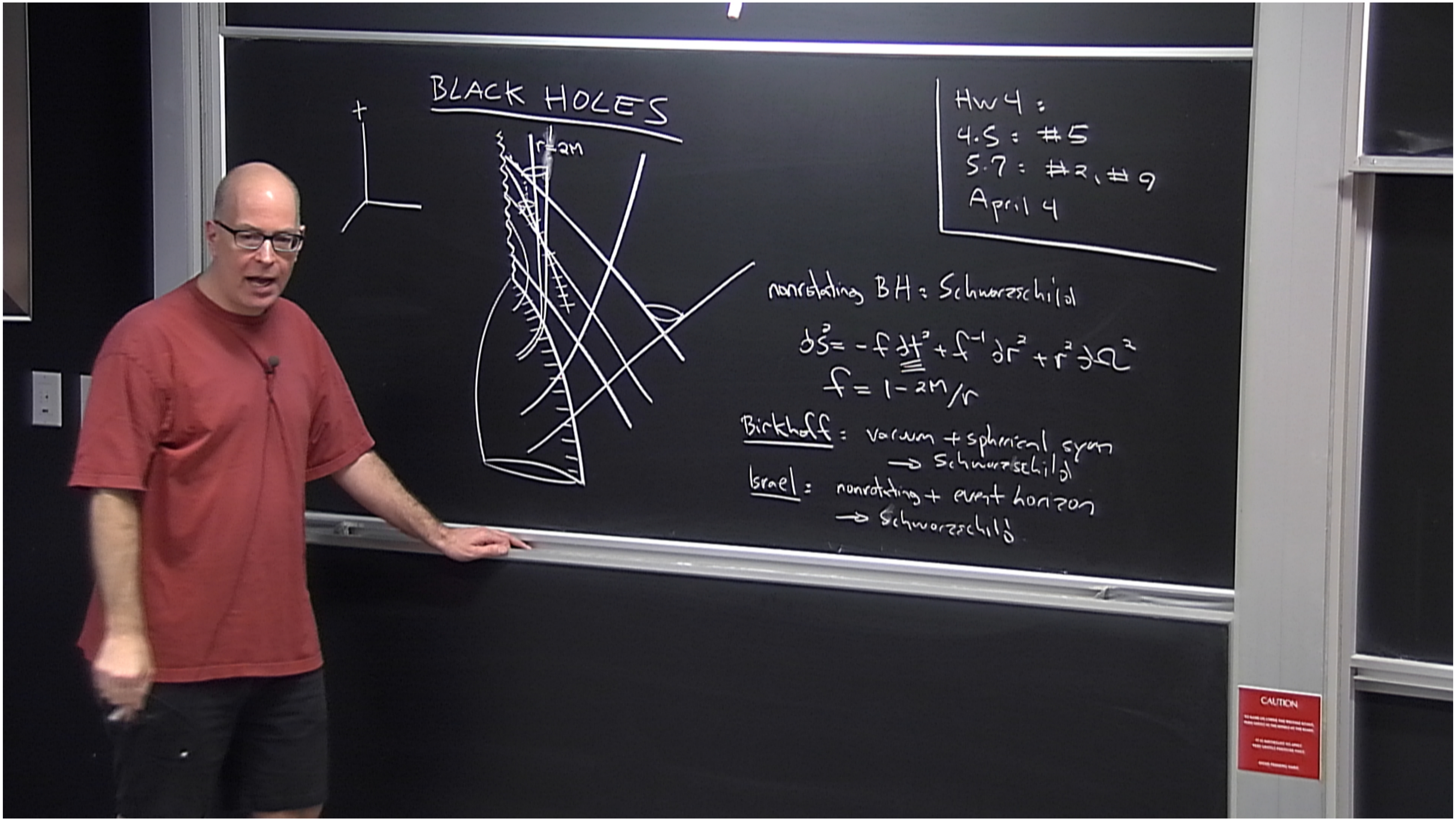
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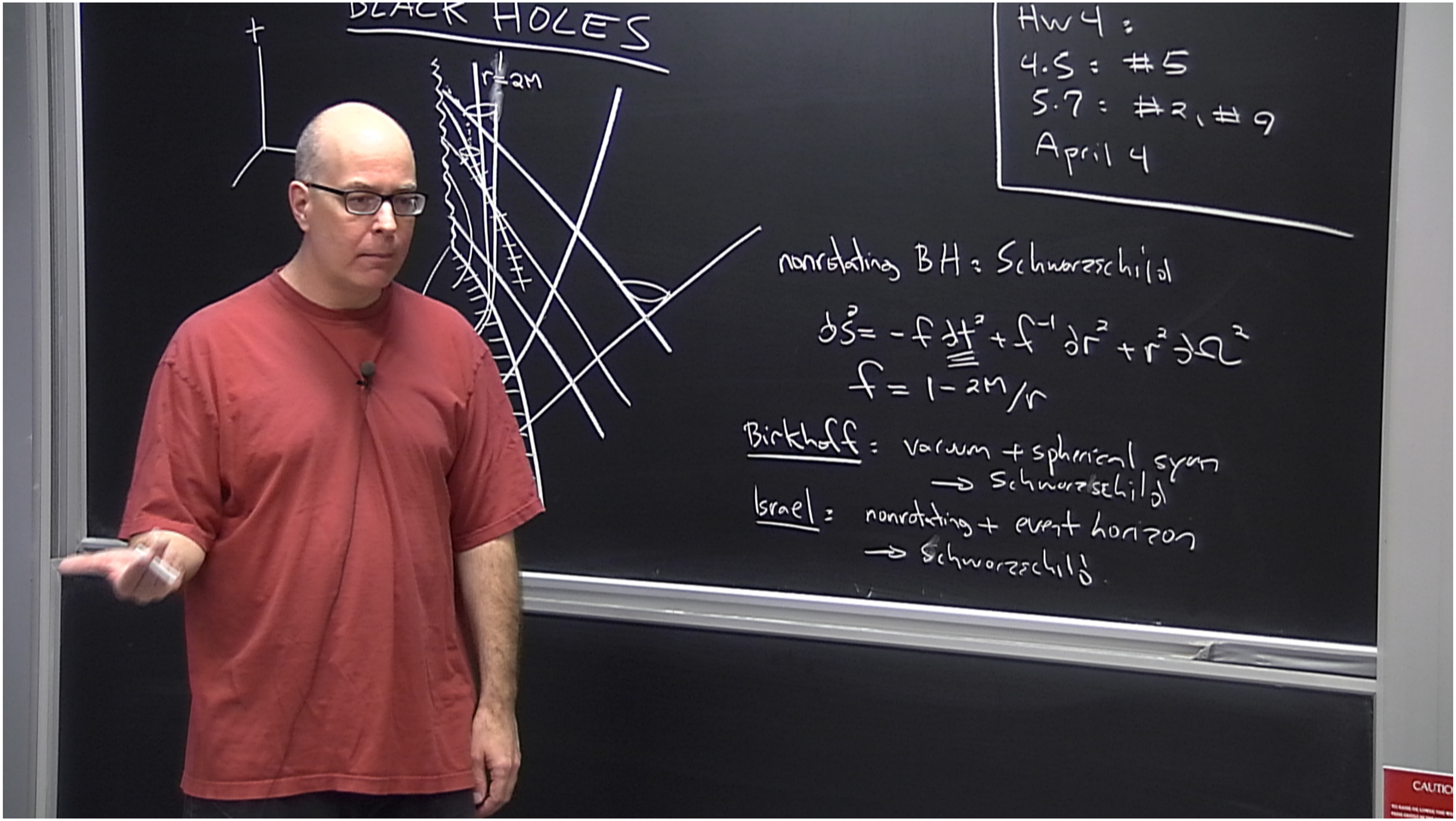
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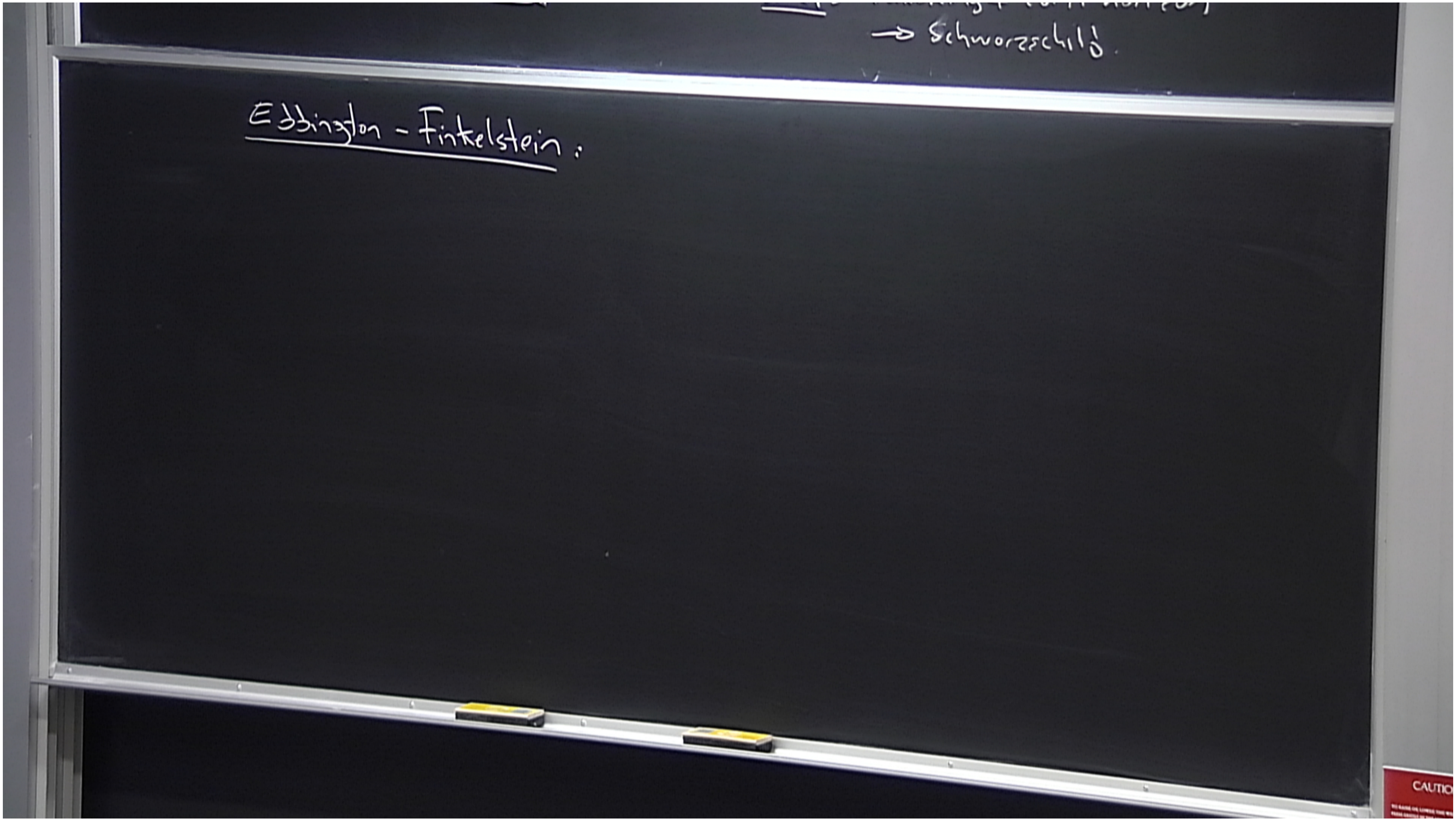
→ Schwarzschild.
Eddington - Finkelstein :

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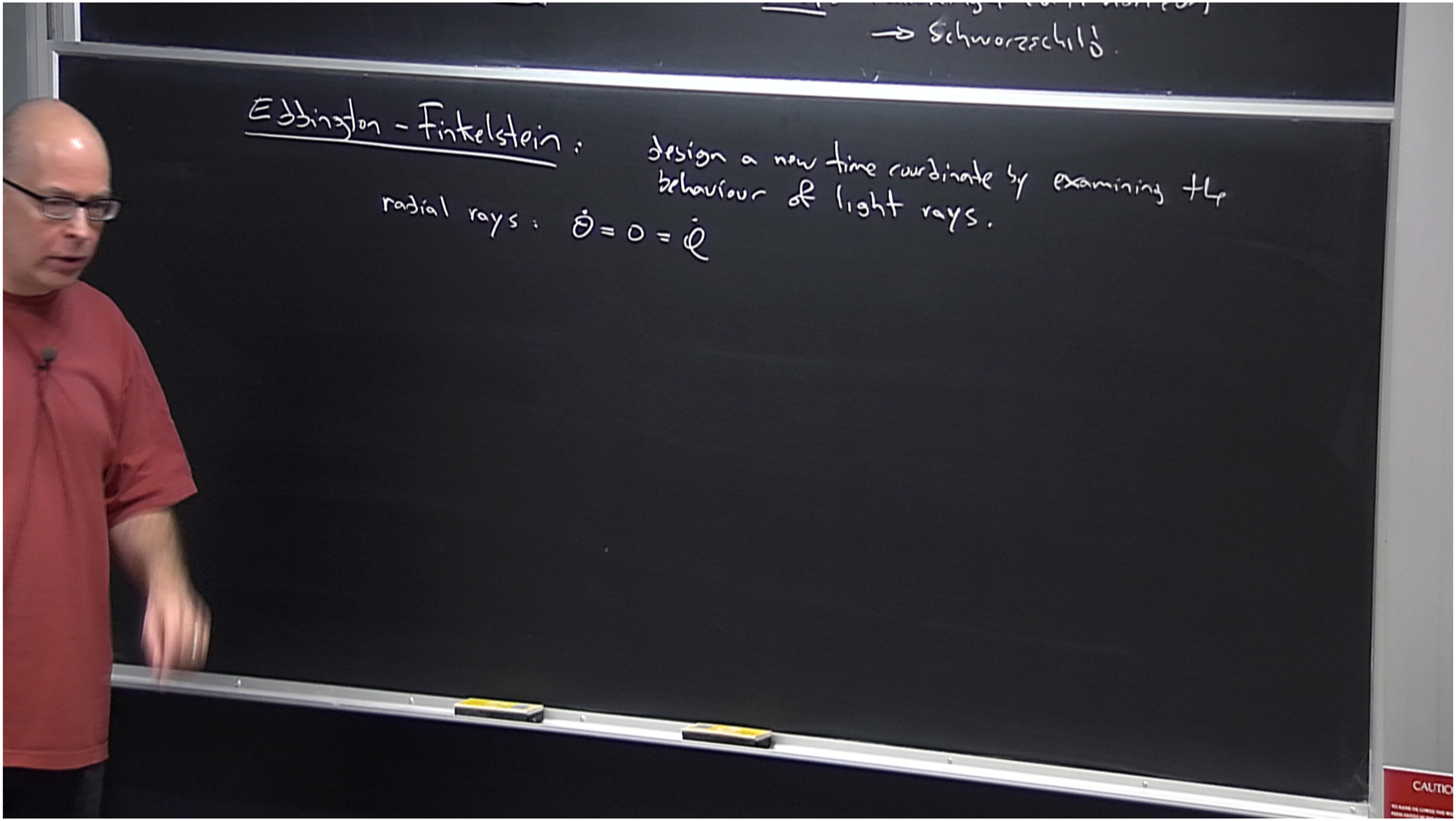
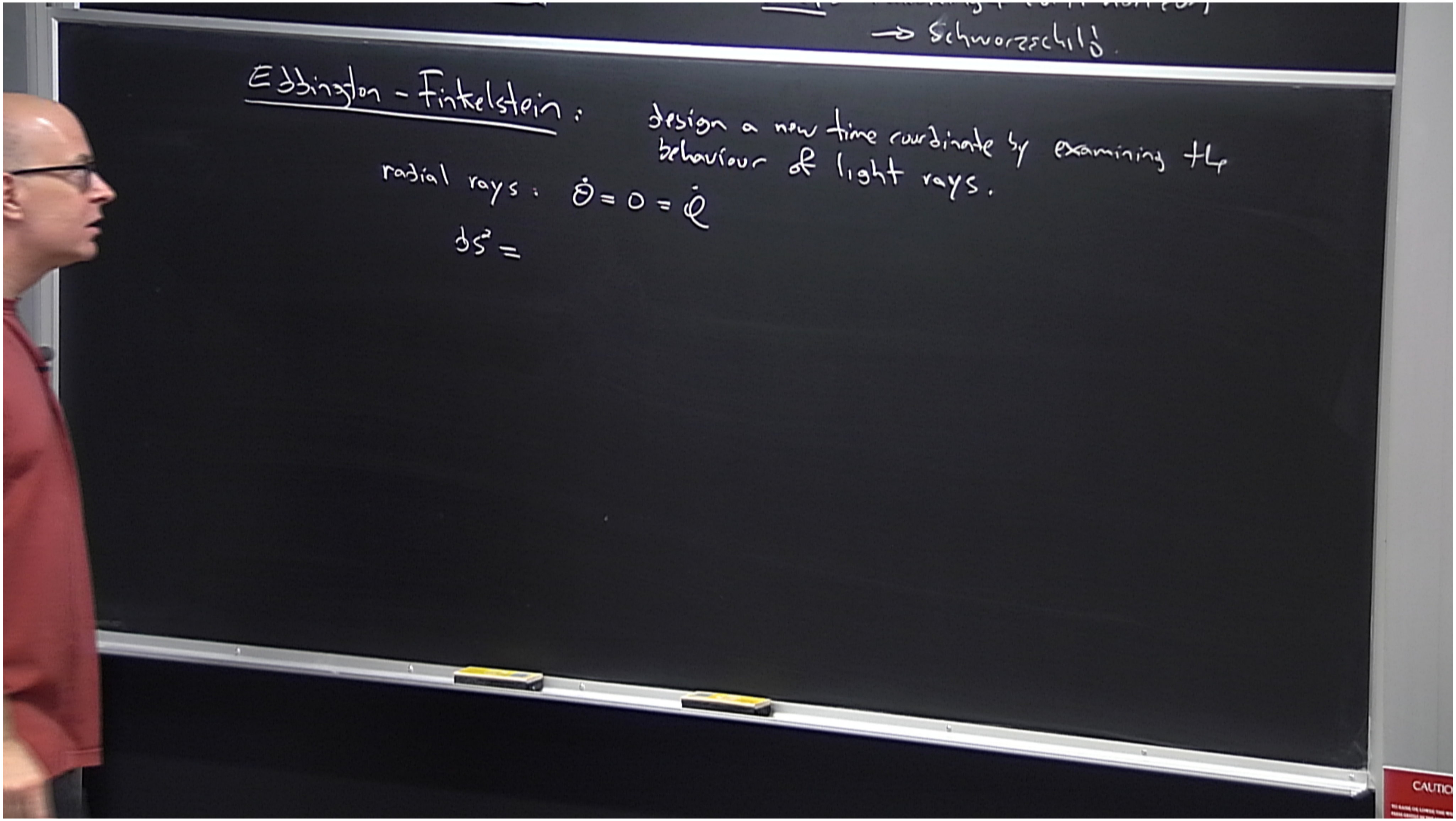
Eddington - Finkelstein: design a new time coordinate by examining the behaviour of light rays.


→ Schwarzschild.
Eddington - Finkelstein:
design a new time coordinate by examining the behaviour of light rays.
radial rays: $\dot{\theta} = 0 = \dot{\varphi}$


→ Schwarzschild.
Eddington - Finkelstein:
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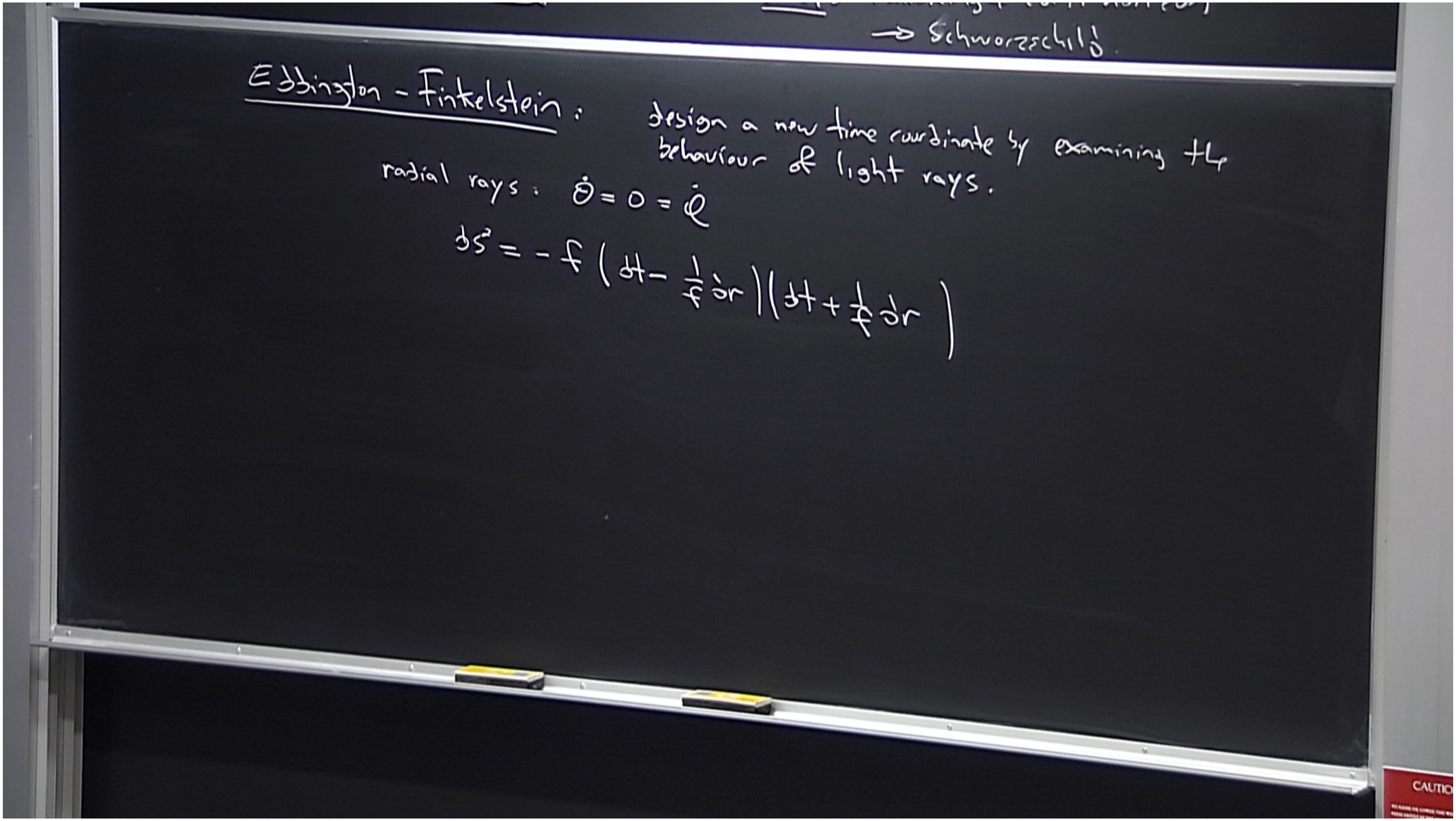
radial rays: $\dot{\theta} = 0 = \dot{\varphi}$
$ds^2 =$
CAUTION


→ Schwarzschild
Eddington - Finkelstein:
design a new time coordinate by examining the behaviour of light rays.
radial rays: $\dot{\theta} = 0 = \dot{\varphi}$
$ds^2 = -f\left(dt - \frac{1}{f}dr\right)\left(dt + \frac{1}{f}dr\right)$

design a new time coordinate by examining the behaviour of light rays.

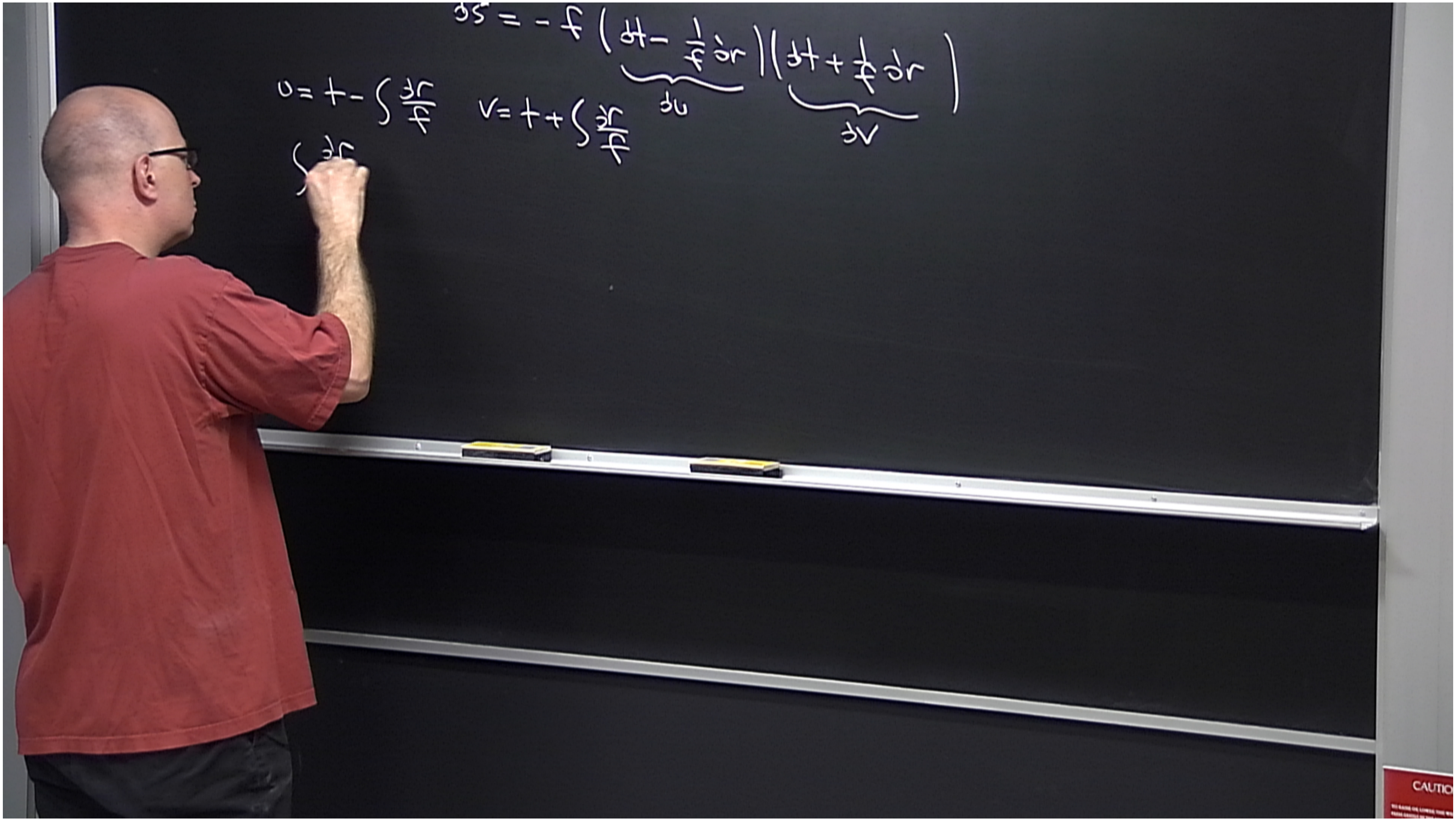
radial rays: $\dot{\theta} = 0 = \dot{\varphi}$

$$ds^2 = -f\Big(\underbrace{dt - \frac{1}{f}\,dr}_{du}\Big)\Big(\underbrace{dt + \frac{1}{f}\,dr}_{dv}\Big)$$

radial rays: $\dot{\theta} = 0 = \dot{\varphi}$

$$ds^2 = -f\left(\underbrace{dt - \frac{1}{f}dr}_{du}\right)\left(\underbrace{dt + \frac{1}{f}dr}_{dv}\right)$$

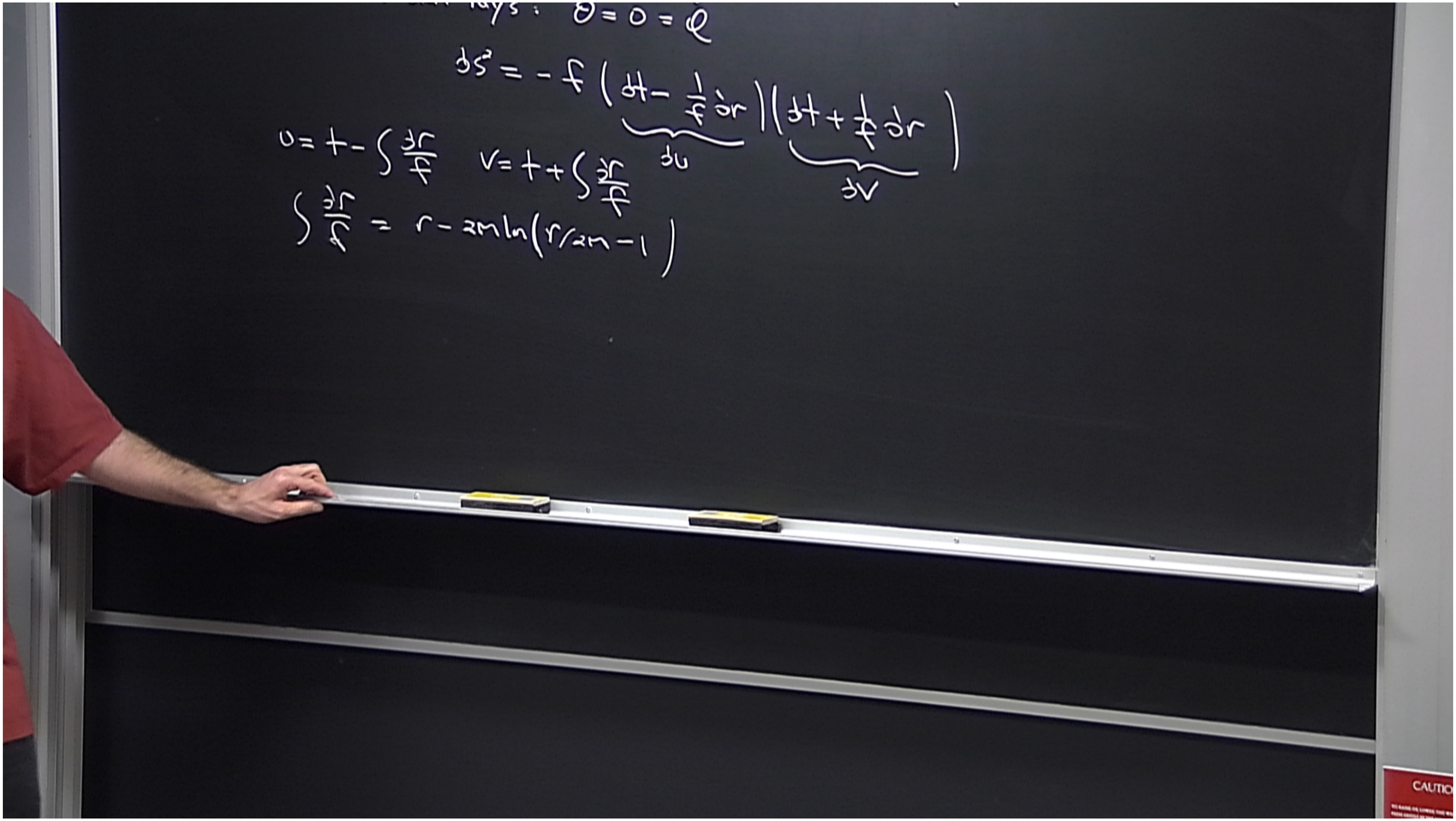
$$u = t - \int \frac{dr}{f} \qquad v = t + \int \frac{dr}{f}$$


$ds^2 = -f\,\underbrace{\left(dt - \frac{1}{f}dr\right)}_{du}\underbrace{\left(dt + \frac{1}{f}dr\right)}_{dv}$
$u = t - \int \frac{dr}{f}$
$v = t + \int \frac{dr}{f}$
$\int \frac{dr}{}$

$$f\,\Big(\underbrace{dt - \tfrac{1}{f}\,dr}_{du}\Big)\Big(\underbrace{dt + \tfrac{1}{f}\,dr}_{dv}\Big)$$

$$u = t - \int \frac{dr}{f} \qquad v = t + \int \frac{dr}{f}$$

$$\int \frac{dr}{f} = r - 2m\ln\left(r/2m - 1\right)$$



rays: $\theta = 0 = \varphi$
$ds^2 = -f\,\underbrace{\left(dt - \frac{1}{f}dr\right)}_{du}\underbrace{\left(dt + \frac{1}{f}dr\right)}_{dv}$
$u = t - \int \frac{dr}{f} \quad v = t + \int \frac{dr}{f}$
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CAUTION

rays : $\theta = 0 = \ell$

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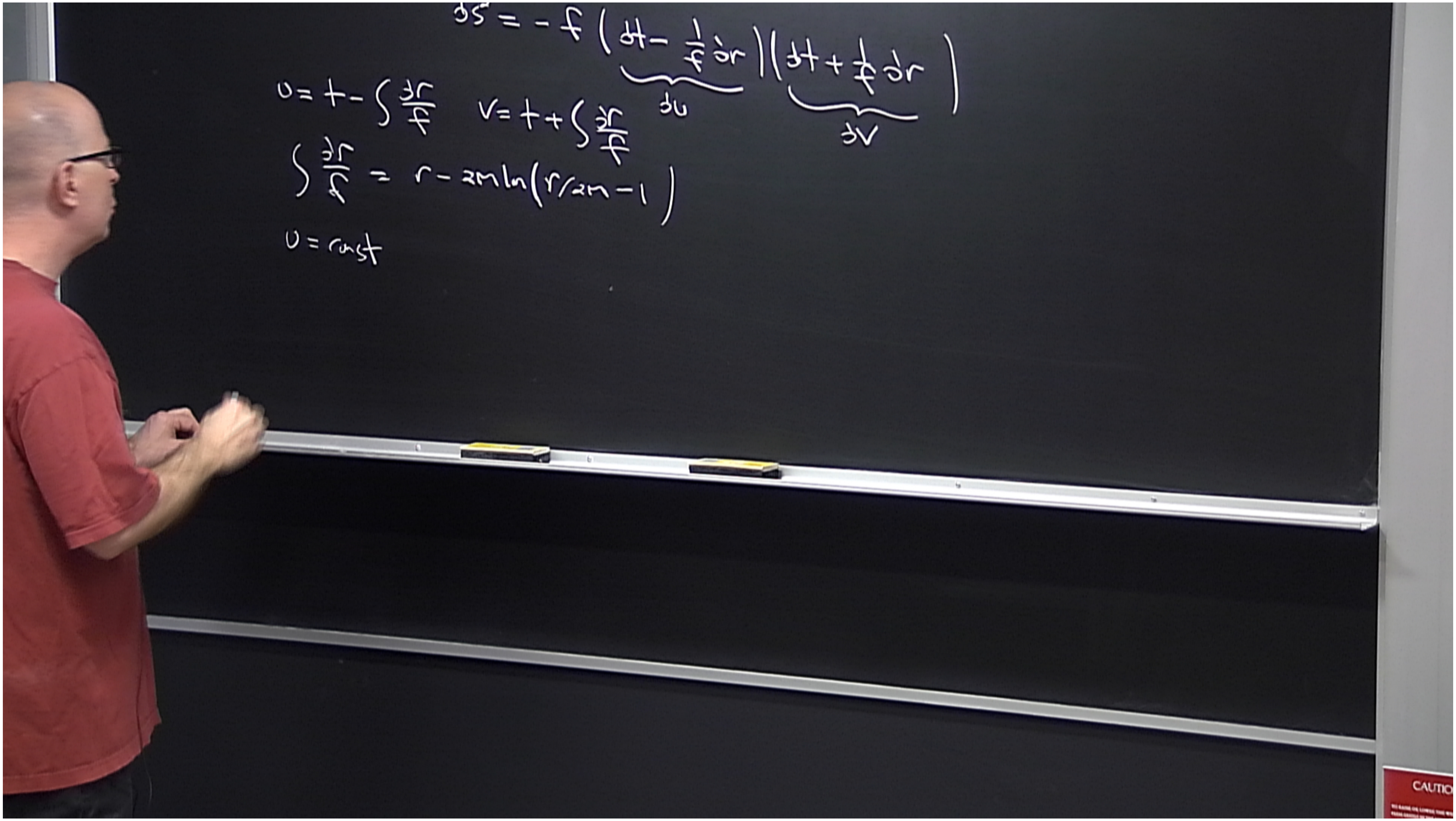
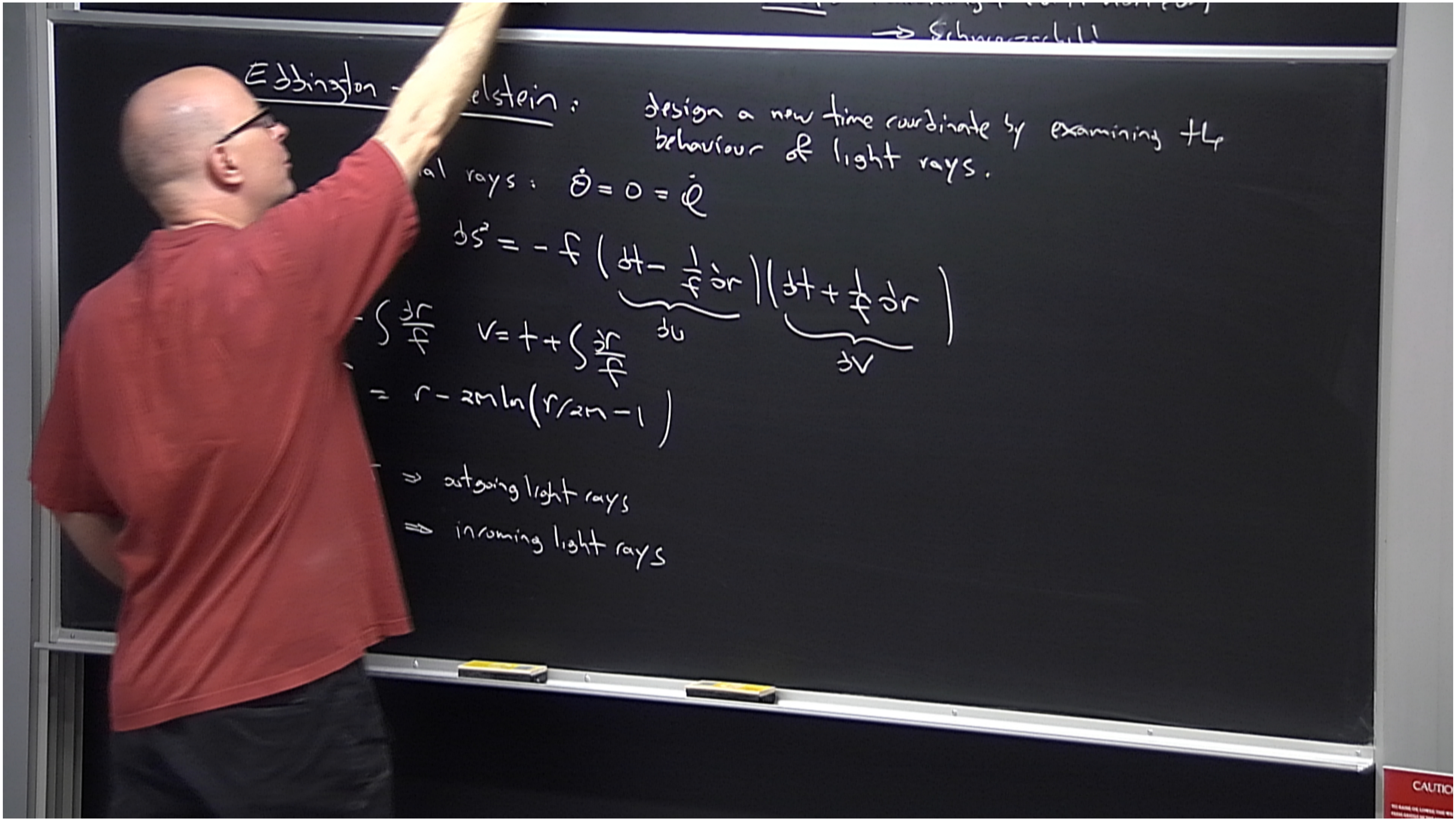

$ds^2 = -f\,\underbrace{\left(dt - \frac{1}{f}dr\right)}_{du}\underbrace{\left(dt + \frac{1}{f}dr\right)}_{dv}$
$u = t - \int \frac{dr}{f}$
$v = t + \int \frac{dr}{f}$
$\int \frac{dr}{f} = r - 2m\ln(r/2m - 1)$
$u = \text{const}$



→ Schwarzschild!
Eddington – …elstein:
design a new time coordinate by examining the behaviour of light rays.
…al rays: θ̇ = 0 = φ̇
ds² = − f (dt − 1/f dr)(dt + 1/f dr)
du
dv
∫ dr/f   v = t + ∫ dr/f
= r − 2m ln(r/2m − 1)
⇒ outgoing light rays
⇒ incoming light rays

$\int \frac{dr}{f} \qquad v = t + \int \frac{dr}{f} \qquad \partial u \qquad \partial v$

$$\int \frac{dr}{f} = r - 2m \ln\left(r/2m - 1\right)$$

$u = \text{const} \Rightarrow$ outgoing light rays

$V = \text{const} \Rightarrow$ incoming light rays

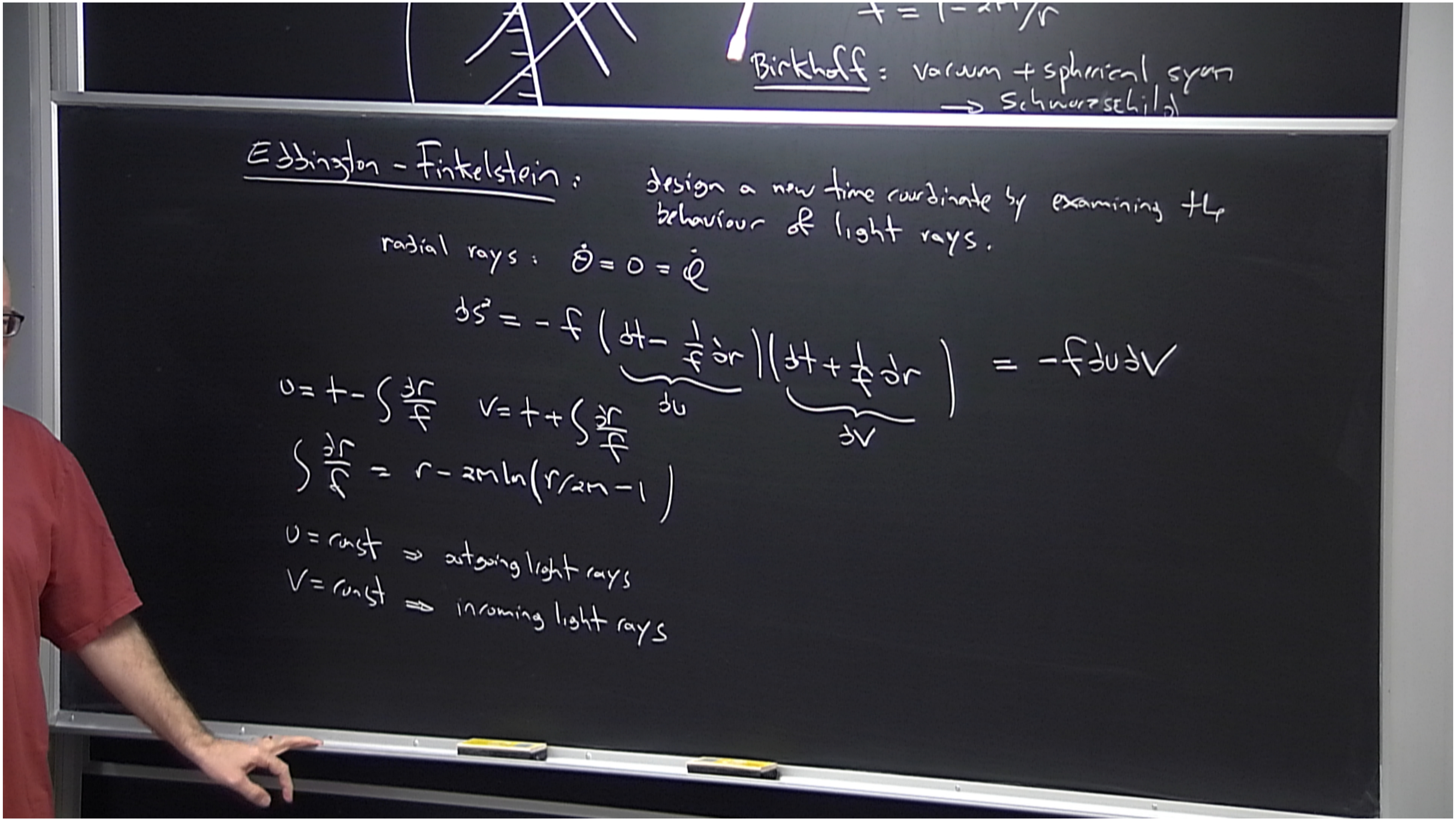
$$ds^2 = -f\,\underbrace{\left(dt - \frac{1}{f}dr\right)}_{du}\underbrace{\left(dt + \frac{1}{f}dr\right)}_{dv} = -f\,du\,dv$$

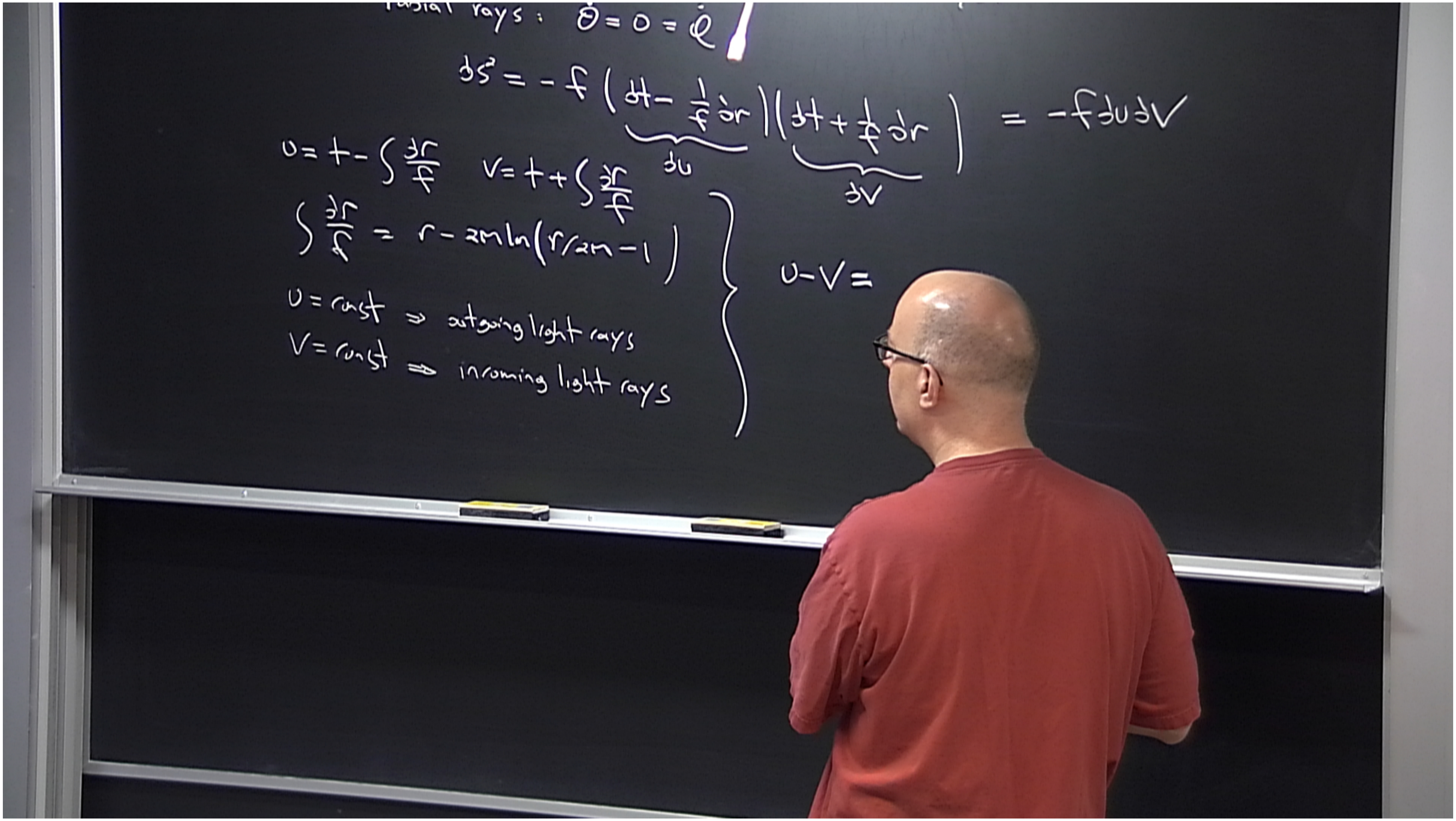
$$u = t - \int \frac{dr}{f} \qquad v = t + \int \frac{dr}{f}$$

$$\int \frac{dr}{f} = r - 2m\ln\left(r/2m - 1\right)$$

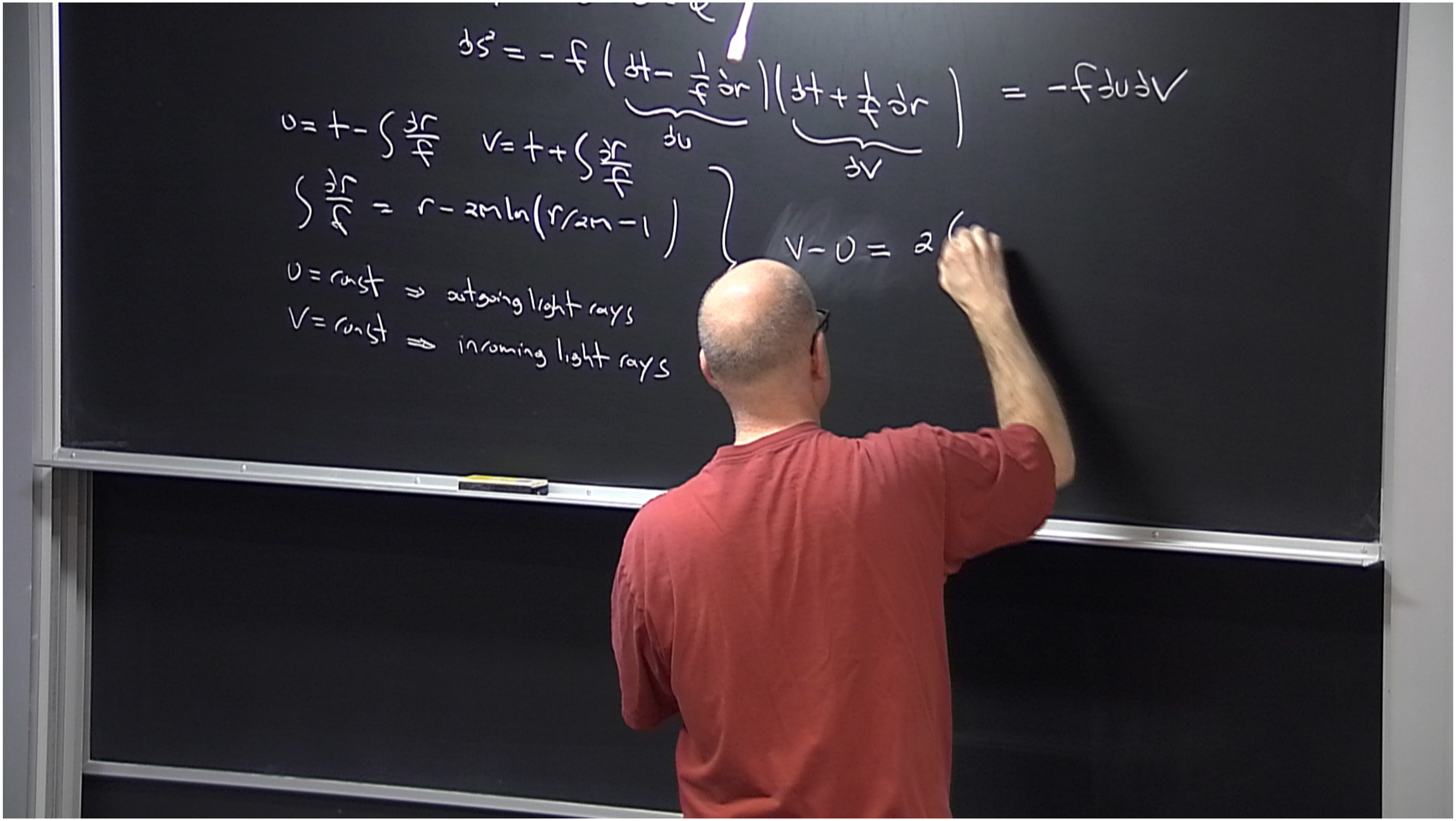
$u = \text{const} \Rightarrow$ outgoing light rays

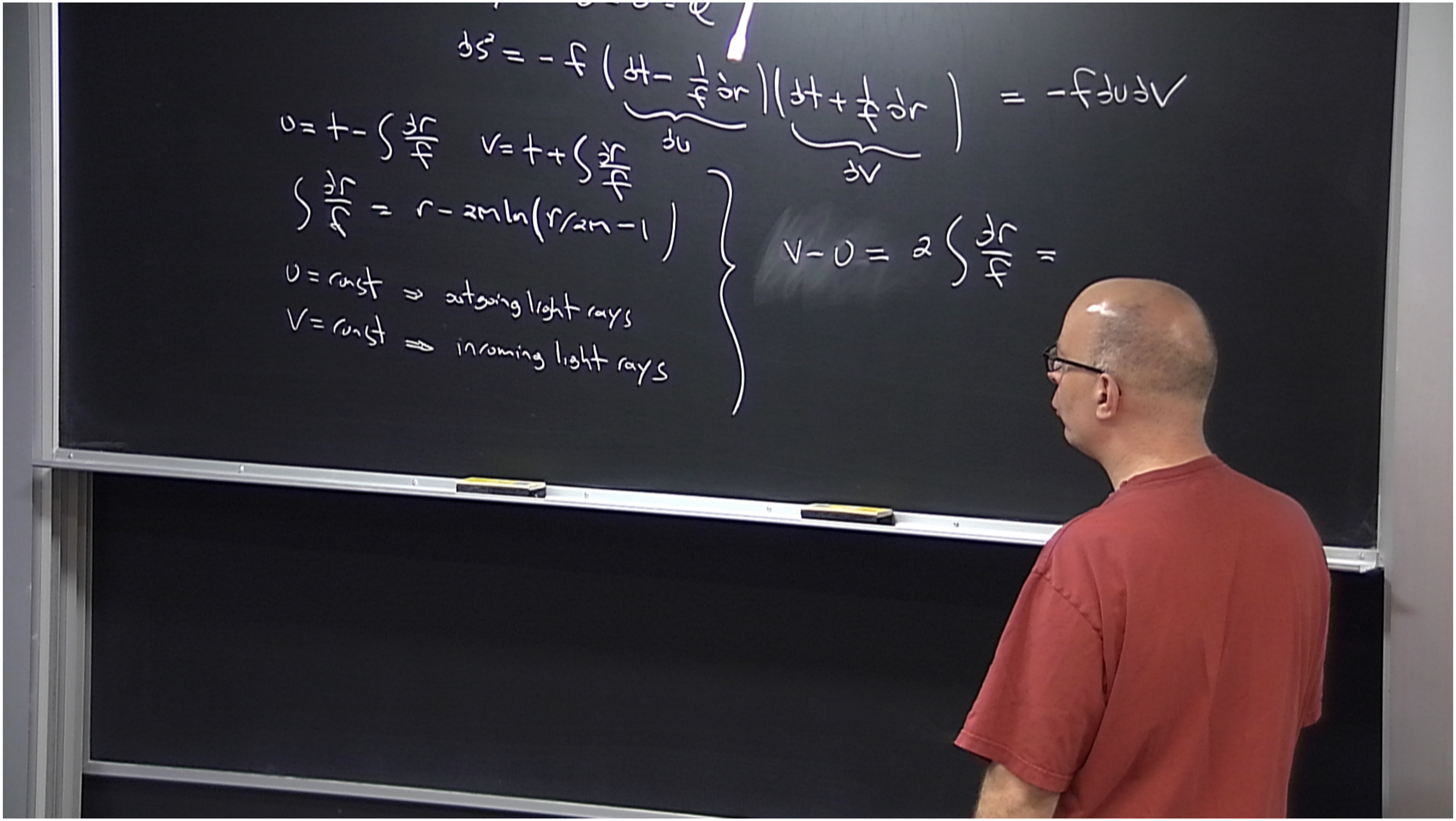
$v = \text{const} \Rightarrow$ incoming light rays

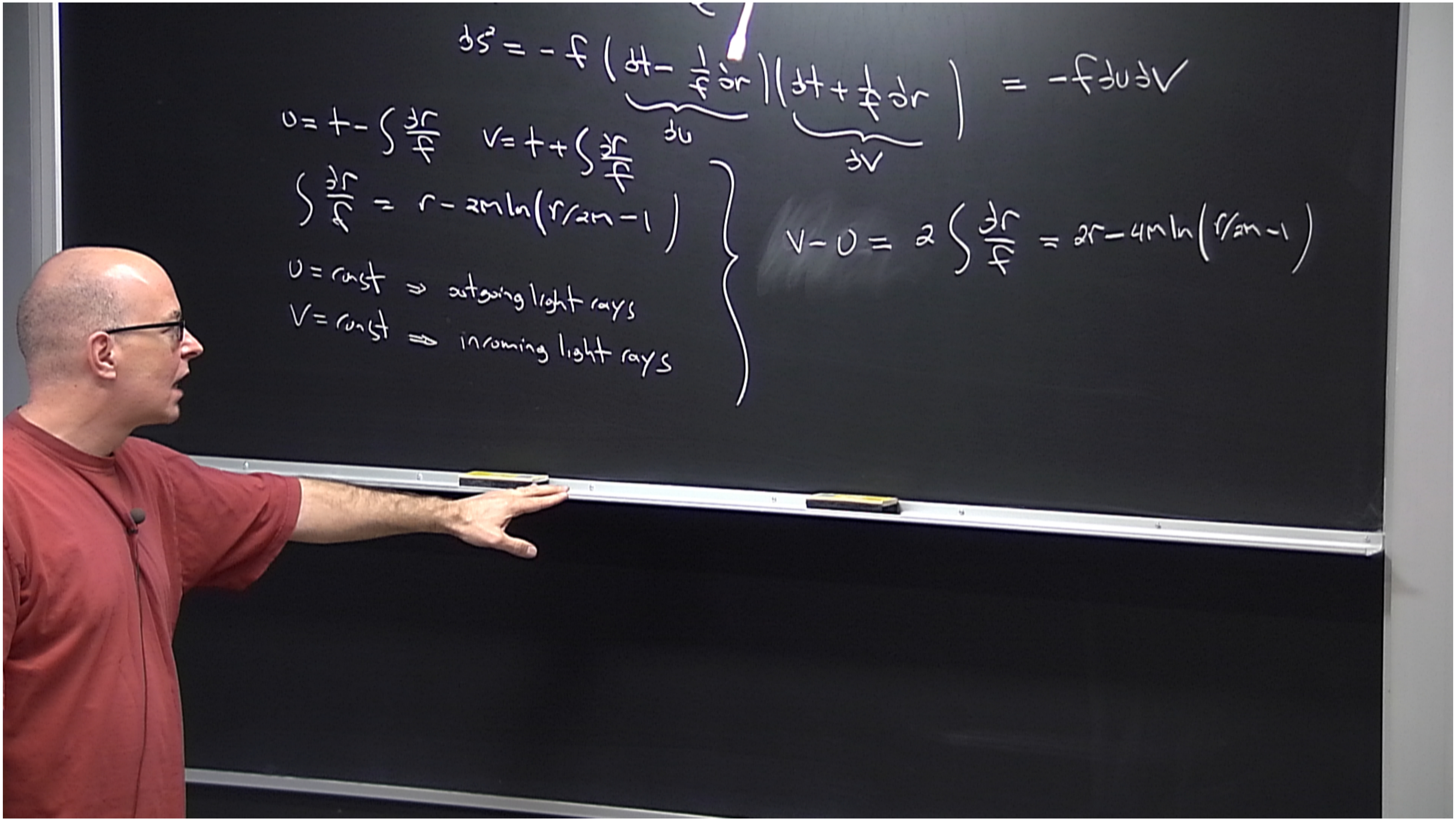

$f = 1 - 2M/r$
Birkhoff: vacuum + spherical sym
$\Rightarrow$ Schwarzschild
Eddington - Finkelstein: design a new time coordinate by examining the behaviour of light rays.
radial rays: $\dot\theta = 0 = \dot\varphi$
$ds^2 = -f\,(\underbrace{dt - \frac{1}{f}dr}_{du})(\underbrace{dt + \frac{1}{f}dr}_{dv}) = -f\,du\,dv$
$u = t - \int \frac{dr}{f}$ $v = t + \int \frac{dr}{f}$
$\int \frac{dr}{f} = r - 2m\ln(r/2m - 1)$
u = const $\Rightarrow$ outgoing light rays
v = const $\Rightarrow$ incoming light rays

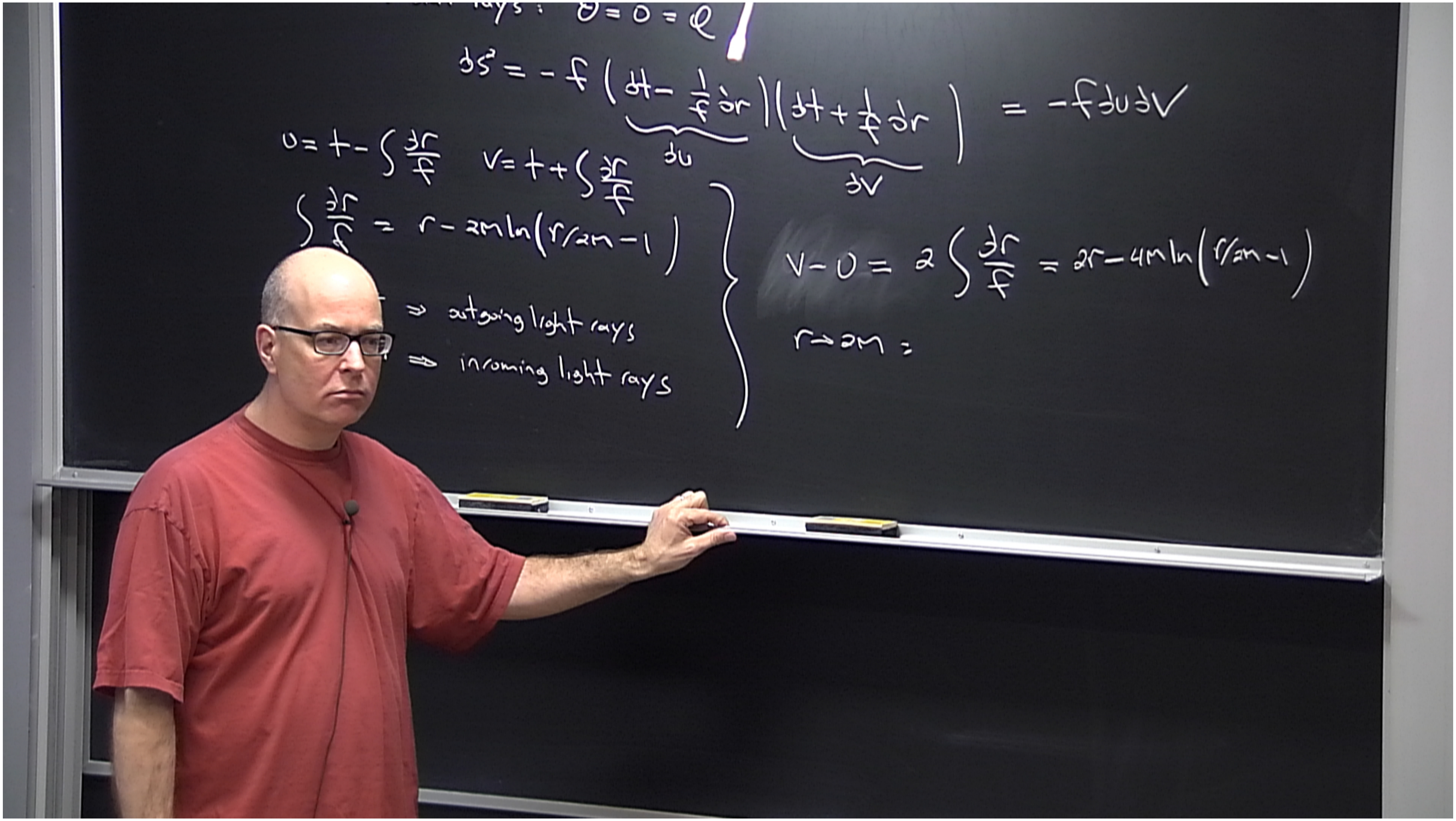


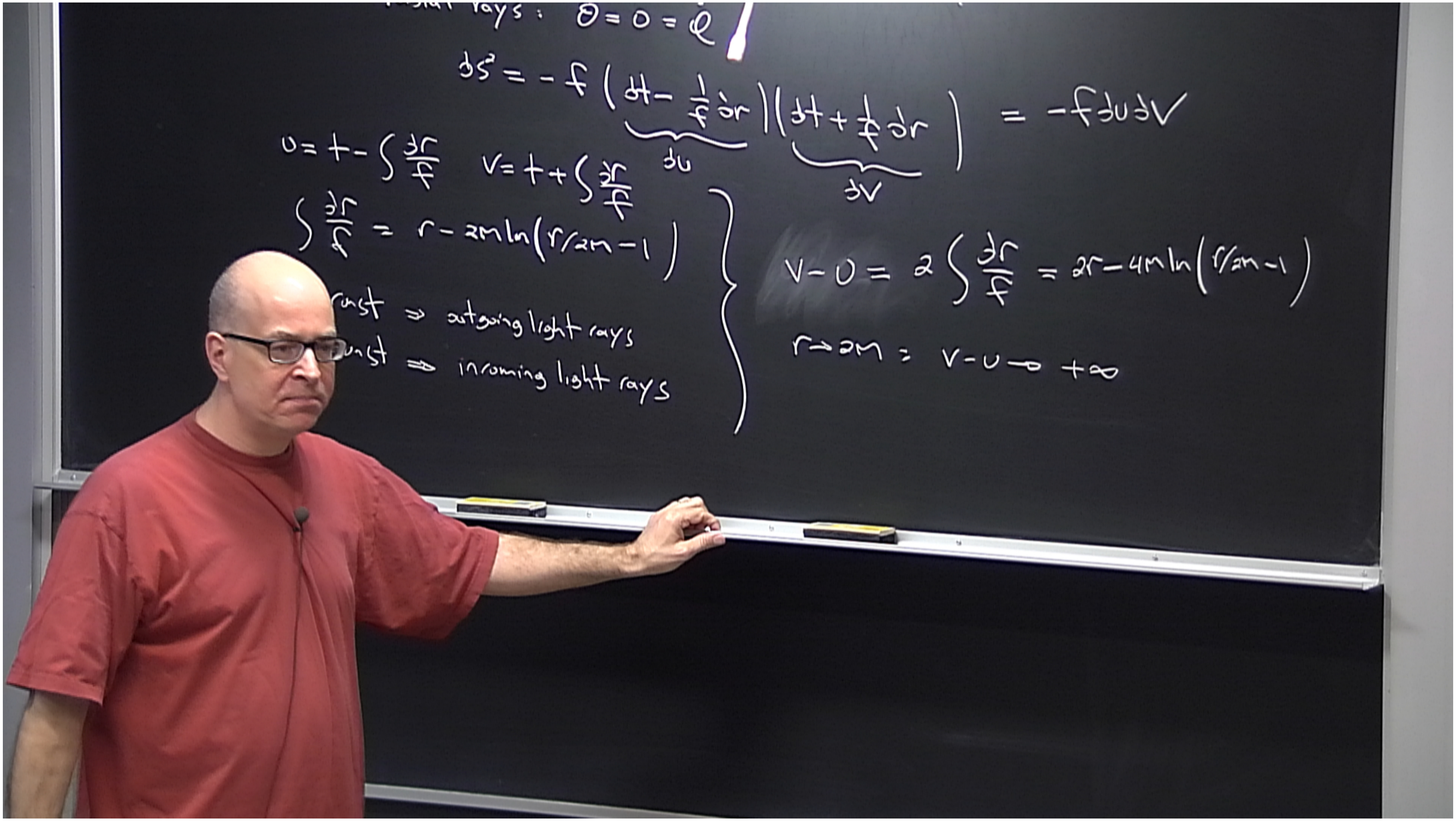
radial rays: $\theta = 0 = \dot{\varphi}$
$ds^2 = -f\Big(\underbrace{dt - \frac{1}{f}dr}_{du}\Big)\Big(\underbrace{dt + \frac{1}{f}dr}_{dv}\Big) = -f\,du\,dv$
$u = t - \int \frac{dr}{f}$ $v = t + \int \frac{dr}{f}$
$\int \frac{dr}{f} = r - 2m \ln(r/2m - 1)$
u = const ⇒ outgoing light rays
v = const ⇒ incoming light rays
$u - v =$

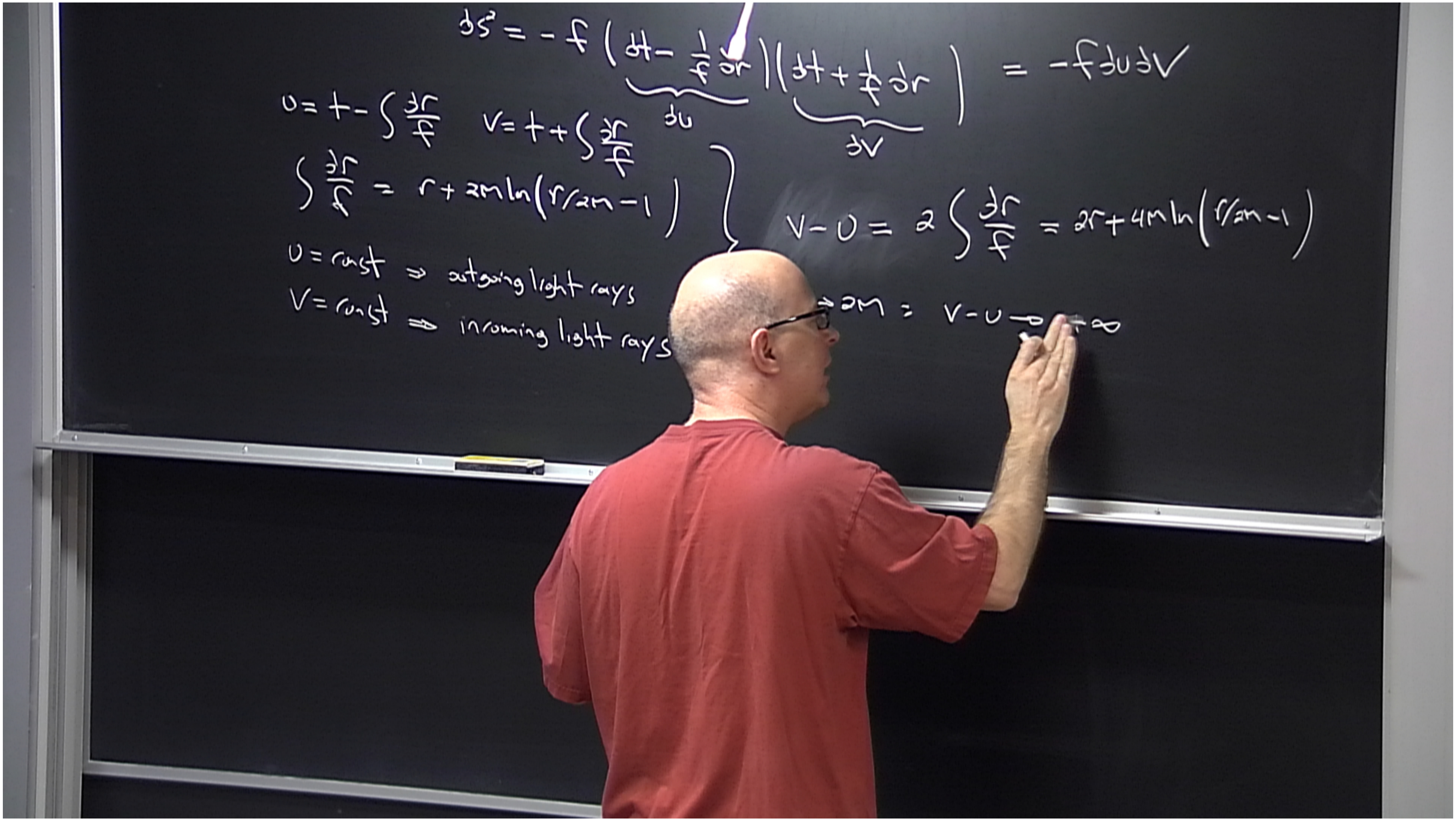

ds² = − f (dt − 1/f dr)(dt + 1/f dr) = − f du dv
du
dv
u = t − ∫ dr/f   v = t + ∫ dr/f
∫ dr/f = r − 2m ln(r/2m − 1)
u = const ⇒ outgoing light rays
V = const ⇒ incoming light rays
V − U = 2 (

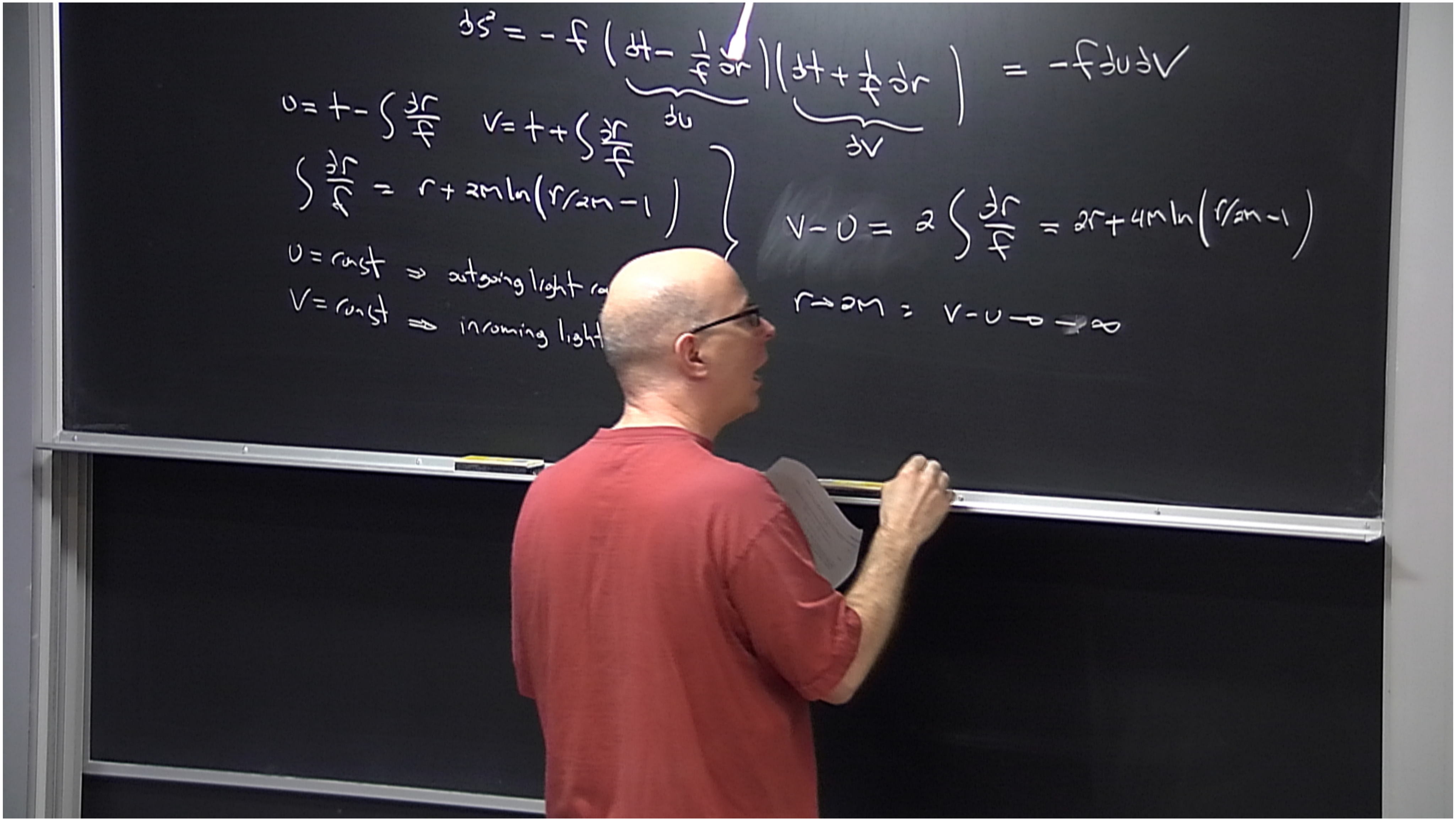

$ds^2 = -f\left(\underbrace{dt - \frac{1}{f}dr}_{du}\right)\left(\underbrace{dt + \frac{1}{f}dr}_{dv}\right) = -f\,du\,dv$
$u = t - \int \frac{dr}{f}$ $v = t + \int \frac{dr}{f}$
$\int \frac{dr}{f} = r - 2m\ln(r/2m - 1)$
u = const ⇒ outgoing light rays
v = const ⇒ incoming light rays
$v - u = 2\int \frac{dr}{f} =$

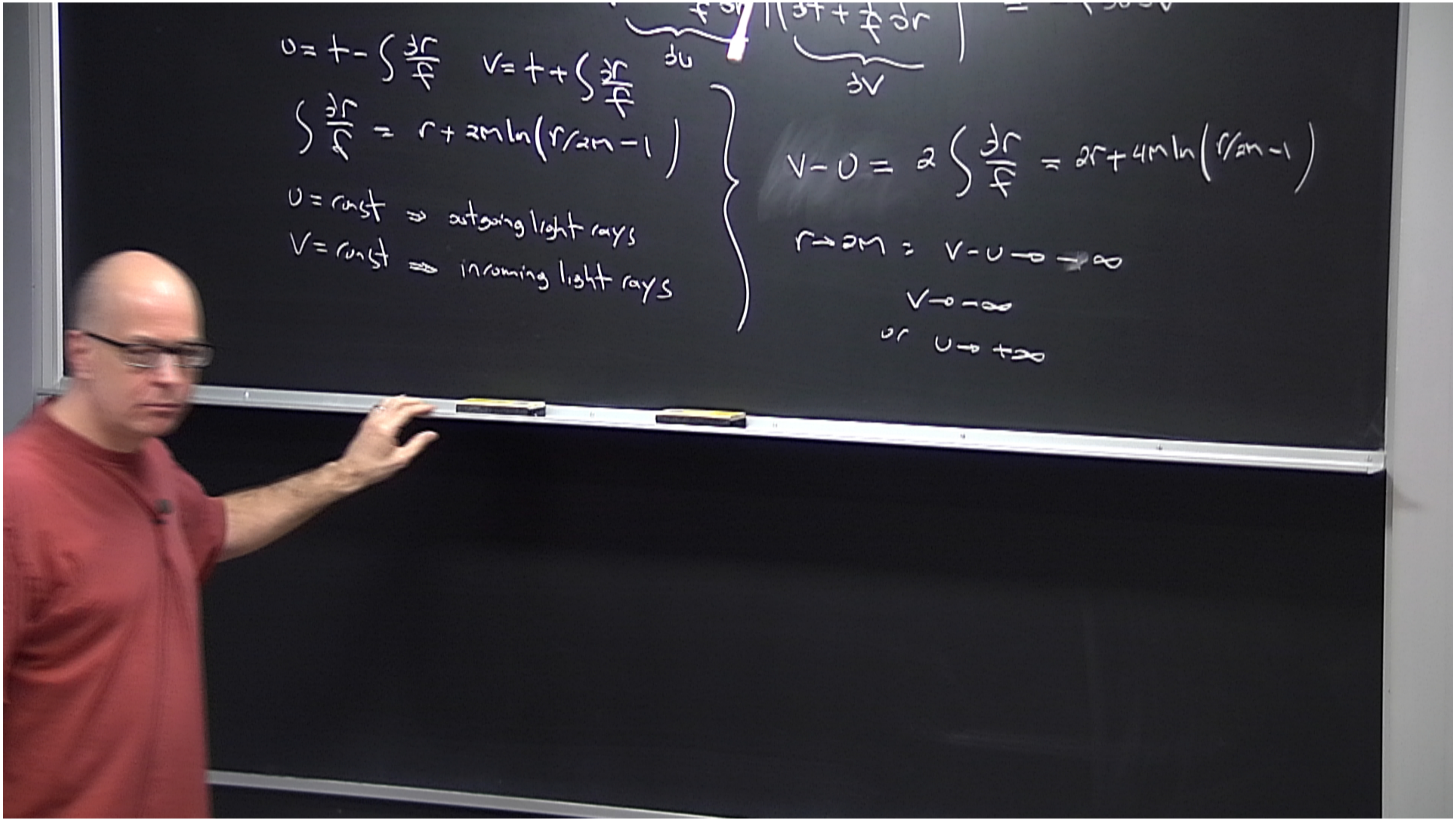

$ds^2 = -f\left(\underbrace{dt - \frac{1}{f}dr}_{du}\right)\left(\underbrace{dt + \frac{1}{f}dr}_{dv}\right) = -f\,du\,dv$
$u = t - \int \frac{dr}{f}$ $v = t + \int \frac{dr}{f}$
$\int \frac{dr}{f} = r - 2m\ln(r/2m - 1)$
$v - u = 2\int \frac{dr}{f} = 2r - 4m\ln(r/2m - 1)$
u = const ⇒ outgoing light rays
v = const ⇒ incoming light rays


θ = 0 = φ
ds² = −f (dt − 1/f dr)(dt + 1/f dr) = −f du dv
du
dv
u = t − ∫ dr/f   v = t + ∫ dr/f
∫ dr/f = r − 2m ln(r/2m − 1)
⇒ outgoing light rays
⇒ incoming light rays
v − u = 2 ∫ dr/f = 2r − 4m ln(r/2m − 1)
r → 2m =

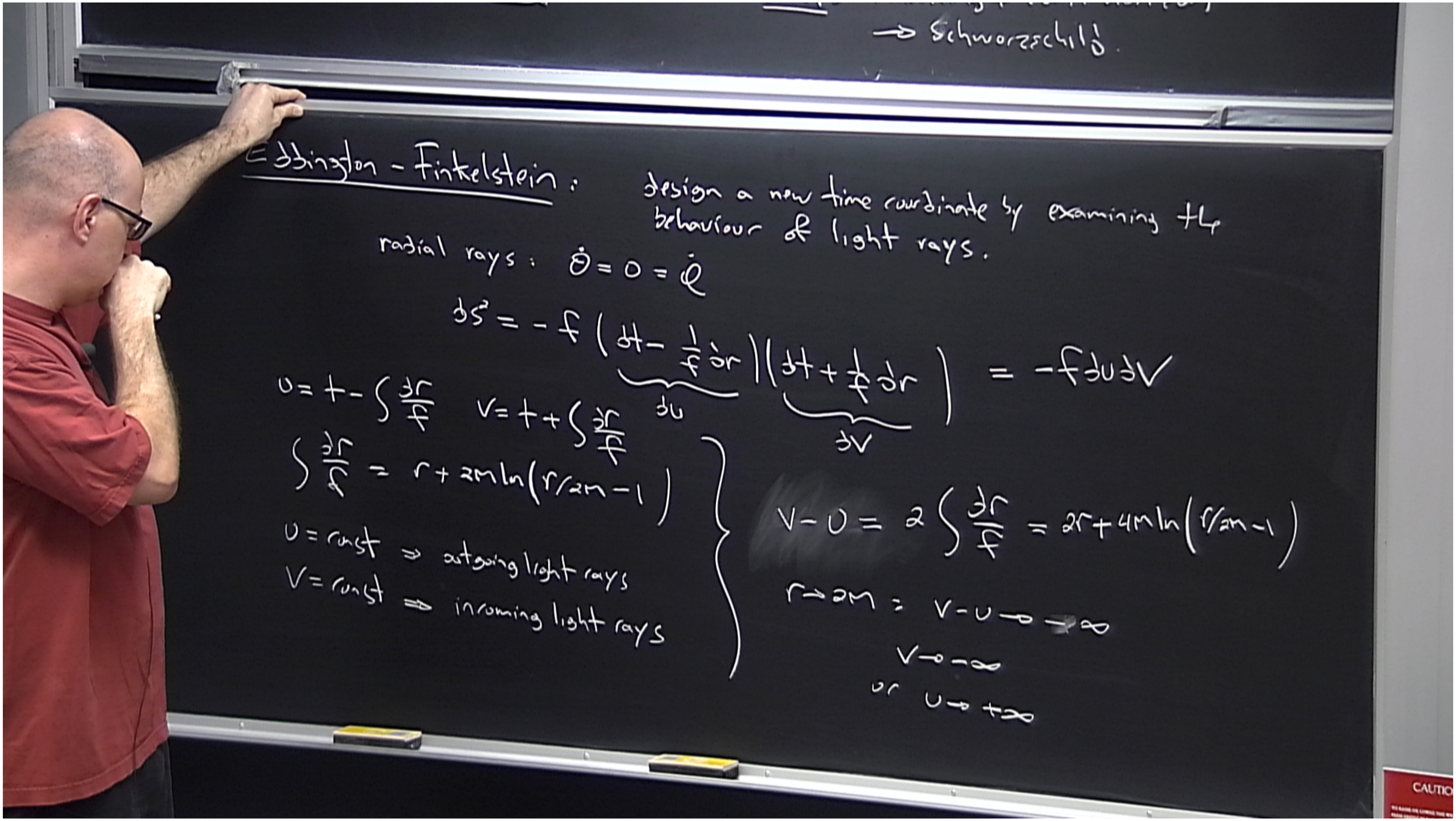

rays : $\theta = 0 = \ell$
$ds^2 = -f\left(\underbrace{dt - \frac{1}{f}dr}_{du}\right)\left(\underbrace{dt + \frac{1}{f}dr}_{dv}\right) = -f\,du\,dv$
$u = t - \int \frac{dr}{f}$ $v = t + \int \frac{dr}{f}$
$\int \frac{dr}{f} = r - 2m \ln(r/2m - 1)$
$v - u = 2\int \frac{dr}{f} = 2r - 4m\ln(r/2m - 1)$
$r \to 2m$: $v - u \to +\infty$
const $\Rightarrow$ outgoing light rays
const $\Rightarrow$ incoming light rays

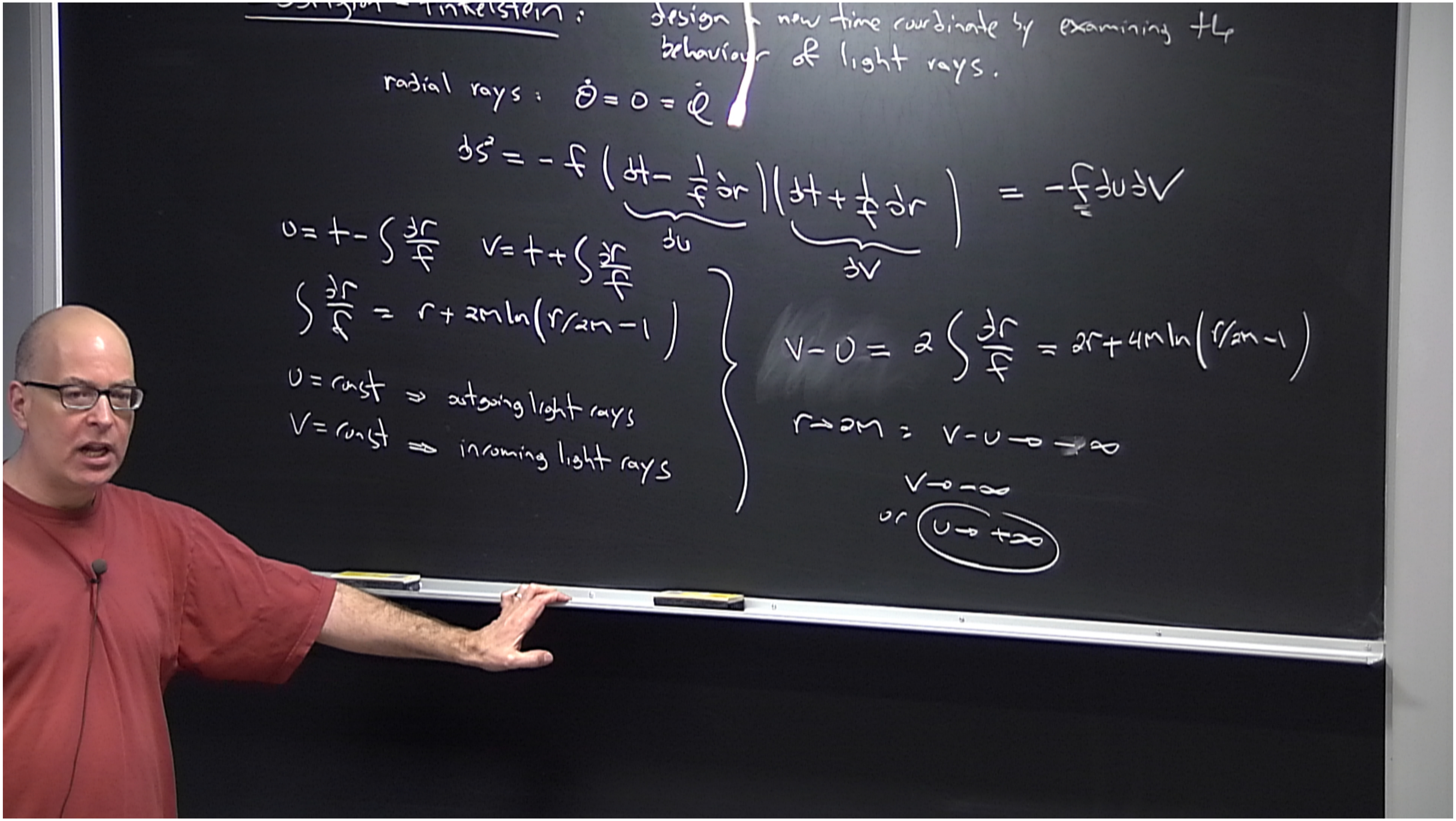

$ds^2 = -f\left(dt - \frac{1}{f}dr\right)\left(dt + \frac{1}{f}dr\right) = -f\,du\,dv$
du
dv
$u = t - \int \frac{dr}{f}$ $\quad v = t + \int \frac{dr}{f}$
$\int \frac{dr}{f} = r + 2m \ln(r/2m - 1)$
$v - u = 2\int \frac{dr}{f} = 2r + 4m\ln(r/2m - 1)$
u = const ⇒ outgoing light rays
V = const ⇒ incoming light rays
$\Rightarrow 2m: \; v - u \to -\infty$


$ds^2 = -f\left(\underbrace{dt - \frac{1}{f}dr}_{du}\right)\left(\underbrace{dt + \frac{1}{f}dr}_{dV}\right) = -f\,du\,dV$
$u = t - \int \frac{dr}{f}$ $\quad V = t + \int \frac{dr}{f}$
$\int \frac{dr}{f} = r + 2M\ln(r/2M - 1)$
$V - U = 2\int \frac{dr}{f} = 2r + 4M\ln(r/2M - 1)$
$U = \text{const} \Rightarrow$ outgoing light
$V = \text{const} \Rightarrow$ incoming light
$r \to 2M$: $V - U \to -\infty$



$u = t - \int \frac{dr}{f}$ $v = t + \int \frac{dr}{f}$
$\int \frac{dr}{f} = r + 2m \ln(r/2m - 1)$
$u = \text{const} \Rightarrow$ outgoing light rays
$v = \text{const} \Rightarrow$ incoming light rays
$v - u = 2\int \frac{dr}{f} = 2r + 4m \ln(r/2m - 1)$
$r \to 2m : v - u \to -\infty$
$v \to -\infty$
or $u \to +\infty$


→ Schwarzschild.
Eddington - Finkelstein:
design a new time coordinate by examining the behaviour of light rays.
radial rays: $\dot\theta = 0 = \dot\varphi$
$ds^2 = -f\left(\underbrace{dt - \frac{1}{f}dr}_{du}\right)\left(\underbrace{dt + \frac{1}{f}dr}_{dv}\right) = -f\,du\,dv$
$u = t - \int\frac{dr}{f}$ $v = t + \int\frac{dr}{f}$
$\int\frac{dr}{f} = r + 2M\ln(r/2M - 1)$
u = const ⇒ outgoing light rays
v = const ⇒ incoming light rays
$v - u = 2\int\frac{dr}{f} = 2r + 4M\ln(r/2M - 1)$
$r \to 2M$: $v - u \to -\infty$
$v \to -\infty$
or $u \to +\infty$
CAUTION


design a new time coordinate by examining the behaviour of light rays.
radial rays: $\dot{\theta} = 0 = \dot{\varphi}$
$ds^2 = -f\left(\underbrace{dt - \frac{1}{f}dr}_{du}\right)\left(\underbrace{dt + \frac{1}{f}dr}_{dv}\right) = -f\,du\,dv$
$u = t - \int \frac{dr}{f}$ $v = t + \int \frac{dr}{f}$
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u = const ⇒ outgoing light rays
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$v - u = 2\int \frac{dr}{f} = 2r + 4m\ln(r/2m - 1)$
$r \to 2m$: $v - u \to -\infty$
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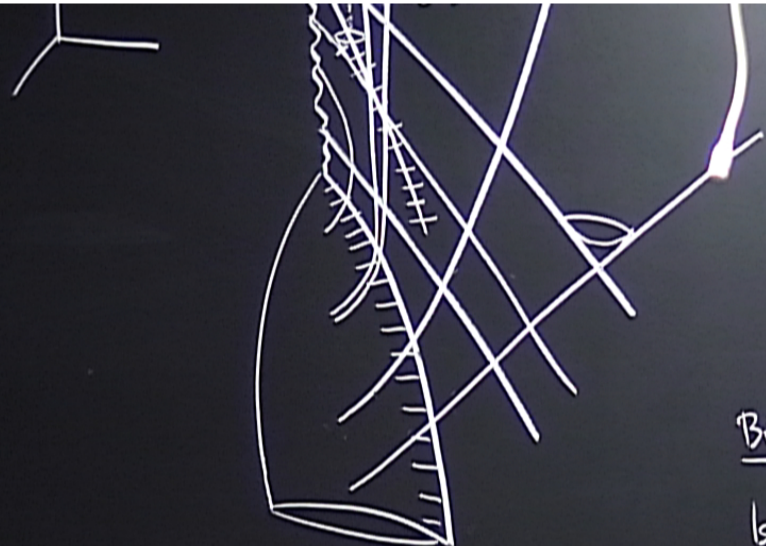


April 9

nonrotating BH = Schwarzschild

$$ds^2 = -f\,dt^2 + f^{-1}dr^2 + r^2 d\Omega^2$$

$$f = 1 - 2M/r$$

Birkhoff: vacuum + spherical sym $\rightarrow$ Schwarzschild

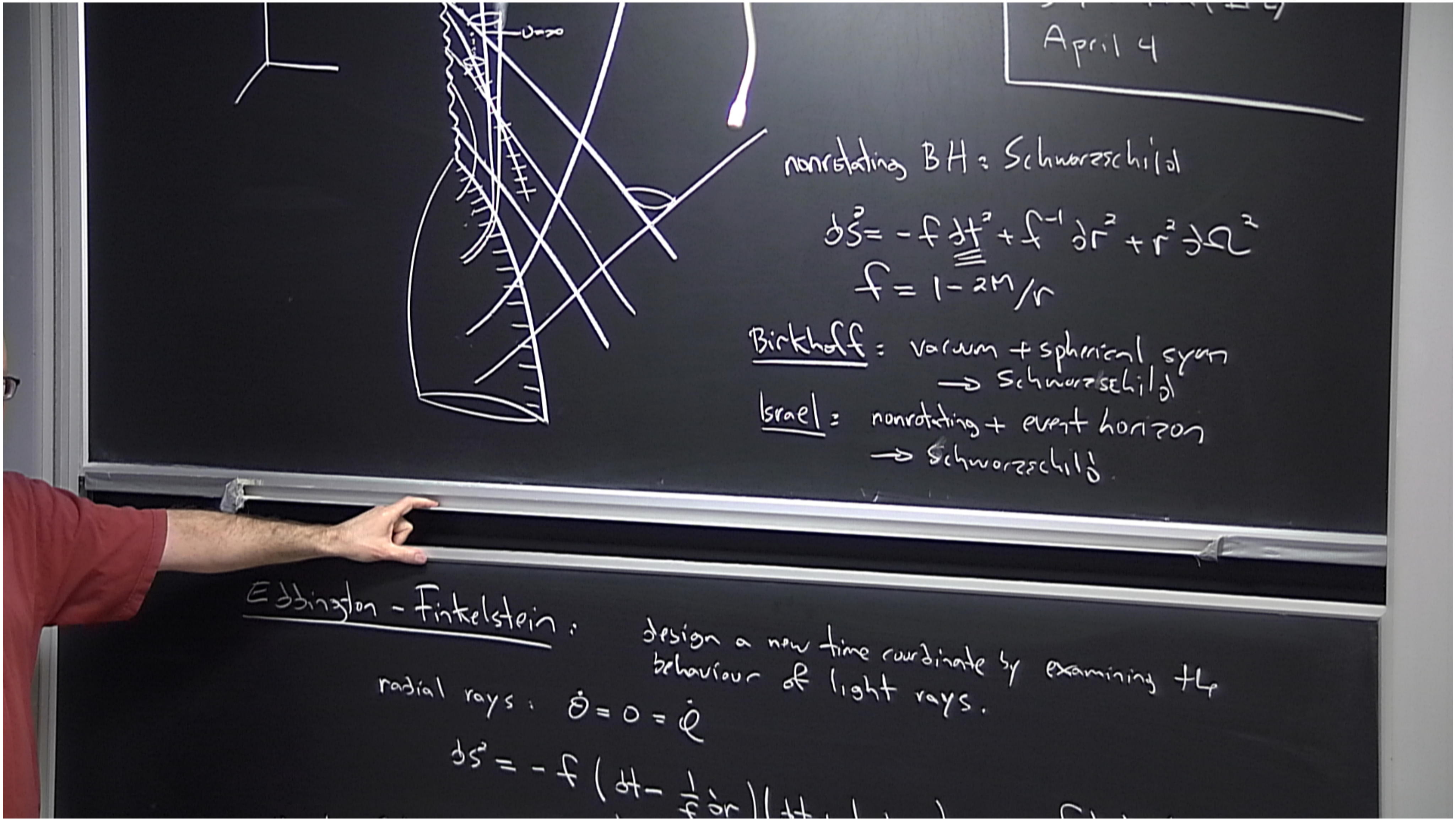
Israel: nonrotating + event horizon $\rightarrow$ Schwarzschild

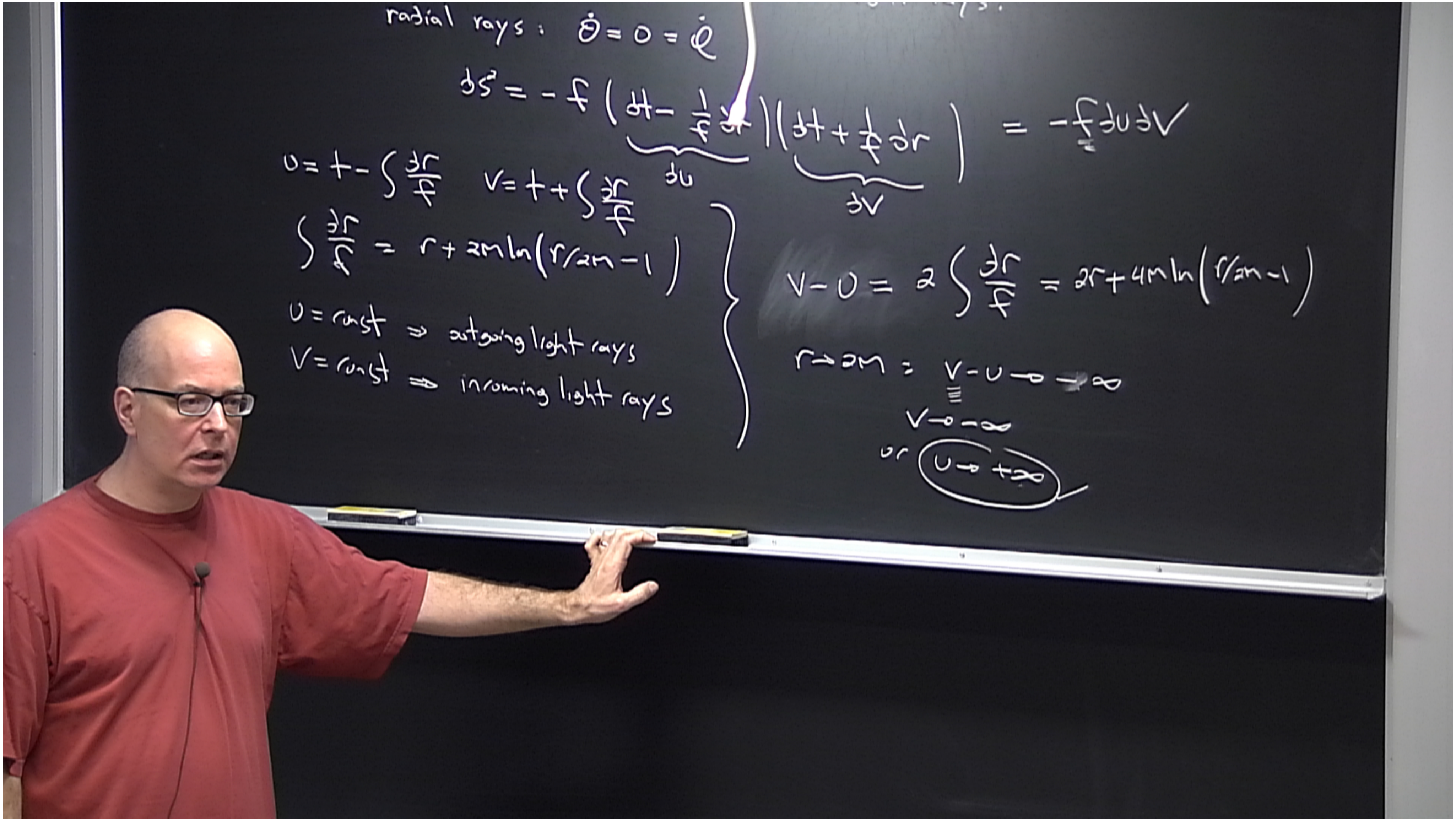
Eddington - Finkelstein: design a new time coordinate by examining the behaviour of light rays.

radial rays: $\dot\theta = 0 = \dot\varphi$

$$ds^2 = -f\,\Big(\underbrace{dt - \tfrac{1}{f}dr}_{du}\Big)\Big(\underbrace{dt + \tfrac{1}{f}dr}\Big) = -f\,du\,dv$$

$u = t - \int \frac{dr}{f}$ $\quad v = t + \int \frac{dr}{\ldots}$


April 4
nonrotating BH = Schwarzschild
ds² = − f dt² + f⁻¹ dr² + r² dΩ²
f = 1 − 2M/r
Birkhoff: vacuum + spherical sym
→ Schwarzschild
Israel: nonrotating + event horizon
→ Schwarzschild
Eddington - Finkelstein:
design a new time coordinate by examining the behaviour of light rays.
radial rays: θ̇ = 0 = φ̇
ds² = − f (dt − 1/f dr)


radial rays: $\dot{\theta} = 0 = \dot{\varphi}$
$ds^2 = -f\underbrace{\left(dt - \frac{1}{f}dr\right)}_{du}\underbrace{\left(dt + \frac{1}{f}dr\right)}_{dv} = -f\,du\,dv$
$u = t - \int \frac{dr}{f}$
$v = t + \int \frac{dr}{f}$
$\int \frac{dr}{f} = r + 2m\ln(r/2m - 1)$
u = const ⇒ outgoing light rays
v = const ⇒ incoming light rays
$v - u = 2\int \frac{dr}{f} = 2r + 4m\ln(r/2m - 1)$
$r \to 2m$: $v - u \to -\infty$
$v \to -\infty$
or $u \to +\infty$

$U = \text{const} \Rightarrow$ outgoing light rays

$V = \text{const} \Rightarrow$ incoming light rays

$V - U = 2\int \frac{dr}{f} = \ldots$

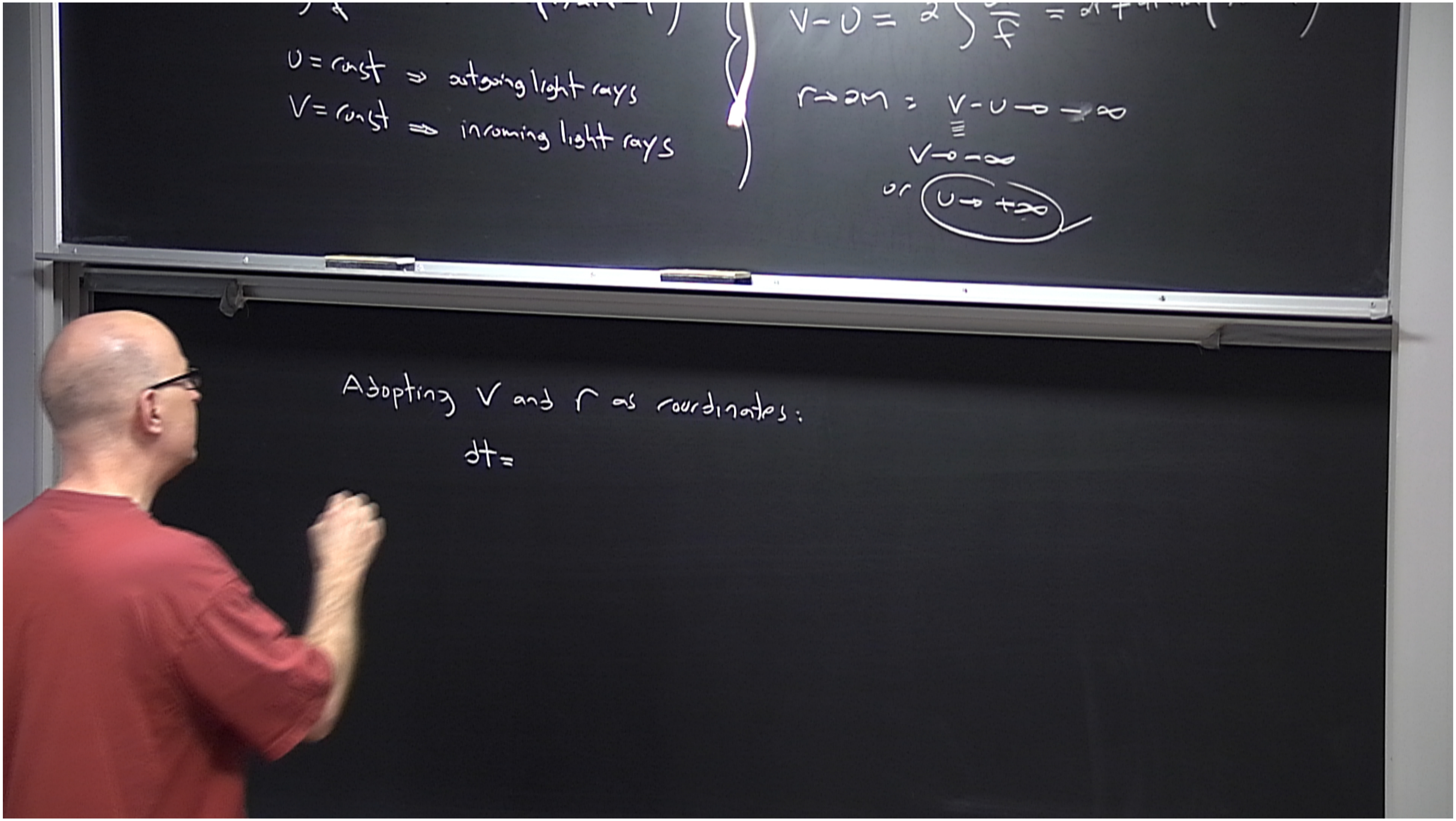
$r \to 2M : \; V - U \to -\infty$

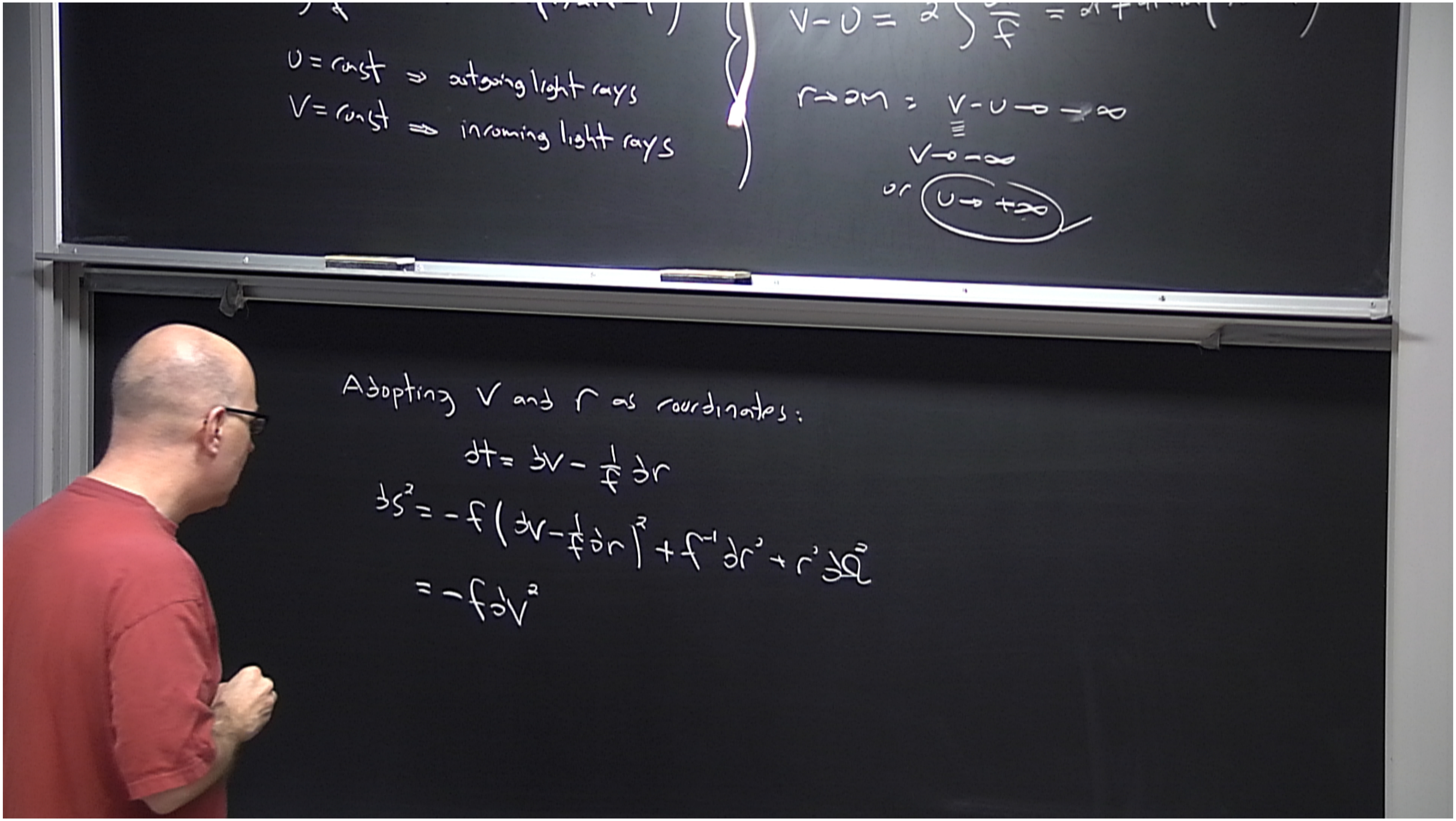
$\equiv$

$V \to -\infty$

or $U \to +\infty$

Adopting V and r as coordinates:


U = const ⇒ outgoing light rays
V = const ⇒ incoming light rays
r → 2M : V − U → −∞
V → −∞
or U → +∞
Adopting V and r as coordinates:
dt =


$U = \text{const} \Rightarrow$ outgoing light rays
$V = \text{const} \Rightarrow$ incoming light rays
$V - U = 2\int \frac{dr}{f}$
$r \to 2M : V - U \to -\infty$
$\equiv V \to -\infty$
or $U \to +\infty$
Adopting V and r as coordinates:
$dt = dV - \frac{1}{f} dr$
$ds^2 = -f\left(dV - \frac{1}{f}dr\right)^2 + f^{-1}dr^2 + r^2 d\Omega^2$
$= -f dV^2$

Adopting v and r as coordinates:

$$dt = dv - \frac{1}{f} dr$$

$$ds^2 = -f\left(dv - \frac{1}{f} dr\right)^2 + f^{-1} dr^2 + r^2 d\Omega^2$$

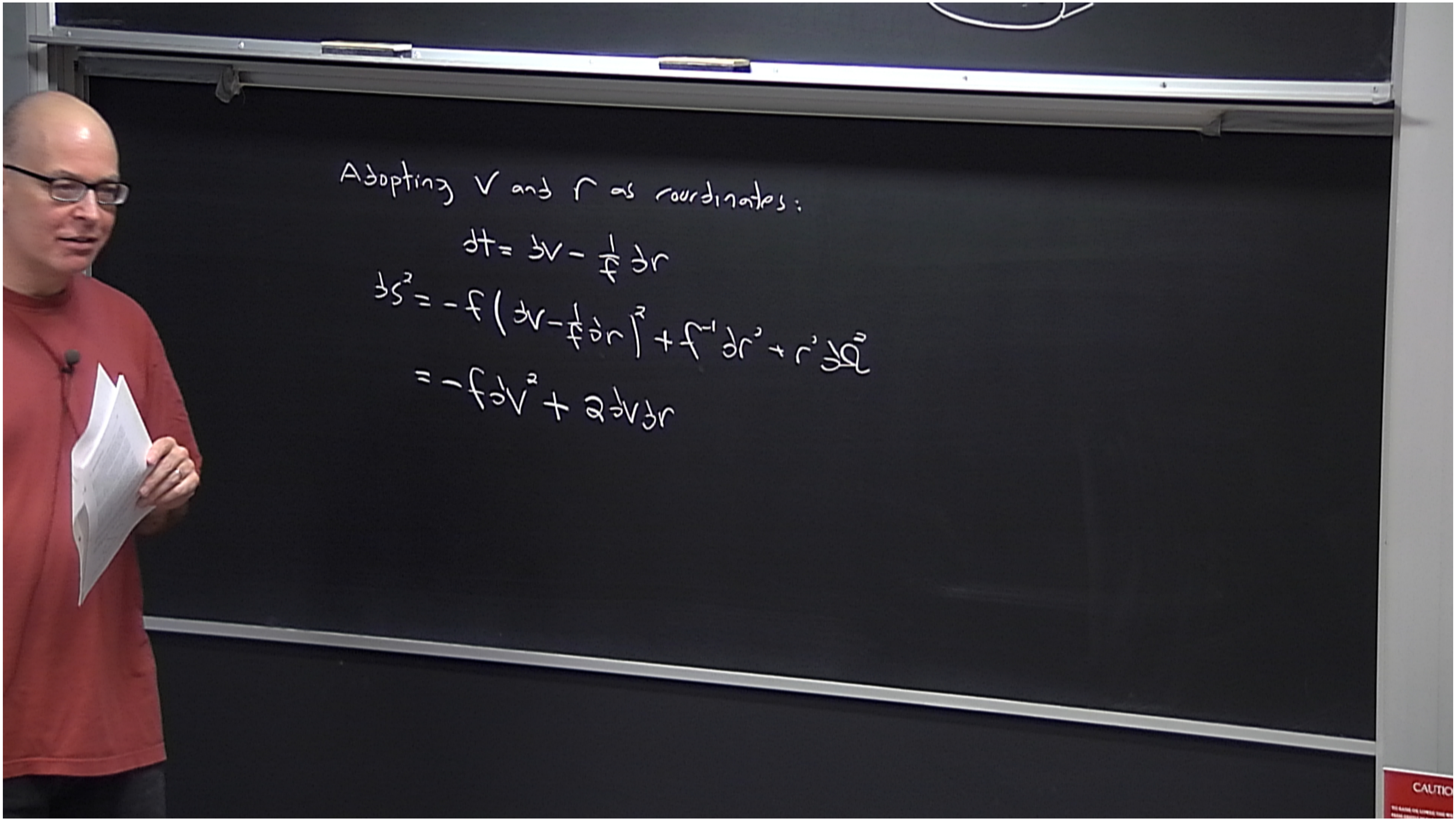
$$= -f dv^2 - 2 dv dr$$

Adopting v and r as coordinates:

$$dt = dv - \frac{1}{f} dr$$

$$ds^2 = -f\left(dv - \frac{1}{f} dr\right)^2 + f^{-1} dr^2 + r^2 d\Omega^2$$

$$= -f dv^2 + 2 dv dr$$


Adopting v and r as coordinates:
$dt = dv - \frac{1}{f} dr$
$ds^2 = -f\left(dv - \frac{1}{f} dr\right)^2 + f^{-1} dr^2 + r^2 d\Omega^2$
$= -f dv^2 + 2 dv dr$
CAUTIO

Adopting v and r as coordinates:

$$dt = dv - \frac{1}{f} dr$$

$$ds^2 = -f\left(dv - \frac{1}{f} dr\right)^2 + f^{-1} dr^2 + r^2 d\Omega^2$$

$$\boxed{ds^2 = -f dv^2 + 2 dv dr + r^2 d\Omega^2}$$

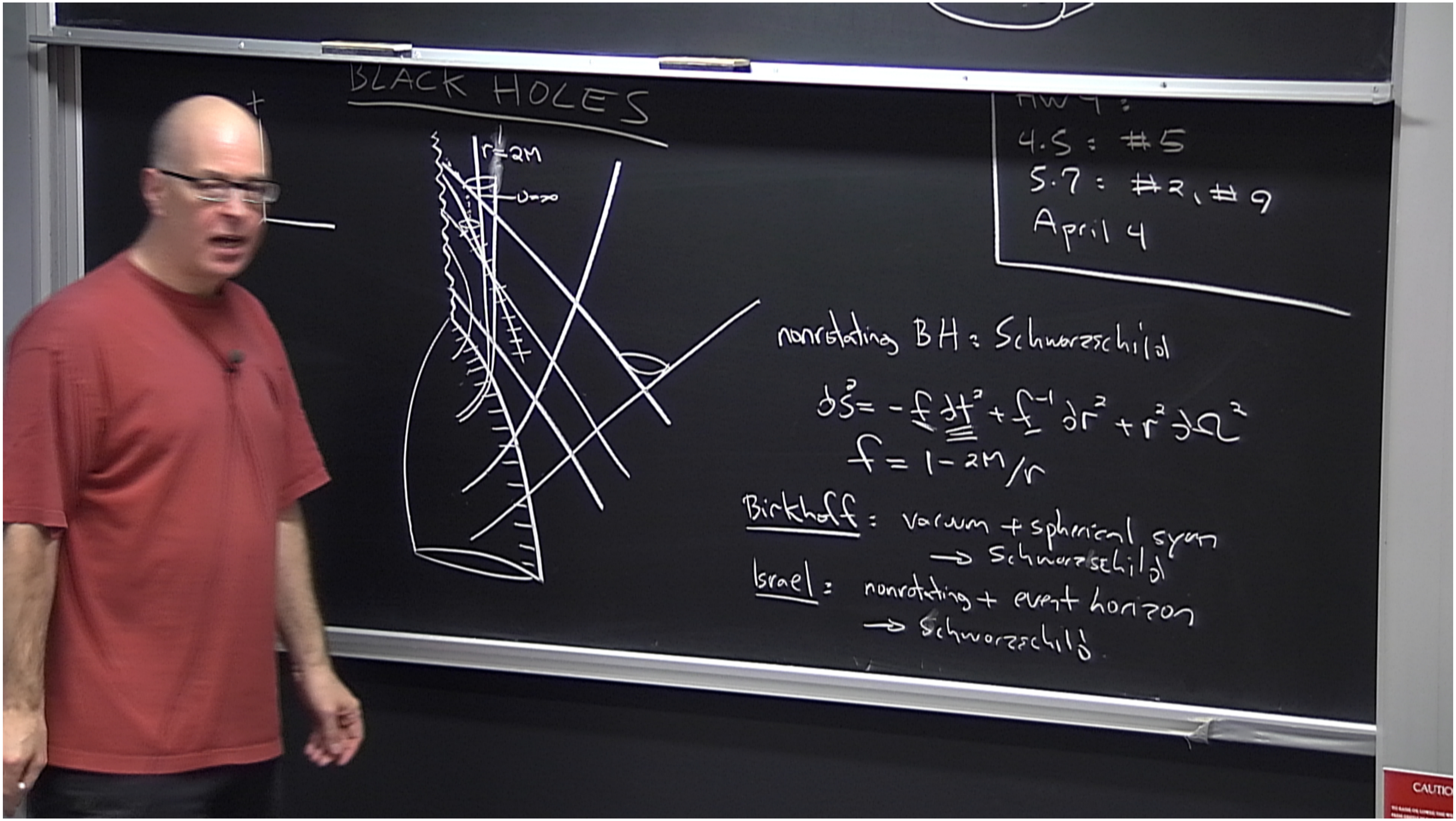
Adopting v and r as coordinates:

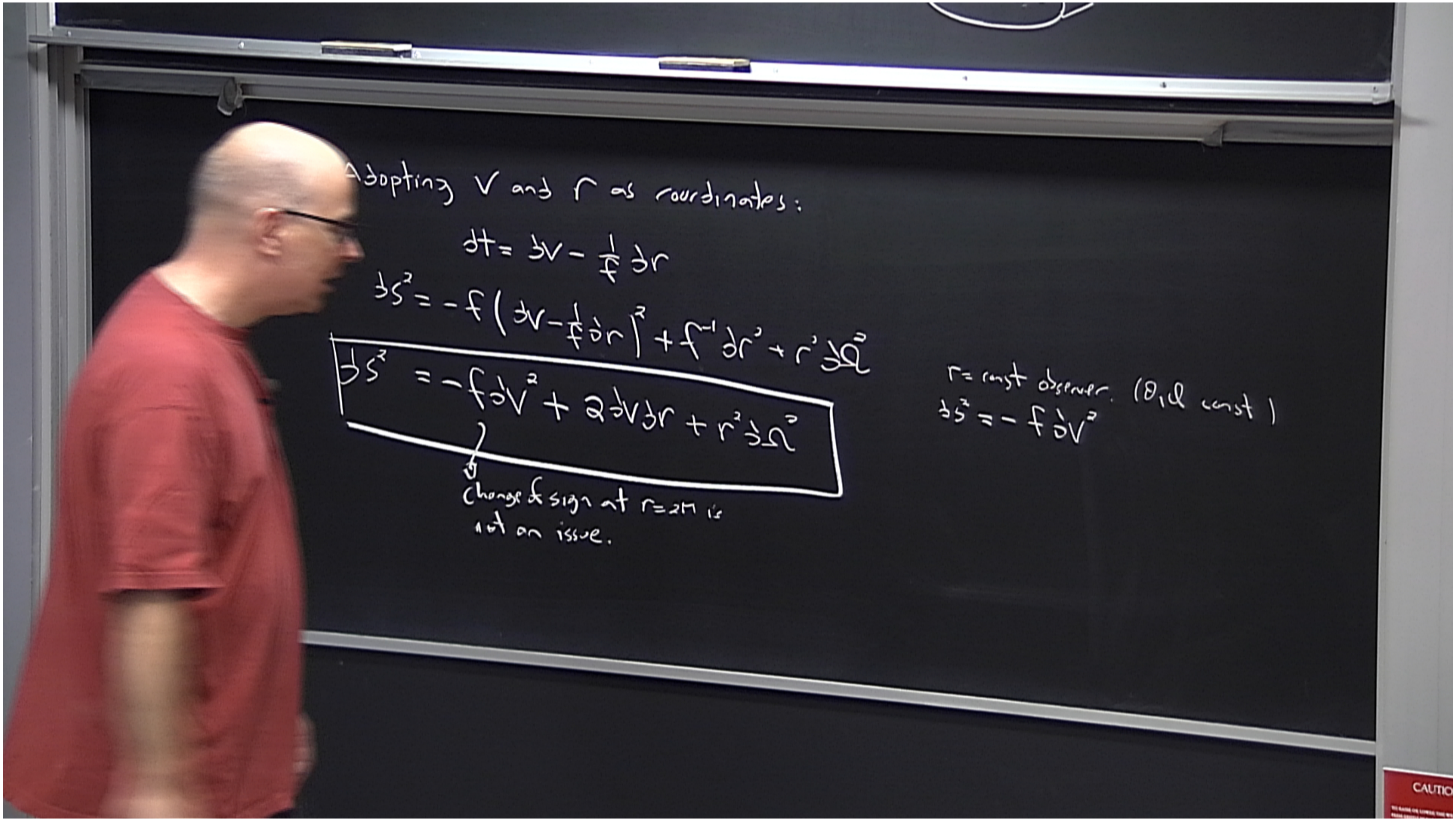
$$dt = dv - \frac{1}{f} dr$$

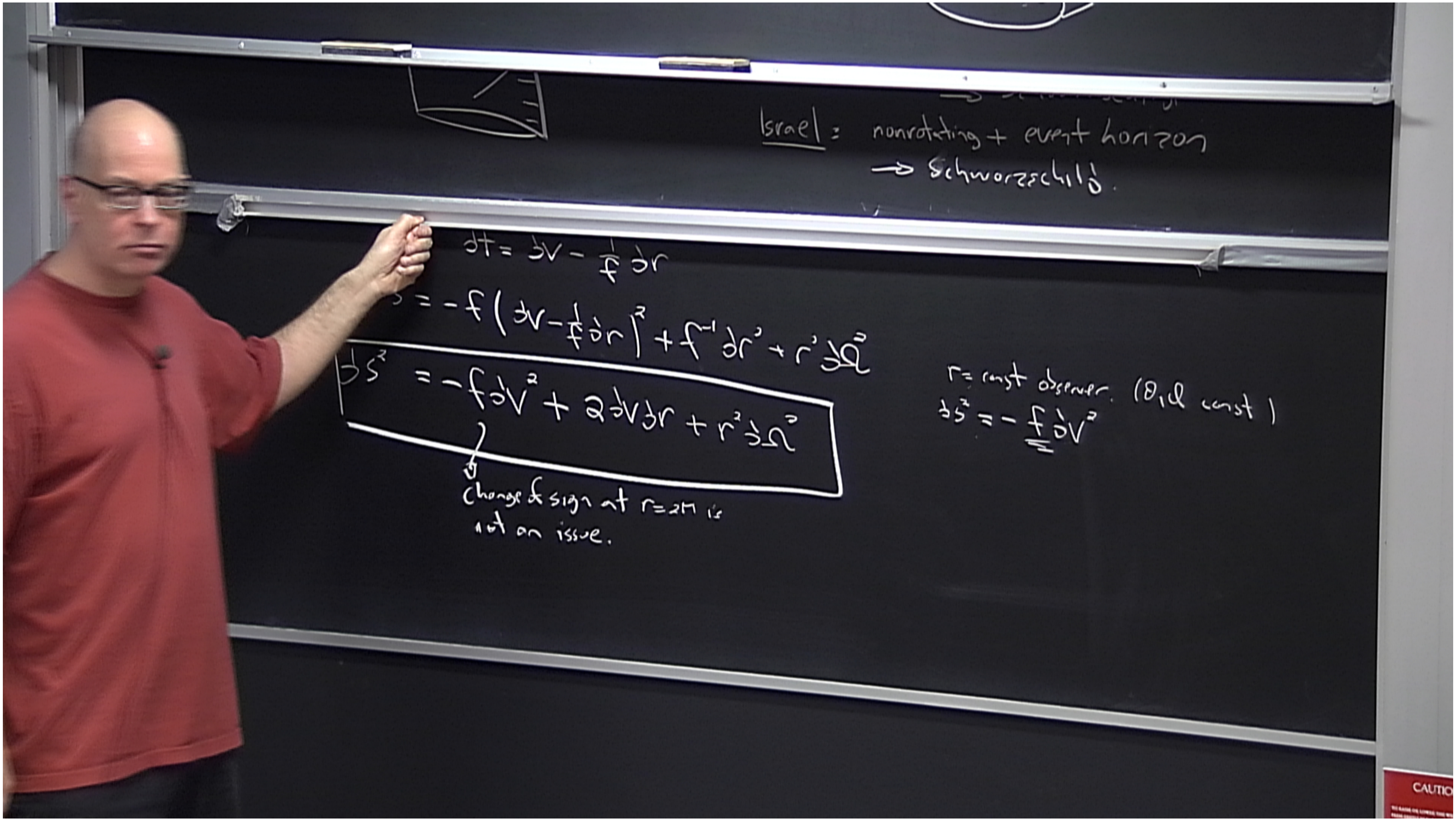
$$ds^2 = -f\left(dv - \frac{1}{f} dr\right)^2 + f^{-1} dr^2 + r^2 d\Omega^2$$

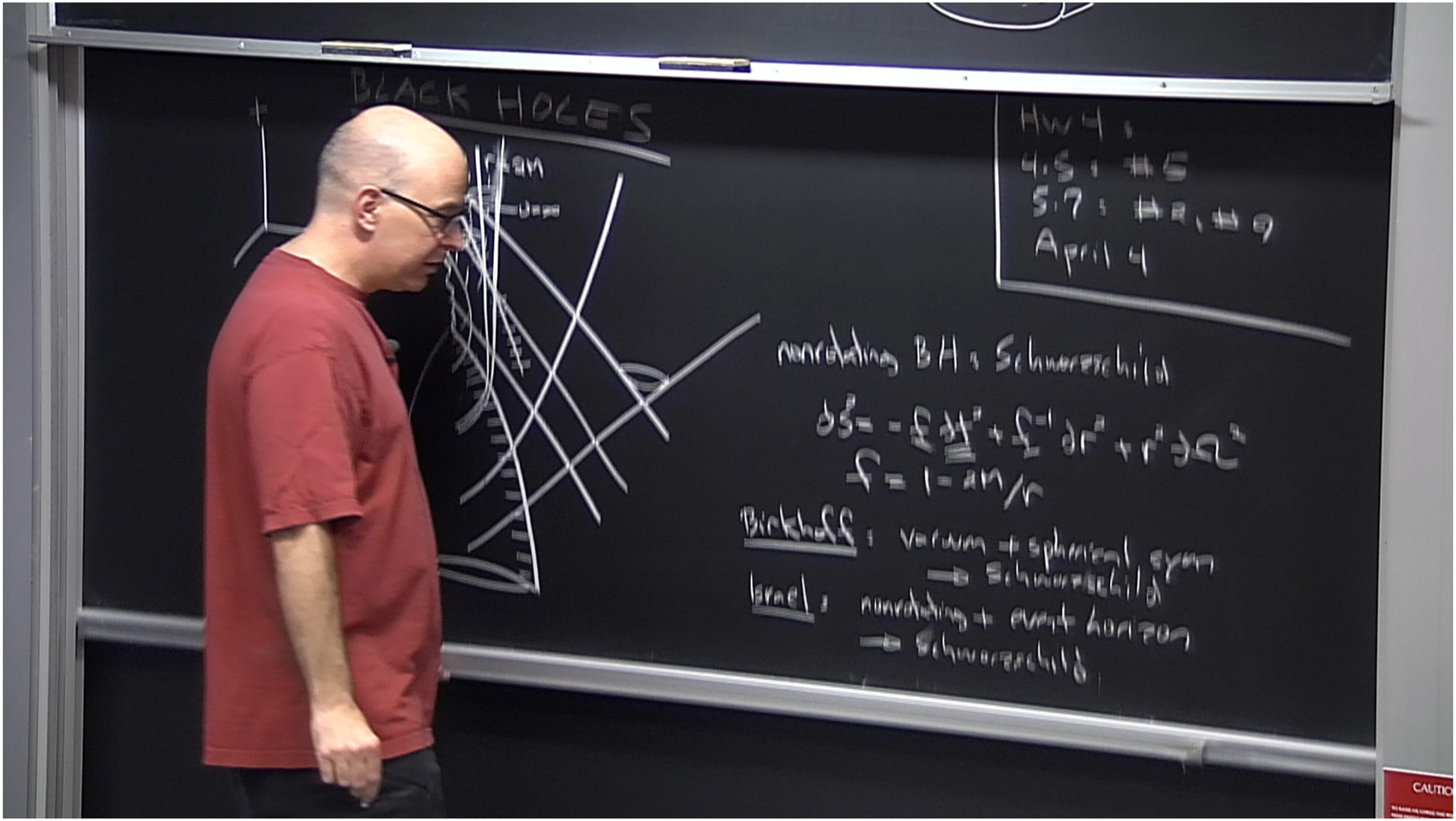
$$\boxed{ds^2 = -f\,dv^2 + 2\,dv\,dr + r^2 d\Omega^2}$$

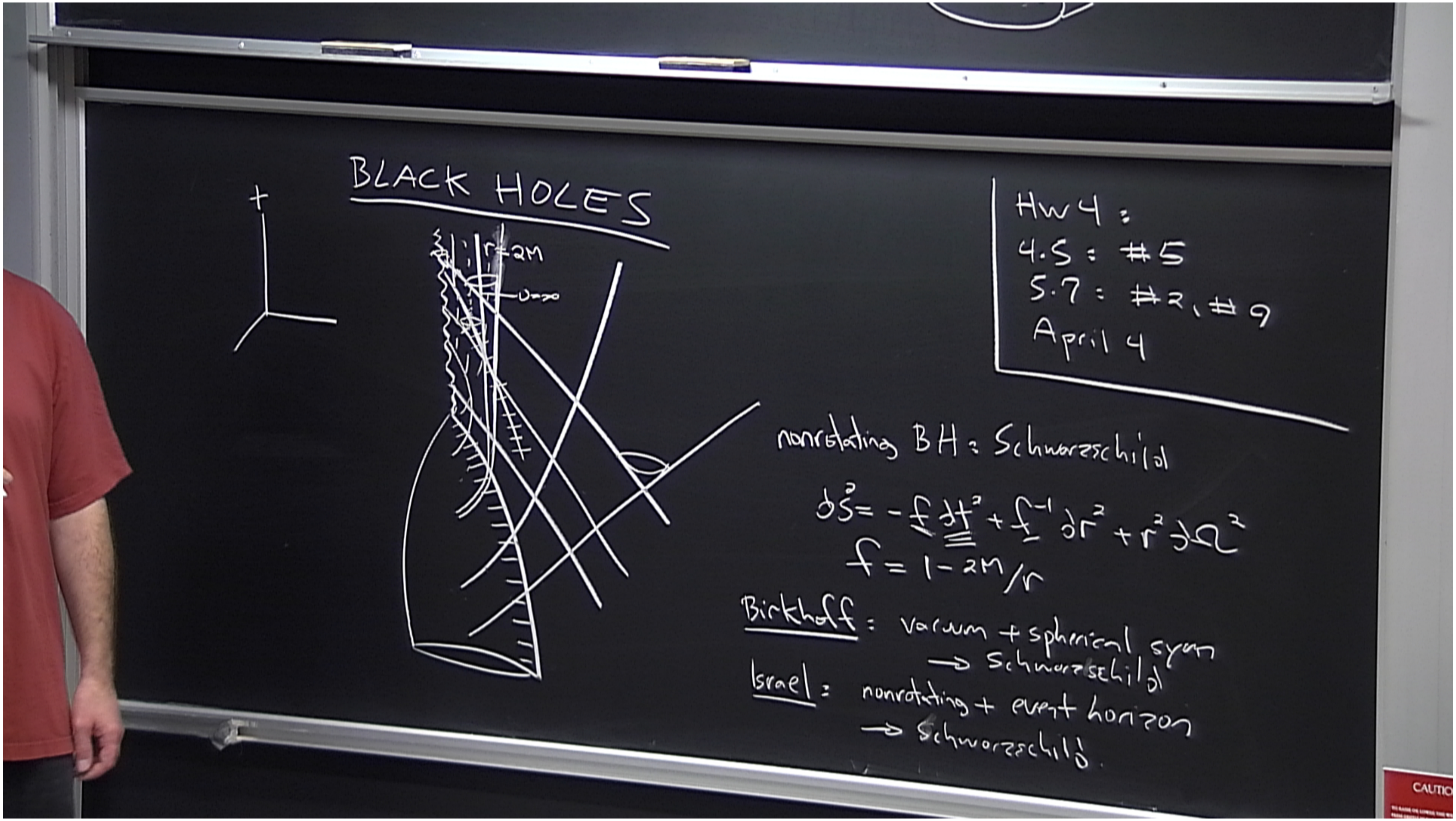
Change of sign at $r = 2M$ is not an issue.

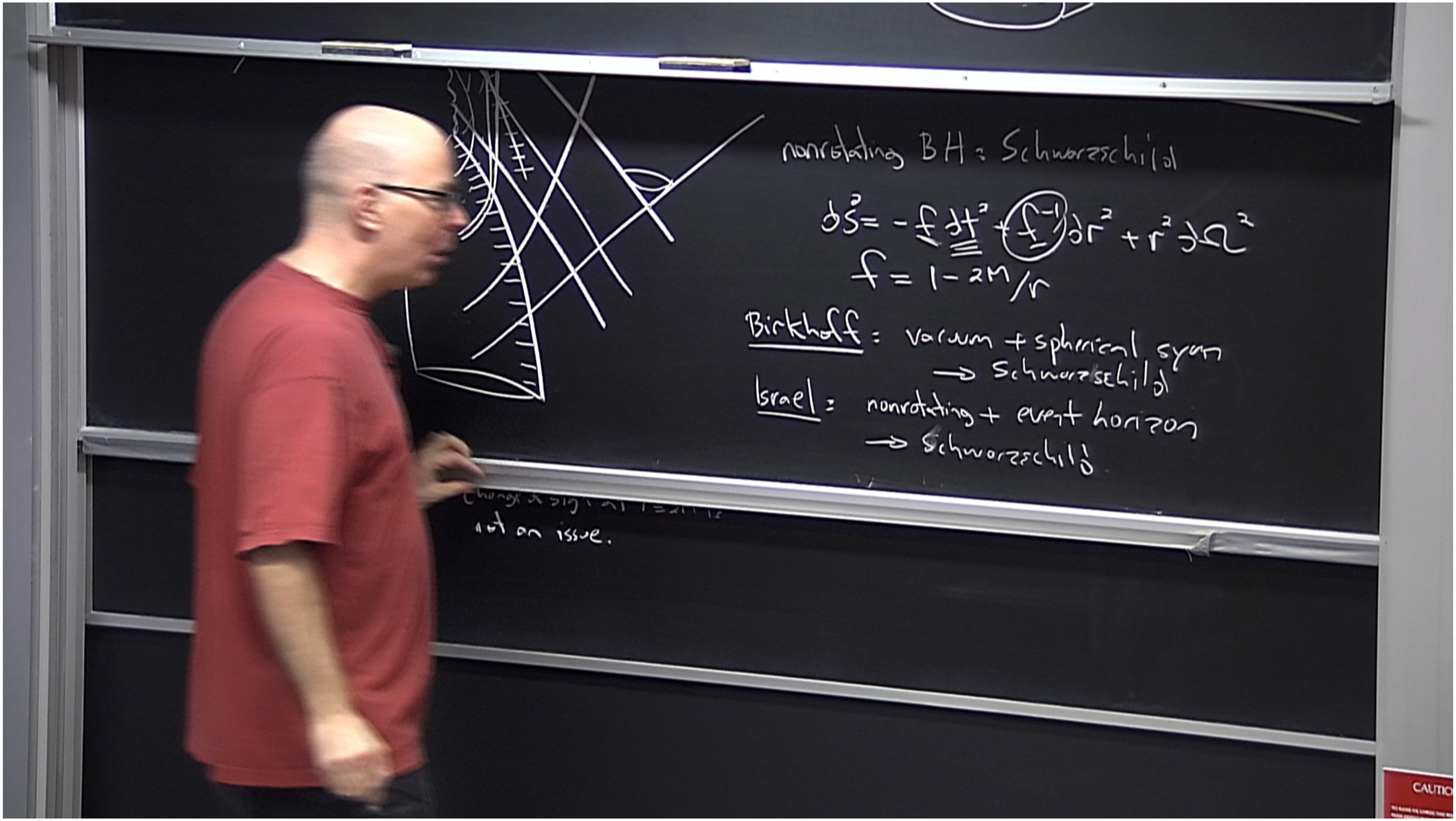

BLACK HOLES
r=2M
HW 4:
4.5: #5
5.7: #2, #9
April 4
nonrotating BH: Schwarzschild
$ds^2 = -f\,dt^2 + f^{-1}dr^2 + r^2 d\Omega^2$
$f = 1 - 2M/r$
Birkhoff: vacuum + spherical sym → Schwarzschild
Israel: nonrotating + event horizon → Schwarzschild
CAUTION


Adopting v and r as coordinates:
$dt = dv - \frac{1}{f} dr$
$ds^2 = -f\left(dv - \frac{1}{f} dr\right)^2 + f^{-1} dr^2 + r^2 d\Omega^2$
$\boxed{ds^2 = -f dv^2 + 2 dv dr + r^2 d\Omega^2}$
Change of sign at $r=2M$ is not an issue.
$r=$ const observer. (θ, φ const)
$ds^2 = -f dv^2$


Israel: nonrotating + event horizon
→ Schwarzschild.
dt = dv − 1/f dr
= −f (dv − 1/f dr)² + f⁻¹ dr² + r² dΩ²
ds² = −f dv² + 2 dv dr + r² dΩ²
Change of sign at r=2M is not an issue.
r=const observer. (θ, φ const)
ds² = −f dv²
CAUTION


BLACK HOLES
HW 4:
4.5: #5
5.7: #2, #9
April 4
nonrotating BH: Schwarzschild
ds² = -f dt² + f⁻¹ dr² + r² dΩ²
f = 1 - 2M/r
Birkhoff: vacuum + spherical symm
⇒ Schwarzschild
Israel: nonrotating + event horizon
⇒ Schwarzschild


BLACK HOLES
t
r=2M
v=∞
Hw 4:
4.5: #5
5.7: #2, #9
April 4
nonrotating BH = Schwarzschild
ds² = −f dt² + f⁻¹ dr² + r² dΩ²
f = 1 − 2M/r
Birkhoff: vacuum + spherical sym → Schwarzschild
Israel: nonrotating + event horizon → Schwarzschild.


nonrotating BH = Schwarzschild
$ds^2 = -f\,dt^2 + f^{-1}dr^2 + r^2 d\Omega^2$
$f = 1 - 2M/r$
Birkhoff: vacuum + spherical sym
→ Schwarzschild
Israel: nonrotating + event horizon
→ Schwarzschild
not an issue.

→ Schwarzschild

Adopting v and r as coordinates:

$$dt = dv - \frac{1}{f}\,dr$$

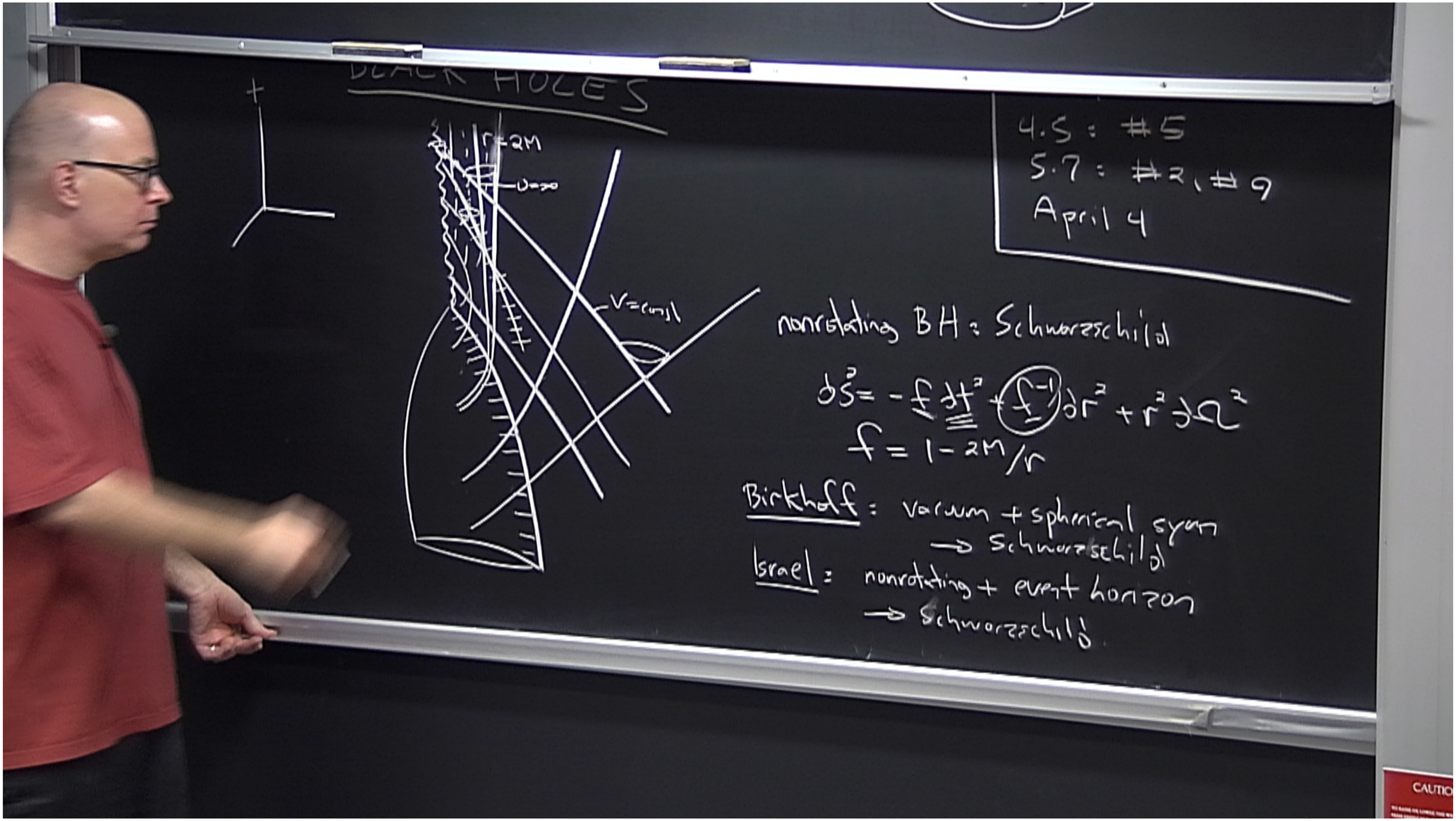
$$ds^2 = -f\left(dv - \frac{1}{f}\,dr\right)^2 + f^{-1}dr^2 + r^2 d\Omega^2$$

$$\boxed{ds^2 = -f\,dv^2 + 2\,dv\,dr + r^2 d\Omega^2}$$

Change of sign at $r=2M$ is not an issue.

$r=$ const observer. (θ, φ const)

$$ds^2 = -\underline{\underline{f}}\,dv^2$$


BLACK HOLES
r=2M
v=∞
v=const
4.5 : #5
5.7 : #2, #9
April 4
nonrotating BH = Schwarzschild
$ds^2 = -f\,dt^2 + f^{-1}dr^2 + r^2 d\Omega^2$
$f = 1 - 2M/r$
Birkhoff : vacuum + spherical sym → Schwarzschild
Israel : nonrotating + event horizon → Schwarzschild

→ Schwarzschild.

Adopting v and r as coordinates:

$$\boxed{dt = dv - \frac{1}{f}dr} \qquad t = v - r - 2M\ln(r/2M - 1)$$

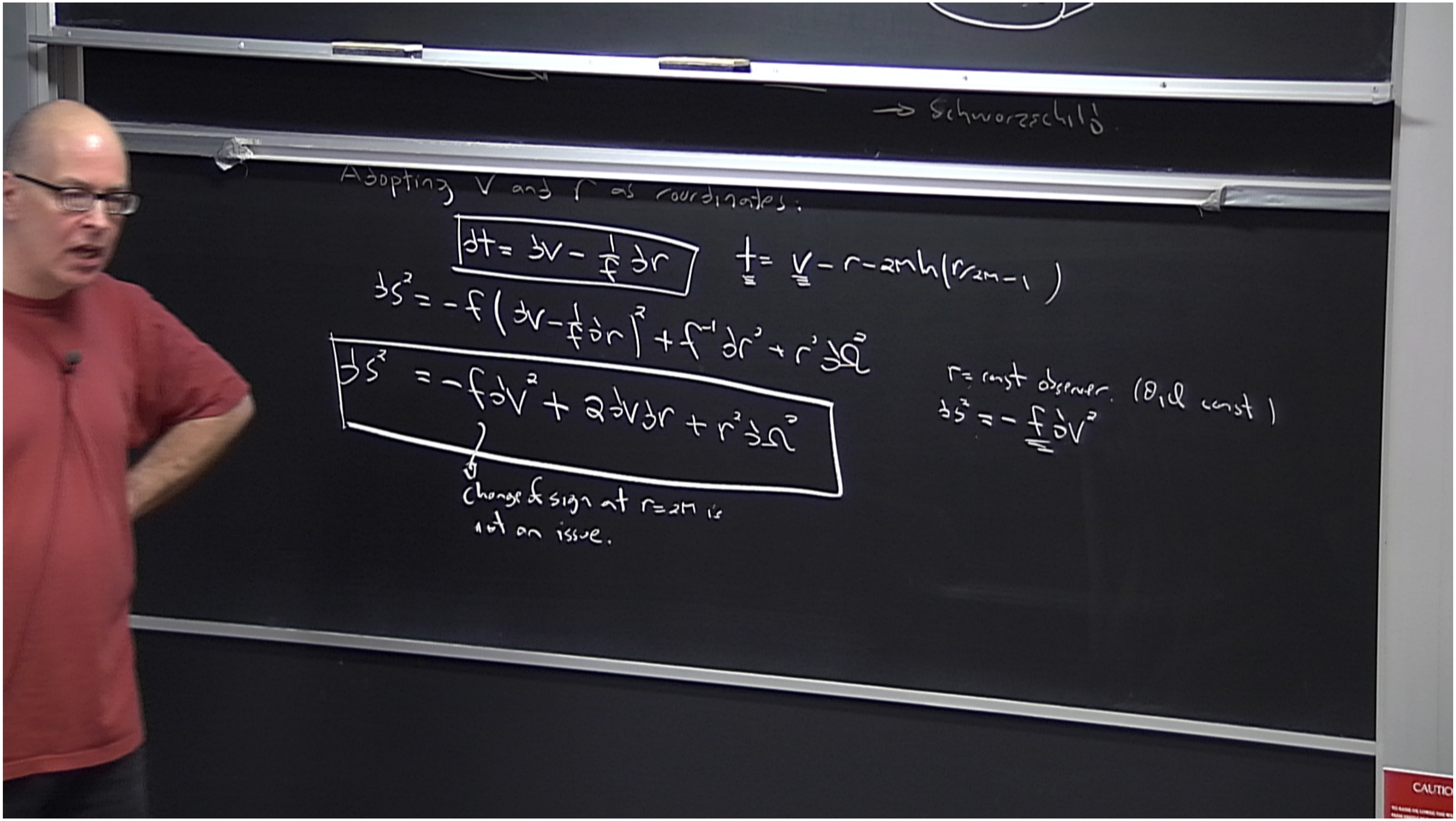
$$ds^2 = -f\left(dv - \frac{1}{f}dr\right)^2 + f^{-1}dr^2 + r^2 d\Omega^2$$

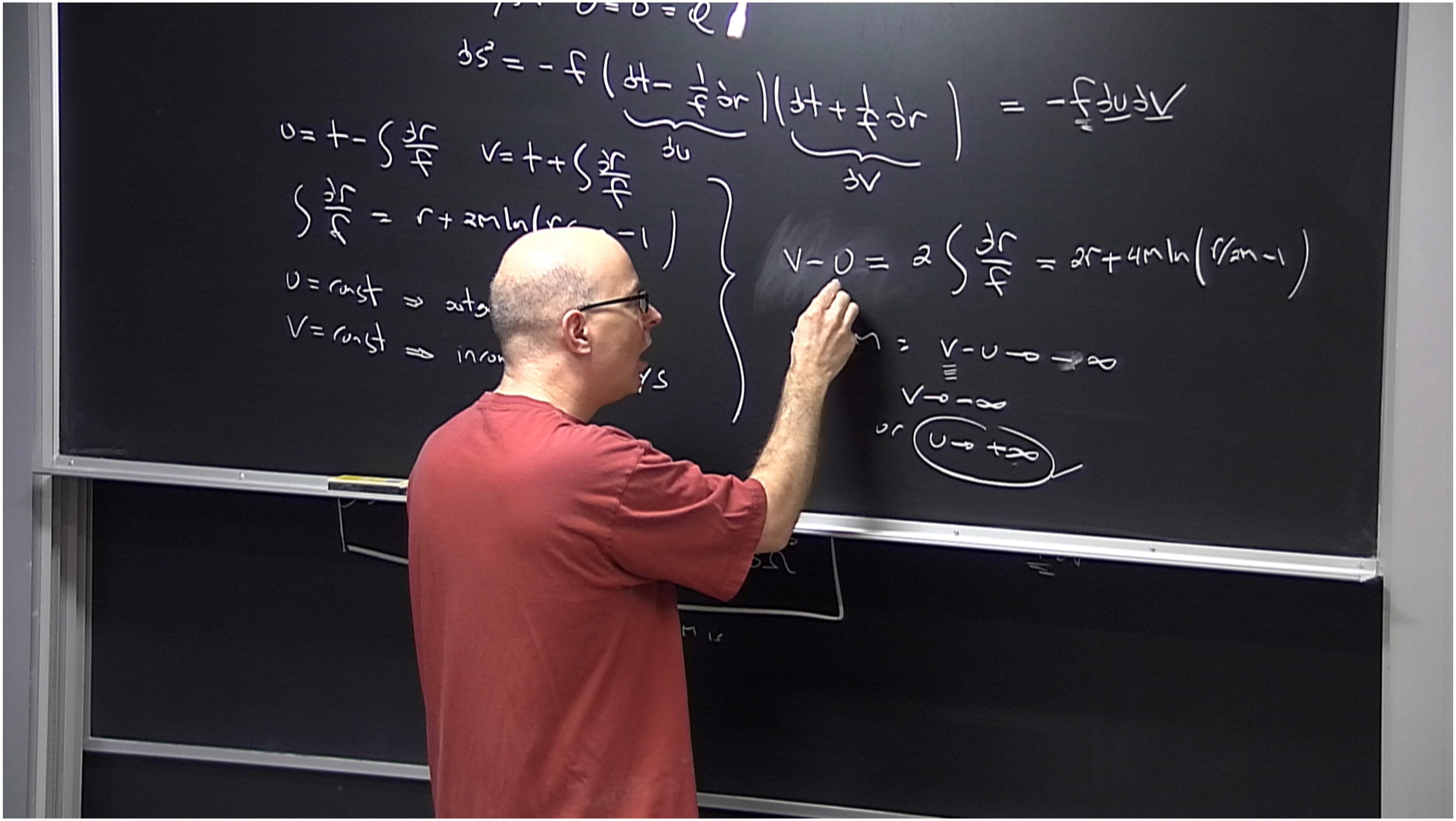
$$\boxed{ds^2 = -f\,dv^2 + 2\,dv\,dr + r^2 d\Omega^2}$$

Change of sign at $r=2M$ is not an issue.

$r=$ const observer. (θ,φ const)

$$ds^2 = -f\,dv^2$$

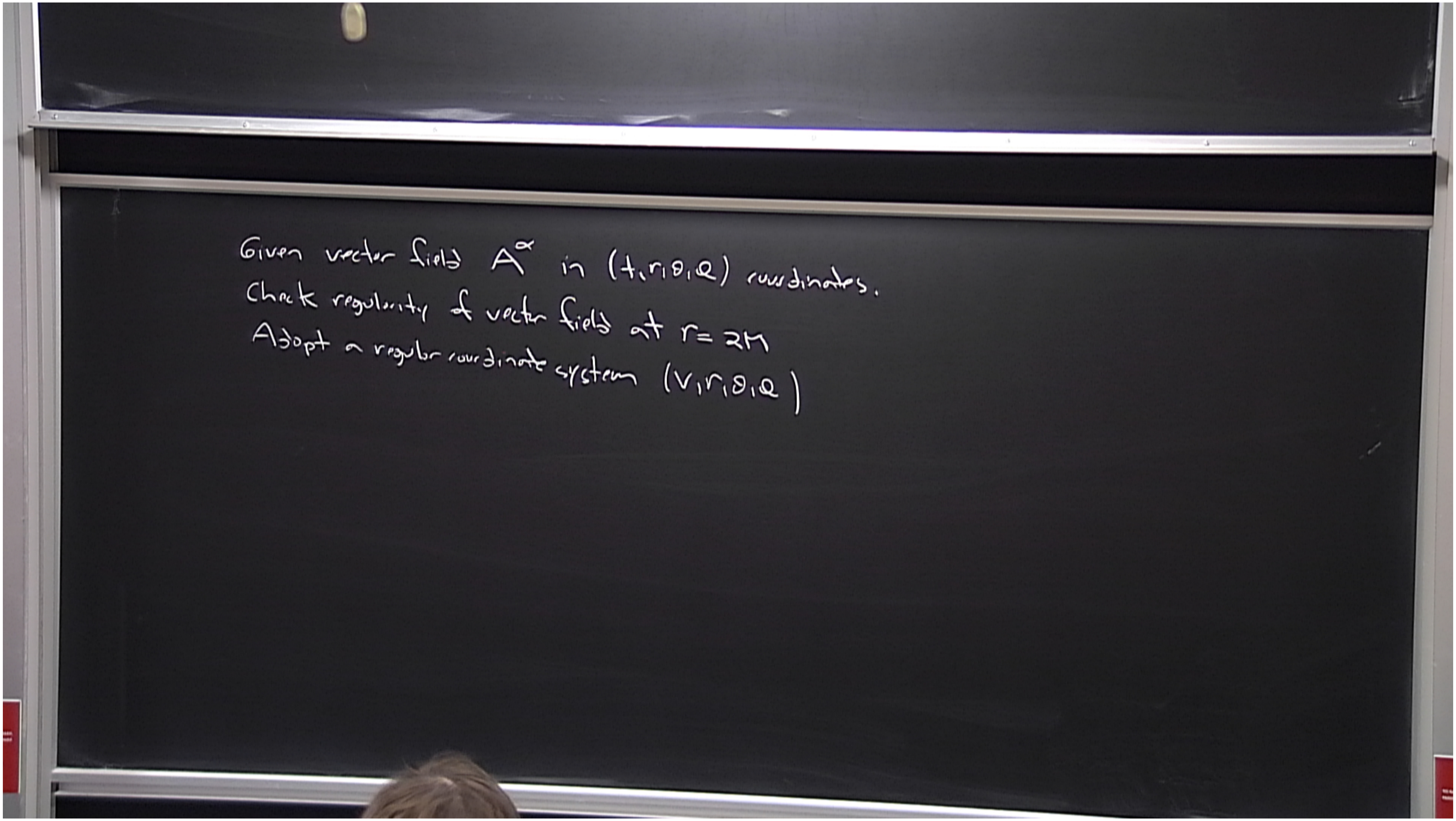

→ Schwarzschild
Adopting v and r as coordinates:
$dt = dv - \frac{1}{f} dr$
$t = v - r - 2M \ln(r/2M - 1)$
$ds^2 = -f\left(dv - \frac{1}{f} dr\right)^2 + f^{-1} dr^2 + r^2 d\Omega^2$
$ds^2 = -f dv^2 + 2 dv dr + r^2 d\Omega^2$
Change of sign at r=2M is not an issue.
r = const observer. (θ, φ const)
$ds^2 = -f dv^2$

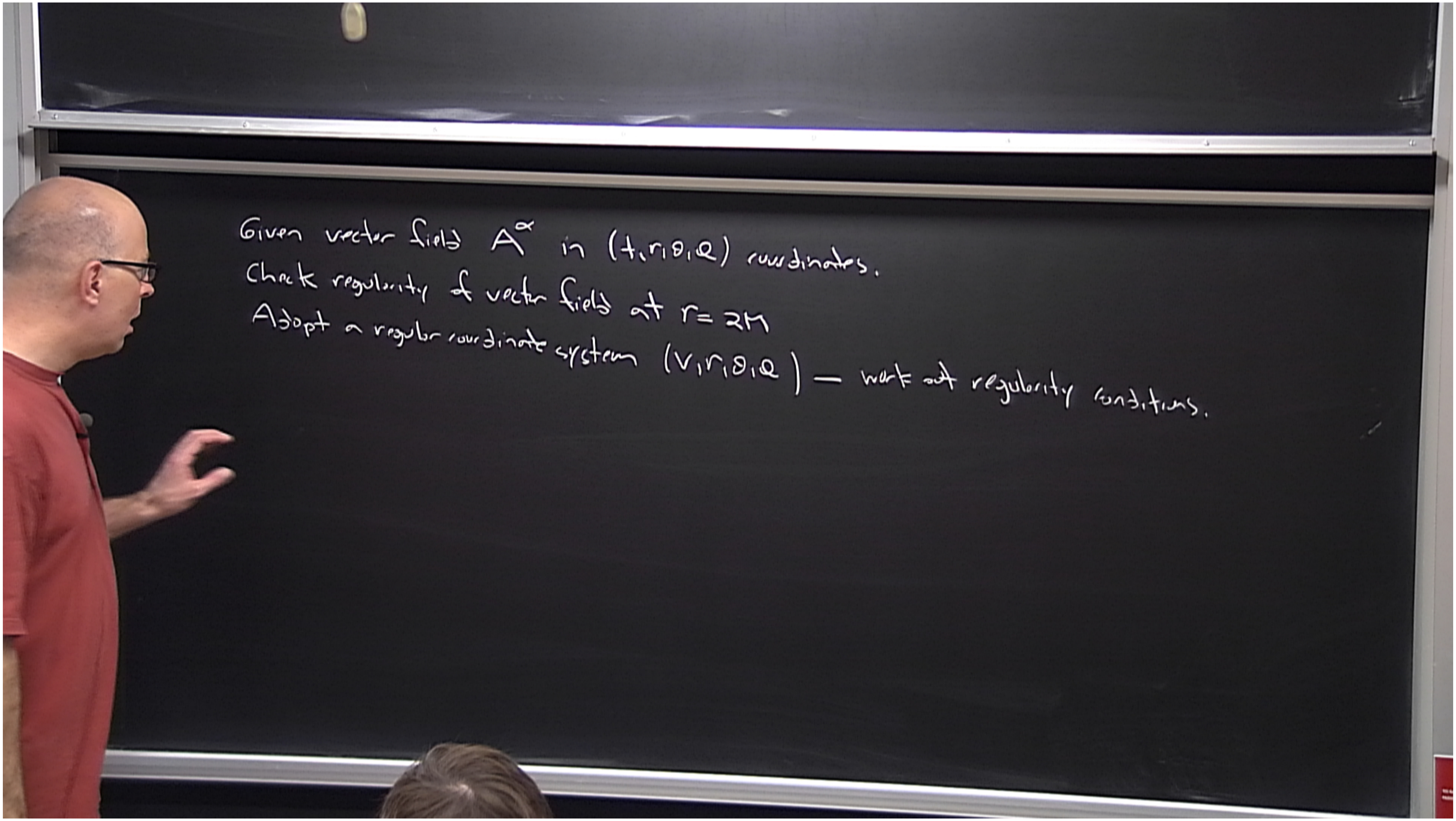


$ds^2 = -f\left(dt - \frac{1}{f}dr\right)\left(dt + \frac{1}{f}dr\right) = -f\,du\,dv$
$u = t - \int \frac{dr}{f}$
$v = t + \int \frac{dr}{f}$
$u = \text{const}$
$v = \text{const}$
$v - u = 2\int \frac{dr}{f} = 2r + 4m\ln(r/2m - 1)$
$v \to -\infty$
or $u \to +\infty$

Given vector field A^α in (t, r, θ, φ) coordinates.

Given vector field A^α in (t, r, θ, φ) coordinates.
Check regularity of vector field at $r = 2M$


Given vector field A^α in (t,r,θ,φ) coordinates.
Check regularity of vector field at $r=2M$
Adopt a regular coordinate system (v,r,θ,φ)


Given vector field A^α in (t,r,θ,φ) coordinates.
Check regularity of vector field at $r=2M$
Adopt a regular coordinate system (v,r,θ,φ) — work out regularity conditions.

Given vector field A^α in (t, r, θ, φ) coordinates.

Check regularity of vector field at $r = 2M$

Adopt a regular coordinate system (v, r, θ, φ) — work out regularity conditions.

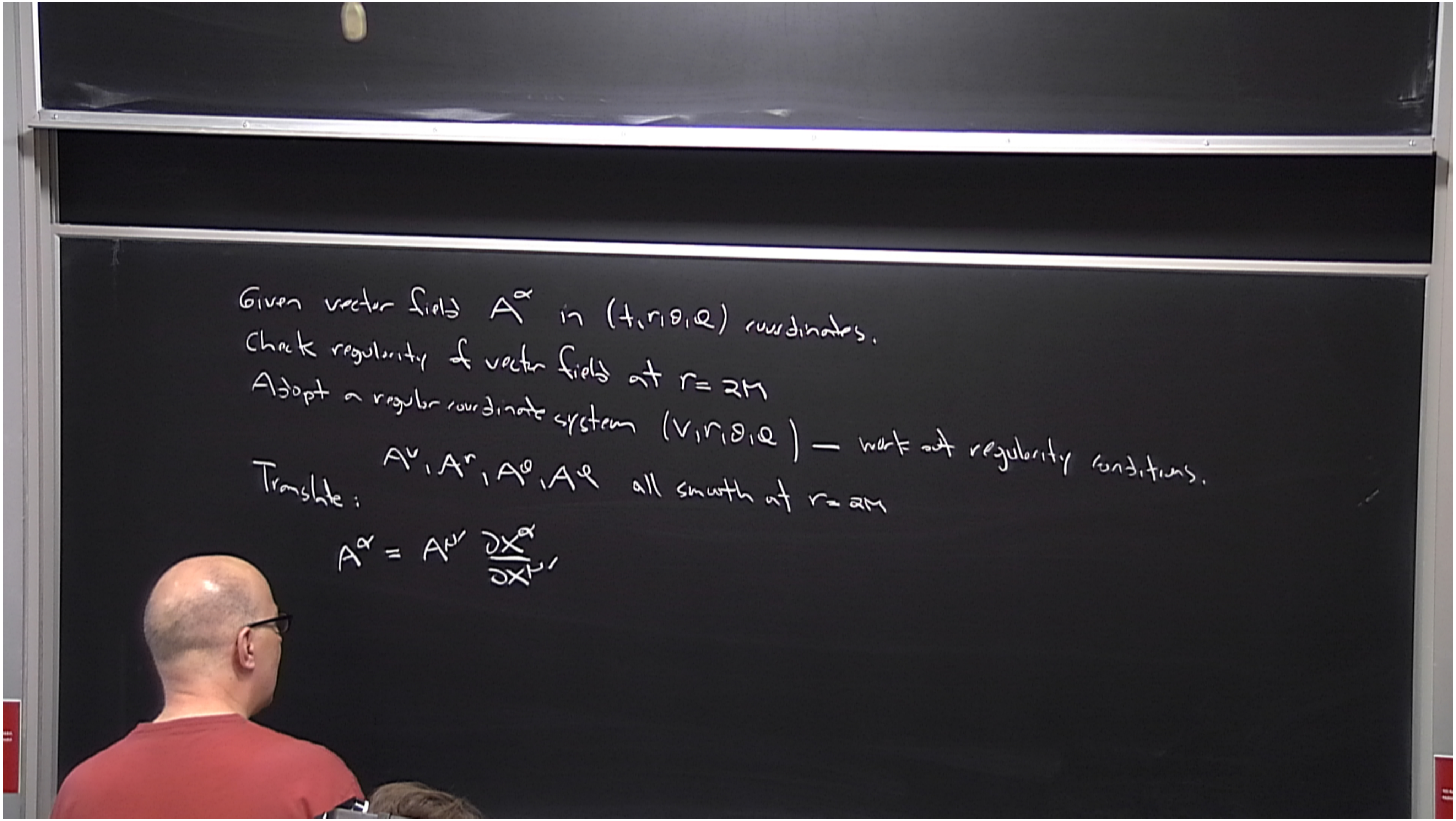
$A^v, A^r, A^\theta, A^\varphi$

Given vector field A^α in (t,r,θ,φ) coordinates.

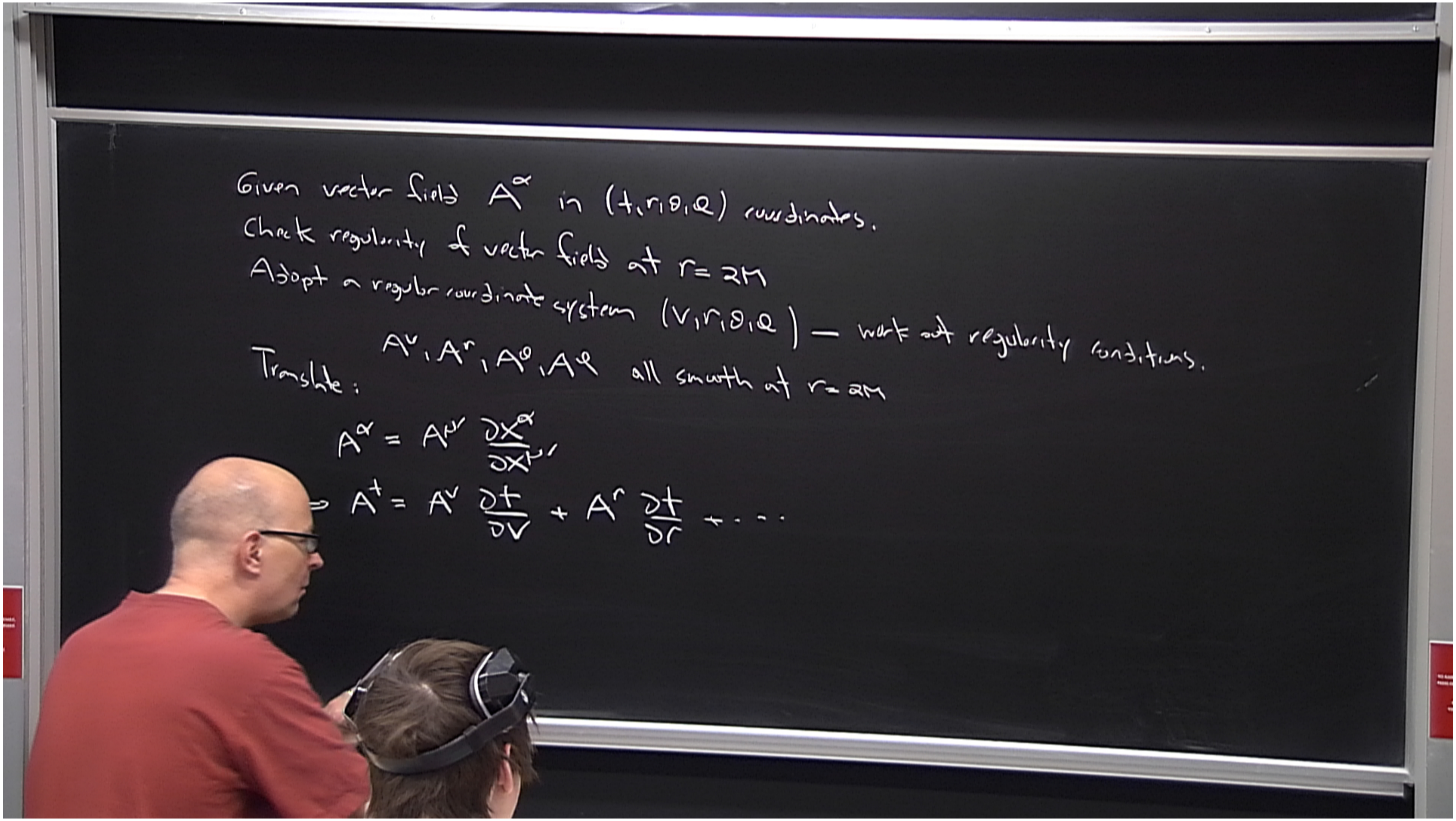
Check regularity of vector field at $r=2M$

Adopt a regular coordinate system (v,r,θ,φ) — work out regularity conditions.

$A^v, A^r, A^\theta, A^\varphi$ all smooth at $r=2M$



Given vector field A^α in (t,r,θ,φ) coordinates.
Check regularity of vector field at $r=2M$
Adopt a regular coordinate system (v,r,θ,φ) — work out regularity conditions.
Translate: $A^v, A^r, A^\theta, A^\varphi$ all smooth at $r=2M$
$A^\alpha = A^{\mu'} \frac{\partial x^\alpha}{\partial x^{\mu'}}$


Given vector field A^α in (t,r,θ,φ) coordinates.
Check regularity of vector field at $r=2M$
Adopt a regular coordinate system (v,r,θ,φ) — work out regularity conditions.
Translate: $A^v, A^r, A^\theta, A^\varphi$ all smooth at $r=2M$
$A^\alpha = A^{\mu'} \frac{\partial x^\alpha}{\partial x^{\mu'}}$
$\Rightarrow A^t = A^v \frac{\partial t}{\partial v} + A^r \frac{\partial t}{\partial r} + \dots$

Given vector field A^α in (t, r, θ, φ) coordinates.

Check regularity of vector field at $r = 2M$

Adopt a regular coordinate system (v, r, θ, φ) — work out regularity conditions.

Translate: $A^v, A^r, A^\theta, A^\varphi$ all smooth at $r = 2M$

$$A^\alpha = A^{\mu'} \frac{\partial x^\alpha}{\partial x^{\mu'}}$$

$$\Rightarrow A^t = A^v \frac{\partial t}{\partial v} + A^r \frac{\partial t}{\partial r}$$

Given vector field A^α in (t, r, θ, φ) coordinates.

Check regularity of vector field at $r = 2M$

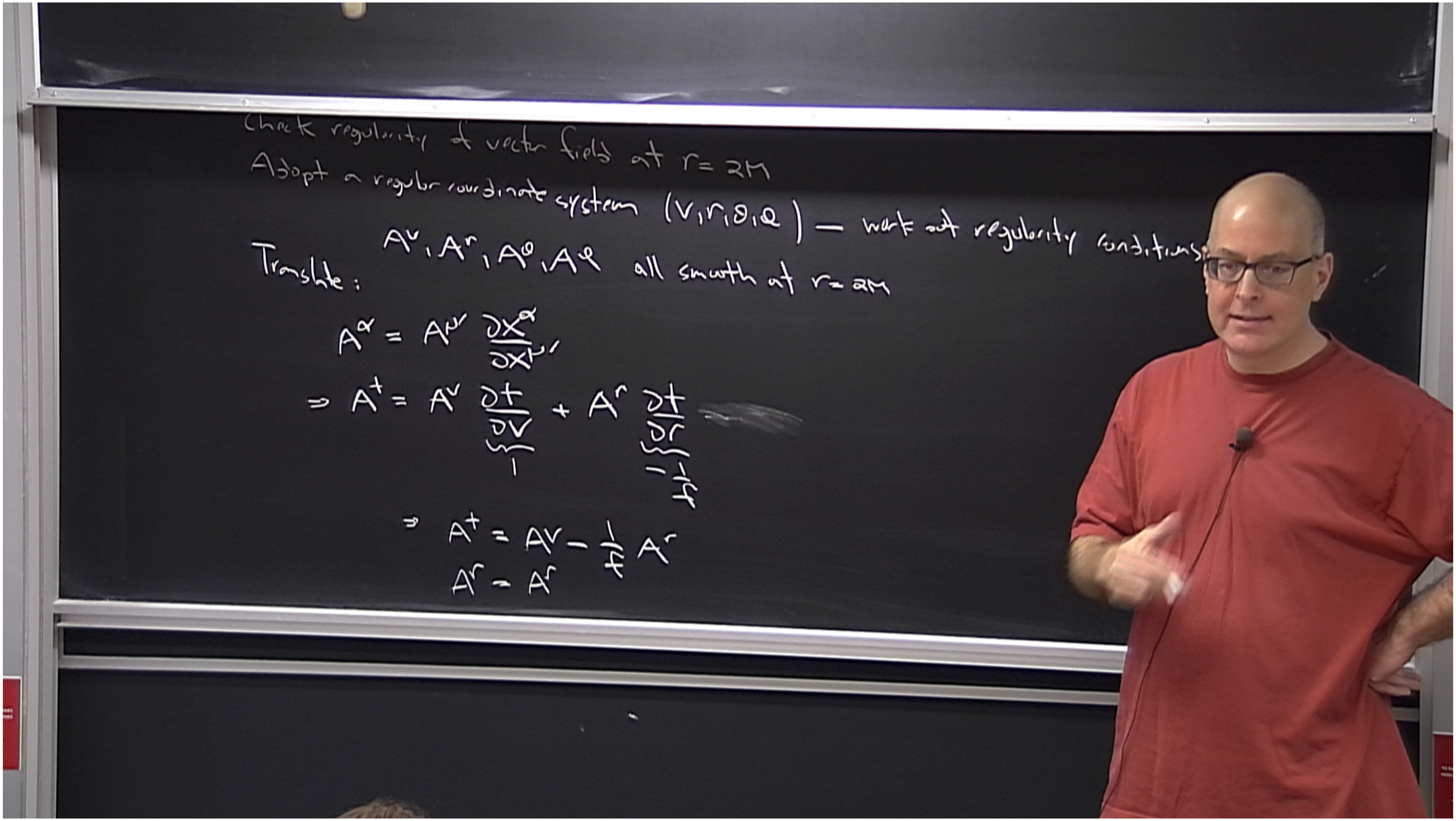
Adopt a regular coordinate system (v, r, θ, φ) — work out regularity conditions.

Translate: $A^v, A^r, A^\theta, A^\varphi$ all smooth at $r = 2M$

$$A^\alpha = A^{\mu'} \frac{\partial x^\alpha}{\partial x^{\mu'}}$$

$$\Rightarrow A^t = A^v \underbrace{\frac{\partial t}{\partial v}}_{1} + A^r \underbrace{\frac{\partial t}{\partial r}}_{-\frac{1}{f}}$$

$$\Rightarrow A^t = A^v - \frac{1}{f} A^r$$


Check regularity of vector fields at r = 2M
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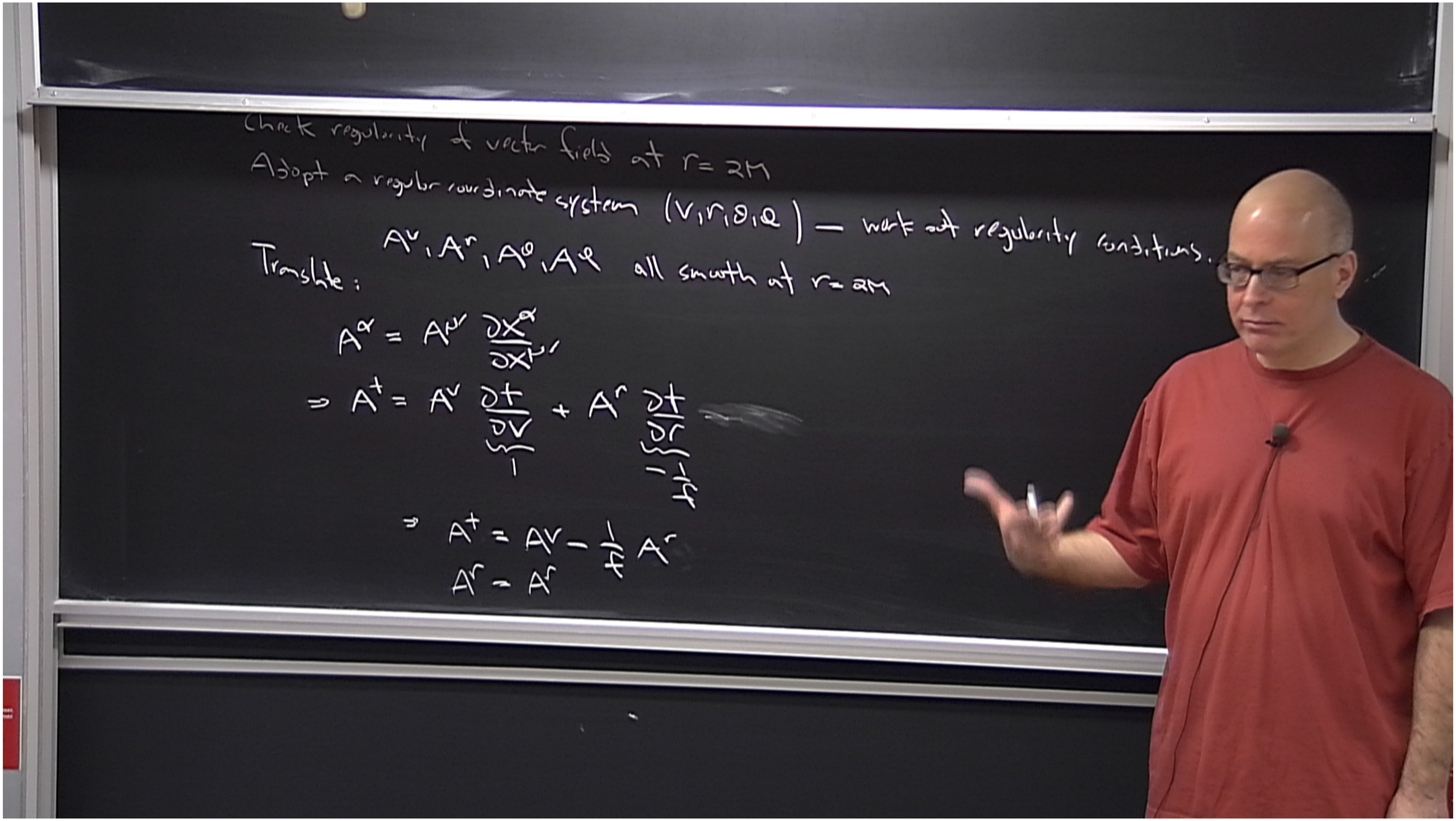
Translate: A^v, A^r, A^θ, A^φ all smooth at r = 2M
$A^\alpha = A^{\mu'} \frac{\partial x^\alpha}{\partial x^{\mu'}}$
$\Rightarrow A^t = A^v \underbrace{\frac{\partial t}{\partial v}}_{1} + A^r \underbrace{\frac{\partial t}{\partial r}}_{-\frac{1}{f}}$
$\Rightarrow A^t = A^v - \frac{1}{f} A^r$
$A^r = A^r$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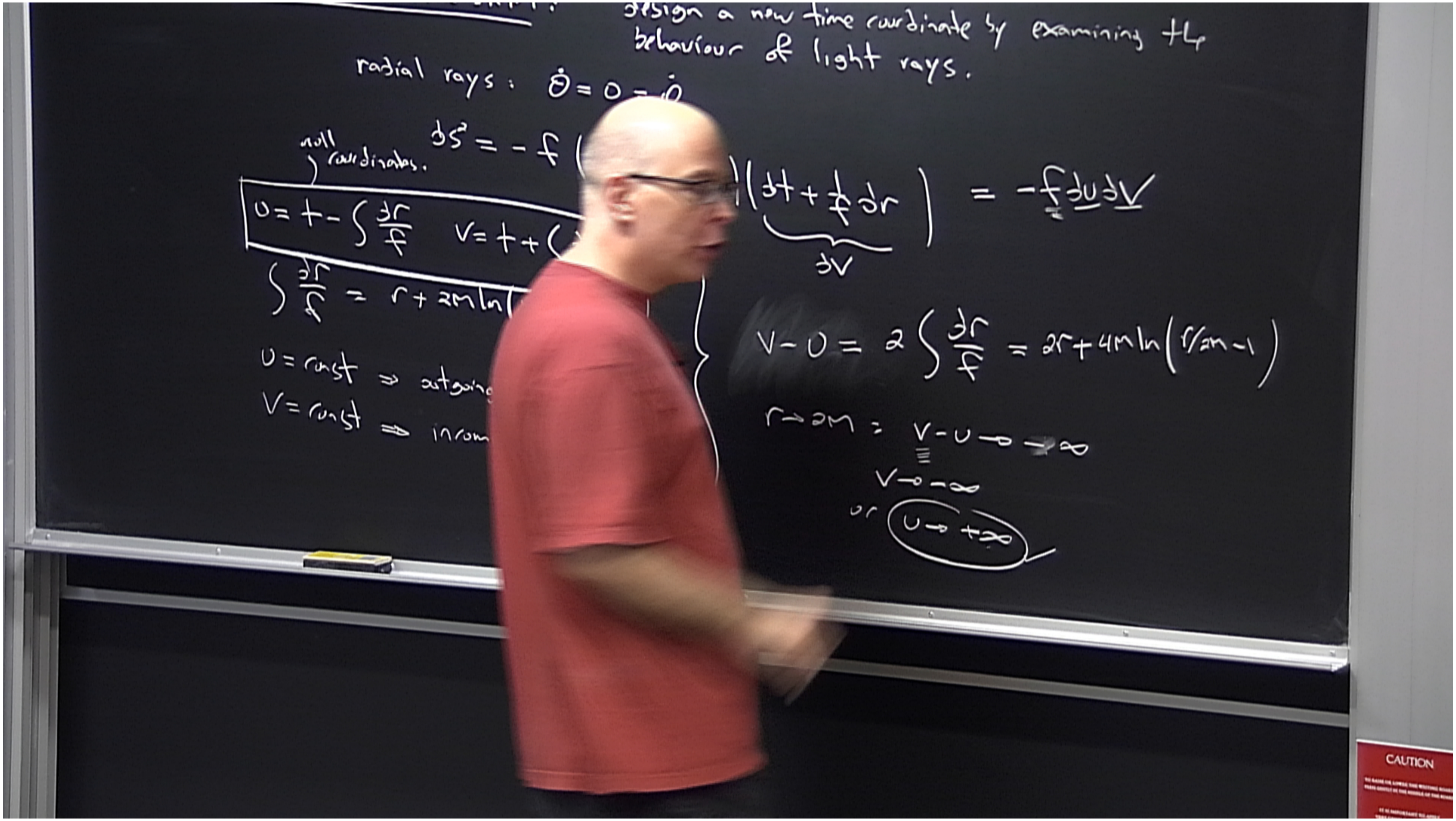
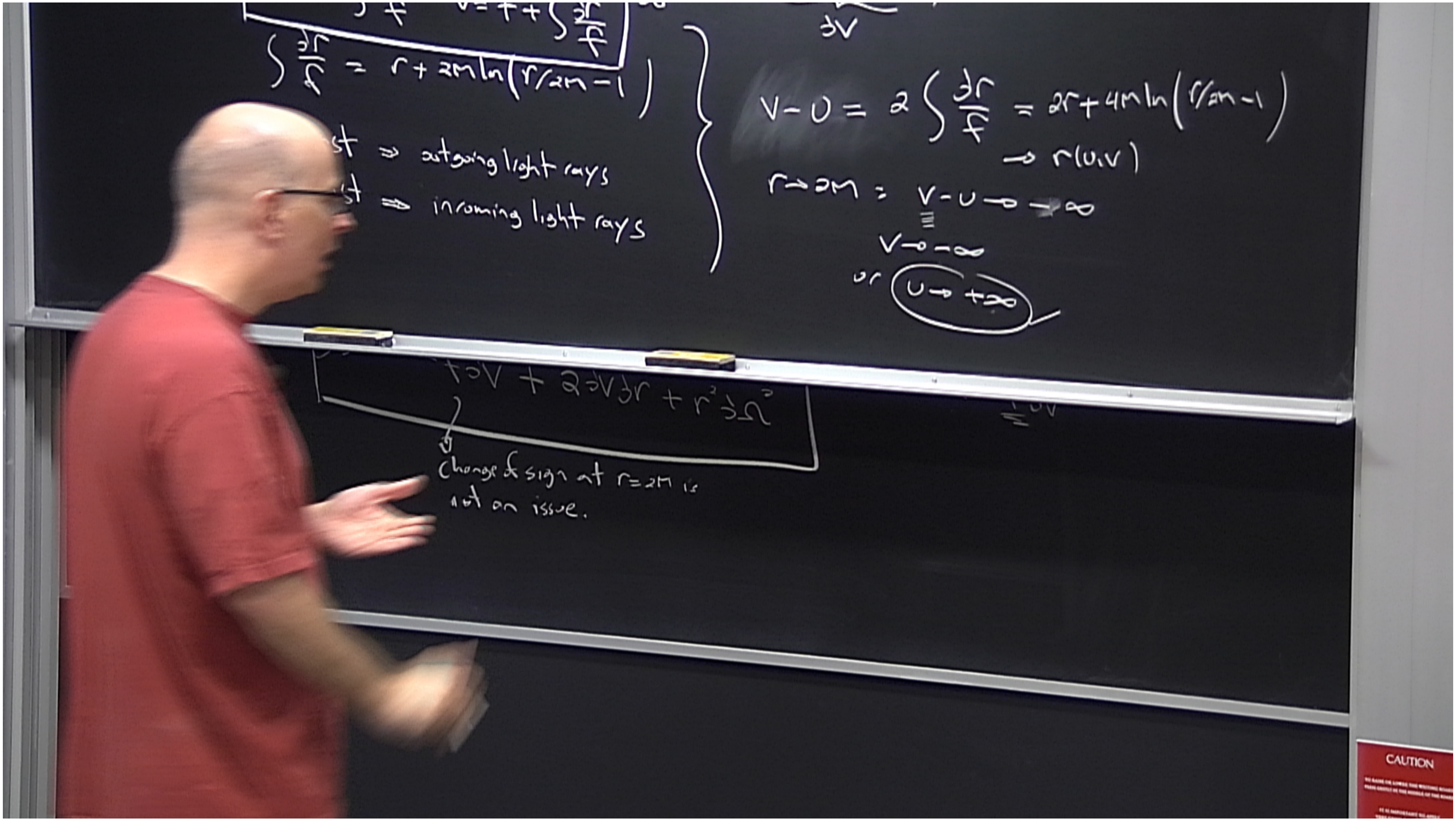

design a new time coordinate by examining the
behaviour of light rays.
radial rays: $\dot\theta = 0 = \dot\phi$
null coordinates.
$U = t - \int \frac{dr}{f}$
$V = t + ($
$\int \frac{dr}{f} = r + 2m\ln($
$U = \text{const} \Rightarrow$ outgoing
$V = \text{const} \Rightarrow$ incom
$= -f\,dU\,dV$
$V - U = 2\int \frac{dr}{f} = 2r + 4m\ln(r/2m - 1)$
$r \to 2m$: $V - U \to -\infty$
$V \to -\infty$
or $U \to +\infty$
CAUTION


$\int \frac{dr}{f} = r + 2M\ln(r/2M - 1)$
st ⇒ outgoing light rays
st ⇒ incoming light rays
$V - U = 2\int \frac{dr}{f} = 2r + 4M\ln(r/2M - 1)$
$\to r(U,V)$
$r \to 2M : V - U \to -\infty$
$V \to -\infty$
or $U \to +\infty$
Change of sign at $r=2M$ is not an issue.
CAUTION

$$\Rightarrow A^{+} = A^{v} - \frac{1}{f} A^{r}$$
$$A^{r} = A^{r}$$

Kruskal coordinates

Coordinate redefs: $U = U(u)$
$V = V(v)$

$$\Rightarrow \quad A^{+} = A^{v} - \frac{1}{f} A^{r}$$

$$A^{r} = A^{r}$$

Kruskal coordinates

Coordinate redefs : $U = U(u)$

$V = V(v)$

$$U = - e^{-\kappa u}$$

$$\Rightarrow \quad A^{+} = A^{v} - \frac{1}{f} A^{r}$$
$$A^{r} = A^{r}$$

Kruskal coordinates

$$U = -e^{-\kappa u}$$
$$V =$$

Coordinate relabels: $U = U(u)$, $V = V(v)$

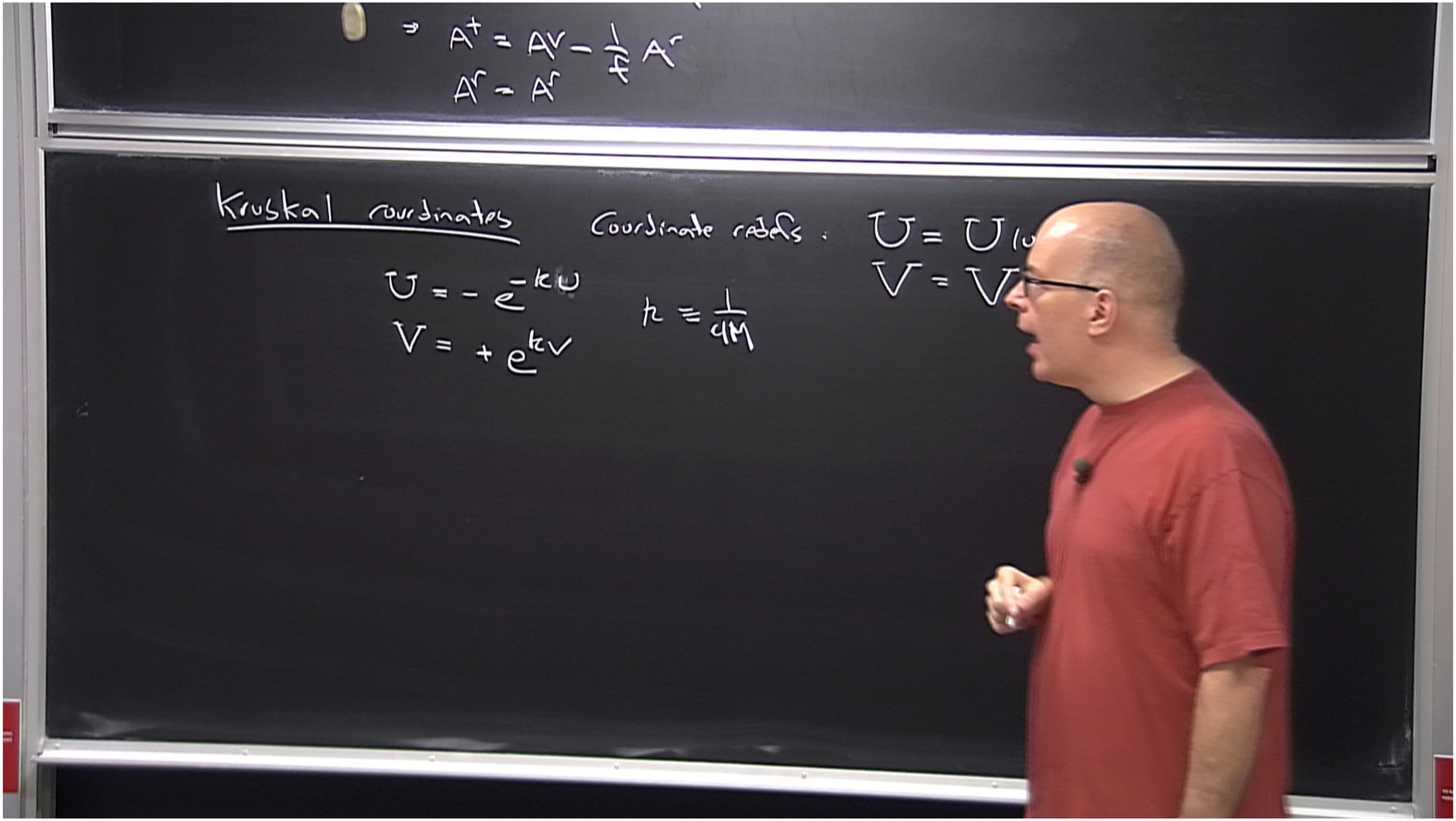
$$\Rightarrow A^+ = A^v - \frac{1}{f} A^r$$
$$A^r = A^r$$

Kruskal coordinates

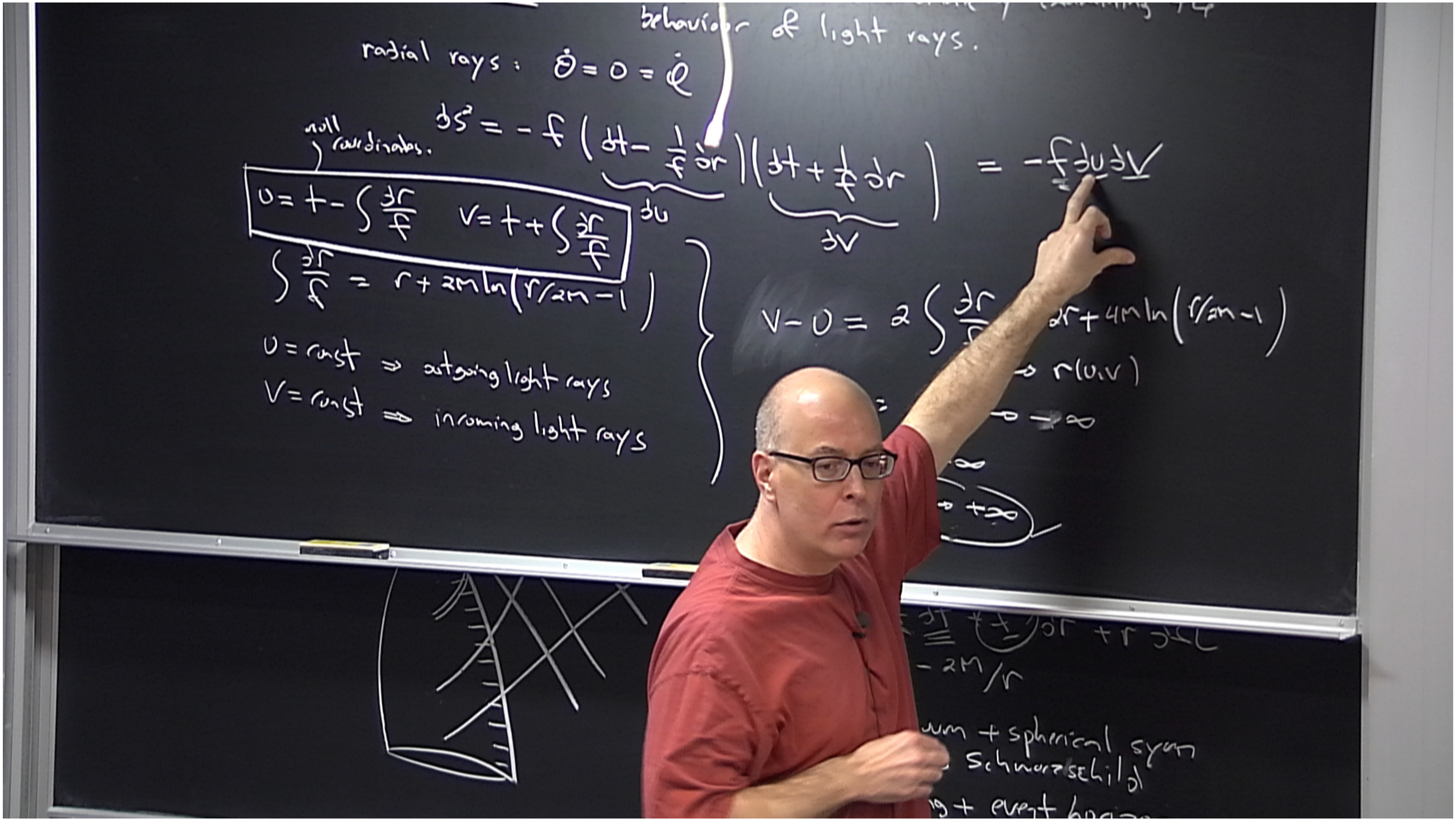
Coordinate relabels: $U = U(u)$

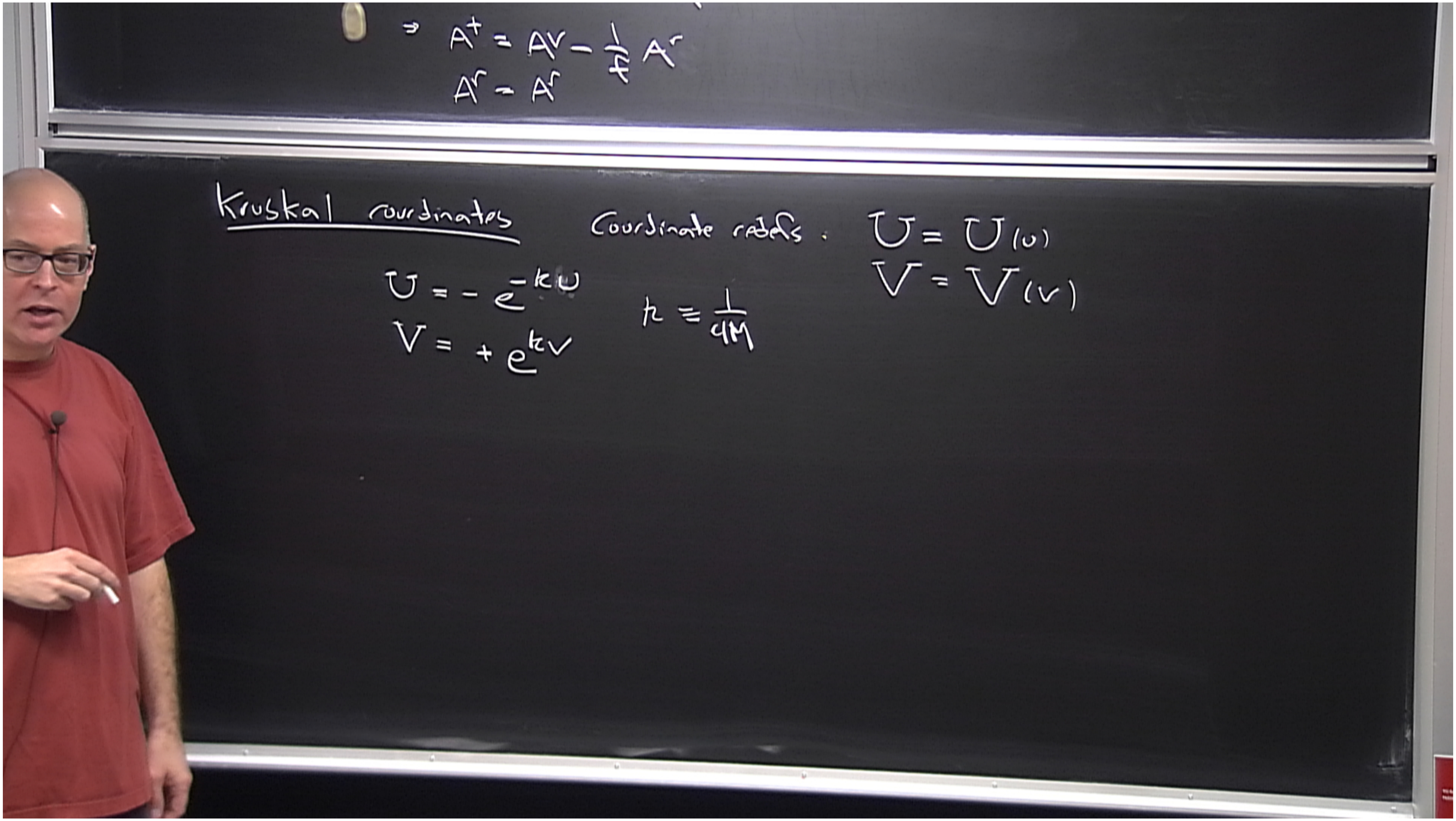
$V = V(v)$

$$U = -e^{-\kappa u}$$
$$V = +e^{\kappa v}$$



⇒ $A^t = A^v - \frac{1}{f} A^r$
$A^r = A^r$
Kruskal coordinates
Coordinate redefs :
$U = U(u$
$V = V$
$U = -e^{-ku}$
$V = +e^{kv}$
$k \equiv \frac{1}{4M}$


behavior of light rays.
radial rays: θ̇ = 0 = φ̇
null coordinates.
ds² = −f (dt − 1/f dr)(dt + 1/f dr) = −f du dv
du
dv
u = t − ∫ dr/f   v = t + ∫ dr/f
∫ dr/f = r + 2m ln(r/2m − 1)
u = const ⇒ outgoing light rays
v = const ⇒ incoming light rays
v − u = 2 ∫ dr/f = 2r + 4m ln(r/2m − 1)
r(u,v)
− 2M/r
Schwarzschild


$\Rightarrow A^{+} = A^{v} - \frac{1}{f} A^{r}$
$A^{r} = A^{r}$
Kruskal coordinates
Coordinate redefs:
$U = U(u)$
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$U = -e^{-ku}$
$V = +e^{kv}$
$k \equiv \frac{1}{4M}$

$$\Rightarrow \quad A^{+} = A^{v} - \frac{1}{f} A^{r}$$
$$A^{r} = A^{r}$$

Kruskal coordinates

Coordinate redefs: $U = U(u)$, $V = V(v)$

$$U = -e^{-ku} \qquad k \equiv \frac{1}{4M}$$
$$V = +e^{kv}$$

$$ds^2 = -\frac{32M^3}{r} e^{-r/2M}\, dU\, dV + r^2 d\Omega^2$$

$\Rightarrow A^{+} = A^{v} - \frac{1}{f} A^{r}$

$A^{r} = A^{r}$

Kruskal coordinates

Coordinate redefs: $U = U(u)$

$V = V(v)$

$U = -e^{-ku}$

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$$\Rightarrow A^{+} = A^{v} - \frac{1}{f} A^{r}$$
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<u>Kruskal coordinates</u>

Coordinate redefs : $U = U(u)$, $V = V(v)$

$$U = -e^{-\kappa u} \qquad \kappa = \frac{1}{4M}$$
$$V = +e^{\kappa v}$$

$$ds^2 = -\frac{32M^3}{r} e^{-r/2M}\, dU\, dV + r^2 d\Omega^2$$

$$e^{r/2M}\left(\frac{r}{2M} - 1\right) = -UV$$

$$\Rightarrow \quad A^{+} = A^{v} - \frac{1}{f} A^{r}$$

$$A^{r} = A^{r}$$

Kruskal coordinates

Coordinate rebels: $U = U(u)$

$V = V(v)$

$$U = -e^{-ku}$$

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$$\Rightarrow \quad A^{+} = A^{v} - \frac{1}{f} A^{r}$$
$$A^{r} = A^{r}$$

Kruskal coordinates

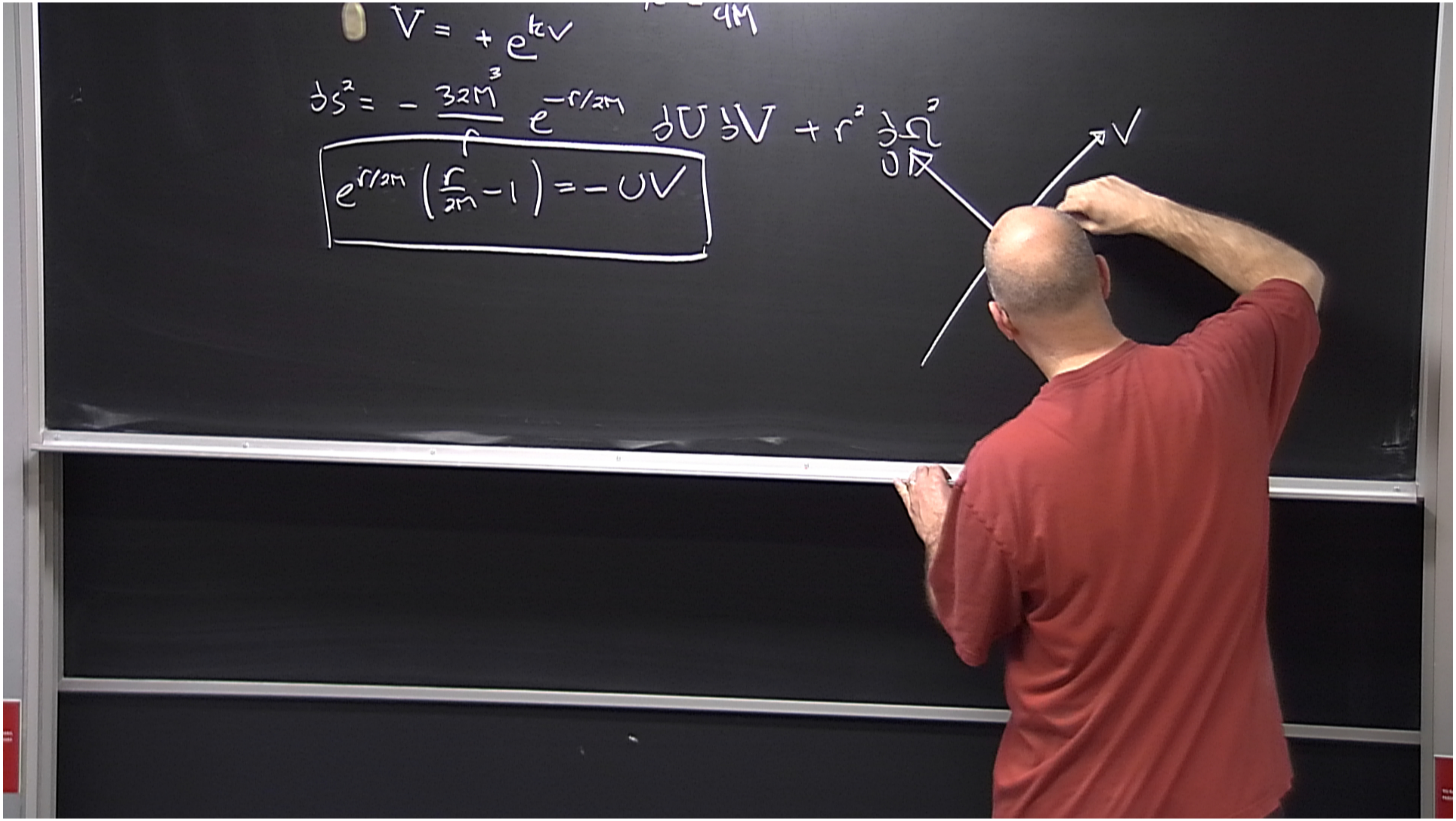
Coordinate redefs: $U = U(u)$, $V = V(v)$

$W = 1$

$$U = -e^{-\kappa u} \qquad \kappa = \frac{1}{4M}$$
$$V = +e^{\kappa v}$$

$$ds^2 = -\frac{32M^3}{r} e^{-r/2M} \, dU \, dV + r^2 d\Omega^2$$

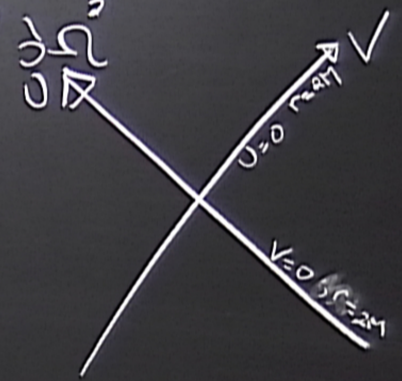
$$\boxed{e^{r/2M}\left(\frac{r}{2M} - 1\right) = -UV}$$


V = + e^kV
ds² = − 32M³/r e^−r/2M dU dV + r² dΩ²
e^r/2M (r/2M − 1) = − UV
V
U

$$V = +e^{\kappa v}$$

$$ds^2 = -\frac{32M^3}{r} e^{-r/2M}\, dU\, dV + r^2\, d\Omega^2$$

$$\boxed{e^{r/2M}\left(\frac{r}{2M} - 1\right) = -UV}$$



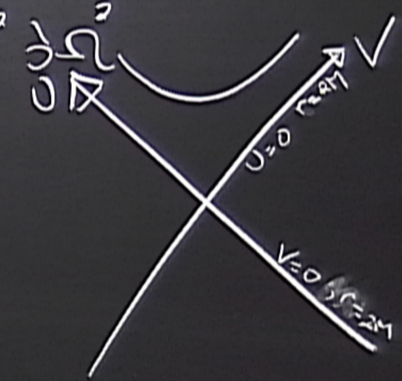
$\omega = 1$

$$U = -e^{-kU} \qquad k \equiv \frac{1}{4M} \qquad V = V(v)$$

$$V = +e^{kV}$$

$$ds^2 = -\frac{32M^3}{r} e^{-r/2M} \, dU \, dV + r^2 d\Omega^2$$

$$\boxed{e^{r/2M}\left(\frac{r}{2M} - 1\right) = -UV}$$



$$U = -e^{-kU}$$

$$V = +e^{kV}$$

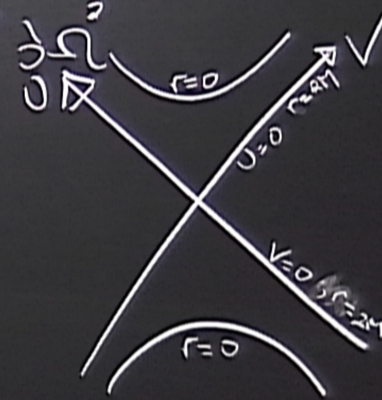
$$k = \frac{1}{4M}$$

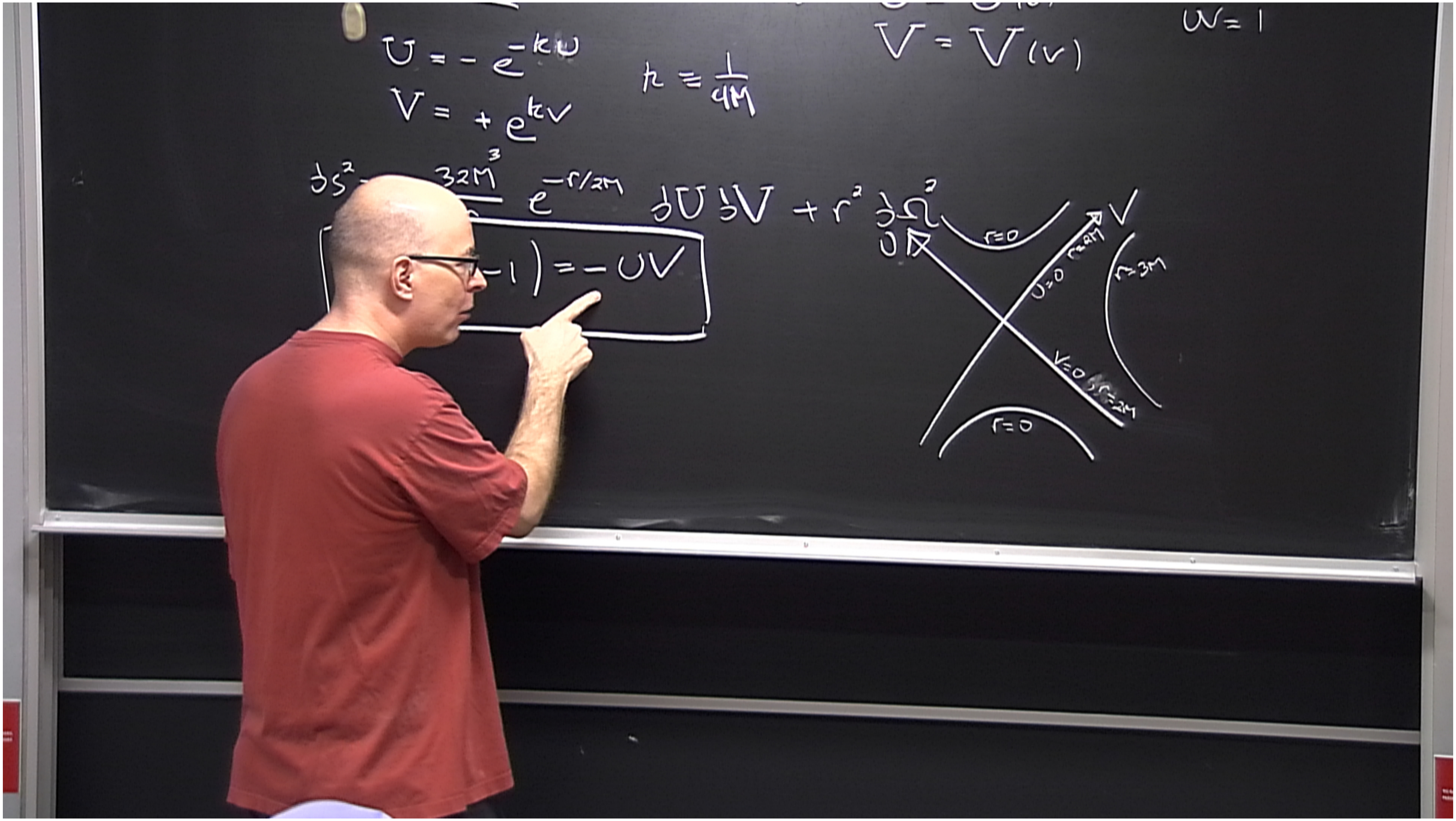
$$V = V(v)$$

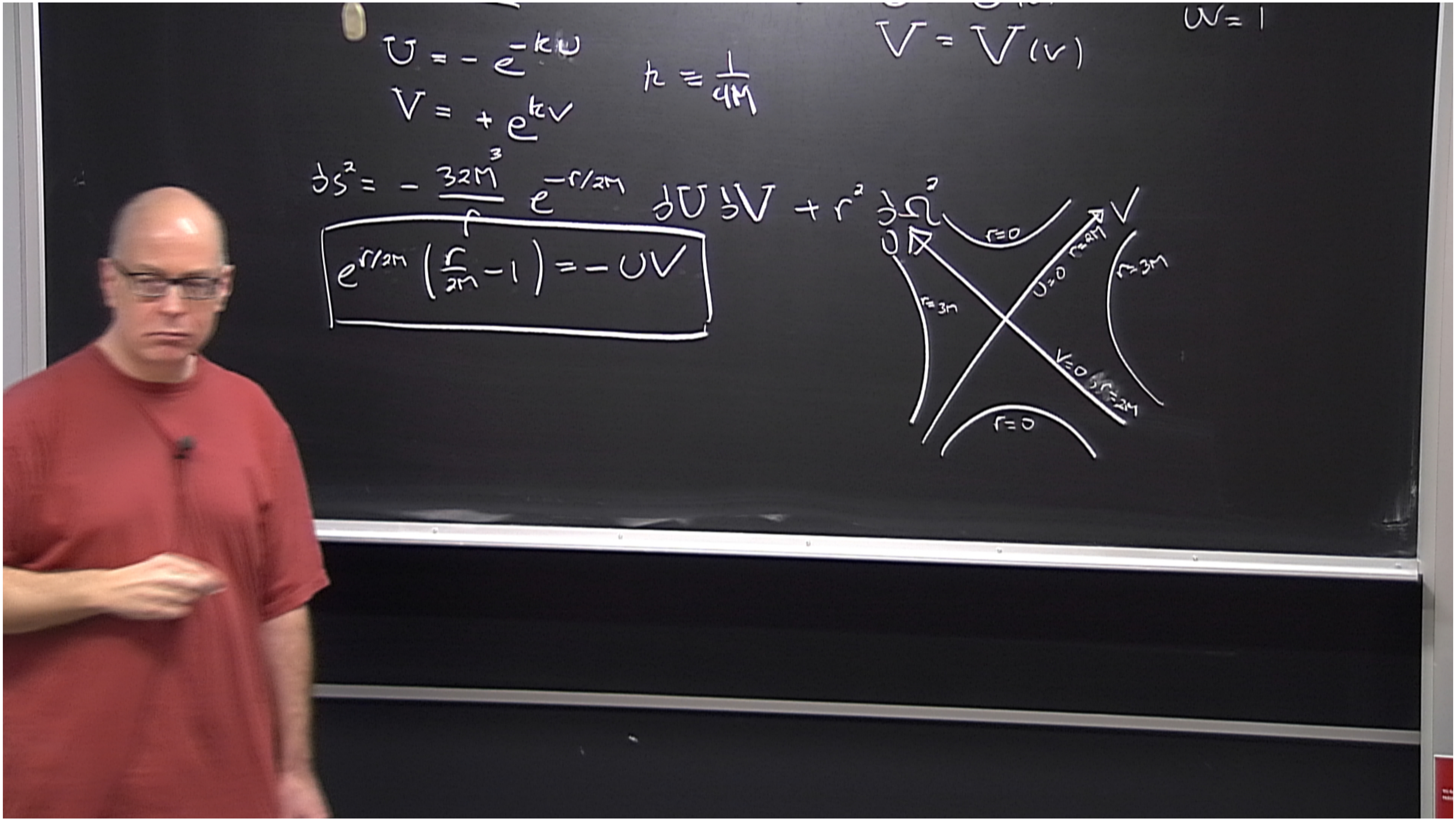
$$\omega = 1$$

$$ds^2 = -\frac{32M^3}{r} e^{-r/2M}\, dU\, dV + r^2\, d\Omega^2$$

$$\boxed{e^{r/2M}\left(\frac{r}{2M} - 1\right) = -UV}$$




U = - e^-ku
V = + e^kv
k ≡ 1/4M
V = V(v)
w = 1
ds² = 32M³ e^-r/2M dU dV + r² dΩ²
-1) = - UV
V
U
r=0
r=2M
U=0
r=3M
V=0
r=2M
r=0

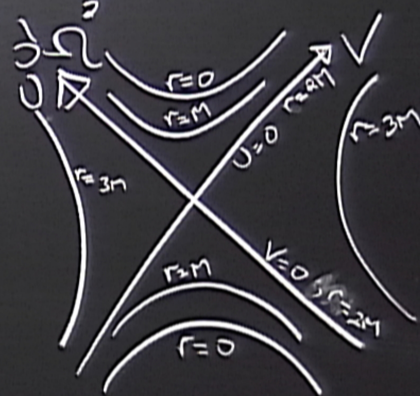

U = − e^{−kU}
V = + e^{kV}
k = 1/4M
V = V(v)
ω = 1
ds² = − 32M³/r e^{−r/2M} dU dV + r² dΩ²
e^{r/2M} (r/2M − 1) = − UV
U
V
r=0
r=2M
r=3M
U=0
V=0
r=3M
r=2M
r=0

$$U = -e^{-ku} \qquad k = \frac{1}{4M} \qquad V = V(v) \qquad w = 1$$

$$V = +e^{kv}$$

$$ds^2 = -\frac{32M^3}{r} e^{-r/2M}\, dU\, dV + r^2 d\Omega^2$$

$$\boxed{e^{r/2M}\left(\frac{r}{2M} - 1\right) = -UV}$$

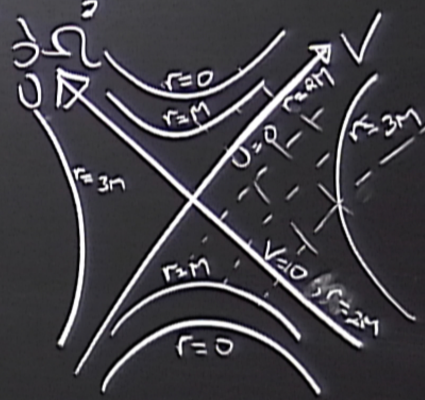


$$U = -e^{-kU} \qquad k = \frac{1}{4M} \qquad V = V(v)$$

$$V = +e^{kV}$$

$$ds^2 = -\frac{32M^3}{r} e^{-r/2M}\, dU\, dV + r^2 d\Omega^2$$

$$\boxed{e^{r/2M}\left(\frac{r}{2M} - 1\right) = -UV}$$



$$U = -e^{-kU} \qquad k = \frac{1}{4M} \qquad V = V(v)$$

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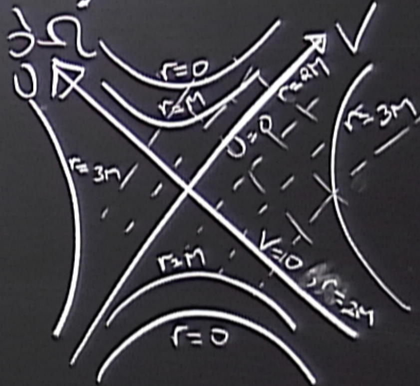
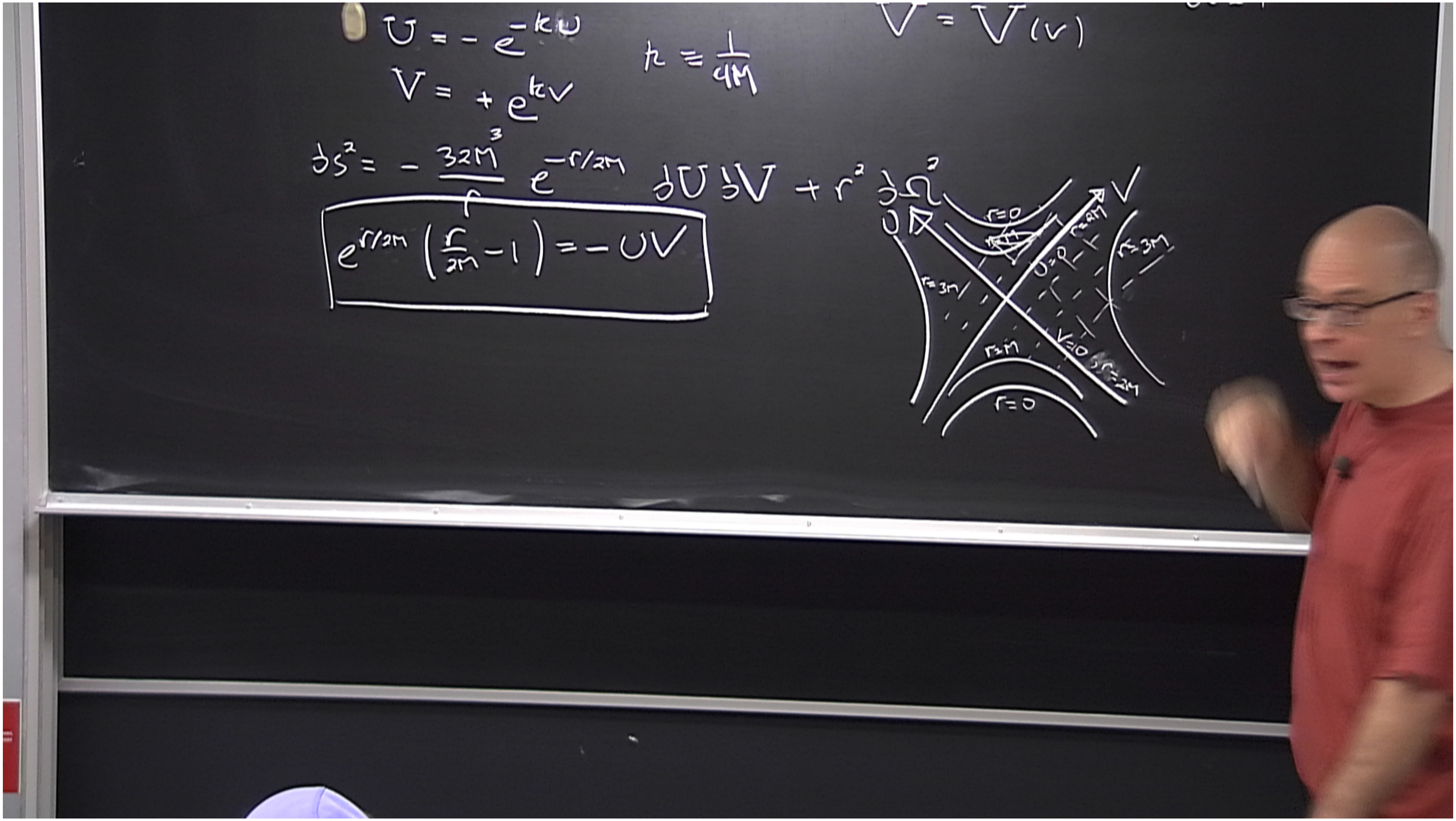
$$ds^2 = -\frac{32M^3}{r} e^{-r/2M}\, dU\, dV + r^2 d\Omega^2$$

$$\boxed{e^{r/2M}\left(\frac{r}{2M} - 1\right) = -UV}$$




$U = -e^{-kU}$
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$ds^2 = -\frac{32M^3}{r} e^{-r/2M} dU\, dV + r^2 d\Omega^2$
$e^{r/2M}\left(\frac{r}{2M} - 1\right) = -UV$
U
V
r=0
r=2M
r=3M
U=0
V=0
r=M
r=2M
r=3M
r=0

$$U = -e^{-ku} \qquad k \equiv \frac{1}{4M} \qquad V = V(v)$$

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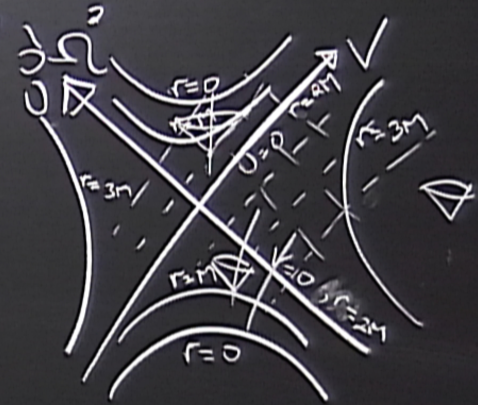
$$\boxed{e^{r/2M}\left(\frac{r}{2M} - 1\right) = -UV}$$



$$U = -e^{-kU} \qquad k \equiv \frac{1}{4M} \qquad V = V(v)$$

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$$ds^2 = -\frac{32M^3}{r} e^{-r/2M}\, dU\, dV + r^2\, d\Omega^2$$

$$\boxed{e^{r/2M}\left(\frac{r}{2M} - 1\right) = -UV}$$