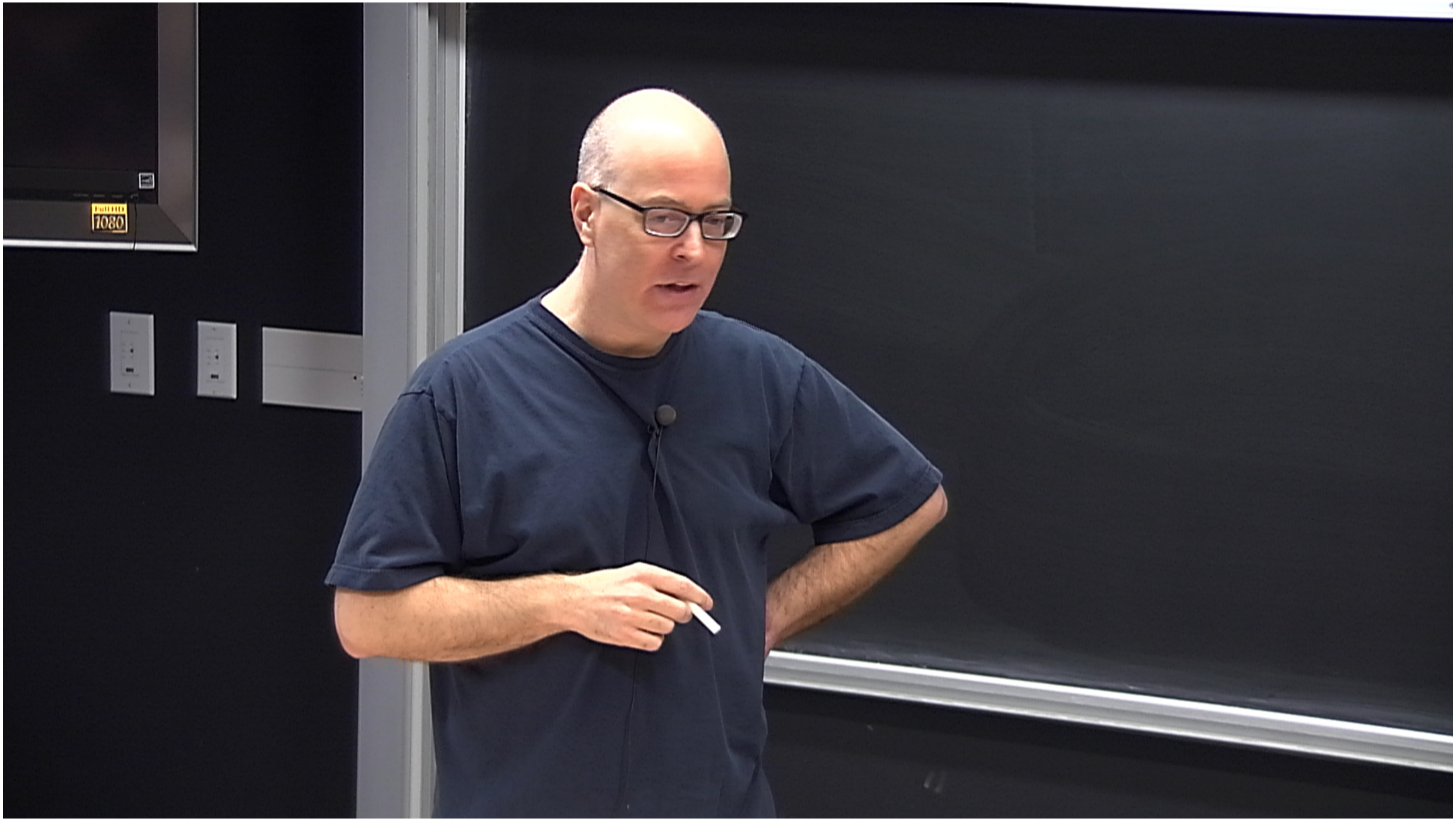
Title: Advanced General Relativity - Lecture 15

Date: Mar 07, 2012 10:00 AM

URL: http://pirsa.org/12030068

Abstract:



HAMILTONIAN FORMULATION

mechanics: $L(q, \dot{q}) \rightarrow p = \dfrac{\partial L}{\partial \dot{q}}$

HAMILTONIAN FORMULATION

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$$H = p\dot{q} - L$$

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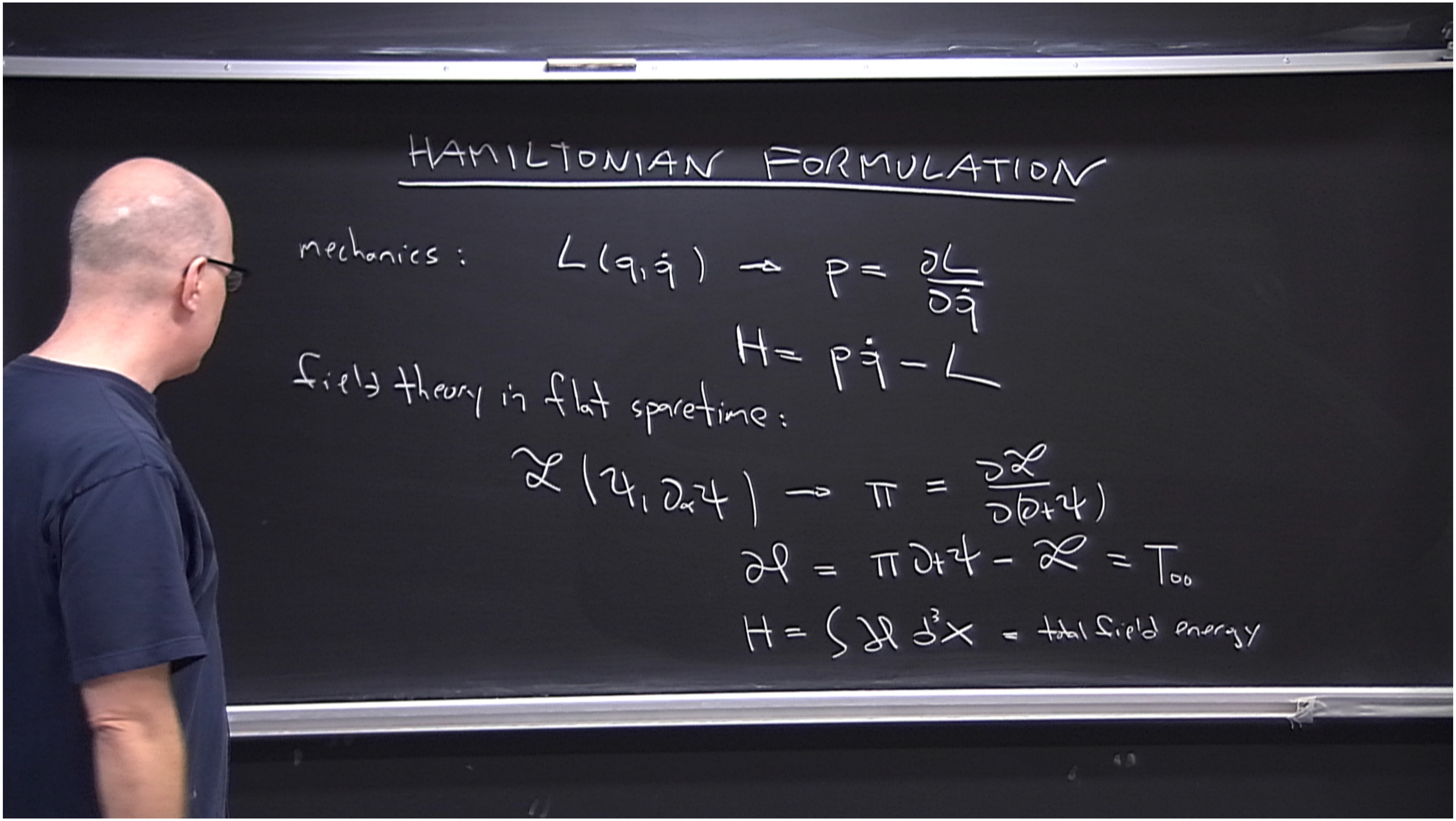
field theory in flat spacetime:

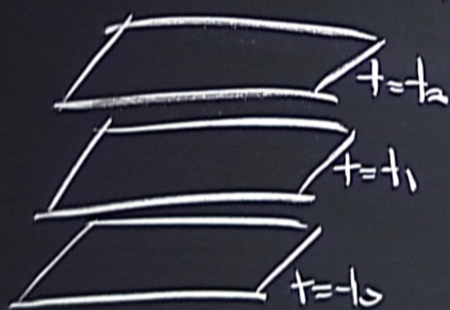
$$\mathcal{L}(\psi, \partial_\alpha \psi) \rightarrow \pi = \frac{\partial \mathcal{L}}{\partial(\partial_t \psi)}$$


HAMILTONIAN FORMULATION
mechanics: $L(q,\dot{q}) \rightarrow p = \frac{\partial L}{\partial \dot{q}}$
$H = p\dot{q} - L$
field theory in flat spacetime:
$\mathcal{L}(\psi, \partial_\alpha \psi) \rightarrow \pi = \frac{\partial \mathcal{L}}{\partial(\partial_t \psi)}$
$\mathcal{H} = \pi \partial_t \psi - \mathcal{L} = T_{00}$
$H = \int \mathcal{H} \, d^3x$ = total field energy

$$\overline{\partial \dot q}$$

$$H = p\dot q - L$$

field theory in <u>flat</u> spacetime:



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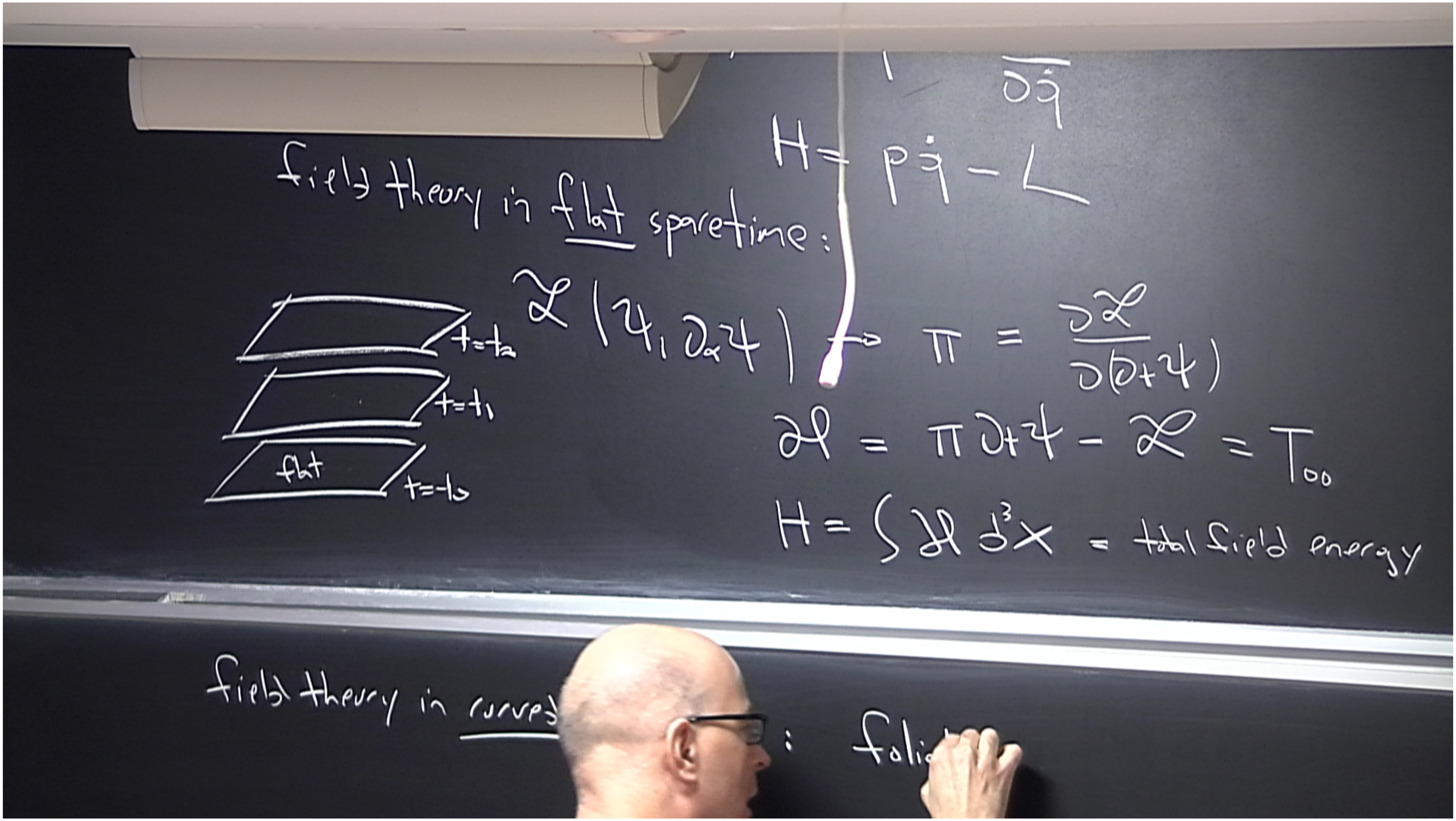
field theory in <u>curved</u> spacetime:


$H = p\dot{q} - L$
field theory in flat spacetime:
$\mathcal{L}(\psi, \partial_\alpha \psi) \to \pi = \frac{\partial \mathcal{L}}{\partial(\partial_t \psi)}$
t=t2
t=t1
flat
t=t0
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field theory in curve
: foli

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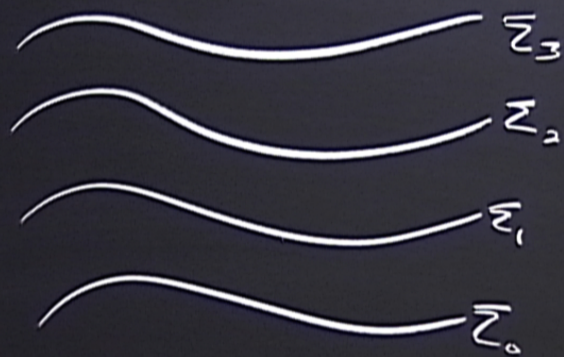
field theory in curved spacetime:



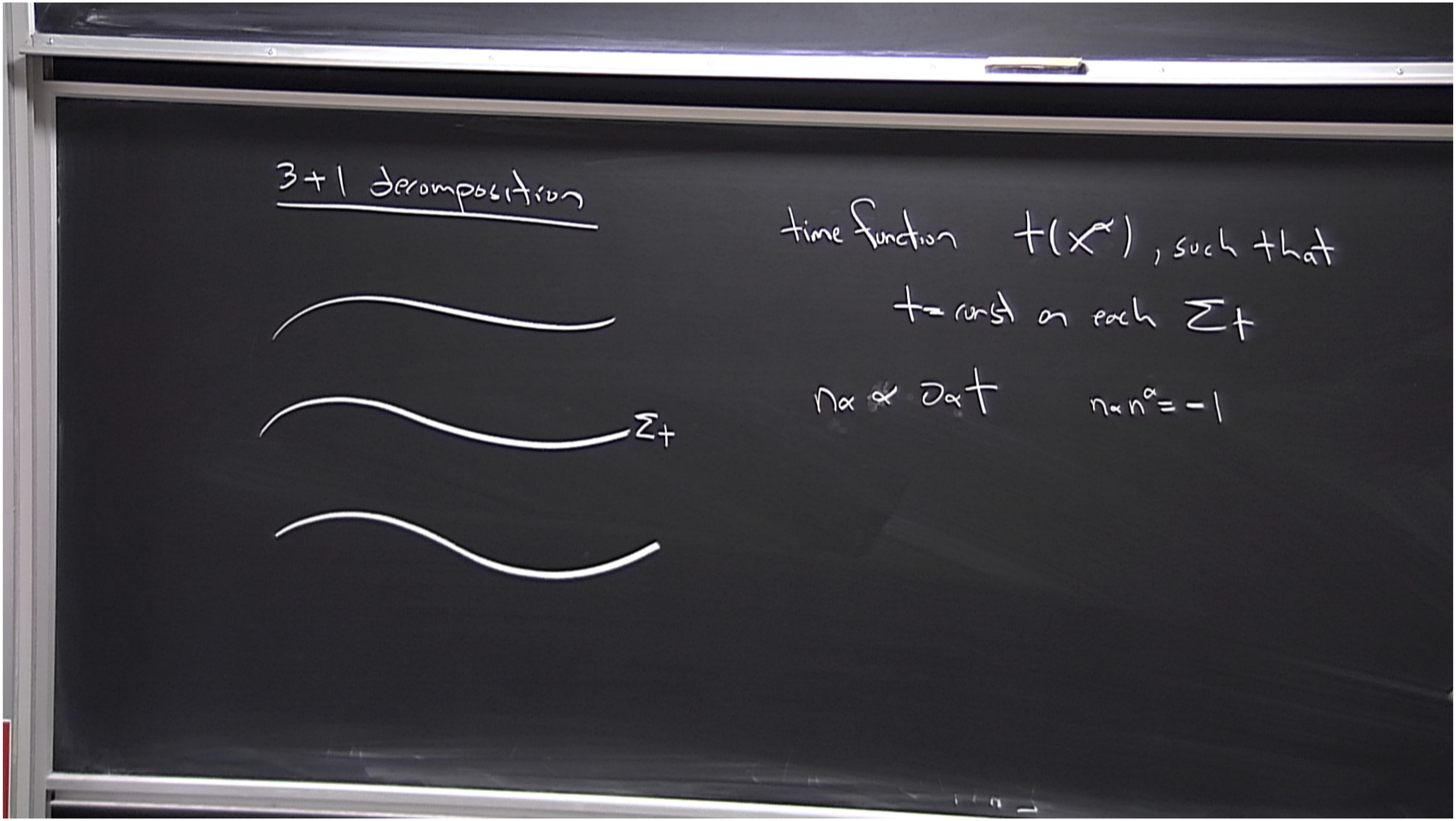
foliate spacetime with arbitrary spacelike hypersurfaces.

$\partial_t \psi \to n^\alpha \partial_\alpha \psi$

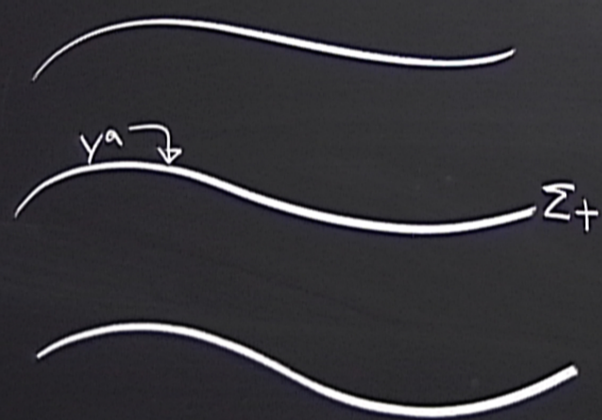
3+1 decomposition

time function $t(x^\alpha)$, such that

$t =$ const on each Σ_t


3+1 decomposition
time function $t(x^\alpha)$, such that
$t =$ const on each Σ_t
$n_\alpha \propto \partial_\alpha t$
$n_\alpha n^\alpha = -1$
Σ_t

3+1 decomposition



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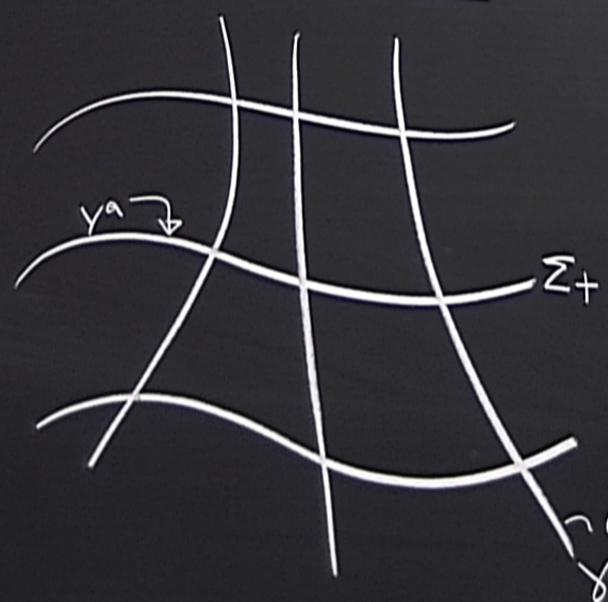
3+1 decomposition



congruence of curves; $y^a = \text{const}$

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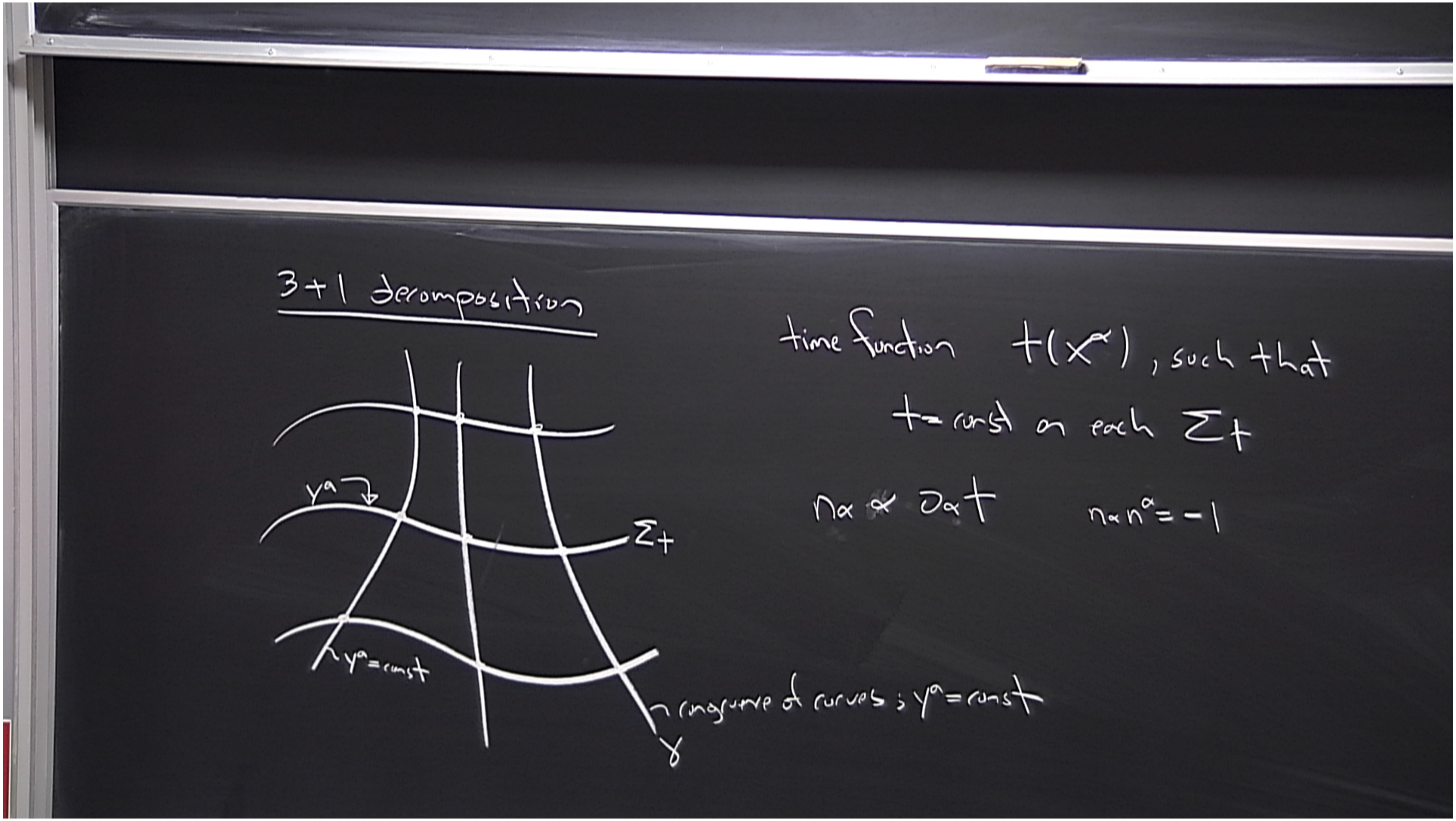
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y^a
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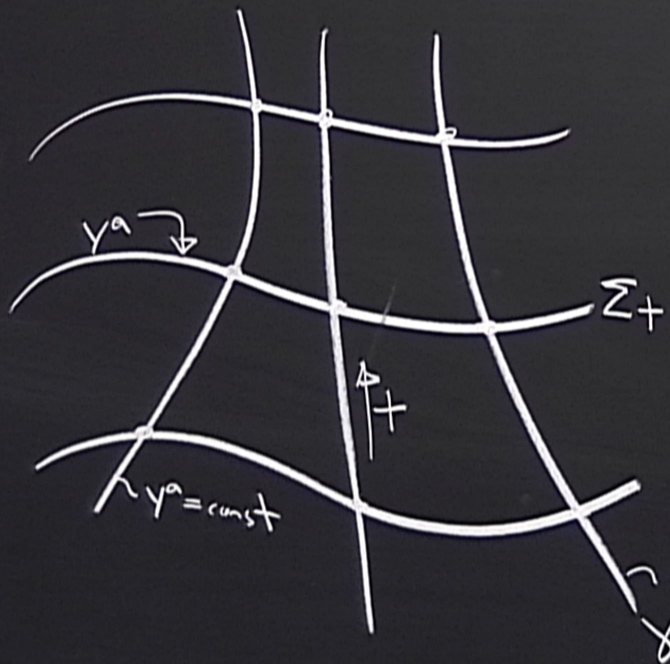
3+1 decomposition



time function $t(x^\alpha)$, such that

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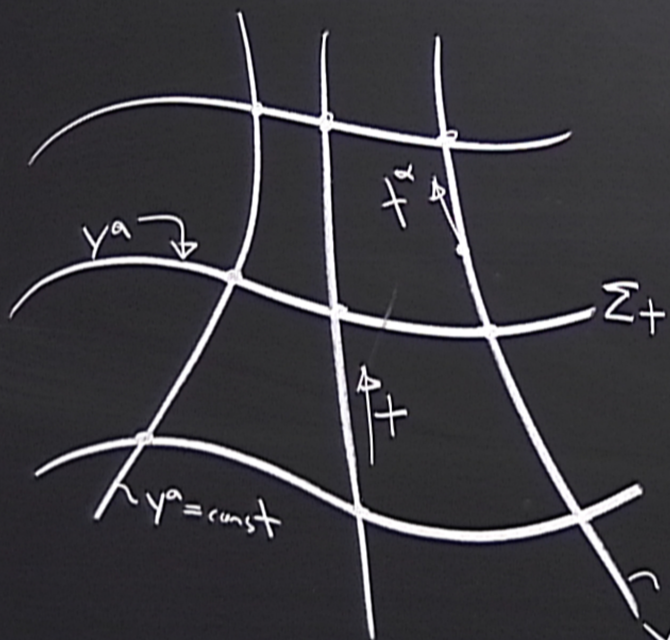
congruence of curves; $y^a = \text{const}$; parametrized by t.

3+1 decomposition



time function $t(x^\alpha)$, such that

$t = \text{const}$ on each Σ_t

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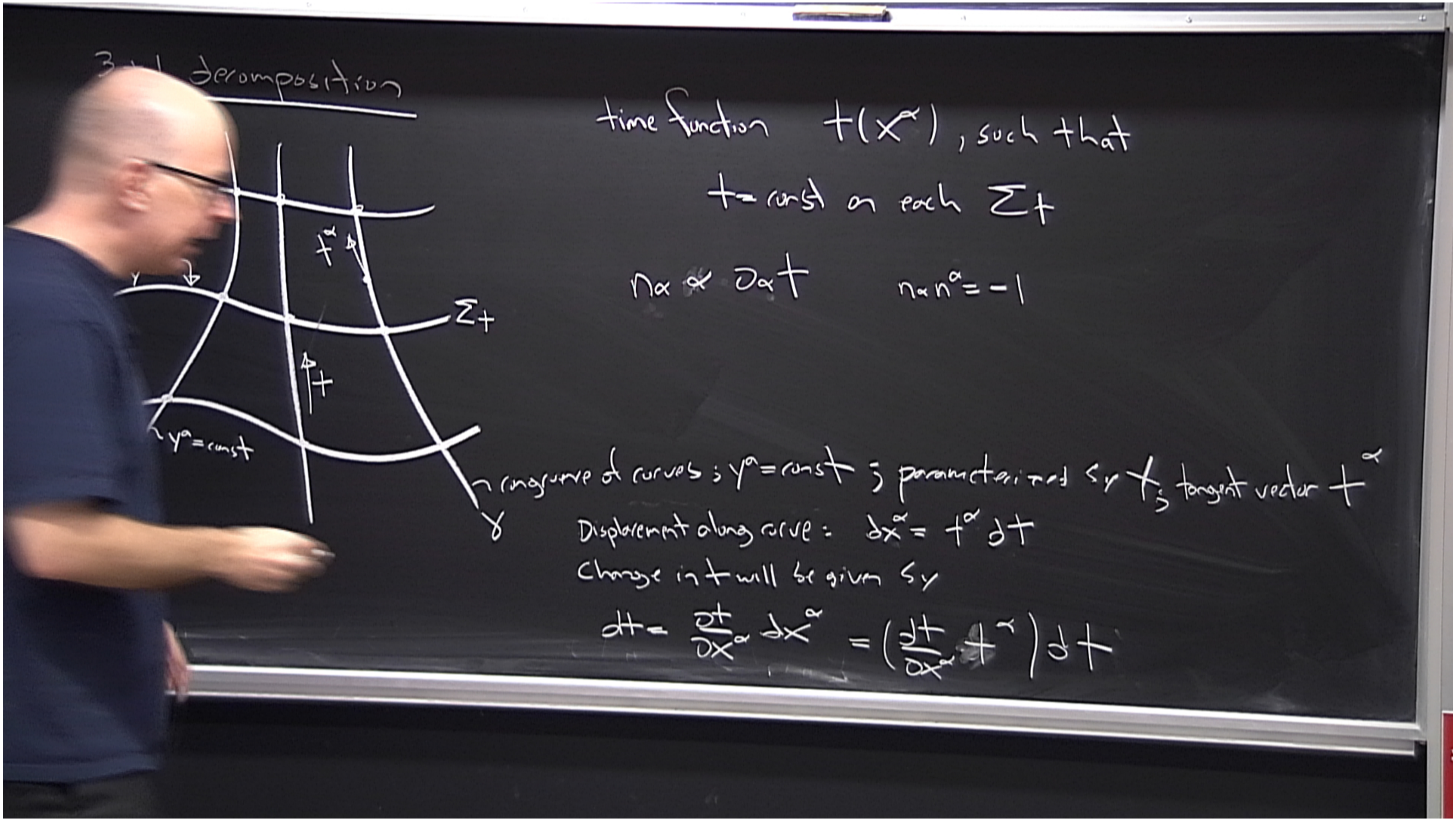
congruence of curves; $y^a = \text{const}$; parametrized by t.

γ Displacement along curve: $dx^\alpha =$



decomposition
time function $t(x^\alpha)$, such that
t = const on each Σ_t
$n_\alpha \propto \partial_\alpha t$
$n_\alpha n^\alpha = -1$
t^α
Σ_t
∂_t
y^a = const
γ
congruence of curves; y^a = const; parameterized by t; tangent vector t^α
Displacement along curve: $dx^\alpha = t^\alpha dt$
Change in t will be given by
$dt = \frac{\partial t}{\partial x^\alpha} dx^\alpha = \left(\frac{\partial t}{\partial x^\alpha} t^\alpha\right) dt$

congruence of curves ; $y^a = \text{const}$; parameterized by t ; tangent v

Displacement along curve: $dx^\alpha = t^\alpha dt$

Change in t will be given by

$$dt = \frac{\partial t}{\partial x^\alpha} dx^\alpha = \left(\frac{\partial t}{\partial x^\alpha} t^\alpha \right) dt$$

Original coordinate system x^α

Alternative : (t, y^a)

Relation $x^\alpha = x^\alpha(t, y^a)$: parametric relations for γ

$$\boxed{t^\alpha \partial_\alpha t = 1}$$

Original

Alterna

Relati

$$\left(\frac{\partial x^\alpha}{\partial t}\right)_{y^a} = t^\alpha$$

$$\left(\frac{\partial x^\alpha}{\partial y^a}\right)_{t} = e^\alpha_{\ a}$$

$$\boxed{t^\alpha \partial_\alpha t = 1}$$

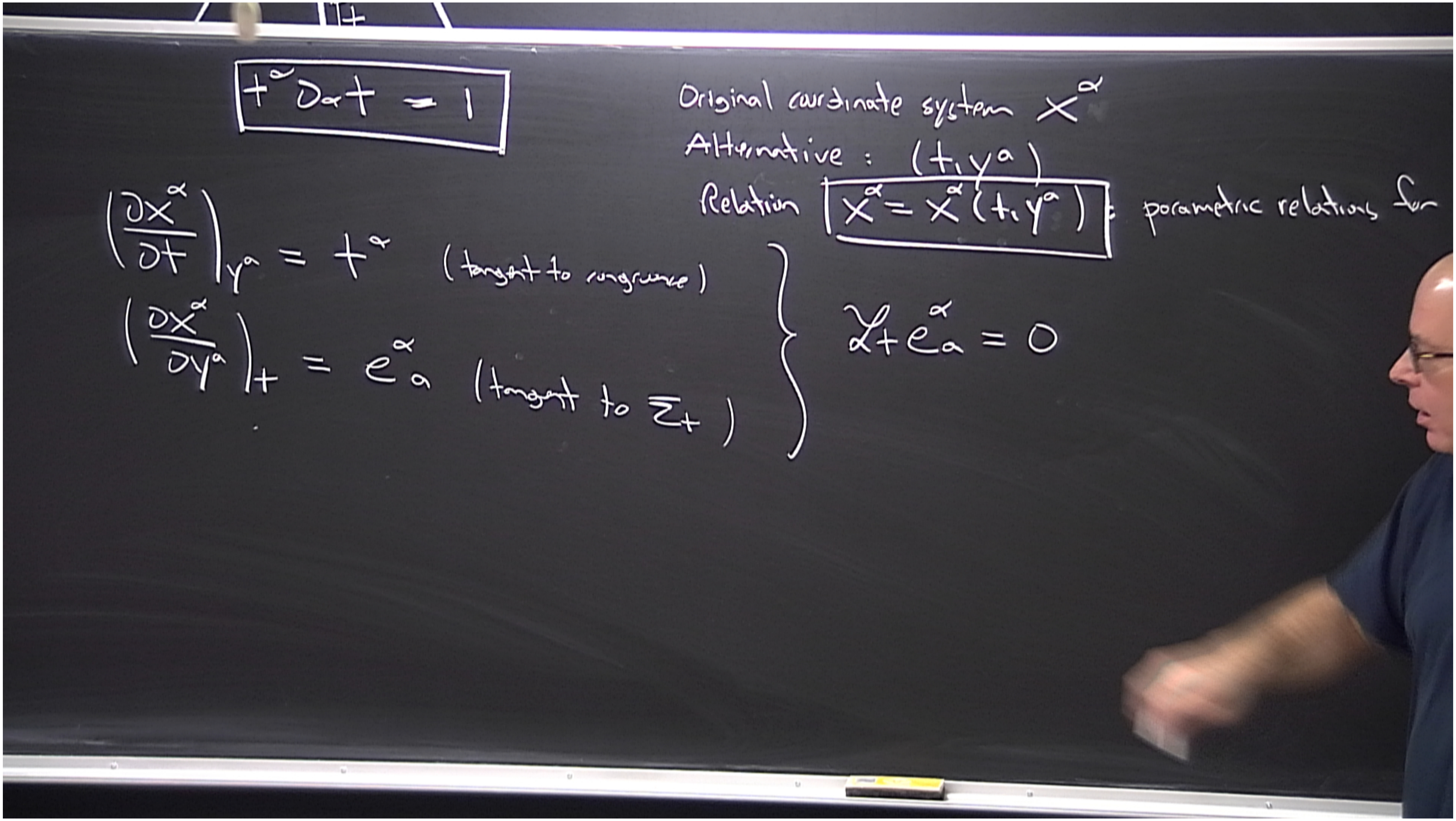
Original coordinate sy

Alternative : (t

Relation $\boxed{x^\alpha = x}$

$$\left(\frac{\partial x^\alpha}{\partial t}\right)_{y^a} = t^\alpha \quad \text{(tangent to congruence)}$$

$$\left(\frac{\partial x^\alpha}{\partial y^a}\right)_t = e^\alpha_a \quad (\text{tangent to } \Sigma_t)$$


$t^\alpha \nabla_\alpha t = 1$
Original coordinate system x^α
Alternative: (t, y^a)
Relation $x^\alpha = x^\alpha(t, y^a)$: parametric relations for
$\left(\frac{\partial x^\alpha}{\partial t}\right)_{y^a} = t^\alpha$ (tangent to congruence)
$\left(\frac{\partial x^\alpha}{\partial y^a}\right)_t = e^\alpha_a$ (tangent to Σ_t)
$\mathcal{L}_t e^\alpha_a = 0$

$$\boxed{t^\alpha \partial_\alpha t = 1}$$

Original coordinate system x^α

Alternative : (t, y^a)

Relation $\boxed{x^\alpha = x^\alpha(t, y^a)}$: parametric relations for γ

$$\left(\frac{\partial x^\alpha}{\partial t}\right)_{y^a} = t^\alpha \quad \text{(tangent to congruence)}$$

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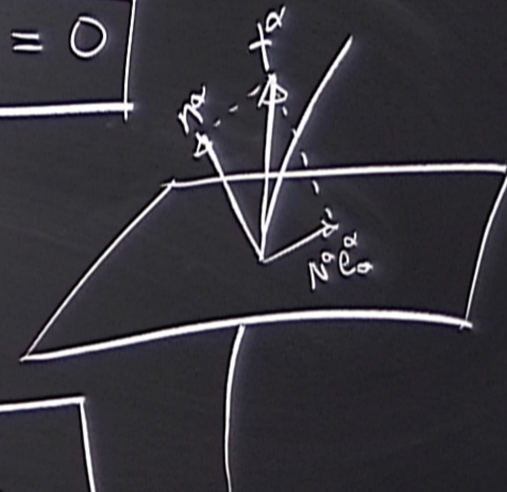
$$\boxed{\mathcal{L}_t e^\alpha_a = 0}$$



$$\boxed{n_\alpha = -N \partial_\alpha t}$$

N ↳ lapse function.

$$\boxed{n_\alpha e^\alpha_a = 0}$$

$$\boxed{t^\alpha = N n^\alpha + N^a e^\alpha_a}$$

N^a ↳ shift vector

$$t^\alpha = A n^\alpha + N^a e_a^\alpha$$

$$1 = t^\alpha \partial_\alpha t = t^\alpha \left(-\frac{1}{N} n_\alpha \right) = \left(A n^\alpha + \dots \right) \left(-\frac{1}{N} n_\alpha \right) = \frac{A}{N}$$

$$t^\alpha = A n^\alpha + N^a e_a^\alpha$$

$$1 = t^\alpha \partial_\alpha t = t^\alpha \left(-\frac{1}{N} n_\alpha\right) = \left(A n^\alpha + \dots\right)\left(-\frac{1}{N} n_\alpha\right) = \frac{A}{N}$$

metric in coordinates (t, y^a):

$$dx^\alpha = t^\alpha dt + e^\alpha_a dy^a$$

$$= (N n^\alpha + N^a e^\alpha_a) dt + e^\alpha_a dy^a$$

$$= (N dt) n^\alpha + (N^a dt + dy^a) e^\alpha_a$$

metric in coordinates (t, y^a):

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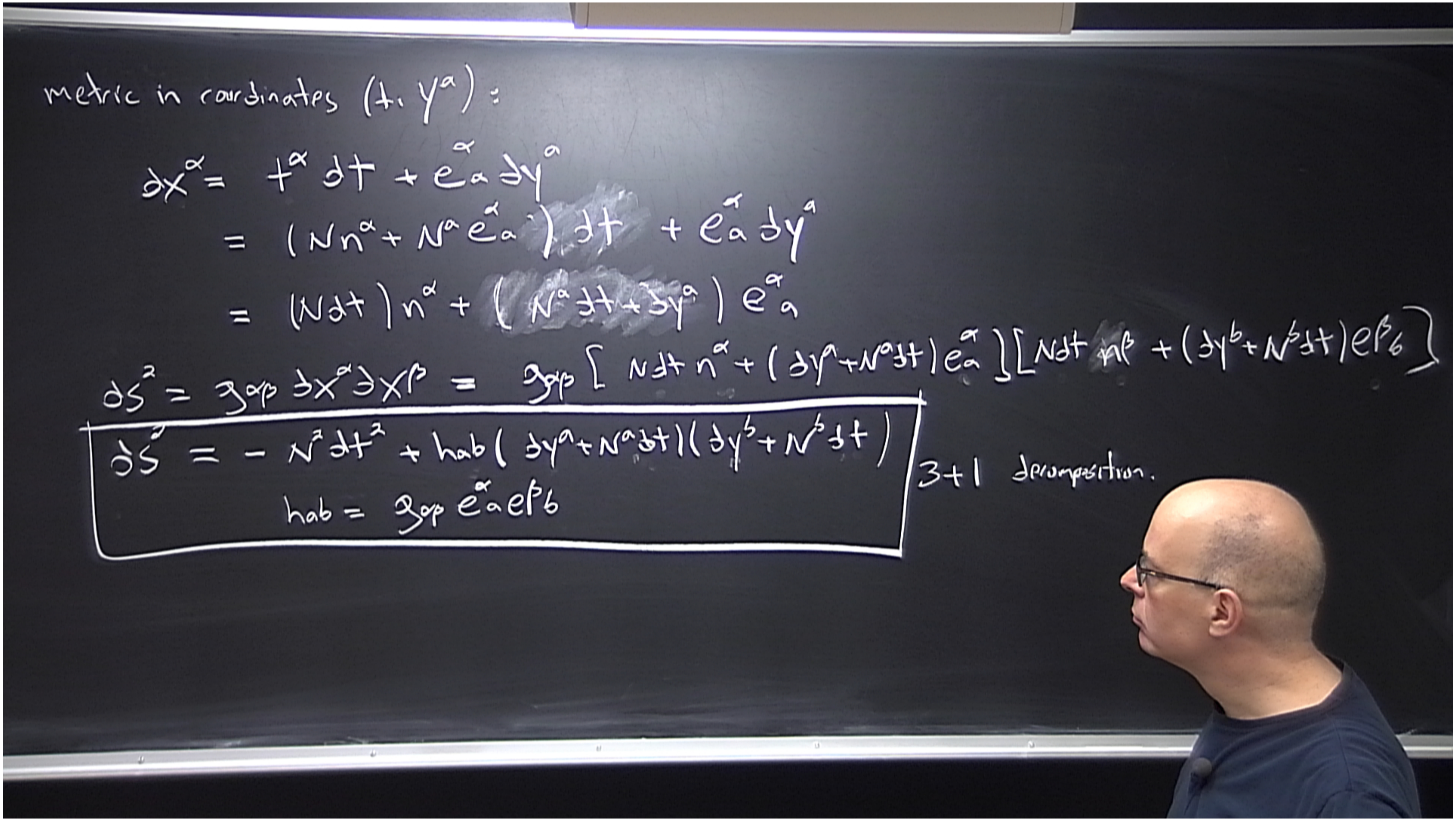
$$= (N dt) n^\alpha + (N^a dt + dy^a) e^\alpha_a$$

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta = g_{\alpha\beta} [N dt\, n^\alpha + (dy^a + N^a dt) e^\alpha_a] [N dt\, n^\beta + (dy^b + N^b dt) e^\beta_b]$$

$$= - N^2 dt^2$$


metric in coordinates (t, y^a):
$dx^\alpha = t^\alpha dt + e^\alpha_a dy^a$
$= (N n^\alpha + N^a e^\alpha_a) dt + e^\alpha_a dy^a$
$= (N dt) n^\alpha + (N^a dt + dy^a) e^\alpha_a$
$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta = g_{\alpha\beta} [N dt\, n^\alpha + (dy^a + N^a dt) e^\alpha_a][N dt\, n^\beta + (dy^b + N^b dt) e^\beta_b]$
$\boxed{ds^2 = -N^2 dt^2 + h_{ab}(dy^a + N^a dt)(dy^b + N^b dt) \qquad h_{ab} = g_{\alpha\beta} e^\alpha_a e^\beta_b}$
3+1 decomposition.

Metric in coordinates (t, y^a):

$$dx^\alpha = t^\alpha dt + e^\alpha_a dy^a$$

$$= (N n^\alpha + N^a e^\alpha_a)\, dt + e^\alpha_a dy^a$$

$$= (N dt)\, n^\alpha + (N^a dt + dy^a)\, e^\alpha_a$$

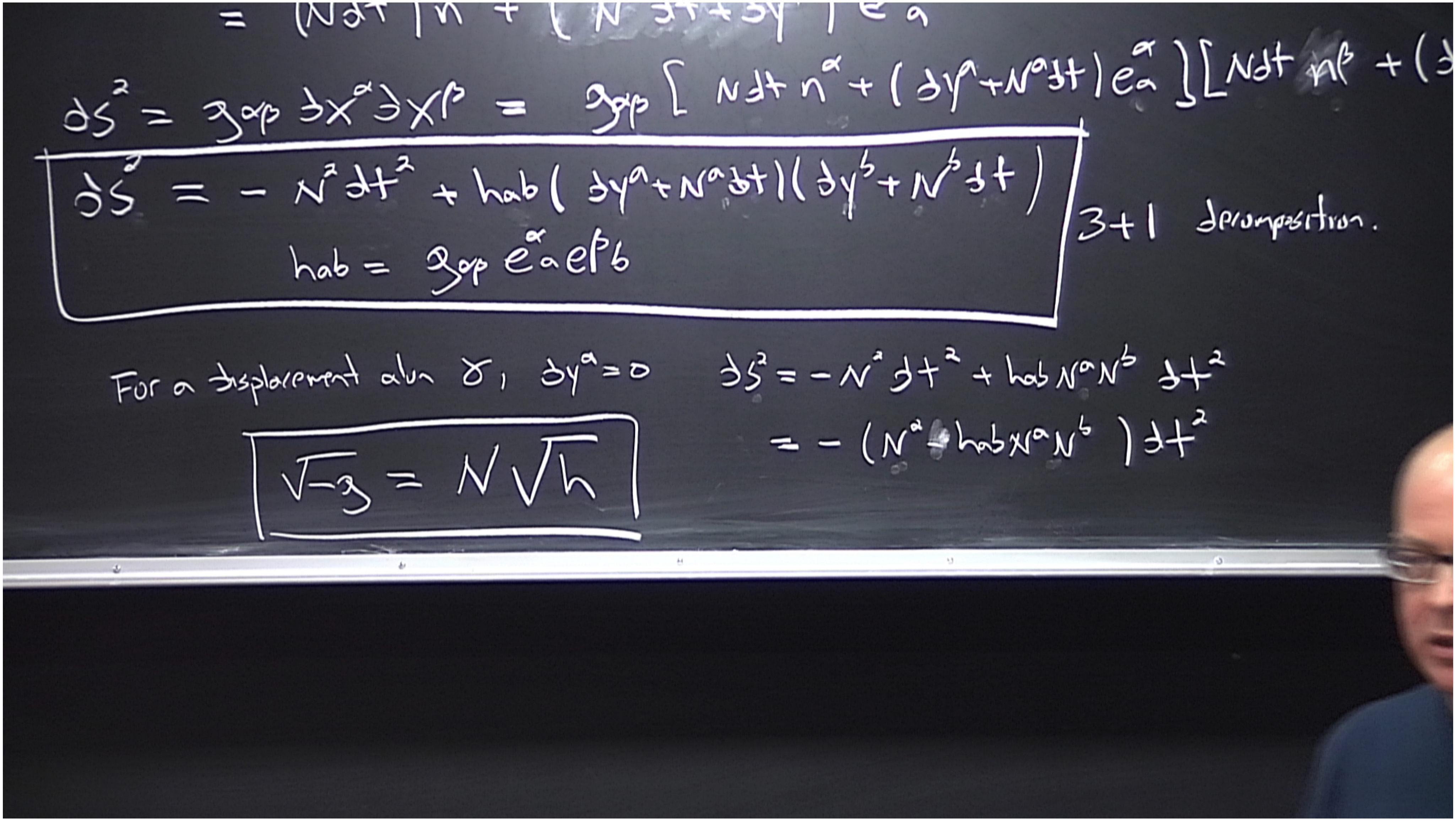
$$ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta = g_{\alpha\beta} \left[N dt\, n^\alpha + (dy^a + N^a dt)\, e^\alpha_a \right] \left[N dt\, n^\beta + (dy^b + N^b dt)\, e^\beta_b \right]$$

$$\boxed{\begin{aligned} ds^2 &= -N^2 dt^2 + h_{ab}(dy^a + N^a dt)(dy^b + N^b dt) \\ h_{ab} &= g_{\alpha\beta}\, e^\alpha_a e^\beta_b \end{aligned}}$$ 3+1 decomposition.

For a displacement along γ, $dy^a = 0$

$$ds^2 = -N^2 dt^2 + h_{ab} N^a N^b dt^2$$

$$= -(N^2 - h_{ab} N^a N^b)\, dt^2$$


$= (N\,dt)\,n^\alpha + (N^a dt + dy^a)\,e^\alpha_a$
$ds^2 = g_{\alpha\beta}\,dx^\alpha dx^\beta = g_{\alpha\beta}[N\,dt\,n^\alpha + (dy^a + N^a dt)\,e^\alpha_a][N\,dt\,n^\beta + (d$
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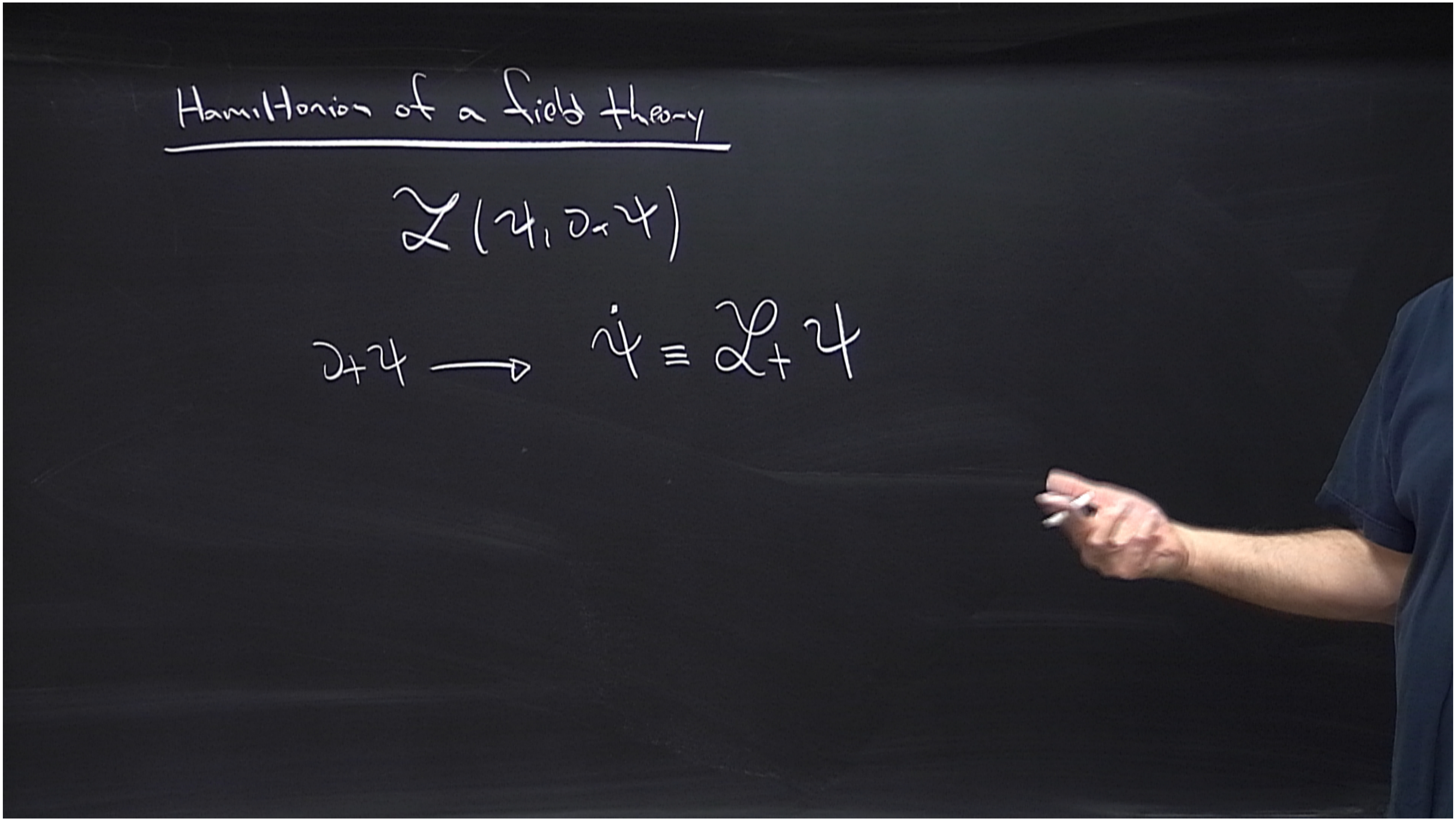
3+1 decomposition.
For a displacement along γ, $dy^a = 0$
$ds^2 = -N^2 dt^2 + h_{ab} N^a N^b dt^2$
$= -(N^2 - h_{ab} N^a N^b)\,dt^2$
$\sqrt{-g} = N\sqrt{h}$

Hamiltonian of a field theory

$$\mathcal{L}(\psi, \partial_\mu \psi)$$


Hamiltonian of a field theory
$\mathcal{L}(\psi, \partial_t \psi)$
$\partial_t \psi \longrightarrow \dot{\psi} \equiv \mathcal{L}_t \psi$

For a displacement along δt, $\delta y = 0$ $\quad ds^2 = -N^2 dt^2 + h_{ab} N^a N^b dt^2$

$$\boxed{\sqrt{-g} = N\sqrt{h}}$$

$$= -(N^2 - h_{ab} N^a N^b)\, dt^2$$

$\mathcal{L}(\psi, \partial_\alpha \psi)$

$$\partial_t \psi \longrightarrow \boxed{\dot{\psi} \equiv \mathcal{L}_t \psi}$$

(derivative along t^α, not n^α)

$$\boxed{\pi \equiv \frac{\partial}{\partial \dot{\psi}}\left(\sqrt{-g}\,\mathcal{L}\right)}$$

canonical momentum.

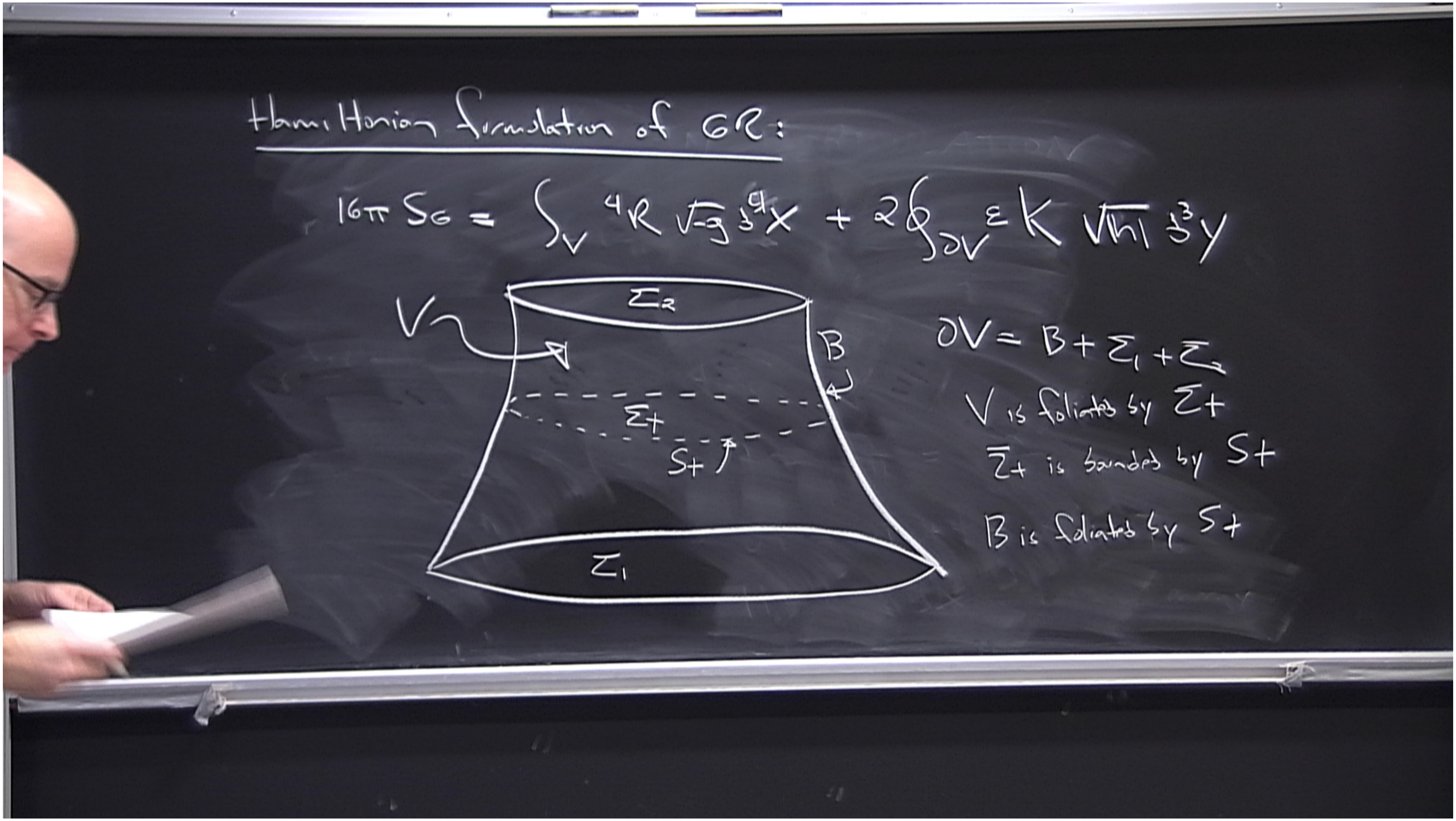
π not a scalar.

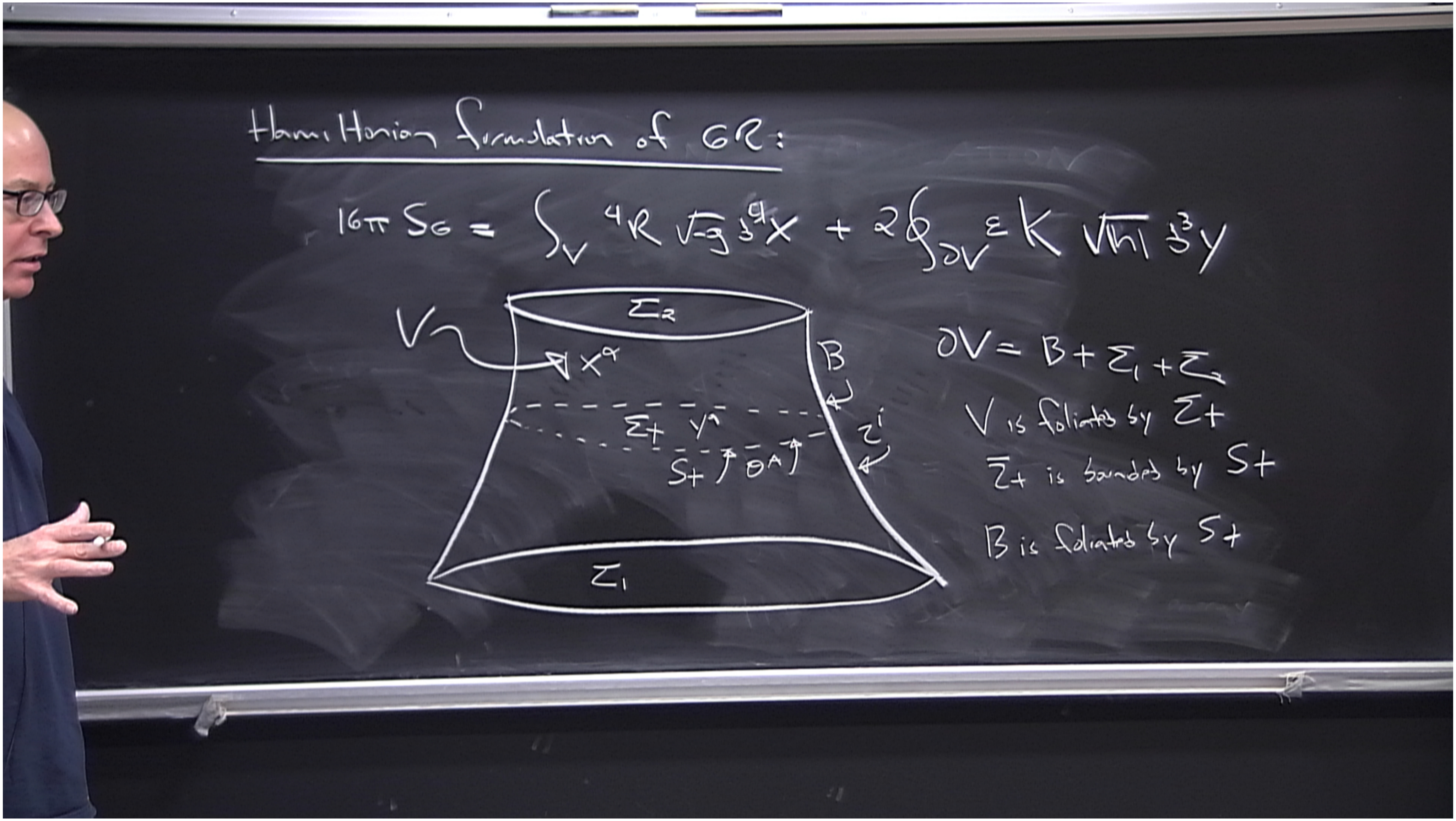
Hamiltonian density

$$\boxed{\mathcal{H} \equiv \pi \dot{\psi} - \sqrt{-g}\,\mathcal{L}}$$

Hamiltonian.

$$\boxed{H = \int \mathcal{H}\, d^3y}$$


Hamiltonian formulation of GR:
$16\pi S_G = \int_V {}^4R \sqrt{-g}\, d^4x + 2\oint_{\partial V} \varepsilon K \sqrt{|h|}\, d^3y$
V
Σ_2
B
Σ_t
S_t
Σ_1
$\partial V = B + \Sigma_1 + \Sigma_2$
V is foliated by Σ_t
Σ_t is bounded by S_t
B is foliated by S_t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Σ_2
V
x^α
B
Σ_t y^a
z^i
S_t θ^A
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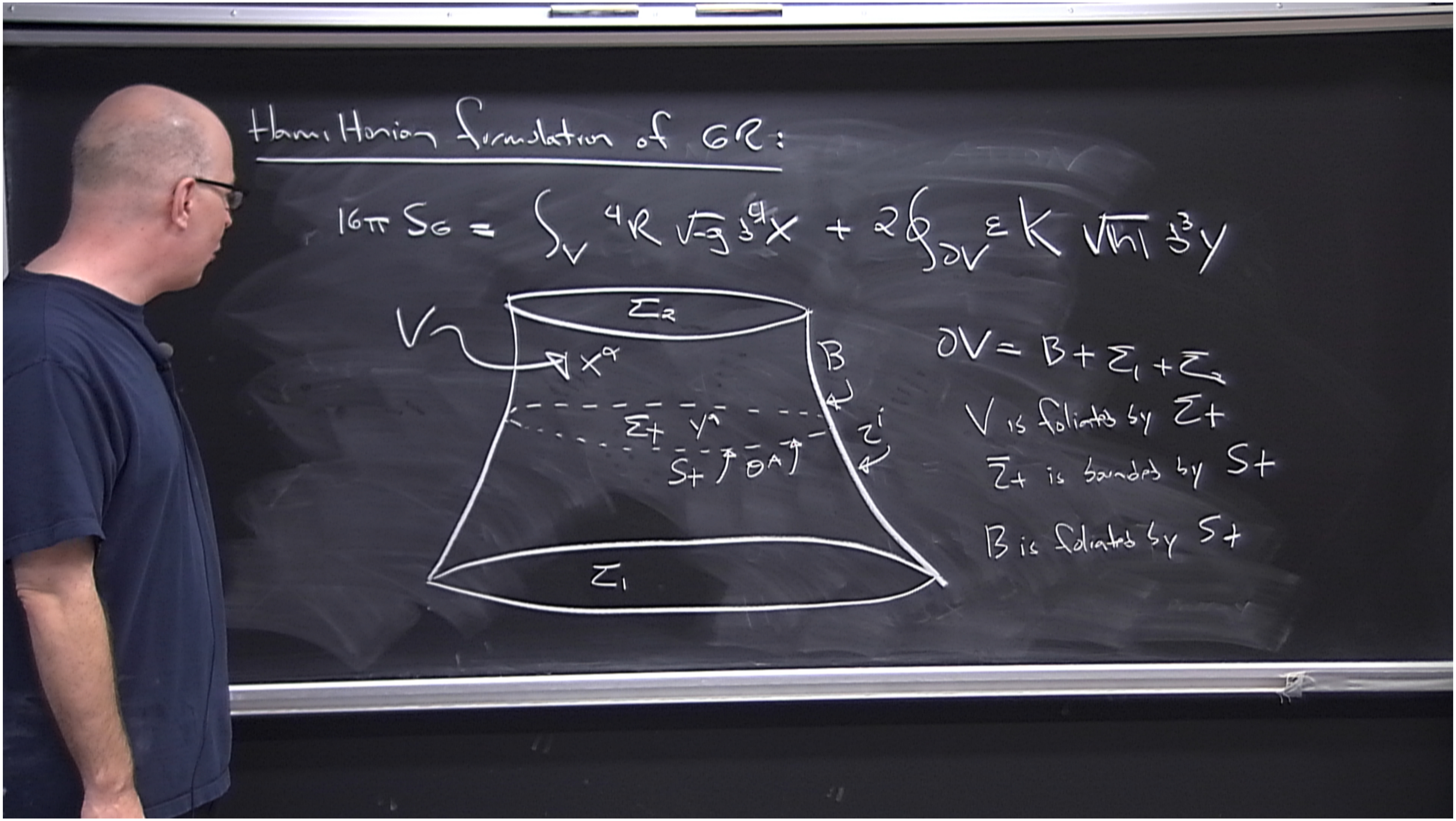
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$\partial V = B + \Sigma_1 + \Sigma_2$
V is foliated by Σ_t
Σ_t is bounded by S_t
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