Title: Explorations in String Theory - Lecture 9

Date: Mar 22, 2012 11:30 AM

URL: http://www.pirsa.org/12030052

Abstract:

In a Conformal Field Theory

(for scalar operators) $$\langle \mathcal{O}_A(x_1)\,\mathcal{O}_B(x_2)\rangle = \frac{\delta_{AB}}{(x_1-x_2)^{2\Delta_A}}$$

conformal data

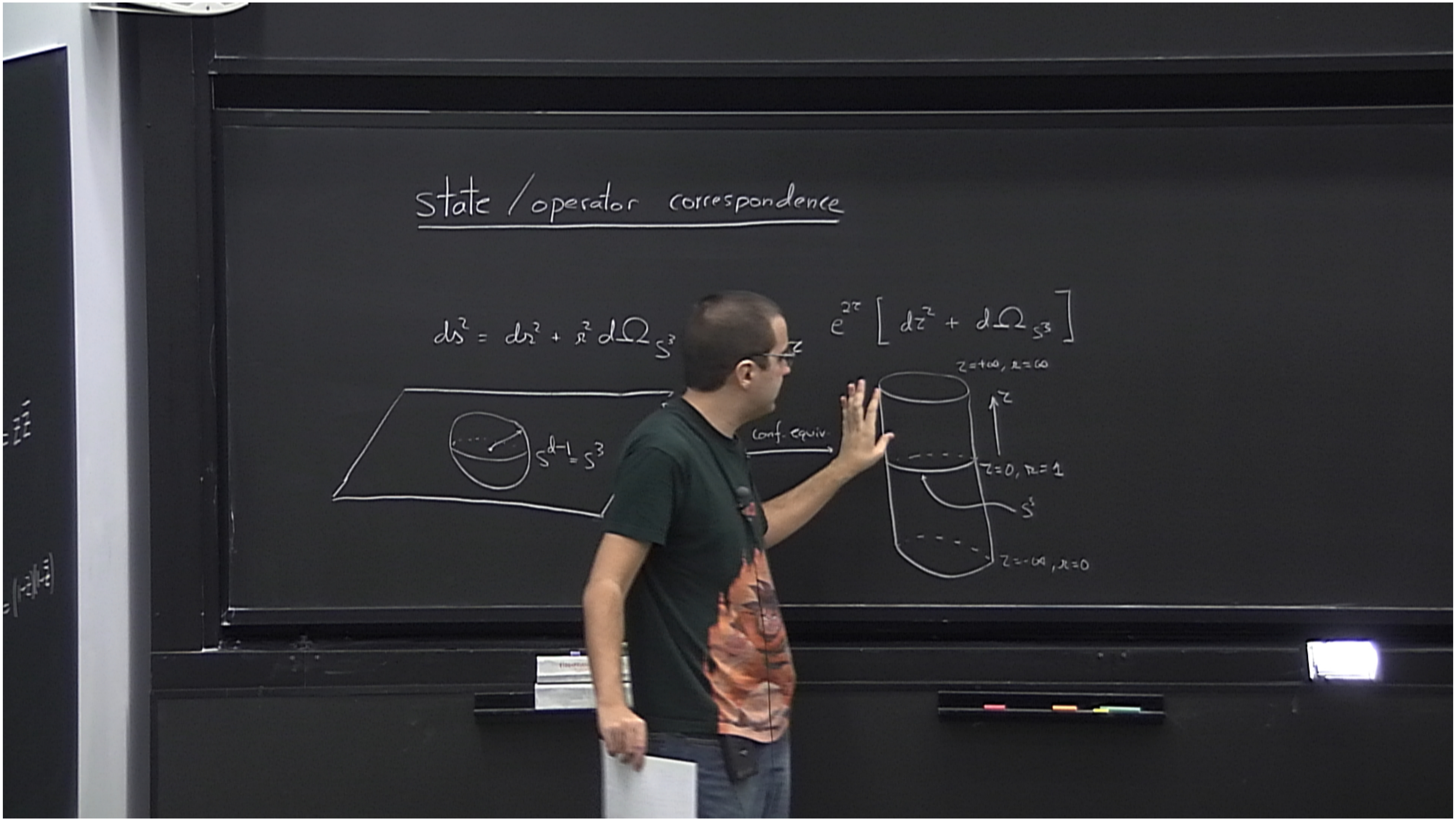
$$\{\Delta_A,\ C_{ABC}\} \xrightarrow[\text{TODAY}]{} \text{fixes } \mathcal{F}(z,\bar{z})$$

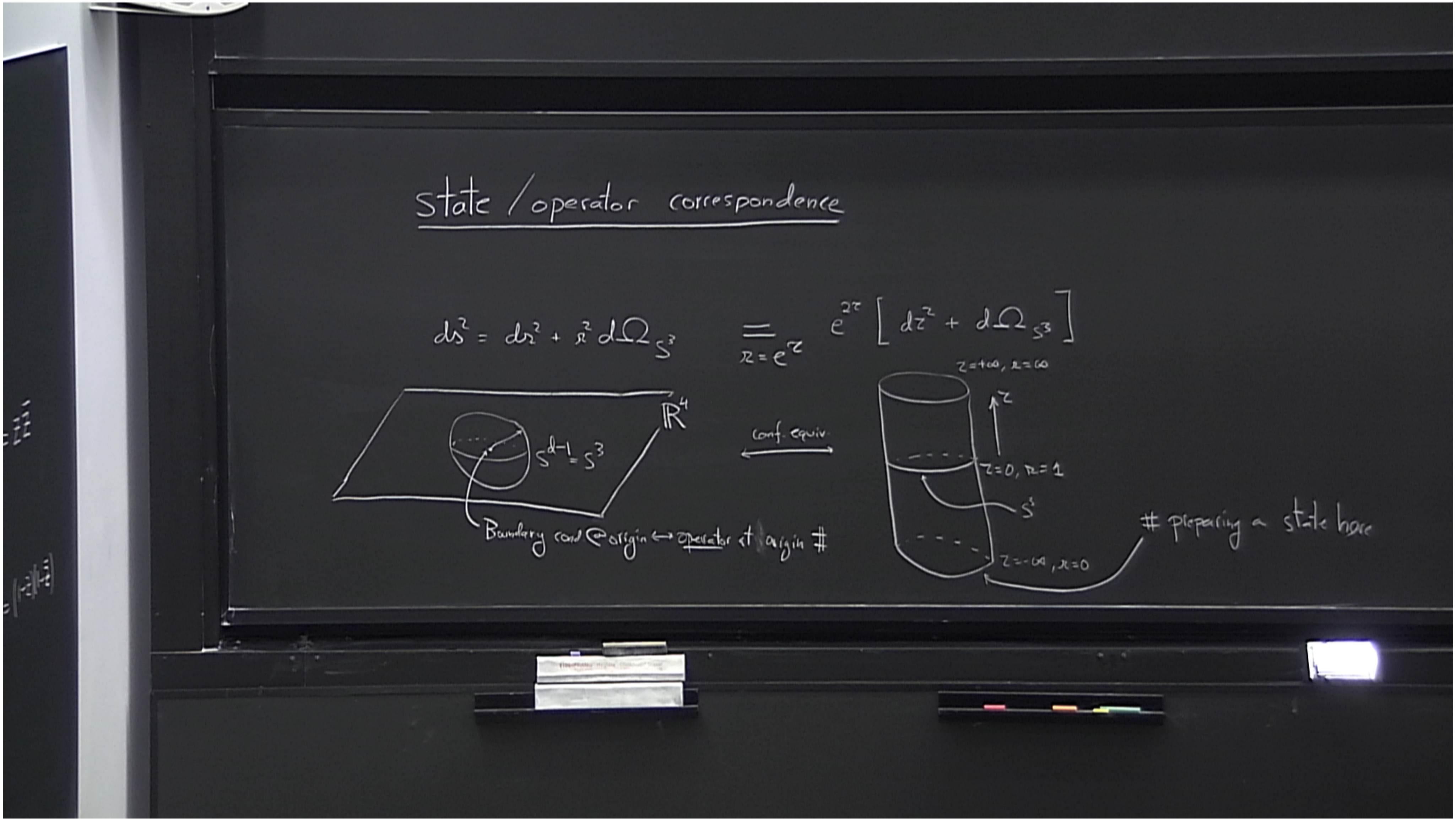
$$\langle \mathcal{O}_1(x_1)\,\mathcal{O}_2(x_2)\,\mathcal{O}_3(x_3)\rangle = \frac{C_{123}}{|x_1-x_2|^{\Delta_1+\Delta_2-\Delta_3}\,|x_1-x_3|^{\Delta_1+\Delta_3-\Delta_2}\,|x_2-x_3|^{\Delta_2+\Delta_3-\Delta_1}}$$

$$\langle \mathcal{O}_1(x_1)\,\mathcal{O}_2(x_2)\,\mathcal{O}_3(x_3)\,\mathcal{O}_4(x_4)\rangle = \frac{1}{|x_1-x_2|^{\Delta_1+\Delta_2}\,|x_3-x_4|^{\Delta_3+\Delta_4}}\left|\frac{x_2-x_4}{x_1-x_4}\right|^{\Delta_1-\Delta_2}\left|\frac{x_1-x_4}{x_1-x_3}\right|^{\Delta_3-\Delta_4}\mathcal{F}(z,\bar{z})$$

$$\frac{(x_1-x_2)^2(x_3-x_4)^2}{(x_1-x_3)^2(x_2-x_4)^2}$$

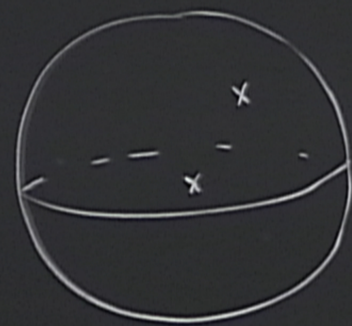
$$\frac{(x_1-x_4)^2(x_2-x_3)^2}{(x_1-x_3)^2(x_2-x_4)^2}$$


state / operator correspondence
$ds^2 = dr^2 + r^2 d\Omega_{S^3}$
$e^{2\tau}\left[d\tau^2 + d\Omega_{S^3} \right]$
$S^{d-1} = S^3$
conf. equiv.
$\tau = +\infty,\ r = \infty$
τ
$\tau = 0,\ r = 1$
S^3
$\tau = -\infty,\ r = 0$


state / operator correspondence
$ds^2 = dr^2 + r^2 d\Omega_{S^3} \underset{r=e^\tau}{=} e^{2\tau}\left[d\tau^2 + d\Omega_{S^3} \right]$
$\mathbb{R}^4$
$S^{d-1} = S^3$
conf. equiv.
$\tau=+\infty, r=\infty$
τ
$\tau=0, r=1$
S^3
$\tau=-\infty, r=0$
preparing a state here
Boundary cond @ origin ↔ operator at origin #

$$\Big\}_{\mathbb{R}\times S^3}$$

OPE

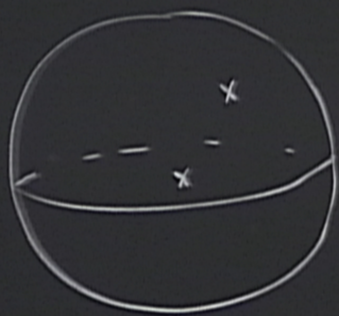


$$= \mathcal{O}_1(0)\,\mathcal{O}_2(x)$$

$$= \sum_i \tilde{C}_i(x)\,\mathcal{O}_i(0)$$

$$= \sum_{\substack{\text{primaries}\\ K}} \Big(C_K(x)\,\mathcal{O}_K(0) + C_K^{\mu}(x)\,\partial_\mu \mathcal{O}_K(0) + C_K^{\mu_1 \mu_2}(x)\,\partial_{\mu_1}\partial_{\mu_2}\mathcal{O}_K(0) + \dots \Big)$$

OPE



$$= \mathcal{O}_1(0)\,\mathcal{O}_2(x)$$

$$= \sum_i \tilde{C}_i(x)\,\mathcal{O}_i(0)$$

$$= \sum_{\substack{\text{primaries}\\ k}} \Big(C_k(x)\,\mathcal{O}_k(0) + C_k^{\mu}(x)\,\partial_\mu \mathcal{O}_k(0) + C_k^{\mu_1\mu_2}(x)\,\partial_{\mu_1}\partial_{\mu_2}\mathcal{O}_k(0) + \ldots \Big)$$

under dilatation $x \to \lambda x$

$$\mathcal{O}_i \longrightarrow \lambda^{-\Delta_i} \mathcal{O}_i$$

$$\text{lhs} \to \lambda^{-\Delta-\Delta_2} \, \text{lhs}$$

$$C_k^{\mu_1 \cdots \mu_n} \longrightarrow \lambda^{-\Delta_1 - \Delta_2 + \Delta_k + n} \, C_k^{\mu_1 \cdots \mu_n}$$

under dilatation $x \to \lambda x$

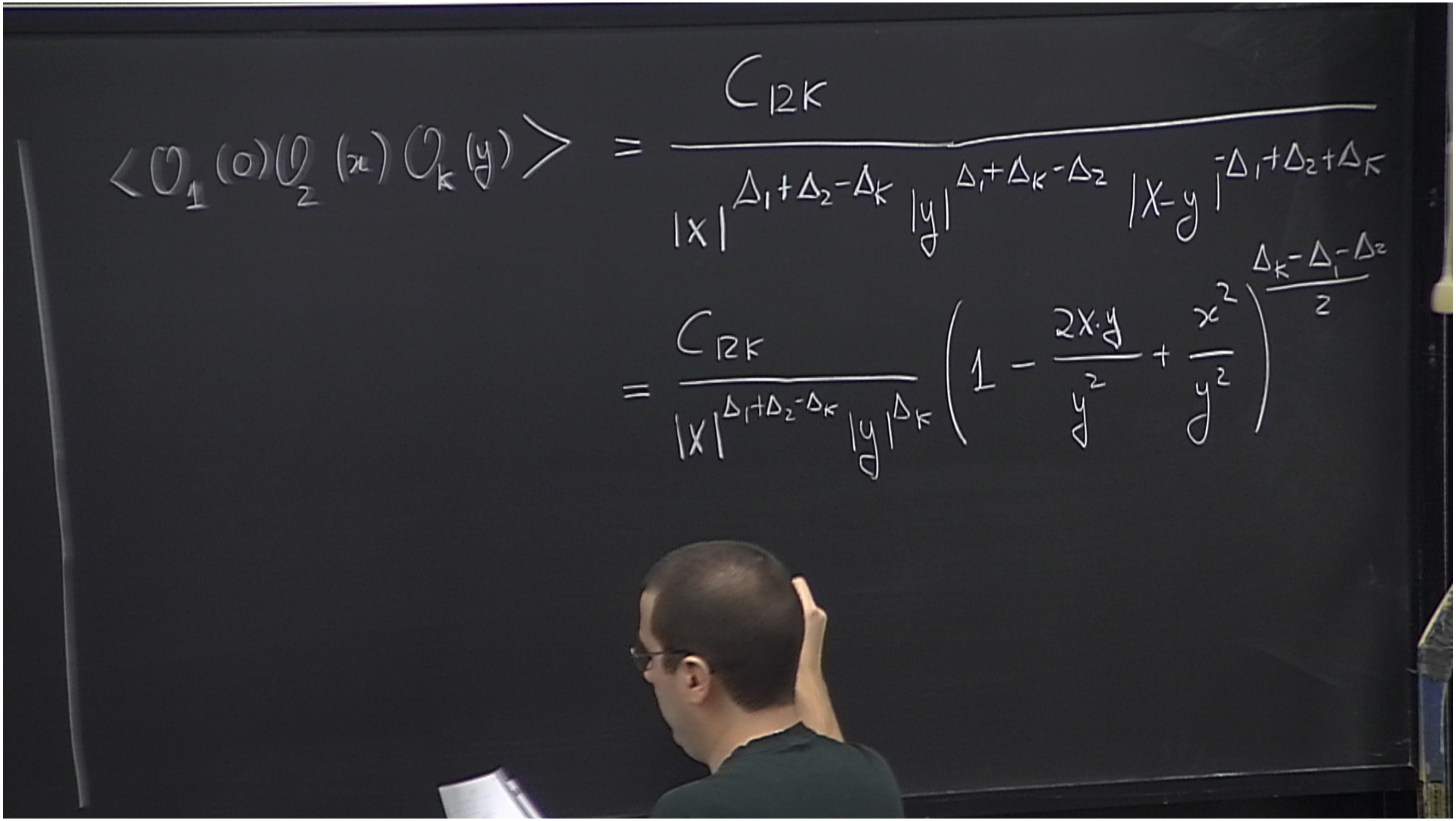
$$\mathcal{O}_i \longrightarrow \lambda^{-\Delta_i} \mathcal{O}_i$$

$$\text{lhs} \to \lambda^{-\Delta_1-\Delta_2} \text{lhs}$$

more and more suppressed as $x \to 0$ when n increases

$$C_k^{\mu_1 \cdots \mu_n} \longrightarrow \lambda^{-\Delta_1 - \Delta_2 + \Delta_k + n} C_k^{\mu_1 \cdots \mu_n}$$

Lets fix the $C_k^{\mu_1 \cdots \mu_n}$ using conf. sym.


$\langle \mathcal{O}_1(0)\, \mathcal{O}_2(x)\, \mathcal{O}_k(y) \rangle = \dfrac{C_{12k}}{|x|^{\Delta_1+\Delta_2-\Delta_k}\, |y|^{\Delta_1+\Delta_k-\Delta_2}\, |x-y|^{-\Delta_1+\Delta_2+\Delta_k}}$
$= \dfrac{C_{12k}}{|x|^{\Delta_1+\Delta_2-\Delta_k}\, |y|^{\Delta_k}} \left(1 - \dfrac{2x\cdot y}{y^2} + \dfrac{x^2}{y^2}\right)^{\frac{\Delta_k-\Delta_1-\Delta_2}{2}}$

$$\langle \mathcal{O}_1(0)\,\mathcal{O}_2(x)\,\mathcal{O}_k(y)\rangle = \frac{\quad}{|x|^{\Delta_1+\Delta_2-\Delta_k}\,|y|^{\Delta_1+\Delta_k-\Delta_2}\,|x-y|^{-\Delta_1+\Delta_2+\Delta_k}}$$

$$= \frac{C_{12k}}{|x|^{\Delta_1+\Delta_2-\Delta_k}\,|y|^{\Delta_k}}\underbrace{\left(1-\frac{2x\cdot y}{y^2}+\frac{x^2}{y^2}\right)^{\frac{\Delta_k-\Delta_1-\Delta_2}{2}}}_{1+\frac{\Delta_k-\Delta_1-\Delta_2}{2}\frac{y\cdot x}{y^2}+\cdots\cdots}$$

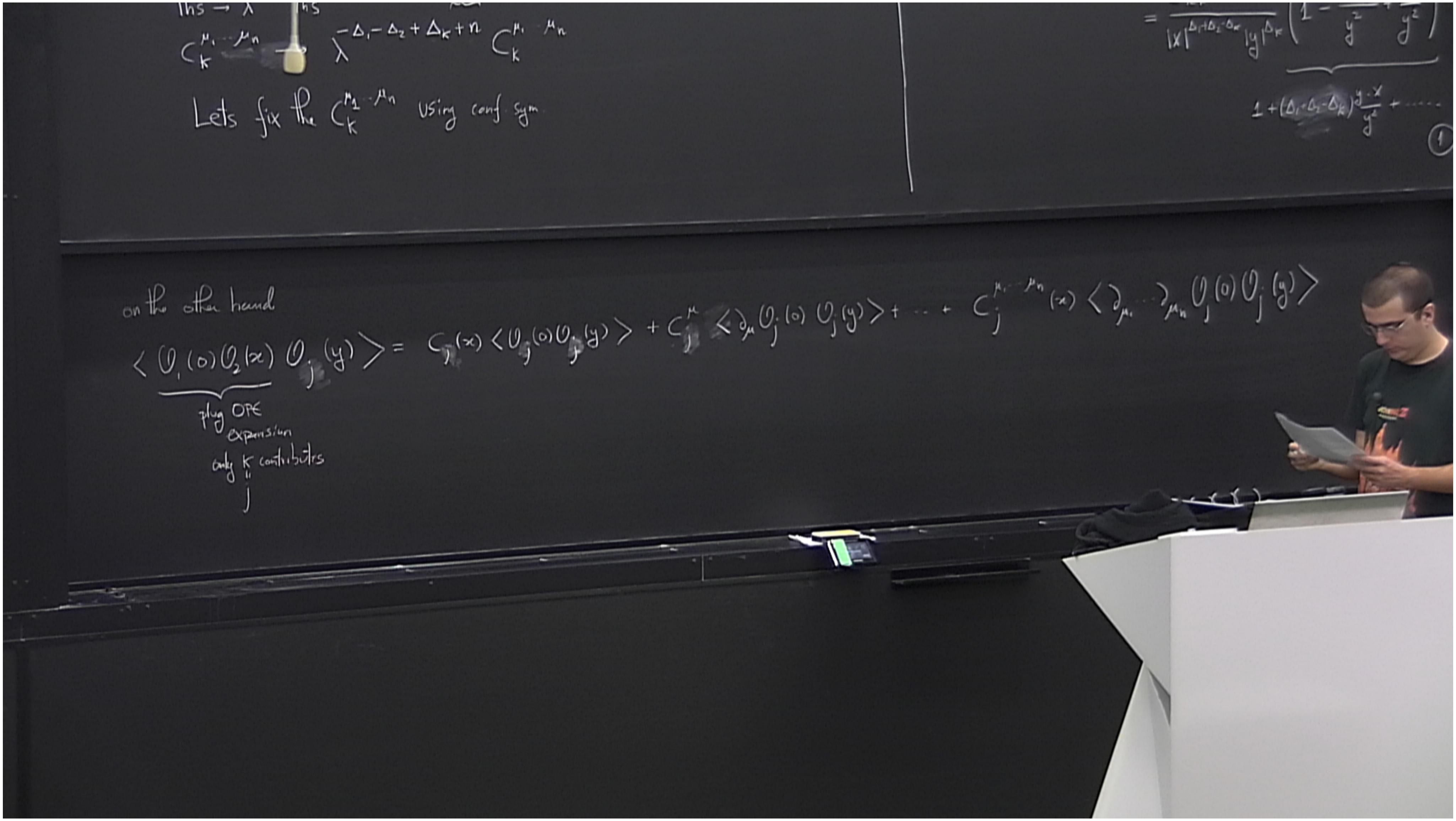
$$\langle \mathcal{O}_1(0)\mathcal{O}_2(x)\mathcal{O}_k(y)\rangle = \frac{C_{12k}}{|x|^{\Delta_1+\Delta_2-\Delta_k}\,|y|^{\Delta_1+\Delta_k-\Delta_2}\,|x-y|^{-\Delta_1+\Delta_2+\Delta_k}}$$

$$= \frac{C_{12k}}{|x|^{\Delta_1+\Delta_2-\Delta_k}\,|y|^{\Delta_k}} \underbrace{\left(1-\frac{2x\cdot y}{y^2}+\frac{x^2}{y^2}\right)^{\frac{\Delta_k-\Delta_1-\Delta_2}{2}}}_{1+\frac{\Delta_k-\Delta_1-\Delta_2}{2}\frac{y\cdot x}{y^2}+\cdots\cdots}$$

1

$$= \frac{C_{12k}}{|x|^{\Delta_1+\Delta_2-\Delta_k}\,|y|^{\Delta_k}} \underbrace{\left(1 - \frac{2x\cdot y}{y^2} + \frac{x^2}{y^2}\right)^{\frac{\cdots}{2}}}_{1 + (\Delta_1+\Delta_2-\Delta_k)\frac{y\cdot x}{y^2} + \cdots\cdots}$$

1


Lets fix the $C_k^{\mu_1 \dots \mu_n}$ using conf sym
on the other hand
plug OPE expansion
only k contributes

$$\langle \mathcal{O}_1(0)\,\mathcal{O}_2(x)\,\mathcal{O}_k(y)\rangle = \frac{C_{12k}}{|x|^{\Delta_1+\Delta_2-\Delta_k}\,|y|^{\Delta_1+\Delta_k-\Delta_2}\,|x-y|^{-\Delta_1+\Delta_2+\Delta_k}}$$

$$= \frac{C_{12k}}{|x|^{\Delta_1+\Delta_2-\Delta_k}\,|y|^{2\Delta_k}}\underbrace{\left(1-\frac{2x\cdot y}{y^2}+\frac{x^2}{y^2}\right)^{\frac{\Delta_k-\Delta_1-\Delta_2}{2}}}_{1+(\Delta_1+\Delta_2-\Delta_k)\frac{y\cdot x}{y^2}+\cdots\cdots}$$

1

under dilatation $x \to \lambda x$

$\mathcal{O}_i \to \lambda^{-\Delta_i} \mathcal{O}_i$

$\text{lhs} \to \lambda^{-\Delta_1-\Delta_2}\,\text{lhs}$

more and more suppressed as $x \to 0$ when n increases

$C_k^{\mu_1\cdots\mu_n} \longrightarrow \lambda^{-\Delta_1-\Delta_2+\Delta_k+n}\, C_k^{\mu_1\cdots\mu_n}$

Lets fix the $C_k^{\mu_1\cdots\mu_n}$ using conf sym

$$\langle \mathcal{O}_1(0)\mathcal{O}_2(x)\mathcal{O}_k(y)\rangle = \frac{C_{12k}}{|x|^{\Delta_1+\Delta_2-\Delta_k}\,|y|^{\Delta_1+\Delta_k-\Delta_2}\,|x-y\cdots}$$

$$= \frac{C_{12k}}{|x|^{\Delta_1+\Delta_2-\Delta_k}\,|y|^{2\Delta_k}}\left(1-\frac{2x\cdot y}{y^2}+\frac{x\cdots}{y\cdots}\right.$$

$$1+(\Delta_1+\Delta_2-\Delta_k)\frac{y\cdot x}{y^2}+\cdots$$

on the other hand

$$\underbrace{\langle \mathcal{O}_1(0)\mathcal{O}_2(x)}_{\substack{\text{plug OPE}\\ \text{expansion}\\ \text{only } k \text{ contributes}\\ \parallel \\ j}}\mathcal{O}_j(y)\rangle = C_j(x)\langle \mathcal{O}_j(0)\mathcal{O}_j(y)\rangle + C_j^{\mu}\langle \partial_\mu \mathcal{O}_j(0)\,\mathcal{O}_j(y)\rangle + \cdots + C_j^{\mu_1\cdots\mu_n}(x)\langle \partial_{\mu_1}\cdots\partial_{\mu_n}\mathcal{O}_j(0)\,\mathcal{O}_j(y)\rangle \quad (2)$$

$$= C_j(x)\frac{1}{|y|^{2\Delta_j}} + C_j^{\mu}(x)\frac{2\Delta_j\, y_\mu}{|y|^{2\Delta_j}} + \cdots$$

Comparing 1 and 2

$$C_K(x) = \frac{C_{12K}}{|x|^{\Delta_1+\Delta_2-\Delta_K}}$$

$$C_K^{\mu}(x) = \frac{C_{12K}}{|x|^{\Delta_1+\Delta_2-\Delta_K}} \frac{\Delta_1+\Delta_2-\Delta_K}{2\Delta_K} x^{\mu}$$

$$C_K^{\mu_1 \cdots \mu_n}(x) = \frac{C_{12K}}{|x|^{\Delta_1+\Delta_2-\Delta_K}} \times \left(\text{Something totally fixed by conf. sym}\right)$$

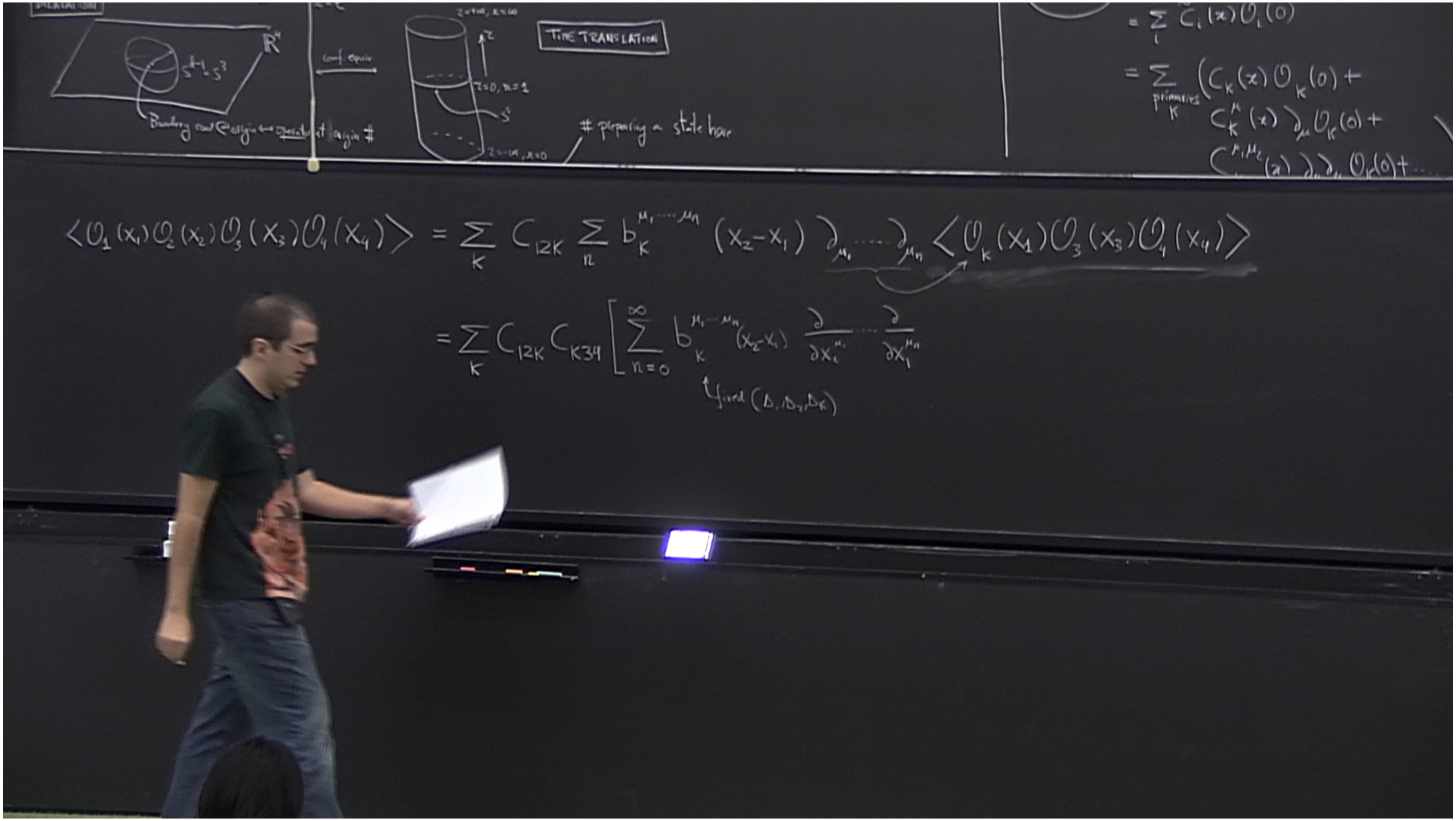
Comparing 1 and 2

$$C_k(x) = \frac{C_{12k}}{|x|^{\Delta_1+\Delta_2-\Delta_k}}$$

← depend C_{12k}, $\{\Delta_k\}$

$$C_k^{\mu}(x) = \frac{C_{12k}}{|x|^{\Delta_1+\Delta_2-\Delta_k}} \frac{\Delta_1+\Delta_2-\Delta_k}{2\Delta_k} x^{\mu}$$

$$C_k^{\mu_1\cdots\mu_n}(x) = \frac{C_{12k}}{|x|^{\Delta_1+\Delta_2-\Delta_k}} \times (\text{Something totally fixed by conf. sym})$$


TIME TRANSLATION
preparing a state here
primaries
fixed (Δ1, Δ2, Δk)

DILATATION

$\mathbb{R}^4$

$S^{d-1} = S^3$

Boundary cond @ origin ↔ operator at origin #

conf equiv

$\tau = +\infty,\ r = \infty$

τ

$\tau = 0,\ r = 1$

S^3

$\tau = -\infty,\ r = 0$

TIME TRANSLATION

preparing a state here

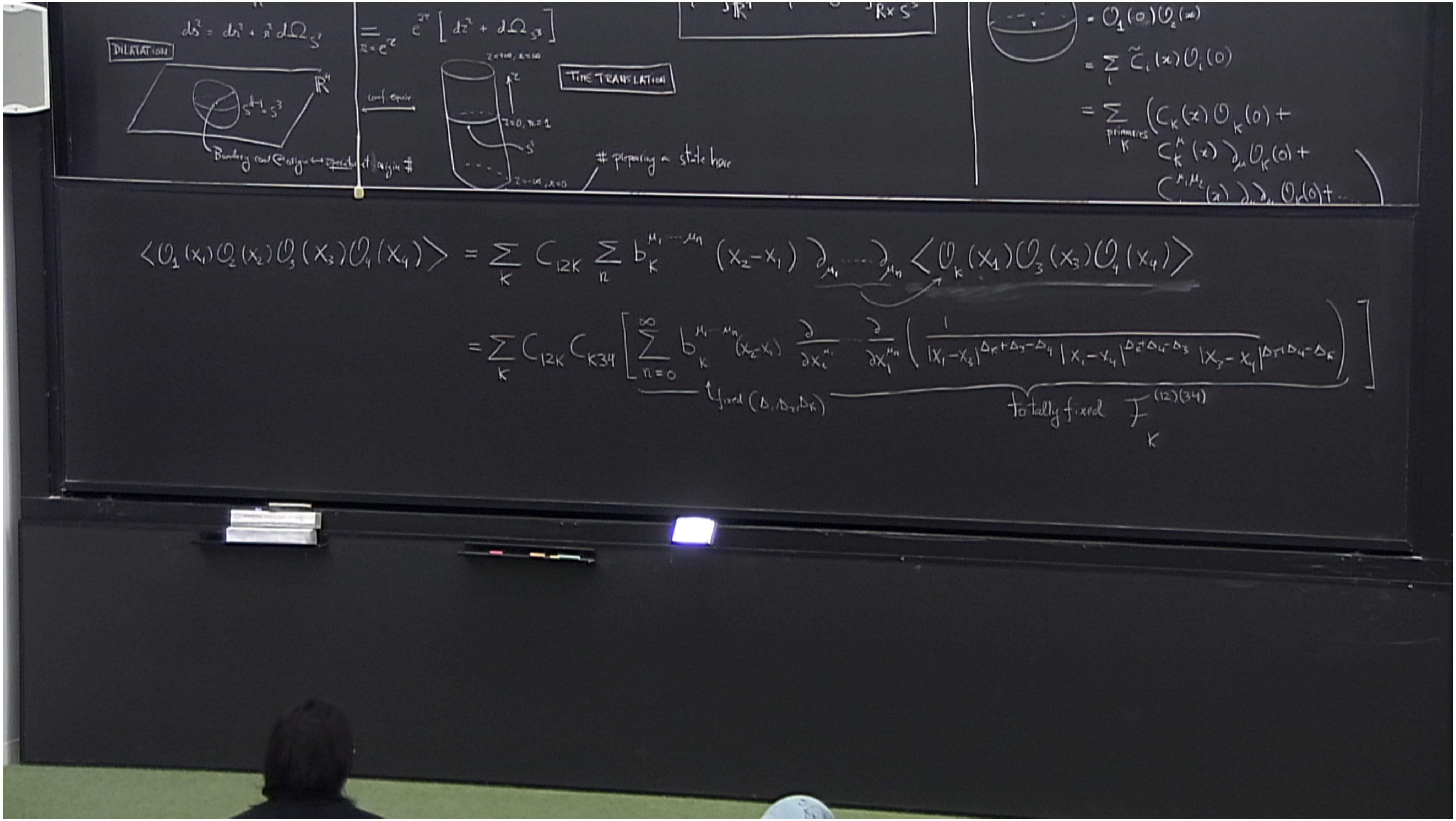
$$= \sum_i C_i(x)\, \mathcal{O}_i(0)$$

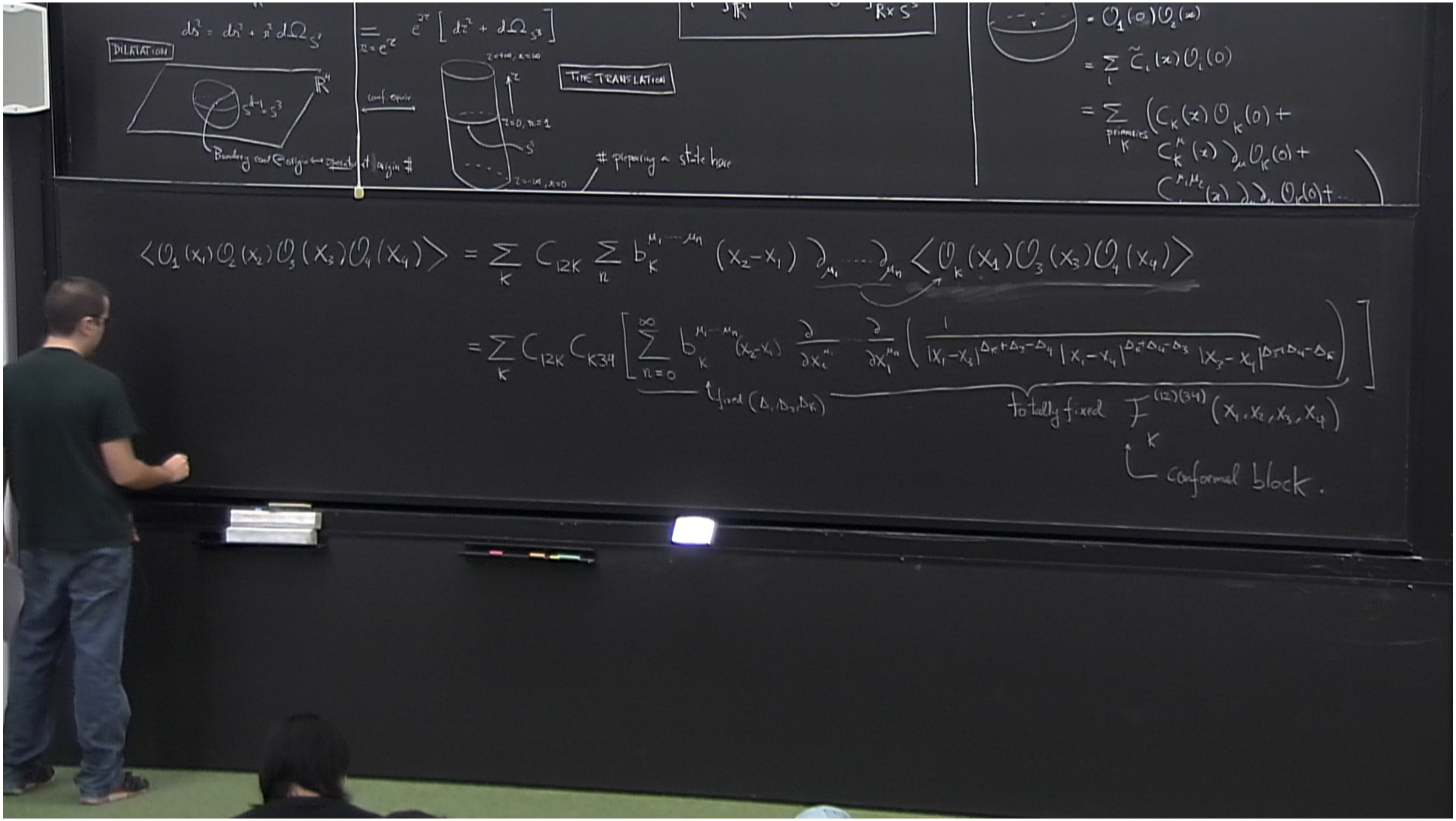
$$= \sum_{\substack{\text{primaries} \\ K}} \Big(C_K(x)\, \mathcal{O}_K(0) + C_K^{\mu}(x)\, \partial_\mu \mathcal{O}_K(0) + C_K^{\mu_1 \mu_2}(x)\, \partial_{\mu_1}\partial_{\mu_2} \mathcal{O}_K(0) + \cdots$$

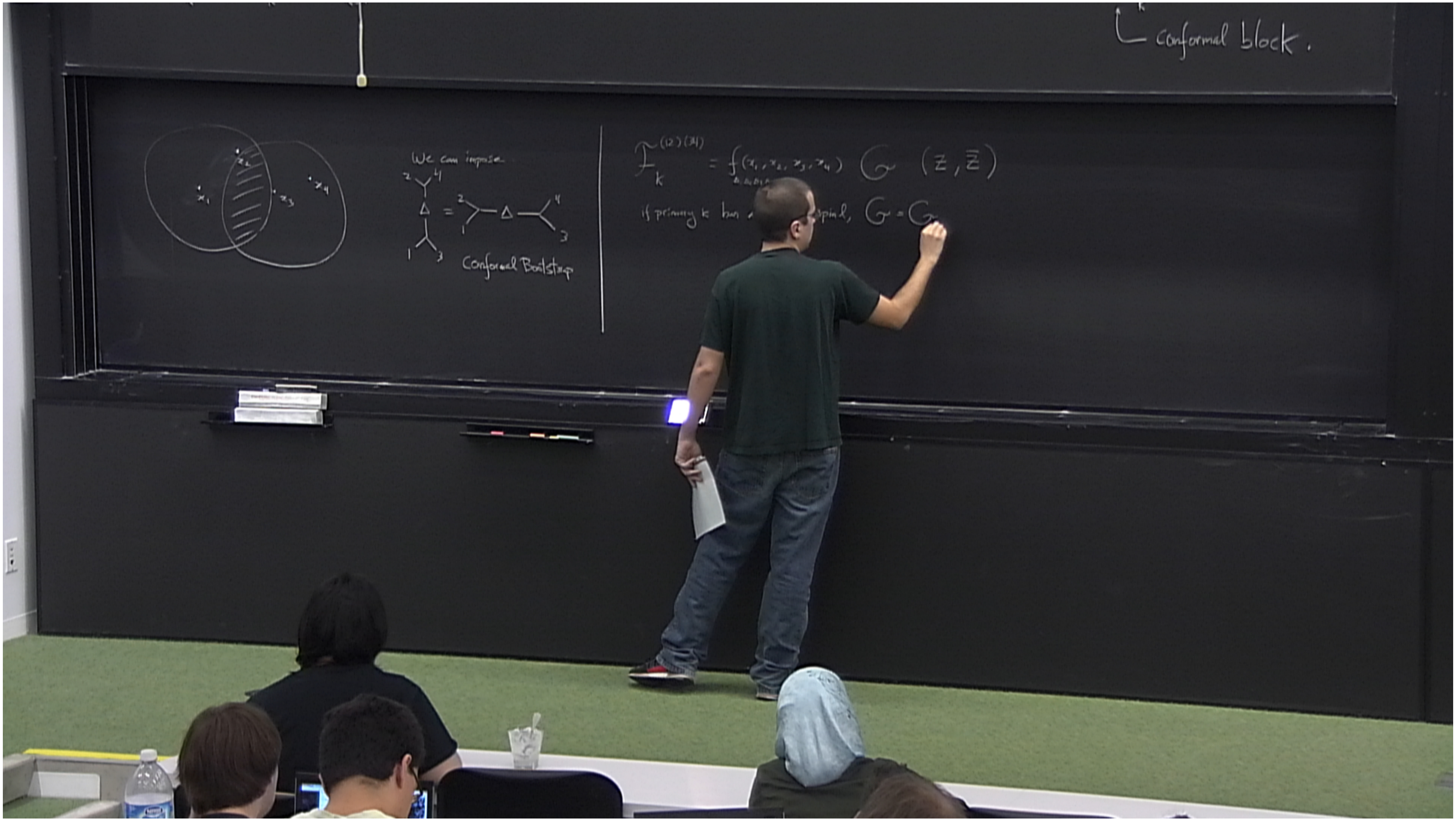
$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle = \sum_k C_{12k} \sum_n b_k^{\mu_1 \cdots \mu_n}(x_2 - x_1)\, \partial_{\mu_1} \cdots \partial_{\mu_n} \langle \mathcal{O}_k(x_1)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle$$

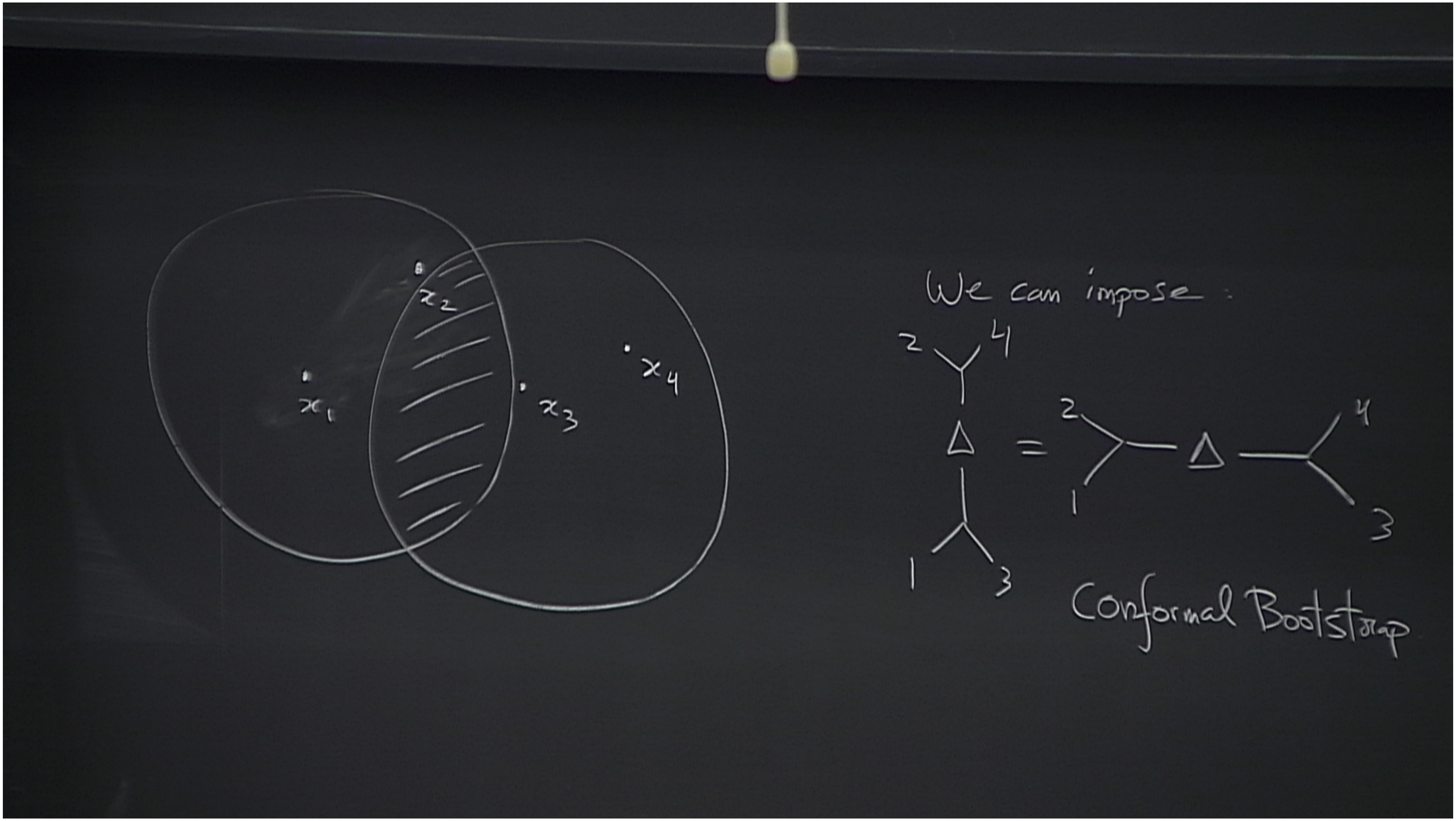
$$= \sum_k C_{12k} C_{k34} \left[\sum_{n=0}^{\infty} b_k^{\mu_1 \cdots \mu_n}(x_2 - x_1)\, \frac{\partial}{\partial x_1^{\mu_1}} \cdots \frac{\partial}{\partial x_1^{\mu_n}} \left(\frac{1}{|x_1 - x_3|^{\Delta_k + \Delta_3 - \Delta_4}\, |x_1 - x_4|^{\Delta_k + \Delta_4 - \Delta_3}\, |x_3 - x_4|^{\Delta_3 + \Delta_4 - \Delta_k}} \right) \right]$$

fixed $(\Delta_1, \Delta_2, \Delta_k)$


DILATATION
$ds^2 = dr^2 + r^2 d\Omega_{S^3}$
$\mathbb{R}^4$
$S^{d-1} = S^3$
Boundary cond @ origin ⟷ operator at origin
$r = e^{\tau}$
$e^{2\tau}[d\tau^2 + d\Omega_{S^3}]$
conf. equiv
TIME TRANSLATION
$\tau = 0, r = 1$
S^3
preparing a state here
$R \times S^3$
$= \mathcal{O}_1(0)\mathcal{O}_2(x)$
$= \sum_i \tilde{C}_i(x)\mathcal{O}_i(0)$
$= \sum_{\text{primaries } k} \big(C_k(x)\mathcal{O}_k(0) + C_k^{\mu}(x)\partial_\mu \mathcal{O}_k(0) + C_k^{\mu_1\mu_2}(x)\partial_{\mu_1}\partial_{\mu_2}\mathcal{O}_k(0) + \dots \big)$
$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle = \sum_k C_{12k} \sum_n b_k^{\mu_1\cdots\mu_n}(x_2-x_1)\,\partial_{\mu_1}\cdots\partial_{\mu_n}\langle \mathcal{O}_k(x_1)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle$
$= \sum_k C_{12k}C_{k34}\Big[\sum_{n=0}^{\infty} b_k^{\mu_1\cdots\mu_n}(x_2-x_1)\frac{\partial}{\partial x_1^{\mu_1}}\cdots\frac{\partial}{\partial x_1^{\mu_n}}\Big(\frac{1}{|x_1-x_3|^{\Delta_k+\Delta_3-\Delta_4}|x_1-x_4|^{\Delta_k+\Delta_4-\Delta_3}|x_3-x_4|^{\Delta_3+\Delta_4-\Delta_k}}\Big)\Big]$
fixed $(\Delta,\Delta_2,\Delta_k)$
totally fixed $F_k^{(12)(34)}$


$ds^2 = dr^2 + r^2 d\Omega_{S^3}$
DILATATION
$\mathbb{R}^4$
$S^{d-1} = S^3$
Boundary cond @ origin ↔ operator at origin #
$r = e^{\tau}$
$e^{2\tau}\left[d\tau^2 + d\Omega_{S^3}\right]$
conf. equiv.
TIME TRANSLATION
$\tau = +\infty,\ r = \infty$
$\tau = 0,\ r = 1$
S^3
$\tau = -\infty,\ r = 0$
preparing a state here
$= \mathcal{O}_1(x)\mathcal{O}_2(0)$
$= \sum_i \tilde{C}_i(x)\mathcal{O}_i(0)$
$= \sum_{\text{primaries } K}\Big(C_K(x)\mathcal{O}_K(0) + C_K^{\mu}(x)\partial_\mu\mathcal{O}_K(0) + C_K^{\mu_1\mu_2}(x)\partial_{\mu_1}\partial_{\mu_2}\mathcal{O}_K(0) + \dots\Big)$
$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle = \sum_K C_{12K}\sum_n b_K^{\mu_1\cdots\mu_n}(x_2-x_1)\,\partial_{\mu_1}\cdots\partial_{\mu_n}\langle\mathcal{O}_K(x_1)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle$
$= \sum_K C_{12K}C_{K34}\left[\sum_{n=0}^{\infty} b_K^{\mu_1\cdots\mu_n}(x_2-x_1)\frac{\partial}{\partial x_1^{\mu_1}}\cdots\frac{\partial}{\partial x_1^{\mu_n}}\left(\frac{1}{|x_1-x_3|^{\Delta_K+\Delta_3-\Delta_4}|x_1-x_4|^{\Delta_K+\Delta_4-\Delta_3}|x_3-x_4|^{\Delta_3+\Delta_4-\Delta_K}}\right)\right]$
fixed $(\Delta_1,\Delta_2,\Delta_K)$
totally fixed $F_K^{(12)(34)}(x_1,x_2,x_3,x_4)$
conformal block.


Conformal block.
We can impose
Conformal Bootstrap
if primary k has
spin,


x_1
x_2
x_3
x_4
We can impose:
2
4
Δ
1
3
=
2
4
Δ
1
3
Conformal Bootstrap

fixed $(\Delta_1, \Delta_2, \Delta_K)$

totally fixed f (x_1, x_2, x_3, x_4)

K

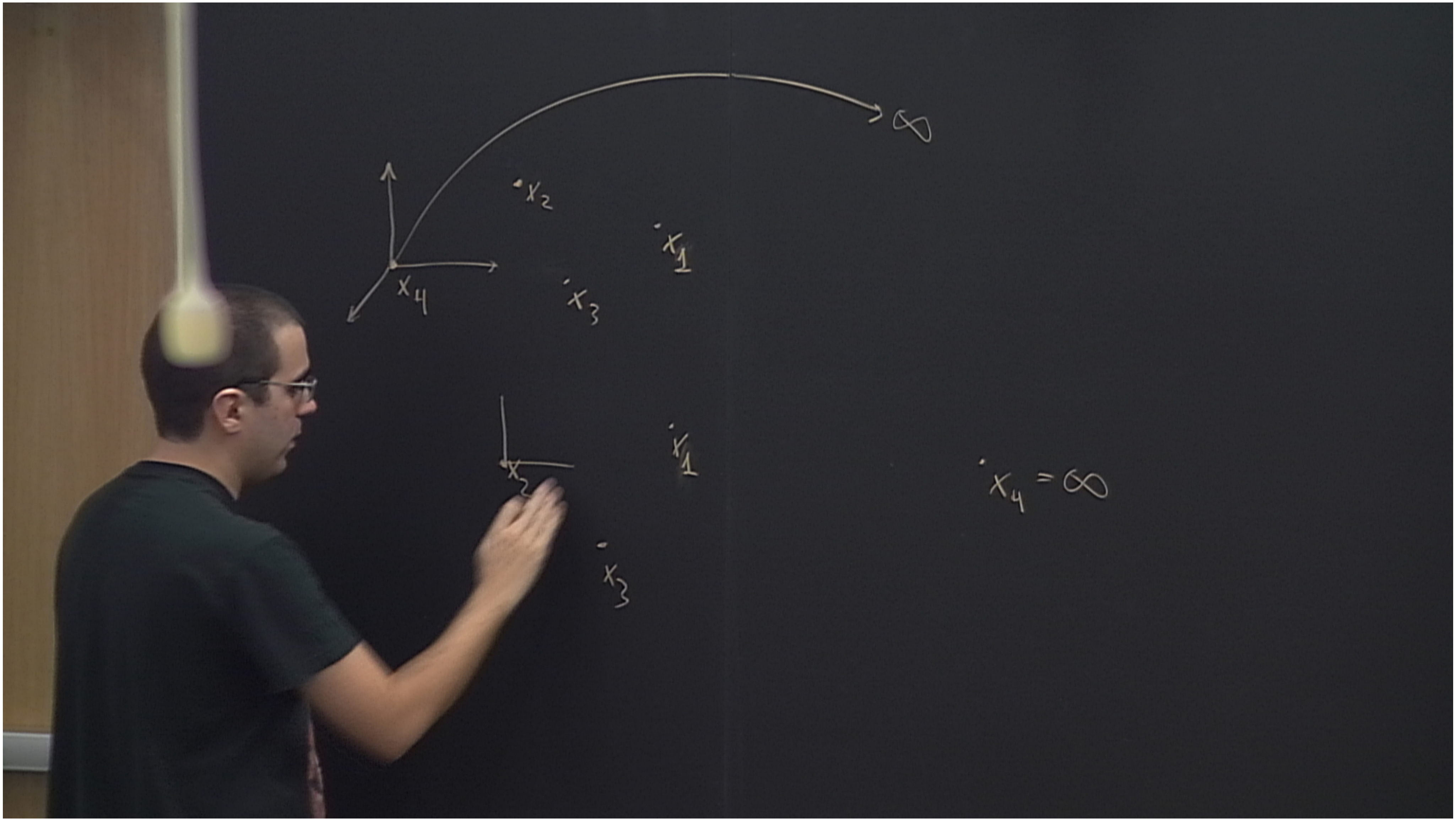
conformal block.

$$\mathcal{F}_k^{(12)(34)} = f_{\Delta_1 \Delta_2 \Delta_3 \Delta_4}(x_1, x_2, x_3, x_4)\; G\,(z, \bar{z})$$

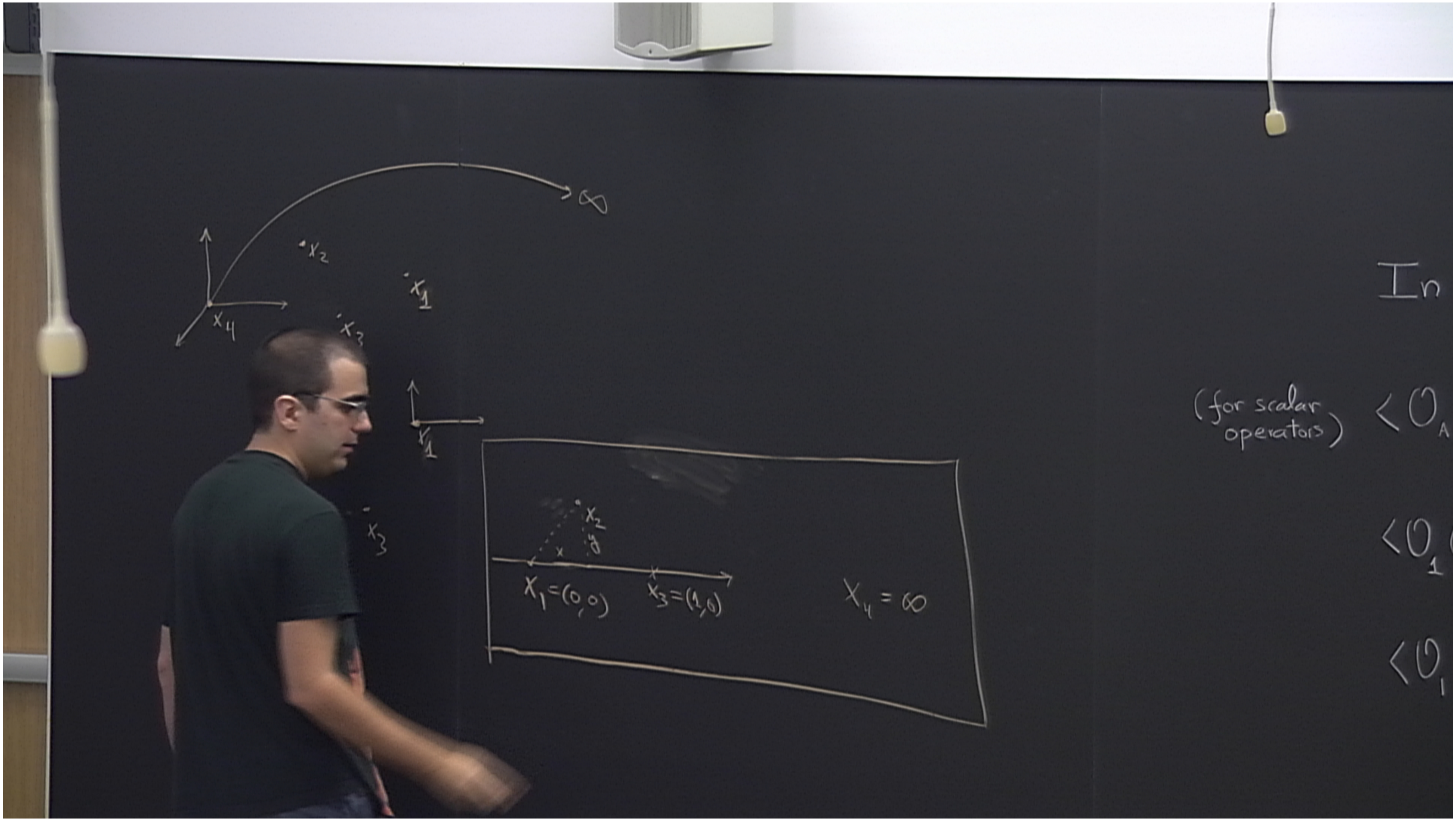
if primary k has dim Δ and spin ℓ, $G = G_{\Delta,\ell}$

$$G_{\Delta,\ell}(z,\bar{z}) = \frac{z\bar{z}}{z-\bar{z}}\left(k_{\frac{\Delta+\ell}{2}}(z)\, k_{\frac{\Delta-\ell}{2}}(\bar{z}) + z \leftrightarrow \bar{z}\right),$$

where $k_\lambda(z) = z^\lambda \times {}_2F_1\left(\lambda + \frac{\Delta_2-\Delta_1}{2}, \lambda + \frac{\Delta_3-\Delta_4}{2}, 2\lambda, z\right)$



∞
x_2
x_1
x_4
x_3
x_1
x_2
$x_4 = \infty$
x_3



∞
x_2
x_1
x_4
x_3
x_1
x_2
x_3
x
y
$x_1 = (0,0)$
$x_3 = (1,0)$
$x_4 = \infty$
(for scalar operators)
In