Title: QF Meeting - Erik Curiel

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Abstract:

Outline

1. Classical and Dynamical Systems
2. Possible Interactions and the Structure of the Space of States
3. Classical Mechanics Is Lagrangian
4. Classical Mechanics Is Not Hamiltonian
5. How Lagrangian and Hamiltonian Mechanics Respectively Represent Classical Systems

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1 Classical and Dynamical Systems

Atheoretical Adumbration of a Classical System

1. something one can interact with
2. bears quantities: magnitudes measurable by apparatuses exploiting particular couplings
3. in states: consistent aggregations of values for all its quantities, sufficient for id at a moment
4. evolves: changes state (in general) over time

Atheoretical Characterization of a Dynamical System

space of states set of identifiable states
quantities scalar/tensorial fields on space of states
evolutions family of vector fields ("kinematically possible")
isolation distinguished vector field ("free kinematical vector field")

comprises representations of all physically significant structure required to investigate classical systems

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Physical Meaning of Elements of a Dynamical System

quantities individuate and identify states; define topology, differential structure on space of states

space of states arcwise-connected = all states of "same sys"; $\exists n$ (even or ∞), minimum quantities needed to individuate and identify states

evolutions "first-order differential equations appropriate for classical systems"

isolation "we know how to shield system"

Why “Atheoretical”

no claim or interpretation depends on fixation of framework or theory (*e.g.*, no “configuration” or “momentum” distinguished)

reaches down to, represents structure at very deep level of our understanding of classical systems

2 Possible Interactions and the Structure of the Space of States

Ends

1. recovery of intrinsic geometry of space of kinematically possible vector fields ("evolutions")
2. intrinsic construction of configuration space
3. intrinsic characterization of space of states' differential topology: tangent bundle of configuration space

Means

1. characterize family of possible interactions
2. intrinsic differentiation of configurative from velocital quantities
3. derive intrinsic geometry of family of possible interactions

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Configurative Quantities?

- where does configuration space come from? (not handed down by Prometheus with fire)
- why do we need it? (probing dynamical systems gets one only space of states and kinematically possible vector fields)
- what, *e.g.*, to choose as configuration for Lagrangian formulation of electromagnetic field, $\mathbf{E}$ or $\mathbf{B}$? why?

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Example: Free Newtonian Particle

- parametrize space of states by $\mathbf{x}$ and $\mathbf{v}$ ("natural coords")
- equations of motion:

$$\dot{\mathbf{x}} = \mathbf{v}$$
$$\dot{\mathbf{v}} = \mathbf{0}$$

- kinematical vector field: $(\mathbf{v}, \mathbf{0})$ ("free kinematical field")

Example: Hit Newtonian Particle with a Stick

- equations of motion:

$$\dot{\mathbf{x}} = \mathbf{v}$$
$$\dot{\mathbf{v}} = \mathbf{F}_{\text{stick}}$$

- kinematical vector field: $(\mathbf{v}, \mathbf{F}_{\text{stick}})$
  ("kinematically possible vector field")

empirical observation

classical systems “couple” to external systems (the environment) only by way of the equation of motion for velocities; equation of motion for position never changes

roughly: velocities need not evolve continuously as interactions turn on and off—can happen as abruptly as one likes; position always evolves continuously

Example: Free Electromagnetic Field

- parametrize space of states by $\nabla \cdot \mathbf{B}$, $\dot{\mathbf{B}}$, $\nabla \cdot \mathbf{E}$, $\dot{\mathbf{E}}$ ("natural coords")
- equations of motion (Maxwell's Equations):

$$\begin{aligned} \nabla \cdot \mathbf{B} &= 0 \\ \dot{\mathbf{B}} &= -\nabla \times \mathbf{E} \\ \nabla \cdot \mathbf{E} &= 0 \\ \dot{\mathbf{E}} &= \nabla \times \mathbf{B} \end{aligned}$$

- kinematical vector field: $(0, -\nabla \times \mathbf{E}, 0, \nabla \times \mathbf{B})$ ("free kinematical field")

empirical observation

again, only half the quantities directly "couple" to external systems; the other half don't ("couple only to second-order")

Configurative Quantities

A brute fact about the physical world

for all classical systems, only some physical quantities can be "directly pushed around via allowed interactions", whereas others can't

One generalizes

configurative quantities are those one cannot directly push around

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Kinematical Constraints

$$\dot{\mathbf{x}} = \mathbf{v}$$

is a **kinematical constraint**: theories don't predict them; they require them as a precondition for their own applicability

The Form of Allowed Interactions

existence of kinematical constraint implies difference of two kinematically possible vector fields always has special form:

- $(\mathbf{v},\, \mathbf{F}_2) - (\mathbf{v},\, \mathbf{F}_1) = (\mathbf{0},\, \mathbf{F}_2 - \mathbf{F}_1)$
- "$\mathsf{MF}_2 - \mathsf{MF}_1$" $= ((0,\, \mathbf{0}),\, (\rho_2 - \rho_1,\, \boldsymbol{J}_2 - \boldsymbol{J}_1))$

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Interaction Vector Fields

difference-vectors of allowed evolutions point only in "velocital directions", encode only rates of change for velocital quantities: **accelerations**

Thus

a vector field of the form $(\mathbf{0}, \mathbf{F})$ represents a **possible interaction**

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The Geometry of Allowed Interactions and Kinematically Possible Vector Fields

1. $(\mathbf{0},\ \mathbf{F}_1) + (\mathbf{0},\ \mathbf{F}_2)$ again an interaction vector field: **vector space**
2. $(\mathbf{v},\ \mathbf{F}_1) + (\mathbf{0},\ \mathbf{F}_2)$ again a kinematically possible vector field: **affine space** modeled on interactions

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Configuration Space

1. divide space of states into equivalence classes: "is connected by an interaction vector field to"
2. by construction, all points in equivalence class have same configuration
3. space of equivalence classes is configuration space
4. in natural way, point of space of states becomes point of configuration space plus tangent vector at that point (= velocity)

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Physical Meaning of Configuration Space

surprising

what counts as "configuration" for a classical system is not intrinsic to the system, but rather depends on the structure of its family of allowed interactions with other systems

The Space of States Is the Tangent Bundle of Configuration Space

1. fix point of configuration space
2. free kinematical vector field at that point takes all possible velocity values
3. $\Rightarrow$ natural, one-to-one, onto mapping of space of states to tangent bundle of configuration space

(theorem of R. Geroch)

3 Classical Mechanics Is Lagrangian

The math starts to get hard.

Ends

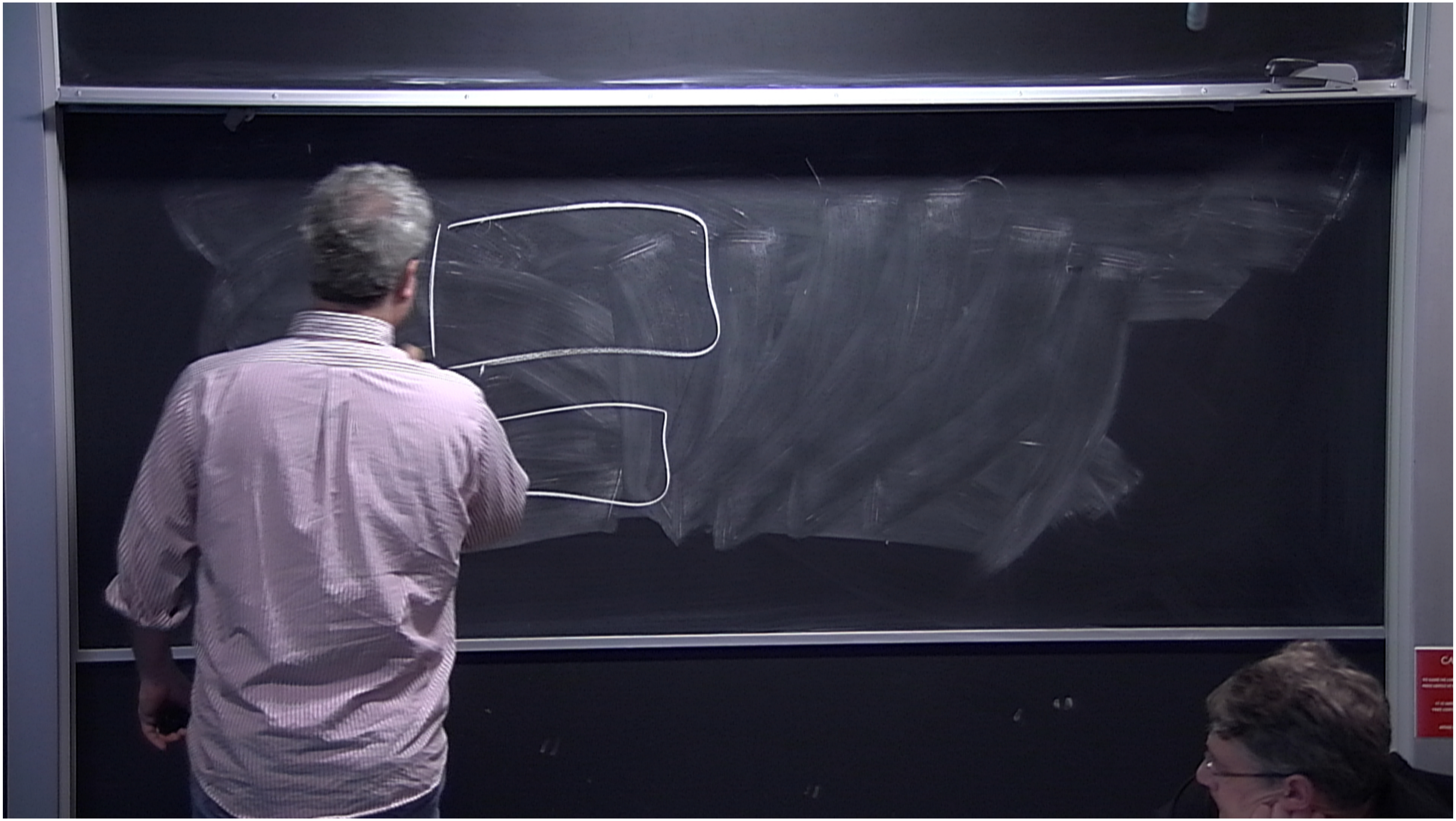
Classical Systems Are Lagrangian structures of an abstract dynamical system, when pushed to the tangent bundle, allow one to construct a Lagrangian representation

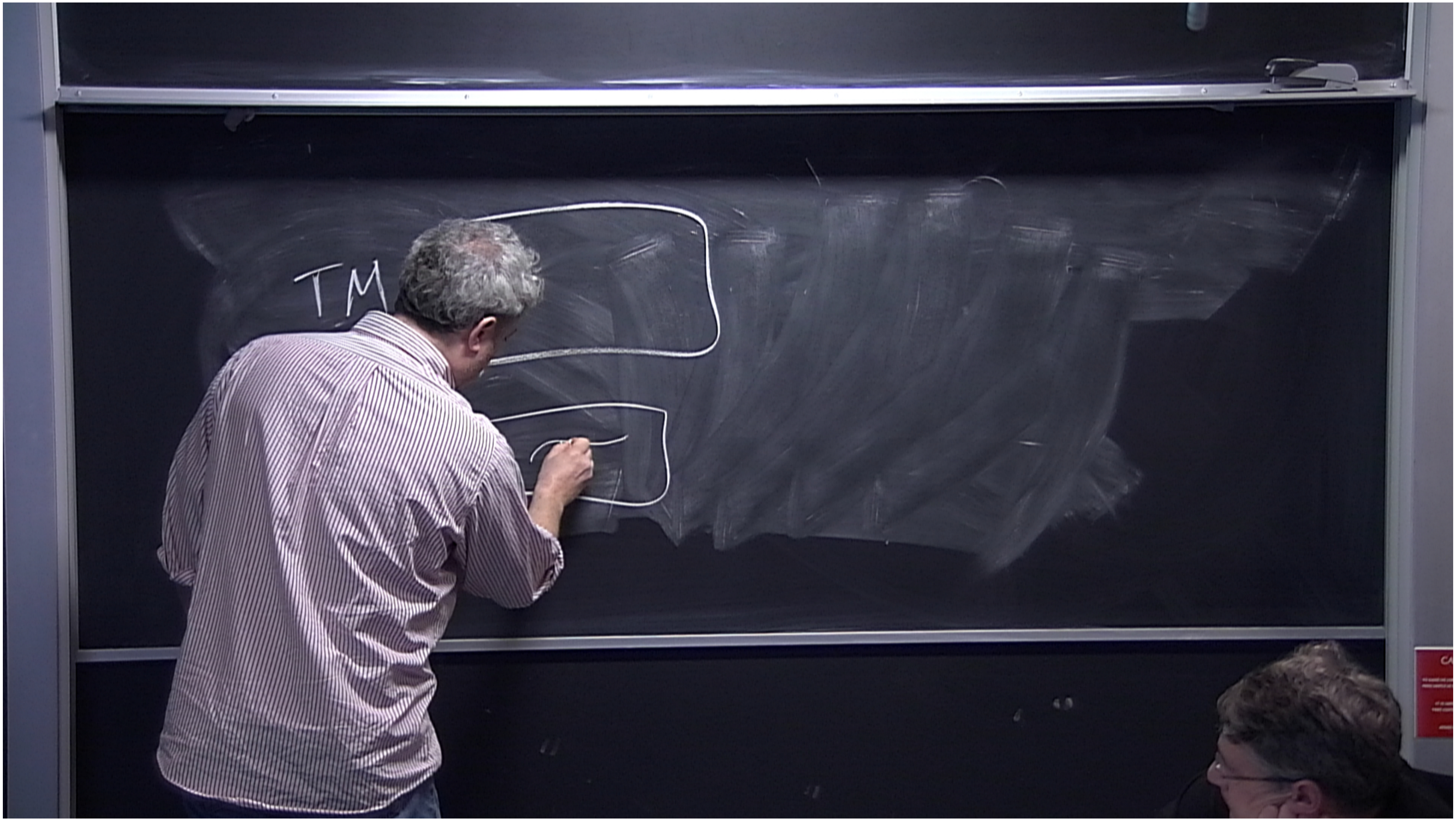
Lagrangian Systems Are Classical a Lagrangian representation of a classical system, in the most minimal sense, allows one to construct its abstract dynamical representation

Means

exploit intrinsic geometry of tangent bundles

1. distinguished vector space of vector fields, “vertical vectors”, possible generalized forces in Lagrangian mechanics (encoded in fiber-bundle structure)
2. distinguished affine space of vector fields modeled on vertical vector fields, possible solutions to Euler-Lagrange equation (encodes **almost-tangent structure**, $\mathbf{J}^{\mathrm{a}}{}_{\mathrm{b}}$)

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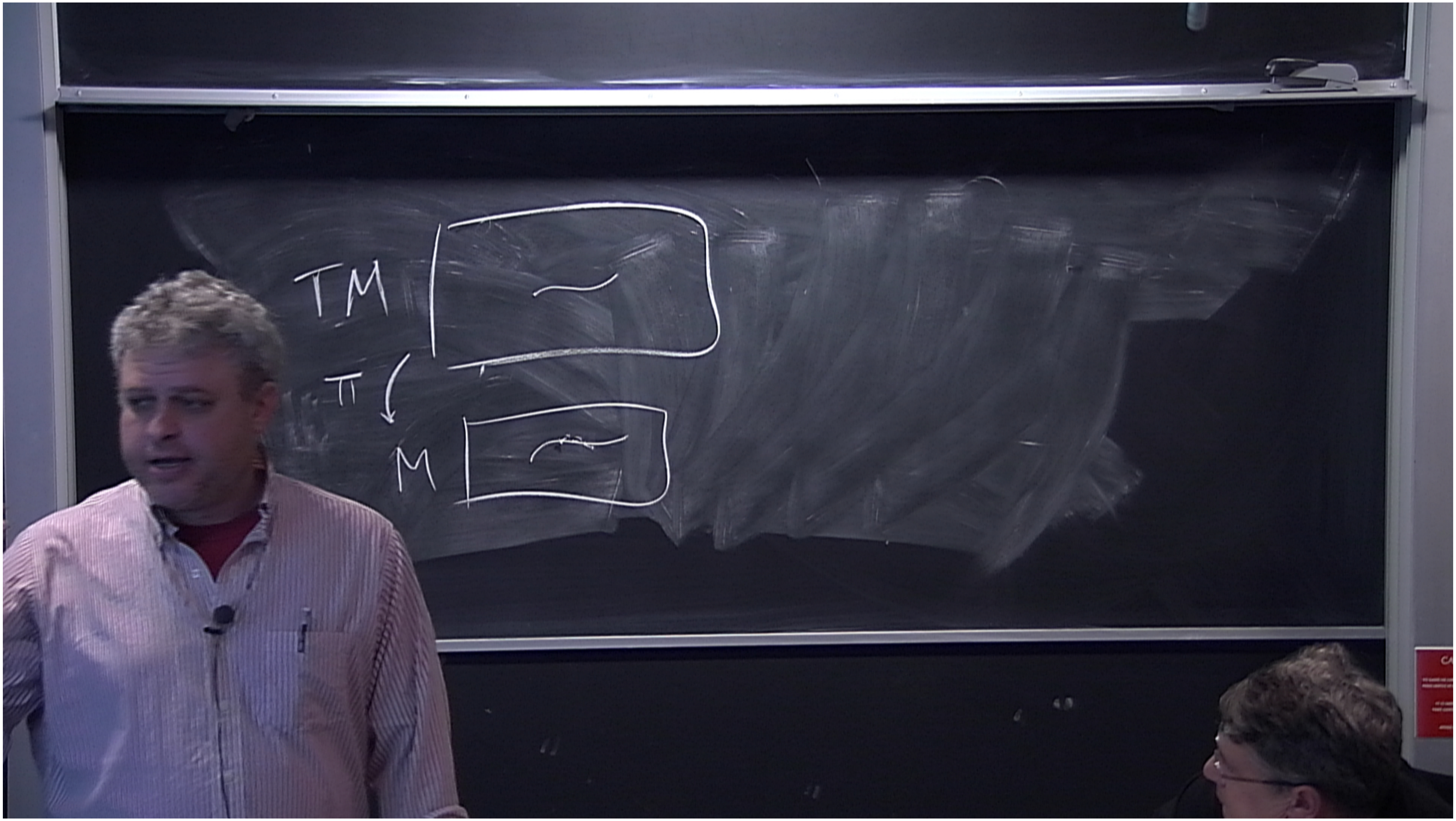
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Almost-Tangent Structure

a smooth tensor field $J^a{}_b$ on a $2n$-dimensional manifold $\mathcal{N}$ satisfying the following conditions:

1. considered as a linear operator on the tangent planes of $\mathcal{N}$, $J^a{}_b$ has rank n everywhere
2. $J^a{}_n J^n{}_b = 0$

It is not difficult to see that, as a linear operator, the range of $J^a{}_b$ equals its kernel, an n-dimensional distribution on $\mathcal{N}$

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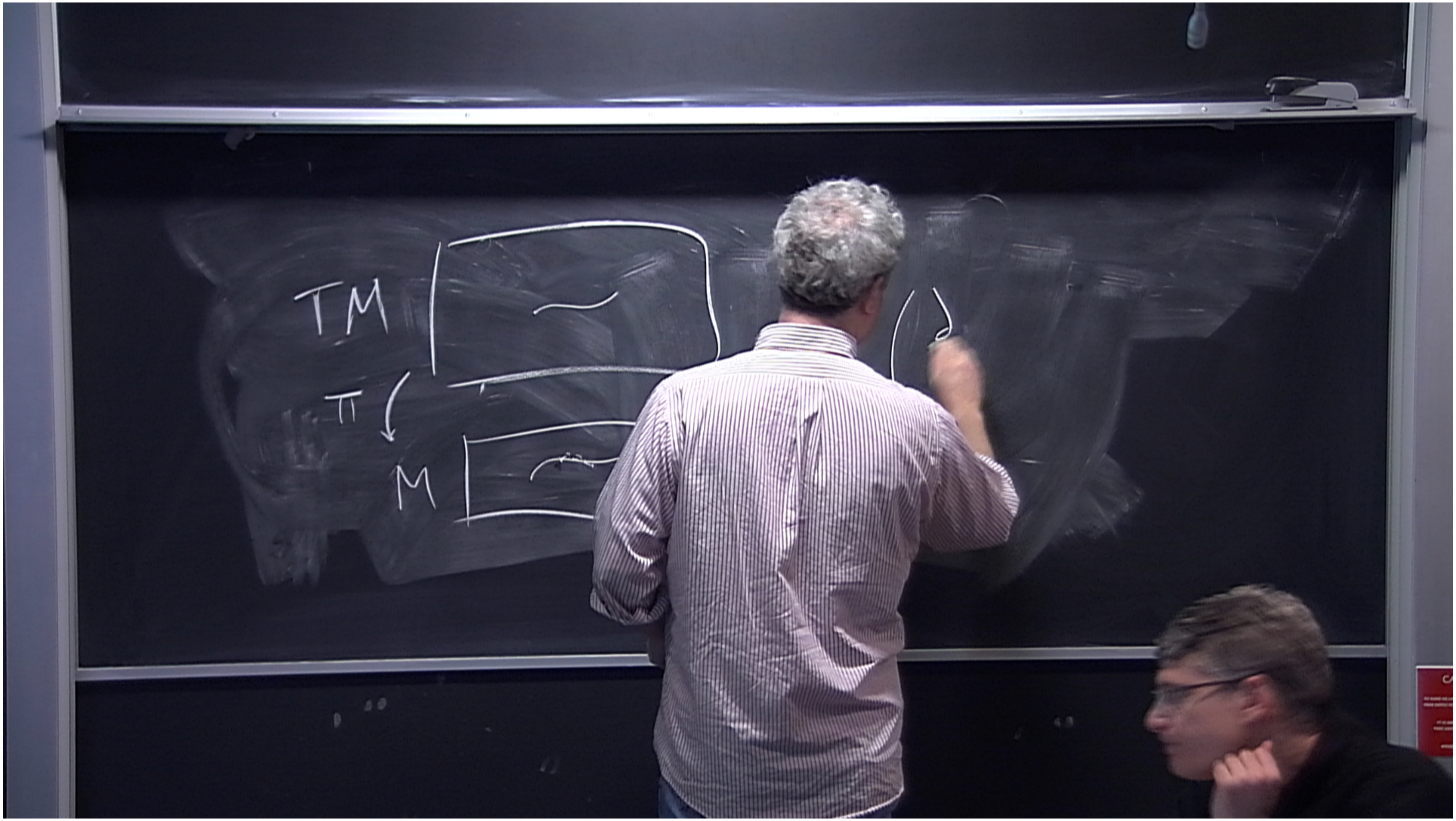
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Lagrangian Vector Fields

second-order vector fields are tangent to lifts of curves from base to tangent bundle; they form an affine space over vector space of vertical vector fields

Theorem

Lagrangian vector fields are always second-order vector fields.

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The Euler-Lagrange Equation

To formulate Euler-Lagrange equation in invariant, geometrical terms, for Lagrangian L with solution ξ,

$$\mathcal{L}_{\xi}(J^{n}{}_{a}\nabla_{n}L) - \nabla_{a}L = 0$$

one needs (and only needs) almost-tangent structure, $J^{a}{}_{b}$

implicitly contains **Lagrangian 2-form** $\boldsymbol{\nabla_{[a}(J^{n}{}_{b]}\nabla_{n}L)}$

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Lagrangian kinematical constraints

1. knowledge of family of second-order vector fields allows reconstruction of $J^a{}_b$
2. $J^a{}_b$ encodes classical kinematical constraint $\dot{\mathbf{x}} = \mathbf{v}$:

$$J^n{}_a \nabla_n q_i = \mathbf{0}$$
$$J^n{}_a \nabla_n v_i = (\mathrm{d}q_i)_a$$

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The Euler-Lagrange Equation from Dynamical System Structure

using canonical isomorphism of dynamical space of states with tangent bundle over configuration space:

1. push kinematical vector fields over $\Rightarrow$ affine space of Lagrangian vector fields (same physical meaning)
2. push interaction vector fields over $\Rightarrow$ vector space of vertical vector fields (same physical meaning)
3. $\Longrightarrow J^a{}_b$

Classical Systems Are Lagrangian

The Euler-Lagrange Equation from Dynamical System Structure

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Classical Systems Are Lagrangian

Dynamical System Structure from the Euler-Lagrange Equation

an **Euler-Lagrange operator** is a (non-linear) functional that takes a scalar field to its associated Lagrangian vector field on a manifold that supports the formulation of the Euler-Lagrange equation (has proper $J^a{}_b$)

Theorem (Curiel)

A manifold has an Euler-Lagrange operator if and only if it is a tangent bundle; the operator's action allows one to recover the space over which it is the tangent bundle.

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Theorem (Curiel)

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using canonical isomorphism of tangent bundle over configuration space with dynamical space of states:

1. push affine space of Lagrangian vector fields over $\Rightarrow$ kinematical vector fields (same physical meaning)
2. push vertical vector fields over $\Rightarrow$ vector space of interaction vector fields (same physical meaning)

Lagrangian Systems Are Classical

Physical Meaning of Structures

almost-tangent structure encodes kinematical constraint $\dot{\mathbf{x}} = \mathbf{v}$, and affine space structure of solutions (**kinematics**)

Lagrangian 2-form ensures uniqueness and existence of solutions, when symplectic (**kinematics**)

Lagrangian L encodes dynamical evolution; ensures Lagrangian 2-form is symplectic, when regular (**mixture of kinematics and dynamics**)

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4 Classical Mechanics Is Not Hamiltonian

Hamilton's Equation

To formulate Hamilton's equation in invariant, geometrical terms,

$$\Omega^{an} \nabla_n H = \xi^a$$

one needs (and only needs):

1. fixed, canonical symplectic structure, Ω^{ab}
2. Hamiltonian H

Kinematical Constraints in Hamiltonian Mechanics

recall Lagrangian Mechanics, "half of canonical quantities are dynamical derivatives of other half"; in Hamiltonian mechanics, canonical quantities must satisfy **Poisson Brackets**

$$\{q_i,\, q_j\} = 0$$
$$\{q_i,\, p_j\} = \delta_{ij}$$
$$\{p_i,\, p_j\} = 0$$

not necessary that $\dot{\mathrm{q}} = \mathrm{p}$

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Closest Analogue to Lagrangian Theorem

Theorem

Fix an even-dimensional, orientable manifold with a vector space of vector fields on it and a Poisson bracket structure. Then the Poisson bracket arises from a symplectic structure and the vector space includes all and only solutions to Hamilton's equation formulated with it if and only if the vector space spans the tangent planes, and the manifold has a group of coordinate systems whose coordinate functions satisfy the canonical Poission bracket relations, and whose associated coordinate vector fields leave the vector space invariant under the action of the Lie bracket.

Physical Meaning of Structures

canonical symplectic structure ensures existence and uniqueness of solutions; encodes kinematical constraints (Poisson brackets); ensures conservation of energy (**kinematics**)

Hamiltonian H encodes dynamical evolution (**dynamics**)

5 How Lagrangian and Hamiltonian Mechanics Respectively Represent Classical Systems

Four Deep Questions

1. If Hamiltonian mechanics does not respect the kinematical constraints intrinsic to dynamical systems, how can it provide adequate representations of classical systems (*e.g.*, the simple harmonic oscillator)?
2. Why does Lagrangian mechanics always respect the constraints of dynamical systems?
3. Because we know the Hamiltonian and Lagrangian formulations to be related by the Legendre transform, what happens in the passage from Lagrangian to Hamiltonian mechanics that expunges respect for those constraints?
4. Is any structure in Hamiltonian mechanics isomorphic to any structure in Lagrangian mechanics?

How Hamiltonian Mechanics Respects Classical Kinematical Constraints

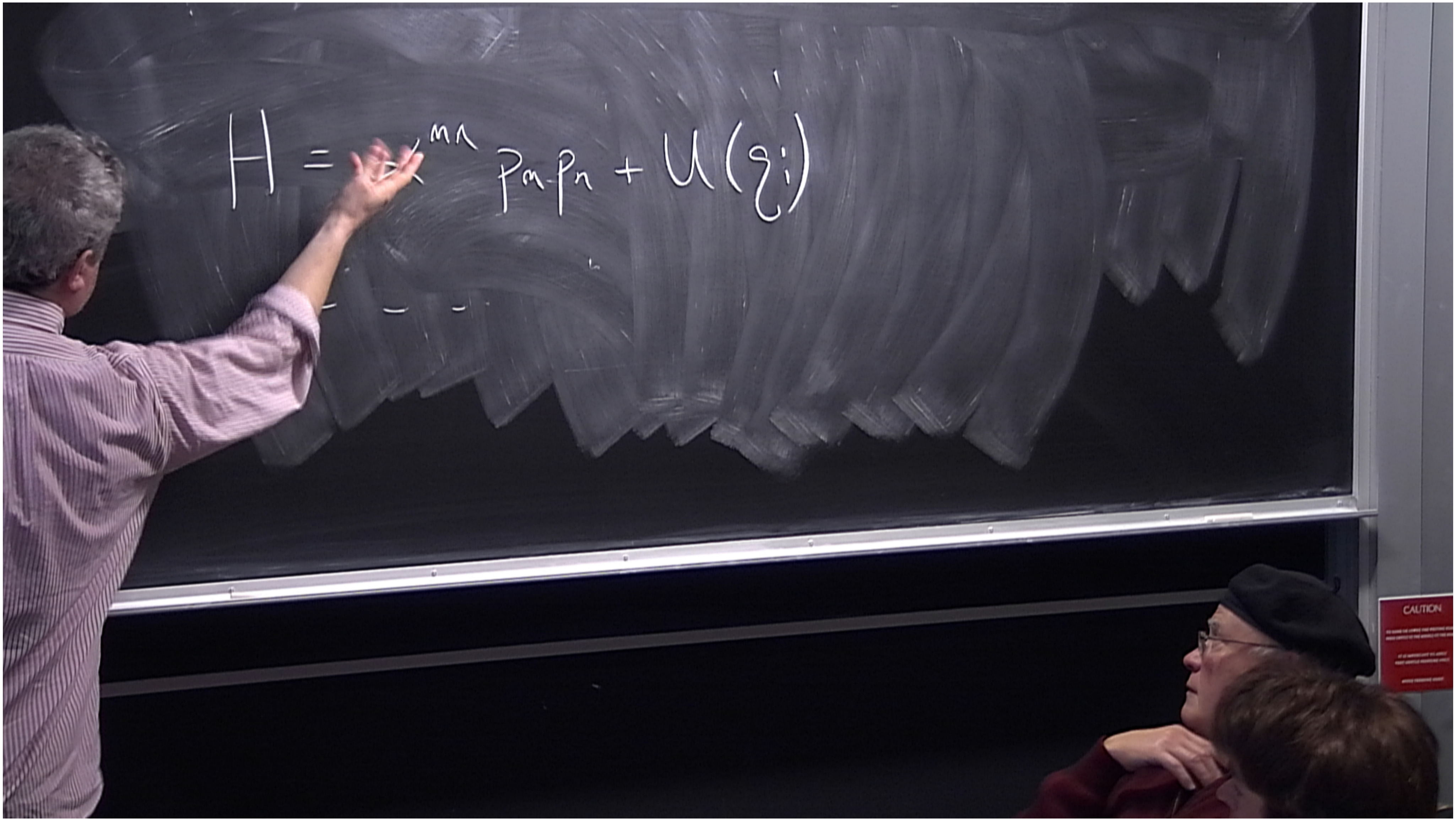
by ad hoc fiat

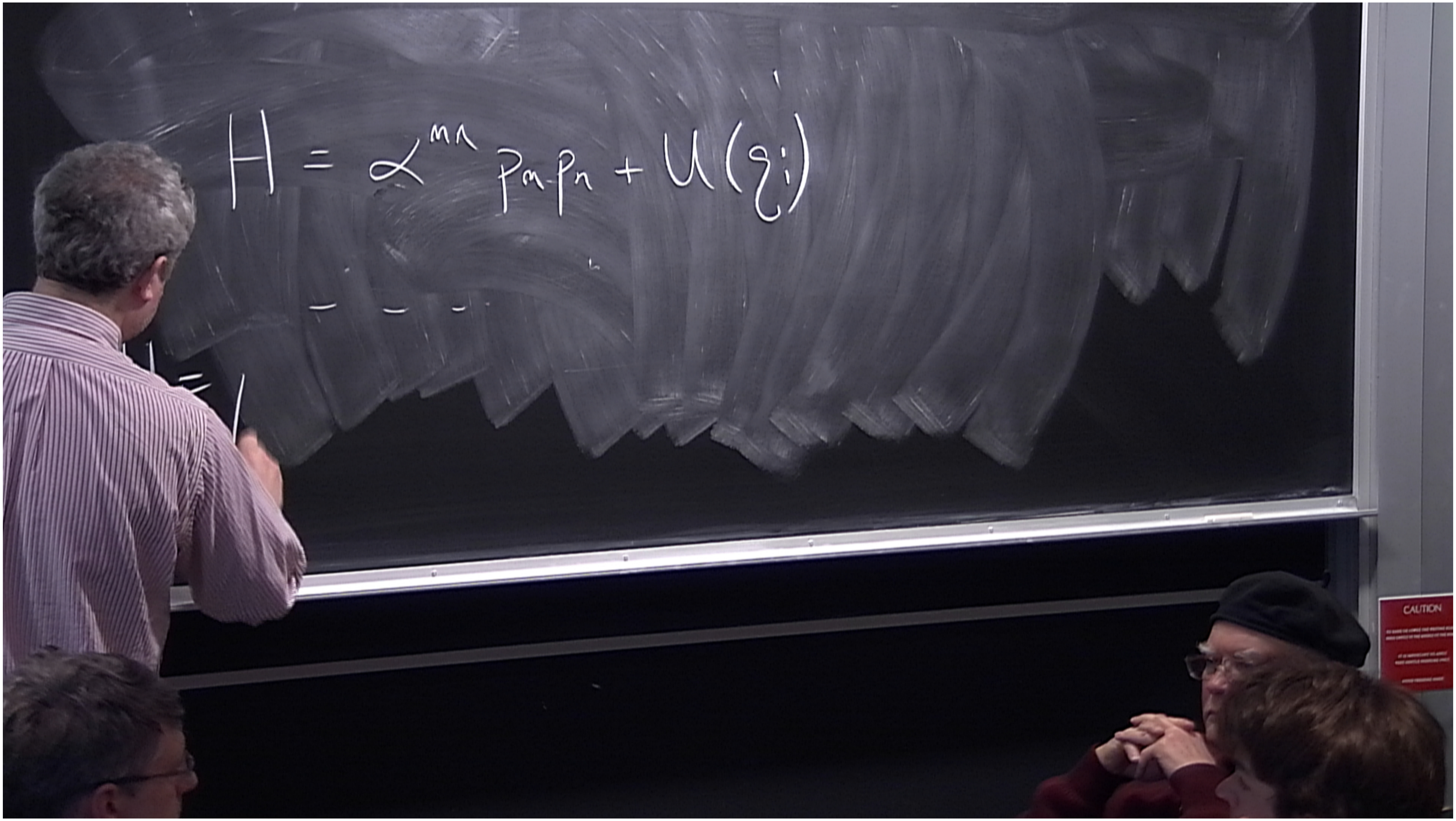
demand that only Hamiltonians H used be quadratic in momenta, with no other momental dependence; then

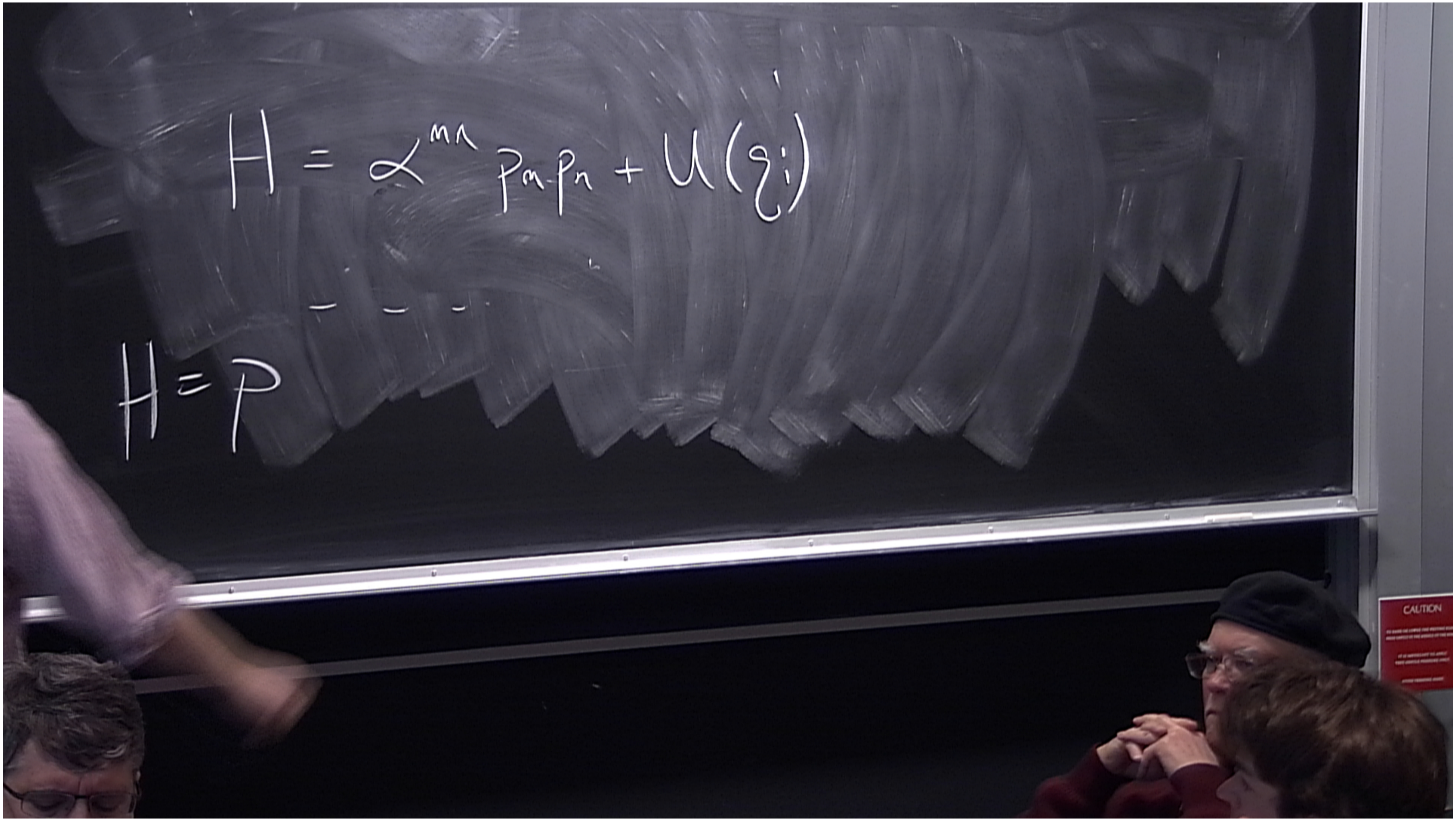
$$v_i =_{\text{df}} \dot{q}_i = \frac{\partial H}{\partial p^i}$$

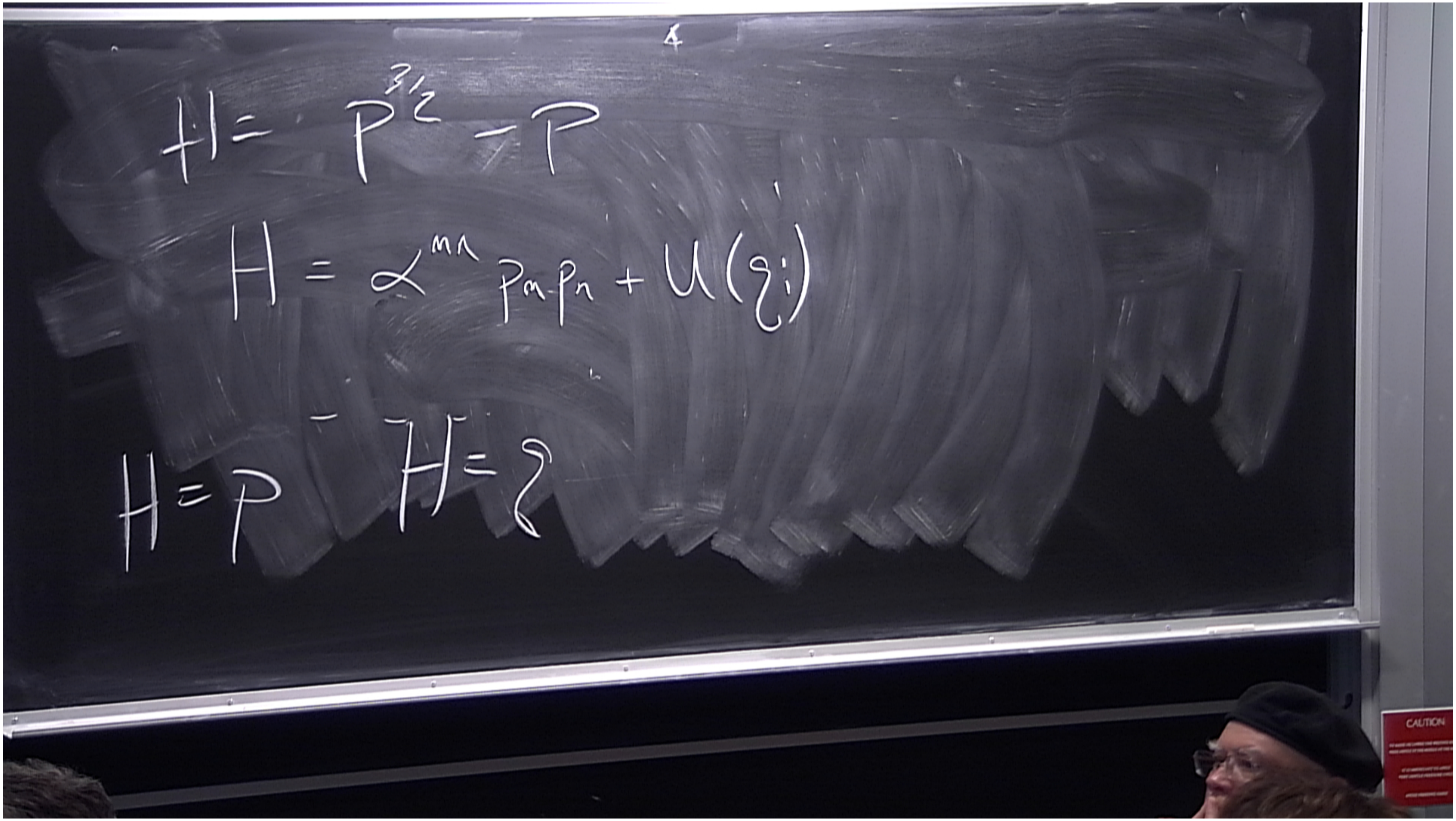
is an identity

justifiable by nothing intrinsic to the theory, not a kinematical constraint


H = mn pm pn + U(qi)
CAUTION


H = α^mn p_m p_n + U(q_i)


$H = \alpha^{mn} p_m p_n + U(q_i)$
$H = P$


H = p^{3/2} - P
H = α^{mn} p_m p_n + U(q_i)
H = P
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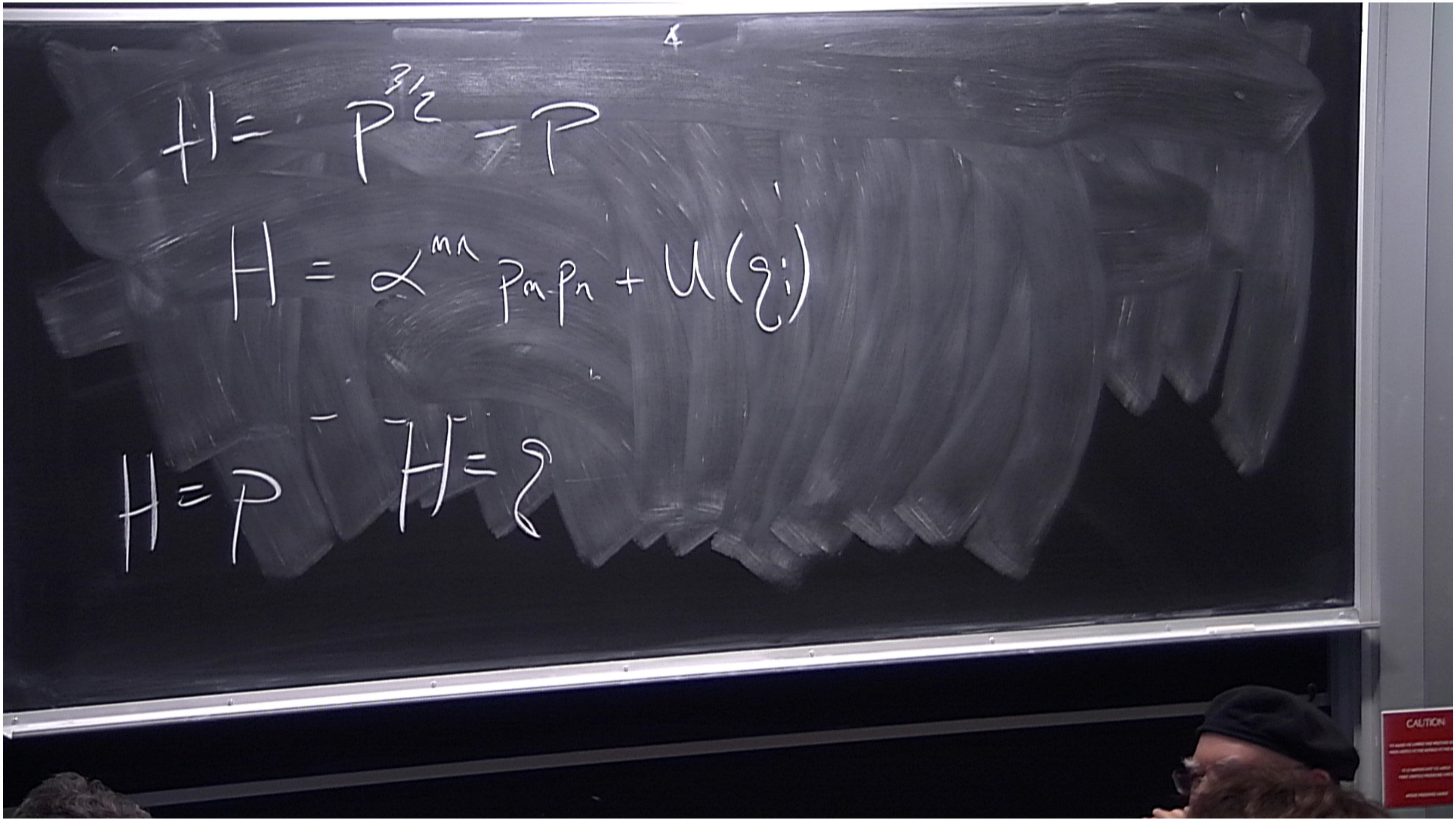
Why Lagrangian Mechanics Always Respects Classical Kinematical Constraints

built in from the start

1. standard variational problem not solvable unless kinematical constraint imposed from the start (hidden by usual formulation)
2. geometric formulation of Euler-Lagrange equation requires the kinematical constraints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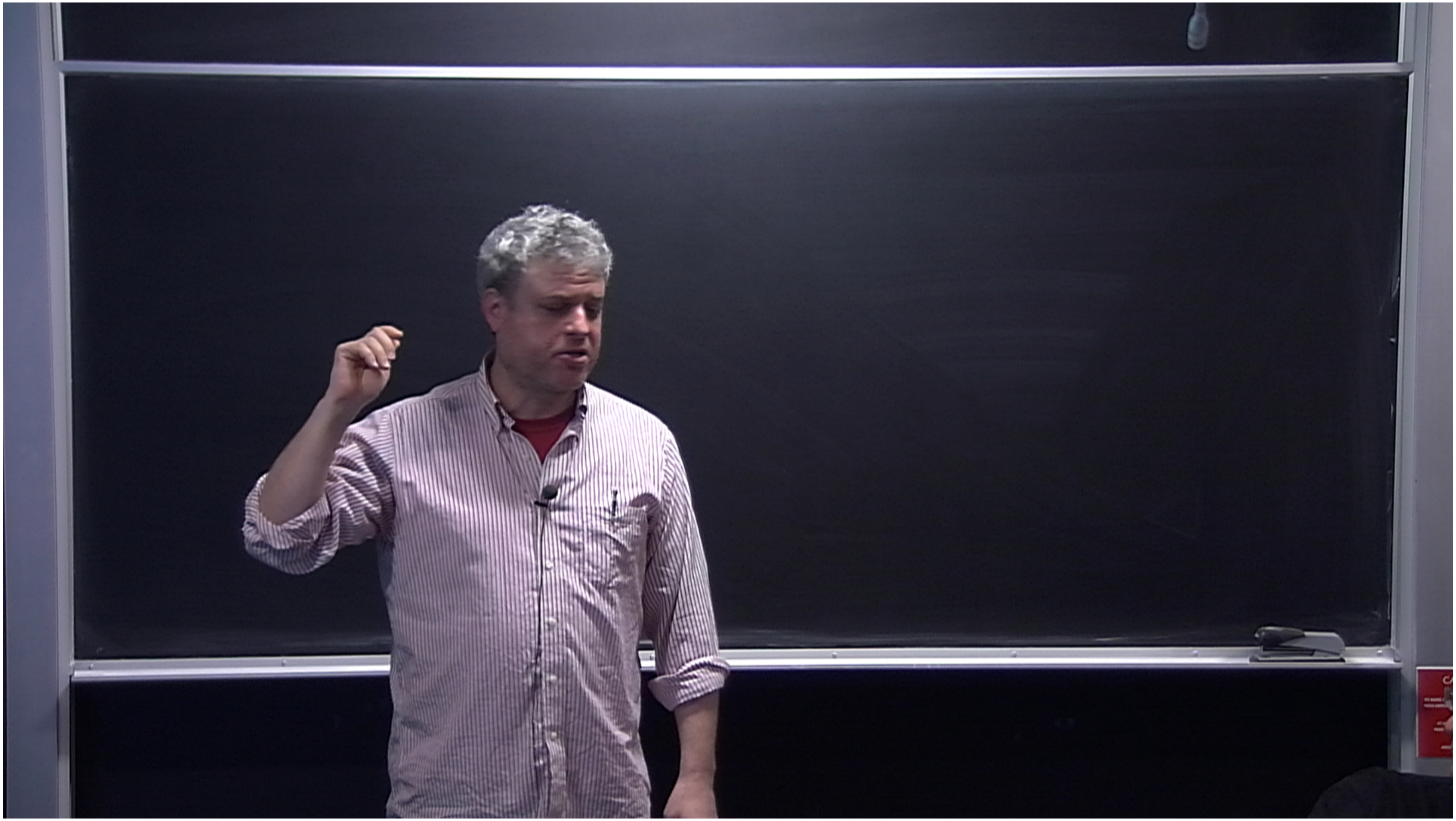
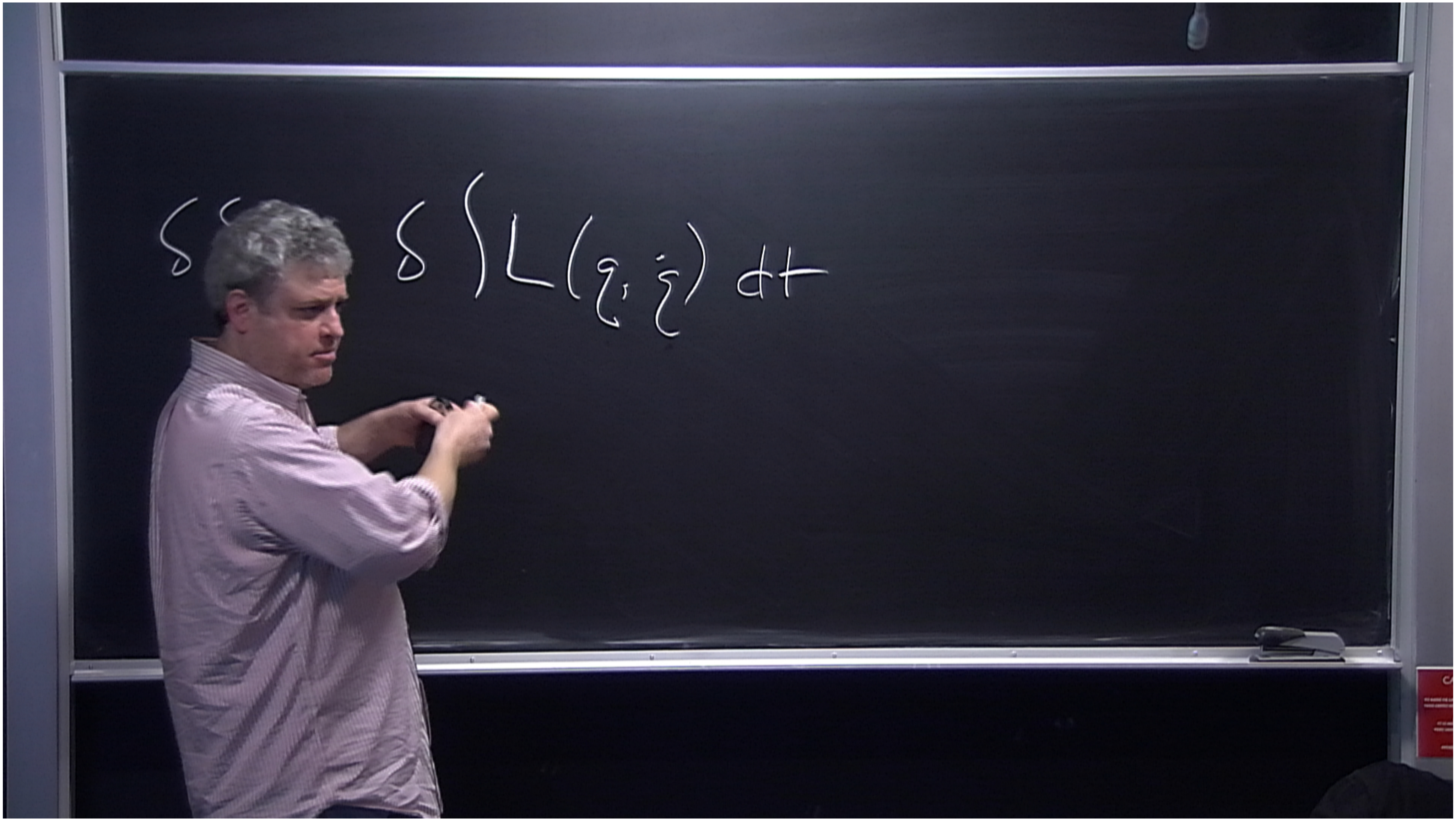
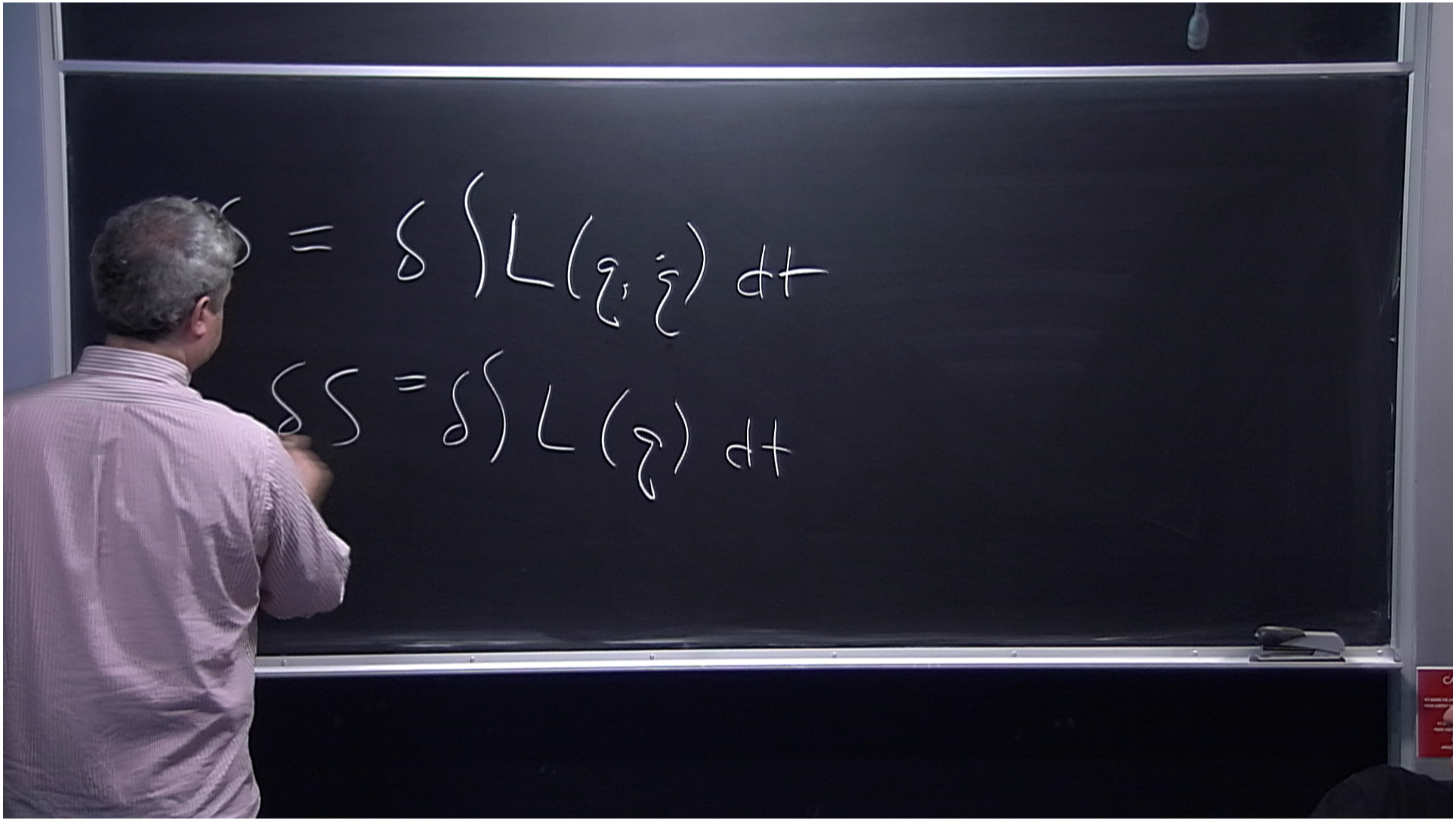
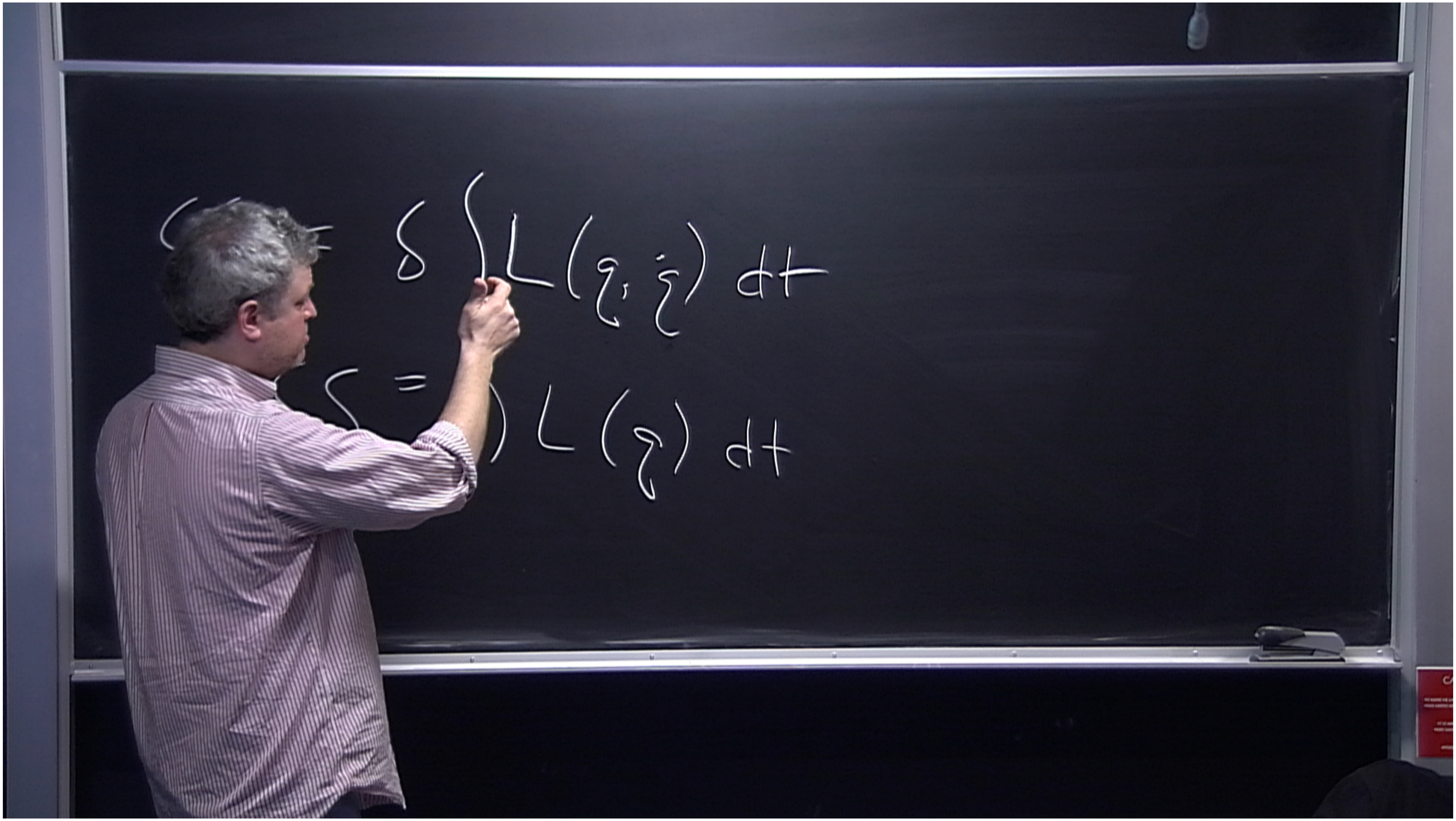
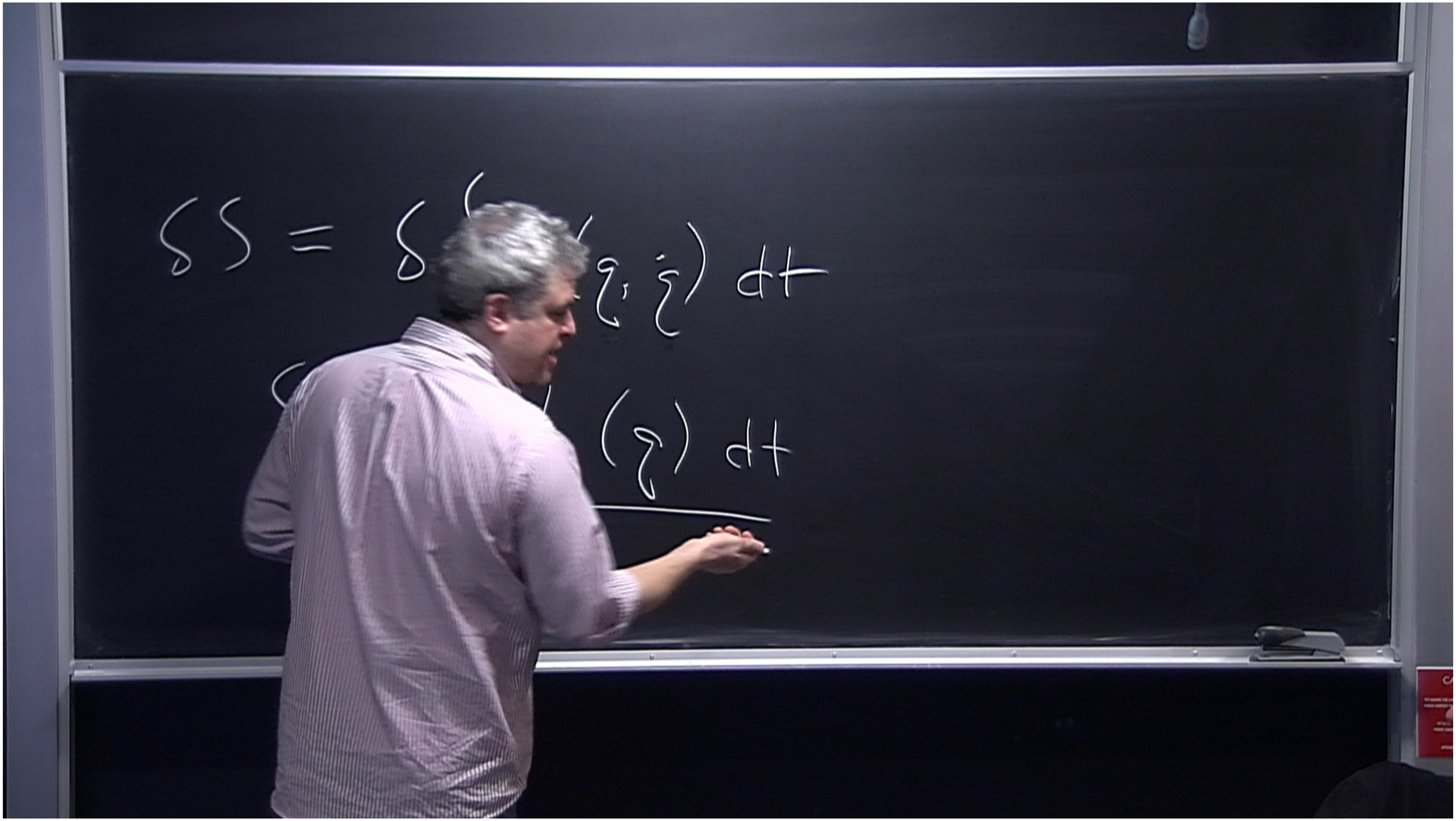
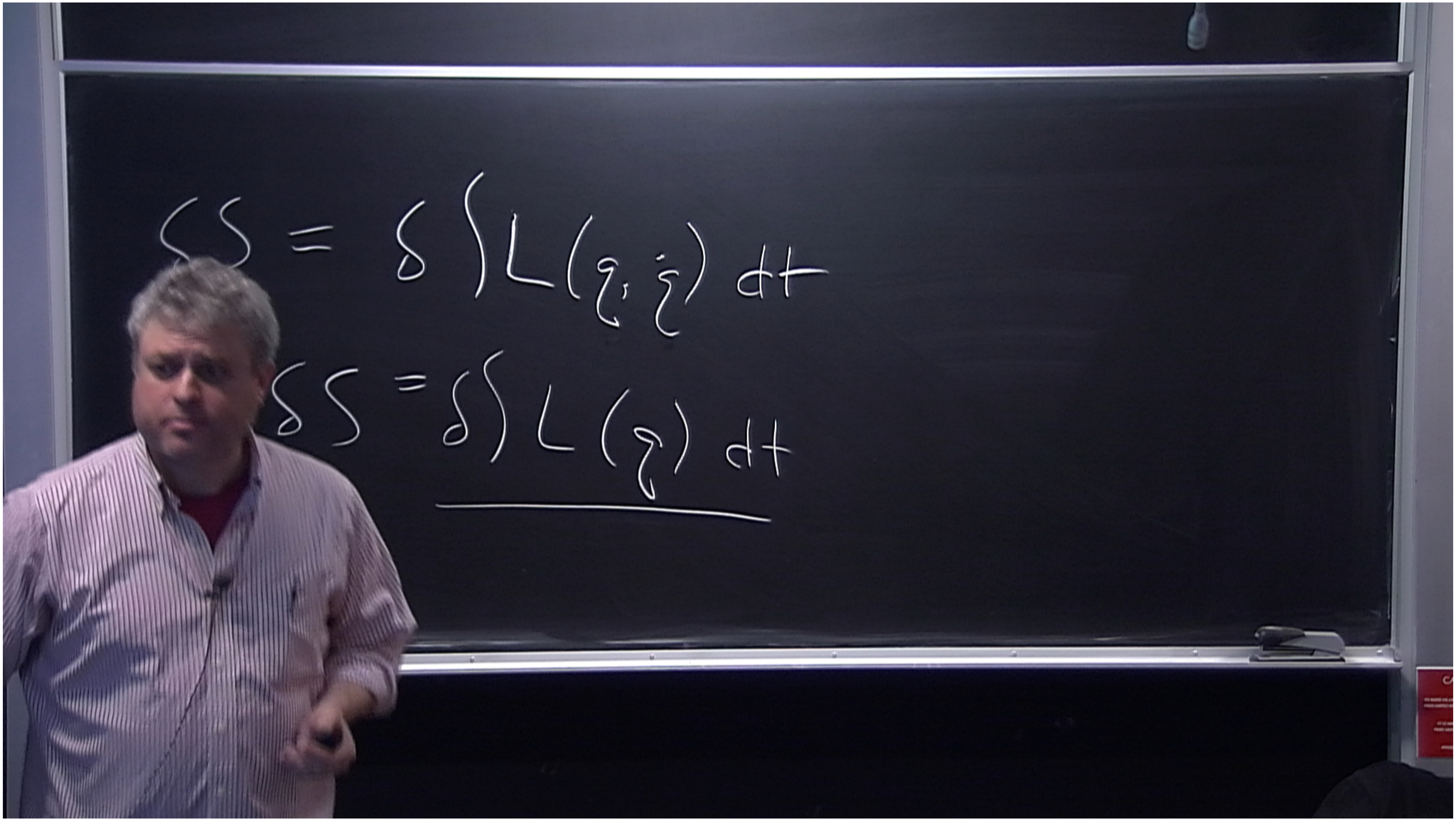



δ ∫ L(q, q̇) dt


$= \delta \int L(q, \dot{q})\, dt$
$\delta S = \delta \int L(q)\, dt$


= δ ∫ L(q, q̇) dt
S = ∫ L(q) dt

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Legendre Transform

1. Legendre transform is (in special cases) diffeomorphism of tangent bundle to phase space
2. does not map second-order vector fields to Hamiltonian vector fields, not even to affine sub-space of them
3. does not preserve kinematical constraints
4. is not an isomorphism of any kinematical structure

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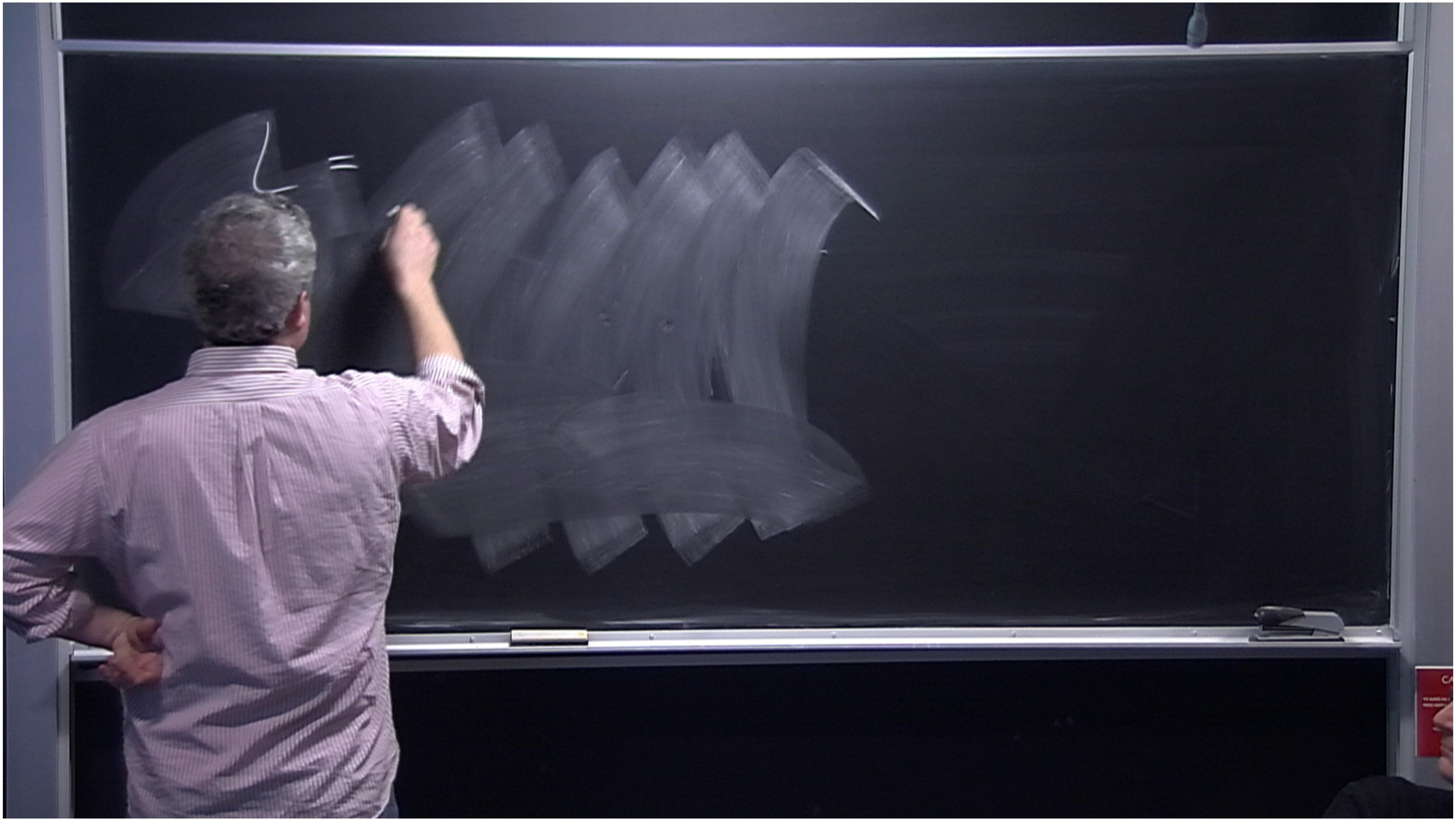
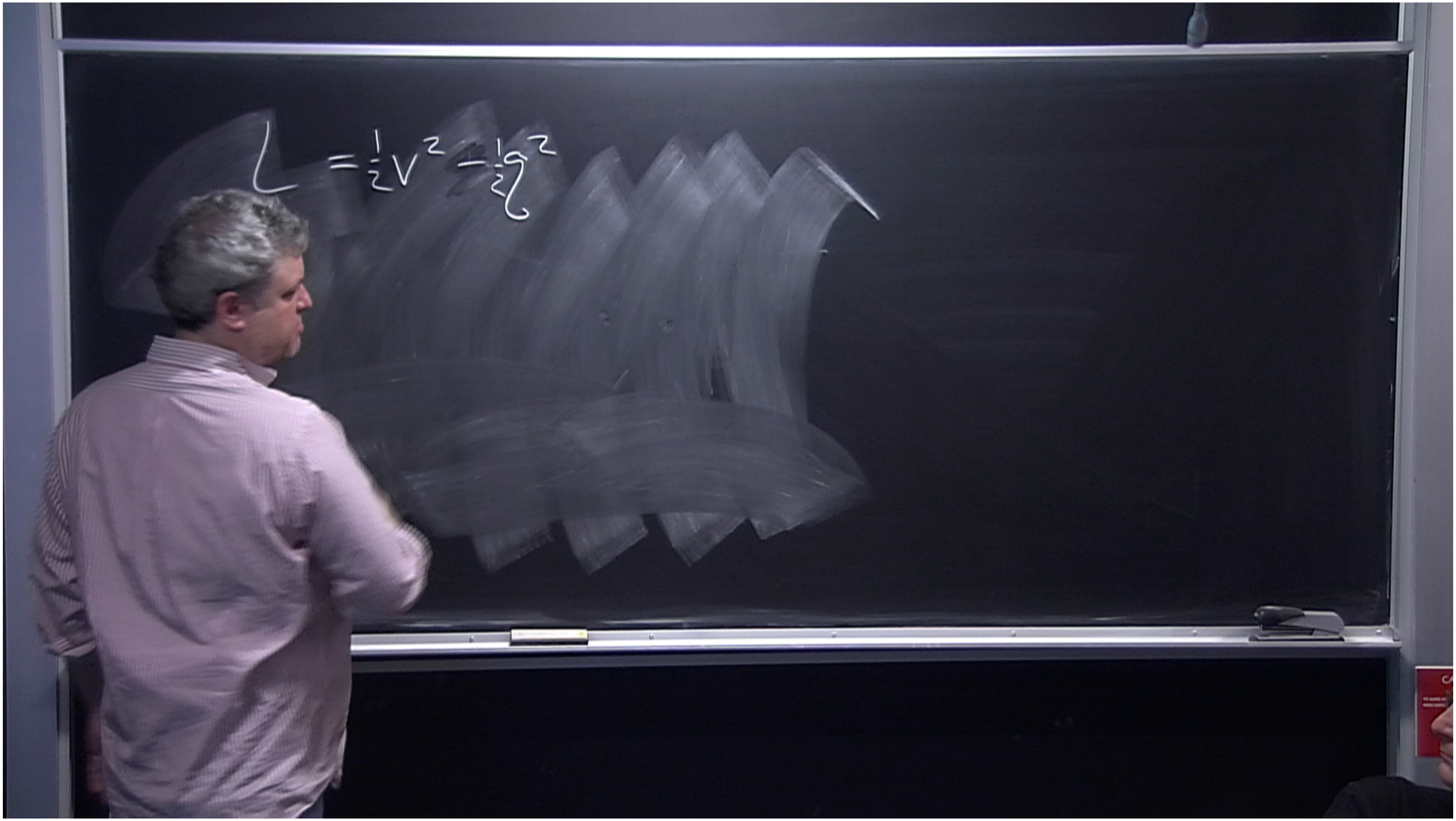
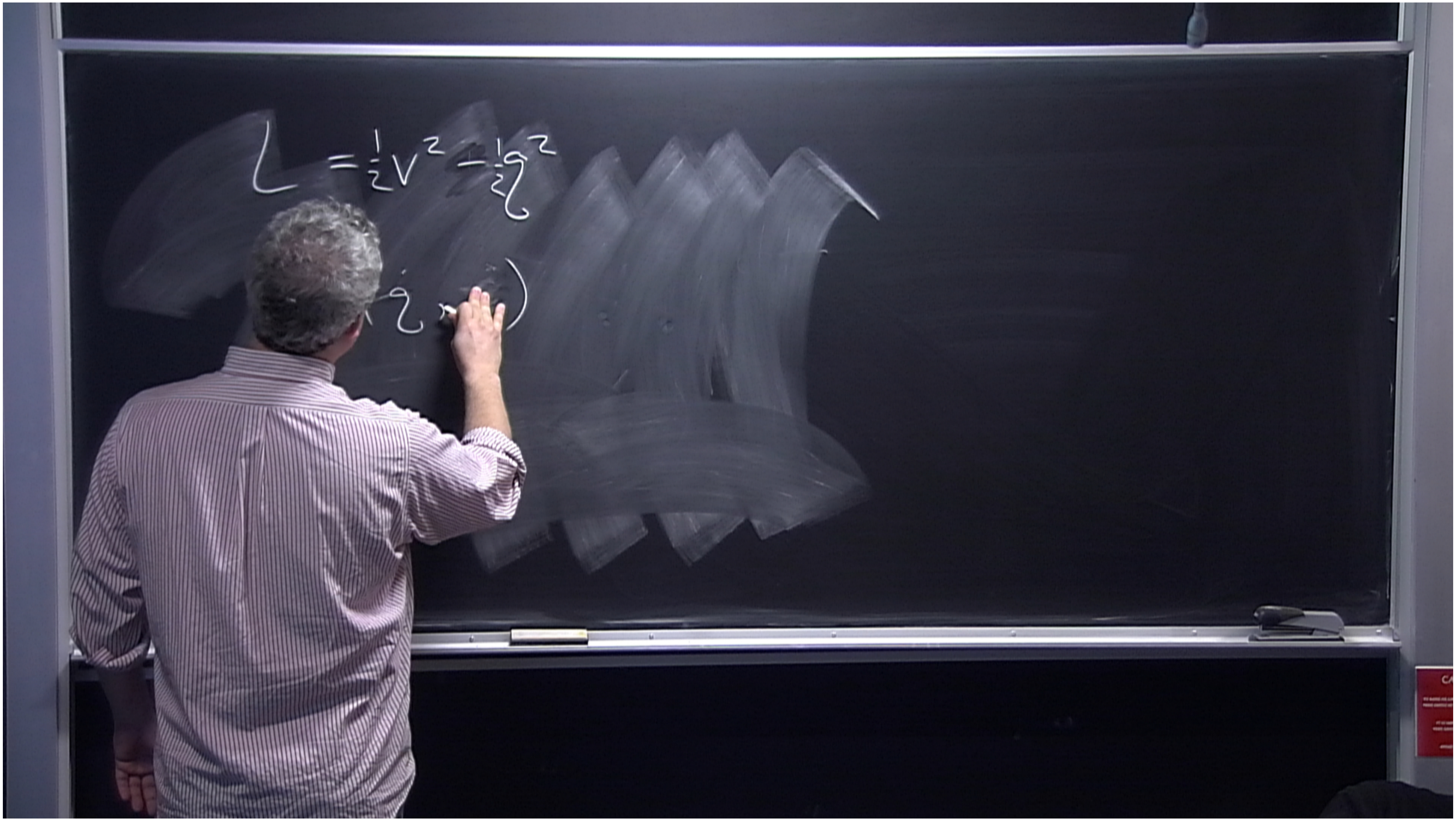
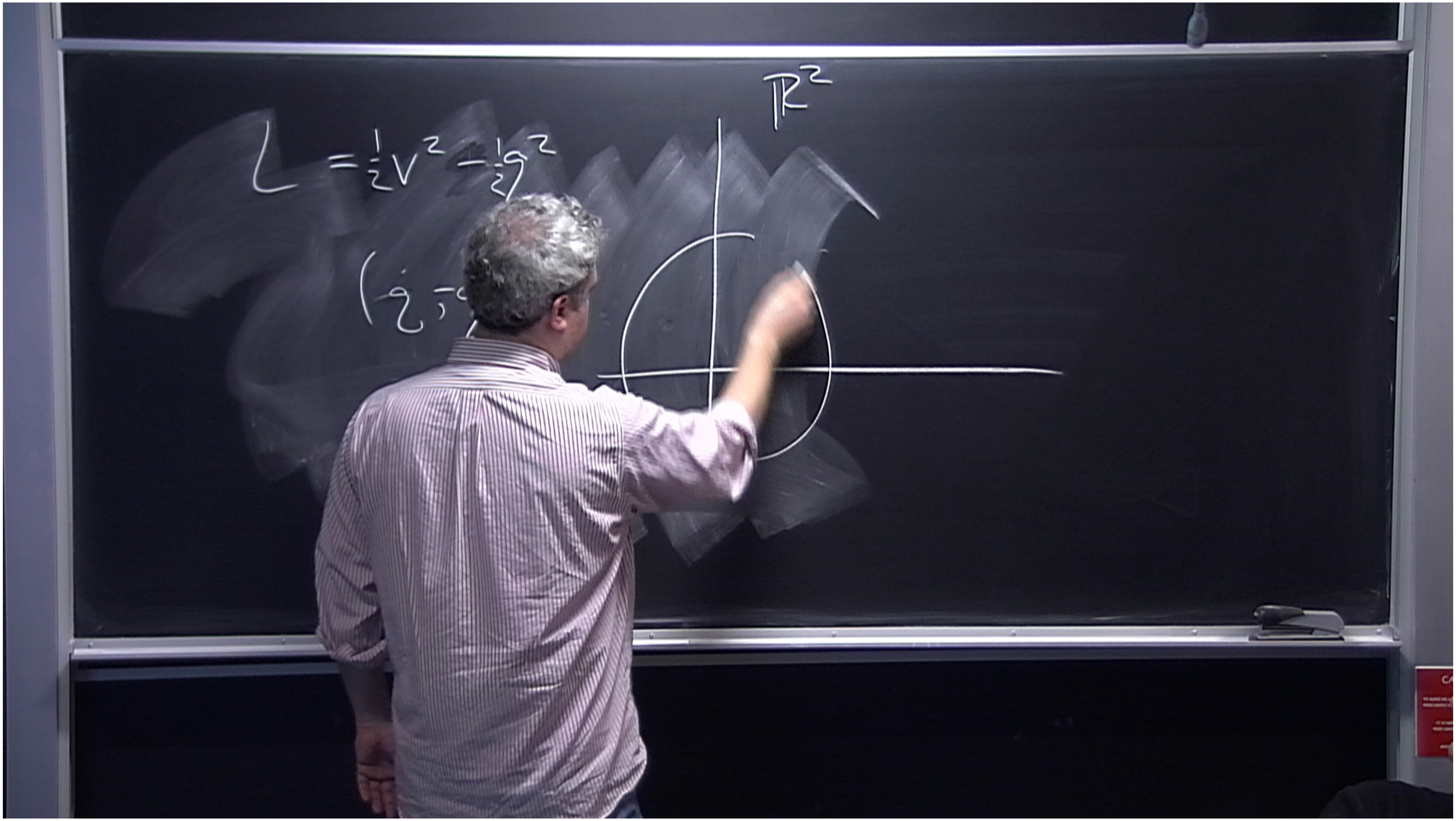
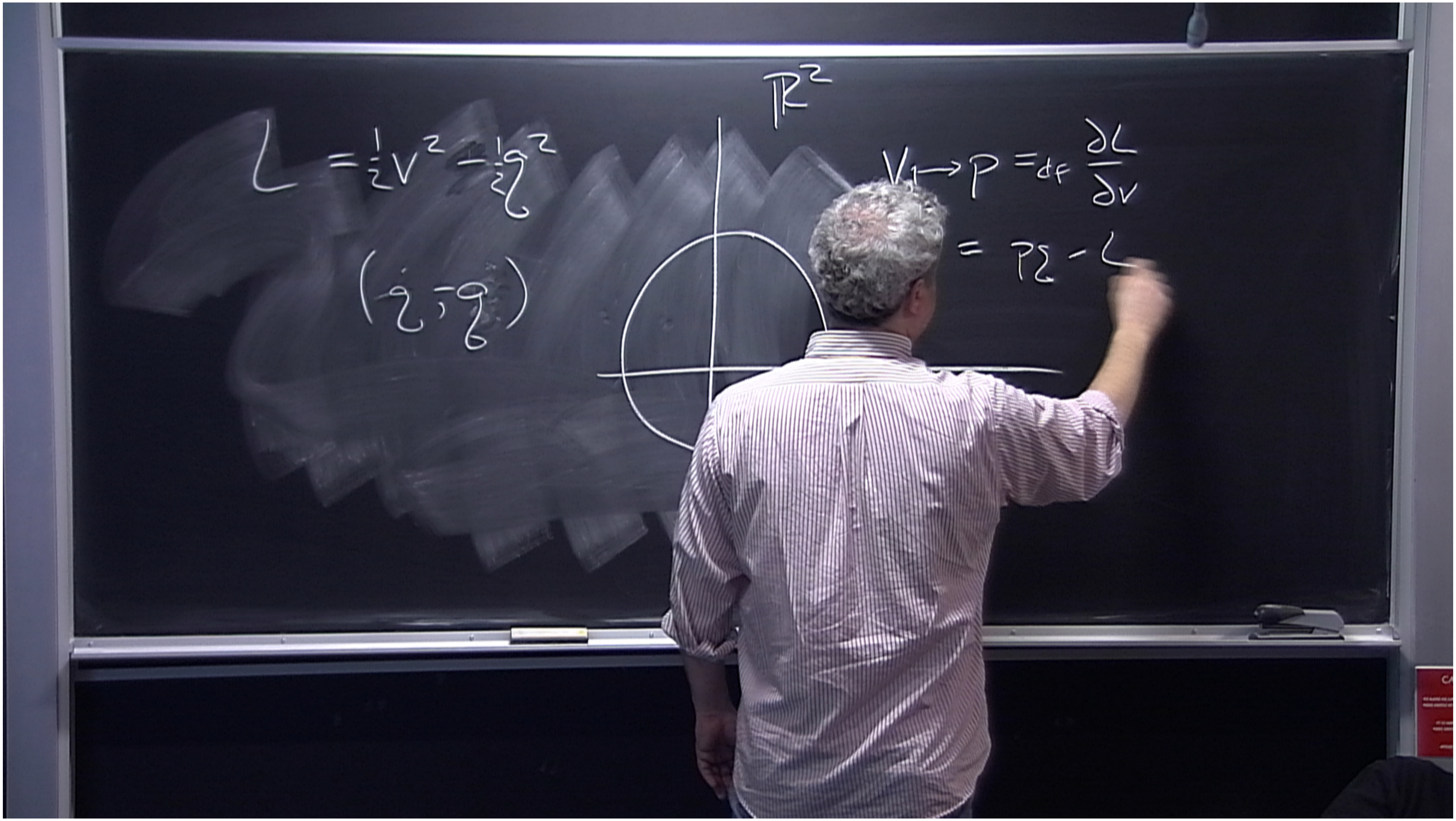


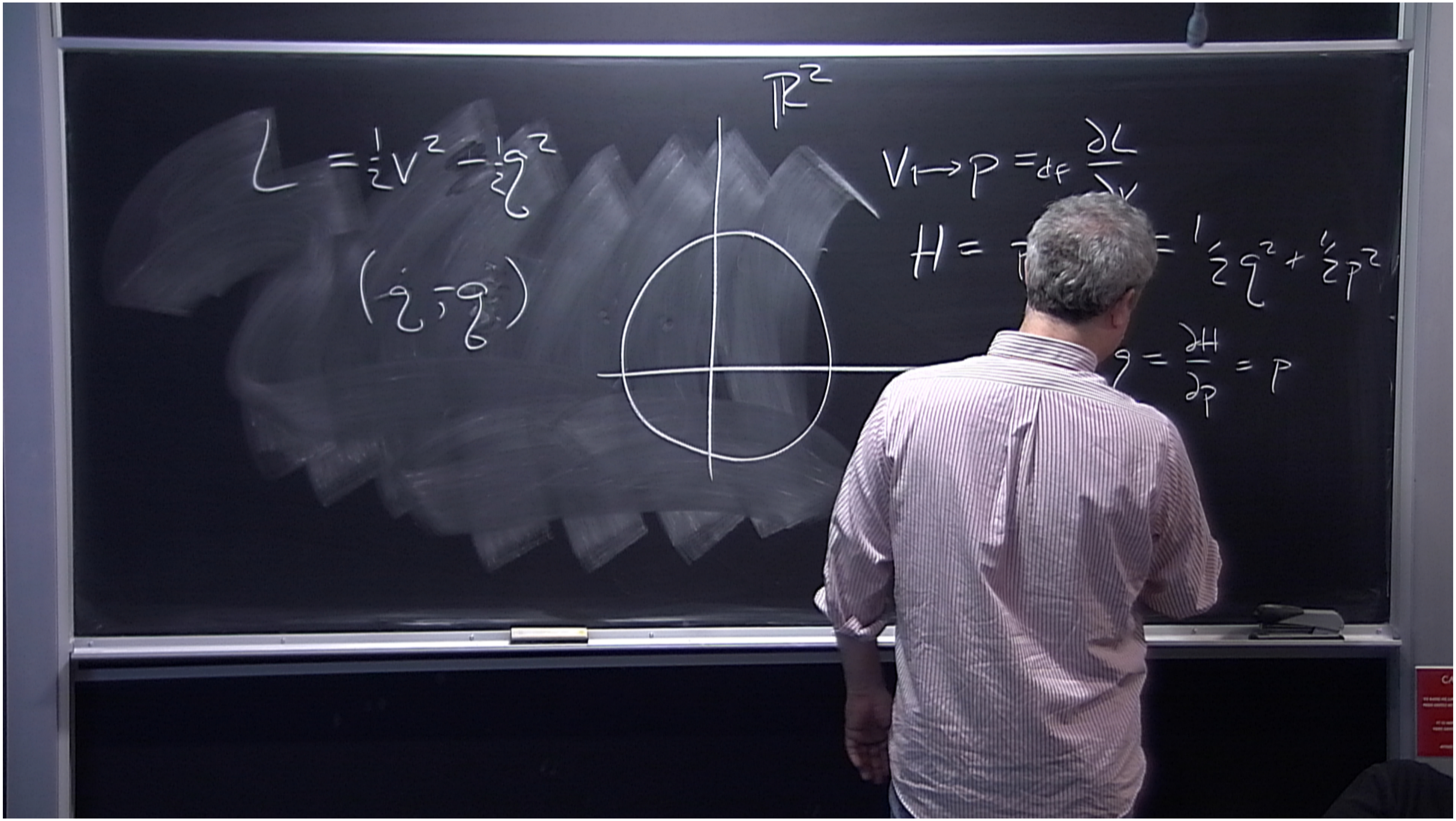

$L = \frac{1}{2}v^2 - \frac{1}{2}q^2$

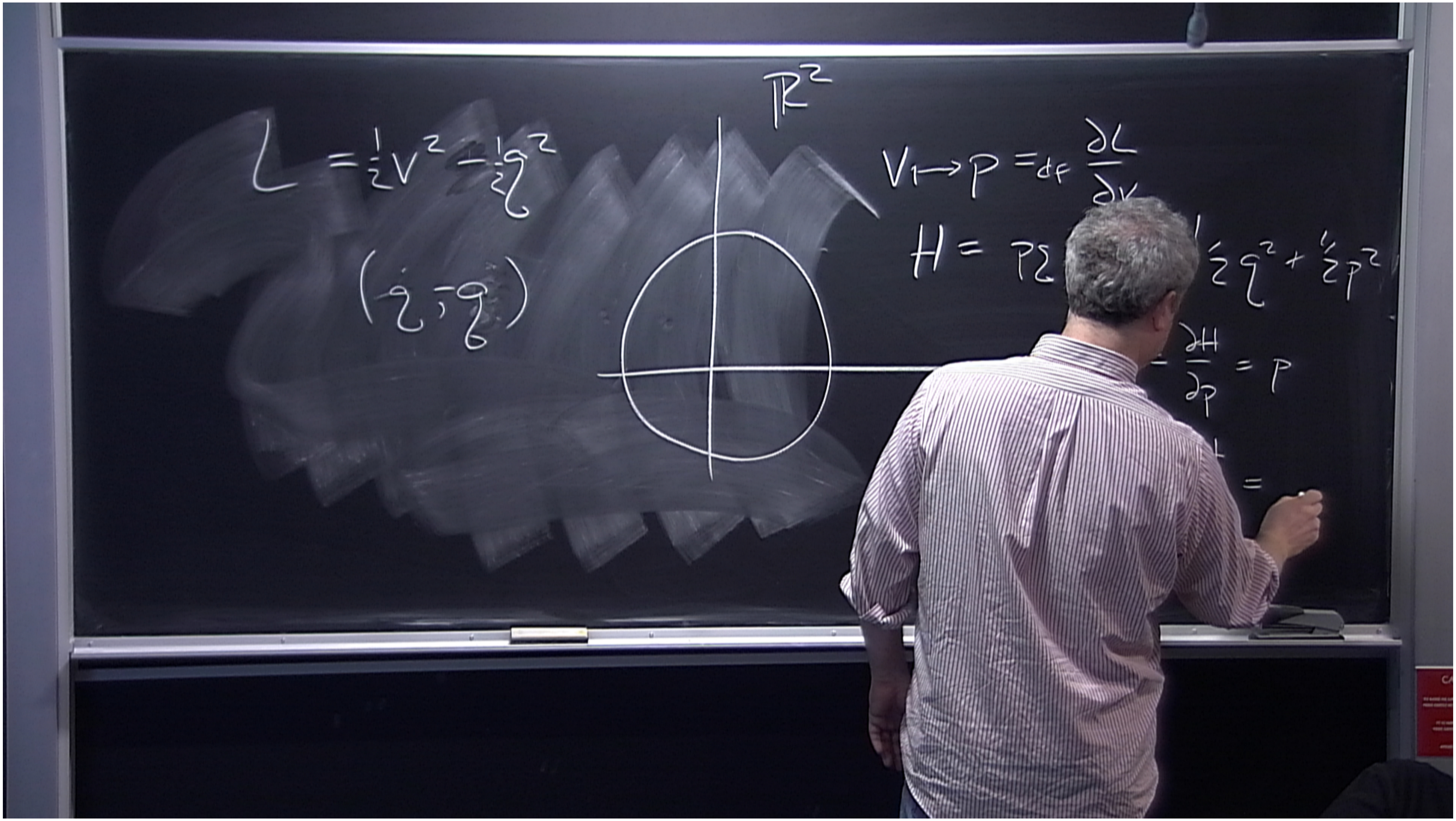

L = ½v² − ½q²



$L = \frac{1}{2}v^2 - \frac{1}{2}q^2$
$\mathbb{R}^2$

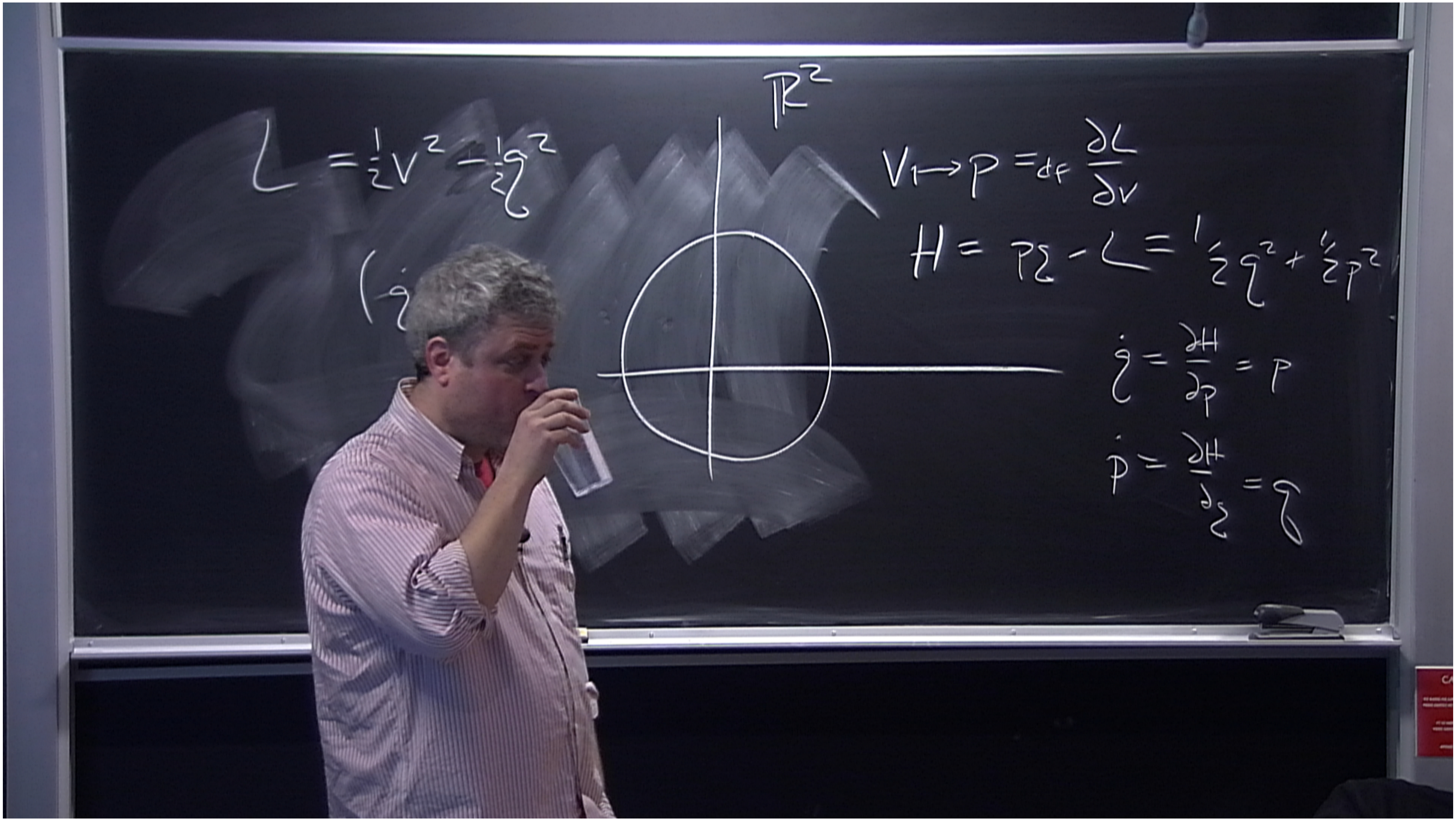

$L = \frac{1}{2}v^2 - \frac{1}{2}q^2$
$(\dot{q}, -q)$
$\mathbb{R}^2$
$v \mapsto p =_{df} \frac{\partial L}{\partial v}$
$= p\dot{q} - L$


$L = \frac{1}{2}v^2 - \frac{1}{2}q^2$
$(\dot{q}, -q)$
$\mathbb{R}^2$
$V \mapsto p =_{df} \frac{\partial L}{\partial v}$
$= \frac{1}{2}q^2 + \frac{1}{2}p^2$
$= \frac{\partial H}{\partial p} = p$

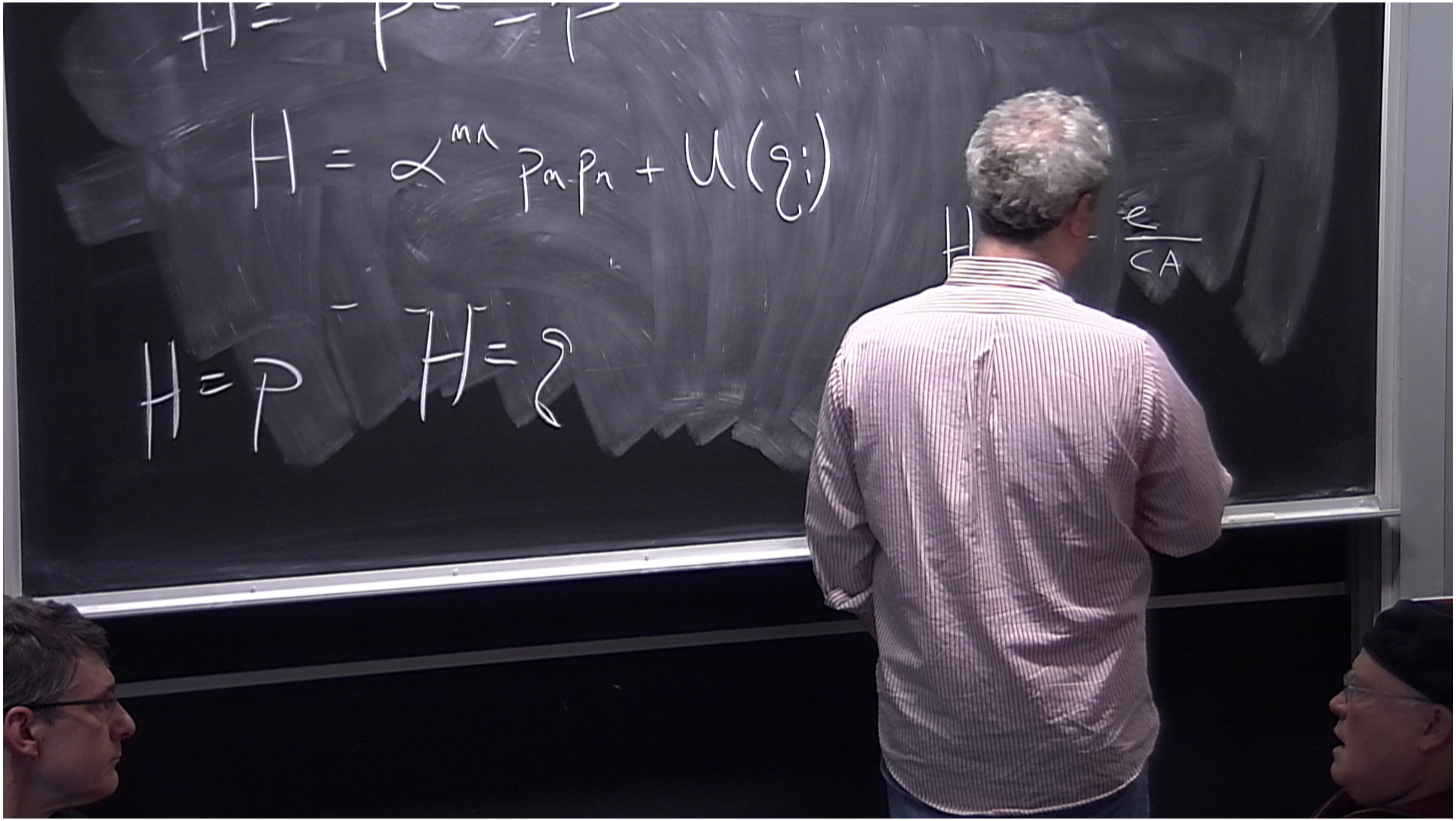

$\mathbb{R}^2$
$L = \frac{1}{2}v^2 - \frac{1}{2}q^2$
$(\dot{q}, -q)$
$v \mapsto p =_{df} \frac{\partial L}{\partial v}$
$H = pq$
$\frac{1}{2}q^2 + \frac{1}{2}p^2$
$\frac{\partial H}{\partial p} = p$

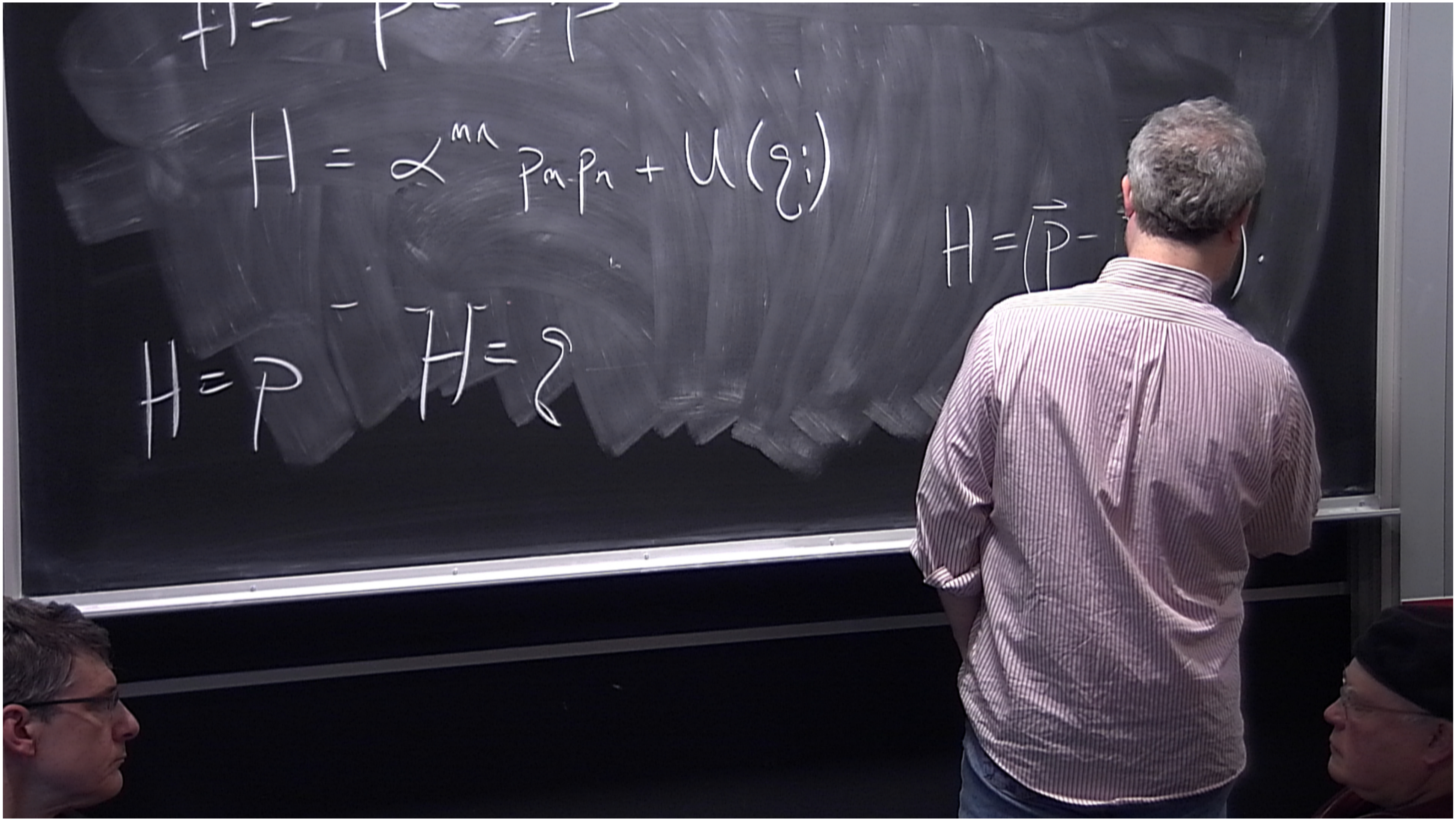
Four Deep Questions

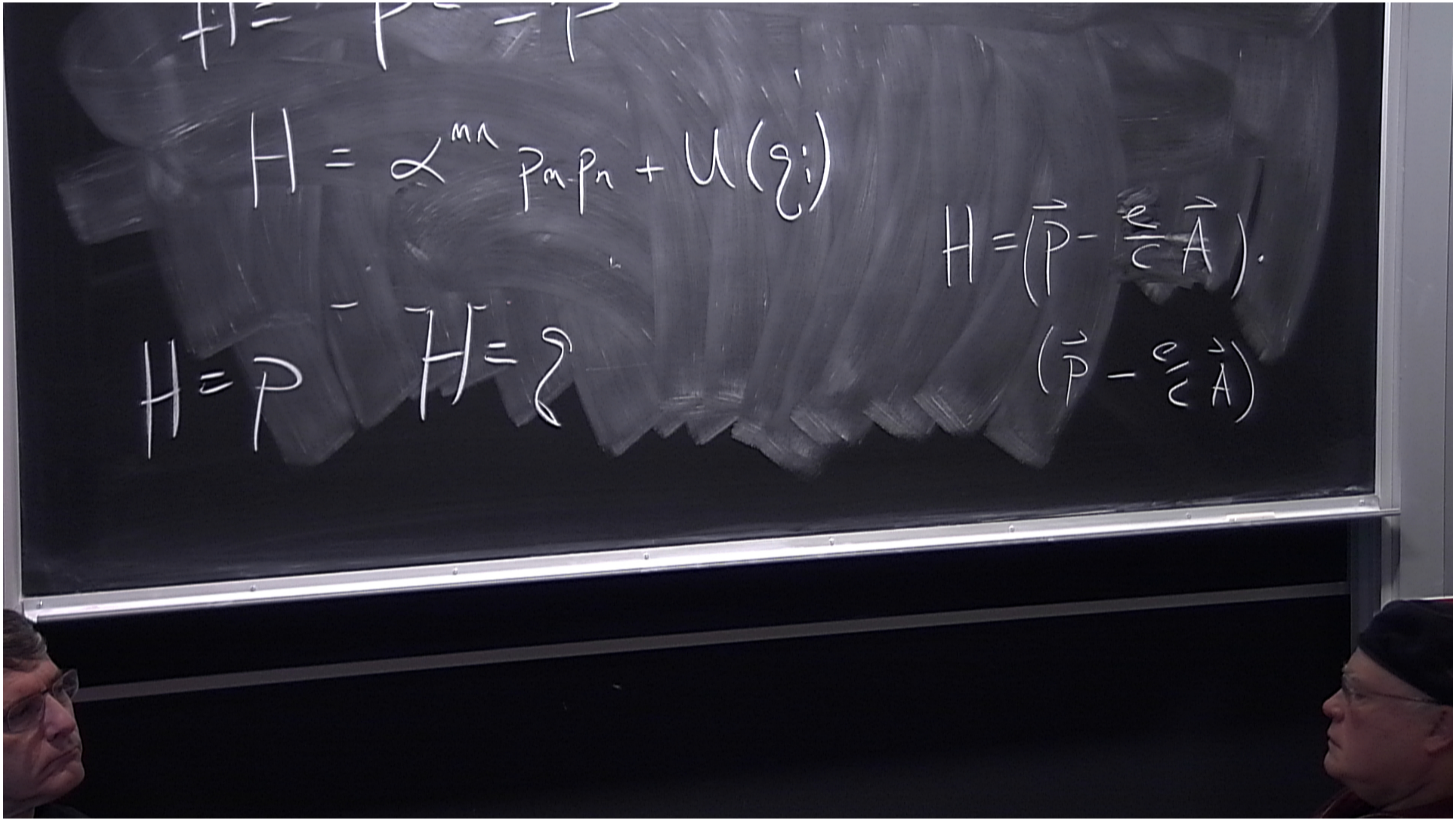
1. If Hamiltonian mechanics does not respect the kinematical constraints intrinsic to dynamical systems, how can it provide adequate representations of classical systems (*e.g.*, the simple harmonic oscillator)?
2. Why does Lagrangian mechanics always respect the constraints of dynamical systems?
3. Because we know the Hamiltonian and Lagrangian formulations to be related by the Legendre transform, what happens in the passage from Lagrangian to Hamiltonian mechanics that expunges respect for those constraints?
4. Is any structure in Hamiltonian mechanics isomorphic to any structure in Lagrangian mechanics?



$\mathbb{R}^2$
$L = \frac{1}{2}v^2 - \frac{1}{2}q^2$
$v \mapsto p =_{df} \frac{\partial L}{\partial v}$
$H = pq - L = \frac{1}{2}q^2 + \frac{1}{2}p^2$
$\dot{q} = \frac{\partial H}{\partial p} = p$
$\dot{p} = \frac{\partial H}{\partial q} = q$


H = α^mn p_m p_n + U(q_i)
H = p
H = q
e
CA


H = α^mn pm pn + U(qi)
H = p
H = q


H = α^mn p_m p_n + U(q_i)
H = (p⃗ − e/c A⃗)·
(p⃗ − e/c A⃗)
H = p
H = q