Title: Instanton Corrections to Hypermultiplet Moduli Spaces

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Abstract: Finding the exact, quantum corrected metric on the hypermultiplet moduli space in Type II string compactifications on Calabi-Yau threefolds is still an open problem. We address this issue by relating the quaternionic-KÃ¤hler metric on the hypermultiplet moduli space to the complex contact geometry on its twistor space. In this framework, Euclidean D-brane instantons are captured by contact transformations between different patches. We derive those by recasting the previously known A-type D2-instanton corrections in the language of contact geometry, covariantizing the result under electro-magnetic duality, and using mirror symmetry. As a result, we are able to express the effects of all D-instantons in Type II compactifications concisely as a sum of dilogarithm functions.

Instanton corrections to hypermultiplet moduli spaces

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S. Alexandrov, B. Pioline, F.S., S. Vandoren, Commun. Math. Phys. 296 (2010) 353

S. Alexandrov, B. Pioline, F.S., S. Vandoren, JHEP 0903 (2009) 044

S. Alexandrov, F.S., JHEP 0909 (2009) 108

String Seminar

Perimeter Institute, February 21, 2012

Outline

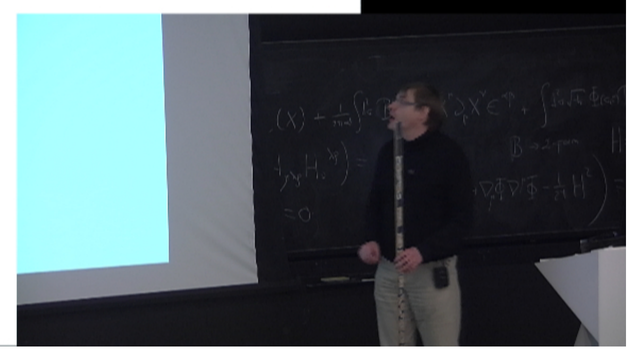
- introduction
  - $N = 2, d = 4$ supergravity
  - Type II string compactifications
  - space-time instantons
- geometry part I
  - hyperkähler manifolds from projective superspace
- instanton corrections part I: exact results
- geometry part II
  - the twistor view on hyperkähler manifolds
- instanton corrections part II: twistor formulation
- summary of further developments

Compactify Type II string theory on Calabi-Yau threefolds

classical effective action:

$\Rightarrow$ compactify $d = 10$ type II supergravity on $\mathrm{M}_4 \times \mathrm{CY}_3$

- low energy physics: $d = 4, N = 2$ supergravity
- special properties: symmetries (e.g. SL(2, $\mathbb{Z}$) in IIB)



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quantum corrections to the classical effective action:

Worldsheet conformal field theory
α'

- Perturbative world sheet corrections
- World sheet instantons

String coupling constant
g_s

- String loop corrections
- Instantons
  Euclidean branes wrapping cycles

$N=2, d=4$ supergravity in a nutshell

building blocks:

- supergravity multiplet: $e_\mu{}^a$, A_μ
- vector multiplets (VM): X, $\bar{X}$, A_μ
- hypermultiplets (HM): q^1, q^2, q^3, q^4

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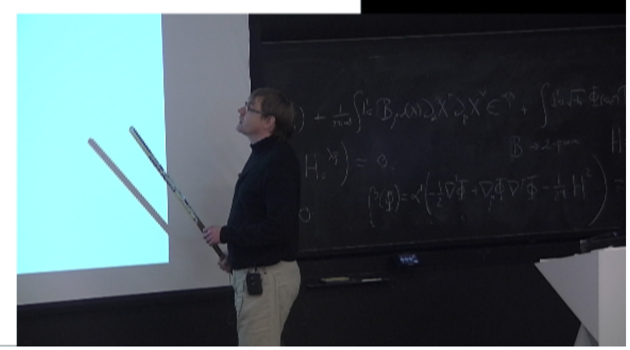
supersymmetry $\Longrightarrow$ factorization of scalar manifolds

$$\mathcal{M} = \mathcal{M}_{\rm VM} \otimes \mathcal{M}_{\rm HM}$$

- metrics on $\mathcal{M}_{\rm VM}$, $\mathcal{M}_{\rm HM}$ determine action completely!



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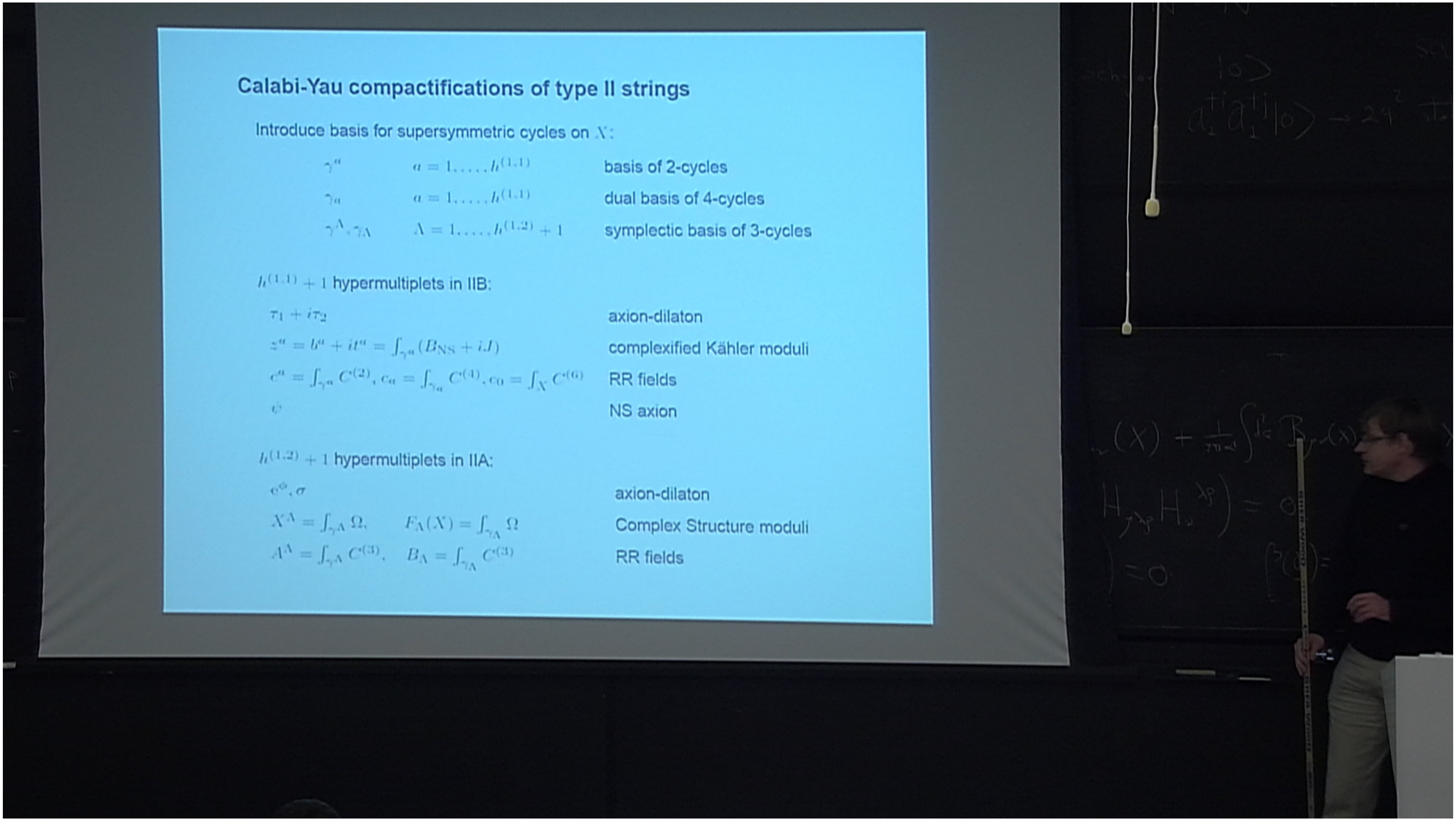
$\mathcal{M}_{\rm VM}$ is special Kähler:

- determined by holomorphic prepotential $F(X)$
- scalar kinetic terms: Kähler potential $K = i(\bar{X}^\Lambda F_\Lambda - X^\Lambda \bar{F}_\Lambda)$

$\mathcal{M}_{\rm HM}$ is quaternion-Kähler:

- notoriously complicated...

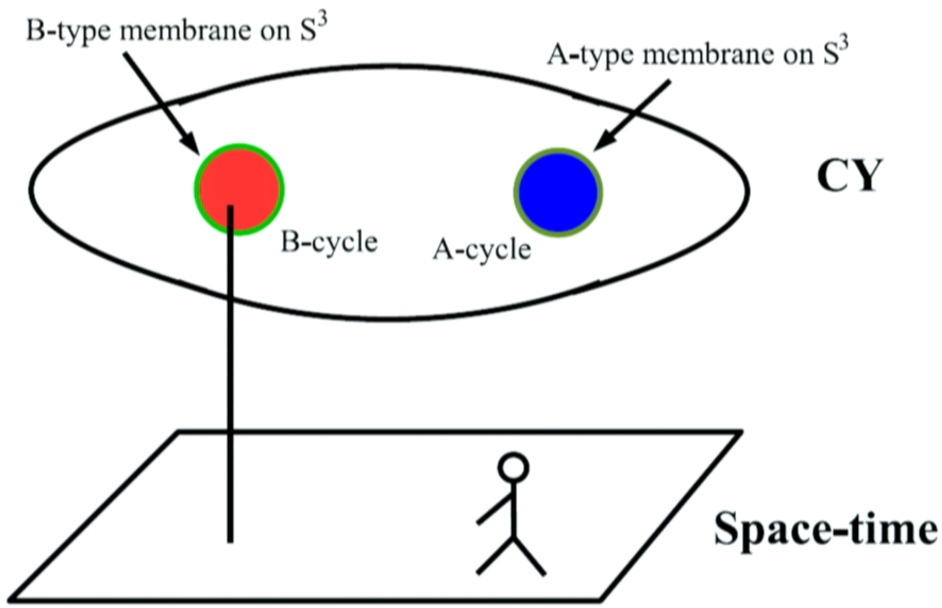

Calabi-Yau compactifications of type II strings
Introduce basis for supersymmetric cycles on X:
γ^a $a = 1, \ldots, h^{(1,1)}$ basis of 2-cycles
γ_a $a = 1, \ldots, h^{(1,1)}$ dual basis of 4-cycles
$\gamma^\Lambda, \gamma_\Lambda$ $\Lambda = 1, \ldots, h^{(1,2)} + 1$ symplectic basis of 3-cycles
$h^{(1,1)} + 1$ hypermultiplets in IIB:
$\tau_1 + i\tau_2$ axion-dilaton
$z^a = b^a + it^a = \int_{\gamma^a}(B_{\rm NS} + iJ)$ complexified Kähler moduli
$c^a = \int_{\gamma^a} C^{(2)}, c_a = \int_{\gamma_a} C^{(4)}, c_0 = \int_X C^{(6)}$ RR fields
ψ NS axion
$h^{(1,2)} + 1$ hypermultiplets in IIA:
e^ϕ, σ axion-dilaton
$X^\Lambda = \int_{\gamma^\Lambda} \Omega, \quad F_\Lambda(X) = \int_{\gamma_\Lambda} \Omega$ Complex Structure moduli
$A^\Lambda = \int_{\gamma^\Lambda} C^{(3)}, \quad B_\Lambda = \int_{\gamma_\Lambda} C^{(3)}$ RR fields

Instanton corrections from D-branes: IIA strings

Instantons: Euclidean Dp-branes wrapping $p+1$-dimensional submanifolds

- IIA: Dp-even: $D0$, $D2$ (membranes), $D4$, $NS5$

space-time instanton configurations in type IIA



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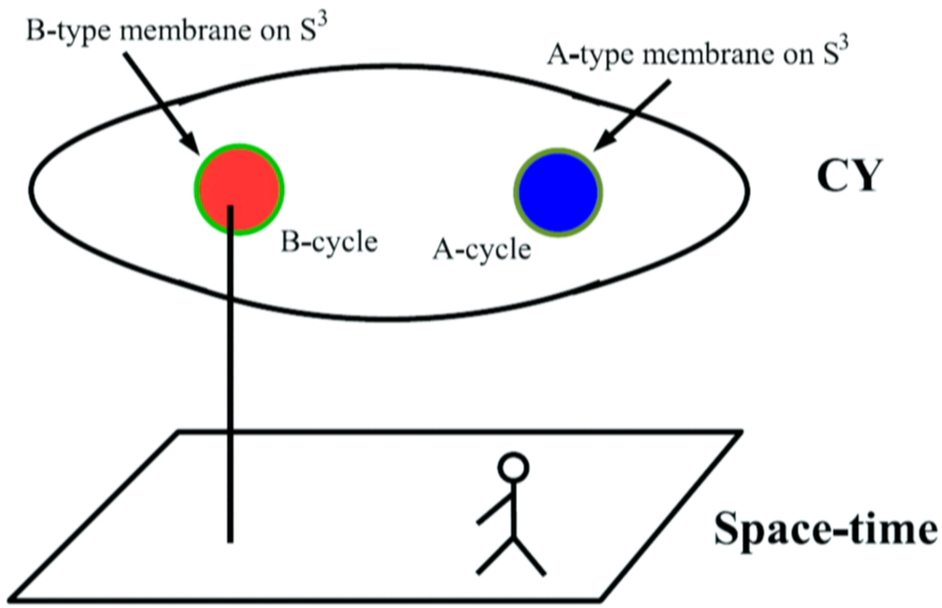

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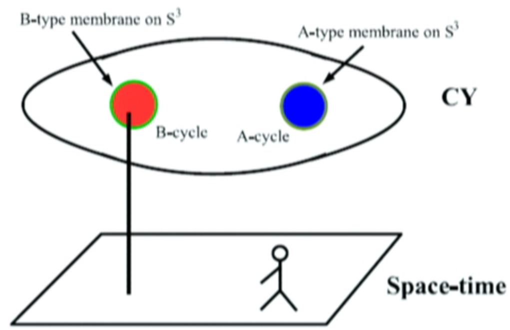
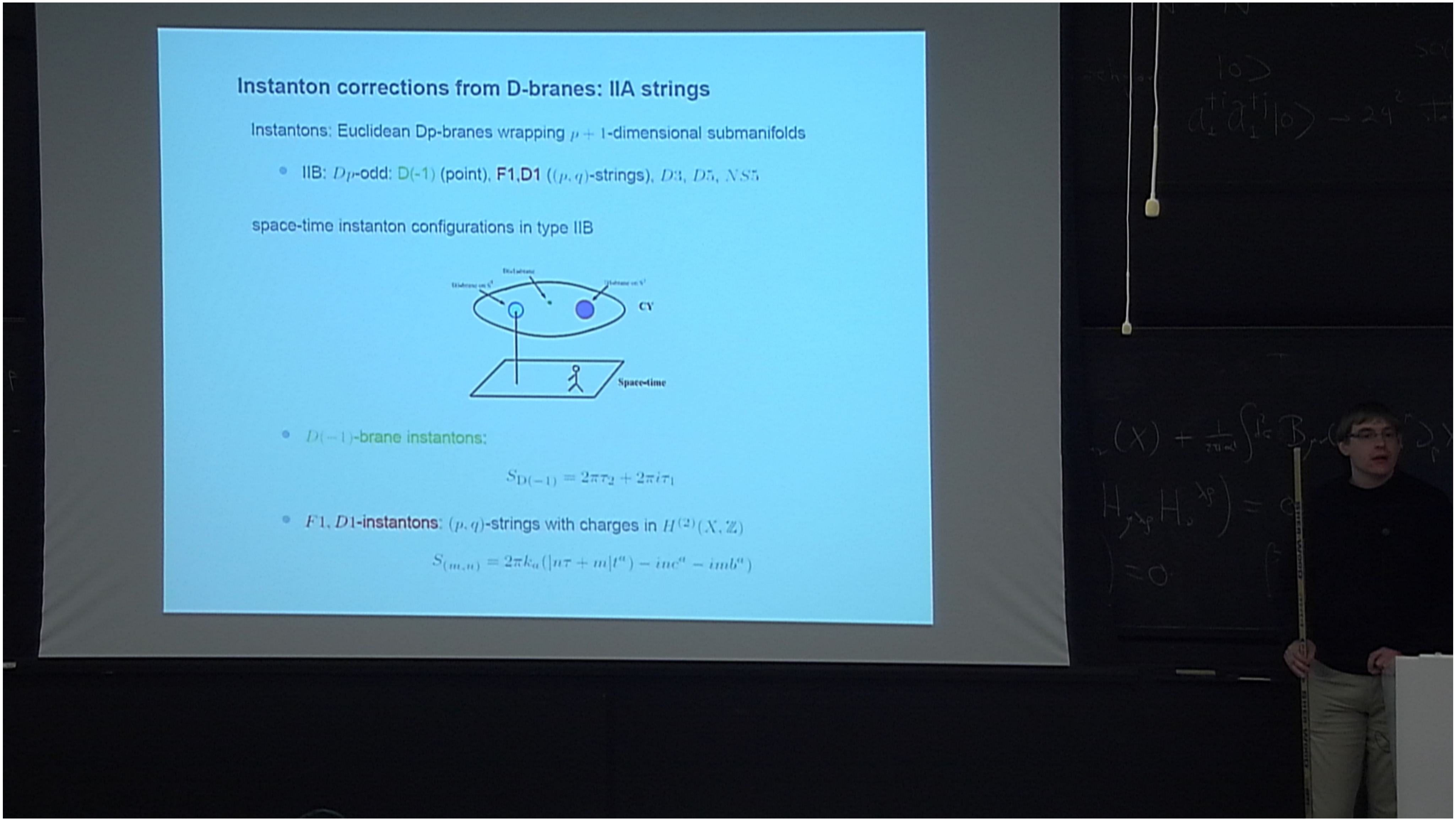


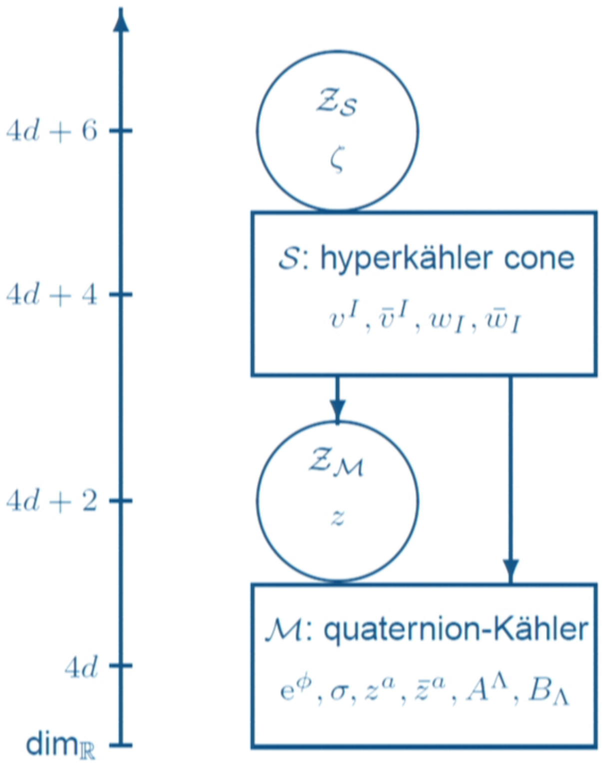
- instantons: D2 (E/M-membranes) wrapping electric/magnetic 3-cycles

$$S_{\rm D2} = 2\pi g_s^{-1} \left| z^\Lambda q_\Lambda - F_\Lambda p^\Lambda \right| + 2\pi i (A^\Lambda q_\Lambda - B_\Lambda p^\Lambda)$$

  - q_Λ, p^Λ = electric/magnetic charge of instanton configuration
  - break shift-symmetries in A^Λ, B_Λ


Instanton corrections from D-branes: IIA strings
Instantons: Euclidean Dp-branes wrapping $p+1$-dimensional submanifolds
IIB: Dp-odd: D(-1) (point), F1,D1 ((p,q)-strings), $D3$, $D5$, $NS5$
space-time instanton configurations in type IIB
CY
Space-time
$D(-1)$-brane instantons:
$S_{\mathrm{D}(-1)} = 2\pi\tau_2 + 2\pi i\tau_1$
$F1, D1$-instantons: (p,q)-strings with charges in $H^{(2)}(X,\mathbb{Z})$
$S_{(m,n)} = 2\pi k_a(|n\tau + m|t^a) - inc^a - imb^a)$

The twistor view on supergravity



twistor space $\mathcal{Z}_{\mathcal{S}}$ over $\mathcal{S}$

projective superspace description of $\mathcal{S}$

hypermultiplets in $N=2$ conformal sugra

related to $\mathcal{M}$ by superconformal quotient

Twistor space $\mathcal{Z}_{\mathcal{M}}$ over $\mathcal{M}$

use contact geometry to encode metric in $\mathcal{M}$

$\mathcal{M}$: hypermultiplet couplings

in $N=2$ supergravity

Hyperkähler manifolds, ...

Hyperkähler manifold $\mathcal{S}$ (g, J^1, J^2, J^3) of real dimension $4d+4$

- triplet of complex structures $J^i, i = 1, 2, 3$

$$J^i\, J^j = -\delta^{ij} + \epsilon^{ijk} J^k$$

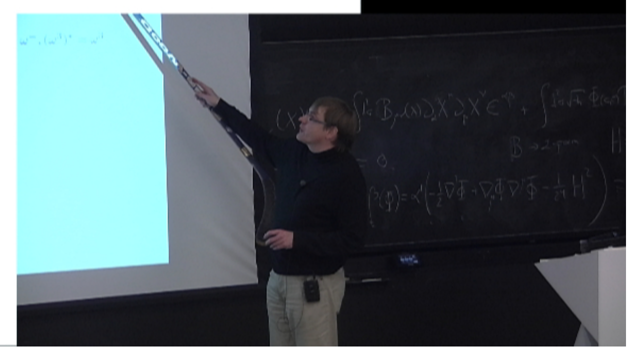
- triplet of Kähler forms

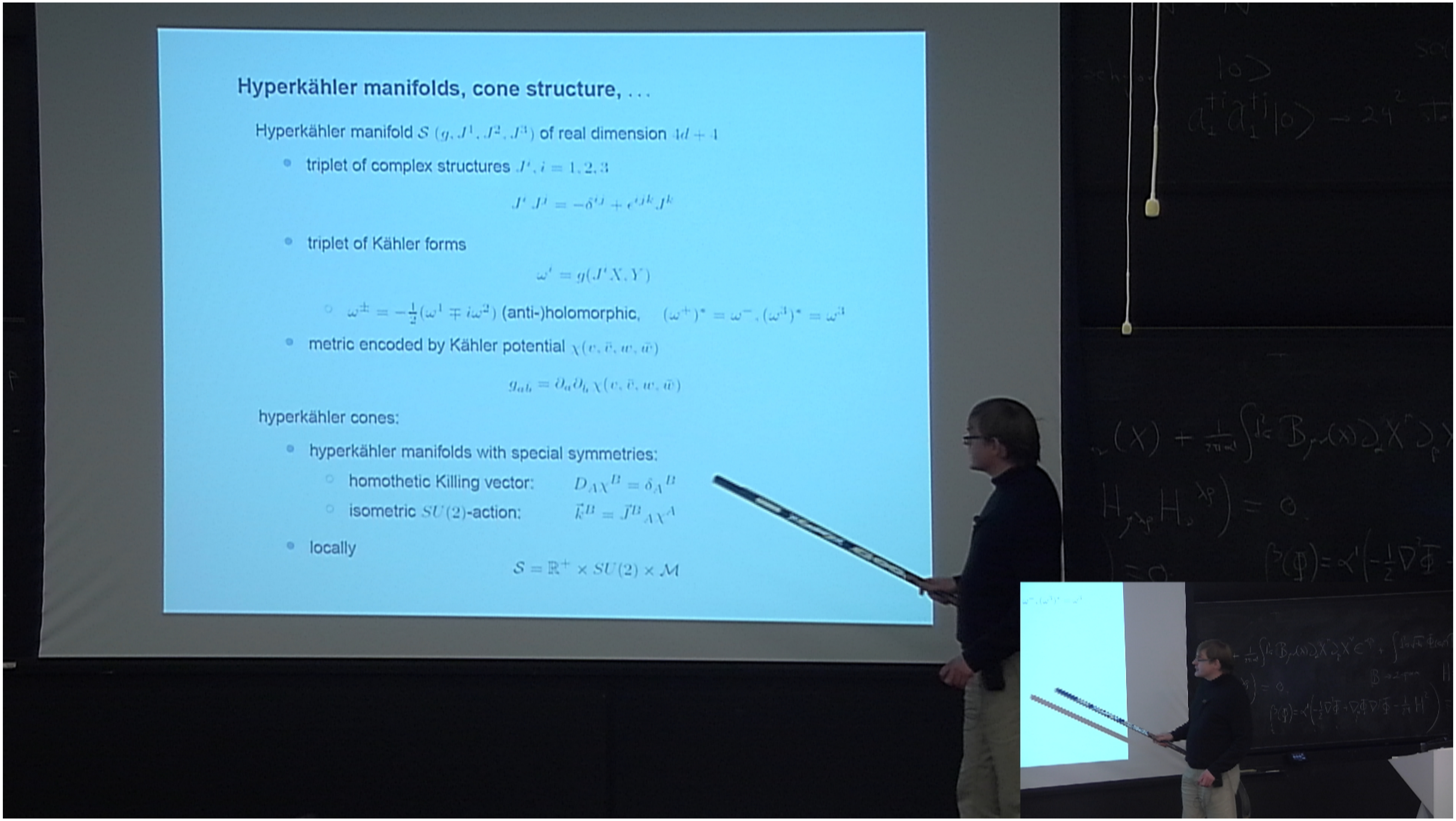
$$\omega^i = g(J^i X, Y)$$

  - $\omega^{\pm} = -\frac{1}{2}(\omega^1 \mp i\omega^2)$ (anti-)holomorphic, $\quad (\omega^+)^* = \omega^-, (\omega^3)^* = \omega^3$

- metric encoded by Kähler potential $\chi(v, \bar{v}, w, \bar{w})$

$$g_{a\bar{b}} = \partial_a \partial_{\bar{b}}\, \chi(v, \bar{v}, w, \bar{w})$$




Hyperkähler manifolds, cone structure, . . .
Hyperkähler manifold $\mathcal{S}$ (g, J^1, J^2, J^3) of real dimension $4d+4$
triplet of complex structures J^i, $i = 1, 2, 3$
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$g_{a\bar{b}} = \partial_a \partial_{\bar{b}} \chi(v, \bar{v}, w, \bar{w})$
hyperkähler cones:
hyperkähler manifolds with special symmetries:
homothetic Killing vector: $D_A \chi^B = \delta_A{}^B$
isometric $SU(2)$-action: $\vec{k}^B = \vec{J}^B{}_A \chi^A$
locally
$\mathcal{S} = \mathbb{R}^+ \times SU(2) \times \mathcal{M}$

Hyperkähler manifolds, cone structure and twistor spaces

Hyperkähler manifold $\mathcal{S}$ (g, J^1, J^2, J^3) of real dimension $4d+4$

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$\mathcal{Z}_S$: additional holomorphic coordinate $\zeta \in \mathbb{C}P^1$

- whole two-sphere of complex structures:

$$J(\zeta, \bar\zeta) = \frac{1-\zeta\bar\zeta}{1+\zeta\bar\zeta} J^3 + \frac{\zeta+\bar\zeta}{1+\zeta\bar\zeta} J^2 + i\frac{\zeta-\bar\zeta}{1+\zeta\bar\zeta} J^1$$

  - with Kähler form

$$\omega(\zeta, \bar\zeta) = \frac{1}{1+\zeta\bar\zeta}\left[(1-\zeta\bar\zeta)\omega^3 - 2i\zeta\omega^+ + 2i\bar\zeta\omega^-\right]$$

  - and (2,0)-form $\qquad \Omega(\zeta) = \omega^+ - i\zeta\,\omega^3 + \zeta^2\,\omega^-$

Twistor space construction of toric hyperkähler cones

N. Hitchin, A. Karlhede, U. Lindström, M. Roček, Commun. Math. Phys. 108 (1987) 535

- $\mathcal{M}_{\rm HM}$ in perturbative string theory $\Longleftrightarrow$ Heisenberg group of isometries
  - $\mathcal{S}$ has $d+1$ commuting isometries $\Leftrightarrow$ covered by Legendre construction

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- Basic building block: $N=2$ tensor superfield (=$\mathcal{O}(2)$-section on $\mathcal{Z}_{\mathcal{S}}$)

$$\eta^I = \frac{v^I}{\zeta} + x^I - \bar{v}^I \zeta$$

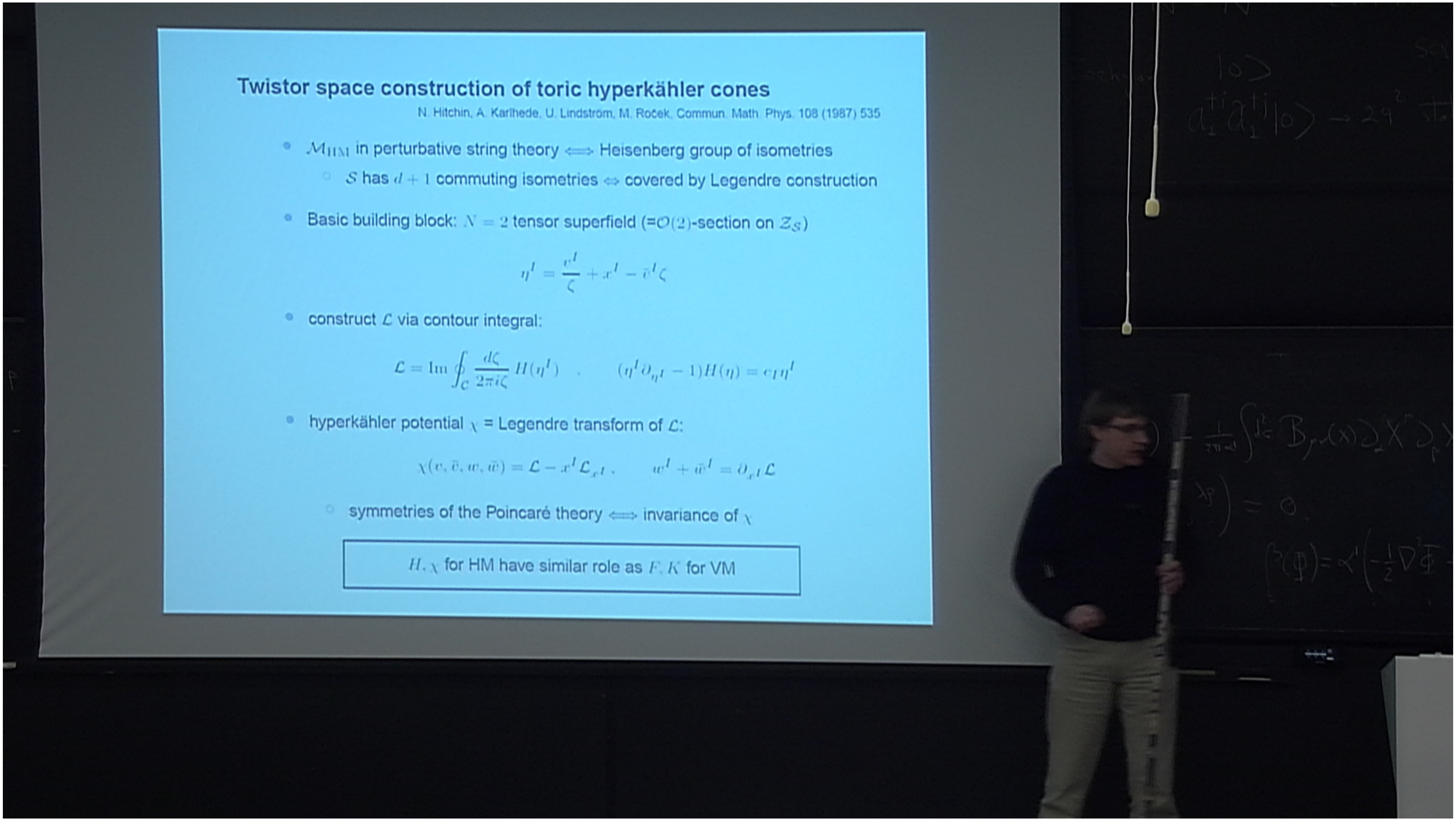
- construct $\mathcal{L}$ via contour integral:

$$\mathcal{L} = \mathrm{Im} \oint_{\mathcal{C}} \frac{d\zeta}{2\pi i \zeta} H(\eta^I) \quad , \qquad (\eta^I \partial_{\eta^I} - 1) H(\eta) = c_I \eta^I$$

- hyperkähler potential χ = Legendre transform of $\mathcal{L}$:

$$\chi(v, \bar{v}, w, \bar{w}) = \mathcal{L} - x^I \mathcal{L}_{x^I} \, , \qquad w^I + \bar{w}^I = \partial_{x^I} \mathcal{L}$$

  - symmetries of the Poincaré theory $\Longleftrightarrow$ invariance of χ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symmetries of the Poincaré theory $\Longleftrightarrow$ invariance of χ
H, χ for HM have similar role as F, K for VM

Dualities of type II string theory

- mirror symmetry (non-perturbative)
  - for X and Y mirror pair of Calabi-Yau manifolds

$$\text{LEEA for type IIA}/X \quad \simeq \quad \text{LEEA for type IIB}/Y$$

Dualities of type II string theory

- mirror symmetry (non-perturbative)
  - for X and Y mirror pair of Calabi-Yau manifolds

    LEEA for type IIA/X $\simeq$ LEEA for type IIB/Y

- T-duality (c-map) (string tree-level)
  - IIA/IIB string theory on S^1 (swap winding $\Leftrightarrow$ momentum modes):

    type IIA/S^1_R $\simeq$ type IIB/$S^1_{1/R}$
  - LEEA:

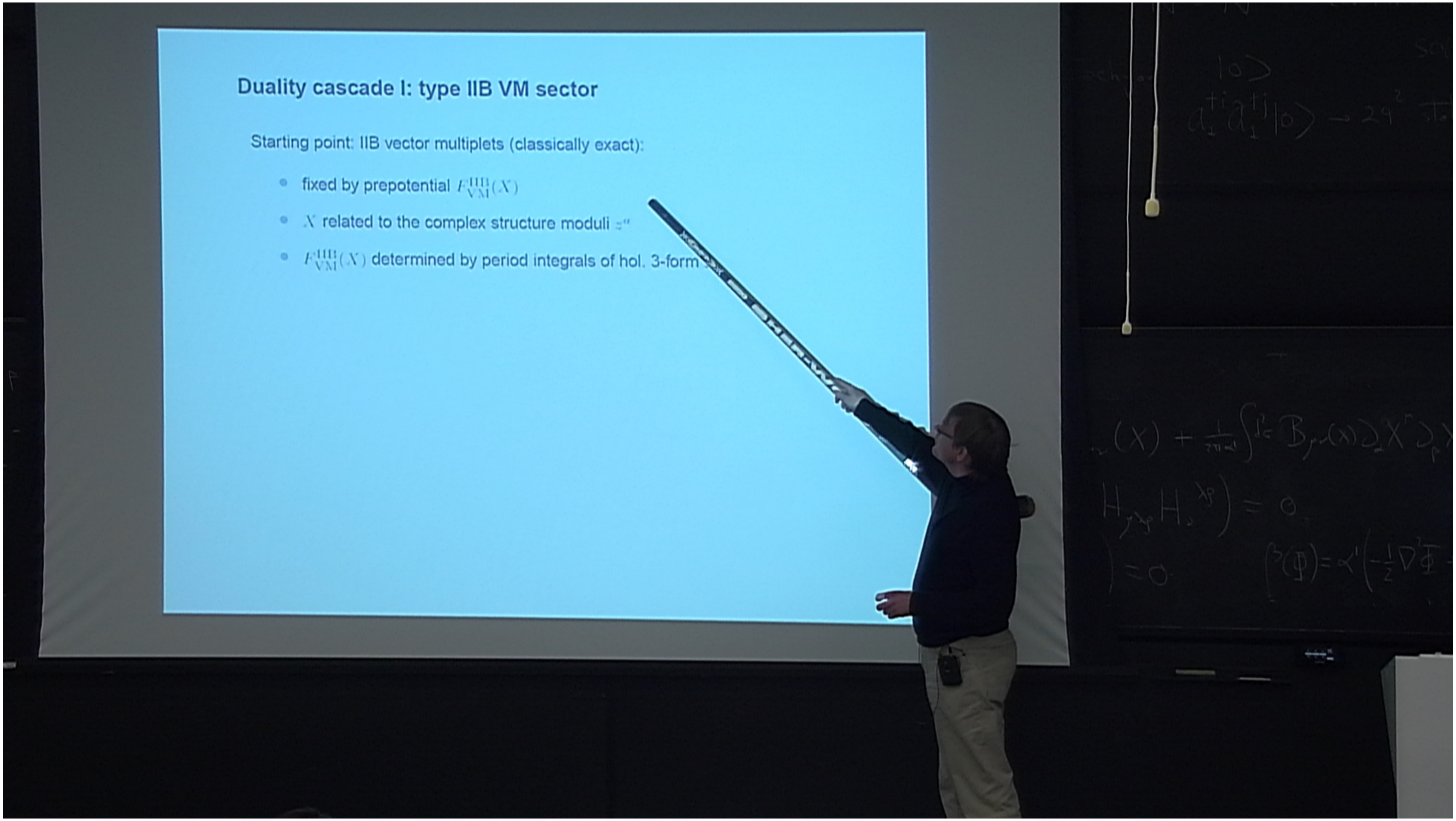
    type IIA/X vector multiplets $\Longleftrightarrow$ type IIB/X hypermultiples

- SL(2,$\mathbb{Z}$)-invariance (non-perturbative)
  - type IIB string has invariance:

    $\tau_1 \to 1/\tau_1$ and $\tau_2 \to \tau_2 + 1$ (+ transformation of tensors)

    relates e.g. fundamental strings and D-strings $\to$ (p, q)-strings
  - LEEA:

    invariant under transformations inherited by 4-dimensional fields


Duality cascade I: type IIB VM sector
Starting point: IIB vector multiplets (classically exact):
fixed by prepotential $F^{\rm IIB}_{\rm VM}(X)$
X related to the complex structure moduli z^a
$F^{\rm IIB}_{\rm VM}(X)$ determined by period integrals of hol. 3-form

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Duality cascade III: IIB HM sector in string perturbation theory

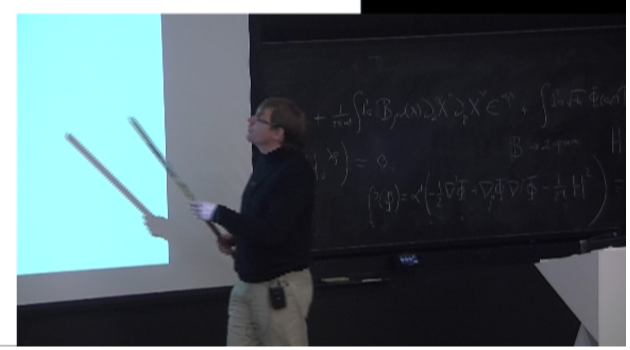
[M. Roček, C. Vafa, S. Vandoren, hep-th/0512206]

[D. Robles-Llana, F.S., S. Vandoren, hep-th/0602164]

perturbative IIB HM sector (classical c-map + 1-loop correction):

$$\mathcal{L}(v,\bar{v},x) = \mathrm{Im} \oint_{\mathcal{C}} \frac{\mathrm{d}\zeta}{2\pi i \zeta} \left[\frac{F(\eta^\Lambda)}{\eta^0} + \frac{i\,\chi_E}{(2\pi)^3}\, 2\zeta(2)\, \eta^0 \, \ln(\eta^0) \right]$$

- $F(\cdot)$ prepotential of dual VM sector
- η^0 conformal compensator (additional TM)
- contour $\mathcal{C}$ taken "zero" of $\zeta\eta^0$



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Determine χ using physical fields ($z^a = b^a + it^a$, t^a = Kähler modulus)

$$\chi_{\mathrm{cl}} = 4\,r^0\,\tau_2^2\,\frac{1}{3!}\,\kappa_{abc}\,t^a\,t^b\,t^c$$

$$\chi_{\mathrm{pt}} = \frac{1}{(2\pi)^3}\,r^0\,\chi_E\,\left[\zeta(3)\tau_2^2 + 2\zeta(2)\right]$$

$$\chi_{\mathrm{ws}} = -\frac{r^0\tau_2^2}{(2\pi)^3}\sum_{k_a} n_{k_a}\left[\mathrm{Li}_3(\mathrm{e}^{2\pi i k_a z^a}) + 2\pi k_a t^a \mathrm{Li}_2(\mathrm{e}^{2\pi i k_a z^a}) + c.c.\right]$$

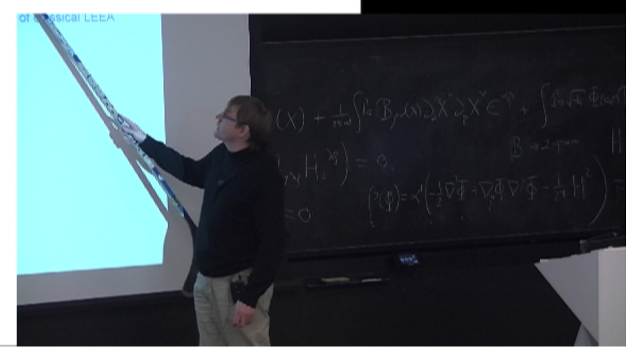
Duality cascade IV: SL(2, $\mathbb{Z}$) invariance of IIB LEEA

4-dimensional fields transform under SL(2,$\mathbb{Z}$) of type IIB string

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}\,,\ t^a \mapsto t^a|c\tau + d|\,,\ b^a \mapsto d\,b^a + c\,c^a\,,\ c^a \mapsto b\,b^a + a\,c^a\,;\ \ (r^0 \mapsto r^0|c\tau + d|)$$

Basic idea: IIB string has non-perturbative SL(2, $\mathbb{Z}$)-invariance

- observation: α', g_s-corrections break SL(2, $\mathbb{Z}$)-invariance of classical LEEA
- $\Longrightarrow$ restore SL(2, $\mathbb{Z}$)-invariance of LEEA
- $\Longleftrightarrow$ complete χ to a modular invariant function



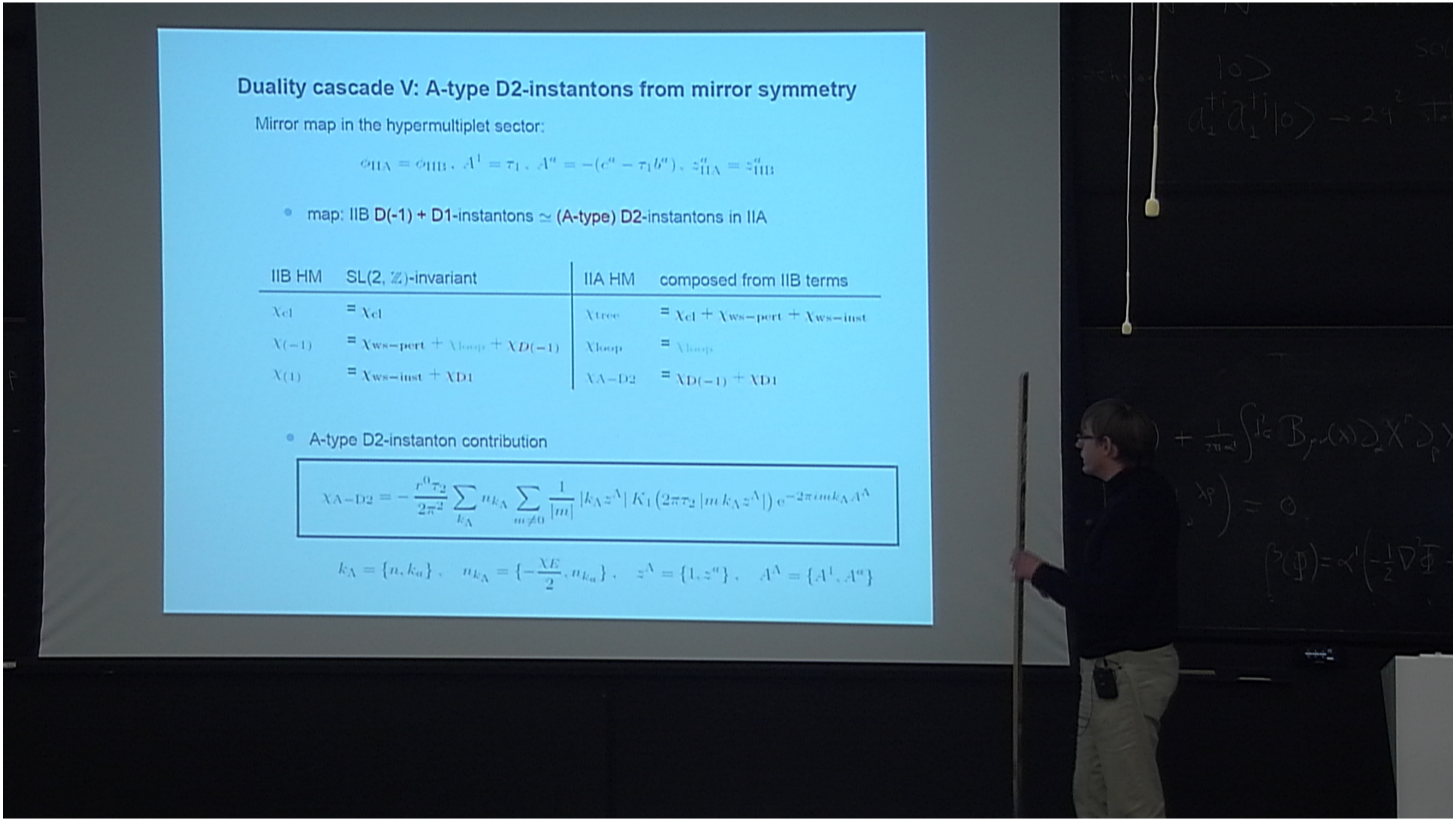
Duality cascade V: A-type D2-instantons from mirror symmetry

Mirror map in the hypermultiplet sector:

$$\phi_{\rm IIA} = \phi_{\rm IIB}\,,\ A^1 = \tau_1\,,\ A^a = -(c^a - \tau_1 b^a)\,,\ z^a_{\rm IIA} = z^a_{\rm IIB}$$

- map: IIB D(-1) + D1-instantons $\simeq$ (A-type) D2-instantons in IIA

IIB HM	SL(2, $\mathbb{Z}$)-invariant	IIA HM	composed from IIB terms
$\chi_{\rm cl}$	$= \chi_{\rm cl}$	$\chi_{\rm tree}$	$= \chi_{\rm cl} + \chi_{\rm ws-pert} + \chi_{\rm ws-inst}$
$\chi_{(-1)}$	$= \chi_{\rm ws-pert} + \chi_{\rm loop} + \chi_{D(-1)}$	$\chi_{\rm loop}$	$= \chi_{\rm loop}$
$\chi_{(1)}$	$= \chi_{\rm ws-inst} + \chi_{\rm D1}$	$\chi_{\rm A-D2}$	$= \chi_{\rm D(-1)} + \chi_{\rm D1}$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IIB HM
SL(2, $\mathbb{Z}$)-invariant
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$= \chi_{\rm loop}$
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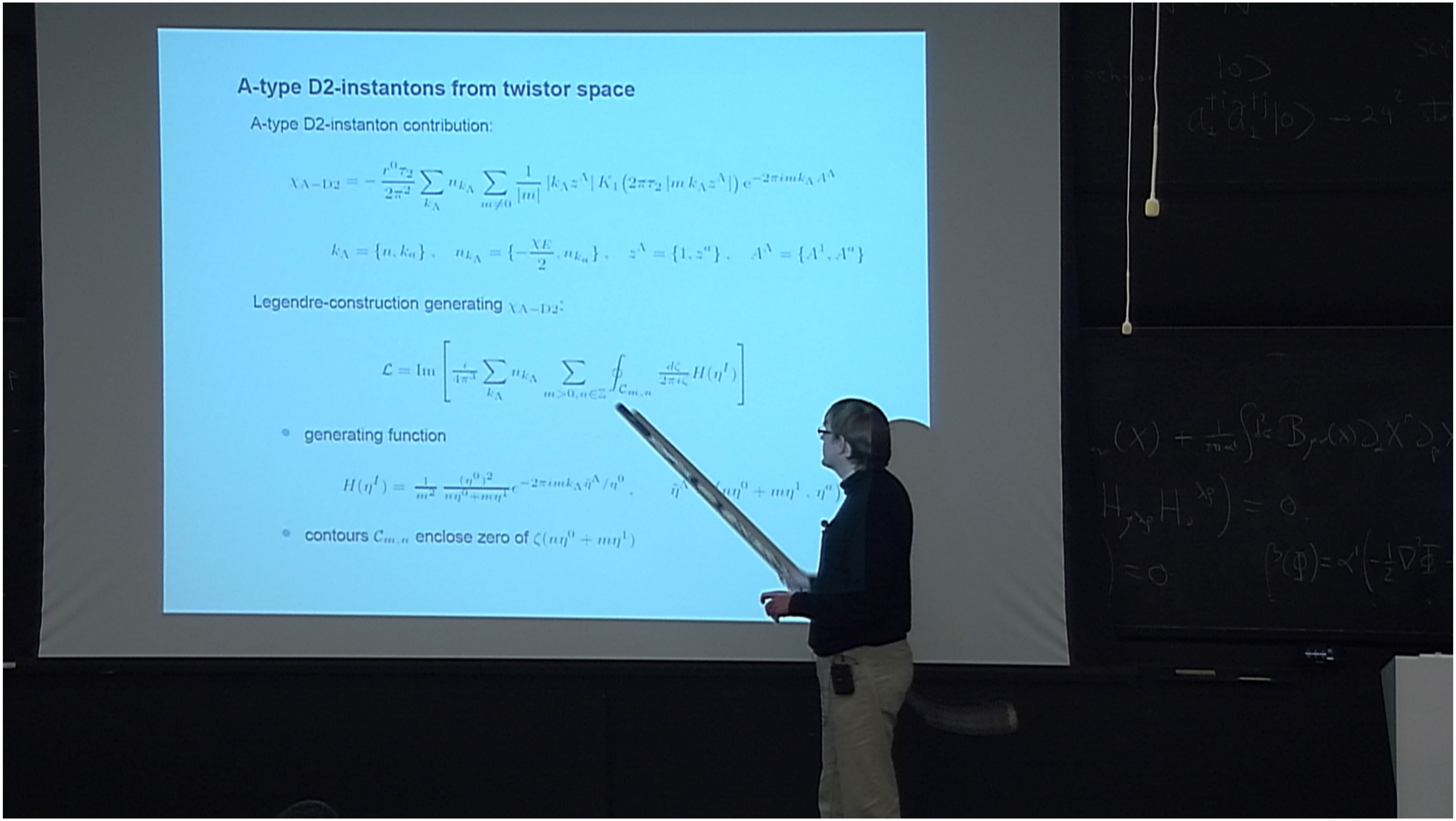
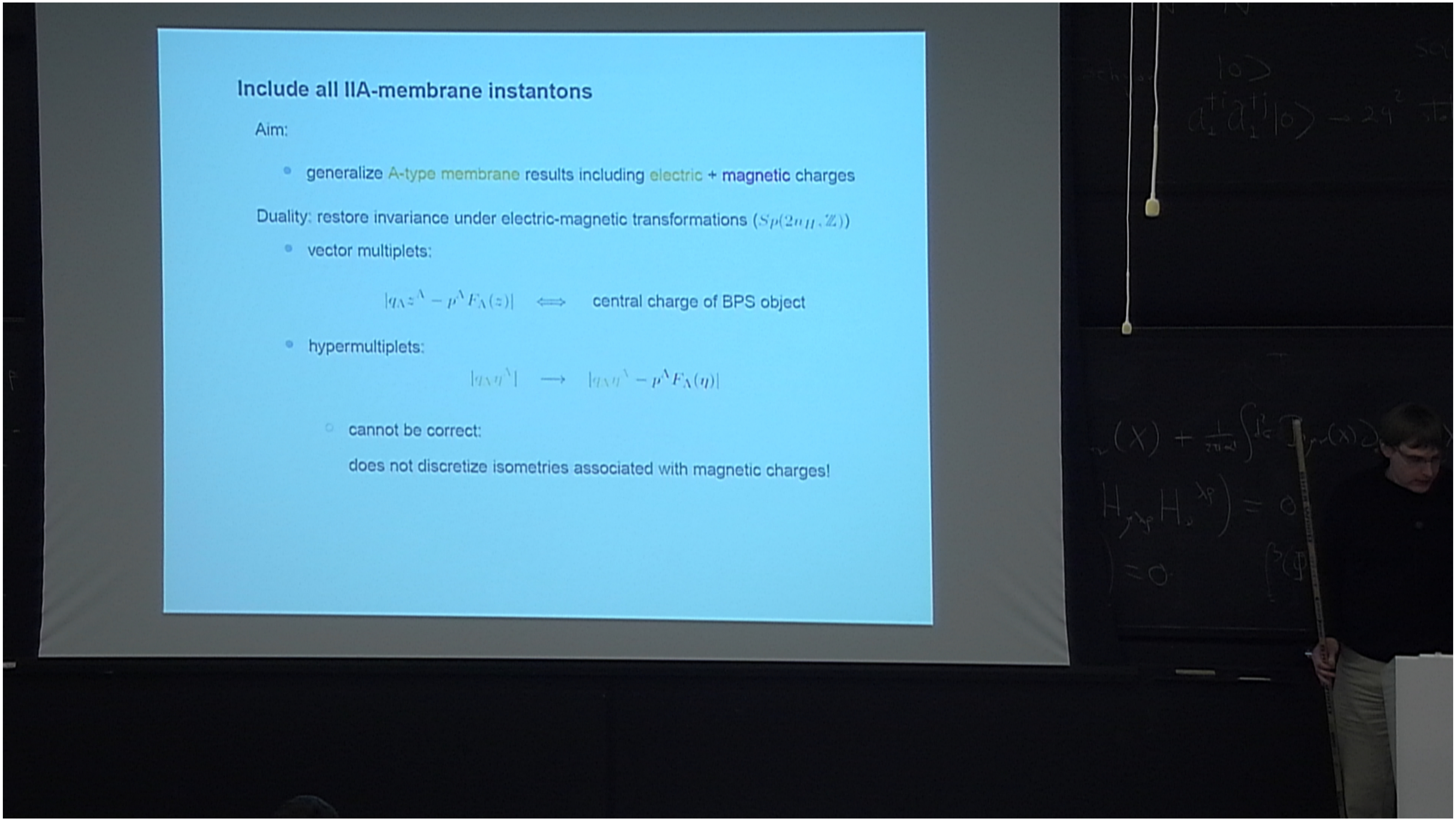
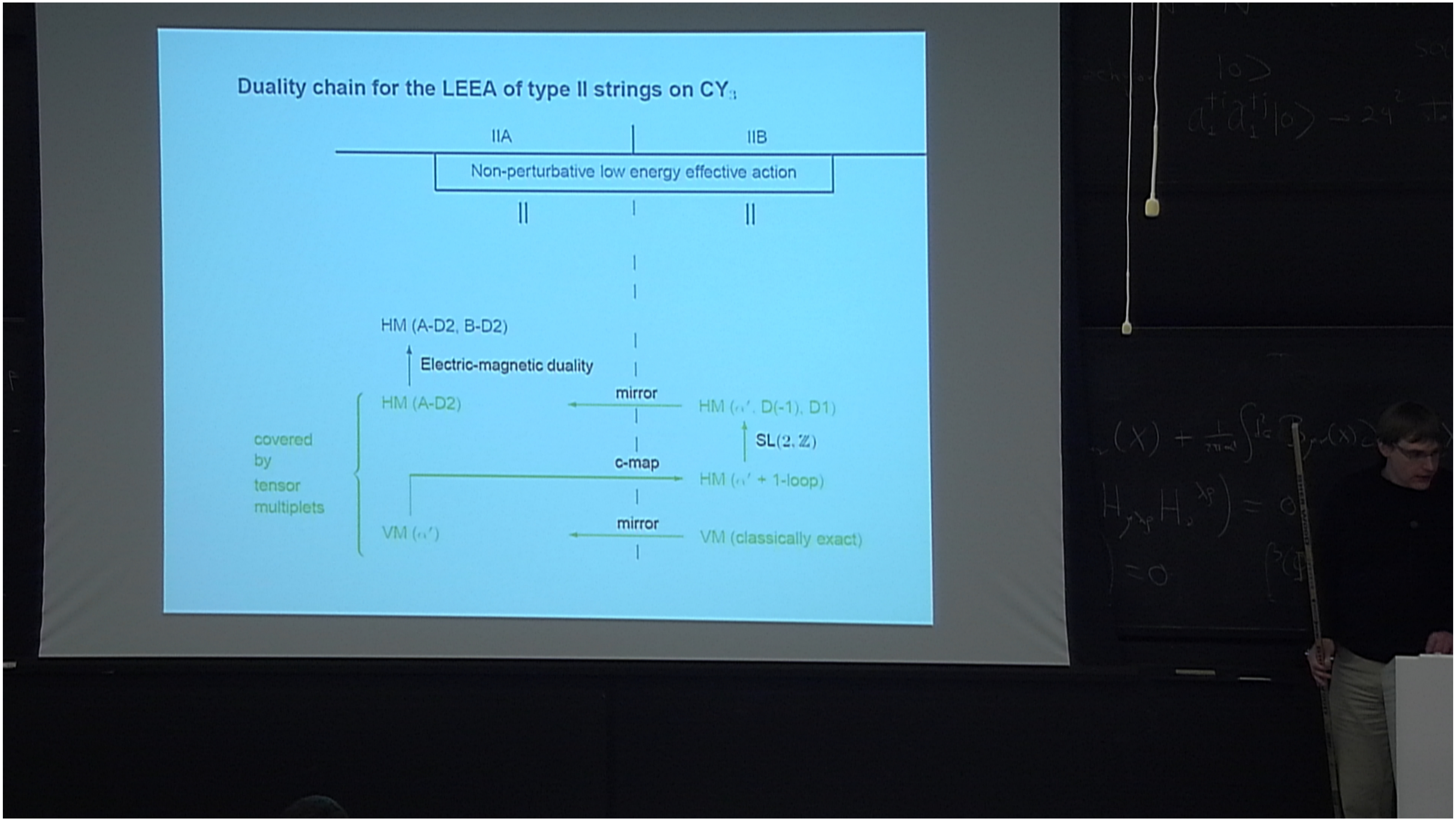
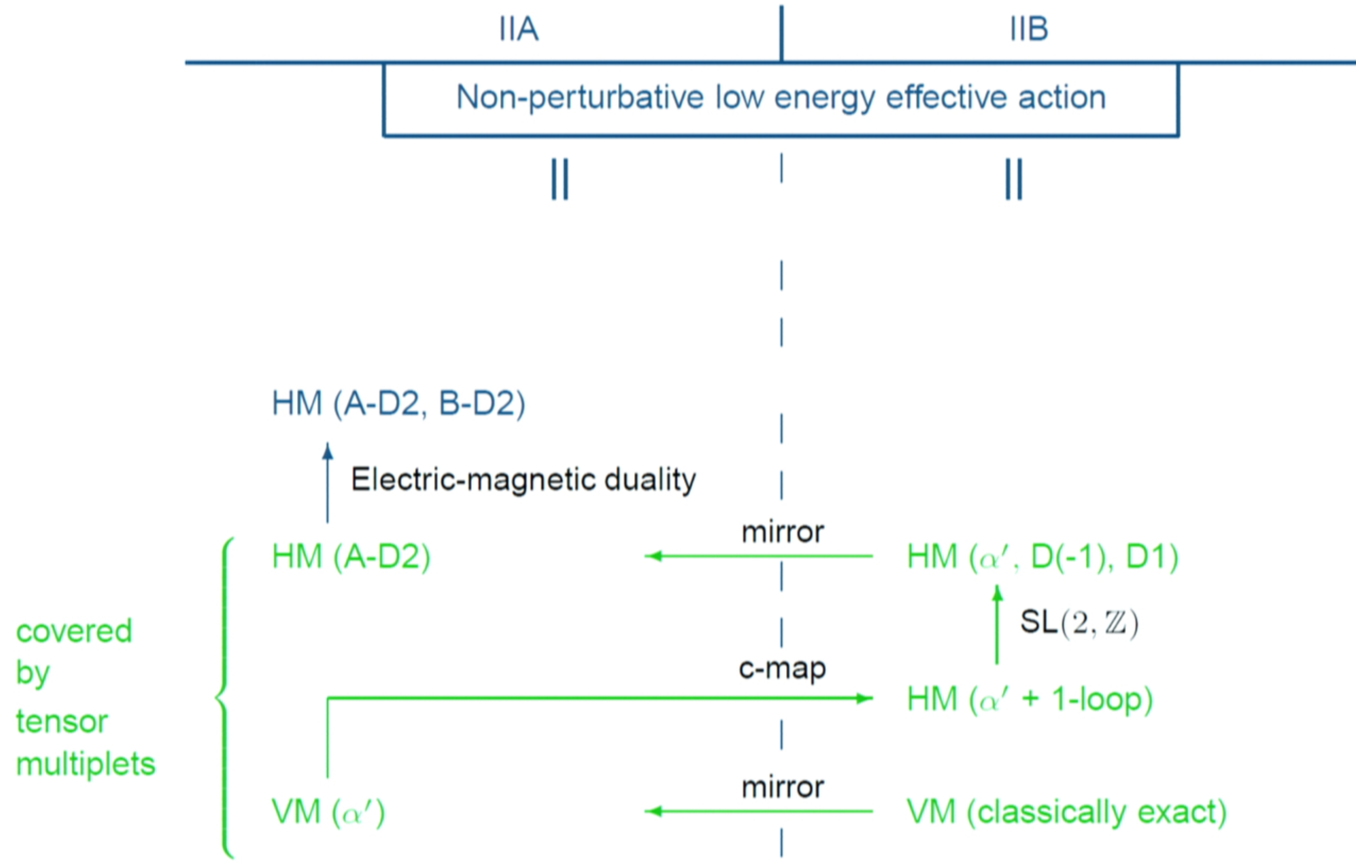
$\chi_{\rm A-D2}$
$= \chi_{\rm D(-1)} + \chi_{\rm D1}$
A-type D2-instanton contribution
$\chi_{\rm A-D2} = -\frac{r^0\tau_2}{2\pi^2}\sum_{k_\Lambda} n_{k_\Lambda}\sum_{m\neq 0}\frac{1}{|m|}\,|k_\Lambda z^\Lambda|\,K_1\left(2\pi\tau_2\,|m\,k_\Lambda z^\Lambda|\right)e^{-2\pi i m k_\Lambda A^\Lambda}$
$k_\Lambda = \{n, k_a\}\,,\quad n_{k_\Lambda} = \{-\frac{\chi_E}{2}, n_{k_a}\}\,,\quad z^\Lambda = \{1, z^a\}\,,\quad A^\Lambda = \{A^1, A^a\}$

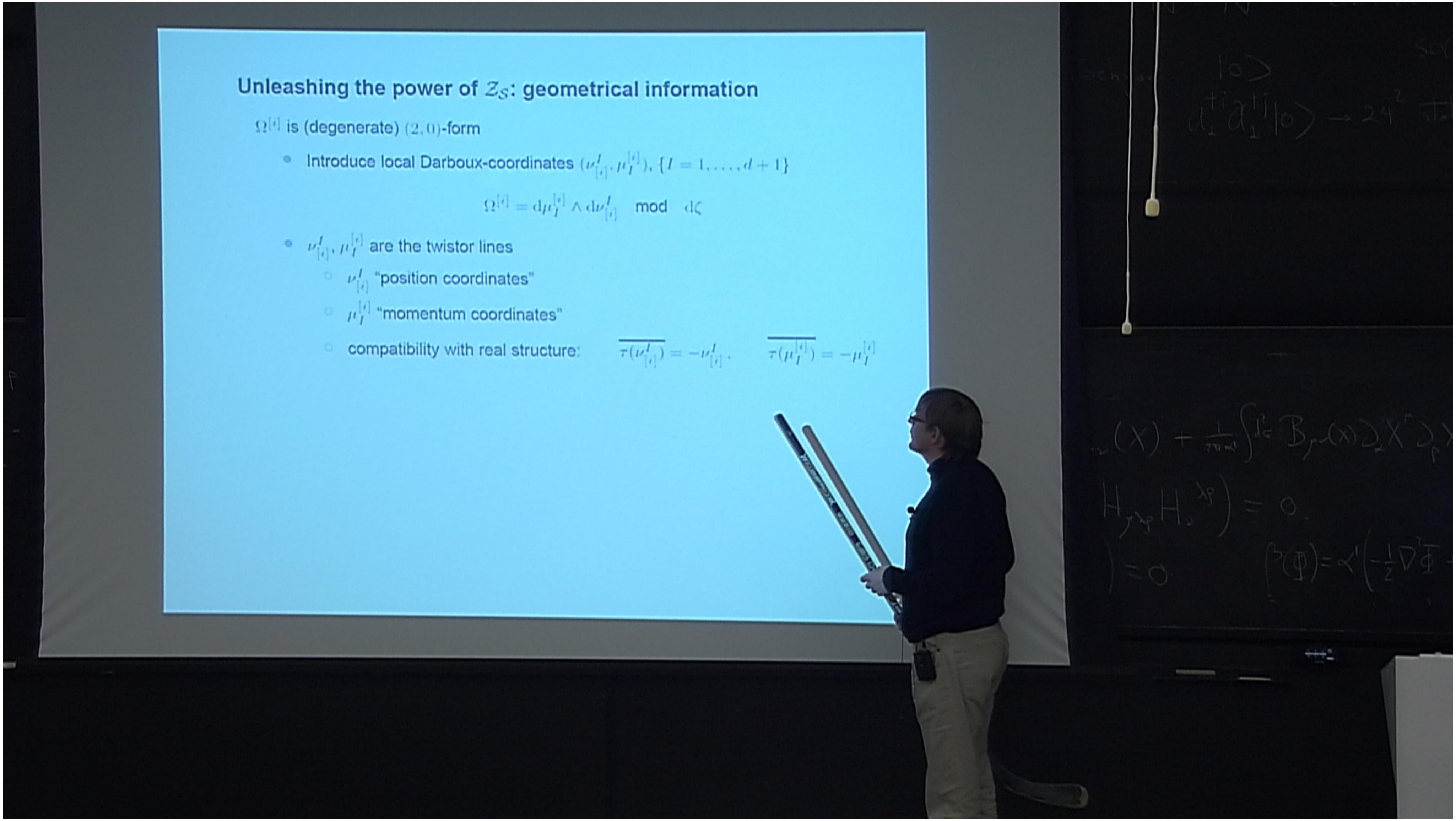

A-type D2-instantons from twistor space
A-type D2-instanton contribution:
$\chi_{A-D2} = -\frac{r^0\tau_2}{2\pi^2}\sum_{k_\Lambda} n_{k_\Lambda} \sum_{m\neq 0} \frac{1}{|m|} |k_\Lambda z^\Lambda| K_1(2\pi\tau_2 |m k_\Lambda z^\Lambda|) e^{-2\pi i m k_\Lambda A^\Lambda}$
$k_\Lambda = \{n, k_a\}, \quad n_{k_\Lambda} = \{-\frac{\chi_E}{2}, n_{k_a}\}, \quad z^\Lambda = \{1, z^a\}, \quad A^\Lambda = \{A^1, A^a\}$
Legendre-construction generating χ_{A-D2}:
$\mathcal{L} = \mathrm{Im}\left[\frac{i}{4\pi^3}\sum_{k_\Lambda} n_{k_\Lambda} \sum_{m>0, n\in\mathbb{Z}} \oint_{C_{m,n}} \frac{d\zeta}{2\pi i\zeta} H(\eta^I)\right]$
generating function
$H(\eta^I) = \frac{1}{m^2}\frac{(\eta^0)^2}{n\eta^0+m\eta^1} e^{-2\pi i m k_\Lambda \tilde{\eta}^\Lambda/\eta^0}$
contours $C_{m,n}$ enclose zero of $\zeta(n\eta^0 + m\eta^1)$


Include all IIA-membrane instantons
Aim:
generalize A-type membrane results including electric + magnetic charges
Duality: restore invariance under electric-magnetic transformations ($Sp(2n_H, \mathbb{Z})$)
vector multiplets:
$|q_\Lambda z^\Lambda - p^\Lambda F_\Lambda(z)| \iff$ central charge of BPS object
hypermultiplets:
$|q_\Lambda \eta^\Lambda| \longrightarrow |q_\Lambda \eta^\Lambda - p^\Lambda F_\Lambda(\eta)|$
cannot be correct:
does not discretize isometries associated with magnetic charges!


Duality chain for the LEEA of type II strings on CY$_3$
IIA
IIB
Non-perturbative low energy effective action
HM (A-D2, B-D2)
Electric-magnetic duality
HM (A-D2)
mirror
HM (α', D(-1), D1)
SL$(2, \mathbb{Z})$
covered
by
tensor
multiplets
c-map
HM (α' + 1-loop)
VM (α')
mirror
VM (classically exact)

Duality chain for the LEEA of type II strings on CY_3




Unleashing the power of $\mathcal{Z}_{\mathcal{S}}$: geometrical information
$\Omega^{[i]}$ is (degenerate) $(2,0)$-form
Introduce local Darboux-coordinates $(\nu^I_{[i]}, \mu_I^{[i]})$, $\{I = 1, \ldots, d+1\}$
$\Omega^{[i]} = \mathrm{d}\mu_I^{[i]} \wedge \mathrm{d}\nu^I_{[i]}$ mod $\mathrm{d}\zeta$
$\nu^I_{[i]}, \mu_I^{[i]}$ are the twistor lines
$\nu^I_{[i]}$ "position coordinates"
$\mu_I^{[i]}$ "momentum coordinates"
compatibility with real structure: $\overline{\tau(\nu^I_{[i]})} = -\nu^I_{[\bar{i}]}$, $\overline{\tau(\mu_I^{[i]})} = -\mu_I^{[\bar{i}]}$

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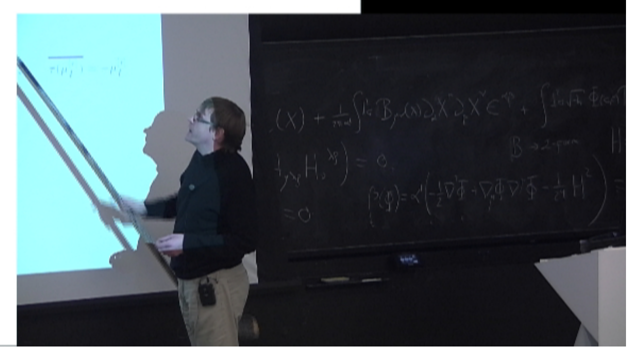

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geometrical data $\Leftrightarrow$ symplectomorphisms on overlap $\mathcal{U}_i \cap \mathcal{U}_j$

- $S^{[ij]}(\nu_{[i]}, \mu^{[j]}, \zeta)$ depended “initial position” and “final momentum”

$$\nu^I_{[j]} = \partial_{\mu_I^{[j]}} S^{[ij]}(\nu_{[i]}, \mu^{[j]}, \zeta)\,, \qquad \mu_I^{[i]} = f_{ij}^2\, \partial_{\nu^I_{[i]}} S^{[ij]}(\nu_{[i]}, \mu^{[j]}, \zeta)$$

- preserves symplectic form of $\Omega^{[i]}$
- guarantees: $\Omega^{[i]}$ transformation as $\mathcal{O}(2)$-section

Unleashing the power of $\mathcal{Z}_{\mathcal{S}}$: cone structure

$N = 2$ supergravity $\Longleftrightarrow$ $\mathcal{S}$ must be hyperkähler cone

constraint on admissible symplectomorphisms:

$$S^{[ij]}\left(\nu^I_{[i]}, \mu^{[j]}_I, \zeta\right) = f_{ij}^{-2}\, \hat{S}^{[ij]}\left(\nu^I_{[i]}, \mu^{[j]}_I\right)$$

- $\hat{S}^{[ij]}$ homogeneous of degree one in its first argument
- no explicit ζ-dependence

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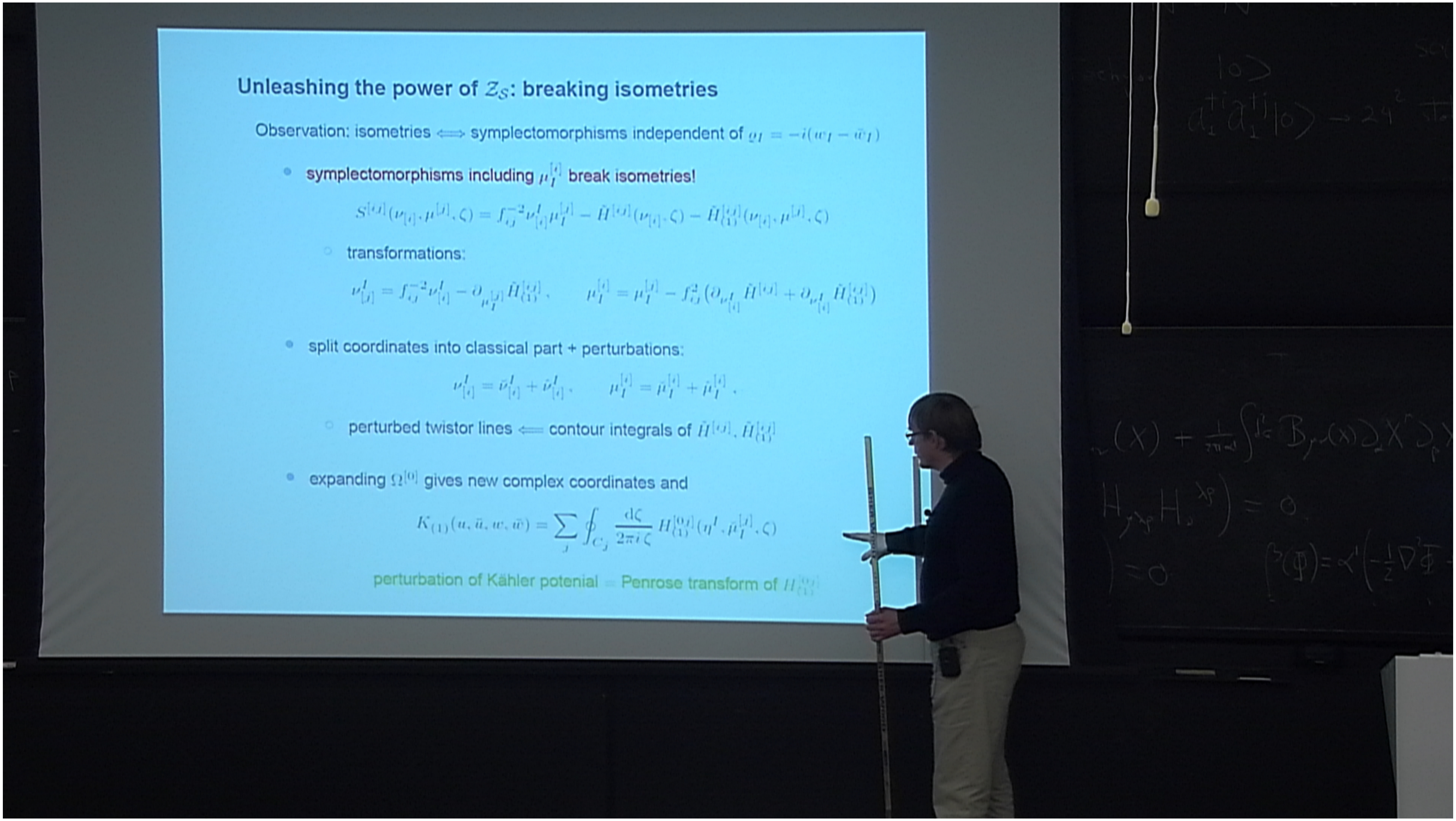
Additional: $\mu^{[i]}_I$ can acquire "anomalous dimensions" $c^{[i]}_I$

$$S^{[ij]} = f_{ij}^{-2}\left[\hat{S}^{[ij]}\left(\nu_{[i]}, \mu^{[j]}_I + c^{[j]}_I \log(f_{ij}^{-2}\nu^\flat_{[i]})\right) - c^{[i]}_I \nu^I_{[i]} \log \nu^\flat_{[i]}\right]$$

- position coordinate $\nu^\flat_{[i]}$ $\Longleftrightarrow$ compensates anomalous $\mu^{[i]}_I$-trafo
- $c^{[i]}_I = -c^{[\bar{i}]}_I$ are real

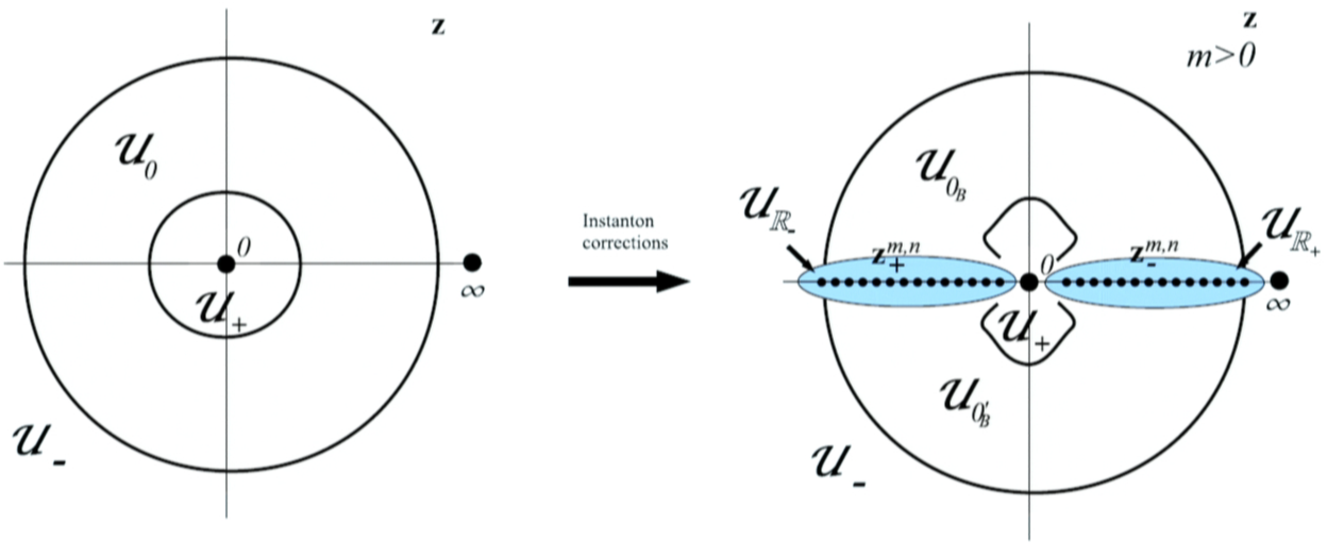

Unleashing the power of $\mathcal{Z}_S$: breaking isometries
Observation: isometries $\Longleftrightarrow$ symplectomorphisms independent of $\varrho_I = -i(w_I - \bar{w}_I)$
symplectomorphisms including $\mu_I^{[i]}$ break isometries!
$S^{[ij]}(\nu_{[i]}, \mu^{[j]}, \zeta) = f_{ij}^{-2}\nu_{[i]}^I \mu_I^{[j]} - \hat{H}^{[ij]}(\nu_{[i]}, \zeta) - \hat{H}_{(1)}^{[ij]}(\nu_{[i]}, \mu^{[j]}, \zeta)$
transformations:
$\nu_{[j]}^I = f_{ij}^{-2}\nu_{[i]}^I - \partial_{\mu_I^{[j]}}\hat{H}_{(1)}^{[ij]}, \qquad \mu_I^{[i]} = \mu_I^{[j]} - f_{ij}^2\left(\partial_{\nu_{[i]}^I}\hat{H}^{[ij]} + \partial_{\nu_{[i]}^I}\hat{H}_{(1)}^{[ij]}\right)$
split coordinates into classical part + perturbations:
$\nu_{[i]}^I = \bar{\nu}_{[i]}^I + \hat{\nu}_{[i]}^I, \qquad \mu_I^{[i]} = \bar{\mu}_I^{[i]} + \hat{\mu}_I^{[i]}.$
perturbed twistor lines $\Longleftarrow$ contour integrals of $\tilde{H}^{[ij]}, \tilde{H}_{(1)}^{[ij]}$
expanding $\Omega^{[0]}$ gives new complex coordinates and
$K_{(1)}(u, \bar{u}, w, \bar{w}) = \sum_j \oint_{C_j} \frac{d\zeta}{2\pi i\zeta} H_{(1)}^{[0j]}(\eta^I, \bar{\mu}_I^{[j]}, \zeta)$
perturbation of Kähler potenial
Penrose transform of $H_{(1)}^{[0j]}$

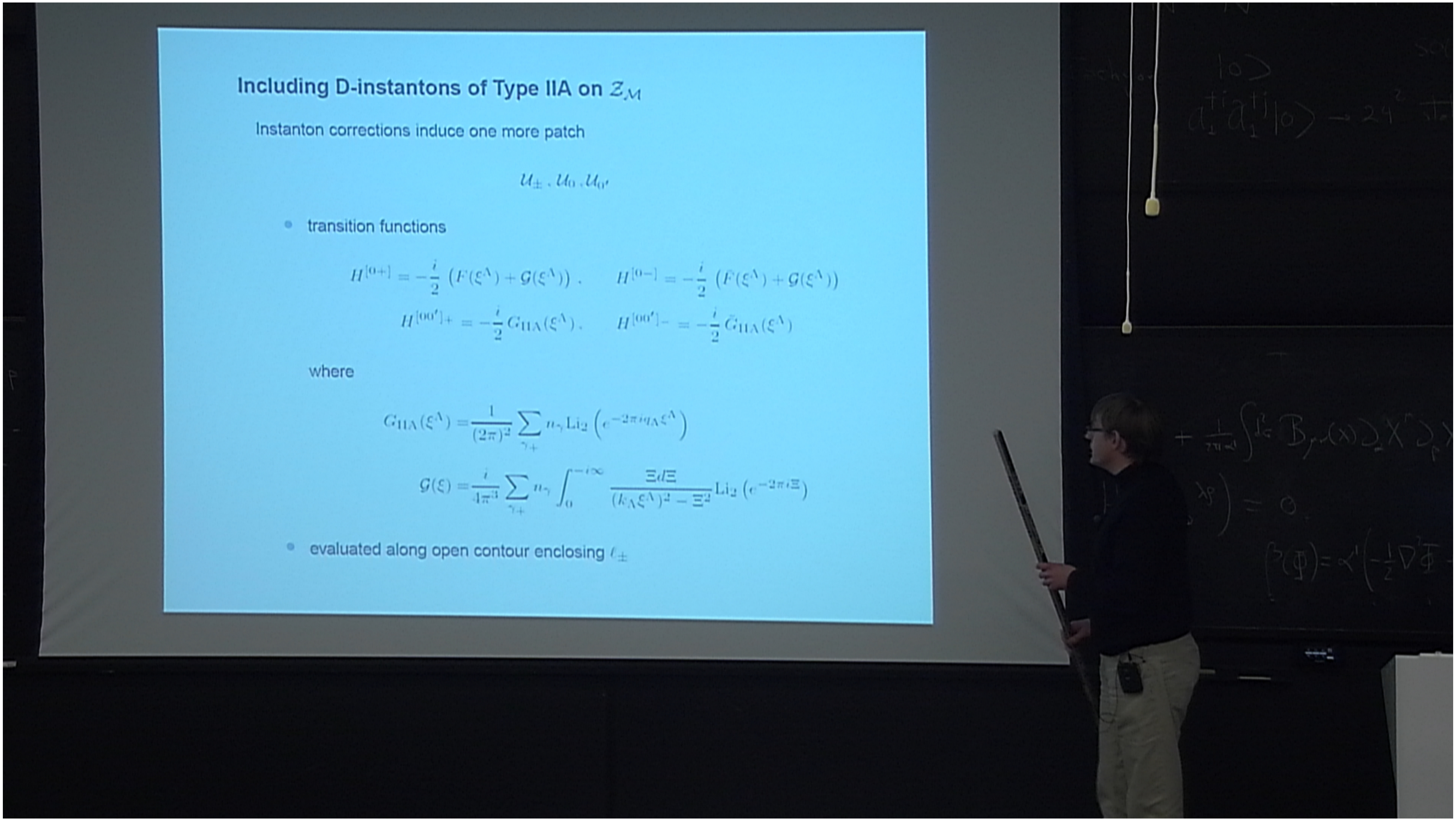
Including D-instantons of Type IIB on $\mathcal{Z}_{\mathcal{M}}$

Instanton corrections induce three more patches

$$\mathcal{U}_{\pm}\ ,\ \mathcal{U}_0\ ,\mathcal{U}_{0'}\ ,\ \mathcal{U}_{\mathbb{R}+}\ ,\ \mathcal{U}_{\mathbb{R}-}$$

- dense set of poles $\mathbf{z}_{\pm}^{m,n}$ on $\mathbb{C}P^1$:




Including D-instantons of Type IIA on $\mathcal{Z}_{\mathcal{M}}$
Instanton corrections induce one more patch
$\mathcal{U}_{\pm}, \mathcal{U}_0, \mathcal{U}_{0'}$
transition functions
$H^{[0+]} = -\frac{i}{2}\left(F(\xi^\Lambda) + \mathcal{G}(\xi^\Lambda)\right), \quad H^{[0-]} = -\frac{i}{2}\left(F(\xi^\Lambda) + \mathcal{G}(\xi^\Lambda)\right)$
$H^{[00']+} = -\frac{i}{2} G_{\mathrm{IIA}}(\xi^\Lambda), \quad H^{[00']-} = -\frac{i}{2} G_{\mathrm{IIA}}(\xi^\Lambda)$
where
$G_{\mathrm{IIA}}(\xi^\Lambda) = \frac{1}{(2\pi)^2} \sum_{\gamma_+} n_\gamma \mathrm{Li}_2\left(e^{-2\pi i q_\Lambda \xi^\Lambda}\right)$
$\mathcal{G}(\xi) = \frac{i}{4\pi^3} \sum_{\gamma_+} n_\gamma \int_0^{-i\infty} \frac{\Xi d\Xi}{(k_\Lambda \xi^\Lambda)^2 - \Xi^2} \mathrm{Li}_2\left(e^{-2\pi i \Xi}\right)$
evaluated along open contour enclosing $\ell_{\pm}$