Title: Mathematical Modelling and Applications; Einstein-Maxwell-Boltzmann system on a Bianchi type 1 space-time

Date: Feb 14, 2012 11:00 AM

URL: http://pirsa.org/12020136

Abstract: First part: The research group in Yaounde (Cameroon), working on Mathematical Modelling and Applications is introduced. Second part: Global existence of solutions to the spatially homogeneous Einstein-Maxwell-Boltzmann system on a Bianchi type 1 space-time is proved.


MATHEMATICAL MODELLING AND APPLICATIONS (MMA)
(Norbert NOUTCHEGUEME)
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(Yaoundé, CAMEROON)
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# MATHEMATICAL MODELLING AND APPLICATIONS (MMA)

**(Norbert NOUTCHEGUEME)**

**(UNIVERSITY OF YAOUNDE 1)**

**(Yaoundé, CAMEROON)**

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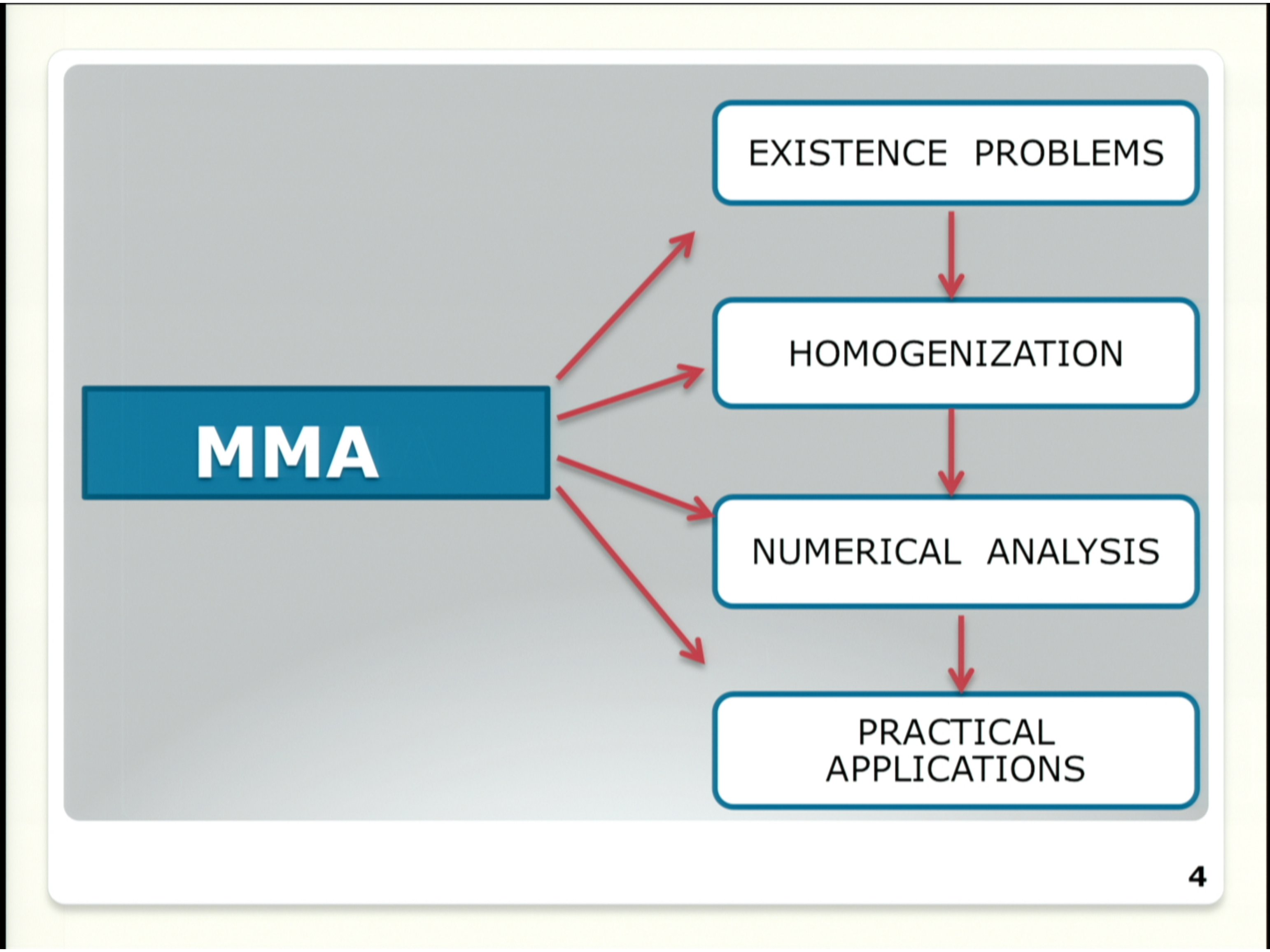
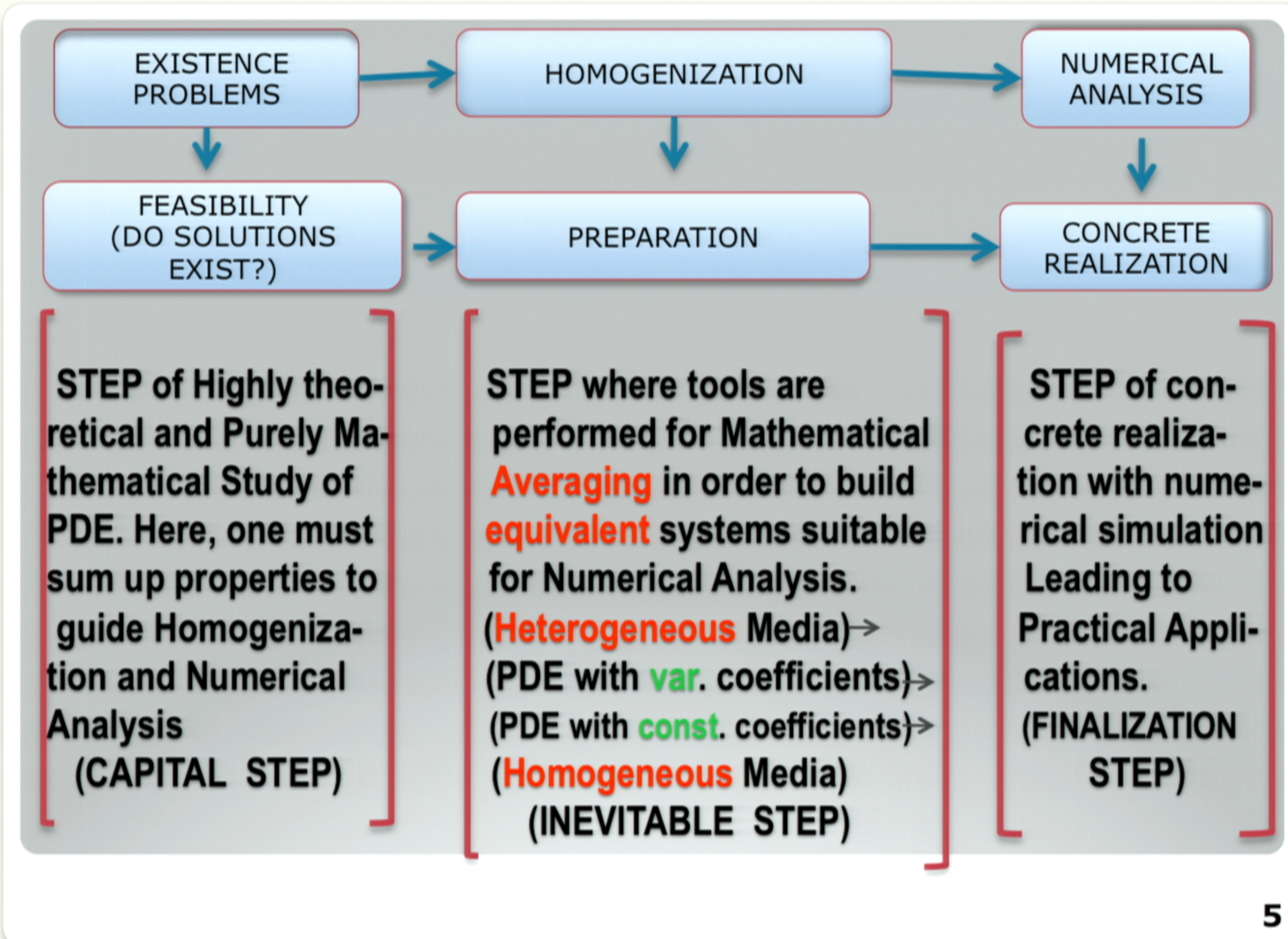
| | |
|---|---|
| **OBJECTIVES =** | **MATHEMATICAL MODELLING**<br>**WITH PARTIAL DIFFERENTIAL EQUATIONS (PDE)**<br>**OF**<br>**SEVERAL KINDS OF NATURAL PHENOMENA** |
| | |


MMA
EXISTENCE PROBLEMS
HOMOGENIZATION
NUMERICAL ANALYSIS
PRACTICAL APPLICATIONS
4


EXISTENCE PROBLEMS
HOMOGENIZATION
NUMERICAL ANALYSIS
FEASIBILITY (DO SOLUTIONS EXIST?)
PREPARATION
CONCRETE REALIZATION
STEP of Highly theoretical and Purely Mathematical Study of PDE. Here, one must sum up properties to guide Homogenization and Numerical Analysis
(CAPITAL STEP)
STEP where tools are performed for Mathematical Averaging in order to build equivalent systems suitable for Numerical Analysis.
(Heterogeneous Media)→
(PDE with var. coefficients)→
(PDE with const. coefficients)→
(Homogeneous Media)
(INEVITABLE STEP)
STEP of concrete realization with numerical simulation Leading to Practical Applications.
(FINALIZATION STEP)
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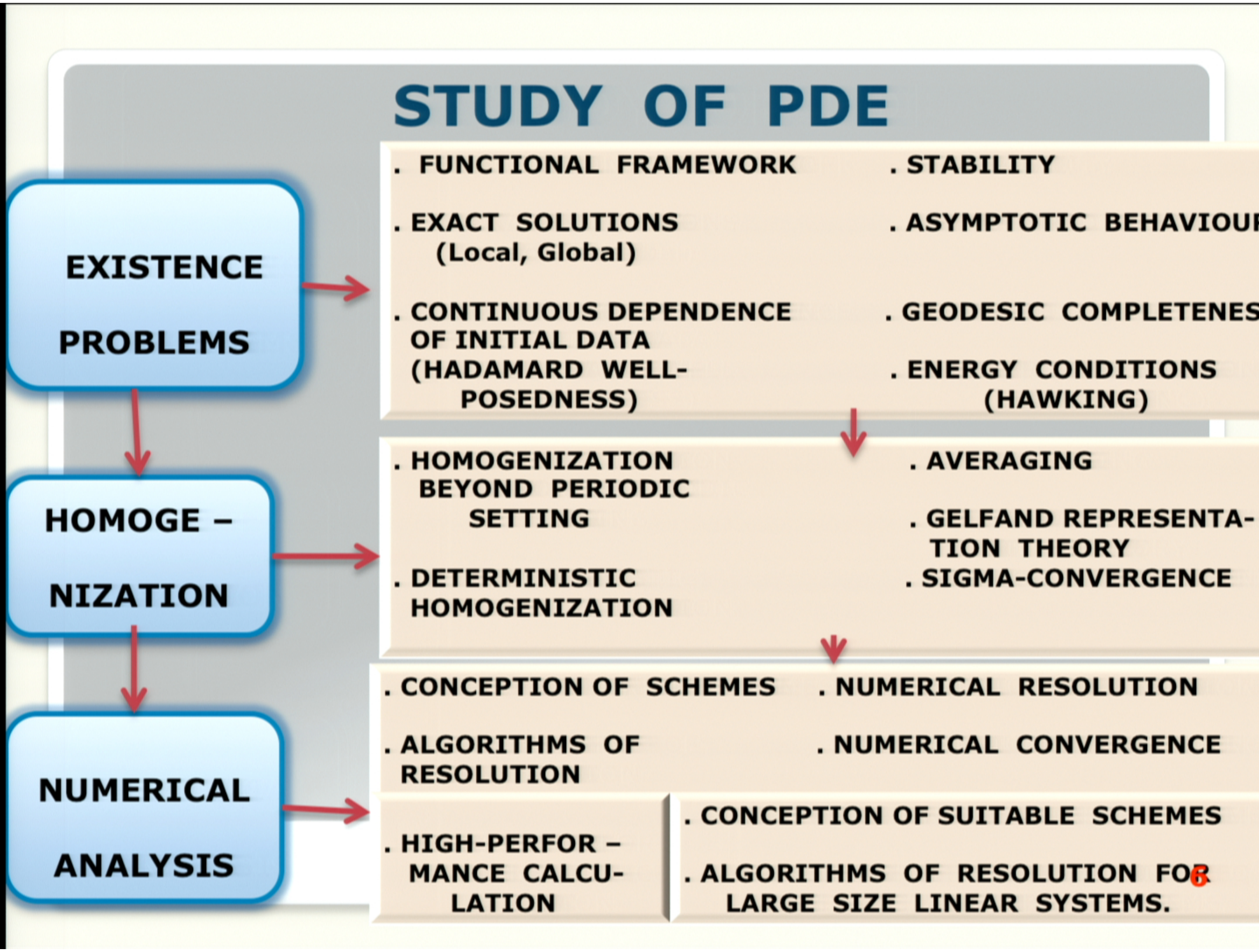
STUDY OF PDE
EXISTENCE PROBLEMS
. FUNCTIONAL FRAMEWORK
. EXACT SOLUTIONS (Local, Global)
. CONTINUOUS DEPENDENCE OF INITIAL DATA (HADAMARD WELL-POSEDNESS)
. STABILITY
. ASYMPTOTIC BEHAVIOUR
. GEODESIC COMPLETENESS
. ENERGY CONDITIONS (HAWKING)
HOMOGE – NIZATION
. HOMOGENIZATION BEYOND PERIODIC SETTING
. DETERMINISTIC HOMOGENIZATION
. AVERAGING
. GELFAND REPRESENTA-TION THEORY
. SIGMA-CONVERGENCE
NUMERICAL ANALYSIS
. CONCEPTION OF SCHEMES
. ALGORITHMS OF RESOLUTION
. NUMERICAL RESOLUTION
. NUMERICAL CONVERGENCE
. HIGH-PERFOR – MANCE CALCU-LATION
. CONCEPTION OF SUITABLE SCHEMES
. ALGORITHMS OF RESOLUTION FOR LARGE SIZE LINEAR SYSTEMS.
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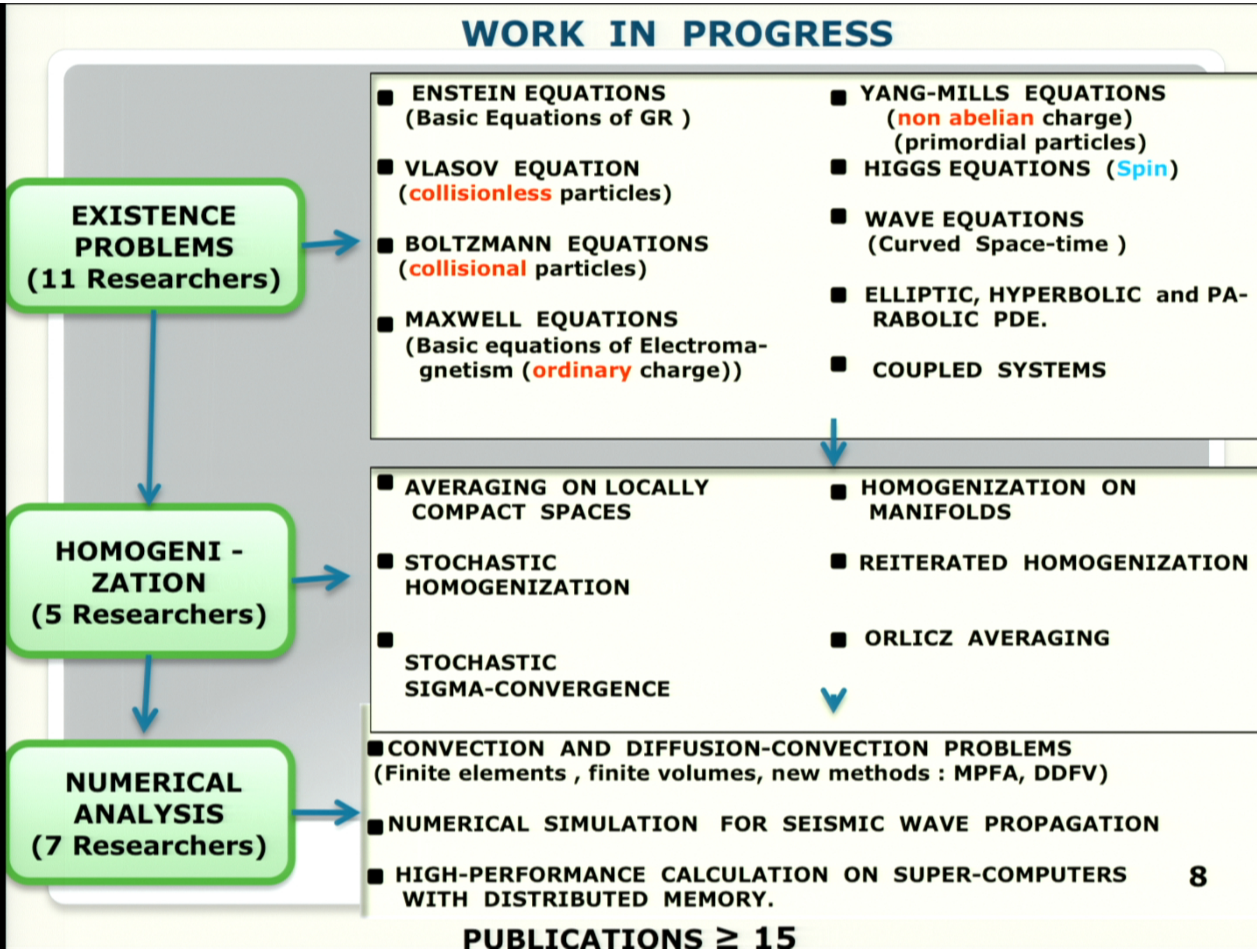
# APPLICATIONS

- **HYDROLOGY (FLOWS IN POROUS MEDIA)**

| Illustration : | |
|---|---|
| DOUALA( Cameroon) (Overpopulated industrial town)<br>➢ Extension of shandy towns<br>➢ Environment Problems<br>➢ Recurent deases (Cholera, …)<br>➢ Provide drinkable water | 1) Interactions between superficial water table and underground flow.<br>[modelization: Nonlinear PDE<br>(elliptic in the case of permanent rate of flow)<br>(hyperbolic in the case of transitory rate of flow)]<br><br>2) Pollutions(chemical wastes, used, dirty water)<br>➢ Transport Phenomenon<br>➢ Measure the Concentration of pollutants. Modelization : Nonlinear hyperbolic PDE<br>Scientists: Solve equations ➢ Master the ground water flow ➢ Submit propositions to decision-makers (Support?) |

- **METEOROLOGY**
- **GEOSEISMICS OF THE GROUND (Simulation of earthquake) (Remember Fukushima)**
- **IMAGE ASSIMILATION (TREATMENT) FOR GEOPHYSIC FLUIDS (Oil prospection)**
- **RESOLUTION OF LARGE SIZE LINEAR SYSTEM AX=B (A= nxn matrix with n very large)**
- **LOCALISATION OF EINGENVALUES FOR LARGE SIZE STRUCTURED MATRICES (Solved $\det(A-\lambda I) = 0$, with A=nxn matrix with n very large )**


WORK IN PROGRESS
EXISTENCE PROBLEMS (11 Researchers)
ENSTEIN EQUATIONS (Basic Equations of GR )
VLASOV EQUATION (collisionless particles)
BOLTZMANN EQUATIONS (collisional particles)
MAXWELL EQUATIONS (Basic equations of Electroma-gnetism (ordinary charge))
YANG-MILLS EQUATIONS (non abelian charge) (primordial particles)
HIGGS EQUATIONS (Spin)
WAVE EQUATIONS (Curved Space-time )
ELLIPTIC, HYPERBOLIC and PA-RABOLIC PDE.
COUPLED SYSTEMS
HOMOGENI - ZATION (5 Researchers)
AVERAGING ON LOCALLY COMPACT SPACES
STOCHASTIC HOMOGENIZATION
STOCHASTIC SIGMA-CONVERGENCE
HOMOGENIZATION ON MANIFOLDS
REITERATED HOMOGENIZATION
ORLICZ AVERAGING
NUMERICAL ANALYSIS (7 Researchers)
CONVECTION AND DIFFUSION-CONVECTION PROBLEMS (Finite elements , finite volumes, new methods : MPFA, DDFV)
NUMERICAL SIMULATION FOR SEISMIC WAVE PROPAGATION
HIGH-PERFORMANCE CALCULATION ON SUPER-COMPUTERS WITH DISTRIBUTED MEMORY.
8
PUBLICATIONS ≥ 15

# MMA Research Team

Localization : **University of Yaounde I**
Lead Researcher : **NOUTCHEGUEME Norbert**

## EXISTENCE PROBLEMS

1 **NOUTCHEGUEME Norbert**
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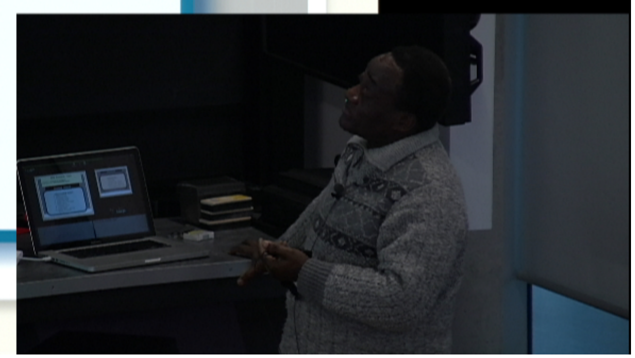
5 TEGANKONG David
6 DONGO David (Dschang)
7 TAKOU Etienne
8 CHENDJOU Gilbert
9 AYISSI Raoûlt
10 NANGUE Alexis (Maroua)
11 TETSADJIO Erick M.



## HOMOGENIZATION

**1 NGUETSENG Gabriel**

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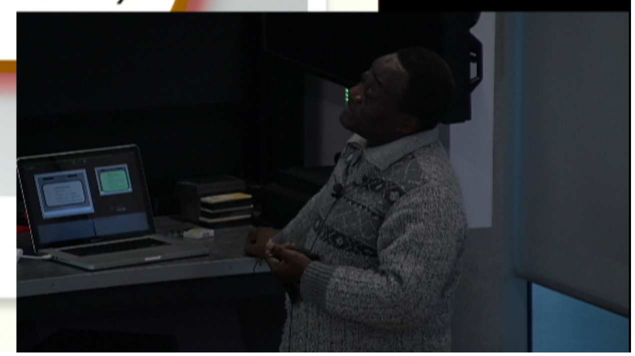
3 WOUKENG J. Louis (Dschang)

4 SIGNING Lazarius ( Ngaoundere)

5 DOUANLA Hermann (Postdoc, Sweden)



## NUMERICAL ANALYSIS

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7 BERNARD Philippe (INRIA, France)

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THANK YOU FOR

YOUR KIND

ATTENTION

# Einstein-Maxwell-Boltzmann system on a Bianchi type I space-time

By

N.Noutchegueme and R.D Ayissi
University
of Yaoundé I, CAMEROON

(A part is published in the Journal: Adv.Studies Theor.Phys.)

# PLAN

1. Introduction
2. The coupled System
3. Study of the Boltzmann equation
4. Study of the Einstein equations
5. Local existence for the coupled Einstein-Boltzmann system.
6. Global existence for the Einstein-Boltzmann system
7. Conclusion et perspectives.

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# PHYSICAL MOTIVATION

The coupled Einstein-Maxwell-Boltzman system governs in relativistic kinetic theory, the **collisional evolution** of a kind of **fast moving massive and charged** particles, under the action of both their common gravitational field and auto-created electromagnetic forces.

Such physical phenomena occur in some media at very high temperature such as: **burning reators**, **nebular galaxies**, **solar winds**... where colliding particles of ionized gas evolve with **very high velocities**, under the action of self-created gravitational and electromagnetic forces.

We consider the Einstein equations with **cosmological constant** which is a appropriate mathematical tool to confirm some astrophysical observations.

# ThE systEm

$$
(1.1)\begin{cases}
R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R + \Lambda g_{\alpha\beta} & = 8\pi(T_{\alpha\beta} + \tau_{\alpha\beta} + \rho u_\alpha u_\beta) \quad (1) \\
\nabla_\alpha F^{\alpha\beta} & = J^\beta \quad (2) \\
\nabla_\alpha F_{\beta\delta} + \nabla_\beta F_{\delta\alpha} + \nabla_\delta F_{\alpha\beta} & = 0 \quad (3) \\
p^\alpha \frac{\partial f}{\partial x^\alpha} + P^\alpha \frac{\partial f}{\partial p^\alpha} & = \mathcal{Q}(f, f) \quad (4)
\end{cases}
$$

where:

$R_{\alpha\beta}$ = Ricci tensor, contracted of the curvature tensor;

$R$ = Scalar curvature, contracted of the Ricci tensor;

$\Lambda$ = **Cosmological constant**

$F_{\alpha\beta}$ = electromagnetic field ( antisymetric closed 2-form )

$f$ = distribution scalar function of the charged particles;

$\mathcal{Q}$ = Nonlinear (binary and elastic) collisions operator.

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# RELATED WORKS (1)

➢**R. T. GLASSEY**, Lectures on Cauchy problems in transport theory. Department of mathematics, Indiana University, Bloomington, In 47405, May (1993).

➢**A. LICHNEROWICZ**, Théories relativistes de la gravitation et de l'électromagnétisme, Paris, Masson, (1955).

➢**D. BANCEL**, **Y. CHOQUET-BRUHAT**, Existence, uniqueness and local stability for Einstein-Maxwell-Boltzmann System, Commun Math Phys, 33, 83-96 (1993).

➢**R. T. GLASSEY**, The Cauchy problem in kinetic theory, SIAM, Philadelphia, (1996).

➢**S. W. HAWKING**, **G. F. R. ELLIS**, The large scale structure of space-time. Cambridge Monographs on Mathematical Physics, (1973).

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# RELATED WORKS (2)

- **T. CARLEMAN**, sur la théorie de l'équation intégrodifférentielle de Boltzmann, Acta Maths 60, 91-146, (1933).

- **R. T. GLASSEY**, **W. STRAUSS**, Asymptotic stability of the relativistic Maxwellian.

- **K. BICHTELER**, On Cauchy problem for the relativistic Boltzmann equation, Comm. Maths Phys. 4, 352-364, 1967.

- **P. HARTMAN**, Ordinary differential equations, Birkhauser, Boston, (1982).

- **P. B. MUCHA**, Global existence for the Einstein-Boltzmann equation in the flat Robertson-Walker space-time. Comm. Maths. Phys.203, 107-118. (1999).

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# RELATED WORKS (4)

- **N. NOUTCHEGUEME**, **E. TAKOU**, Global existence of solutions for the Einstein-Boltzmann system with cosmological constant in a Robertson-Walker space-time. Commun. Math.Sci. at press (pre print gr-qc / 0507042 ), (2005).

- **N. NOUTCHEGUEME**, **D. DONGO**, **E. TAKOU**, Global existence of solutions for the relativistic Boltzmann equation with arbitrarily large initial data on a Bianchi type I space-time. Gen.Rel Grav. at press (preprint gr-qc / 0503048 ), (2005).

- **P. NOUNDJEU**, Système de Yang-Mills-Vlasov pour des particules à densité propre sur un espace-temps courbe. Thèse de Doctorat 3e cycle, Université de Yaoundé I, 1999.

- **N. NOUTCHEGUEME**, **E. TETSADJIO**, (C Q G ) (2009).

- **J. EHLERS**, A Survey of General Relativity theory. Astrophysics and Cosmology. Ed. W. Israel 14-26.

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# THE Eléments (1)

The geometric back-ground is a **Bianchi type1 space-time**, which is an immediate generalisation of the **Friedmann-Lemaitre -Robertson-Walker** space-time considered to be the basic space-time in Cosmology.

The unknown of the system are: **the metric g** of Lorentzian signature with two unknown components $a > 0$ and $b > 0$, subject to the Einstein equations (1) , which are the **basic equations in General Relativity (GR)** and which tell about **gravitational effects;** physically, a and b stand for the
**gravitational potentials.**

g writes, in the canonical coordinates :

$$g = -\mathrm{d}t^2 + a^2(t)(\mathrm{d}x^1)^2 + b^2(t)((\mathrm{d}x^2)^2 + (\mathrm{d}x^3)^2) \qquad (5)$$

where $a > 0$ and $b > 0$ are unknown functions of the single time coordinate $t$.
(**We study an homogeneous phenomenon** ).

# THE Eléments (2)

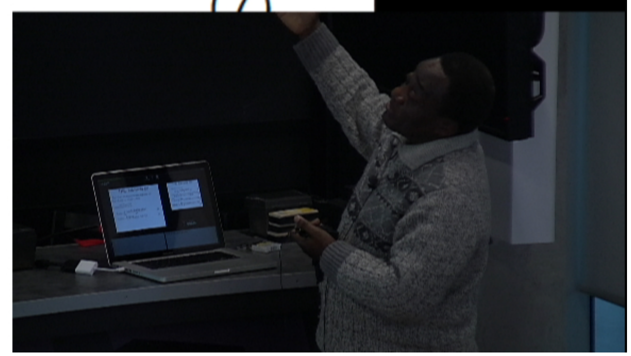
$T_{\alpha\beta} + \tau_{\alpha\beta} + \rho u_\alpha u_\beta$ is the **stress-matter-energy tensor**, with:

$T_{\alpha\beta} = T_{\alpha\beta}(f),\ \tau_{\alpha\beta} = \tau_{\alpha\beta}(F) \quad \rho u_\alpha u_\beta$

where $T_{\alpha\beta}$ and $\tau_{\alpha\beta}$
( Maxwell tensor) are given by :

$$T_{\alpha\beta}(t) = \int_{\mathbb{R}^3} \frac{p_\alpha p_\beta f(t,\overline{p}) ab^2 \mathrm{d}p^1 \mathrm{d}p^2 \mathrm{d}p^3}{p^0} \qquad (6)$$

$$\tau_{\alpha\beta}(t) = -\frac{g_{\alpha\beta}}{4} F^{\lambda\mu} F_{\lambda\mu}(t) + F_{\alpha\lambda} F_\beta{}^\lambda(t) \qquad (7)$$



# THE Eléments (3)

The Maxwell current in (2) is given by:

$$J^\beta(t) = \int_{\mathbb{R}^3} \frac{p^\beta f(t,\overline{p}) ab^2 \mathrm{d}p^1 \mathrm{d}p^2 \mathrm{d}p^3}{p_0} - eu^\beta. \quad (8)$$

The **binary** and **elastic** collision operator due to Lichnerowicz and Chernikov (1940) is:

$$\mathcal{Q}(f,g)(t,\overline{p}) = \mathcal{Q}^+(f,g)(t,\overline{p}) - \mathcal{Q}^-(f,g)(t,\overline{p}), \quad (9)$$

where:

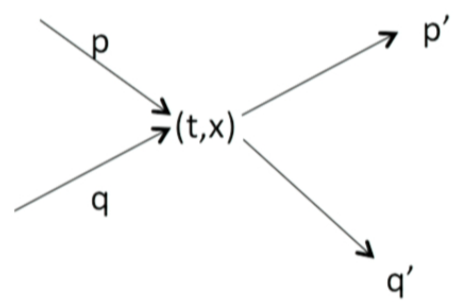
$$\mathcal{Q}^+(f,g)(t,\overline{p}) = \int_{\mathbb{R}^3} \frac{ab^2 \mathrm{d}\overline{q}}{q^0} \int_{S^2} f(t,\overline{p}')g(t,\overline{q}')\sigma(t,a,b,\overline{p},\overline{q},\overline{p}',\overline{q}')\mathrm{d}\omega \quad (10)$$

$$\mathcal{Q}^-(f,g)(t,\overline{p}) = \int_{\mathbb{R}^3} \frac{ab^2 \mathrm{d}\overline{q}}{q^0} \int_{S^2} f(t,\overline{p})g(t,\overline{q})\sigma(t,a,b,\overline{p},\overline{q},\overline{p}',\overline{q}')\mathrm{d}\omega, \quad (11)$$

# THE Eléments (4)

where p and q stand for the momenta of the particles **before** the choc, p' and q' stand for their momenta **after** the choc; the **binary** and **elastic** properties saying that: **only two particles collide at a given position and time, without destructing each other**, only the **sum** of their momenta **before** and **after** the choc being preserved following the scheme:

$$p + q = p' + q'. \qquad (12)$$




(12) Split into:

p0+q0 = p'0 + q'0 (conservation of the elementary energy) and

pi+qi = p'i + q'i which could be interpreted, following R.Glassey by writing:

# THE Eléments (5)

$$\begin{cases} \overline{p}' = \overline{p} + c(\overline{p}, \overline{q}, \omega)\omega \\ \overline{q}' = \overline{q} - c(\overline{p}, \overline{q}, \omega)\omega \end{cases}, \quad \omega \in S^2. \qquad (13)$$

Recall that, the fundamental relation g(p,p) = - 1 gives:

$$p^0 = \sqrt{1 + a^2(p^1)^2 + b^2\left((p^2)^2 + (p^3)^2\right)} \qquad (14)$$

In the intégrals (10) and (11), the scalar function $\sigma$ is a positive regular function of all its arguments, called the **kernel** or the **cross-section of the collisions.**

$u = (u^\alpha)$ is the material unit velocity ; In order to simplify , we consider the particles **spatially at rest (comoving coordinates),** which implies $u^0$=1, $u^i$=0.

e > 0 is an unknown scalar function ,the **charge density**,and $\rho(t)$ > 0 .the **matter density**.

In (13), one computes $c(\overline{p}, \overline{q}, \omega)$ and the jacobian of that transformation(13) , in function of $\overline{p}, \overline{q}, \omega$ so that, finally, (10) and (11) express as functions of only $\overline{p}$

# ThE Eléments (6)

The **trajectories** $s \longmapsto (x^\alpha(s), p^\alpha(s))$ of the charged particles are the solutions of the following differential system , which shows that p becomes **an auxilliary unknown**:

$$\begin{cases} \frac{\mathrm{d}x^\alpha}{\mathrm{d}s} = p^\alpha \\ \frac{\mathrm{d}p^\alpha}{\mathrm{d}s} = P^\alpha \end{cases} \qquad (15)$$

where

$$P^\alpha = -\Gamma^\alpha_{\lambda\mu} p^\lambda p^\mu + e p^\beta F^\alpha_\beta \qquad (16)$$

This enables to write the Boltzmann equation as a differential equation. (2) gives:

$$e(t) = \int_{\mathbb{R}^3} f(t, \overline{p}) d\overline{p} \qquad (17)$$

# THE Eléments (6)

Assuming the invariance of the initial datum $f^0 = f(0)$ of f with respect to a specific subgroup of the **special orthogonal group $O_3$**, yields $J^i = 0$, i= 1,2,3, from where we deduce the required conservation of the Maxwell current:

$$\nabla_\lambda J^\lambda = 0 \qquad (18)$$

The Maxwell equations (2), (3) , then solve at once to give:

$$F^{0i}(t) = \frac{a_0 b_0^2}{ab^2(t)} E^i, \quad F_{ij}(t) = \phi_{ij} = \text{constant} \qquad (19)$$

The **conservation laws or Euler equations write**:

$$\nabla_\alpha T^{\alpha\beta} + \nabla_\alpha \tau^{\alpha\beta} + \nabla_\alpha(\rho u^\alpha u^\beta) = 0 \qquad (20)$$

which gives, since the first term is zero (See Jurgen Ehlers) and using (2):

$$\rho(t) = \frac{\rho_0 a_0 b_0^2}{ab^2(t)} \qquad (21)$$

# fUnctionAl FRAWORK(1)

We are led to choose the function space:

$$\mathrm{L}^1_2(\mathbb{R}^3) := \left\{ f \in \mathrm{L}^1(\mathbb{R}^3) : \left\| \sqrt{1+|\overline{p}|^2} f \right\|_{\mathrm{L}^1(\mathbb{R}^3)} < +\infty \right\}, \quad \text{où,} \quad |\overline{p}|^2 = \|\overline{p}\|^2$$

$\mathrm{L}^1_2(\mathbb{R}^3)$ is endowed with the norm:

$$\|f\| = \left\| \sqrt{1+|\overline{p}|^2} f \right\|_{\mathrm{L}^1(\mathbb{R}^3)} \quad \text{avec } f \in \mathrm{L}^1_2(\mathbb{R}^3)$$

Set, for any real number r > 0 :

$$X_r = \left\{ f \in \mathrm{L}^1_2(\mathbb{R}^3) : \ f > 0 \, \mathrm{p.p}\, , \|f\| \leq r \right\}$$

# fUnctionAl FRAMEWORK(2)

Also set:

$$\mathcal{C}(I, \mathrm{L}^1_2(\mathbb{R}^3)) := \left\{ f : I \longrightarrow \mathrm{L}^1_2(\mathbb{R}^3) : \text{ f continue et bornée} \right\},$$

Endowed with the norm:

$$|||f||| = \sup_{t \in I} ||f(t)||$$

Finally set:

$$\mathcal{C}[I, X_r] = \left\{ f \in \mathcal{C}(I, \mathrm{L}^1_2(\mathbb{R}^3)) : f(t) \in X_r, \forall t \in I \right\}$$

We have the following properties:

**Lemme**

(1) $\mathrm{L}^1_2(\mathbb{R}^3)$ *est un espace de Banach.*

(2) $X_r$ *est un sous-espace métrique complet et connexe de* $\mathrm{L}^1_2(\mathbb{R}^3)$.

(3) $\left( \mathcal{C}(I, \mathrm{L}^1_2(\mathbb{R}^3)); ||| \, ||| \right)$ *est un espace de Banach.*

(4) $\mathcal{C}[I, X_r]$ *muni de la distance induite est un sous-espace métrique complet de l'espace de Banach* $\mathcal{C}(I, \mathrm{L}^1_2(\mathbb{R}^3))$.

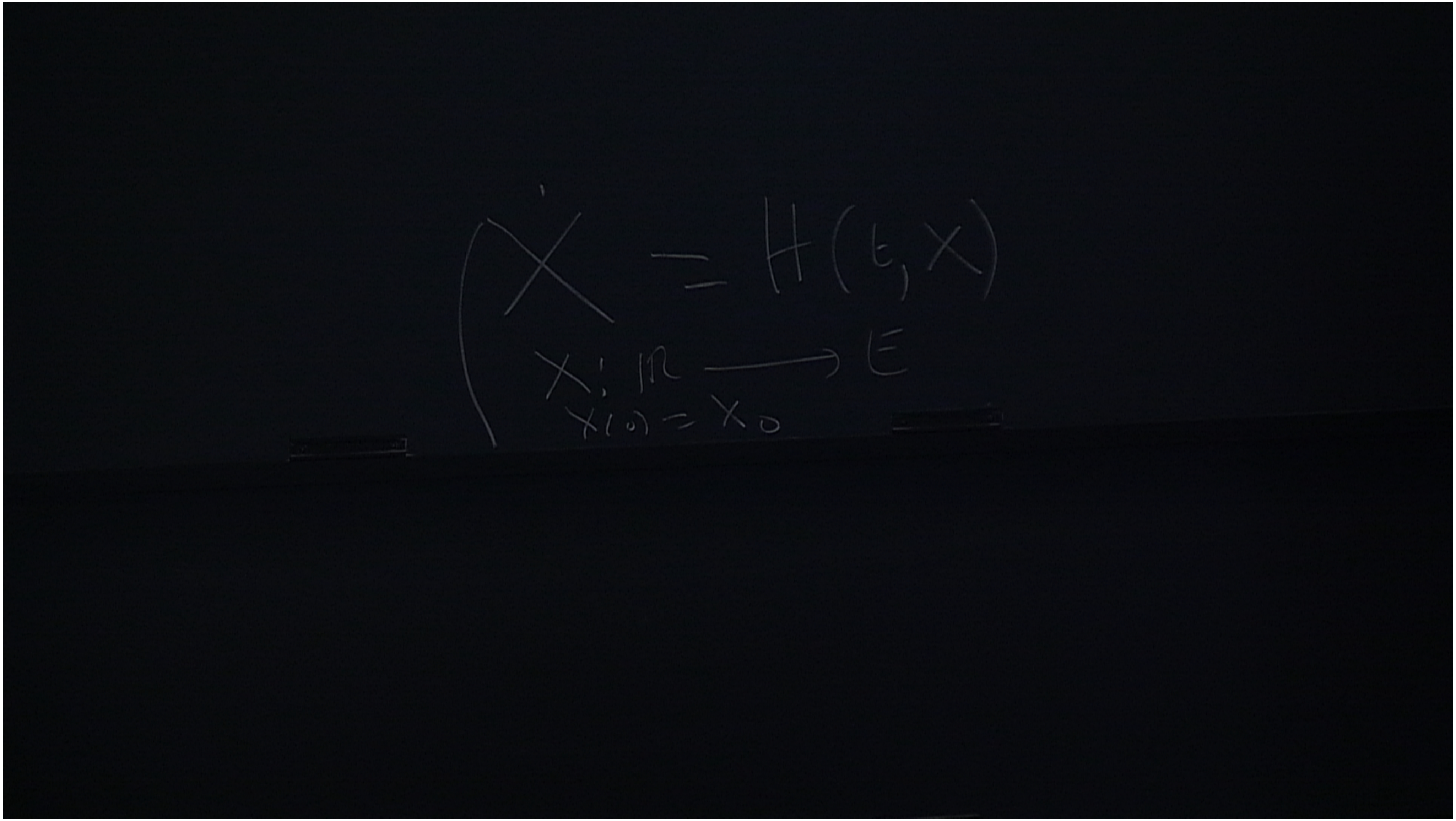
# STUDY OF THE boltzmann EQUATION (1)

**Proposition**

*Soient* $f \in \mathrm{L}^1_2(\mathbb{R}^3)$, *a et b deux fonctions strictement positives,* $\overline{p}_j = (p^i_j)$, $j = 1, 2$. *Alors pour tout entier* $k$ *variant de* 1 *à* 3, *on a:*

$$\left\{ \begin{array}{ll} \left| \dfrac{p^k_1}{p^0_1} - \dfrac{p^k_2}{p^0_2} \right| \leq c_1(a,b) \, \|\overline{p}_1 - \overline{p}_2\| & \text{(i)} \\ \left| \dfrac{1}{p^0_1} - \dfrac{1}{p^0_2} \right| \leq c_2(a,b) \dfrac{\|\overline{p}_1 - \overline{p}_2\|}{p^0_j} & \text{(ii)} \\ \left\| \dfrac{1}{p^0_j} \mathcal{Q}(f, f, \overline{p}_1) - \dfrac{1}{p^0_j} \mathcal{Q}(f, f, \overline{p}_2) \right\| \leq c_3(a,b) \, \|f\|^2 \, \|\overline{p}_1 - \overline{p}_2\| & \text{(iii)} \end{array} \right. \quad (29)$$

*où* $p^0_j$, $j = 1, 2$, *est défini par* (14) *et*
$c_1(a,b) = 5(1 + \frac{b}{a})$, $c_2(a,b) = 2(a+b)$ *et* $c_3(a,b) = \frac{2}{3}\pi a b^2$.


$\dot{x} = H(t, x)$
$x : \mathbb{R} \longrightarrow E$
$x(0) = x_0$

# STUDY OF THE boltzmann EQUATION (2)

We deduce from the above proposition ,using the well-known CAUCHY-LIPSCHITZ theorem :

### Théorème

*Soient a et b deux fonctions strictement positives de classe* $\mathcal{C}^1$ *sur* $\mathbb{R}$*, coefficients de la métrique g. On suppose que le noyau de collisions* $\sigma$ *vérifie les hypothèses* (27)*. Soient* $t_0 \geq 0$ *un réel et* $(\overline{p}_{t_0}, f_{t_0})$ *dans* $\mathbb{R}^3 \times \mathrm{L}^1_2(\mathbb{R}^3)$*. Alors il existe un réel* $\delta > 0$ *tel que le système* (28) *en* $(\overline{p}, f)$ *admette une unique solution locale*
$(\overline{p}, f) \in \mathcal{C}\Big([t_0,\ t_0+\delta]\,,\ \mathbb{R}^3\Big) \times \mathcal{C}\Big([t_0,\ t_0+\delta]\,,\ \mathrm{L}^1_2(\mathbb{R}^3)\Big)$ *avec*
$(\overline{p}, f)(t_0) = (\overline{p}_{t_0}, f_{t_0})$.
*La fonction f satisfait en plus l'inégalité suivante:*

$$|||f||| = \sup_{t \in [t_0, t_0+\delta]} \|f(t)\| \leq \|f_{t_0}\| \,. \qquad (30)$$

# STUDY OF THE einstein EQUATIONS (1)

**Initial data and constraints**

The compatibility of the Einstein Equations (1) requires the constraints:

$$\tau_{22} = \tau_{33}; \tau_{\alpha\beta} = 0, \quad \alpha \neq \beta \tag{31}$$

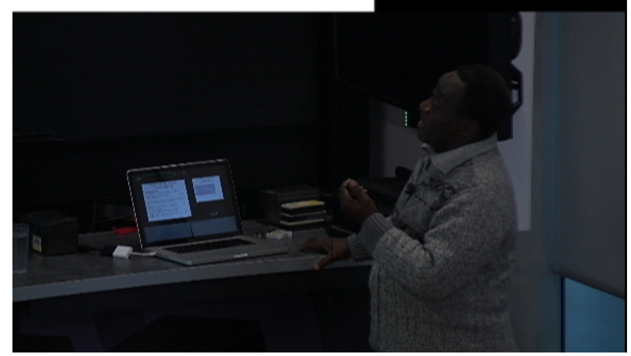
We study the initial values problem for the nonlinear second order differentiel system in a and b , with the initial data : $a_0$; $b_0$ ; $E^i$; i ; j =1; 2 $\dot{a}_0$ $\dot{b}_0$ en ; $\phi_{ij}$

$$\begin{cases} a(0) = a_0, \; b(0) = b_0, \; \dot{a}(0) = \dot{a}_0, \; \dot{b}(0) = \dot{b}_0, \\ F^{0i}(0) = E^i, \; F_{ij}(0) = \phi_{ij}. \end{cases} \tag{32}$$

Which , given the **Hamiltonian Constraint** (**HC**) are submited to :

c

$$2\frac{\dot{a}_0 \dot{b}_0}{a_0 b_0} + \left(\frac{\dot{b}_0}{b}\right)^2 - \Lambda = 8\pi \left( T_{00}(0) + \tau_{00}(0) + \rho_0 \right)$$



# STUDY OF THE EINSTEIN equations (2)

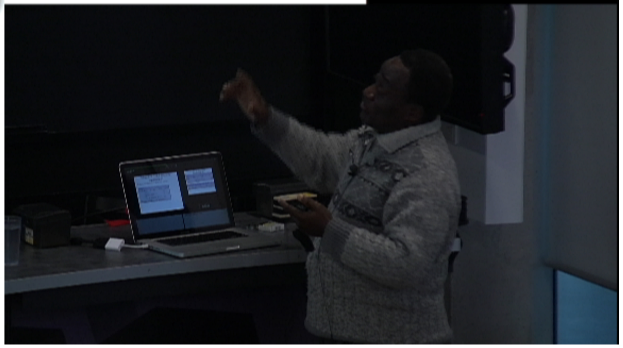
We solve the constraints (31) on the Maxwell tensor as follows::

**Proposition**

*Si les données initiales* $a_0$, $b_0$, $E^i$ *et* $\phi_{ij}$, $i, j = 1, 2, 3$, *en* $t = 0$, *vérifient:*

$$\begin{cases} E^i \phi_{ij} = 0, & (a) \\ \sum_k \phi_{ik}\phi_{jk} - a_0^2 b_0^4 E^i E^j = 0, & (b) \\ \phi_{12}^2 - \phi_{13}^2 - a_0^2 b_0^4 \left( (E^2)^2 - (E^3)^2 \right) = 0. & (c) \end{cases} \quad (34)$$

*Alors le tenseur de Maxwell* $\tau_{\alpha\beta}$ *satisfait les conditions* (31).



# STUDY OF THE equations EQUATIONS (4)

To summarize, the initial data are subject to the following constraints:

$$
\left\{
\begin{array}{ll}
2\dfrac{\dot{a}_0 \dot{b}_0}{a_0 b_0} + \left(\dfrac{\dot{b}_0}{b_0}\right)^2 - \Lambda = 8\pi \displaystyle\int_{\mathbb{R}^3} p^0(0,\bar{p}) f_0(\bar{p}) a_0 b_0^2 \mathrm{d}\bar{p} + & \\
4\pi\Big( (a_0 E^1)^2 + (b_0 E^2)^2 + (b_0 E^3)^2 + & \\
\left(\dfrac{\phi_{12}}{a_0 b_0}\right)^2 + \left(\dfrac{\phi_{13}}{a_0 b_0}\right)^2 + \left(\dfrac{\phi_{23}}{b_0^2}\right)^2 + 2\rho_0 \Big) & (37a) \\
E^j \phi_{ij} = 0 & (37b) \\
\displaystyle\sum_k \phi_{ik}\phi_{jk} - a_0^2 b_0^4 E^i E^j = 0 & (37c) \\
\phi_{12}^2 - \phi_{13}^2 - a_0^2 b_0^4 \Big( (E^2)^2 - (E^3)^2 \Big) = 0, & (37d)
\end{array}
\right. \quad (37)
$$

# STUDY OF THE d'einstein EQUATIONS(5)

where:

$$p^0(0,\overline{p}) = \sqrt{1 + a_0^2(p^1)^2 + b_0^2\Big((p^2)^2 + (p^3)^2\Big)}$$

Following the (3 + 1) formulation of the Einstein equations , interpreted as the time history of the **first and second fundamental forms** of the slices of constant times, the **(HC)** writes:

$$(\mathrm{TrK})^2 - K_{ij}K^{ij} = 2\Lambda + 16\pi(T_{00} + \tau_{00} + \rho), \qquad (41)$$

where ($K_{ij}$ ) is the **second fundamental form** induced by the metric g on the imbedded slices of constant time $S_t$ ={ t } X $R^3$. In the considered case, $K_{ij}$ is defined by:

$$K_{ij} = -\frac{1}{2}\partial_t g_{ij}. \qquad (42)$$

In (41), TrK = $g^{ij} K_{ij}$ , is the **trace** of K and stands for the **mean curvature** of the space-tme (R4; g). Using (5) and (42) yields:

$$\mathrm{TrK} = -\Big(\frac{\dot{a}}{a} + 2\frac{\dot{b}}{b}\Big). \qquad (43)$$

# STUDY OF THE EINSTEIN EQUATIONS (6)

**<u>The evolution system:</u> Compactification of the variables**

We set for the principal unknown a and b , following Rendall et Uggla in [15]:

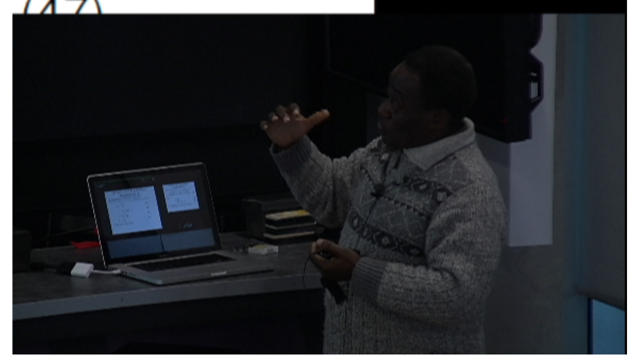
$$H = -\frac{\mathrm{TrK}}{3} \tag{44}$$

$$z = \frac{1}{a^{-2} + 2b^{-2} + 1} \tag{45}$$

$$s = \frac{b^2}{b^2 + 2a^2} \tag{46}$$

$$\Sigma_+ = \frac{1}{H}\frac{\dot{b}}{b} - 1. \tag{47}$$

H is often called the **Hubble variable**.



# STUDY OF THE einstein EQUATIONS (7)

Now set , for the sources terms:

$$\tilde{\rho} = 8\pi(T_{00} + \tau_{00} + \rho), \quad P_1 = \frac{8\pi(T_{11} + \tau_{11})}{a^2},$$

$$P_2 = \frac{8\pi(T_{22} + \tau_{22})}{b^2}, \quad R = \frac{P_1 + 2P_2}{\tilde{\rho}}, \quad R_+ = \frac{P_2 - P_1}{\tilde{\rho}} \tag{48}$$

Finally set:

$$\Omega = \frac{\tilde{\rho}}{3H^2} \tag{49}$$

$$\tilde{q} = 2\Sigma_+^2 + \frac{\Omega}{2}(1 + R). \tag{50}$$

$\Omega$ is the **normalized energy density** and $\tilde{q}$ The **deceleration parameter.**

# STUDY OF THE EINSTEIN EQUATIONS (8)

We have the following properties:

## Lemme

*On a:*

$$0 < z < 1;\ 0 < s < 1;\ a^2 = \frac{z}{s(1-z)};\ b^2 = \frac{2z}{(1-s)(1-z)};$$

$$\Omega = 1 - \Sigma_+^2 - \frac{\Lambda}{3H^2} \qquad (51)$$

$$0 \leq P_1 + 2P_2 \leq \tilde{\rho} - 8\pi\rho;\ 0 \leq R \leq 1;\ \Omega \geq 0 \qquad (52)$$

# STUDY OF THE EINSTEIN equations (9)

We obtain an equivalent first order system in $H,\ s,\ z,\ \Sigma_+$

## Proposition

*Le système des équations d'Einstein* (56)-(57) *en a et b est équivalent au système d'équations différentielles ordinaires non linéaires du premier ordre en* $H,\ s,\ z,\ \Sigma_+$ *suivant:*

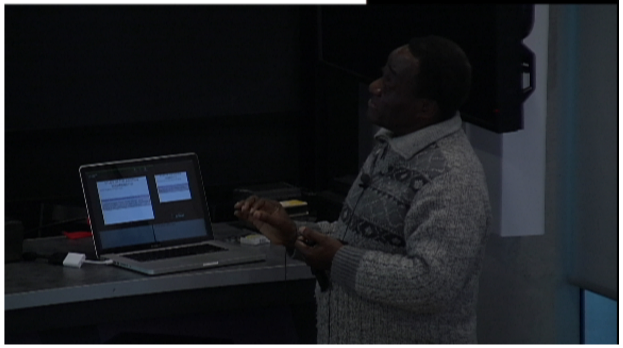
$$
(\text{IV}.2)\left\{
\begin{aligned}
\frac{\mathrm{d}H}{\mathrm{d}t} &= -H^2(1+\tilde{q}) + \frac{\Lambda}{3}, && (58)\\
\frac{\mathrm{d}s}{\mathrm{d}t} &= 6s(1-s)\Sigma_+ H, && (59)\\
\frac{\mathrm{d}z}{\mathrm{d}t} &= 2z(1-z)(1+\Sigma_+ - 3s\Sigma_+)H, && (60)\\
\frac{\mathrm{d}\Sigma_+}{\mathrm{d}t} &= -(2-\tilde{q})\Sigma_+ H + \Omega R_+ H - \frac{\Lambda\Sigma_+}{3H}. && (61)
\end{aligned}
\right.
$$

# STUDY OF THE EINSTEIN EQUATIONS (10)

We first use equation (58) in H to prove:

**Théorème (Non existence globale dans le cas $\Lambda < 0$)**

*Le système homogène d'Einstein-Maxwell-Boltzmann avec la constante cosmologique $\Lambda$ strictement négative, dans l'espace-temps de Bianchi de type* $\mathrm{I}$, *n'admet pas de solution globale sur* $[0\ ;\ +\infty[$.

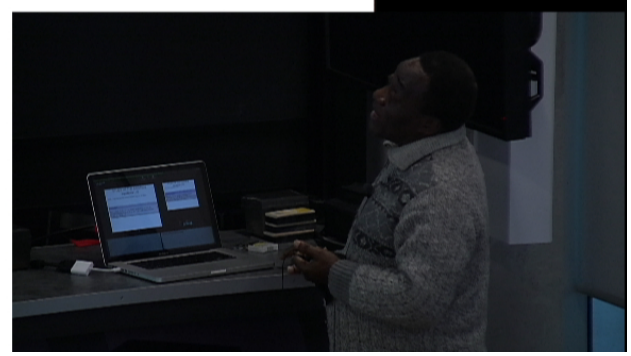


# STUDY OF THE EINSTEIN equations (11)

Finally , using once more the CAUCHY-LIPSCHITZ theorem we obtain:

**Proposition**

*Soit* $f \in \mathcal{C}([0\,,\,T]\,;X_r)$, $T > 0$. *Alors le système* $(\mathrm{IV}.2)$ *avec les données initiales* $(H_0, s_0, z_0, \Sigma_{+0})$ *en* $t = 0$, *a une unique solution* $(H, s, z, \Sigma_+)$ *sur* $[0\,,\,T]$.

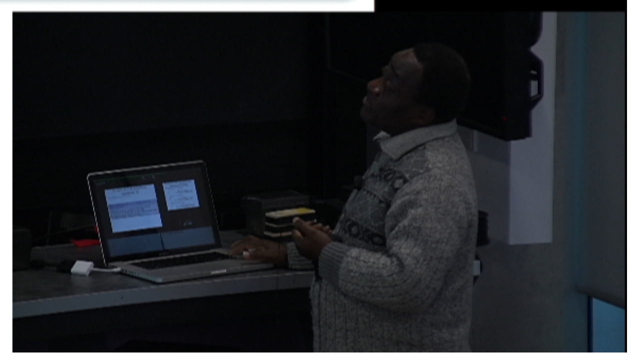


# STUDY OF THE EINSTEIN equations (12)

And, as a direct consequence, we have:

**Théorème (existence sur $[0\,,\ T]$, $T > 0$ de la solution des équations d'Einstein)**

*Soit* $f \in \mathcal{C}([0\,,\ T]\,; X_r)$, $T > 0$. *Alors le système des équations d'Einstein* (22)-(23)-(24), *avec les données initiales* $a_0,\ b_0,\ \dot{a}_0,\ \dot{b}_0,\ f_0,\ \rho_0$, *vérifiant* (38), (39), (40) *et les contraintes initiales* (37), *admet une unique solution* $(a, b)$ *sur* $[0\,,\ T]$.



# STUDY OF THE couplED Einstein-Boltzmann SYSTEM(1)

**Local Existence:** the coupled Einstein-Boltzmann system writes:

$$
\text{(V.1)}\quad
\begin{cases}
\dfrac{\mathrm{d}p^1}{\mathrm{d}t} = \Bigg( -2H(1-2\Sigma_+)p^1 - a_0 b_0^2 E^1 \displaystyle\int_{\mathbb{R}^3} f \mathrm{d}\overline{q} \\
\qquad - \dfrac{2 s^{\frac{1}{2}} z^{\frac{1}{2}} (p^2 \phi_{12} + p^3 \phi_{13}) \int_{\mathbb{R}^3} f \mathrm{d}\overline{q}}{(1-s)(1-z)^{\frac{1}{2}} \sqrt{1 + \frac{z}{s(1-z)}(p^1)^2 + \frac{2z}{(1-s)(1-z)}\left[(p^2)^2 + (p^3)^2\right]}} \Bigg) \\
\dfrac{\mathrm{d}p^2}{\mathrm{d}t} = \Bigg( -2H(1+\Sigma_+)p^2 - a_0 b_0^2 E^2 \displaystyle\int_{\mathbb{R}^3} f \mathrm{d}\overline{q} \\
\qquad - \dfrac{z^{\frac{1}{2}} (p^1 \phi_{21} + p^3 \phi_{23}) \int_{\mathbb{R}^3} f \mathrm{d}\overline{q}}{s^{\frac{1}{2}} (1-z)^{\frac{1}{2}} \sqrt{1 + \frac{z}{s(1-z)}(p^1)^2 + \frac{2z}{(1-s)(1-z)}\left[(p^2)^2 + (p^3)^2\right]}} \Bigg) \\
\dfrac{\mathrm{d}p^3}{\mathrm{d}t} = \Bigg( -2H(1+\Sigma_+)p^3 - a_0 b_0^2 E^3 \displaystyle\int_{\mathbb{R}^3} f \mathrm{d}\overline{q} \\
\qquad - \dfrac{z^{\frac{1}{2}} (p^1 \phi_{31} + p^2 \phi_{32}) \int_{\mathbb{R}^3} f \mathrm{d}\overline{q}}{s^{\frac{1}{2}} (1-z)^{\frac{1}{2}} \sqrt{1 + \frac{z}{s(1-z)}(p^1)^2 + \frac{2z}{(1-s)(1-z)}\left[(p^2)^2 + (p^3)^2\right]}} \Bigg)
\end{cases}
$$

# STUDY OF THE d'Einstein-Boltzmann SYSTEM(2)

$$
\text{(V.2)}\quad
\begin{cases}
\dfrac{\mathrm{d}f}{\mathrm{d}t} = \dfrac{1}{p^0}\mathcal{Q}(f,f) \\[2mm]
\dfrac{\mathrm{d}H}{\mathrm{d}t} = -\dfrac{3}{2}(1+\Sigma_+^2)H^2 - \dfrac{P_1 + 2P_2}{6} + \dfrac{\Lambda}{2} \\[2mm]
\dfrac{\mathrm{d}s}{\mathrm{d}t} = 6s(1-s)\Sigma_+ H \\[2mm]
\dfrac{\mathrm{d}z}{\mathrm{d}t} = 2z(1-z)(1+\Sigma_+ - 3s\Sigma_+)H \\[2mm]
\dfrac{\mathrm{d}\Sigma_+}{\mathrm{d}t} = -\dfrac{3}{2}(1-\Sigma_+^2)H\Sigma_+ + \dfrac{P_1}{6H}(\Sigma_+ - 2) + \dfrac{P_2}{3H}(\Sigma_+ + 1) - \dfrac{\Lambda\Sigma_+}{2H}
\end{cases}
$$

# STUDY OF THE COUPLED EINSTEIN-BOLTZMANN SYSTEM (3)

We establish, by direct calculation, for the differences in the r.h.s:

**Lemme**

*Si* $f_1, f_2 \in \mathrm{L}^1_2(\mathbb{R}^3)$, $s_1, s_2, z_1, z_2 \in \,]0\,,\,1[$ *et* $H_1, H_2 \in \,]0\,,\,H_0]$, *alors pour* $i = 1, 2$ *on a:*

$$|P_i(s_1, z_1, f_1) - P_i(s_2, z_2, f_2)| \leq \frac{C(1 + \|f_1\|)(|s_1 - s_2| + |z_1 - z_2| + \|f_1 - f_2\|)}{\alpha^4(s_1)\alpha^4(s_2)\alpha^6(z_1)\alpha^4(z_2)} \tag{78}$$

$$\left|\frac{P_i(s_1, z_1, f_1)}{H_1} - \frac{P_i(s_2, z_2, f_2)}{H_2}\right| \leq \frac{C(1 + \|f_1\| + \|f_2\|)(|H_1 - H_2| + |s_1 - s_2| + |z_1 - z_2| + \|f_1 - f_2\|)}{\alpha^4(s_1)\alpha^4(s_2)\alpha^6(z_1)\alpha^4(z_2)H_1H_2} \tag{79}$$

STUDY OF THE COUPLED EINSTEIN-BOLTZMANN SYSTEM (4)

And also:

**Lemme**

*Soit* $\overline{p}_1 = (p_1^i)$, $\overline{p}_2 = (p_2^i) \in \mathbb{R}^3$, $f, f_1, f_2 \in \mathrm{L}_2^1(\mathbb{R}^3)$, $s_1, s_2, z_1, z_2 \in ]0\,,\,1[$.
*Alors:*

$$\left\| \left( \frac{1}{p^0(s_1, z_1)} - \frac{1}{p^0(s_2, z_2)} \right) \mathcal{Q}(f_1, f_1)(s_2, z_2) \right\| \leq \frac{C\,\|f_1\|^2 \left( |s_1 - s_2| + |z_1 - z_2| \right)}{\alpha^{\frac{7}{2}}(s_2)\alpha(z_1)\alpha^{\frac{5}{2}}(z_2)} \tag{73}$$

$$\left\| \frac{1}{p^0(s_2, z_2)} (\mathcal{Q}(f_1, f_1)(s_2, z_2) - \mathcal{Q}(f_2, f_2)(s_2, z_2)) \right\| \leq \frac{C \left( \|f_1\| + \|f_2\| \right) \|f_1 - f_2\|}{\alpha^{\frac{3}{2}}(s_2)\alpha^{\frac{3}{2}}(z_2)} \tag{74}$$

$$\left\| \frac{1}{p^0(s_1, z_1)} (\mathcal{Q}(f, f)(s_1, z_1) - \mathcal{Q}(f, f)(s_2, z_2)) \right\| \leq \frac{C\,\|f\|^2 \left( |s_1 - s_2| + |z_1 - z_2| \right)}{\alpha^2(s_1)\alpha^{\frac{7}{2}}(s_2)\alpha^3(z_1)\alpha^{\frac{5}{2}}(z_2)} \tag{75}$$

$$\left| \frac{ab^2 g^{ii}(s_1, z_1)\phi_{ik} p_1^k}{p^0(s_1, z_1, \overline{p}_1)} \int_{\mathbb{R}^3_{\overline{q}}} f_1 \mathrm{d}\overline{q} - \frac{ab^2 g^{ii}(s_2, z_2)\phi_{ik} p_2^k}{p^0(s_2, z_2, \overline{p}_2)} \int_{\mathbb{R}^3_{\overline{q}}} f_2 \mathrm{d}\overline{q} \right| \leq$$

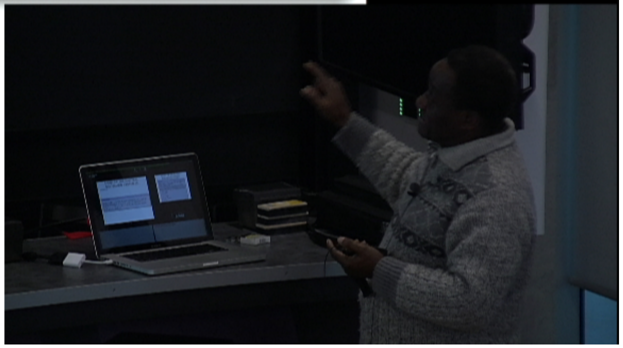
$$\frac{C(1 + \|f_1\|)(\left\| \overline{p}_1 - \overline{p}_2 \right\| + \|f_1 - f_2\| + |s_1 - s_2| + |z_1 - z_2|)}{\alpha^2(s_1)\alpha^{\frac{3}{2}}(s_2)\alpha^4(z_1)\alpha^{\frac{3}{2}}(z_2)} \tag{76}$$

# STUDY OF THE EINSTEIN-BOLTZMANN SYSTEM (5)

And we prove:

**Proposition**

*Il existe un réel* $l > 0$ *tel que le système différentiel* (III.1), *avec les données initiales* $\overline{p}_0, f_0, H_0, s_0, z_0, \Sigma_{+0}$ *en* $t = 0$ *satisfaisant* $H_0 > 0$, $0 < z_0 < 1$, $-1 < \Sigma_{+0} \leq \frac{1}{2}$ *et* $f_0 \in L^1_2(\mathbb{R}^3)$ *ait une unique solution* $(\overline{p}, f, H, s, z, \Sigma_+)$ *sur* $[0\ ,\ l]$.



# STUDY OF THE Einstein-Boltzmann SYSTEM(6)

As a direct consequence we have: ( Local existence theorem)

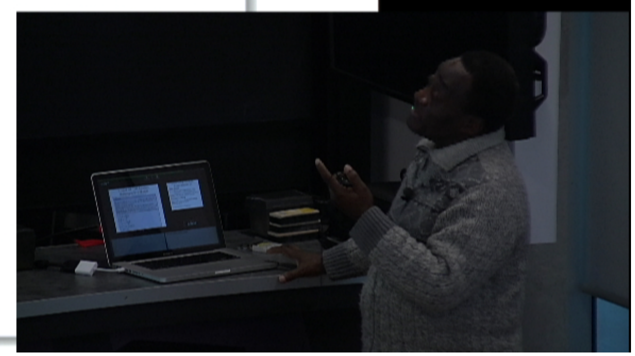
**Théorème**

*Soit* $a_0, b_0, \dot{a}_0$ *et* $\dot{b}_0$, $E^i, \phi_{ij}$, $i, j = 1, 2, 3$, *vérifiant* (37),(38),(39)*et* (40). *Soient* $f_0 \in \mathrm{L}^1_2(\mathbb{R}^3)$ *et* $r > \|f_0\|$ *donnés*, $\overline{p}_0 \in \mathbb{R}^3$. *Alors, il existe un nombre* $l > 0$ *tel que le système d'Einstein-Boltzmann* (22)-(23)- (24)-(28) *ait une unique solution* $(a, b, f)$ *sur* $[0\,,\, l]$. *Cette solution fournit la solution* $(a, b, F^{0i}, F_{ij}, f, \rho)$ *du* ***système complet*** *d'Einstein-Maxwell-Boltzmann* (I.1) *qui prend en* $t = 0$ *la donnée* $(a_0, b_0, E^i, \phi_{ij}, f_0, \rho_0)$ *et vérifie les propriétés suivantes:*

(i) $$\begin{cases} F^{0i} = \dfrac{a_0 b_0^2}{ab^2} E^i \\ F_{ij} = \phi_{ij} \\ \rho = \dfrac{\rho_0 a_0 b_0^2}{ab^2}; \end{cases}$$

(ii) $f \in \mathcal{C}\left([0\,,\, l]\,; \mathrm{L}^1_2(\mathbb{R}^3)\right)$;

(iii) $|||f||| \leq \|f_0\|$.



# STUDY OF THE COUPLED Einstein-Boltzmann (7)

## Global Existence:

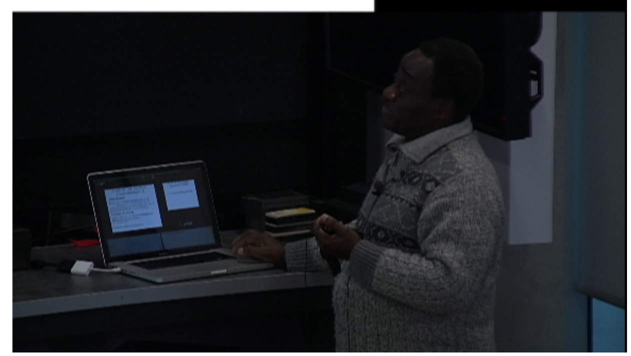
We now study the **global in time existence** i.e. the existence on $[0, +\infty[$ of the solutions of the coupled Einstein-Boltzmann system which, as we saw , is equivalent to the first order system in $s, z, H, \Sigma_+, p^i, f$.

**The strategy is the following:**

Let $[0; T[$, with $0 < T < +\infty$, be the **maximal existence domain** of the solution we denote :

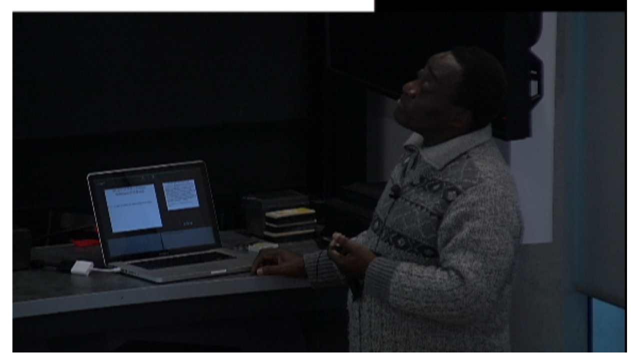
$U = (s, z, H, \Sigma_+, p^i, f)$

and whose existence is proved above.



# STUDY OF THE d'Einstein-Boltzmann SYSTEM(8)

Si $T = +\infty$, alors le problème de l'existence globale est résolu



# STUDY OF THE COUPLED EINSTEIN- Boltzmann SYSTEM (9)

Maintenant si on suppose que $T < +\infty$, on prend $t_0 \in [0, T]$. On construit un cadre fonctionnel approprié dans lequel on prouve l'existence d'une solution locale $u = (s, z, H, \Sigma_+, p^i, f)$ du problème de Cauchy pour le système différentiel du premier ordre, avec la donnée initiale $U(t_0)$ en $t = t_0$, sur un intervalle $[t_0, t_0 + \delta]$, où ce qui est fondamental, $\delta > 0$ **ne dépend pas** de $t_0$.
En prenant alors suffisamment $t_0$ proche de $T$ pour que l'on ait $t_0 + \delta > T$, on en déduit l'existence d'une solution du système sur l'intervalle $[0, t_0 + \delta]$ qui contient strictement l'intervalle $[0, \ T]$ , ce qui contredit la maximalité de $T$. Conclusion $T = +\infty$.

Nous pouvons alors énoncer le principal résultat de notre travail:

# STUDY OF THE d'Einstein-Boltzmann SYSTEM(10)

## Théorème

*Le système homogène d'Einstein-Maxwell-Boltzmann*

$$
\text{(VI.1)}\begin{cases}
R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R + \Lambda g_{\alpha\beta} = 8\pi(T_{\alpha\beta} + \tau_{\alpha\beta} + \rho u_\alpha u_\beta) & (82)\\
\nabla_\alpha F^{\alpha\beta} = J^\beta & (83)\\
\nabla_\alpha F_{\beta\delta} + \nabla_\beta F_{\delta\alpha} + \nabla_\delta F_{\alpha\beta} = 0 & (84)\\
p^\alpha \frac{\partial f}{\partial x^\alpha} + P^\alpha \frac{\partial f}{\partial p^\alpha} = Q(f, f) & (85)
\end{cases}
$$

*avec la constante cosmologique positive ou nulle, dans l'espace-temps de Bianchi type* I, *admet une solution globale* $(a, b, F, \rho)$ *sur* $[0\,,\,+\infty[$ *pour des données initiales* $a_0, b_0, \dot{a}_0, \dot{b}_0, E^i, \phi_{ij}$ *et* $f_0$ *vérifiant* (37), (38), (39) *et* (40).

# Conclusion et perspectives

The global existence of solutions to the coupled **Einstein-Maxwell-Boltzmann with a positive cosmological constant** on a Bianchi type I space-time is interesting since it should help for a better comprehension of some phenomena in the universe . For instance, setting **a=b,** gives, as we pointed out, the **Friedman-Lemaitre-Robertson-Walker** spacetime, which is the basic space-time in Cosmology, in which **a** stands for the **cosmological expansion factor.** We show that : **a(t)>Cexp(Ct), C>0 ,** which proves an **exponential growth of a**. This confirms mathematically, the accelerated expansion of the universe, as revealed by astrophysical observations , based on the study of some far away objects called **Supernovae .**

Recall that the **Nobel Price of Physics 2011** was awarded to **three Astrophysicists** for their **advanced work** on the accelerated expansion of the universe. They are **Saul PERLMUTTER (American),** who received the half part of the Price , **Brian P. SCHMIDT (American) ,**

and **Adam G. REISS ( Australo-American),** who jointly received, the other part of the price.
Also for a better understanding of this phenomenon of accelerated expansion of the universe, the notion ***Dark Energy* ,** was introduced by Cosmologists. That is a kind of energy which is unknown in the laboratories , and whose hypothetical and theoretical **structure** remains an **open question in modern Cosmology ;** so is the **notion of *Dark Ma*tter**. ( Remember Lavoisier)

**The *massive scalar fields*** we introduce in work in progress, is also a huge to understand and to explain accelerated models, **not only** in **inflation,** which is a variant of the **Big-Bang theory,** including a **very short period** of **very high accelerated expansion** during the first moments of the cosmic evolution , **but also** for a better comprehension of the structure the **primordial universe**.
(***See the works of Neil Turok on the question***).

Finally, to complete the study we presented above on the **Einstein-Maxwell-Boltzmann** system, we are going to check the ***asymptotic behaviour,*** the ***geodesic completeness ,*** the ***Energy conditions* : (*weak, strong, dominant*)** dues to **Hawking,** and also the ***well-posedness or continuous dependence of the initial data,*** to master the question of ***stability.***