Title: SUSY Gauge Theories on Squashed 3-spheres

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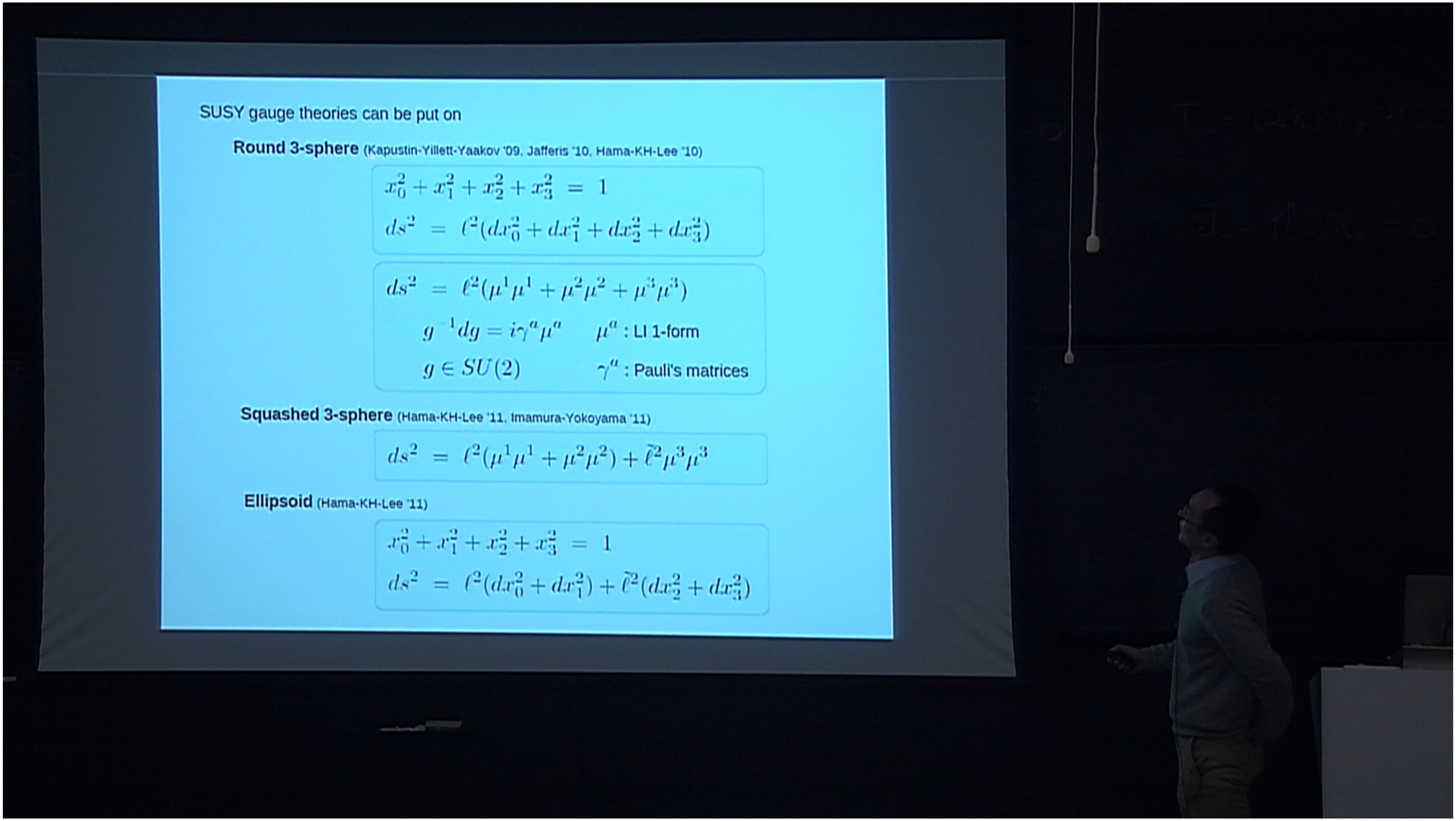
Abstract: I will explain some recent exact developments in N=2 SUSY gauge theories on 3-sphere and its deformations. I will begin by the analysis of Killing spinors and their generalization which are necessary to construct SUSY theories on curved spaces. Then I will sketch the exact computation of partition function using SUSY localization and present a general formula. Some applications to the physics of M5-branes will also be discussed.


SUSY gauge theories can be put on
Round 3-sphere (Kapustin-Yillett-Yaakov '09, Jafferis '10, Hama-KH-Lee '10)
$x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1$
$ds^2 = \ell^2(dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2)$
$ds^2 = \ell^2(\mu^1\mu^1 + \mu^2\mu^2 + \mu^3\mu^3)$
$g^{-1}dg = i\gamma^a\mu^a$
μ^a : LI 1-form
$g \in SU(2)$
γ^a : Pauli's matrices
Squashed 3-sphere (Hama-KH-Lee '11, Imamura-Yokoyama '11)
$ds^2 = \ell^2(\mu^1\mu^1 + \mu^2\mu^2) + \tilde{\ell}^2\mu^3\mu^3$
Ellipsoid (Hama-KH-Lee '11)
$x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1$
$ds^2 = \ell^2(dx_0^2 + dx_1^2) + \tilde{\ell}^2(dx_2^2 + dx_3^2)$

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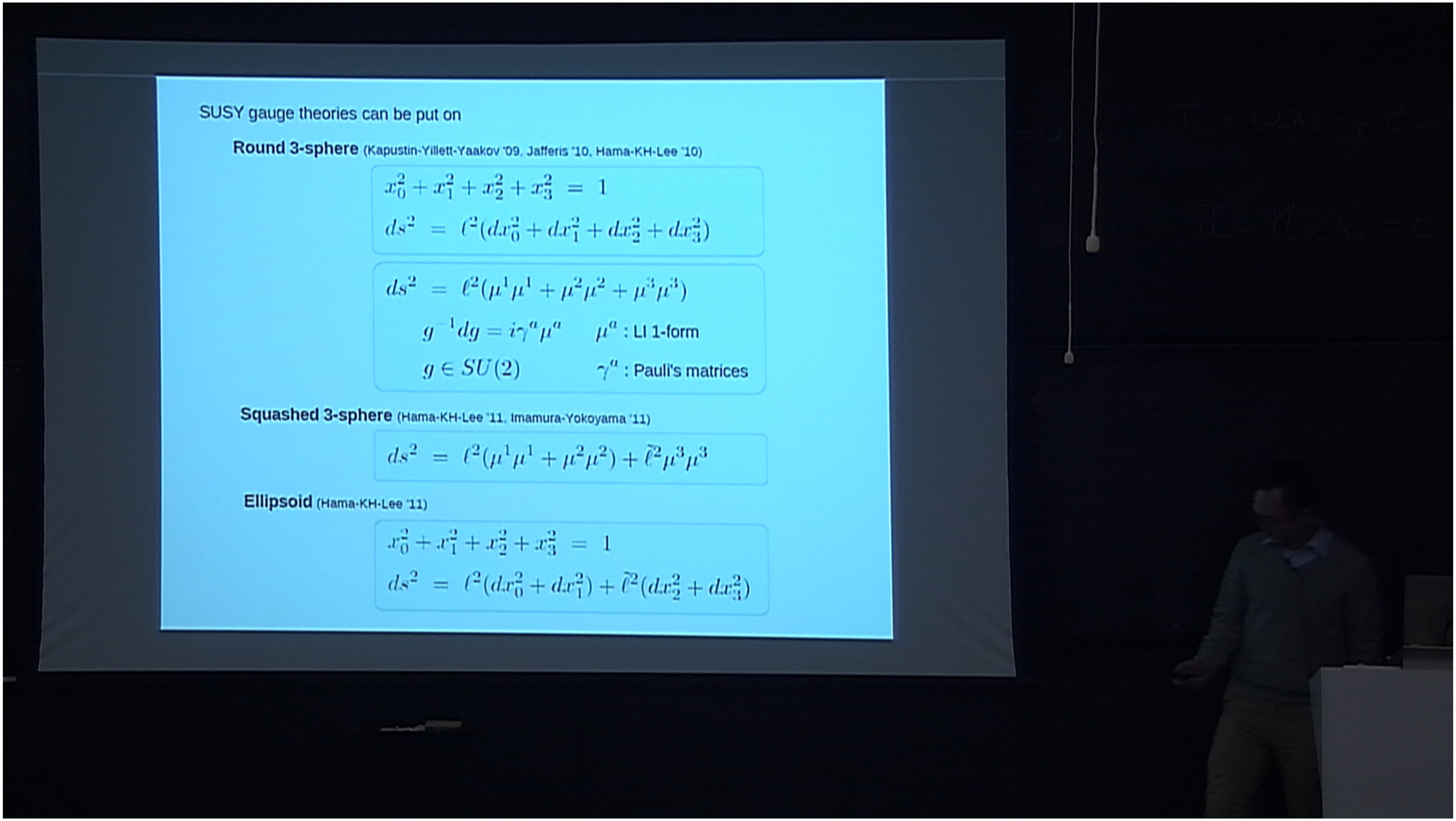
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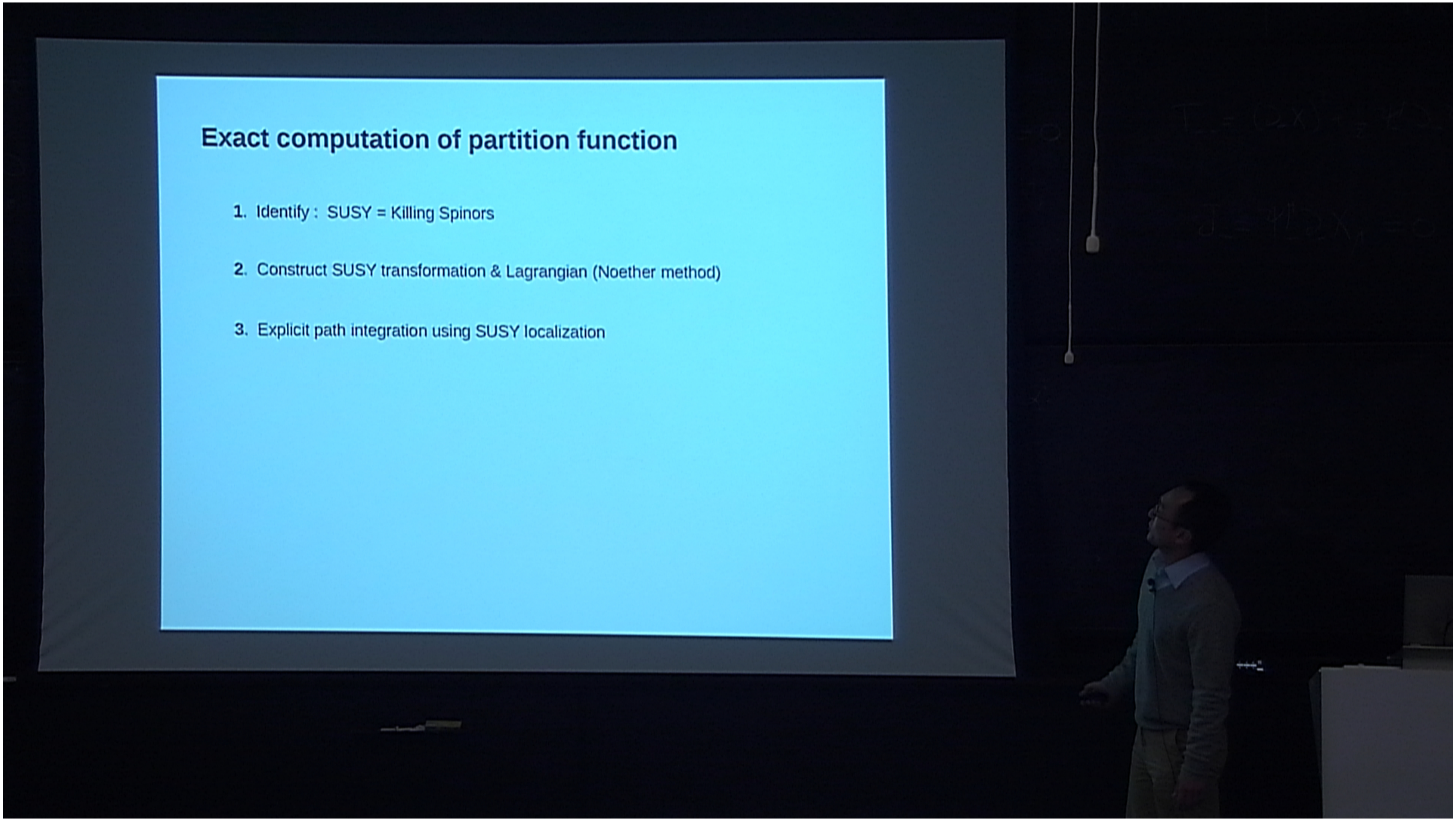

Exact computation of partition function
1. Identify : SUSY = Killing Spinors
2. Construct SUSY transformation & Lagrangian (Noether method)
3. Explicit path integration using SUSY localization

Rigid SUSY on 3-Sphere

is generated by "**conformal Killing spinors**" (KS) ε :

KS equation

$$D_m\varepsilon \;\equiv\; \left(\partial_m + \tfrac{1}{4}\omega_m^{ab}\gamma^{ab}\right)\varepsilon \;=\; \gamma_m\tilde{\varepsilon} \quad \text{for some } \tilde{\varepsilon}$$

On round 3-sphere of radius ℓ there are 4 **KS**s.

2 **KS**s satisfying $D_m\varepsilon = +\frac{i}{2\ell}\gamma_m\varepsilon$

2 **KS**s satisfying $D_m\varepsilon = -\frac{i}{2\ell}\gamma_m\varepsilon$

Any curved space with **KS**s can support rigid SUSY theories.

SUSY theory on $S^2 \times S^1$ → superconformal index.

Generalization 1: (Hama-KH-Lee '11)

Ellipsoid

$$x_0^2 + x_1^2 + x_2^2 + x_3^2 \;=\; 1$$
$$ds^2 \;=\; \ell^2(dx_0^2 + dx_1^2) + \tilde{\ell}^2(dx_2^2 + dx_3^2)$$

Squashing parameter:

$$b \equiv \sqrt{\ell/\tilde{\ell}}$$

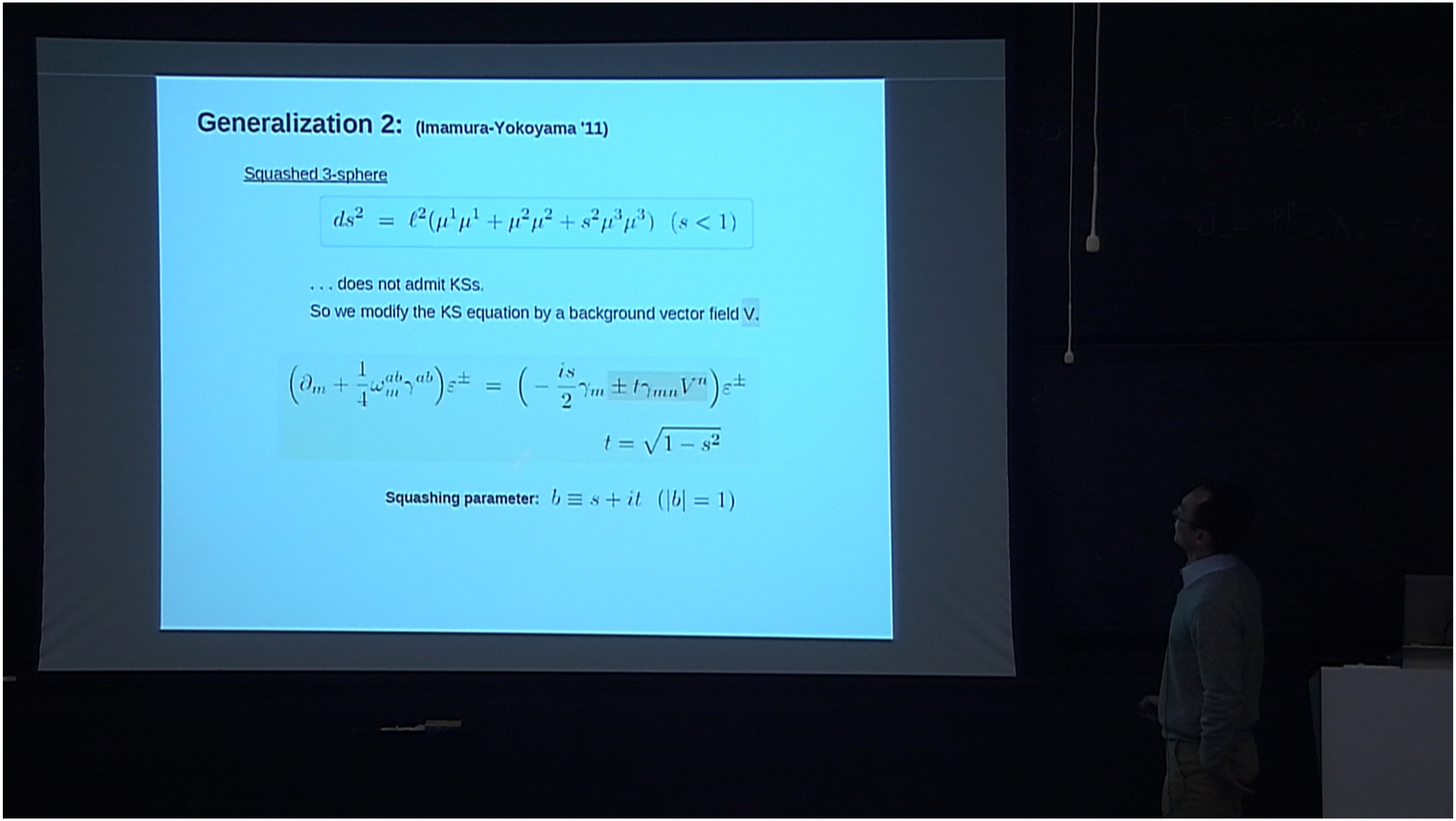
. . . does not admit KSs.

So we modify the KS equation by a background vector field V.

<u>"Charged" KS equation</u>

$$D_m\varepsilon^{\pm} \;\equiv\; \Big(\partial_m + \tfrac{1}{4}\omega_m^{ab}\gamma^{ab} \mp iV_m\Big)\varepsilon^{\pm} \;=\; \frac{i}{2f}\gamma_m\varepsilon^{\pm}$$

* V = background gauge field for U(1) R-symmetry.
* The same works also for squashed 3-spheres.


Generalization 2: (Imamura-Yokoyama '11)
Squashed 3-sphere
$ds^2 = \ell^2(\mu^1\mu^1 + \mu^2\mu^2 + s^2\mu^3\mu^3) \quad (s < 1)$
. . . does not admit KSs.
So we modify the KS equation by a background vector field V.
$\left(\partial_m + \frac{1}{4}\omega_m^{ab}\gamma^{ab}\right)\varepsilon^{\pm} = \left(-\frac{is}{2}\gamma_m \pm t\gamma_{mn}V^n\right)\varepsilon^{\pm}$
$t = \sqrt{1-s^2}$
Squashing parameter: $b \equiv s + it \quad (|b| = 1)$

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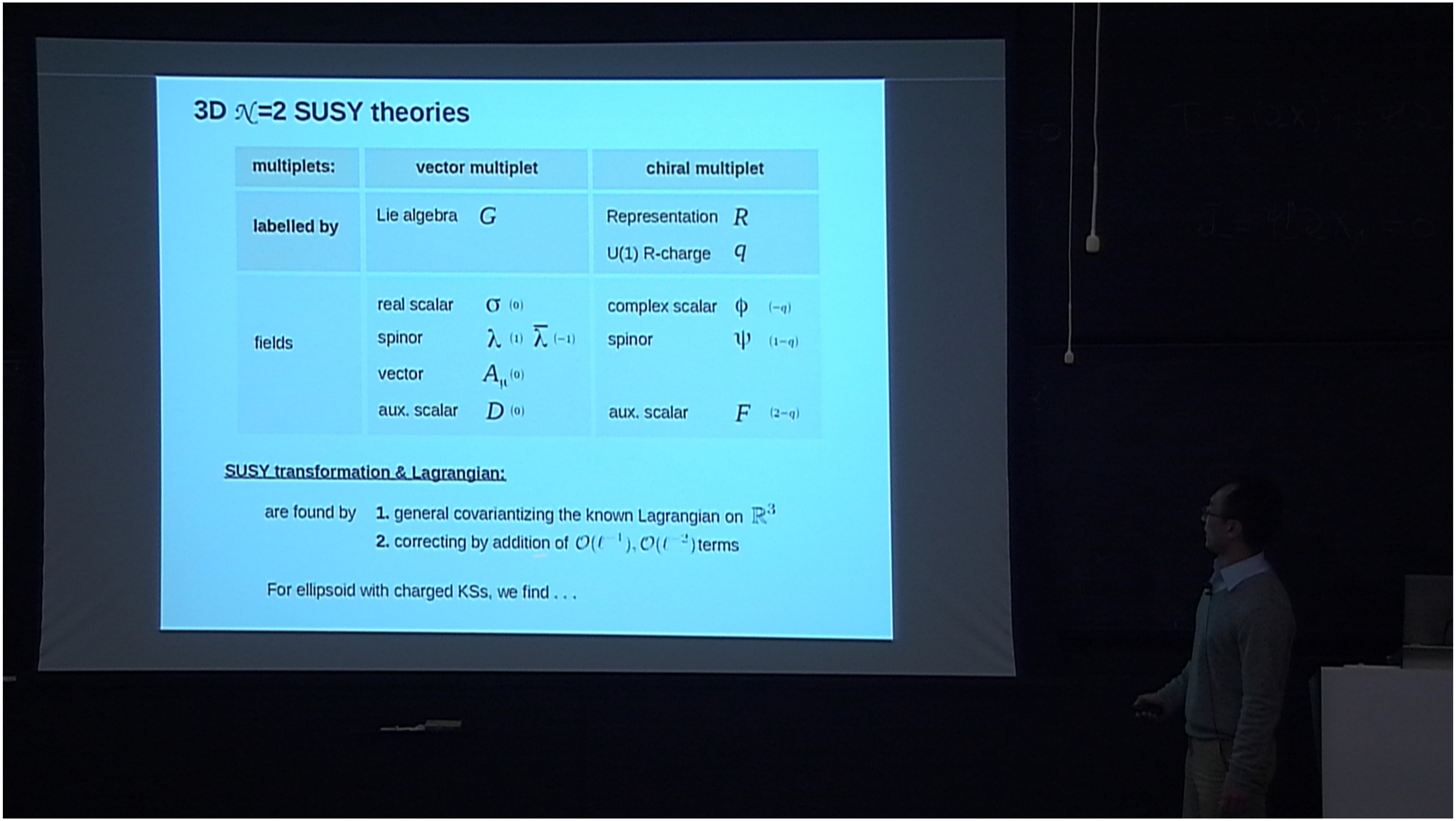
3D $\mathcal{N}$=2 SUSY theories

multiplets:	vector multiplet	chiral multiplet
labelled by	Lie algebra G	Representation R U(1) R-charge q
fields	real scalar $\sigma^{(0)}$ spinor $\lambda^{(1)}$ $\bar{\lambda}^{(-1)}$ vector $A_\mu^{(0)}$ aux. scalar $D^{(0)}$	complex scalar $\phi^{(-q)}$ spinor $\psi^{(1-q)}$ aux. scalar $F^{(2-q)}$

SUSY transformation & Lagrangian:

are found by **1.** general covariantizing the known Lagrangian on $\mathbb{R}^3$
2. correcting by addition of $\mathcal{O}(\ell^{-1}), \mathcal{O}(\ell^{-2})$ terms

For ellipsoid with charged KSs, we find . . .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Vector

$$\delta A_\mu = -\tfrac{i}{2}(\bar\epsilon\gamma_\mu\lambda - \bar\lambda\gamma_\mu\epsilon),$$

$$\delta\sigma = \tfrac{1}{2}(\bar\epsilon\lambda - \bar\lambda\epsilon),$$

$$\delta\lambda = \tfrac{1}{2}\gamma^{\mu\nu}\epsilon F_{\mu\nu} - D\epsilon + i\gamma^\mu\epsilon D_\mu\sigma + \tfrac{2i}{3}\sigma\gamma^\mu D_\mu\epsilon,$$

$$\delta\bar\lambda = \tfrac{1}{2}\gamma^{\mu\nu}\bar\epsilon F_{\mu\nu} + D\bar\epsilon - i\gamma^\mu\bar\epsilon D_\mu\sigma - \tfrac{2i}{3}\sigma\gamma^\mu D_\mu\bar\epsilon,$$

$$\delta D = -\tfrac{i}{2}\bar\epsilon\gamma^\mu D_\mu\lambda - \tfrac{i}{2}D_\mu\bar\lambda\gamma^\mu\epsilon + \tfrac{i}{2}[\bar\epsilon\lambda,\sigma] + \tfrac{i}{2}[\bar\lambda\epsilon,\sigma] - \tfrac{i}{6}(D_\mu\bar\epsilon\gamma^\mu\lambda + \bar\lambda\gamma^\mu D_\mu\epsilon).$$

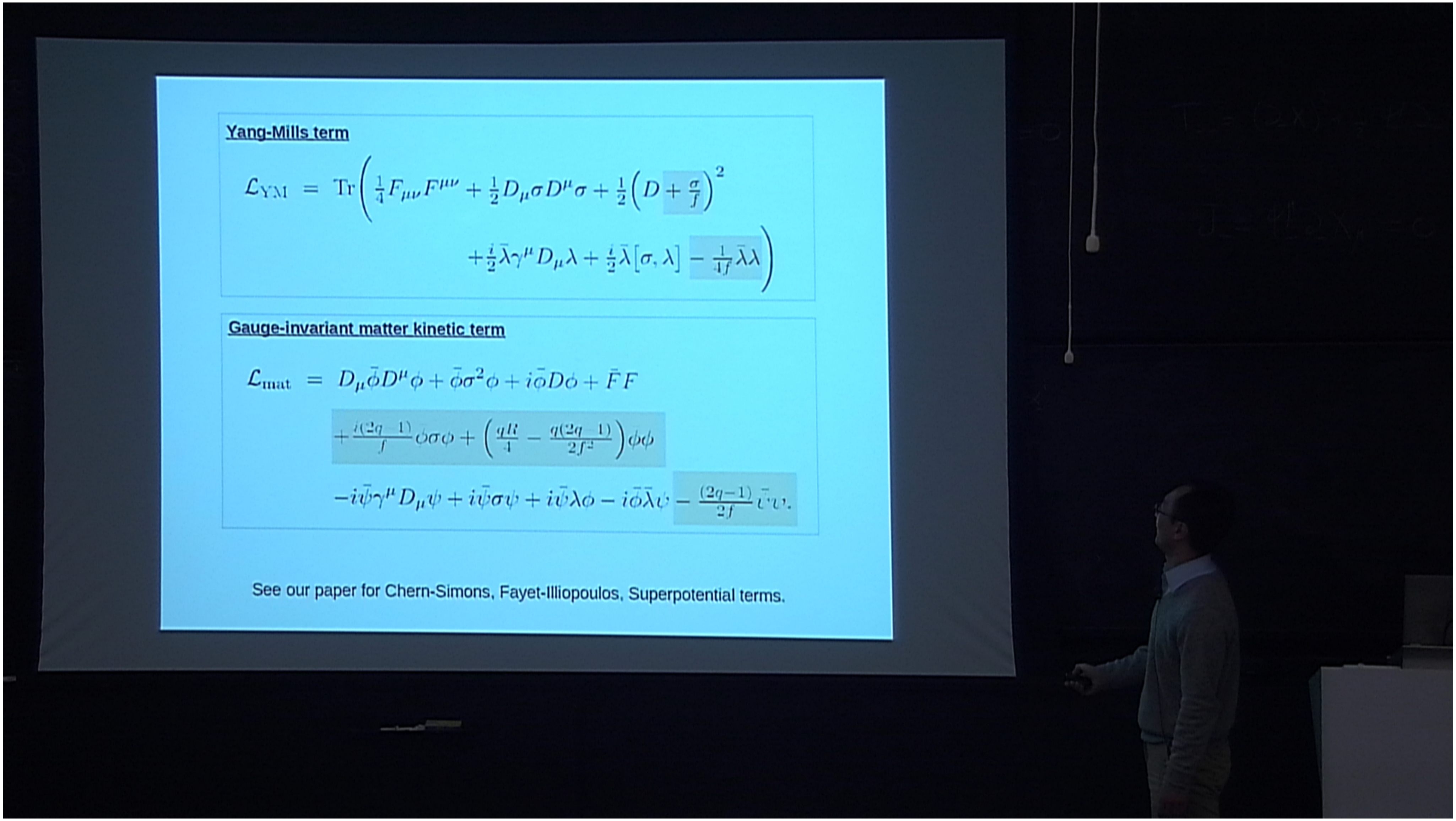
Chiral

$$\delta\phi = \bar\epsilon\psi,$$

$$\delta\psi = i\gamma^\mu\epsilon D_\mu\phi + i\epsilon\sigma\phi + \tfrac{2qi}{3}\gamma^\mu D_\mu\epsilon\phi + \bar\epsilon F,$$

$$\delta F = \epsilon(i\gamma^\mu D_\mu\psi - i\sigma\psi - i\lambda\phi) + \tfrac{i}{3}(2q-1)D_\mu\epsilon\gamma^\mu\psi.$$

$\epsilon, \bar\epsilon$: charged KSs (with charge +1,-1)


Yang-Mills term
$\mathcal{L}_{\rm YM} = {\rm Tr}\Big(\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}D_\mu\sigma D^\mu\sigma + \frac{1}{2}\Big(D + \frac{\sigma}{f}\Big)^2$
$+\frac{i}{2}\bar\lambda\gamma^\mu D_\mu\lambda + \frac{i}{2}\bar\lambda[\sigma,\lambda] - \frac{1}{4f}\bar\lambda\lambda\Big)$
Gauge-invariant matter kinetic term
$\mathcal{L}_{\rm mat} = D_\mu\bar\phi D^\mu\phi + \bar\phi\sigma^2\phi + i\bar\phi D\phi + \bar F F$
$+\frac{i(2q-1)}{f}\bar\phi\sigma\phi + \Big(\frac{qR}{4} - \frac{q(2q-1)}{2f^2}\Big)\bar\phi\phi$
$-i\bar\psi\gamma^\mu D_\mu\psi + i\bar\psi\sigma\psi + i\bar\psi\lambda\phi - i\bar\phi\bar\lambda\psi - \frac{(2q-1)}{2f}\bar\psi\psi.$
See our paper for Chern-Simons, Fayet-Illiopoulos, Superpotential terms.

Remarks:

$\mathcal{L}_{\mathrm{YM}}$ and $\mathcal{L}_{\mathrm{mat}}$ are **SUSY exact**.

For any pair of Killing spinors $\epsilon, \bar{\epsilon}$, one can show

$$\bar{\epsilon}\epsilon \cdot \mathcal{L}_{\mathrm{YM}} = \delta_{\bar{\epsilon}}\delta_{\epsilon}\mathrm{Tr}\Big(\frac{1}{2}\bar{\lambda}\lambda - 2D\sigma\Big),$$

$$\bar{\epsilon}\epsilon \cdot \mathcal{L}_{\mathrm{mat}} = \delta_{\bar{\epsilon}}\delta_{\epsilon}\Big(\bar{\psi}\psi - 2i\bar{\phi}\sigma\phi + \frac{2(q-1)}{f}\bar{\phi}\phi\Big).$$

Calculation of Partition function

SUSY Localization principle:

Nonzero contribution to path-integral arises only from "**saddle points**" = bosonic field configurations satisfying

$$\delta_{\mathrm{SUSY}}\Psi = 0 \quad \text{for all fermions } \Psi$$

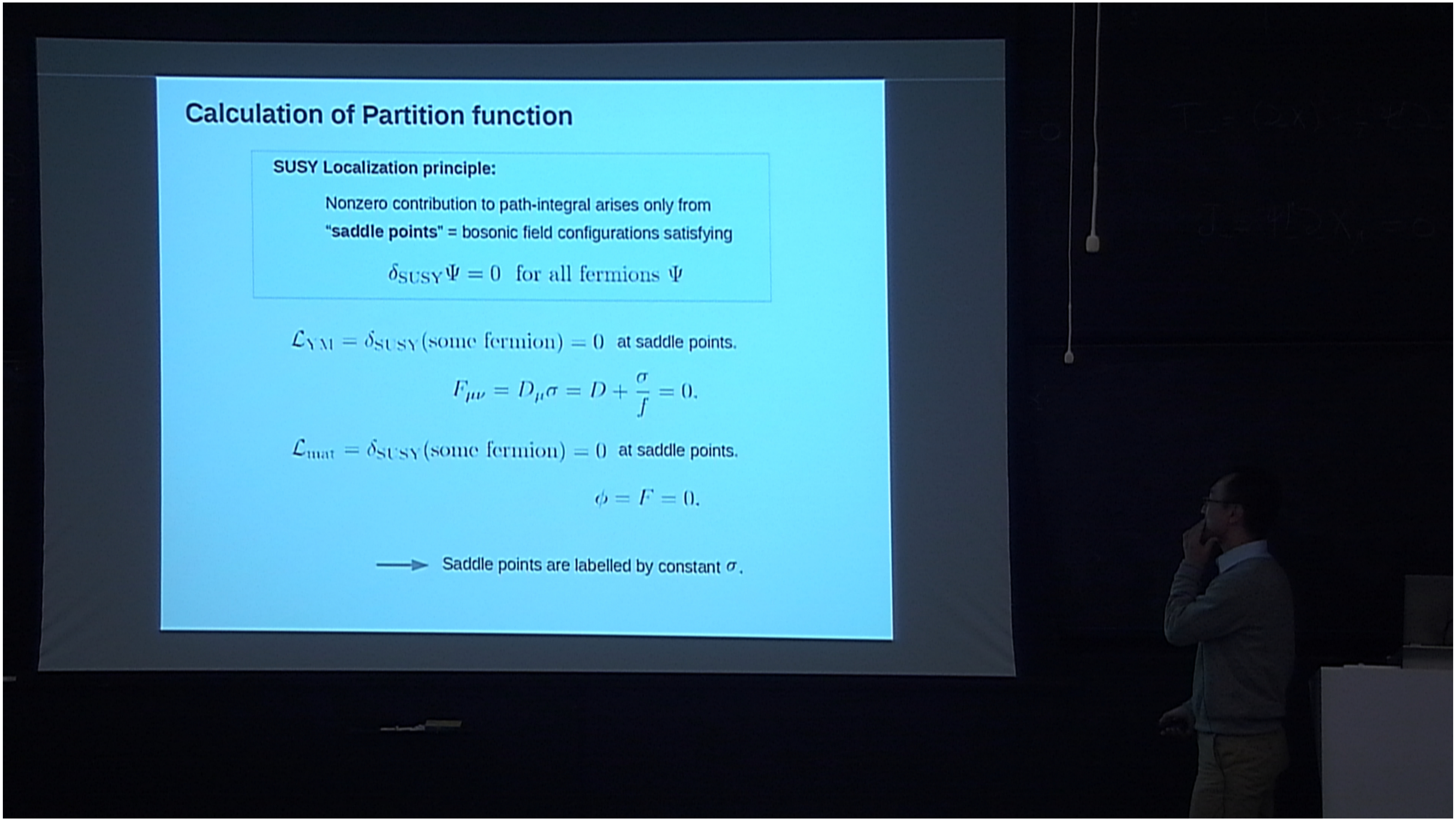
$\mathcal{L}_{\mathrm{YM}} = \delta_{\mathrm{SUSY}}(\text{some fermion}) = 0$ at saddle points.

$$F_{\mu\nu} = D_\mu\sigma = D + \frac{\sigma}{f} = 0.$$

$\mathcal{L}_{\mathrm{mat}} = \delta_{\mathrm{SUSY}}(\text{some fermion}) = 0$ at saddle points.

$$\phi = F = 0.$$

→ Saddle points are labelled by constant σ.



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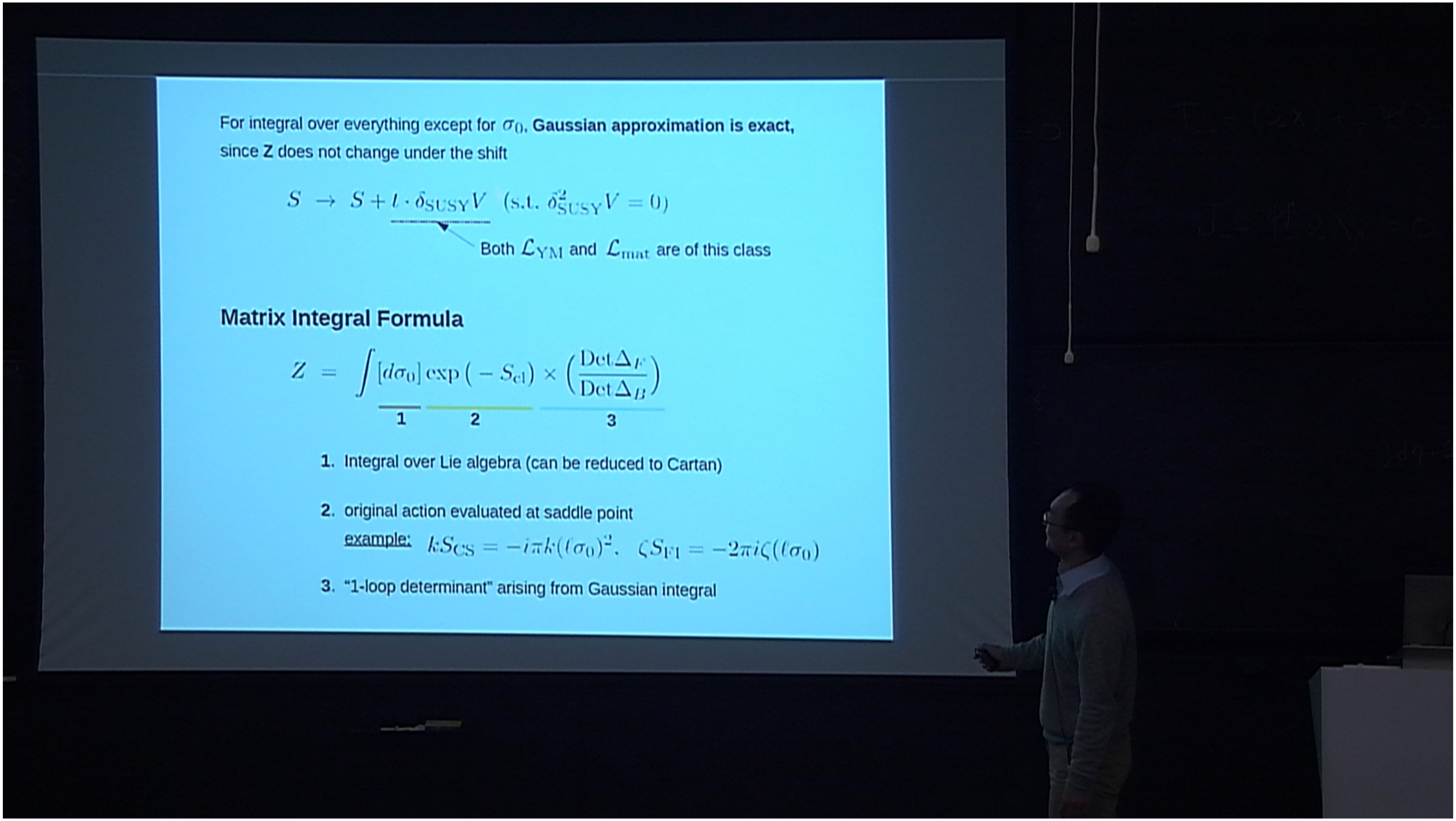
For integral over everything except for σ_0, Gaussian approximation is exact,
since Z does not change under the shift
$S \rightarrow S + t \cdot \delta_{\rm SUSY} V$ (s.t. $\delta^2_{\rm SUSY} V = 0$)
Both $\mathcal{L}_{\rm YM}$ and $\mathcal{L}_{\rm mat}$ are of this class
Matrix Integral Formula
$Z = \int [d\sigma_0] \exp(-S_{\rm cl}) \times \left(\frac{{\rm Det}\Delta_F}{{\rm Det}\Delta_B}\right)$
1 2 3
1. Integral over Lie algebra (can be reduced to Cartan)
2. original action evaluated at saddle point
example: $kS_{\rm CS} = -i\pi k(t\sigma_0)^2, \ \zeta S_{\rm FI} = -2\pi i\zeta(t\sigma_0)$
3. "1-loop determinant" arising from Gaussian integral

Calculation of Partition function

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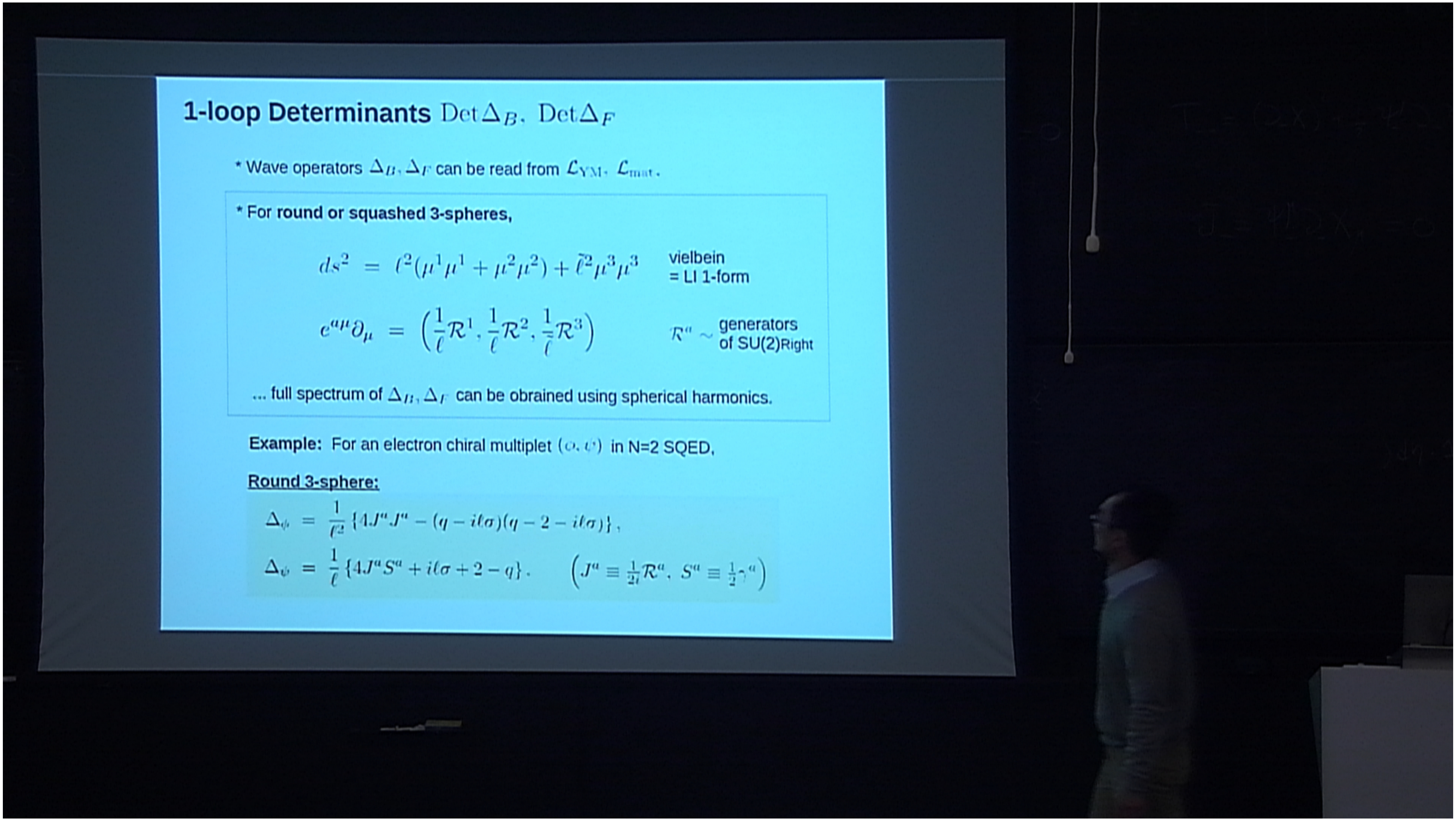
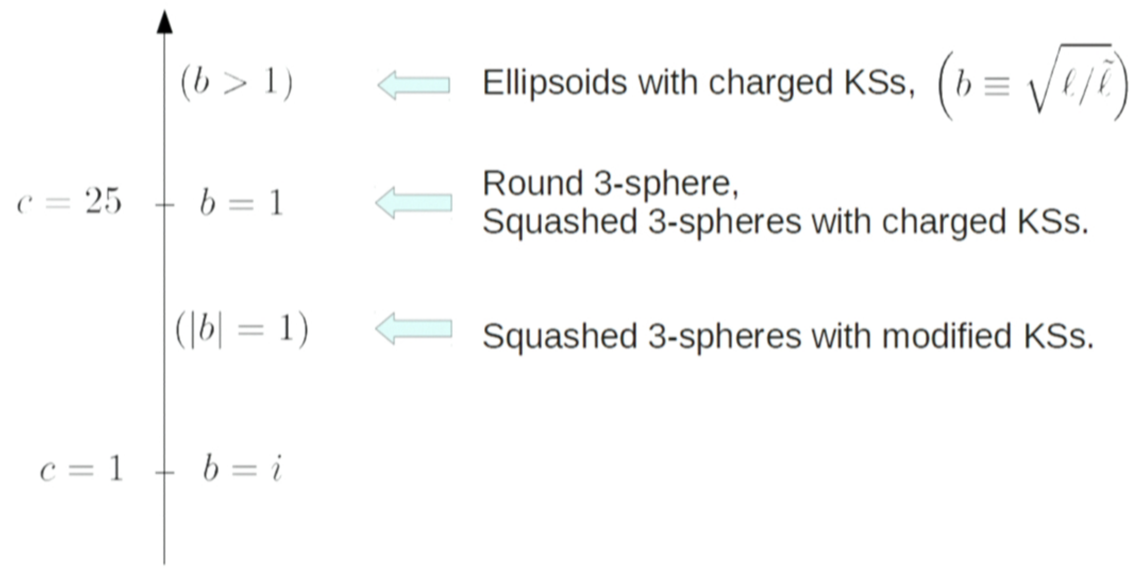
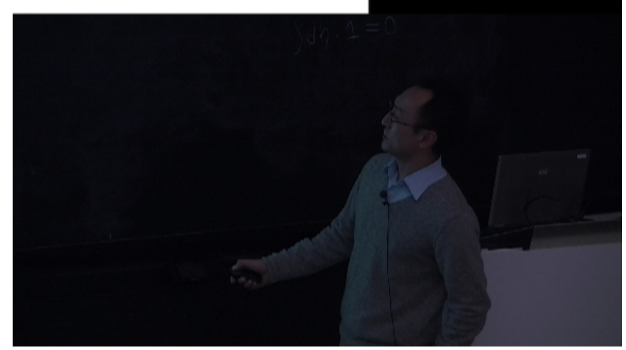

1-loop Determinants $\mathrm{Det}\Delta_B,\ \mathrm{Det}\Delta_F$
* Wave operators Δ_B, Δ_F can be read from $\mathcal{L}_{\rm YM}, \mathcal{L}_{\rm mat}$.
* For round or squashed 3-spheres,
$ds^2 = \ell^2(\mu^1\mu^1 + \mu^2\mu^2) + \tilde{\ell}^2\mu^3\mu^3$
vielbein = LI 1-form
$e^{a\mu}\partial_\mu = \left(\frac{1}{\ell}\mathcal{R}^1, \frac{1}{\ell}\mathcal{R}^2, \frac{1}{\tilde{\ell}}\mathcal{R}^3\right)$
$\mathcal{R}^a \sim$ generators of SU(2)Right
... full spectrum of Δ_B, Δ_F can be obrained using spherical harmonics.
Example: For an electron chiral multiplet (ϕ, ψ) in N=2 SQED,
Round 3-sphere:
$\Delta_\phi = \frac{1}{\ell^2}\{4J^aJ^a - (q - i\ell\sigma)(q - 2 - i\ell\sigma)\},$
$\Delta_\psi = \frac{1}{\ell}\{4J^aS^a + i\ell\sigma + 2 - q\}.$
$\left(J^a \equiv \frac{1}{2i}\mathcal{R}^a,\ S^a \equiv \frac{1}{2}\gamma^a\right)$

Squashing parameter b



($c \equiv 1 + 6(b + b^{-1})^2$: "Liouville central charge")



Overall Normalization

1. **Matter 1-loop determinant** is correct including normalization.

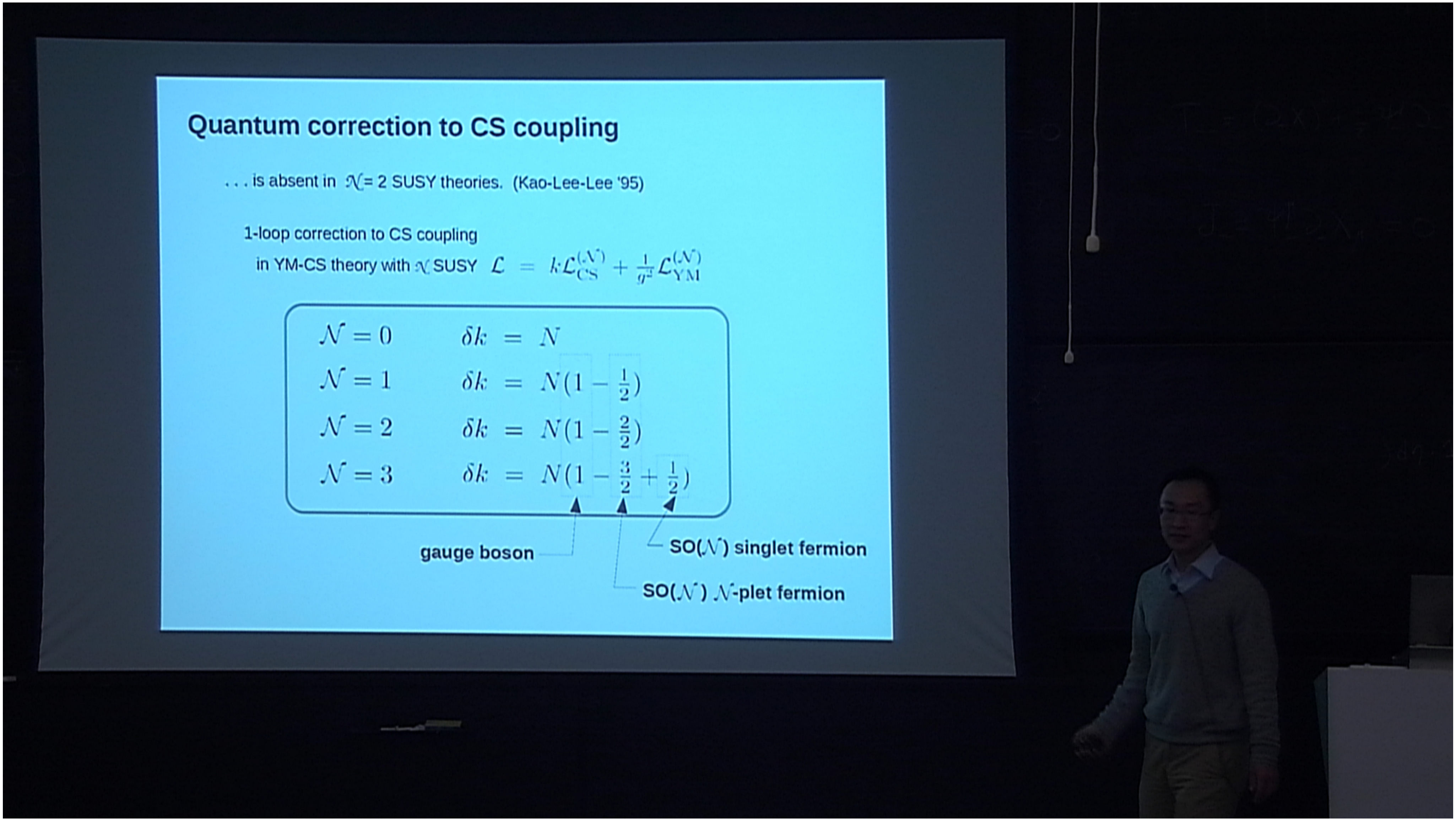
If ϕ, ϕ' are chiral matters of R-charge q, q' and U(1) gauge charge ± 1, their 1-loop determinant is

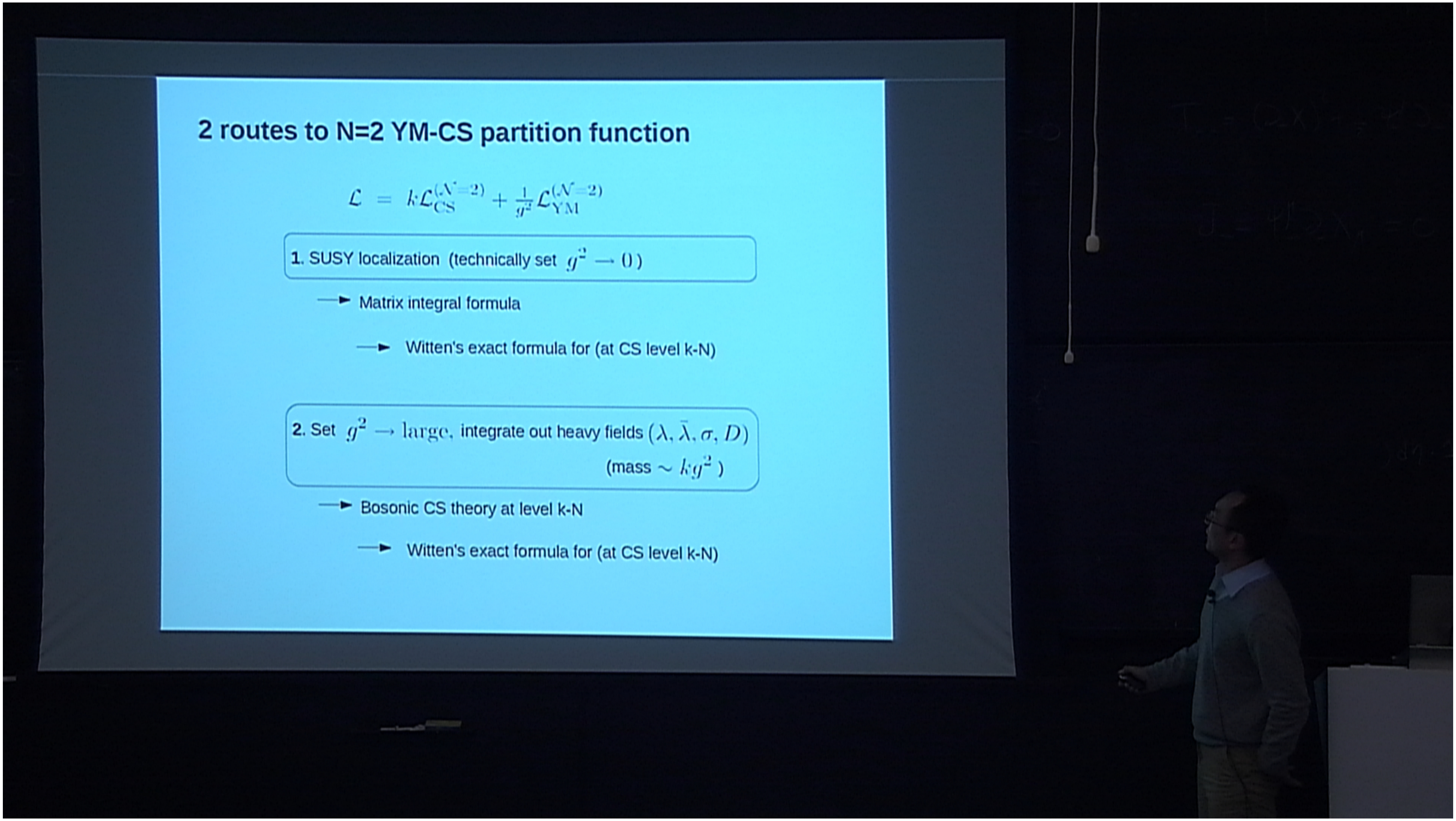
$$\Delta_\phi \Delta_{\phi'} \;=\; s_b(\tfrac{iQ}{2}(1-q)-\sigma)s_b(\tfrac{iQ}{2}(1-q')+\sigma)$$

If $q+q'=2$ they can be decoupled by F-term mass, and accordingly

$$\Delta_\phi \Delta_{\phi'} \;=\; s_b(x)s_b(-x) \;=\; 1.$$

2. **Vector multiplet contribution** needs finite renormalization.
   to reproduce the known exact results in pure CS theory (Witten '89).


Quantum correction to CS coupling
. . . is absent in $\mathcal{N}$ = 2 SUSY theories. (Kao-Lee-Lee '95)
1-loop correction to CS coupling
in YM-CS theory with $\mathcal{N}$ SUSY $\mathcal{L} = k\mathcal{L}_{\rm CS}^{(\mathcal{N})} + \frac{1}{g^2}\mathcal{L}_{\rm YM}^{(\mathcal{N})}$
$\mathcal{N}=0 \quad \delta k = N$
$\mathcal{N}=1 \quad \delta k = N(1-\frac{1}{2})$
$\mathcal{N}=2 \quad \delta k = N(1-\frac{2}{2})$
$\mathcal{N}=3 \quad \delta k = N(1-\frac{3}{2}+\frac{1}{2})$
gauge boson
SO($\mathcal{N}$) singlet fermion
SO($\mathcal{N}$) $\mathcal{N}$-plet fermion


2 routes to N=2 YM-CS partition function
$\mathcal{L} = k\mathcal{L}_{\rm CS}^{(\mathcal{N}=2)} + \frac{1}{g^2}\mathcal{L}_{\rm YM}^{(\mathcal{N}=2)}$
1. SUSY localization (technically set $g^2 \to 0$)
Matrix integral formula
Witten's exact formula for (at CS level k-N)
2. Set $g^2 \to \mathrm{large}$, integrate out heavy fields $(\lambda, \bar{\lambda}, \sigma, D)$
(mass $\sim kg^2$)
Bosonic CS theory at level k-N
Witten's exact formula for (at CS level k-N)

Example : ABJM theory = N M2-branes on $\mathbb{C}^4/\mathbb{Z}_k$

$U(N)_k \times U(N)_{-k}$ Chern-Simons theory with

chiral matters $(X_1, X_2): \ (N, \bar{N}),$

$(Y_1, Y_2): \ (\bar{N}, N)$

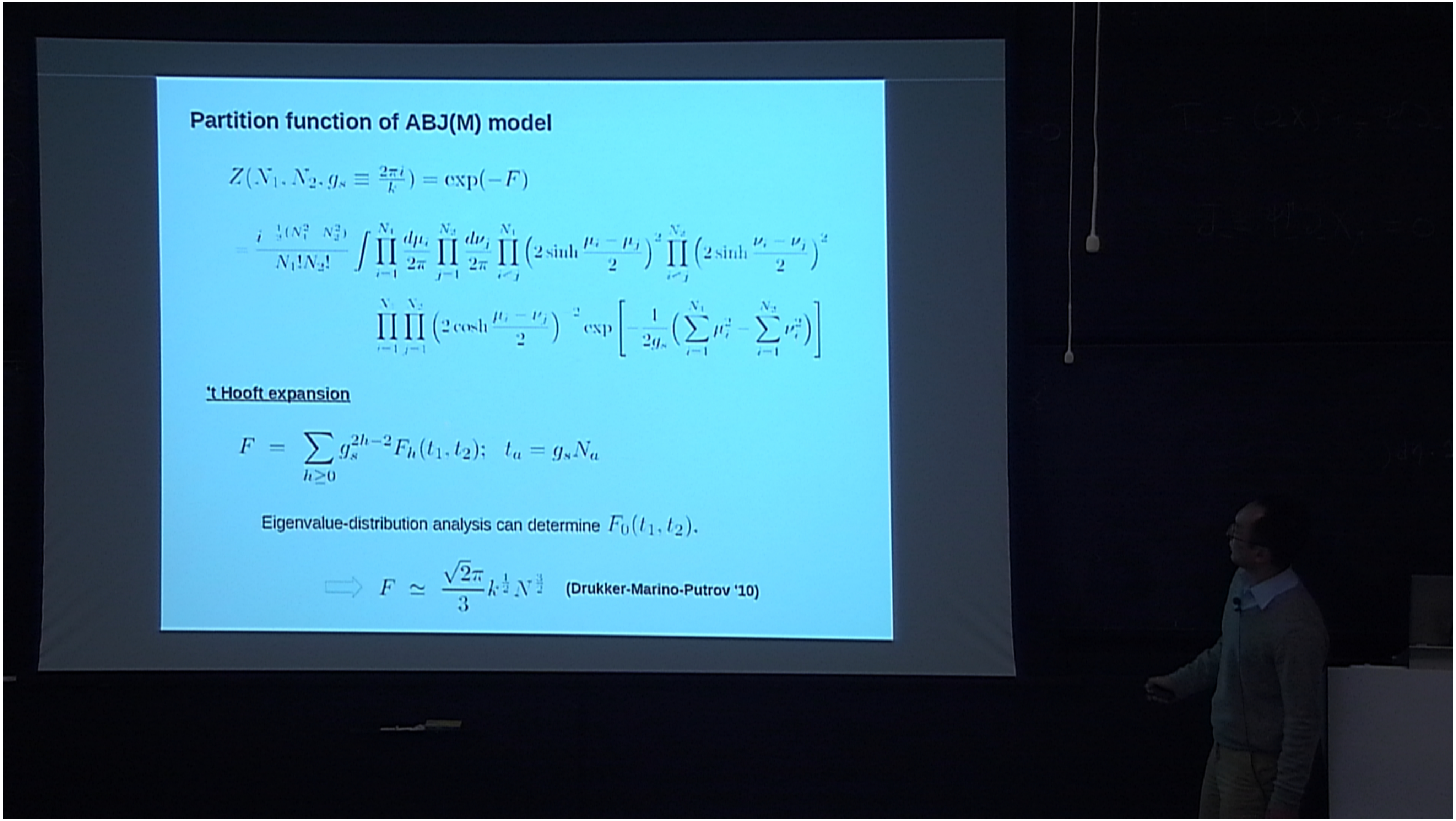
superpotential $W = \frac{2\pi}{k}\epsilon^{ab}\epsilon^{cd}\mathrm{Tr}\big(X_a Y_c X_b Y_d\big)$

At strong coupling ($N/k \gg 1$),

dual to **11d SUGRA** on $AdS_4 \times (S^7/\mathbb{Z}_k) \quad (k \ll N^{1/5})$

IIA SUGRA on $AdS_4 \times \mathbb{CP}^3 \quad (k \gg N^{1/5})$

$$F \sim N^{3/2}$$


Partition function of ABJ(M) model
$Z(N_1, N_2, g_s \equiv \frac{2\pi i}{k}) = \exp(-F)$
$= \frac{i^{-\frac{1}{2}(N_1^2 - N_2^2)}}{N_1! N_2!} \int \prod_{i=1}^{N_1} \frac{d\mu_i}{2\pi} \prod_{j=1}^{N_2} \frac{d\nu_j}{2\pi} \prod_{i<j}^{N_1} \left(2\sinh\frac{\mu_i - \mu_j}{2}\right)^2 \prod_{i<j}^{N_2} \left(2\sinh\frac{\nu_i - \nu_j}{2}\right)^2$
$\prod_{i=1}^{N_1} \prod_{j=1}^{N_2} \left(2\cosh\frac{\mu_i - \nu_j}{2}\right)^{-2} \exp\left[-\frac{1}{2g_s}\left(\sum_{i=1}^{N_1} \mu_i^2 - \sum_{i=1}^{N_2} \nu_i^2\right)\right]$
't Hooft expansion
$F = \sum_{h \geq 0} g_s^{2h-2} F_h(t_1, t_2); \quad t_a = g_s N_a$
Eigenvalue-distribution analysis can determine $F_0(t_1, t_2)$.
$\Rightarrow F \simeq \frac{\sqrt{2}\pi}{3} k^{\frac{1}{2}} N^{\frac{3}{2}}$ (Drukker-Marino-Putrov '10)

Conclusion

Exact 3D partition function and various identities of double-sine function will be useful for a better understanding of

* multiple M2-brane dynamics
* (3d version of) AGT conjecture
* 3d dualities