Title: Spinor Quantisation for Complex Ashtekar Variables

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Abstract: During the last couple of years Dupuis, Freidel, Livine, Speziale and Tambornino developed a twistorial formulation for loop quantum gravity.
Constructed from Ashtekar--Barbero variables, the formalism is restricted to SU(2) gauge transformations.
In this talk, I perform the generalisation to the full Lorentzian case, that is the group SL(2,C).
The phase space of SL(2,C) (i.e. complex or selfdual) Ashtekar variables on a spinnetwork graph is decomposed in terms of twistorial variables. To every link there are two twistors---one to each boundary point---attached. The formalism provides a clean derivation of the solution space of the reality conditions of loop quantum gravity.
Key features of the EPRL spinfoam model are perfectly recovered.
If there is still time, I'll scatch my current project concerning a twistorial path integral for spinfoam gravity as well.

Motivation

- Dupuis, Freidel, Livine, Speziale and Tambornino developed a twistorial formulation for $SU(2)$ Ashtekar–Barbero variables.
- To each link of the boundary spinnetwork they assign a twistor, carrying information on area, angles (normals) and curvature.
- These are $SU(2)$ variables, therefore:
  - How do Lorentz transformations act on them?
  - This could be answered by starting from the true space-time parallel transport, i.e. the $SL(2,\mathbb{C})$ connection $A = \Gamma + \mathrm{i}K$, instead of $A = \Gamma + \beta K$.
- Is there a relation to Penrose's program?
  - Can we speak about the Weyl tensor, i.e. the twistor's curvature?
  - Twistors are light rays incident to points in spacetime. Can we use this to learn more about causality in LQG?

First step towards these goals:

- Generalize the $SU(2)$ twistorial formulation of LQG to $SL(2,\mathbb{C})$.

Outline of the talk

Four to five points:

1. Complex Ashtekar variables for real valued Barbero–Immirzi parameter.
2. Spinors for $SL(2,\mathbb{C})$ phase space on a fixed graph.
3. Spinorial version of the reality conditions.
4. Quantisation.
5. If there's still time: Relation to the $SU(2)$ spinor papers.

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1. Complex Ashtekar variables for real valued Barbero–Immirzi parameter

Complex Ashtekar variables revisited

To speak about $SL(2,\mathbb{C})$ we start from the selfdual decomposition of the Holst action:

$$S_{\mathrm{Holst}} = \frac{\beta + \mathrm{i}}{\mathrm{i}\beta} \int_M \Sigma^A{}_B \wedge R^B{}_A[A] + \mathrm{cc.} \tag{1}$$

Where $\Sigma^A{}_B = \Sigma^i \tau^A{}_{Bi} = \left(\frac{1}{2}\epsilon_l{}^i{}_m e^l \wedge e^m + \mathrm{i} e^0 \wedge e^i\right)\tau^\alpha{}_{\beta i}$ are the selfdual components of the Plebanski 2-form $\Sigma^{\alpha\beta} = e^\alpha \wedge e^\beta$.

And $\beta \in \mathbb{R}$ is the Barbero–Immirzi parameter.

We identify the symplectic structure, e.g.

$$\left\{\Pi_i{}^a(p), A^j{}_b(q)\right\} = \delta_i^j \delta_b^a \tilde{\delta}(p,q) = \left\{\bar{\Pi}_i{}^a(p), \bar{A}^j{}_b(q)\right\} \tag{2}$$

Where

$$\Pi_i{}^a = -\frac{\beta + \mathrm{i}}{4\mathrm{i}\beta} \tilde{\eta}^{abc} \Sigma_{ibc}. \tag{3}$$

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Linear simplicity constraints as reality conditions

Choosing time gauge:

$$e^0 = N\mathrm{d}t \tag{4}$$

we get

$$\Pi_i{}^a = -\frac{\beta+\mathrm{i}}{4\mathrm{i}\beta}\tilde{\eta}^{abc}\Sigma_{ibc} = \frac{\beta+\mathrm{i}}{4\mathrm{i}\beta}\tilde{\eta}^{abc}\epsilon_{ilm}e^l{}_b e^m{}_c. \tag{5}$$

This implies the reality condition:

$$\begin{aligned} C_i{}^a &= \frac{\beta}{\beta+\mathrm{i}}\Pi_i{}^a + \frac{\beta}{\beta-\mathrm{i}}\bar{\Pi}_i{}^a = \\ &= \frac{\beta}{\beta^2+1}\Big(\underbrace{-\mathrm{i}(\Pi_i{}^a - \bar{\Pi}_i{}^a)}_{K} + \beta\underbrace{(\Pi_i{}^a + \bar{\Pi}_i{}^a)}_{L}\Big) = 0 \end{aligned}$$

This constraint is of second class, it is preserved in time only provided the spatial part of the torsion 2-form vanishes.

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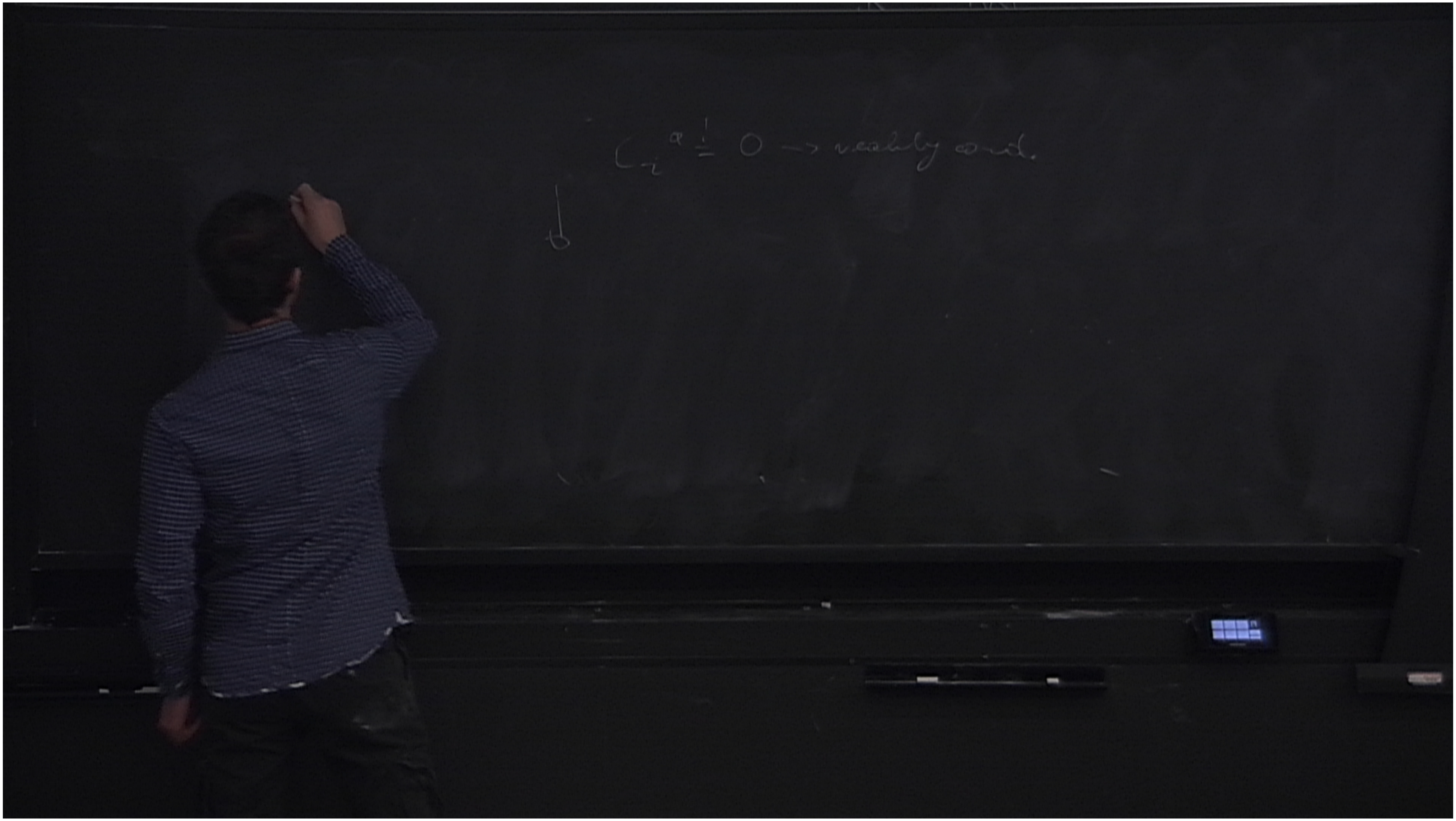
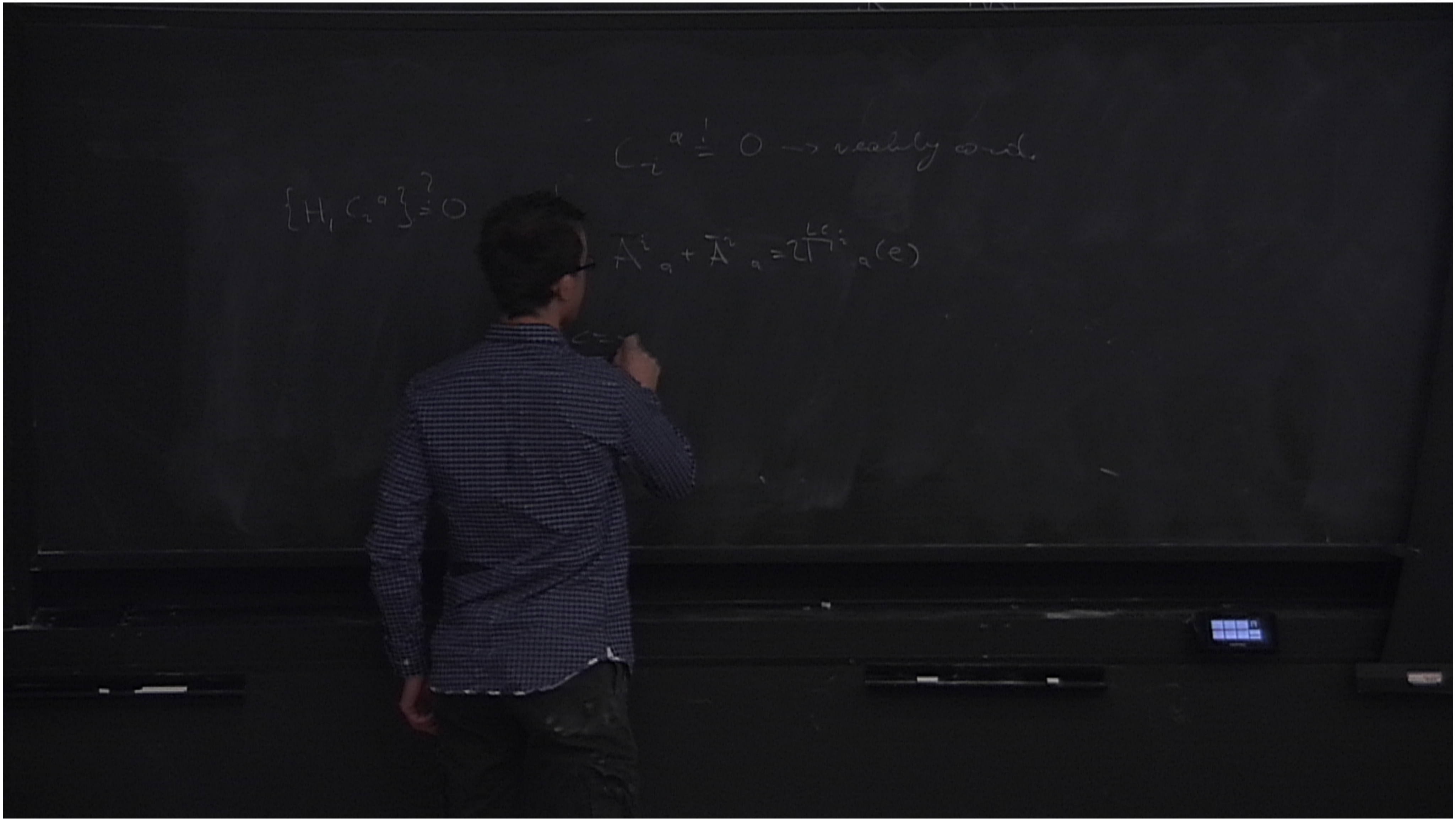
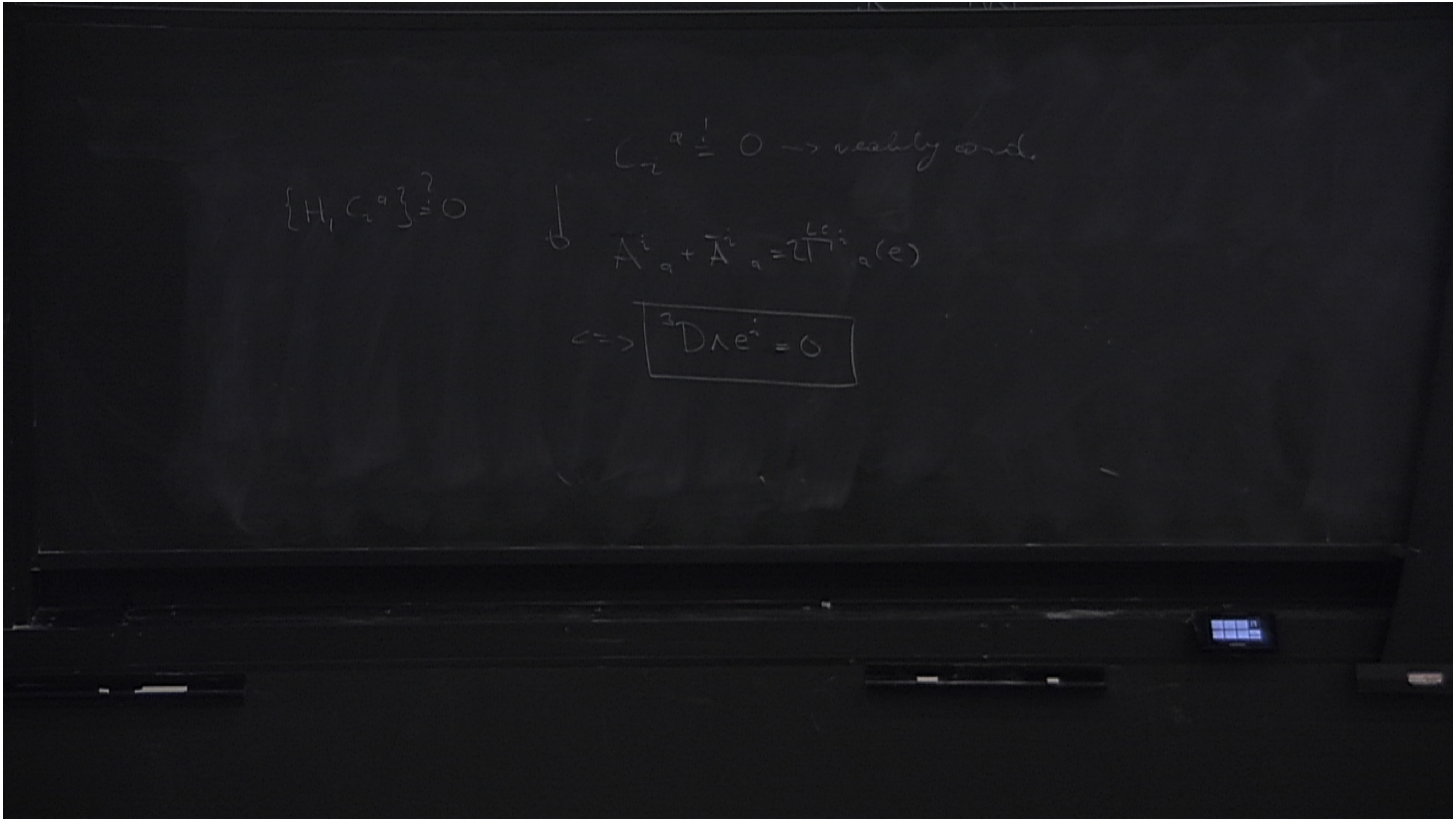
2. Spinorial decomposition of the $SL(2,\mathbb{C})$ phase space on a fixed graph


$C_i^a \overset{!}{=} 0 \rightarrow$ weakly cond.


{H, C_i^a} ?= 0
C_i^a != 0 → weakly const.
A^i_a + Ā^i_a = 2Γ^{LC i}_a(e)


$\{H, C_i^a\} \stackrel{?}{=} 0$
$C_i^a \stackrel{!}{=} 0$ → reality cond.
$A^i_a + \bar{A}^i_a = 2\Gamma^{LC\,i}_a(e)$
$\Leftrightarrow$ $^3D \wedge e^i = 0$

Linear simplicity constraints as reality conditions

Choosing time gauge:

$$e^0 = N\mathrm{d}t \tag{4}$$

we get

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This implies the reality condition:

$$C_i{}^a = \frac{\beta}{\beta + \mathrm{i}}\Pi_i{}^a + \frac{\beta}{\beta - \mathrm{i}}\bar{\Pi}_i{}^a =$$

$$= \frac{\beta}{\beta^2 + 1}\Big(\underbrace{-\mathrm{i}(\Pi_i{}^a - \bar{\Pi}_i{}^a)}_{K} + \beta\underbrace{(\Pi_i{}^a + \bar{\Pi}_i{}^a)}_{L}\Big) = 0$$

This constraint is of second class, it is preserved in time only provided the spatial part of the torsion 2-form vanishes.

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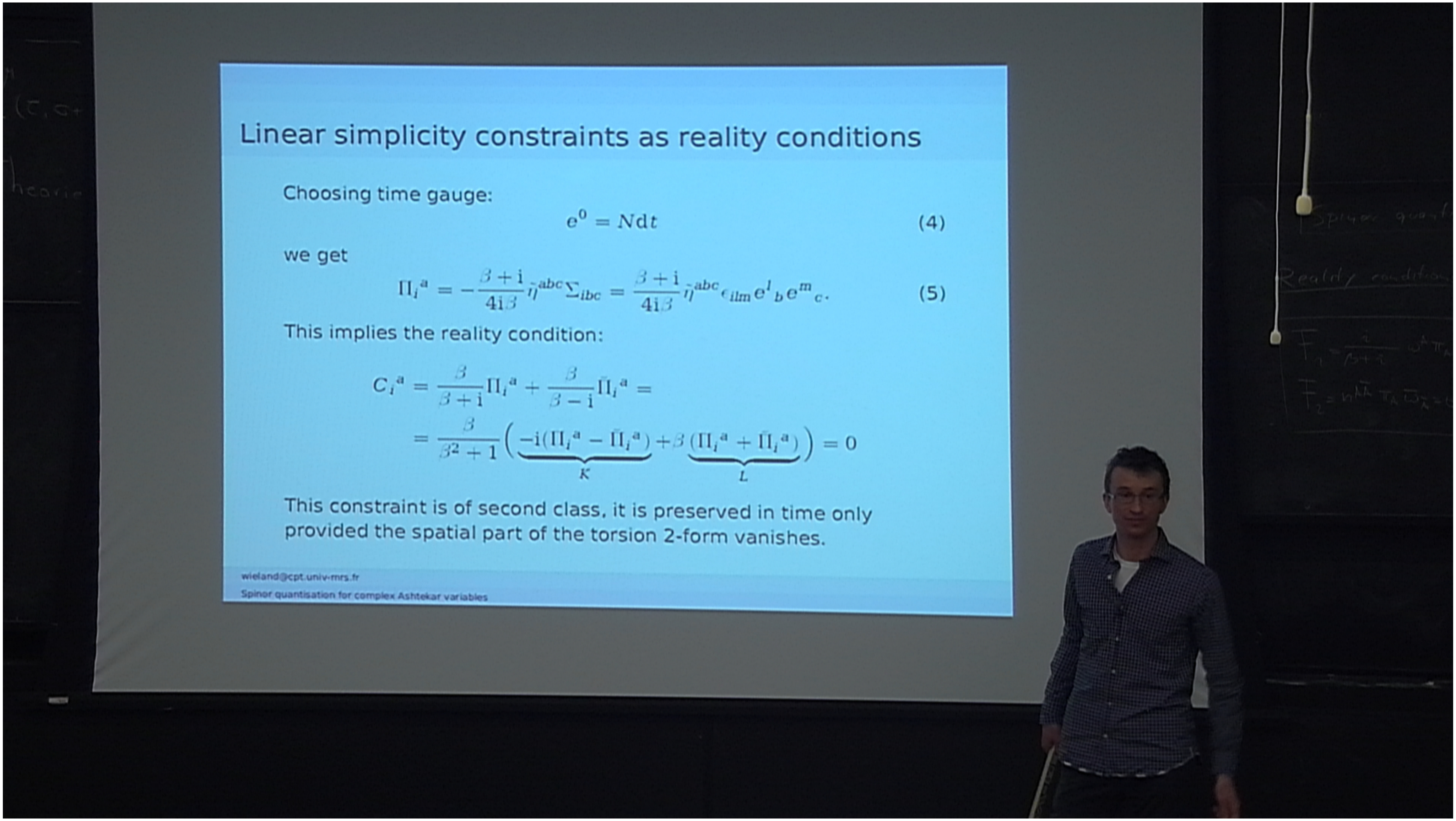
Linear simplicity constraints as reality conditions
Choosing time gauge:
$e^0 = N dt$ (4)
we get
$\Pi_i{}^a = -\frac{\beta + i}{4i\beta}\tilde{\eta}^{abc}\Sigma_{ibc} = \frac{\beta + i}{4i\beta}\tilde{\eta}^{abc}\epsilon_{ilm} e^l{}_b e^m{}_c.$ (5)
This implies the reality condition:
$C_i{}^a = \frac{\beta}{\beta + i}\Pi_i{}^a + \frac{\beta}{\beta - i}\bar{\Pi}_i{}^a = \frac{\beta}{\beta^2 + 1}\Big(\underbrace{-i(\Pi_i{}^a - \bar{\Pi}_i{}^a)}_{K} + \beta\underbrace{(\Pi_i{}^a + \bar{\Pi}_i{}^a)}_{L}\Big) = 0$
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Spinor quantisation for complex Ashtekar variables

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Smeared phase space on a fixed graph

A fixed graph Γ consists of oriented links $\gamma, \gamma', \ldots$, to each of which we assign a dual face $f, f', \ldots$. Introduce smeared variables:

$$SL(2,\mathbb{C}) \ni h[f] = \mathrm{Pexp}\big(-\int_\gamma A\big) \tag{6a}$$

$$\mathfrak{sl}(2,\mathbb{C}) \ni \Pi[f] = \int_{p\in f} h_{p\to\gamma(0)} \Pi_p h_{\gamma(0)\to p} \tag{6b}$$

Holonomy flux algebra, i.e. $T^*SL(2,\mathbb{C})^L$

For a single link:

$$\big\{h[f], h[f']\big\} = 0 \tag{7a}$$

$$\big\{\Pi_i[f], h[f]\big\} = -h[f]\tau_i \tag{7b}$$

$$\big\{\Pi_i[f], \Pi_j[f]\big\} = -\epsilon_{ij}{}^m \Pi_m[f] \tag{7c}$$

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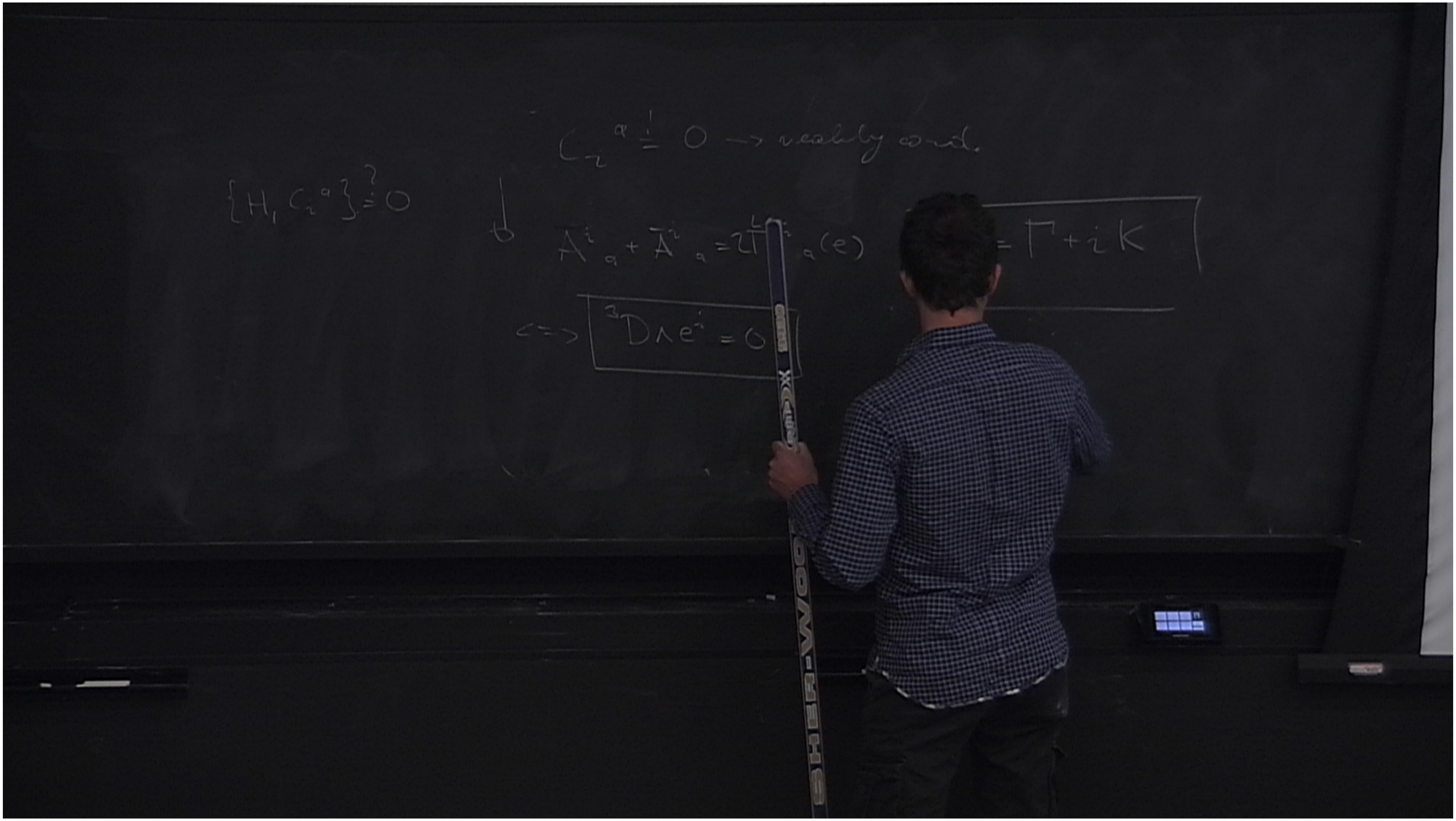


Preliminaries: Twistorial phase space

First: What is a twistor?

1. A twistor Z is a bispinor $Z = (\omega^A, \bar{\pi}_{\bar{A}}) \in \mathbb{C}^2 \oplus (\bar{\mathbb{C}}^2)^*$.
2. $SL(2,\mathbb{C})$ acts in the obvious way:

$$\omega^A \xrightarrow{g} +g^A{}_B\omega^B \quad (8a)$$

$$\bar{\pi}_{\bar{A}} \xrightarrow{g} -\bar{g}_{\bar{A}}{}^{\bar{B}}\bar{\pi}_{\bar{B}} \quad (8b)$$

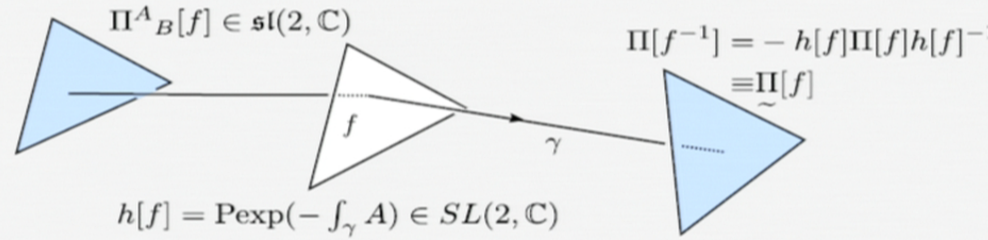
3. There is an $SL(2,\mathbb{C})$ invariant symplectic structure available:

$$\{\pi_A, \omega^B\} = \delta^A_B \quad (9)$$

$$\{\bar{\pi}_{\bar{A}}, \bar{\omega}^{\bar{B}}\} = \bar{\delta}^{\bar{B}}_{\bar{A}} \quad (10)$$

Next: Decompose phase space variables in terms of these bispinors.

Holonomy and fluxes



Fluxes in terms of spinors

$$\Pi^{AB} = -\tfrac{1}{4}\left(\omega^A\pi^B + \omega^B\pi^A\right), \qquad \underset{\sim}{\Pi}^{AB} = \tfrac{1}{4}\left(\underset{\sim}{\omega}^A\underset{\sim}{\pi}^B + \underset{\sim}{\omega}^B\underset{\sim}{\pi}^A\right)$$

Unique up to ordering and rescaling.

Holonomy in terms of spinors

For $\pi_A\omega^A \neq 0 \neq \underset{\sim}{\pi}_A\underset{\sim}{\omega}^A$ we have a basis in $\mathbb{C}^2$, by reordering/-scaling:

$$\omega^A \xrightarrow{h[f]} \underset{\sim}{\omega}^A, \qquad \pi^A \xrightarrow{h[f]} \underset{\sim}{\pi}^A$$

This fixes the holonomy in terms of spinors.

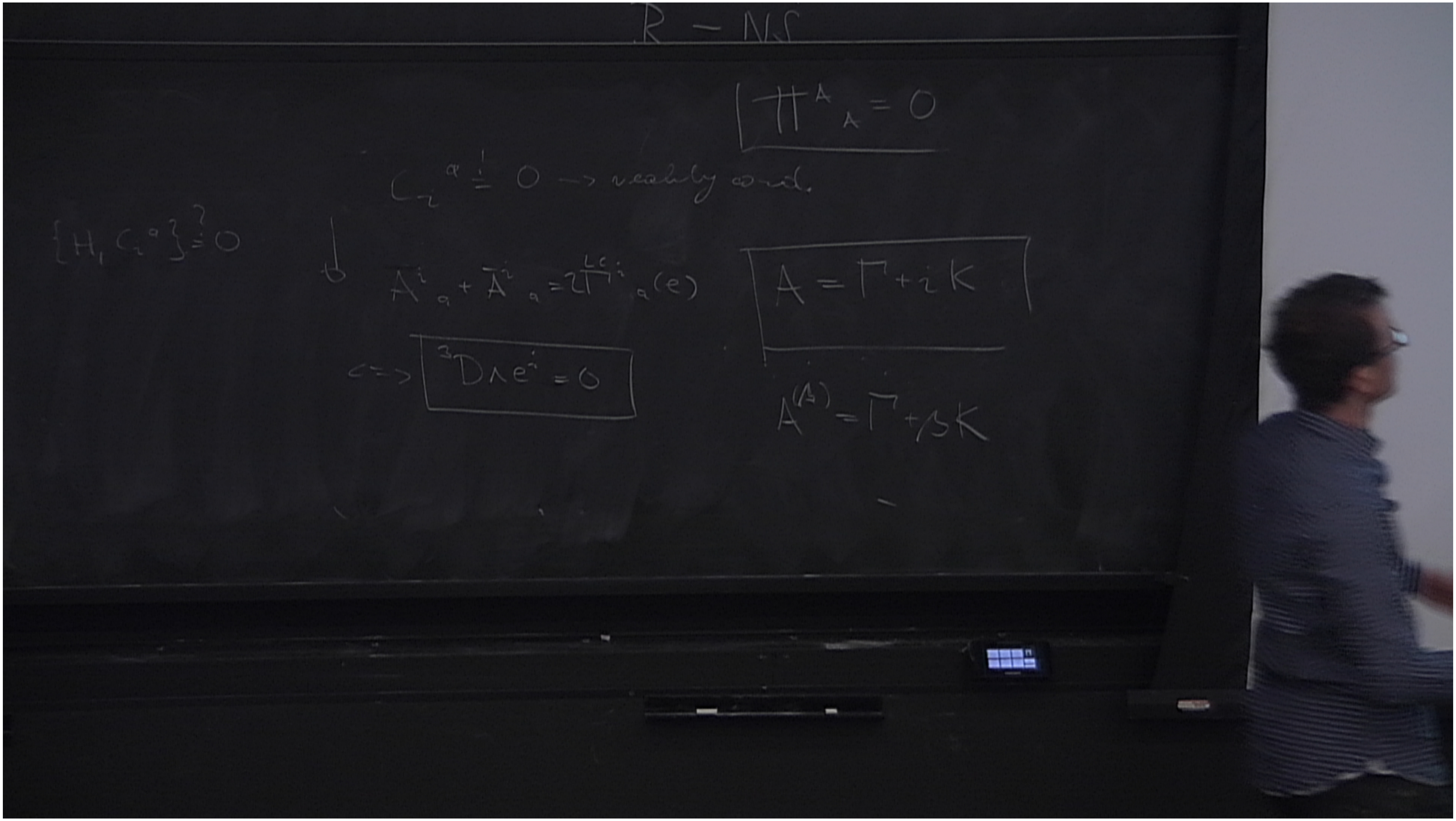
Reverse the logic

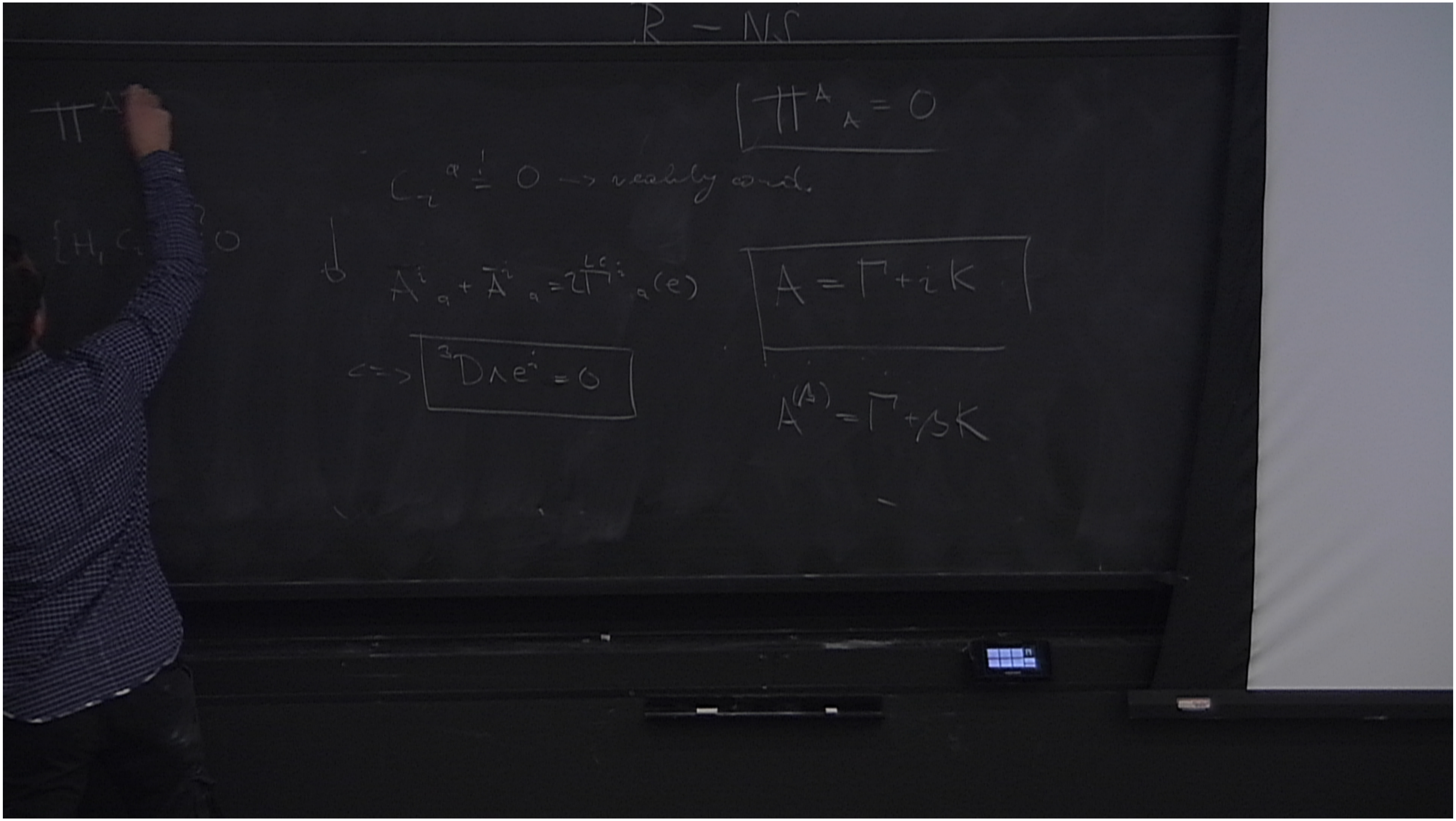
We can parametrize the pase space by $(\pi, \omega, \underset{\sim}{\pi}, \underset{\sim}{\omega}) \in \mathbb{C}^8$ provided:

$\pi_A\omega^A \neq 0$ i.e. f is not null

$\pi_A\omega^A - \underset{\sim}{\pi}_A\underset{\sim}{\omega}^A = 0$ i.e. $h[f]$ has unit determinant

But what about the symplectic structure?


R — NS
Π^A_A = 0
C_i^a ≐ 0 → weakly cond.
{H, C_i^a} ≟ 0
A^i_a + Ā^i_a = 2 Γ^LC i_a(e)
⟺ ³D∧e^i = 0
A = Γ + iK
A^(β) = Γ + βK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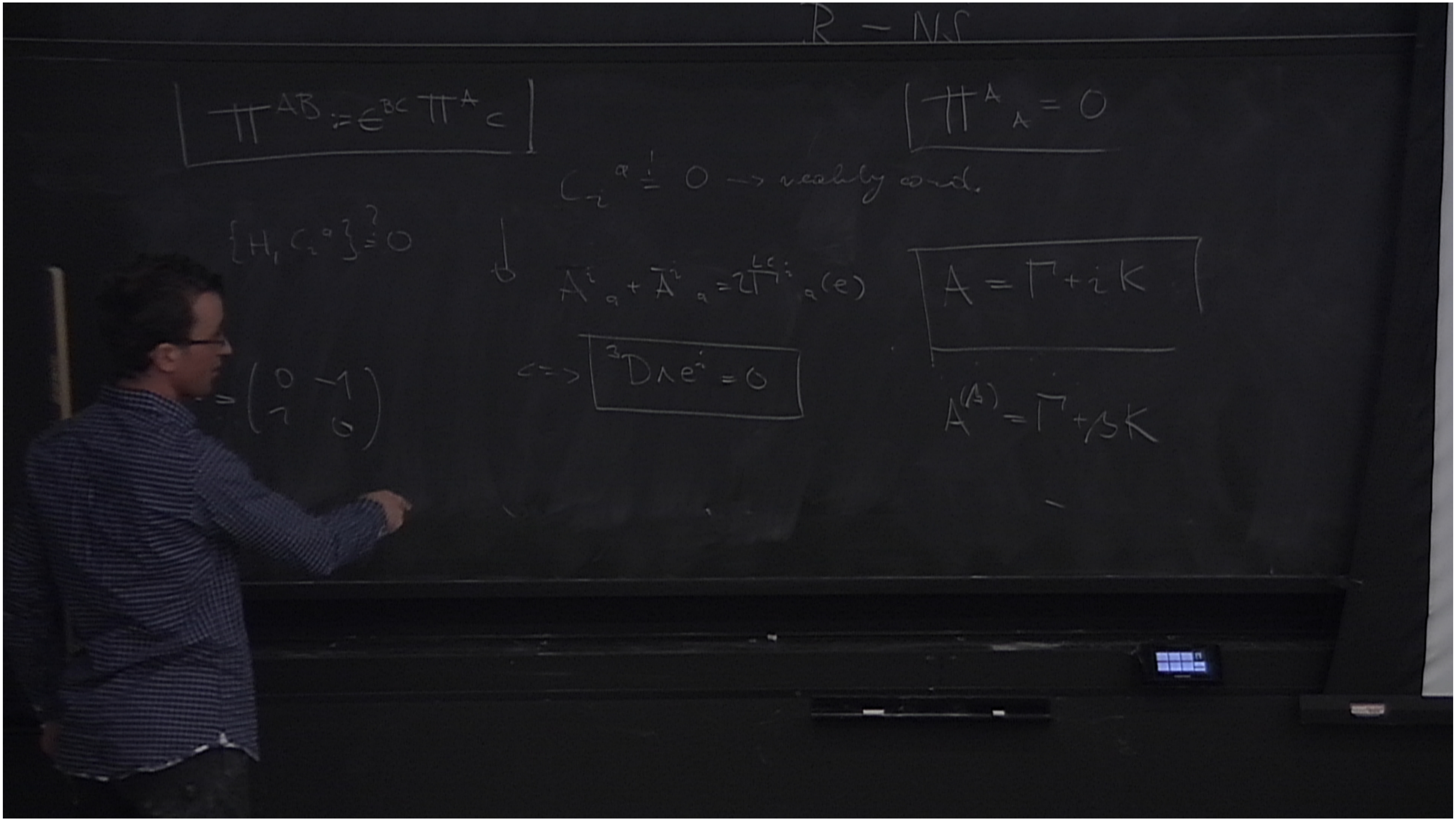
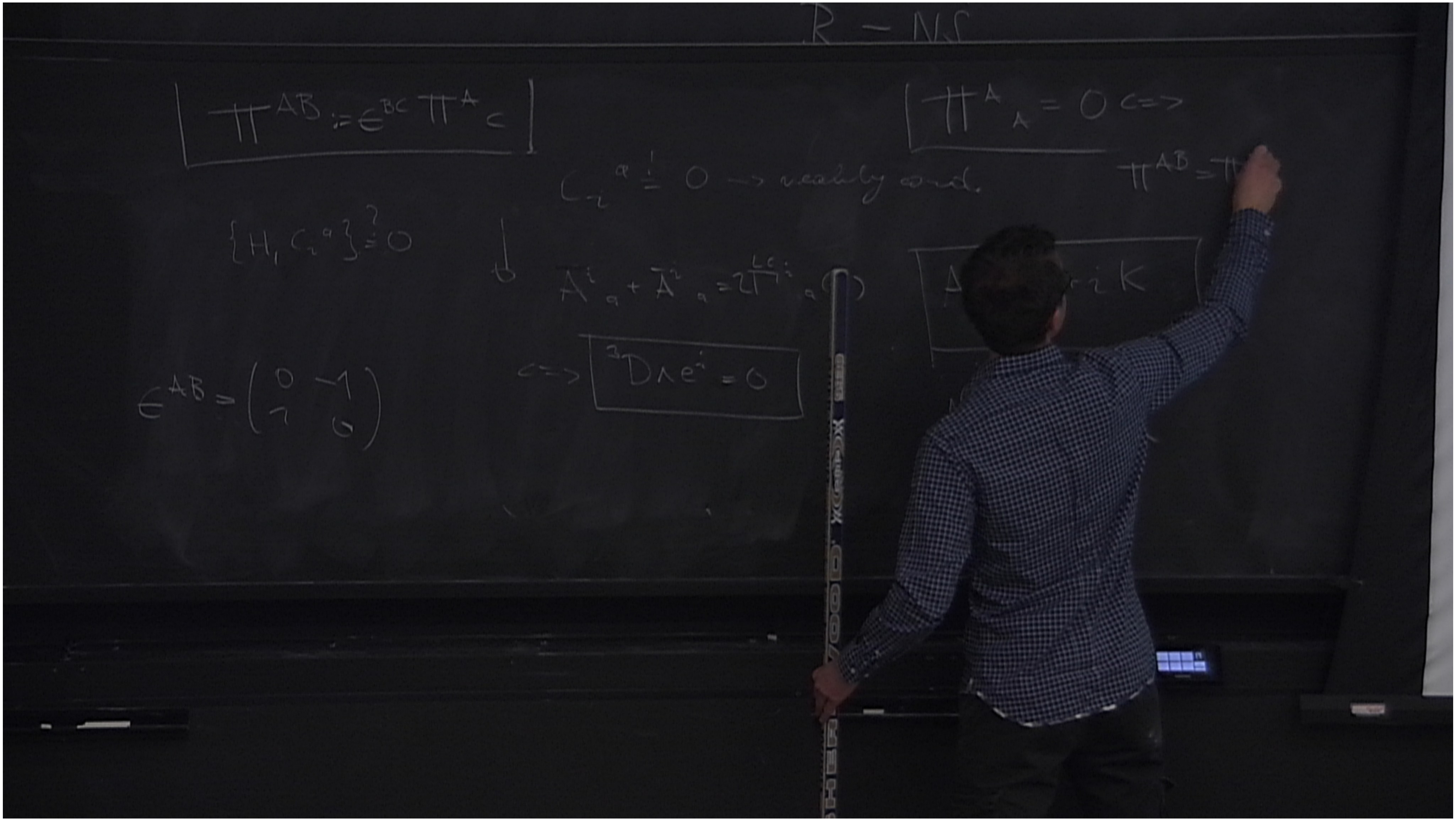
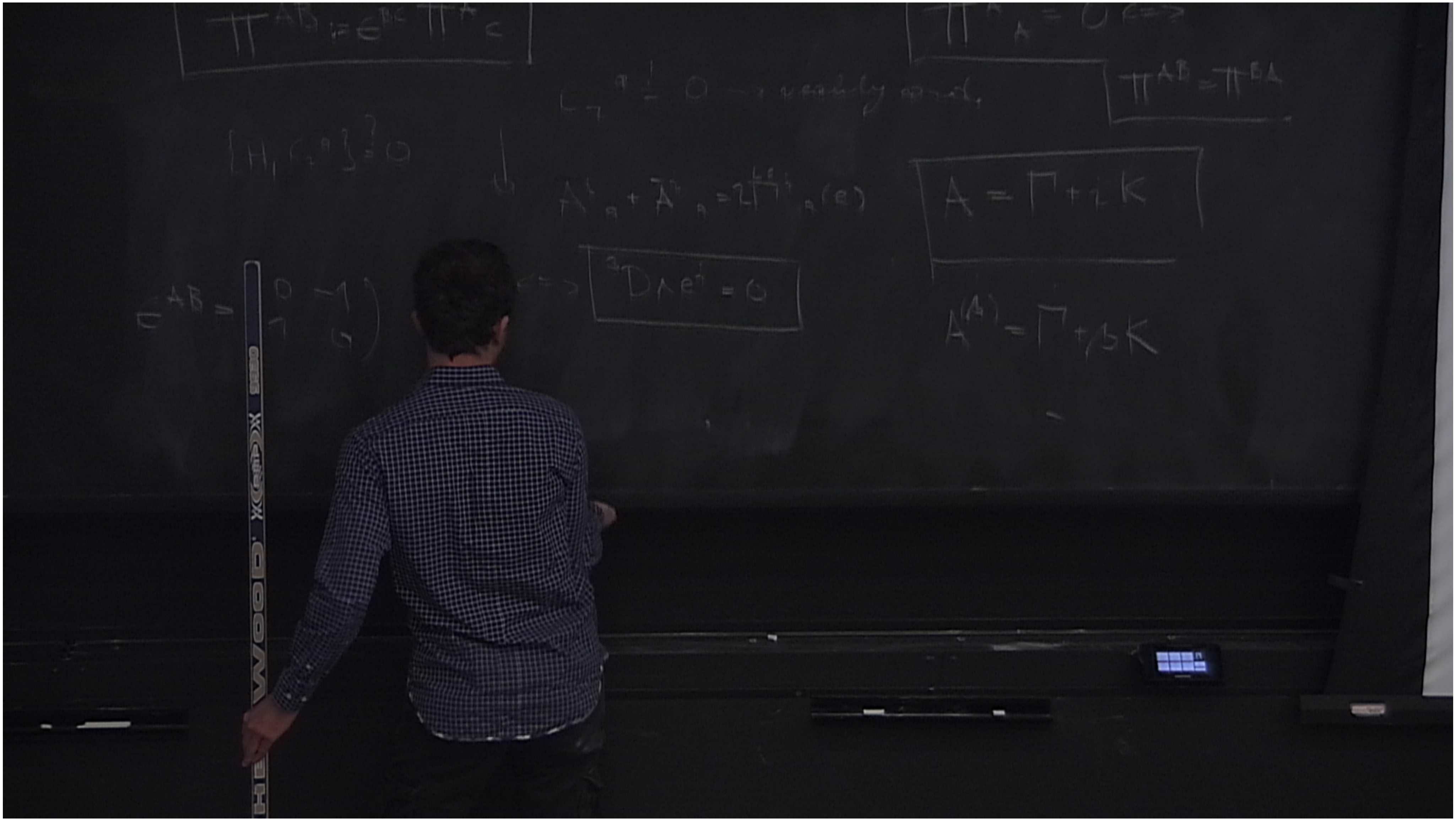
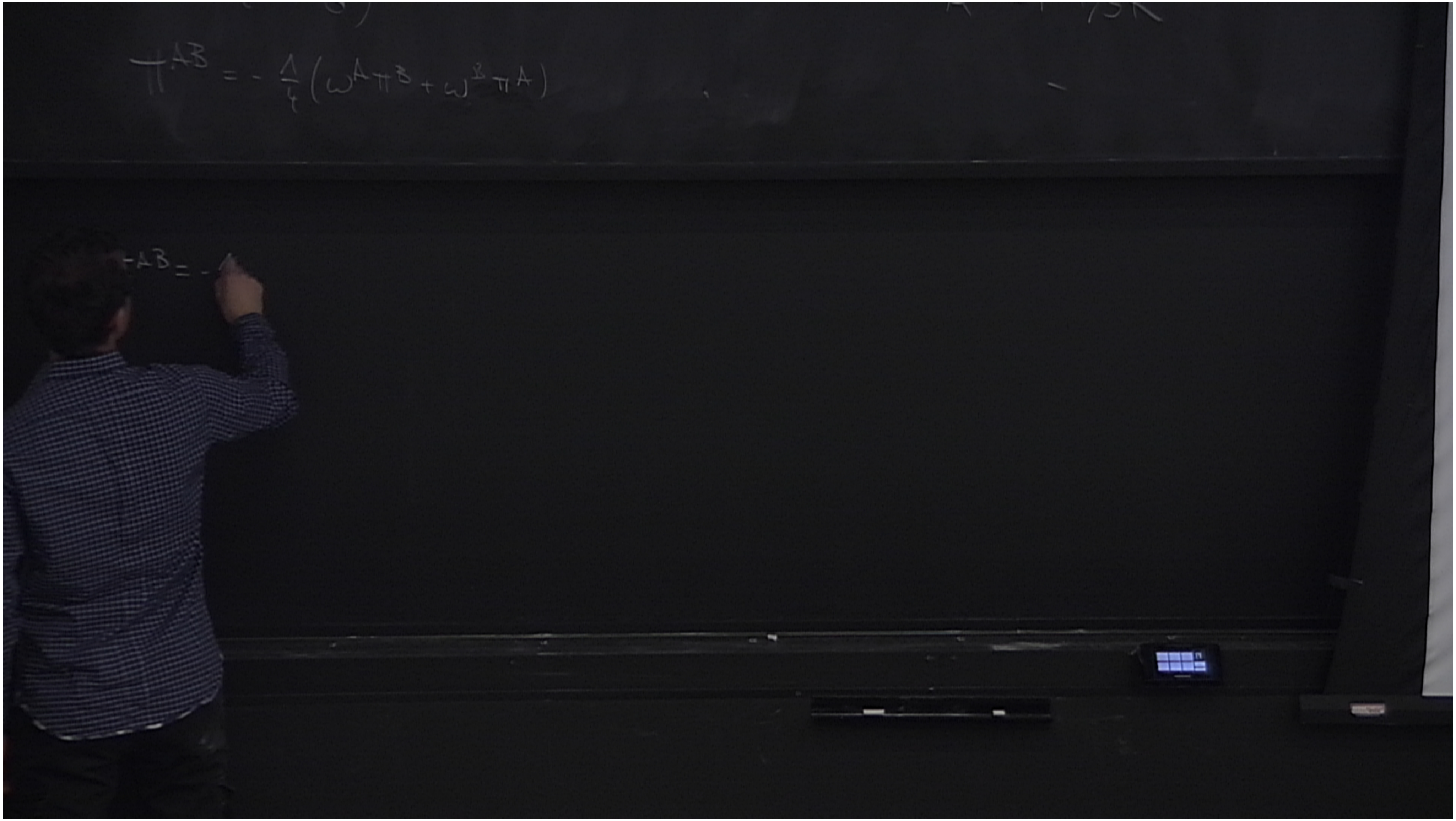
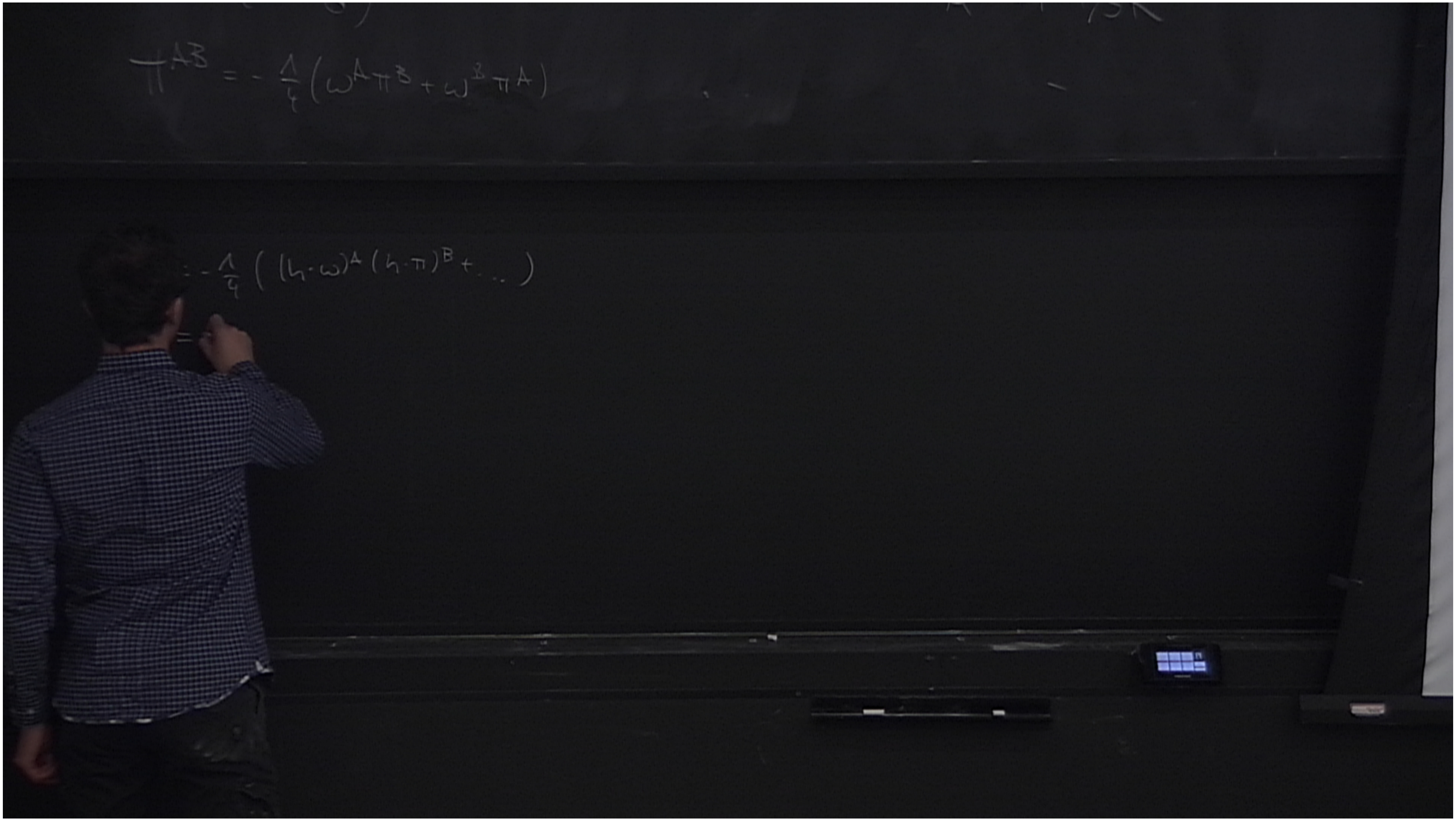
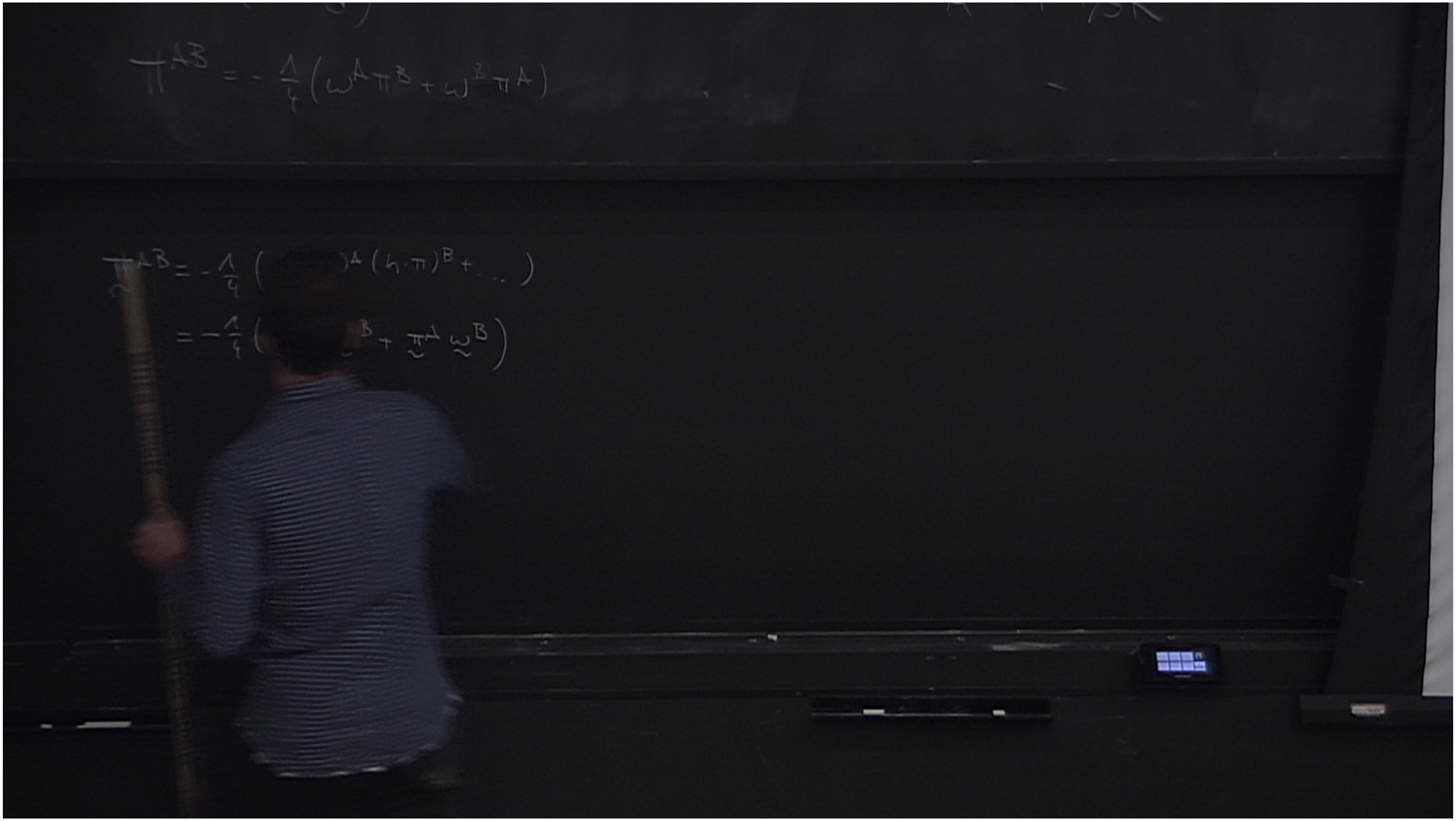

R — N.S
$\pi^{AB} := \epsilon^{BC} \pi^{A}{}_{C}$
$\pi^{A}{}_{A} = 0$
$\{H, C_i{}^a\} \stackrel{?}{=} 0$
$A = \Gamma + iK$
$^{3}D \wedge e^{i} = 0$
$A^{(\beta)} = \Gamma + \beta K$

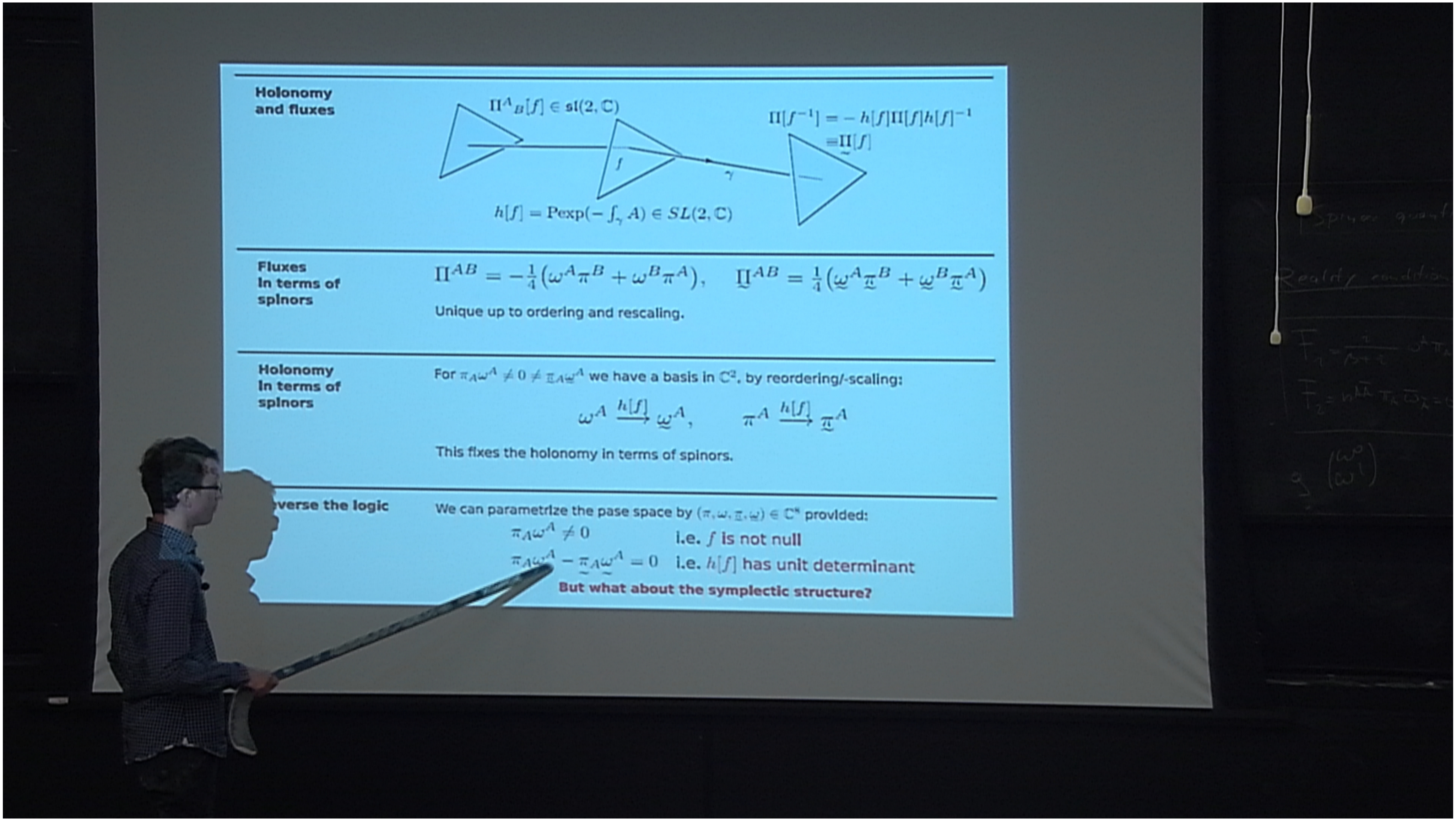

R — NS




π^AB = - Λ/4 (ω^A π^B + ω^B π^A)


Π^AB = - Λ/4 (ω^A π^B + ω^B π^A)
= - Λ/4 ((h·ω)^A (h·π)^B + ...)




Holonomy and fluxes
$\Pi^A{}_B[f] \in \mathfrak{sl}(2,\mathbb{C})$
$\Pi[f^{-1}] = -h[f]\Pi[f]h[f]^{-1}$
$=\underset{\sim}{\Pi}[f]$
f
γ
$h[f] = \mathrm{Pexp}(-\int_\gamma A) \in SL(2,\mathbb{C})$
Fluxes in terms of spinors
$\Pi^{AB} = -\frac{1}{4}(\omega^A\pi^B + \omega^B\pi^A), \quad \underset{\sim}{\Pi}^{AB} = \frac{1}{4}(\underset{\sim}{\omega}^A\underset{\sim}{\pi}^B + \underset{\sim}{\omega}^B\underset{\sim}{\pi}^A)$
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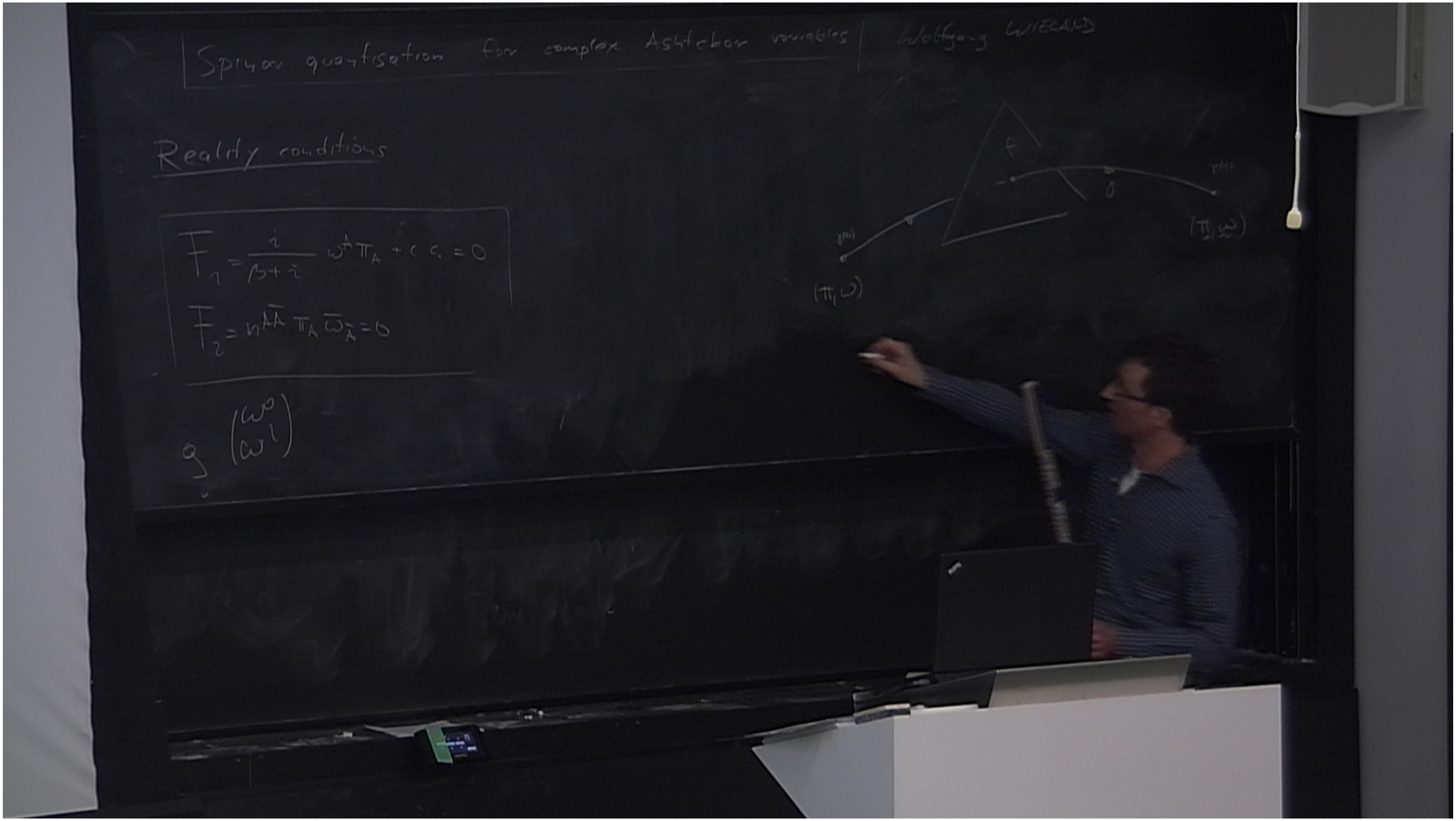
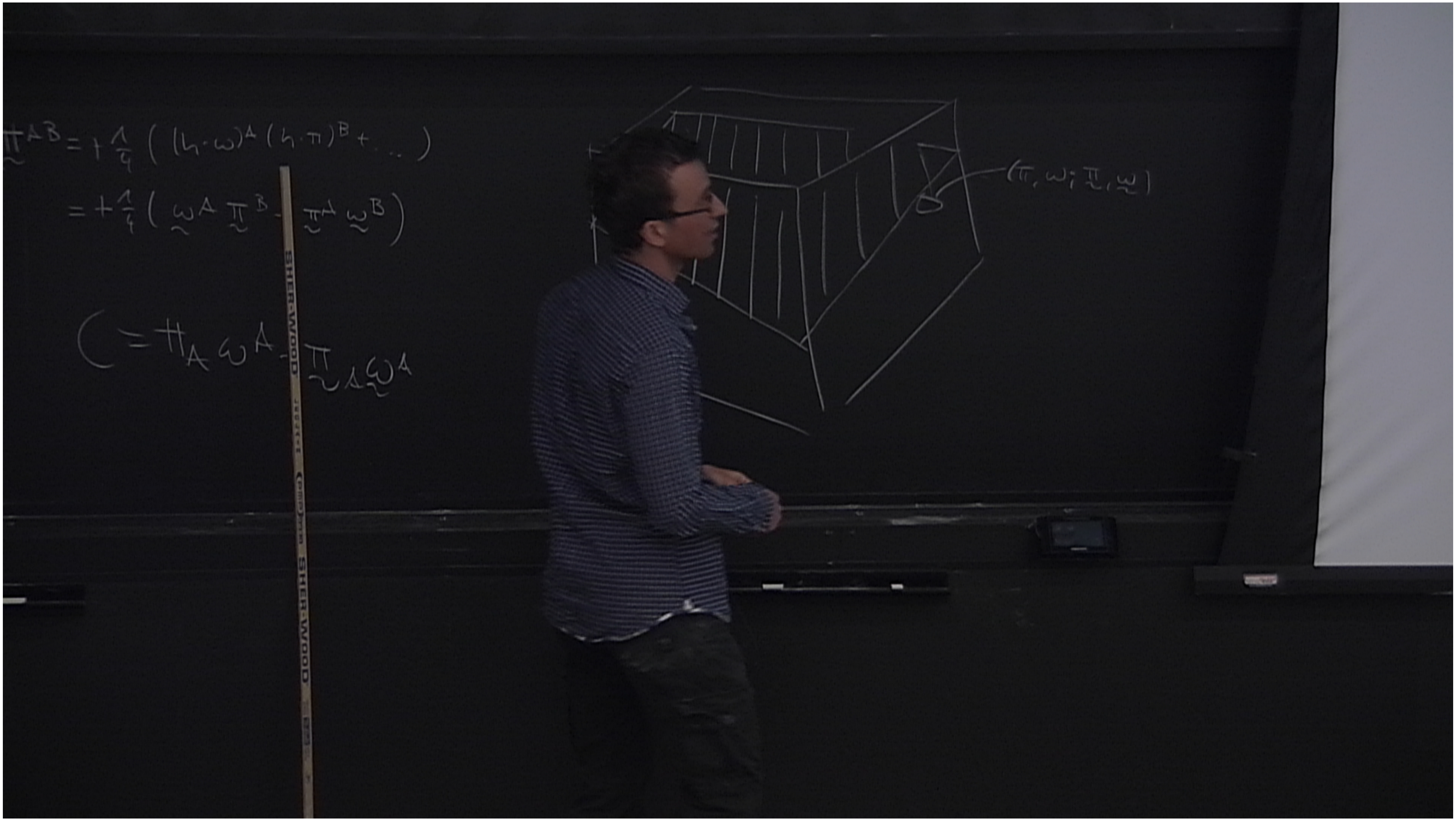
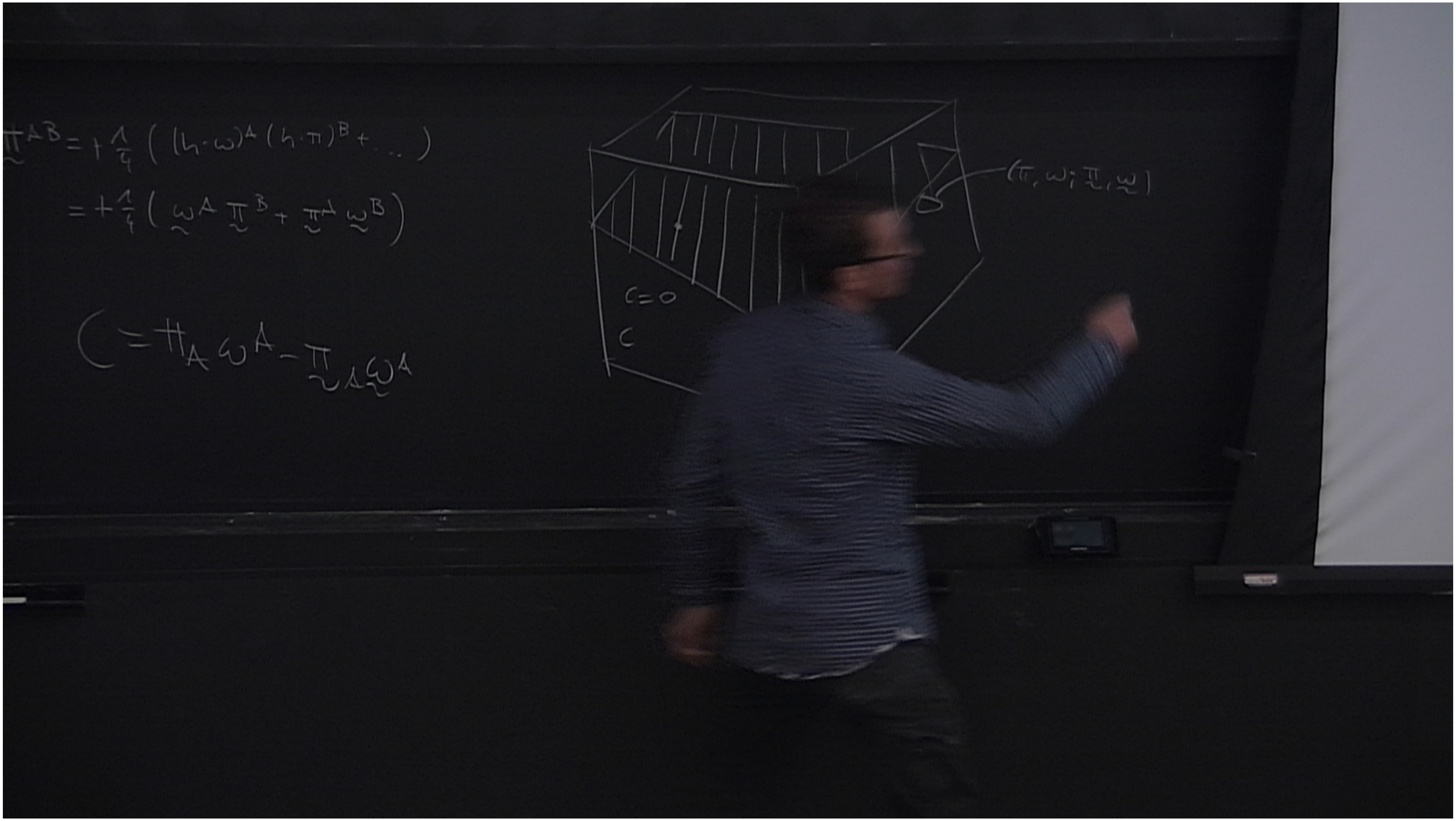
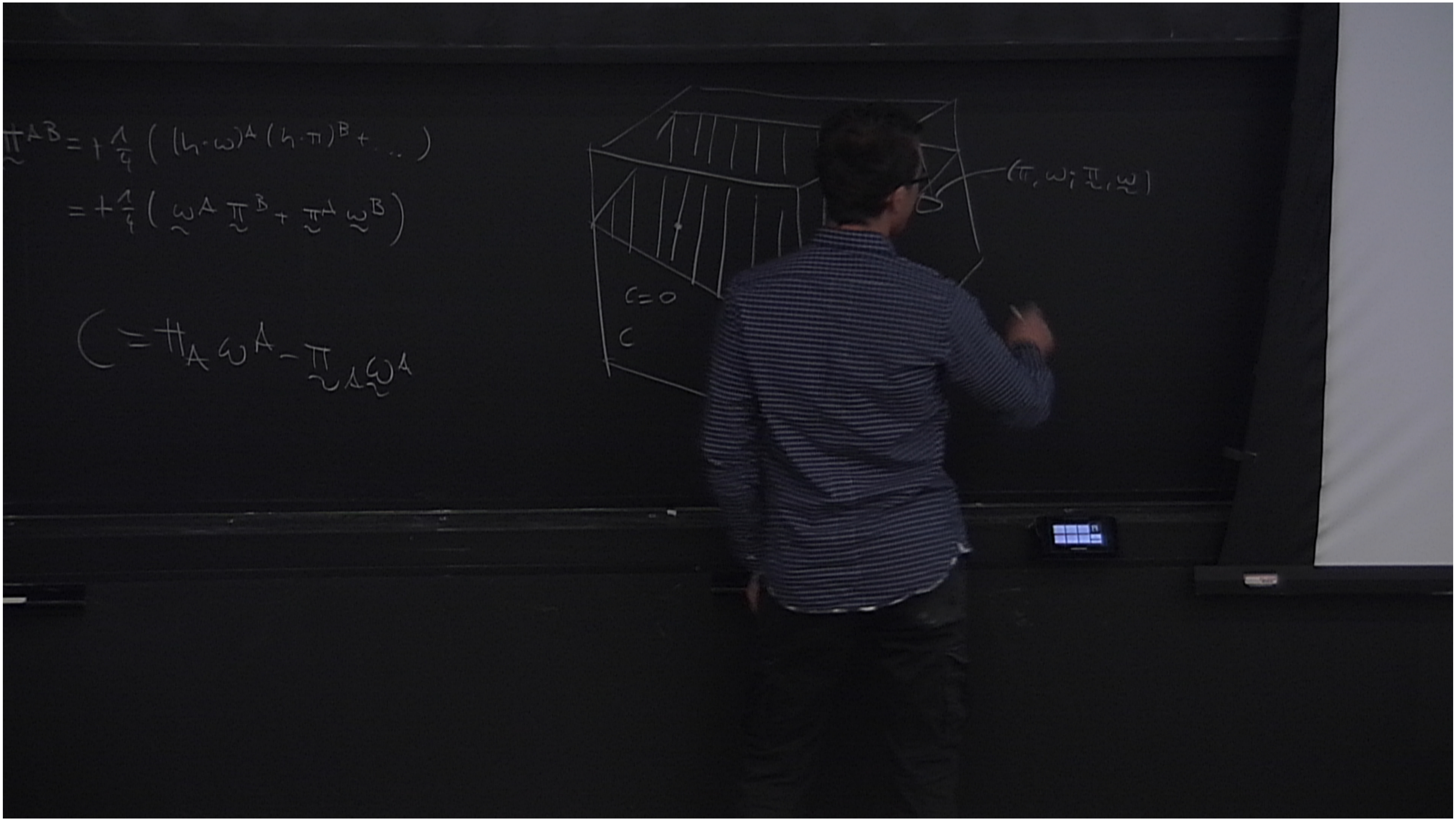
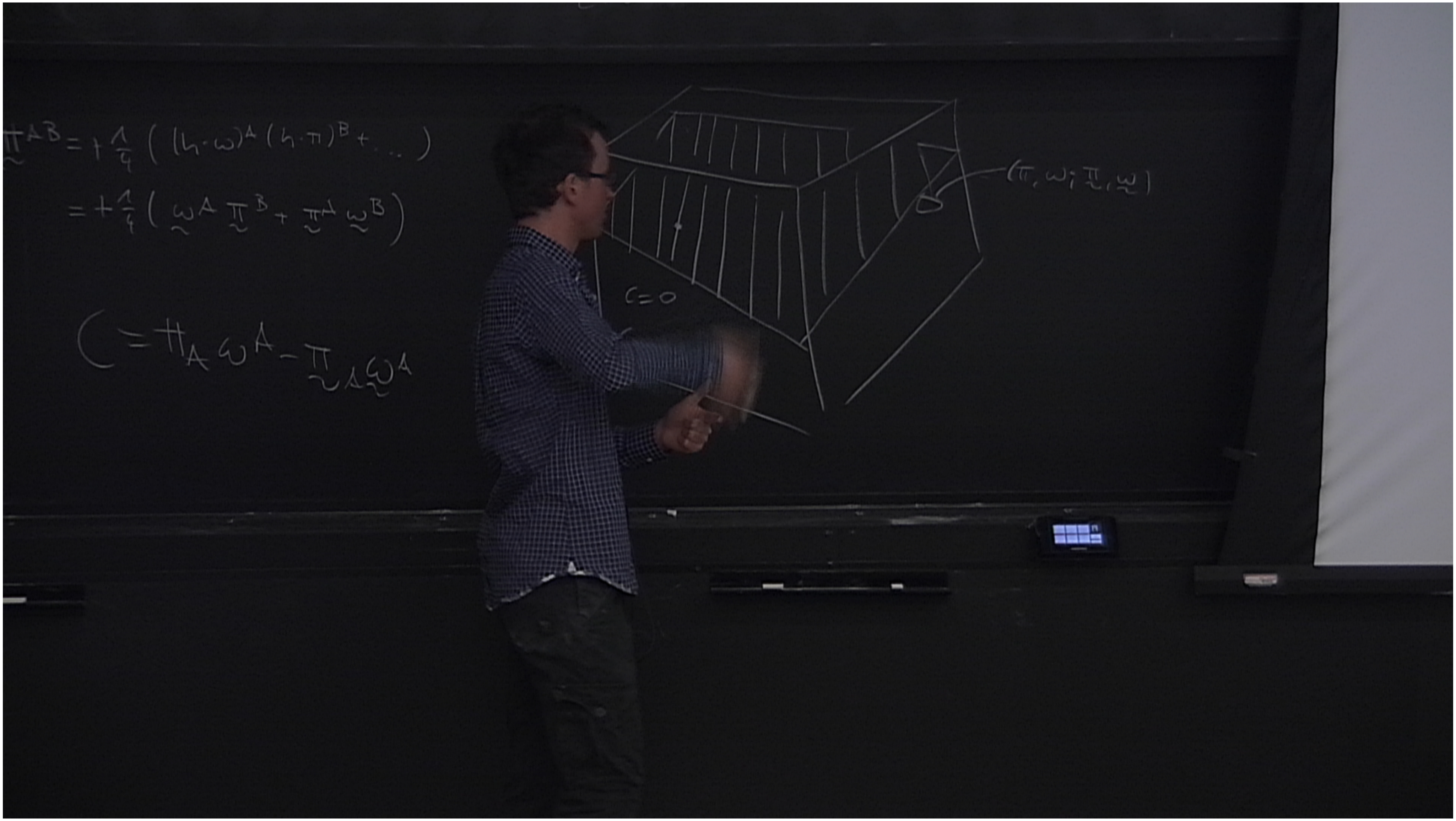

Spinor quantisation for complex Ashtekar variables
Wolfgang WIELAND
Reality conditions
$F_1 = \frac{i}{\beta + i} \omega^A \pi_A + c.c. = 0$
$F_2 = n^{A\bar{A}} \pi_A \bar{\omega}_{\bar{A}} = 0$
(π, ω)
(π, ω)

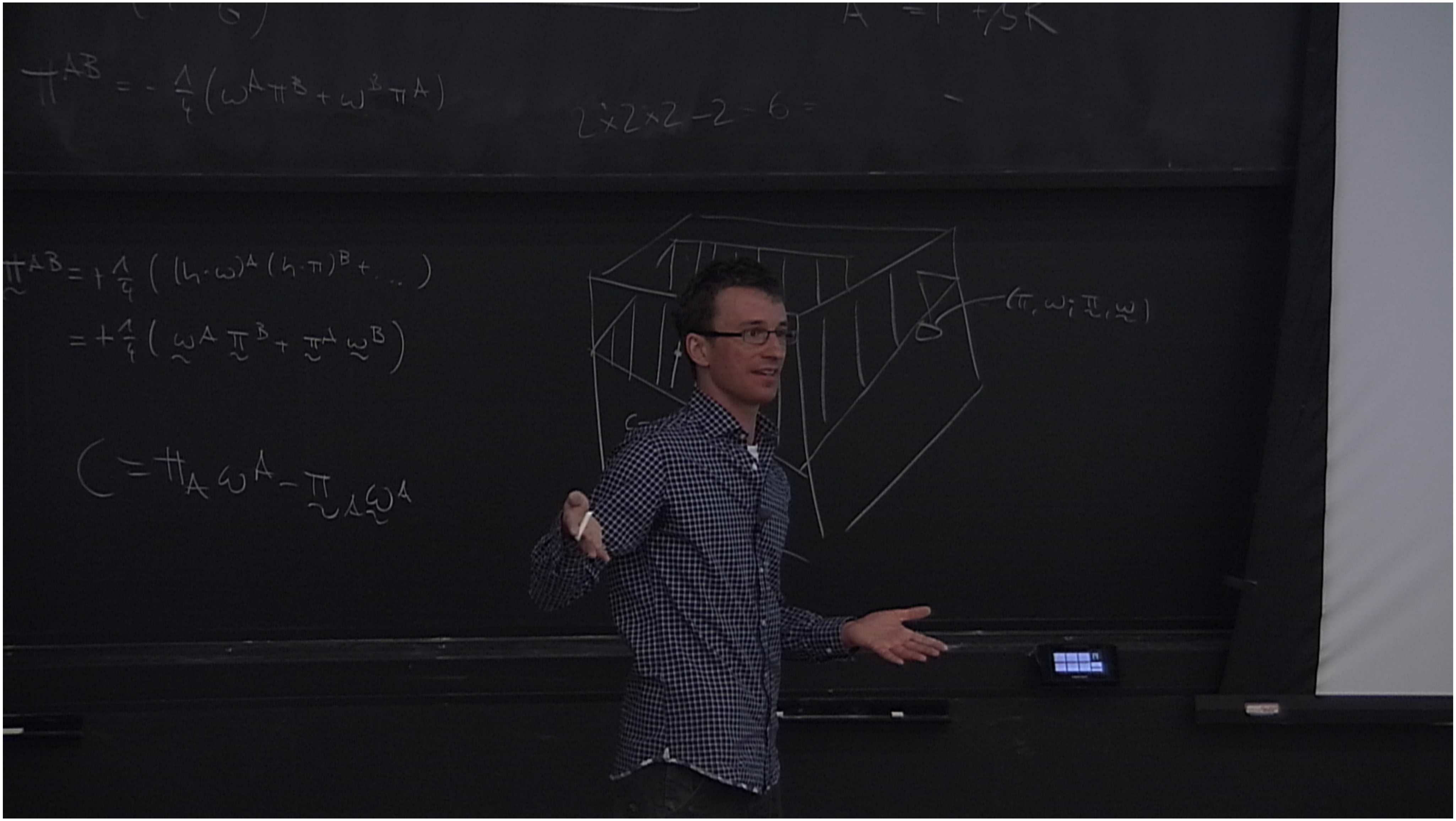

= + 1/4 (ω^A π^B + π^A ω^B)
SHER-WOOD

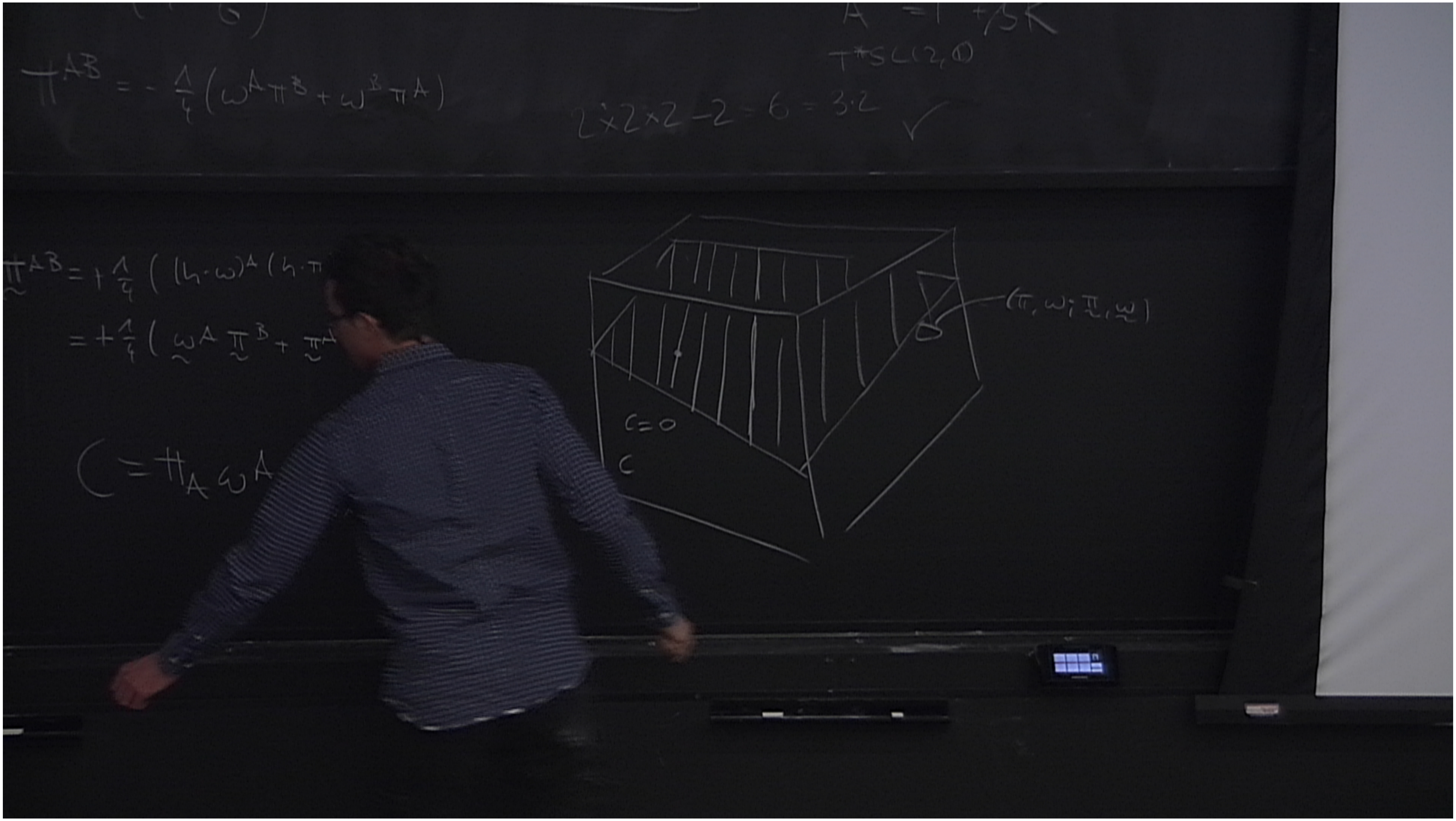

(π, ω; π, ω)


π^AB = + Λ/4 ((h·ω)^A (h·π)^B + . . .)
= + Λ/4 (ω^A π^B + π^A ω^B)
C = π_A ω^A − π_A ω^A
C=0
C
(π, ω; π, ω)




C=0


2×2×2 −2 −6 =
(π, ω; π, ω)



Π^AB = − Λ/4 (ω^A π^B + ω^B π^A)
T*SL(2,ℂ)
2×2×2 − 2 = 6 = 3·2
(π, ω; π, ω)
C=0
C = π_A ω^A

Symplectic reduction

1. The symplectic structure of the holonomy flux algebra is recovered on the constraint hypersurface $C = \pi_A \omega^A - \underset{\sim}{\pi}_A \underset{\sim}{\omega}^A = 0$.
2. The constraint $C = 0$ generates the complex scaling transformations leaving flux and holonomy unchanged.
3. Performing a symplectic reduction the original phase space is recovered. Already plausible from counting $3 \times 2 = 2 \times 2 \times 2 - 2$ complex degrees of freedom.
4. The symplectic structure simplifies. In the holonomy-flux algebra momenta don't commute, here they do: $\{\pi_A, \pi_B\} = 0$.

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Reality conditions in terms of spinors

In terms of spinorial variables:

$$\frac{\beta}{\beta+\mathrm{i}}(\omega_A\pi_B+\omega_B\pi_A)\bar{\epsilon}_{\bar{A}\bar{B}}n^{B\bar{B}}+\mathrm{cc.}=0 \tag{17}$$

- This equations has two free spinor indices.
- But the pair $\omega^A, \delta^{A\bar{A}}\bar{\omega}_{\bar{A}}$ is (unless $\omega=0$) a complete basis in $\mathbb{C}^2$.

Contraction with this basis elements reveals the following two constraints:

$$F_1=\frac{\mathrm{i}}{\beta+\mathrm{i}}\omega^A\pi_A+\mathrm{cc.}=0 \tag{18a}$$

$$F_2=n^{A\bar{A}}\pi_A\bar{\omega}_{\bar{A}}=0 \tag{18b}$$

Notice that F_1 is real but F_2 is complex.

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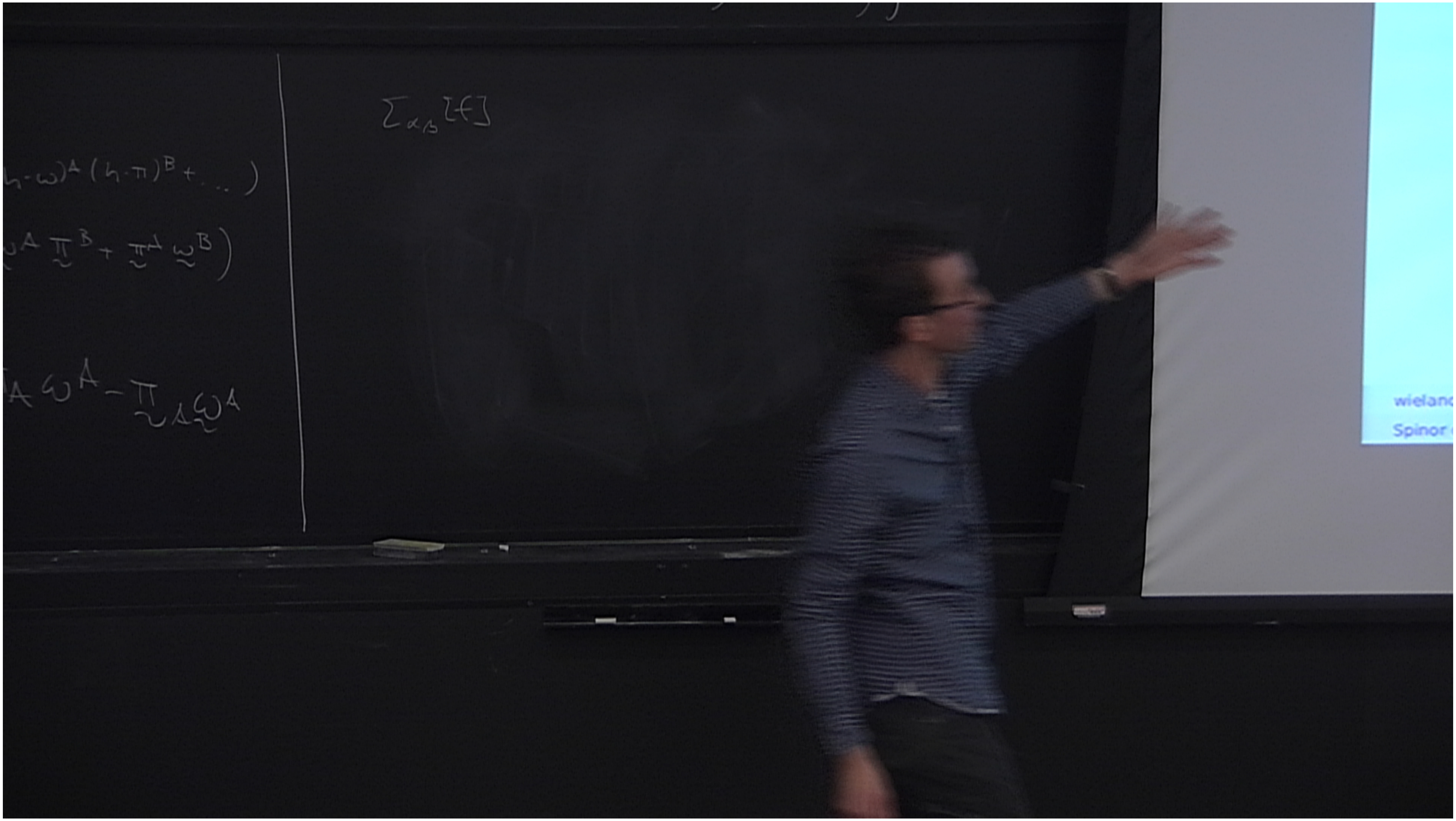
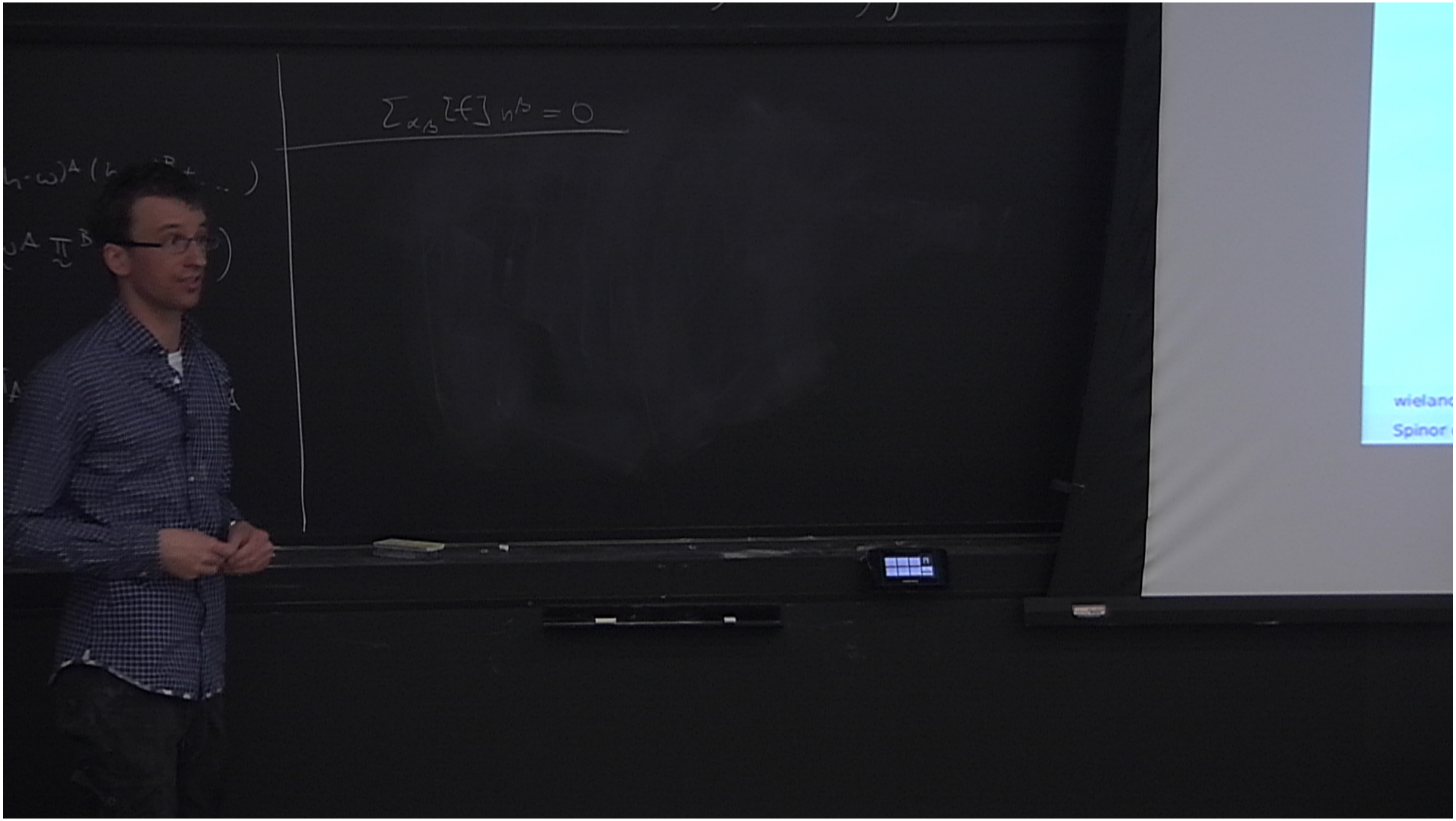
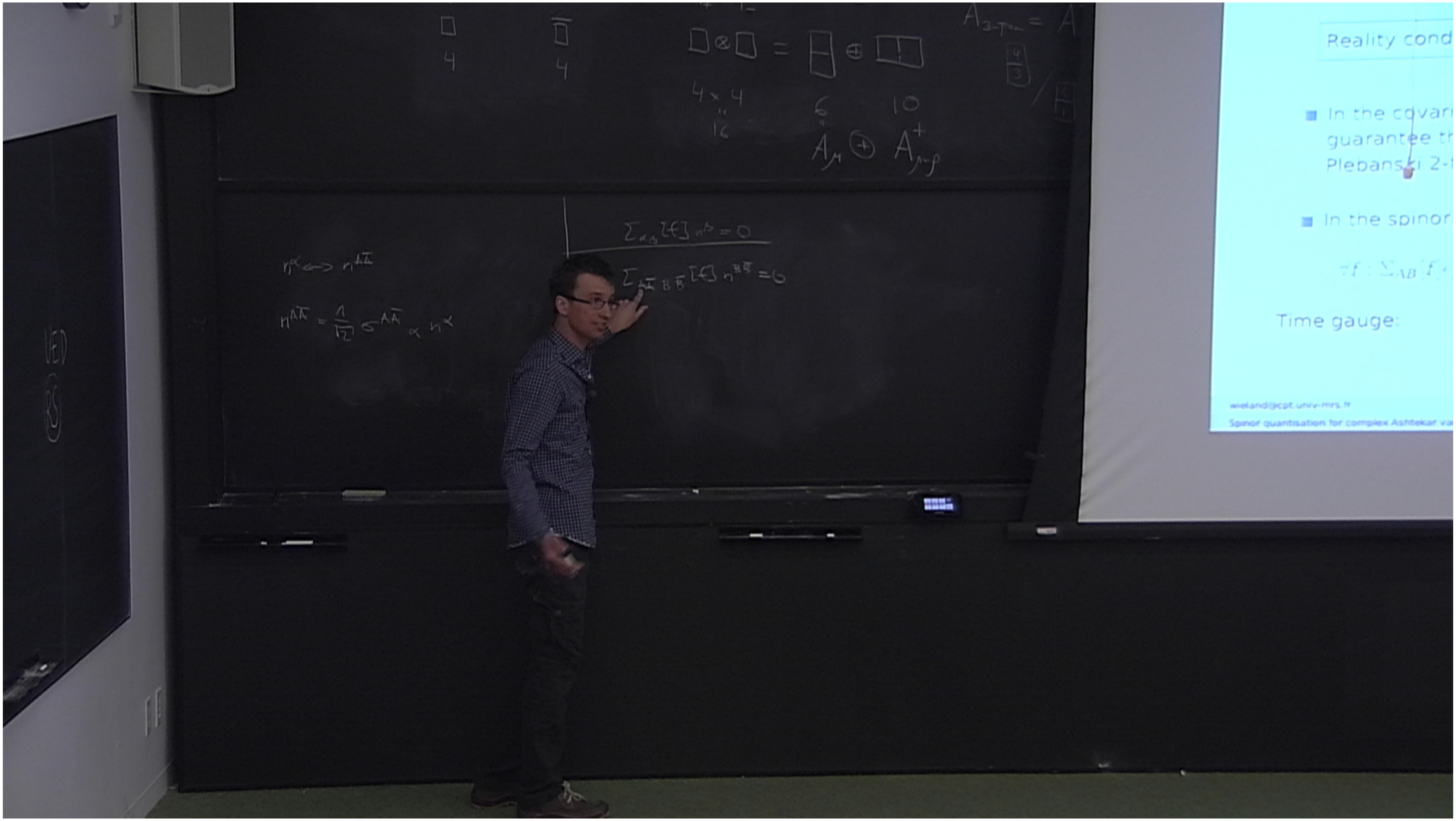



$\Sigma_{\alpha\beta}[f]\, n^{\beta} = 0$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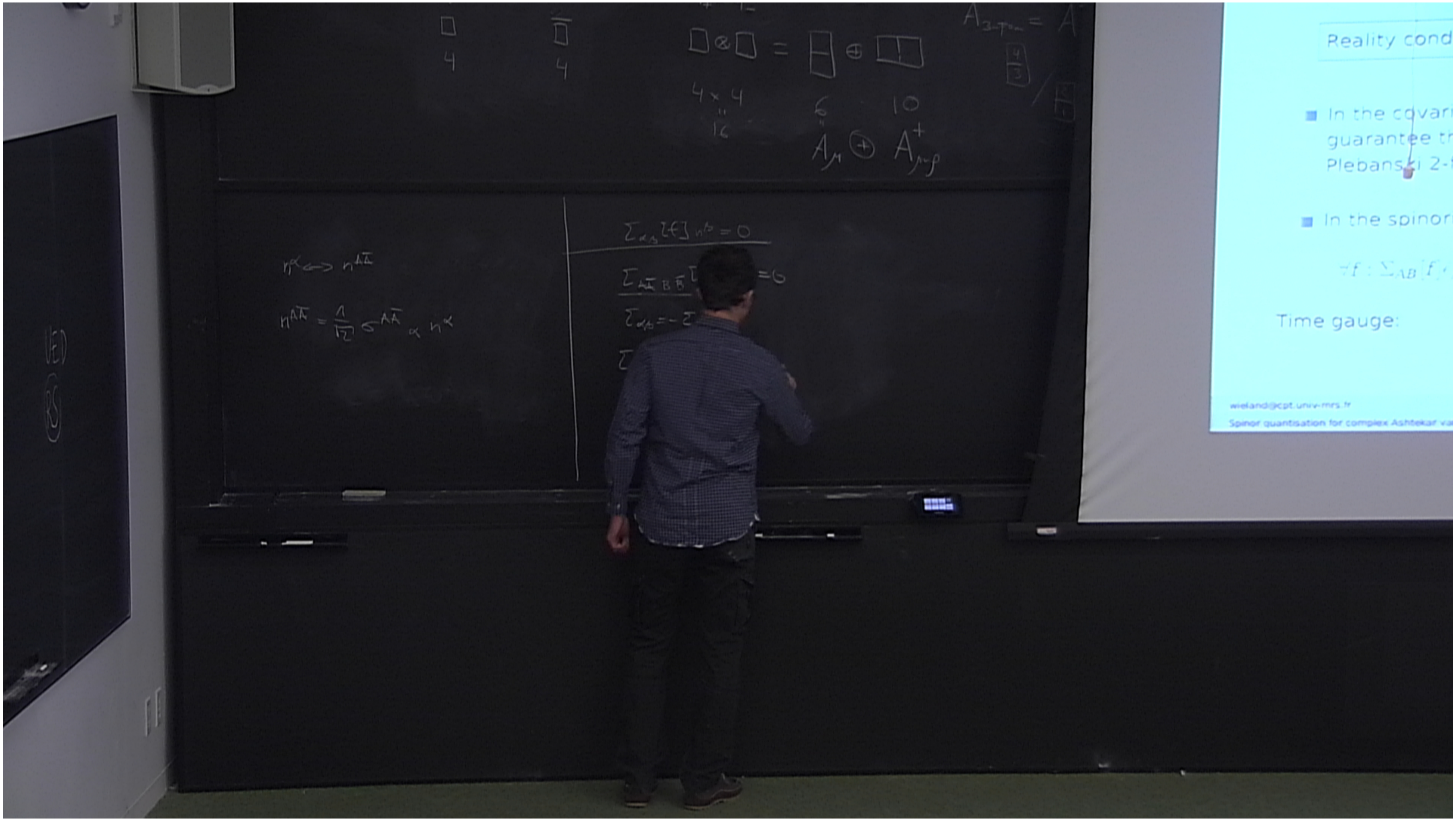
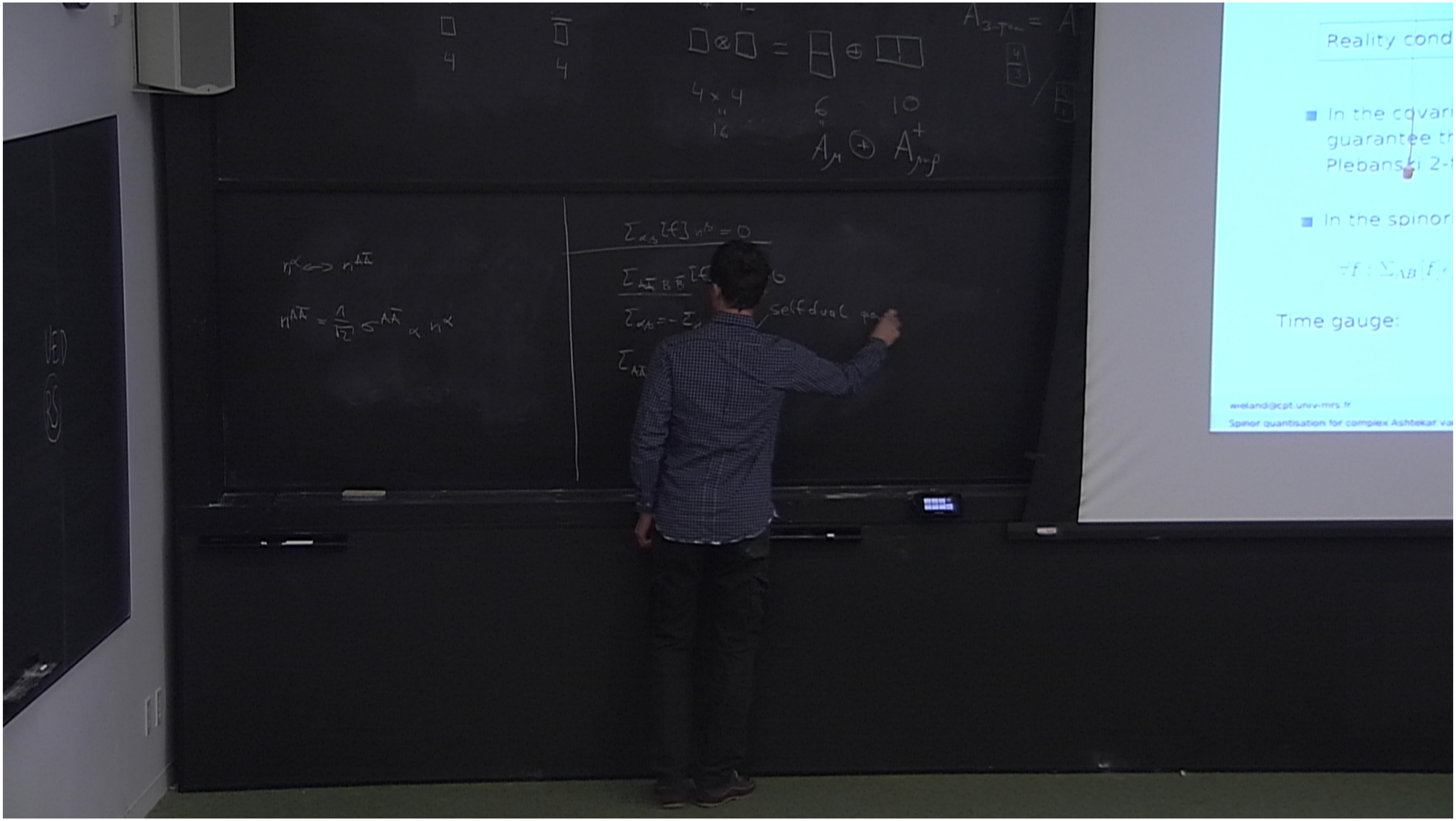
Spinor quantisation for complex Ashtekar va


Time gauge: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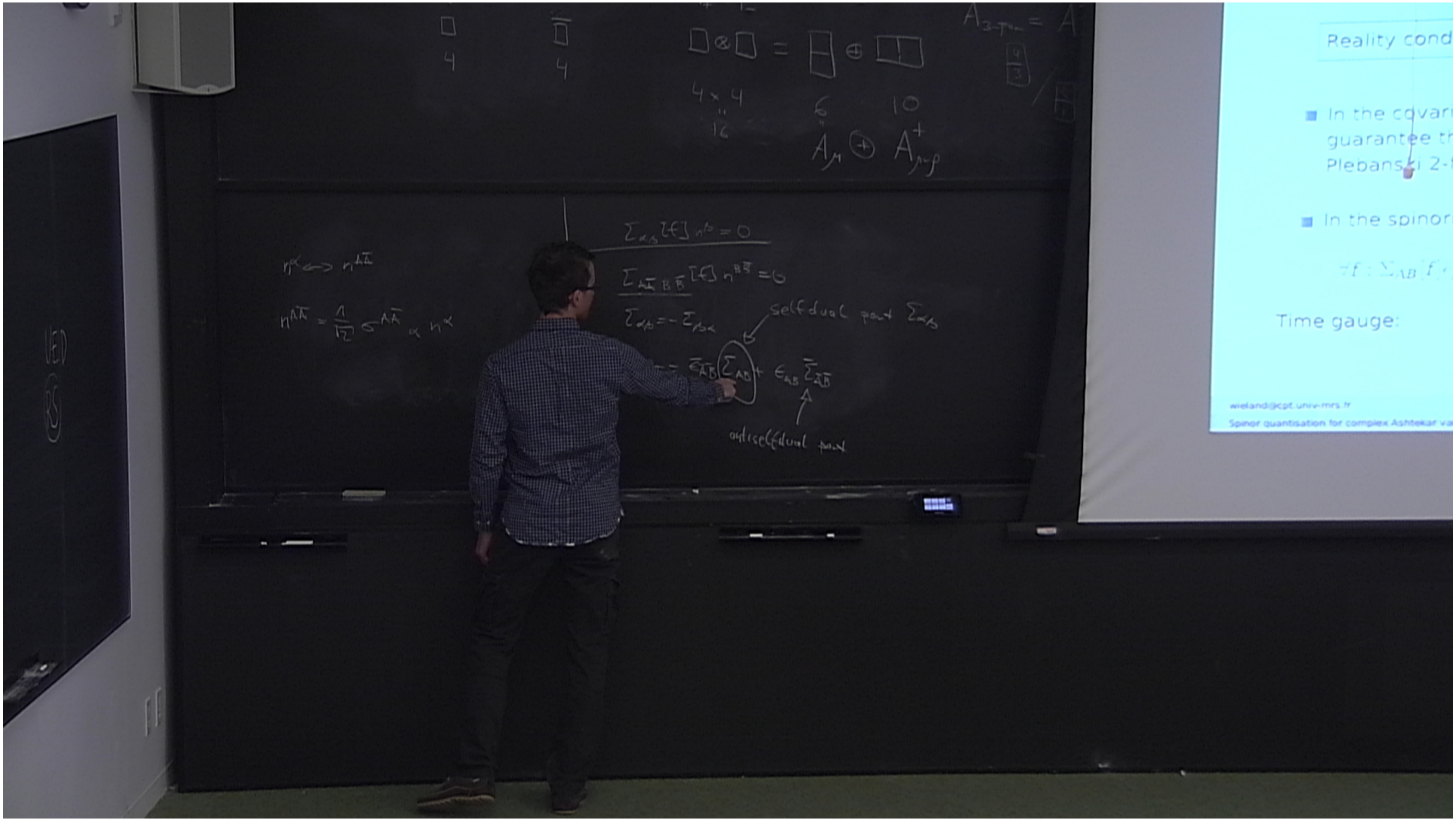
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Spinor quantisation for complex Ashtekar va
selfdual part
antiselfdual part

Reality conditions in terms of spinors

In terms of spinorial variables:

$$\frac{\beta}{\beta+\mathrm{i}}(\omega_A\pi_B+\omega_B\pi_A)\bar{\epsilon}_{\bar{A}\bar{B}}n^{B\bar{B}}+\mathrm{cc.}=0 \tag{17}$$

- This equations has two free spinor indices.
- But the pair $\omega^A, \delta^{A\bar{A}}\bar{\omega}_{\bar{A}}$ is (unless $\omega=0$) a complete basis in $\mathbb{C}^2$.

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Notice that F_1 is real but F_2 is complex.

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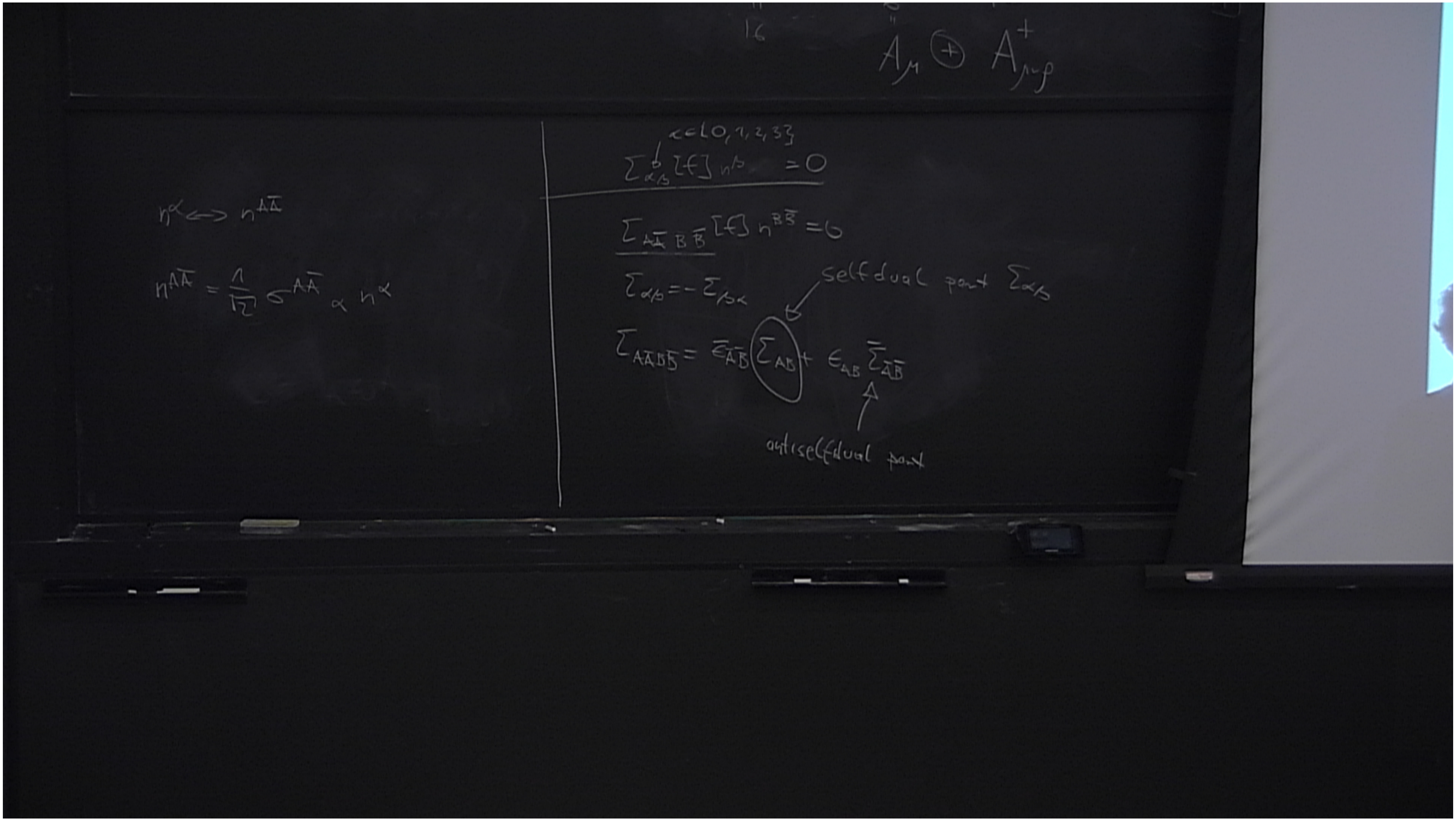
16
A_M ⊕ A^+_{M-P}
α ∈ {0,1,2,3}
Σ^b_{αβ} [f] n^β = 0
n^α <-> n^{AĀ}
Σ_{AĀ BB̄} [f] n^{BB̄} = 0
n^{AĀ} = 1/√2 σ^{AĀ}_α n^α
Σ_{αβ} = - Σ_{βα}
selfdual part Σ_{αβ}
Σ_{AĀBB̄} = ε̄_{ĀB̄} Σ_{AB} + ε_{AB} Σ̄_{ĀB̄}
antiselfdual part

Constraint algebra and master constraint

The corresponding constraint algebra is:

$$\{F_1, F_2\} = -\frac{2\mathrm{i}\beta}{\beta^2+1} F_2 \tag{19a}$$

$$\{F_1, \bar{F}_2\} = +\frac{2\mathrm{i}\beta}{\beta^2+1} \bar{F}_2 \tag{19b}$$

$$\{F_2, \bar{F}_2\} = \frac{1}{2}\left(\pi_A \omega^A - \bar{\pi}_{\bar{A}} \bar{\omega}^{\bar{A}}\right) \tag{19c}$$

F_1 is of first class, but F_2 is second class. Define the master constraint:

$$\mathbf{M} = \bar{F}_2 F_2 \tag{20}$$

And observe

$$\{F_1, \mathbf{M}\} = 0 \tag{21}$$

Right hand side is identically zero!

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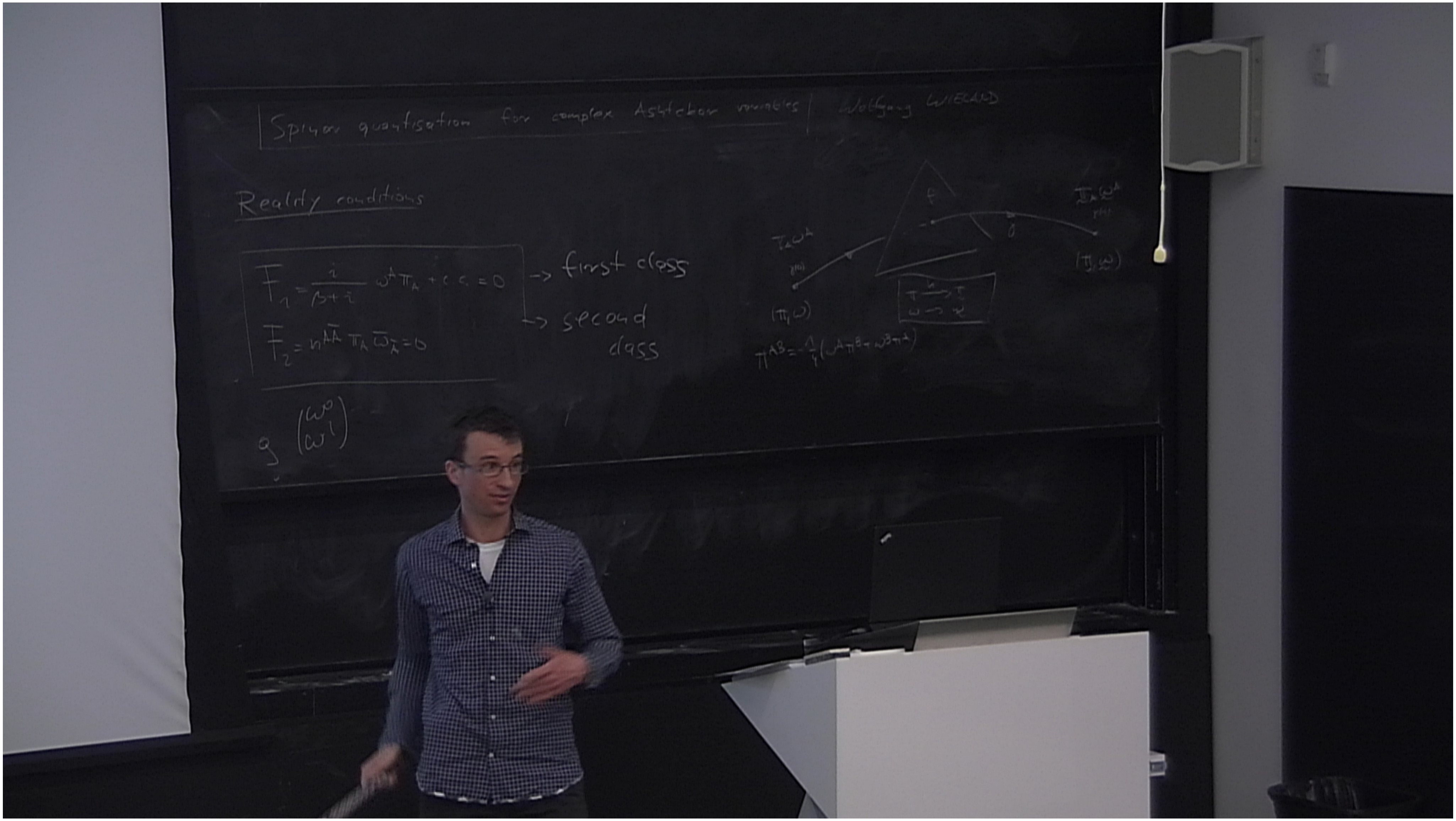
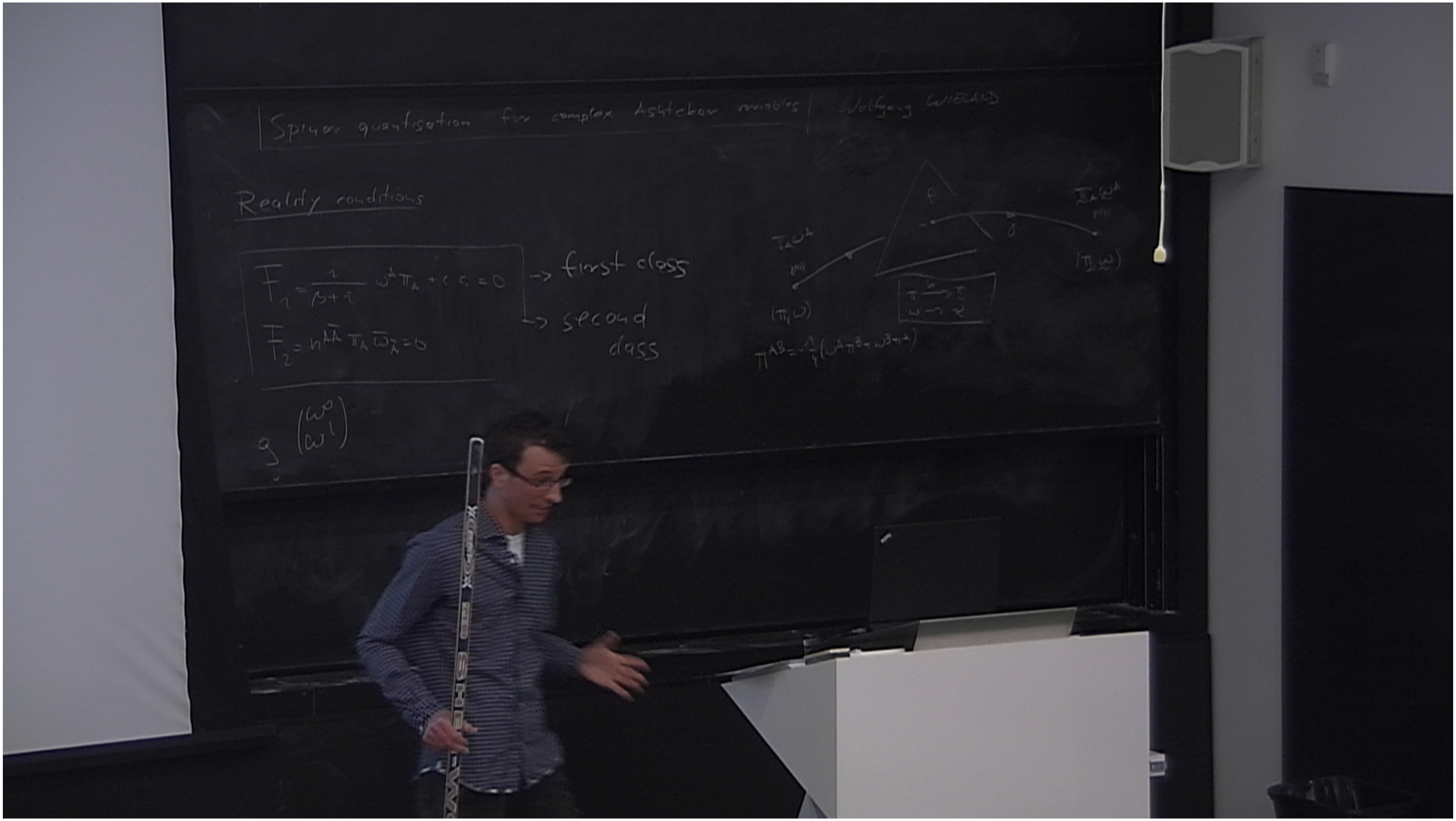

Spinor quantisation for complex Ashtekar variables | Wolfgang WIELAND
Reality conditions
first class
second class



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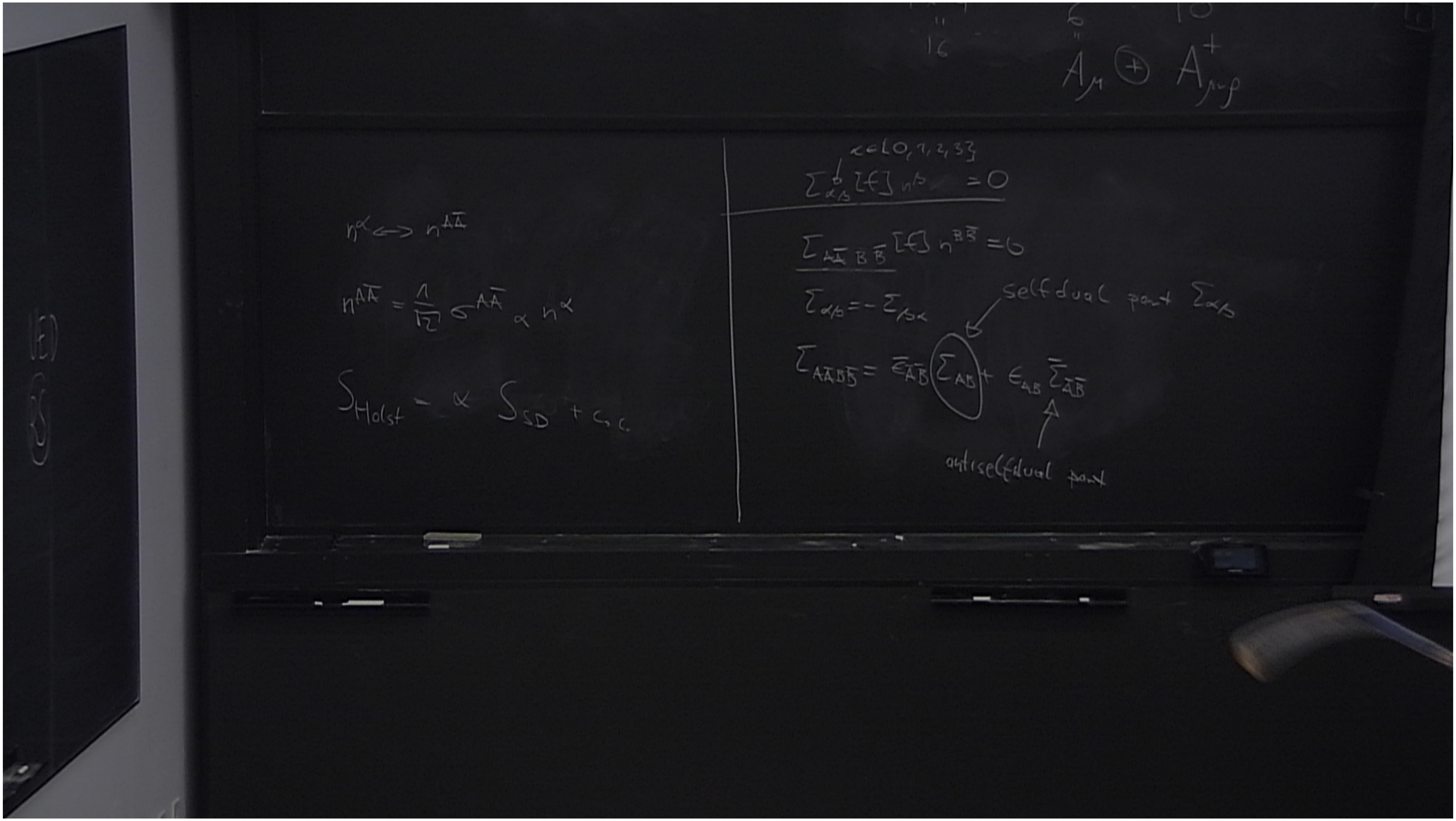
A_μ ⊕ A^+_μνρ
n^α <=> n^AĀ
n^AĀ = 1/√2 σ^AĀ_α n^α
S_Holst = α S_SD + c.c.
α ∈ {0,1,2,3}
Σ_αβ [e] n^β = 0
Σ_AĀBB̄ [e] n^BB̄ = 0
Σ_αβ = − Σ_βα
selfdual part Σ_αβ
Σ_AĀBB̄ = ε̄_ĀB̄ Σ_AB + ε_AB Σ̄_ĀB̄
antiselfdual part

Canonical quantisation of the simplicity constraints

We perform canonical quantisation, e.g.:

$$\big(\pi_A f\big)(\omega) = -\mathrm{i}\frac{\partial}{\partial\omega^A}f(\omega) \qquad (22)$$

And choose normal ordering to find:

$$\widehat{F}_1 = \frac{1}{\beta^2+1}\Big[(\beta-\mathrm{i})\omega^A\frac{\partial}{\partial\omega^A} - (\beta+\mathrm{i})\bar{\omega}^{\bar{A}}\frac{\partial}{\partial\bar{\omega}^{\bar{A}}} - 2\mathrm{i}\Big] \qquad (23a)$$

$$\widehat{F}_2 = -\mathrm{i}n^{A\bar{A}}\bar{\omega}_{\bar{A}}\frac{\partial}{\partial\omega^A} \qquad (23b)$$

$$\widehat{\mathbf{M}} = \widehat{F}_2^\dagger\widehat{F}_2 = \frac{1}{4}\Big[\omega^A\frac{\partial}{\partial\omega^A}\frac{\partial}{\partial\bar{\omega}^{\bar{A}}}\bar{\omega}^{\bar{A}} - \big(\widehat{L}^2 - \widehat{K}^2\big) + 2\widehat{L}^2\Big] \qquad (23c)$$

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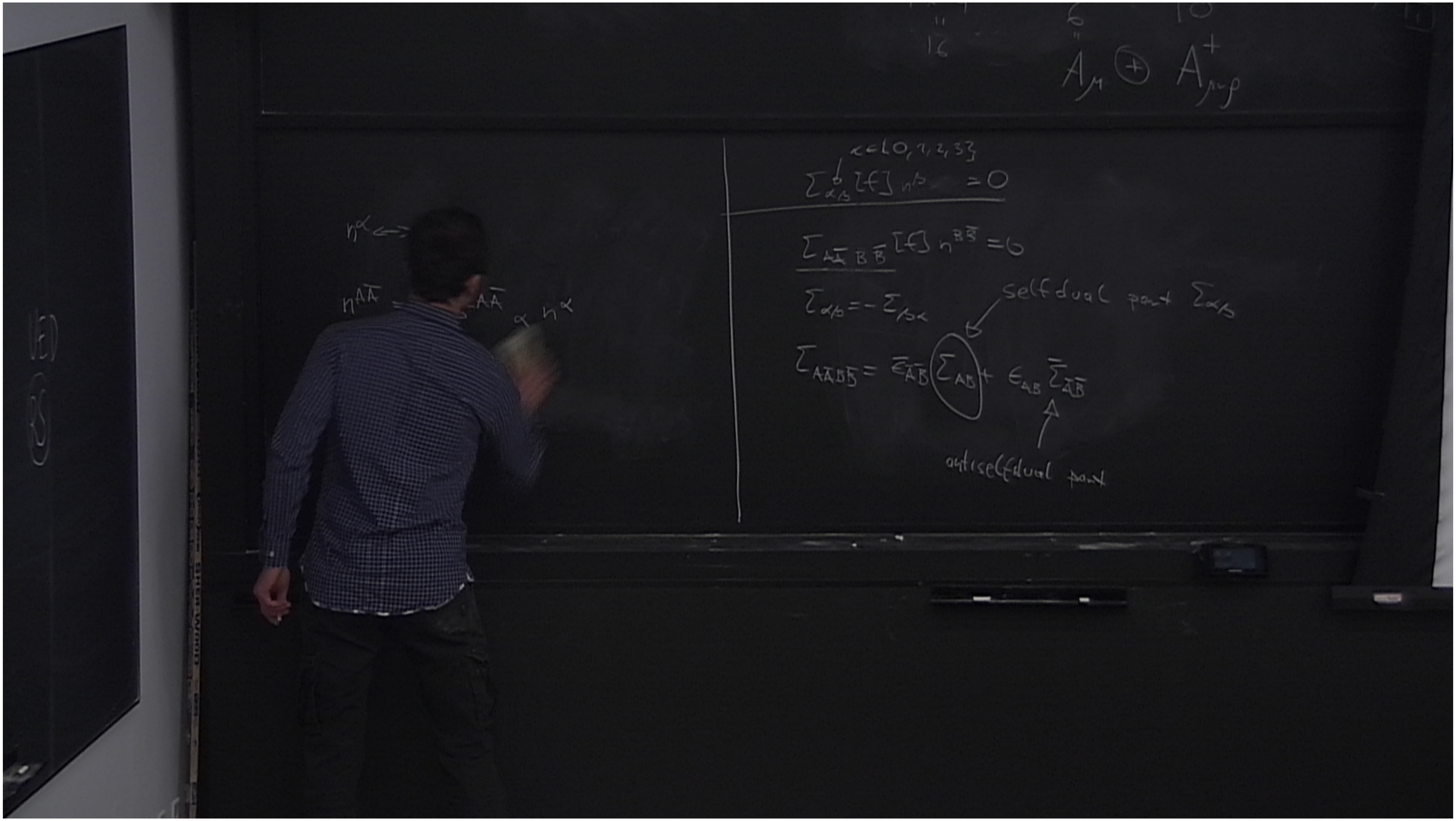
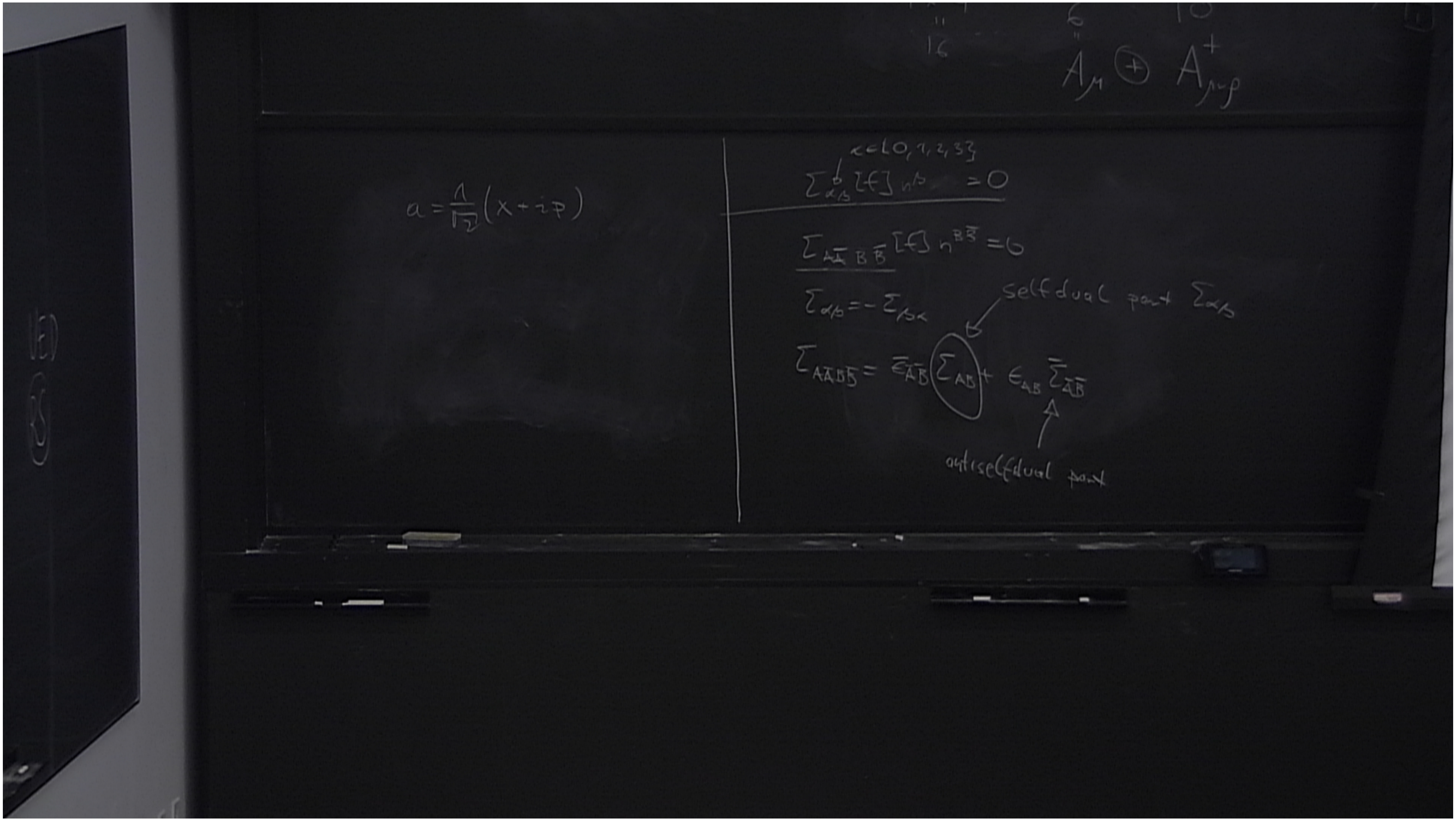

self dual part
antiselfdual part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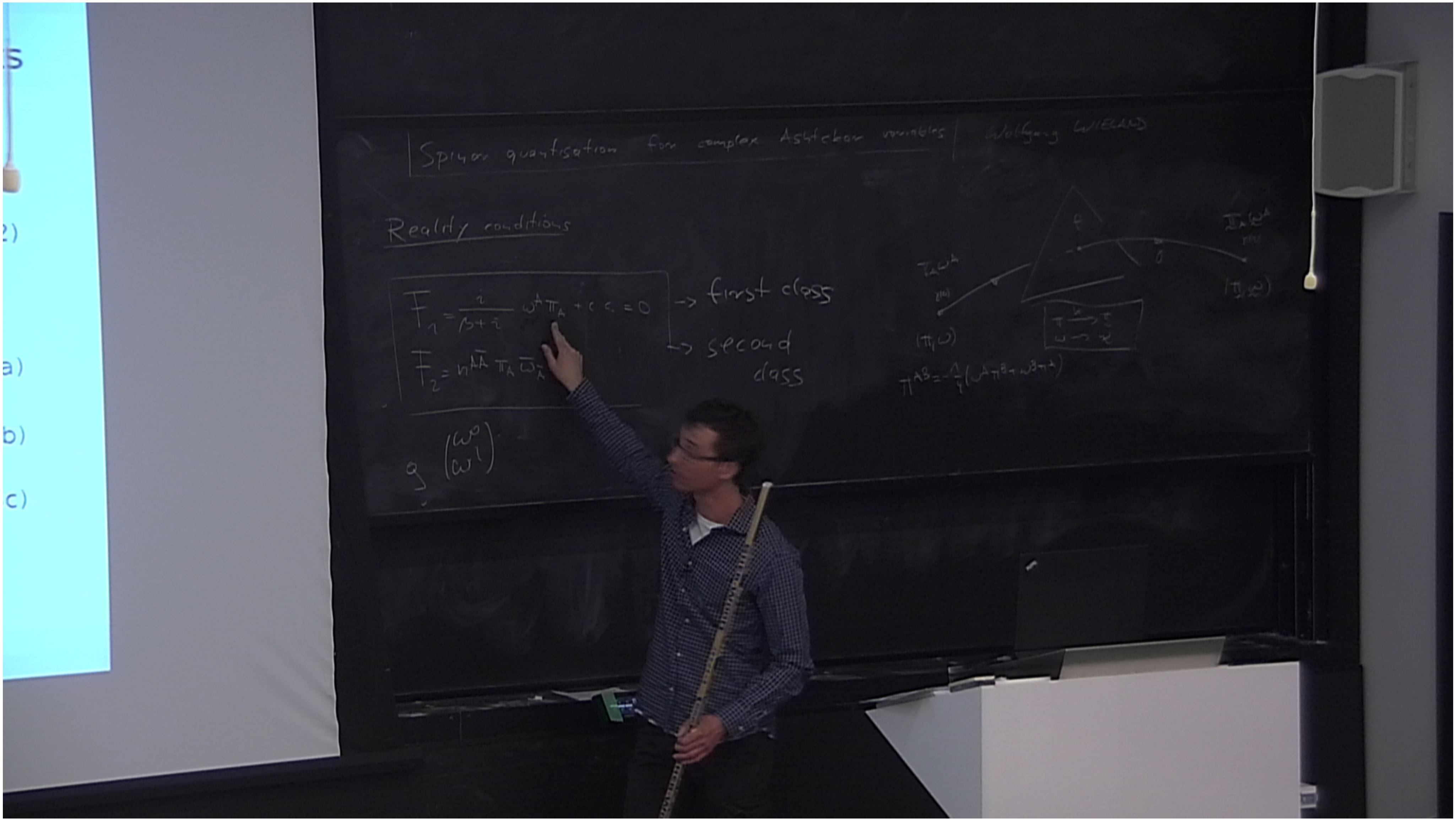
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Reality conditions
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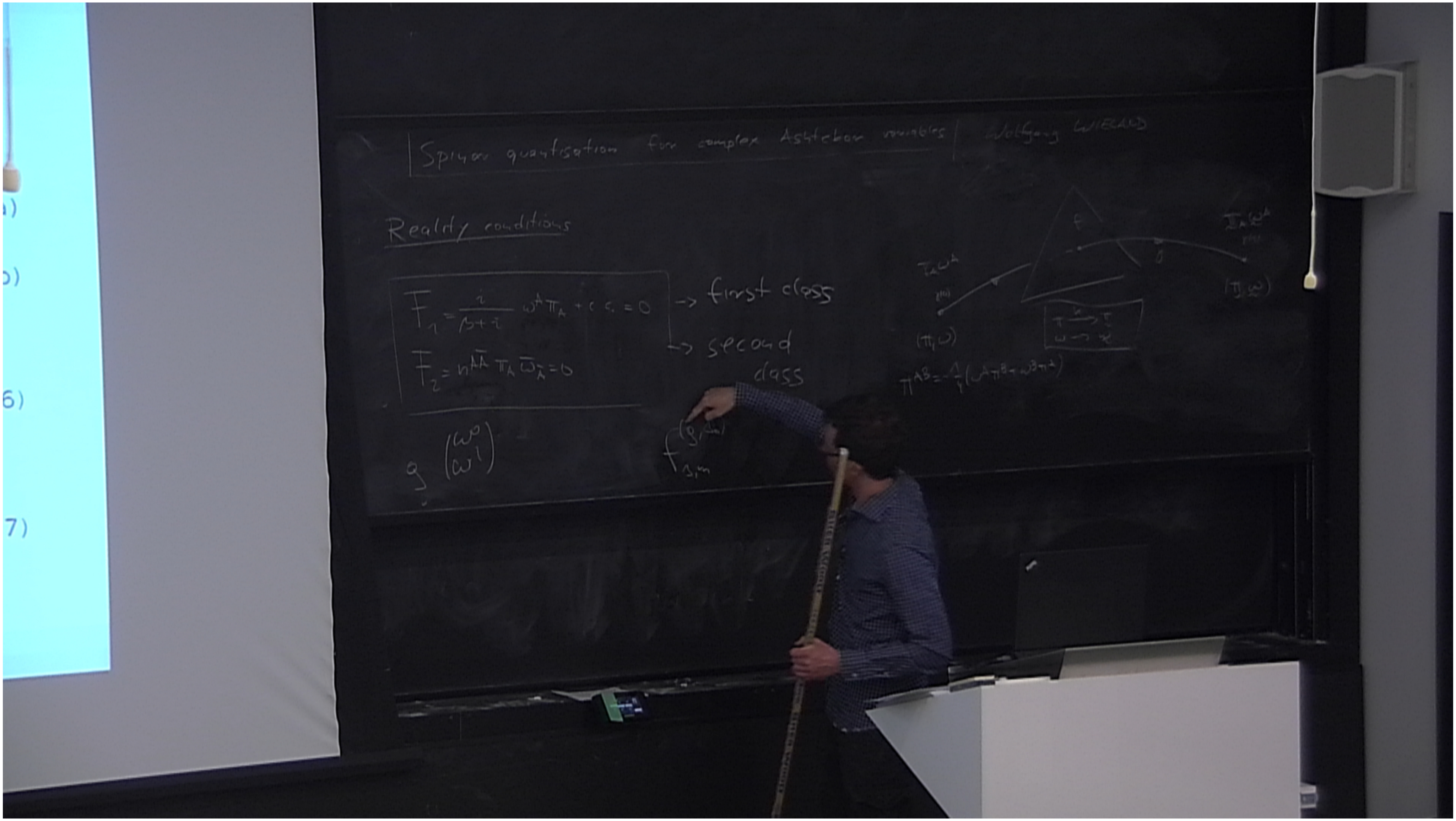
second class



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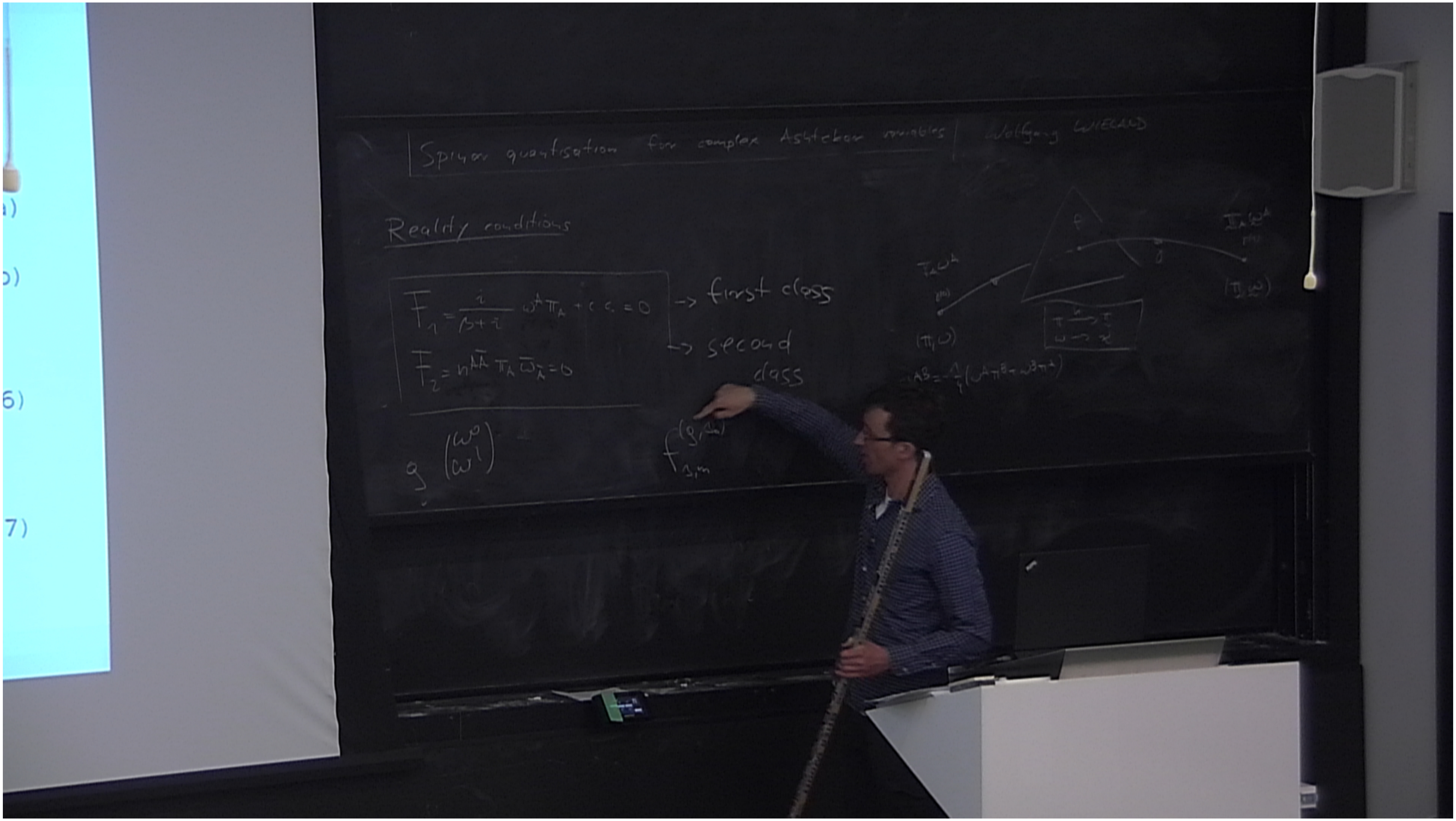
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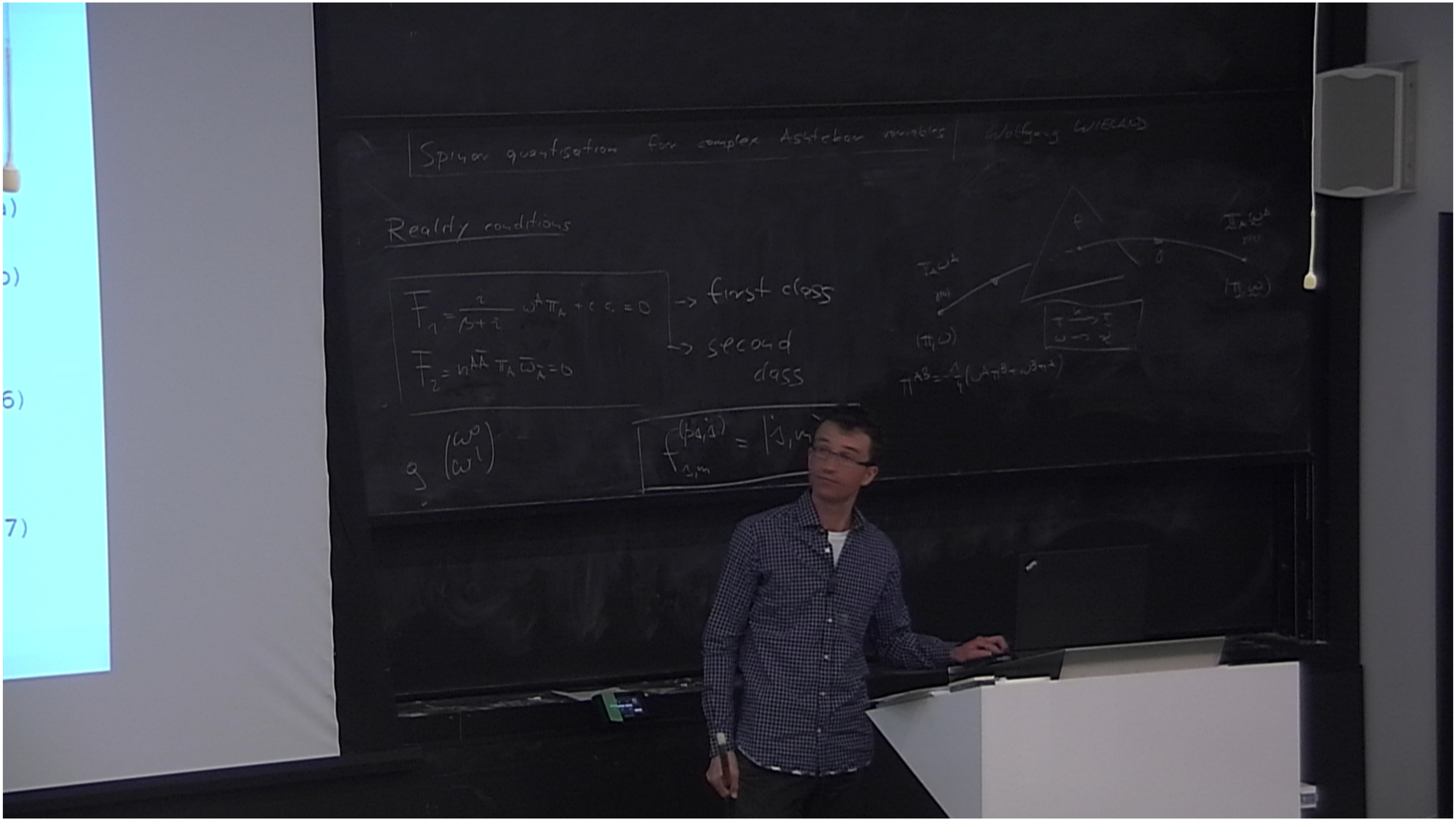
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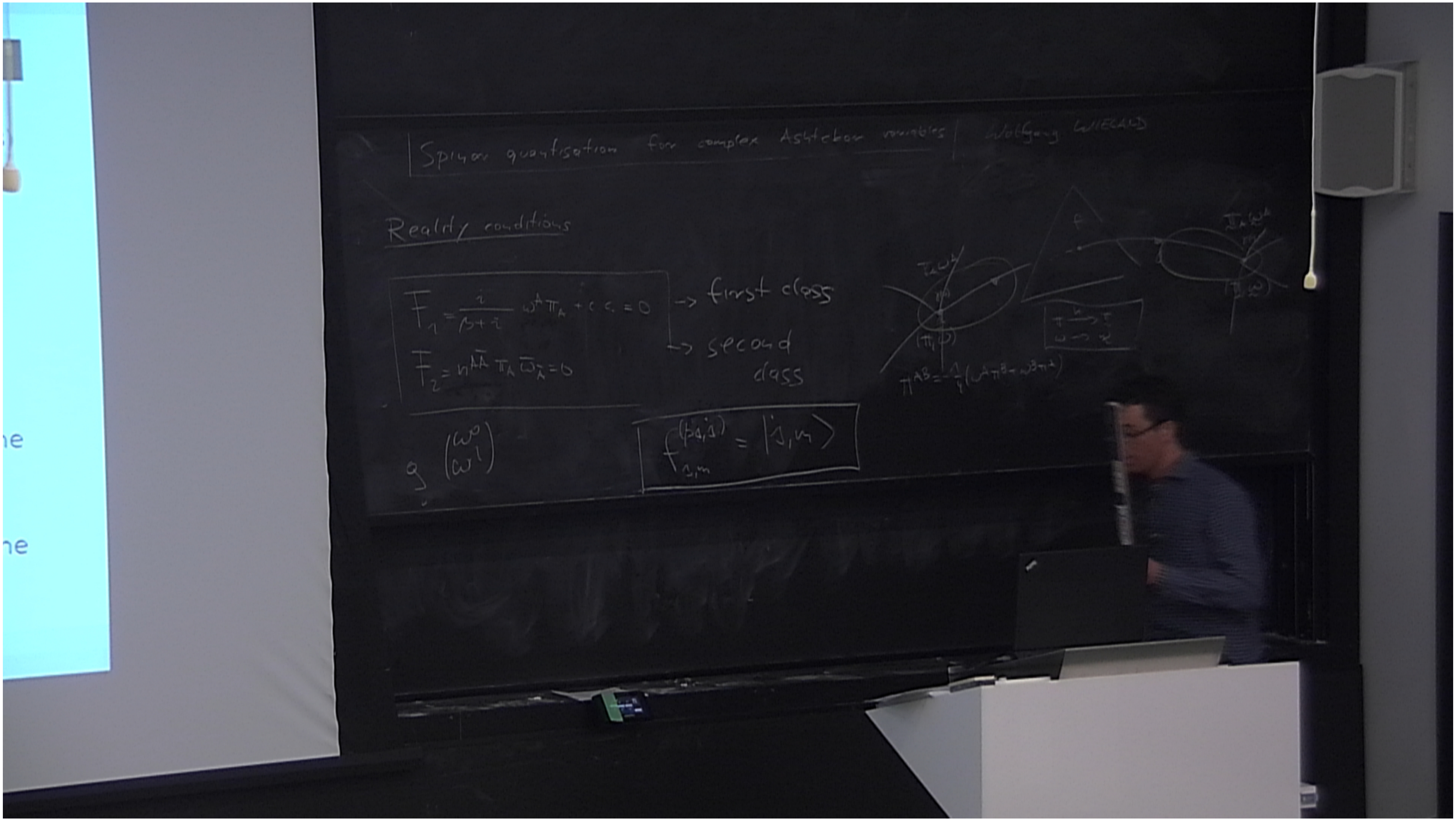
Dupuis–Livine map

$$|j, m\rangle \mapsto f_{j,m}^{(\beta j, j)}(\omega^A) \quad (28)$$

- This map sends $SU(2)$ irreducibles to $SL(2,\mathbb{C})$ unitary irreducibles.
- The resulting Hilbertspace coincides with the space of $SU(2)$ spinnetwork states.

So after all, what is the difference to the $SU(2)$ variables?

- For real variables the reduction from $SL(2,\mathbb{C})$ to $SU(2)$ is in the configuration variable $A = \Gamma + \beta K$.
- Here, the configuration variable is $\omega \in \mathbb{C}^2$, transforms covariantly under $SL(2,\mathbb{C})$, and doesn't know about $SU(2)$. The reduction is in the Dupuis–Livine states $f_{j,m}^{(\beta j, j)}$.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5. Relation to twisted geometries and $SU(2)$ spinors

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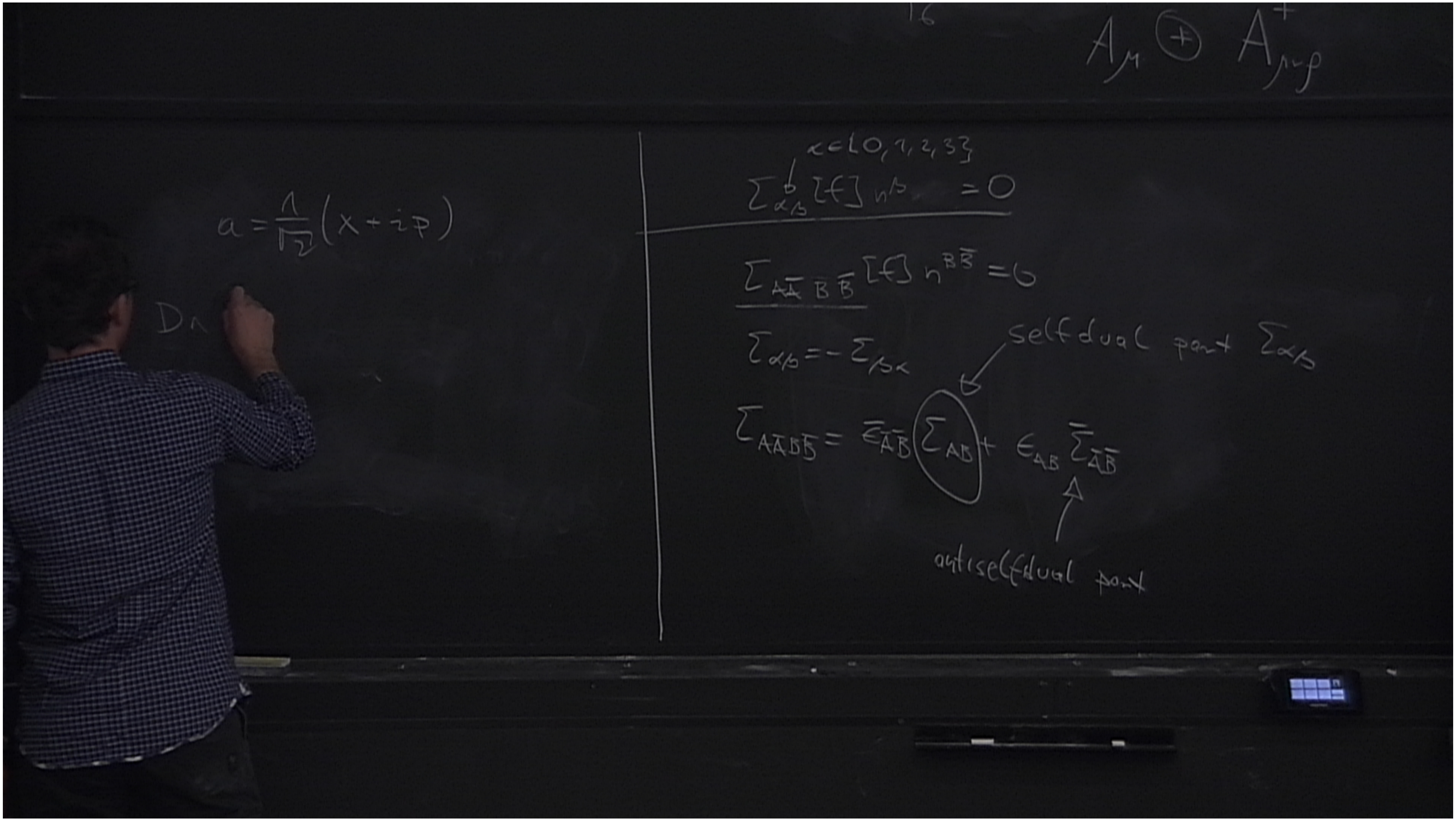
A_μ ⊕ A^+_{μνρ}
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Σ^b_{αβ}[f] n^β = 0
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Σ_{αβ} = − Σ_{βα}
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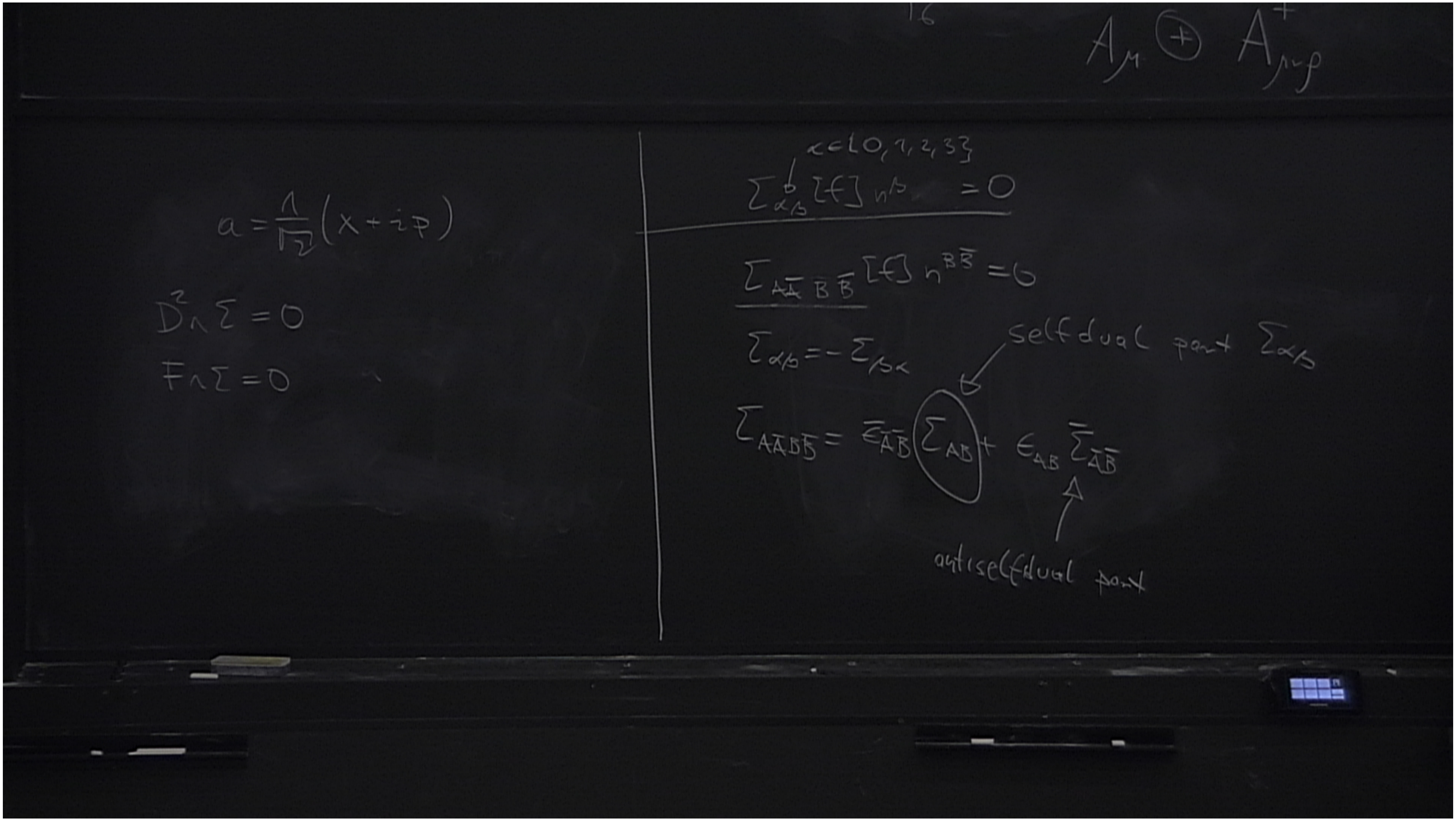
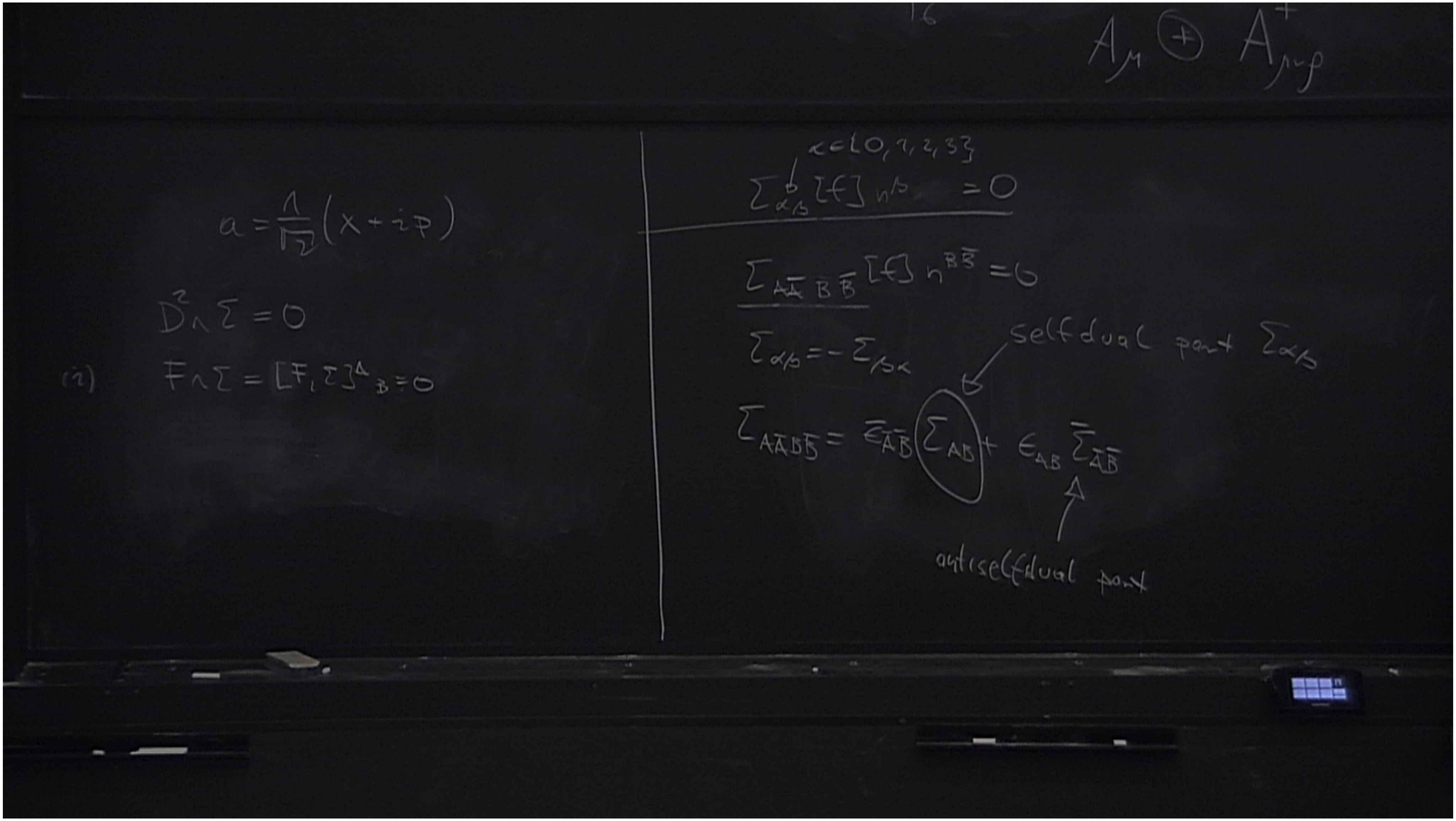


$A_\mu \oplus A^+_{\mu\nu\rho}$
$\alpha \in \{0,1,2,3\}$
$\Sigma^b_{\alpha\beta}[f]\, n^\beta = 0$
$a = \frac{1}{\sqrt{2}}(x + ip)$
$\Sigma_{A\bar{A}B\bar{B}}[f]\, n^{B\bar{B}} = 0$
$D^2 \wedge \Sigma = 0$
$F \wedge \Sigma = 0$
$\Sigma_{\alpha\beta} = -\Sigma_{\beta\alpha}$
selfdual part $\Sigma_{\alpha\beta}$
$\Sigma_{A\bar{A}B\bar{B}} = \epsilon_{\bar{A}\bar{B}}\Sigma_{AB} + \epsilon_{AB}\bar{\Sigma}_{\bar{A}\bar{B}}$
antiselfdual part



$A_\mu \oplus A^+_{\mu\nu\rho}$
$a = \frac{1}{\sqrt{2}}(x + ip)$
$D^2 \wedge \Sigma = 0$
(ii) $F \wedge \Sigma = [F, \Sigma]^A{}_B = 0$
$\alpha \in \{0,1,2,3\}$
$\Sigma^b_{\alpha\beta}[f]\, n^\beta = 0$
$\Sigma_{A\bar{A}B\bar{B}}[f]\, n^{B\bar{B}} = 0$
$\Sigma_{\alpha\beta} = -\Sigma_{\beta\alpha}$
selfdual part $\Sigma_{\alpha\beta}$
$\Sigma_{A\bar{A}B\bar{B}} = \bar{\epsilon}_{\bar{A}\bar{B}} \Sigma_{AB} + \epsilon_{AB} \bar{\Sigma}_{\bar{A}\bar{B}}$
antiselfdual part

Physical interpretation and relation to twisted geometries

Solutions of the reality conditions $F_1 = 0 = F_2$ are parametrised by a single number J, wlog.: $J \in \mathbb{R}_>$

$$\pi_A = (\beta + \mathrm{i}) \frac{J}{\|\omega\|^2} \delta_{A\bar{A}} \bar{\omega}^{\bar{A}} \tag{29}$$

Defining SU(2) spinors

$$z^A := \sqrt{2J} \frac{\omega^A}{\|\omega\|^2}, \quad \underset{\approx}{z}^A := \mathrm{e}^{\mathrm{i}\beta\xi} \sqrt{2\underset{\sim}{J}} \frac{\underset{\approx}{\omega}^A}{\|\underset{\approx}{\omega}\|^2}, \tag{30}$$

- $\xi = \ln(\|\omega\| / \|\underset{\approx}{\omega}\|)$ is the (norm of) the smeared extrinsic curvature.
- $J = \frac{1}{2}\|z\|^2$ is the area of the face.

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Recovering the phase space of twisted geometries

$$\{\bar{z}^{\bar{A}}, z^{A}\} = -\mathrm{i}\delta^{A\bar{A}} = -\{\bar{\underset{\sim}{z}}^{\bar{A}}, \underset{\sim}{z}^{A}\} \tag{31}$$

- Unlike ω and π the $SU(2)$ spinors transform nonlinearly under boosts.
- The parametrisation removes one gauge degree of freedom.
- Residual gauge symmetries are $U(1)$ transformations.
- z^A and $\underset{\sim}{z}^A$ are related by the Ashtekar–Barbero connection:

$$|\underset{\sim}{z}\rangle \approx \mathrm{Pexp}\Big(-\int_{\gamma}(\Gamma + \beta K)\Big)|z\rangle \tag{32}$$

- And parametrize the $\mathfrak{su}(2)$ fluxes:

$$\Sigma^i \tau_i = \frac{\mathrm{i}\beta}{4}\Big(|z\rangle\langle z| - |z][z|\Big) \tag{33}$$

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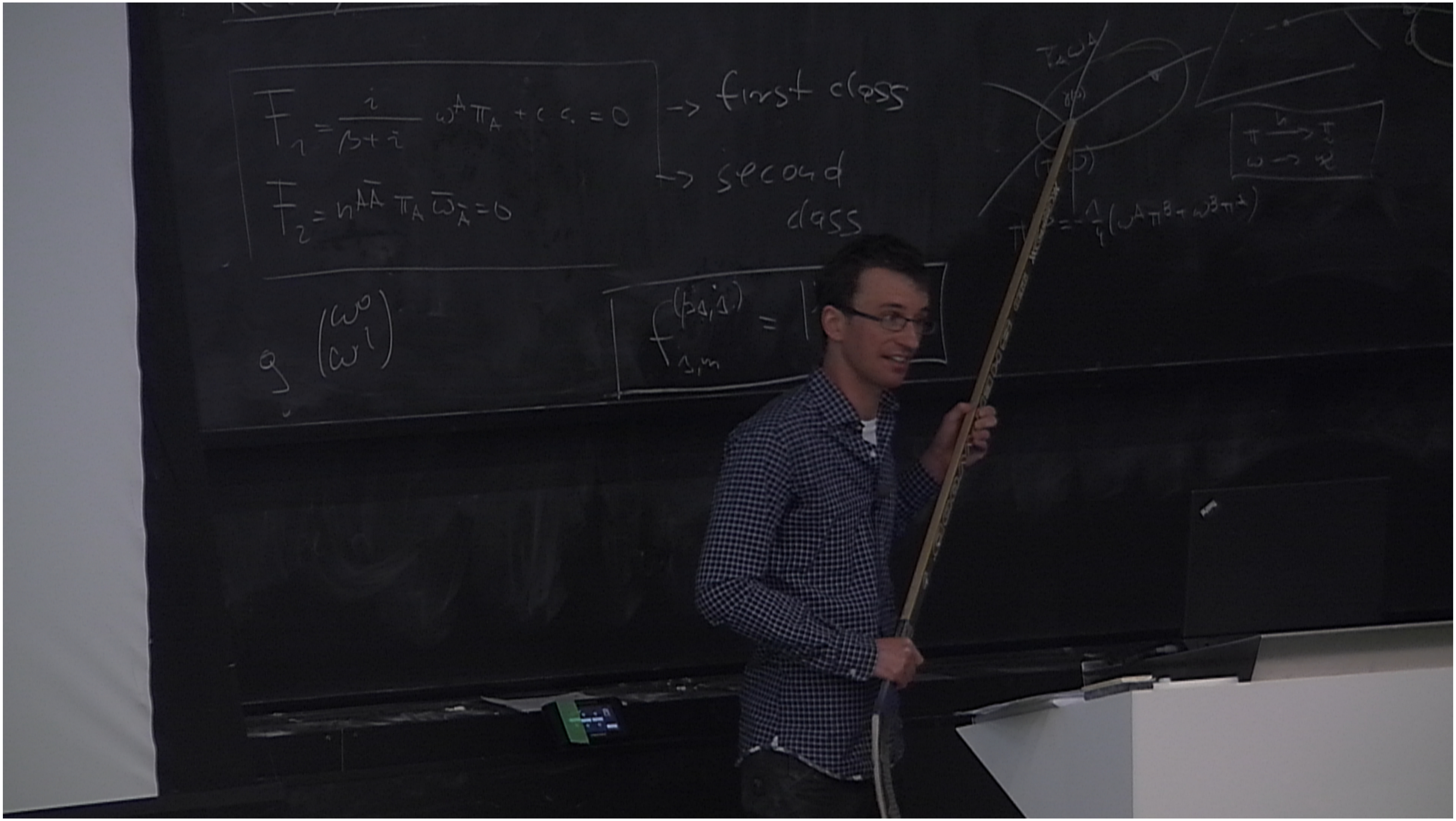

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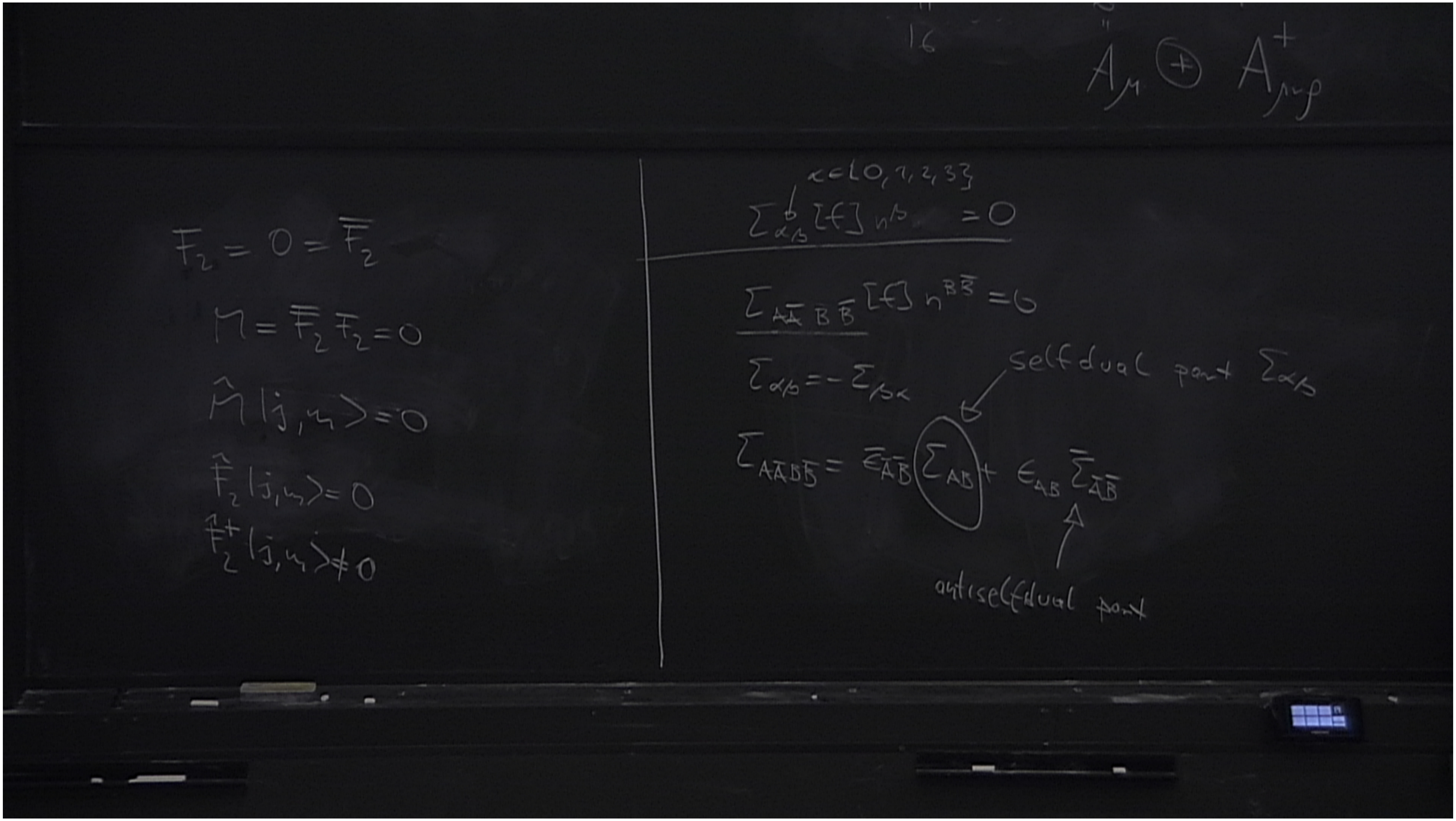
Summary – part 1

- The linear simplicity constraints are reality conditions on the momentum variable.
- The phase space of smeared holonomy-flux variables on a fixed graph was decomposed in terms of twistors. To each link belongs a pair of twistors—one for each of its boundary points.
- This decomposition works as long as
  1. $\Pi[f]^{A}{}_{B}\Pi[f]^{A}{}_{B} \neq 0$, that is unless f is null.
  2. the constraint $C = 0$, generating $\mathbb{C} - \{0\}$ transformations, holds.
- In terms of twistors the reality conditions reduce to $F_1 = 0$ and $\mathbf{M} = \bar{F}_2 F_2 = 0$.

Summary – part 2

- In quantum theory both F_1 and **M** can be imposed strongly.
- The solution space picks the states $|j,m\rangle = f_{j,m}^{(\beta j,j)}$ in the irreducible $(\rho = \beta j, j_o = j)$ unitary representation space of $SL(2,\mathbb{C})$.
- Moreover $\widehat{F}_2|j,m\rangle = 0$ but $\widehat{F}_2^\dagger|j,m\rangle \neq 0$.
- The spinorial method allows for a clean and simple derivation of the Dupuis–Livine map.
- On the classical level the reduction down to $SU(2)$ can explicitly be performed. We arrive at the original phase space of $SU(2)$ spinors introduced by Freidel and Speziale.



16
A_μ ⊕ A^+_{μνρ}
F_2 = 0 = F̄_2
M = F̄_2 F_2 = 0
M̂ |j, m> = 0
F̂_2 |j, m> = 0
F̂_2^+ |j, m> ≠ 0
α ∈ {0, 1, 2, 3}
Σ_{αβ}[f] n^β = 0
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