Title: Newtonian and Relativistic Cosmologies

Date: Feb 09, 2012 01:00 PM

URL: http://pirsa.org/12020128

Abstract: Cosmological N-body simulations are now being performed using Newtonian gravity on scales larger than the Hubble radius. It is well known that a uniformly expanding, homogeneous ball of dust in Newtonian gravity satisfies the same equations as arise in relativistic FLRW cosmology, and it also is known that a correspondence between Newtonian and relativistic dust cosmologies continues to hold in linearized perturbation theory in the marginally bound/spatially flat case. Nevertheless, it is far from obvious that Newtonian gravity can provide a good global description of an inhomogeneous cosmology when there is significant nonlinear dynamical behavior at small scales. We investigate this issue in the light of a perturbative framework that we have recently developed, which allows for such nonlinearity at small scales. We propose a relatively straightforward "dictionary"---which is exact at the linearized level---that maps Newtonian dust cosmologies into general relativistic dust cosmologies, and we use our "ordering scheme" to determine the degree to which the resulting metric and matter distribution solve Einstein's equation. We find that Einstein's equation fails to hold at "order 1" at small scales and at "order ϵ" at large scales. We then find the additional corrections to the metric and matter distribution needed to satisfy Einstein's equation to these orders. While these corrections are of some interest in their own right, our main purpose in calculating them is that their smallness should provide a criterion for the validity of the original dictionary (as well as simplified versions of this dictionary). We expect that, in realistic Newtonian cosmologies, these additional corrections will be very small; if so, this should provide strong justification for the use of Newtonian simulations to describe relativistic cosmologies, even on scales larger than the Hubble radius.

Newtonian and relativistic cosmologies

arXiv:1111.2997 [gr-qc] (accepted by PRD)

Stephen R. Green and Robert M. Wald
University of Chicago

Perimeter Institute
February 9, 2012

Outline

1. Introduction to Newtonian cosmology.
2. Correspondence to GR in linearized regime, and in small-velocity Newtonian regime.
3. General approximation framework for producing a global correspondence.
4. Tentative Newtonian $\Longrightarrow$ GR dictionary: How well does it solve Einstein's equation?
5. Improved dictionary.
6. Conclusions.

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Introduction: Newtonian cosmology

- We can do cosmology entirely within the context of Newtonian gravity, provided we restrict ourselves to "dust" and a cosmological constant.

- How? Begin with Newtonian equations of motion:

$$\partial^i \partial_i \phi + \Lambda = 4\pi\rho \qquad \text{Newton's equation}$$

$$\partial_t \rho + \partial_i(\rho v^i) = 0 \qquad \text{mass conservation}$$

$$\partial_t(\rho v^i) + \partial_j(\rho v^i v^j) = -\rho\partial^i\phi \qquad \text{Euler equation}$$

- **Cosmological background solution:** There exists a solution describing a uniform expanding dust ball, $\rho = \rho_0(t)$, $\phi = \phi_0(t, r)$, $v^i = H(t)x^i$, where

$$H^2 = \frac{8\pi}{3}\rho_0 + \frac{\Lambda}{3} - \frac{k}{a^2}$$

$$\frac{dH}{dt} + H^2 = -\frac{4\pi}{3}\rho_0 + \frac{\Lambda}{3}$$

$$\partial_t \rho_0 = -3H\rho_0$$

Friedmann equations of GR

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Friedmann equations of GR

Introduction: Newtonian cosmology

- This is not surprising:

In both Newtonian gravity and GR, in the presence of spherical symmetry, the behavior of a ball of dust does not depend on the distribution of matter outside of the ball.

$$\Downarrow$$

In both Newtonian gravity and GR, a ball of dust surrounded by vacuum behaves the same as a ball of dust which is part of a homogeneous and isotropic universe.

$$\Downarrow$$

Since a sufficiently small dust ball in GR should behave as in Newtonian gravity, the Friedmann equations should hold in both Newtonian gravity in GR for a homogeneous and isotropic spacetime.

Introduction: Newtonian cosmology

- **Cosmological perturbations:** Re-write the Newtonian equations relative to a background solution. i.e.,

$$v^i = Hx^i + v^i_N$$
$$\rho = \rho_0(1 + \delta_N)$$
$$\phi = \phi_0 + \psi_N$$

  Using comoving co-ordinates and conformal time, the perturbation quantities obey the exact equations

$$\partial^i \partial_i \psi_N = 4\pi a^2 \rho_0 \delta_N$$
$$\dot{\delta}_N + \partial_i \left((1 + \delta_N) v^i_N \right) = 0$$
$$\dot{v}^i_N + v^j_N \partial_j v^i_N + \frac{\dot{a}}{a} v^i_N = -\partial^i \psi_N$$

- These equations are the continuous form of what is used in N-body cosmological simulations. But how are these equations related to Einstein's equation?

Correspondence with GR at linearized level

- Cosmological Newtonian equations

$$\partial^i \partial_i \psi_N = 4\pi a^2 \rho_0 \delta_N$$

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- The linearization of these equations is **identical** to the **scalar** linearized equations in GR for dust with k = 0, when expressed in terms of Bardeen's gauge invariant variables:

$$\Phi_H = -\Phi_A$$

$$\partial^i \partial_i \Phi_A = 4\pi a^2 \rho_0 \epsilon_m$$

$$\dot{\epsilon}_m + \partial_i v^i_s = 0$$

$$\dot{v}^i_s + \frac{\dot{a}}{a} v^i_s = -\partial^i \Phi_A$$

Scalar linearized equations in GR

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Scalar linearized equations in GR

Correspondence with GR at linearized level

- Correspondence continues to hold at the **vector** level if we map the vector part of v_N^i to v_c^i, and we define Bardeen's vector,

$$\partial^j \partial_j \Psi^i = -16\pi a^2 \rho_0 v_c^i$$

Equation for v_c^i already holds in Newtonian gravity: $\dot{v}_c^i + \frac{\dot{a}}{a} v_c^i = 0$

- There is no correspondence at the **tensor** level since GR contains 2 additional tensor degrees of freedom not present in Newtonian gravity.

- Given the Bardeen variables, one can produce a spacetime metric and matter distribution in any coordinate system. It is most straightforward to use the longitudinal gauge, since in this gauge, perturbation variables are algebraically related to the Bardeen variables. We will use this gauge later in the talk.

So are we done?

- The linearized Einstein equation implies linearized matter motion. e.g., off of a flat background,

$$G^{(1)}_{ab} = 8\pi T_{ab} \quad \Longrightarrow \quad \partial^b T_{ab} = 0$$

Linearized Einstein tensor

Partial derivative

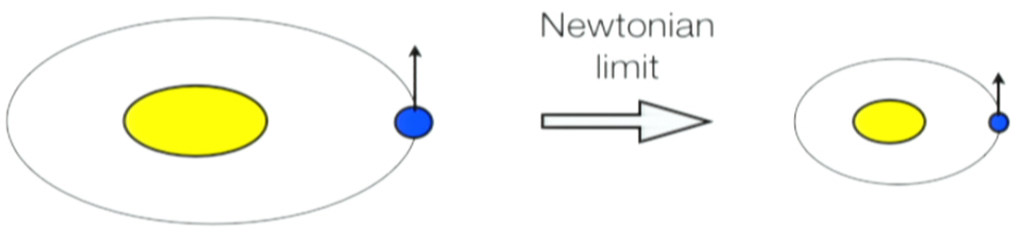
Linearized Bianchi identity

- In other words, dust matter follows geodesics of the background, and matter self-interaction is disallowed. It is inconsistent to assume the linearized Einstein equation, as well as non-linear matter motion.

- A related issue is that in linearized gravity, all perturbative quantities must be small, but at small scales in the universe, we have $|\delta| \gg 1$.

Newtonian limits of GR

- Whereas linearized limits of GR assume all perturbative quantities are small, Newtonian limits take velocities, metric perturbations, and spatial distances to be small, in such a way so as to preserve the Newtonian equations of motion.



- This is the sort of behavior we expect at small scales in the universe. Limit rigorously studied by Oliynyk (2009) in cosmological case.

- As in the linear case, we can find a mapping which takes Newtonian variables into spacetime metric and matter variables, so that Einstein's equation is solved in the limit. However, in longitudinal gauge, the mapping is different from the linearized mapping!

Correspondence with GR at linearized level

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Question

- Cosmological Newtonian equations form the basis of N-body simulations of structure formation. In the case where box sizes exceed the Hubble volume, to what extent can we trust results from such simulations, since the actual universe is relativistic? How do we interpret the results as a relativistic spacetime?

- We have reason to think Newtonian gravity can provide a good description of cosmology:
  background: Friedmann equations hold
  large scale perturbations: exact correspondence with linearized GR
  small scale perturbations: expect Newtonian behavior in GR for small velocities

- However, Newtonian and linearized limits in GR are very different, and not obviously compatible. In a particular gauge, different "dictionaries" are required in Newtonian and linearized correspondences. How can we fit this all together in a consistent approximation scheme?

- This problem can be treated using our generalized framework for analyzing perturbations in cosmology: PRD83, 084020 (2011), arXiv:1011.4920 [gr-qc]

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Relativistic cosmology from Newtonian cosmology

- **Goal:** Given a solution $(\psi_N, \delta_N, v_N^i)$ of the cosmological Newtonian equations of motion on a torus, we wish to generate a spacetime metric g_{ab}, and dust stress energy tensor $T_{ab} = \rho u_a u_b$. And we wish to know to what degree (g_{ab}, T_{ab}) solves Einstein's equation,

$$G_{ab}(g) + \Lambda g_{ab} = 8\pi T_{ab}$$

- Work in a non-linear generalization of "longitudinal gauge" in cosmology

$$ds^2 = a^2(\tau)\left[-(1+2A)d\tau^2 - 2B_i dx^i d\tau + ((1+2H_L)\delta_{ij} + h_{ij})\, dx^i dx^j\right]$$

with $\partial^i B_i = 0,\ \partial^j h_{ij} = 0,\ h^i{}_i = 0$

- Matter variables: $u^\mu \propto (1, v^i)$

$$\rho = \rho_0(1+\delta)$$

- So develop mapping: $\boxed{(\psi_N, \delta_N, v_N^i) \to (A, H_L, B_i, h_{ij}, v^i, \delta)}$

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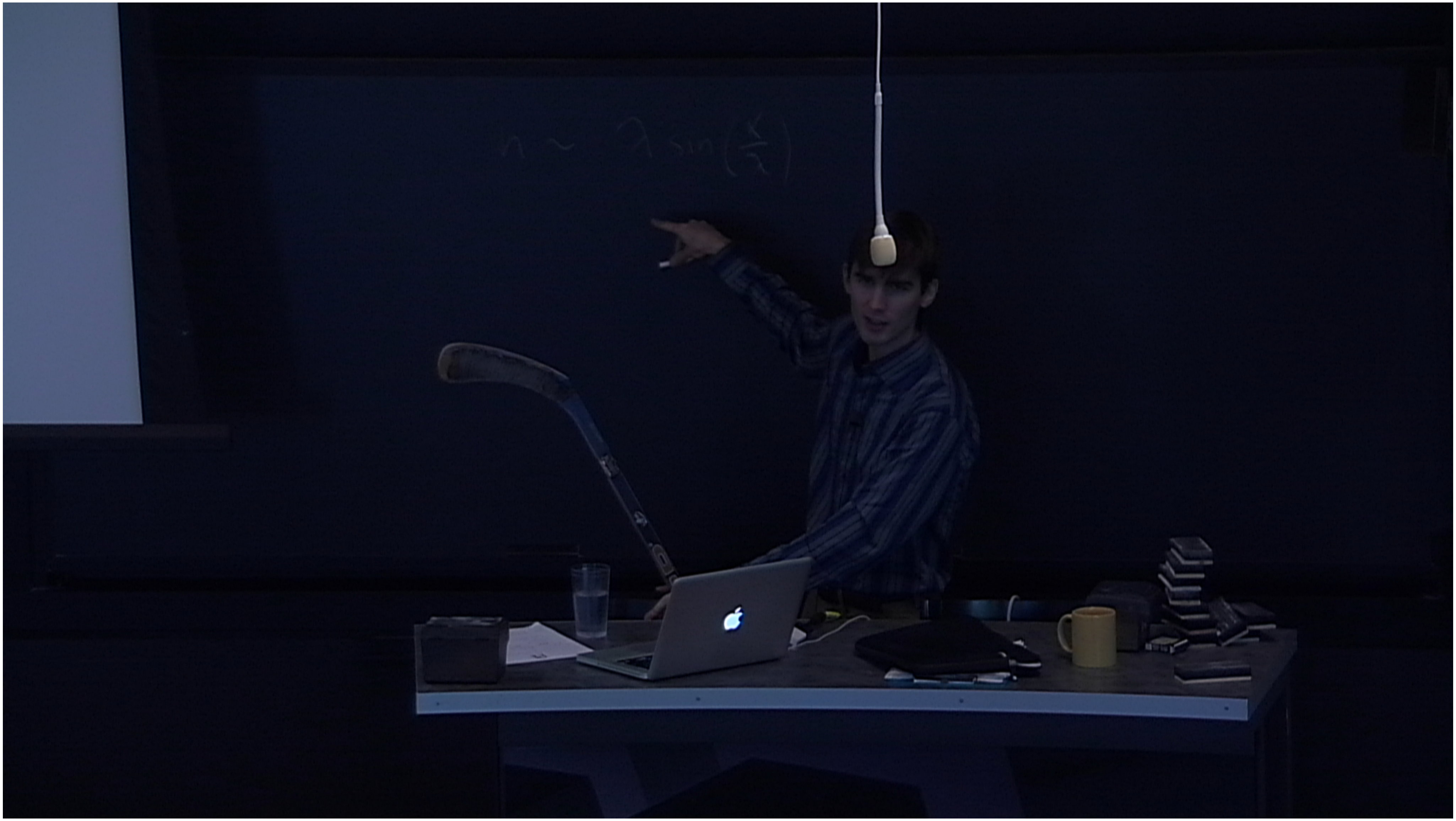
General framework

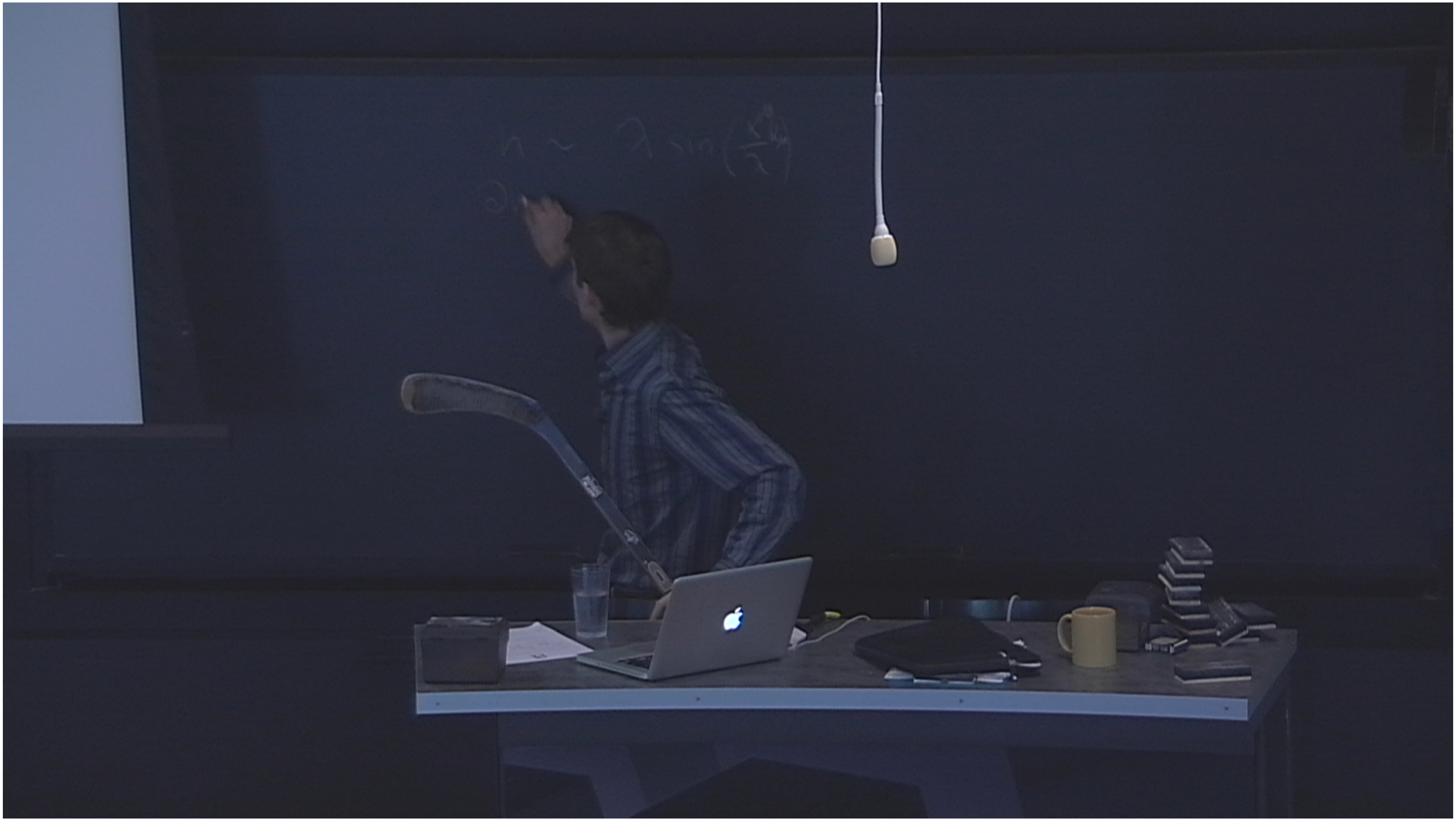
- In our approximation scheme (which extends the "short wave approximation"), there is a small parameter ϵ, that measures the deviation, $\gamma_{ab} = g_{ab} - g^{(0)}_{ab}$, of the metric g_{ab} from the background metric $g^{(0)}_{ab}$, so $\gamma_{ab} = O(\epsilon)$.

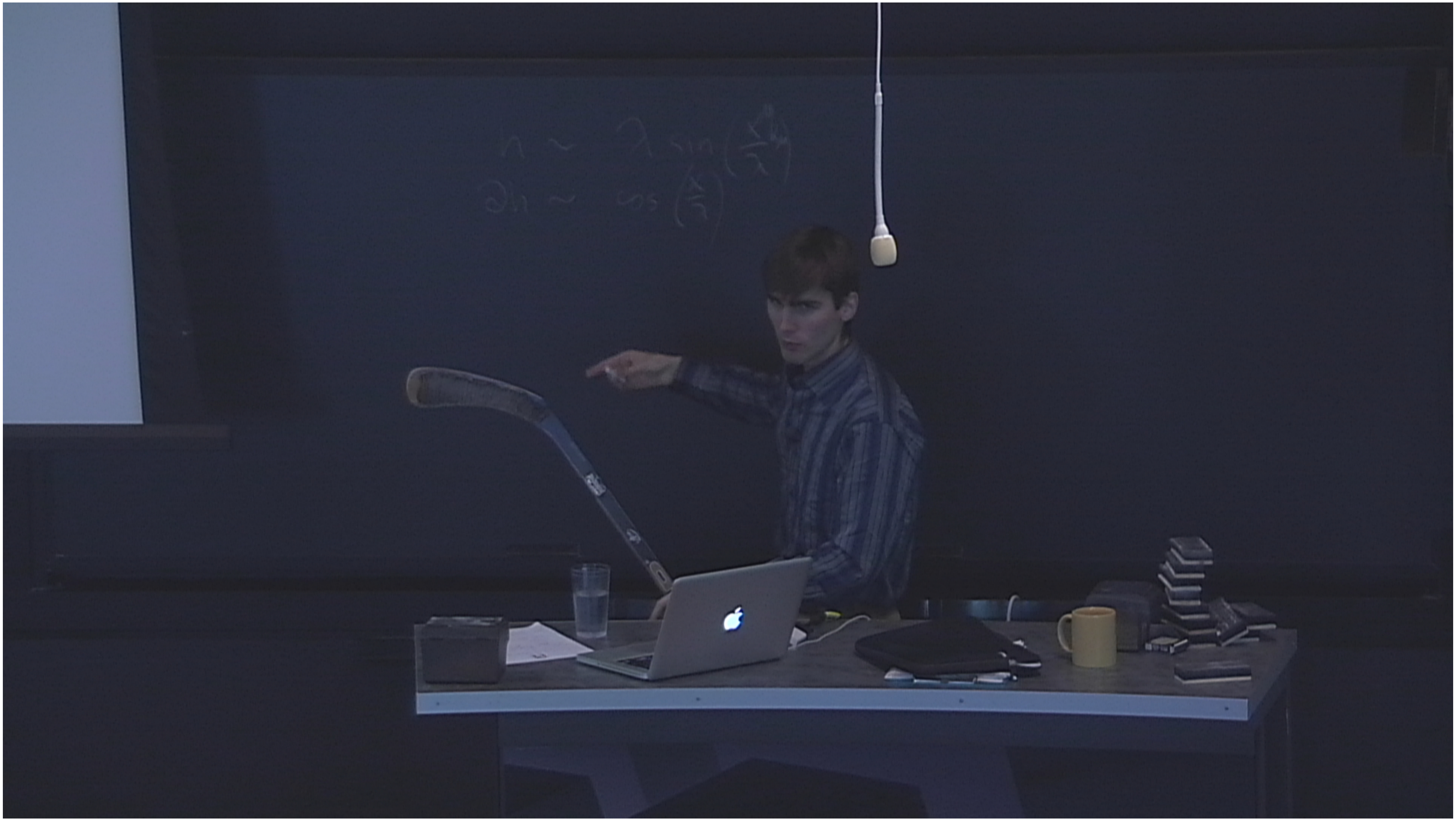
- In contrast to ordinary perturbation theory, first spacetime derivatives of γ_{ab} are allowed to be $O(1)$, and second spacetime derivatives of γ_{ab} are allowed to be $O(1/\epsilon)$. Thus stress energy perturbations can be $O(1/\epsilon)$ as well, via the Einstein equation.

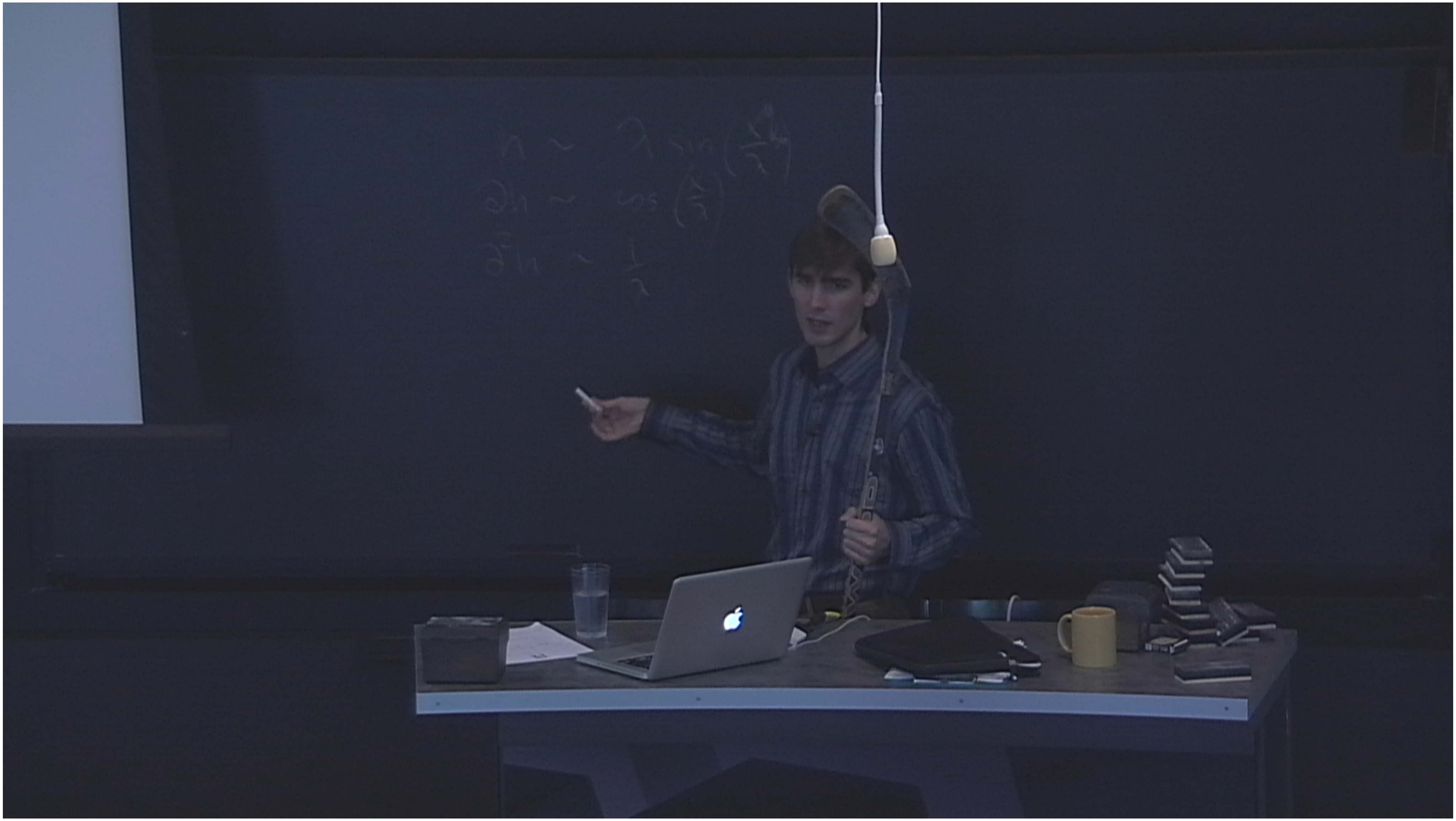
- As a consequence, we have to worry about terms like $\nabla_a \gamma_{bc} \nabla_d \gamma_{ef}$, which appear in the Einstein equation, and are $O(1)$, and can thus potentially backreact on the background metric.

- **Main result of previous work:** Backreaction can only take the form of a traceless effective stress energy tensor with positive energy density (corresponding to gravitational waves), provided the true matter stress energy tensor has positive energy density. In this work we assume negligible radiation content of the universe, so the background metric obeys the ordinary Einstein equation.


$h \sim \lambda \sin\left(\frac{x}{\lambda}\right)$






$h \sim \lambda \sin(\frac{x}{\lambda})$
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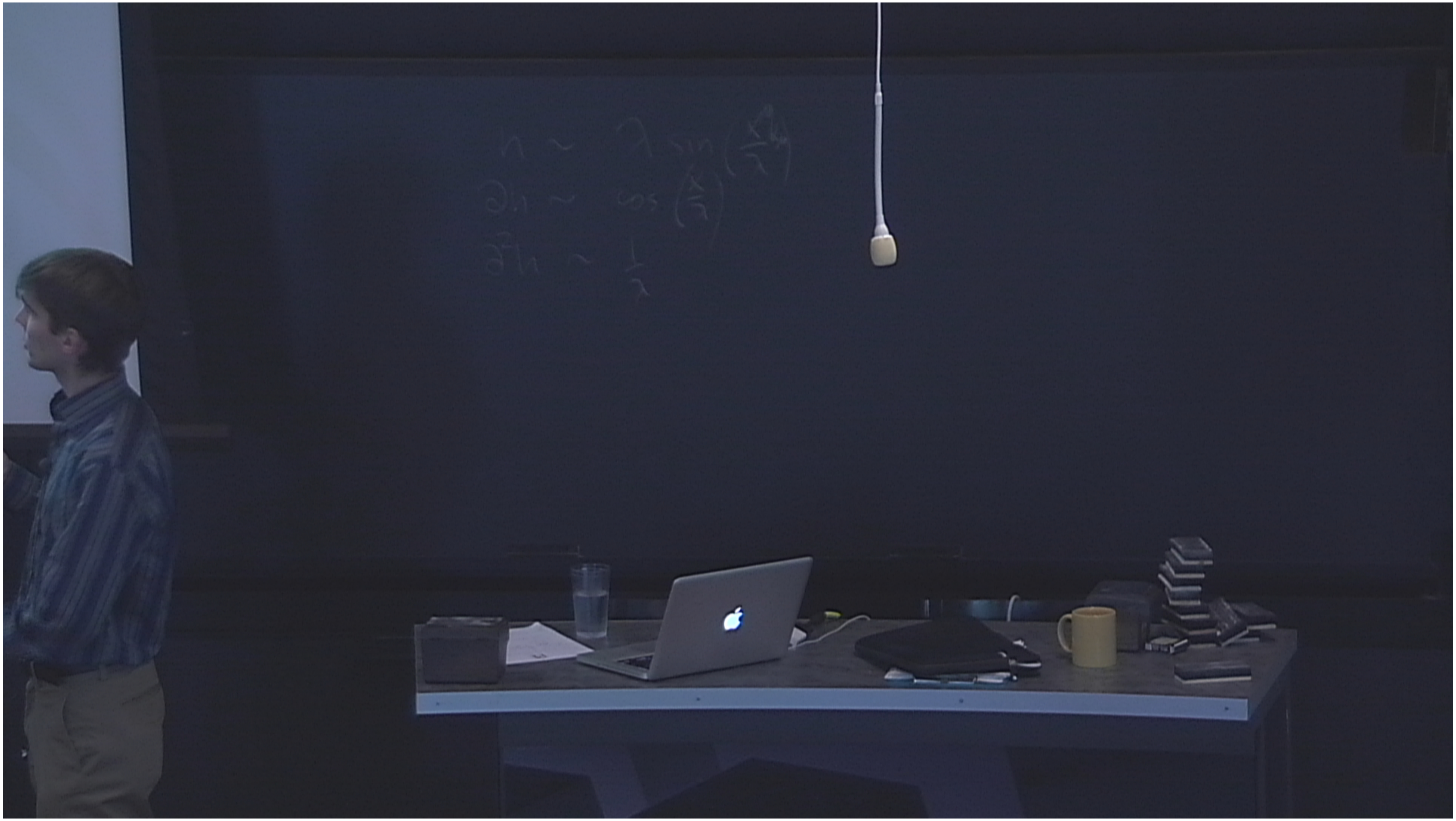
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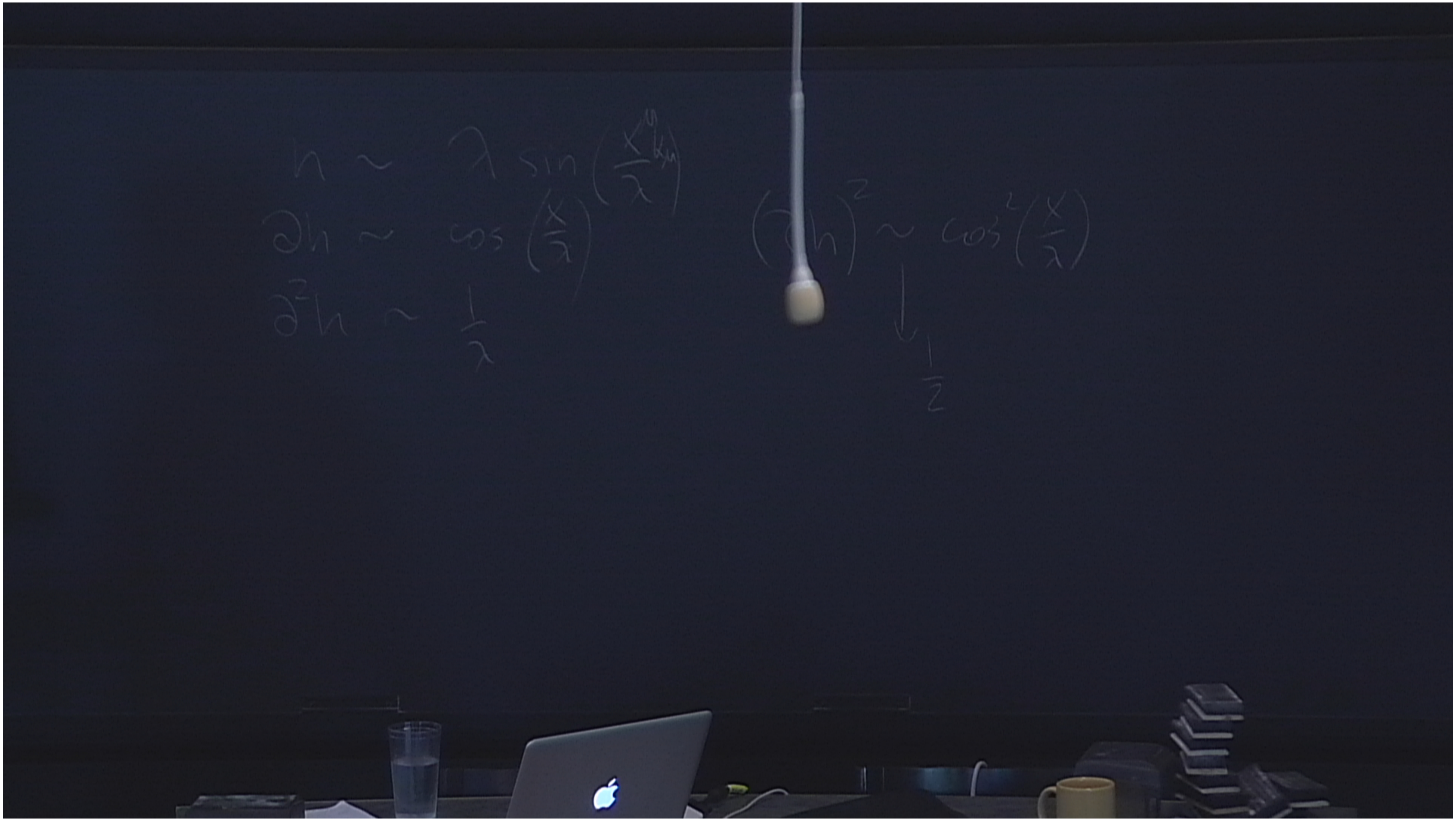
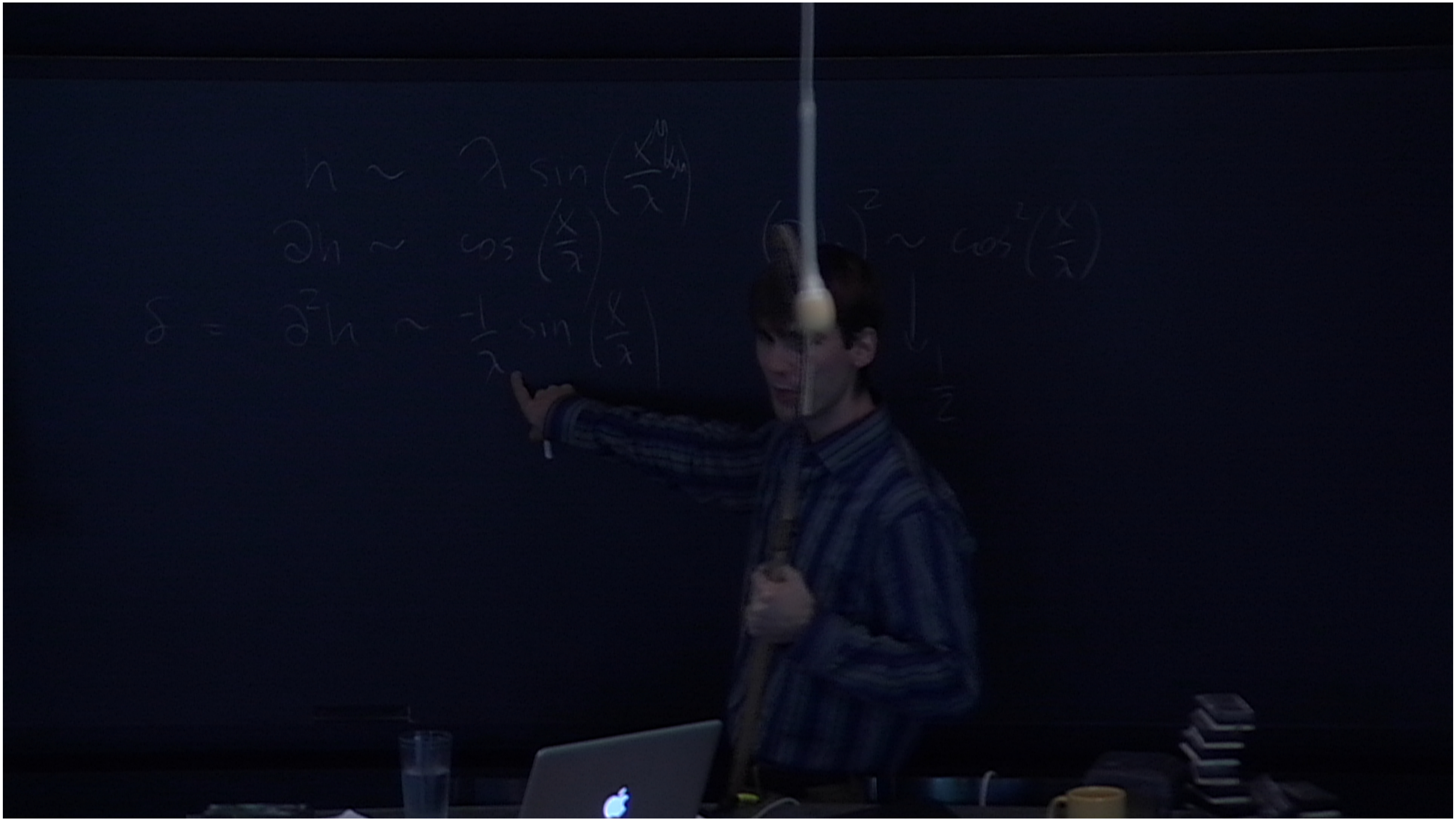
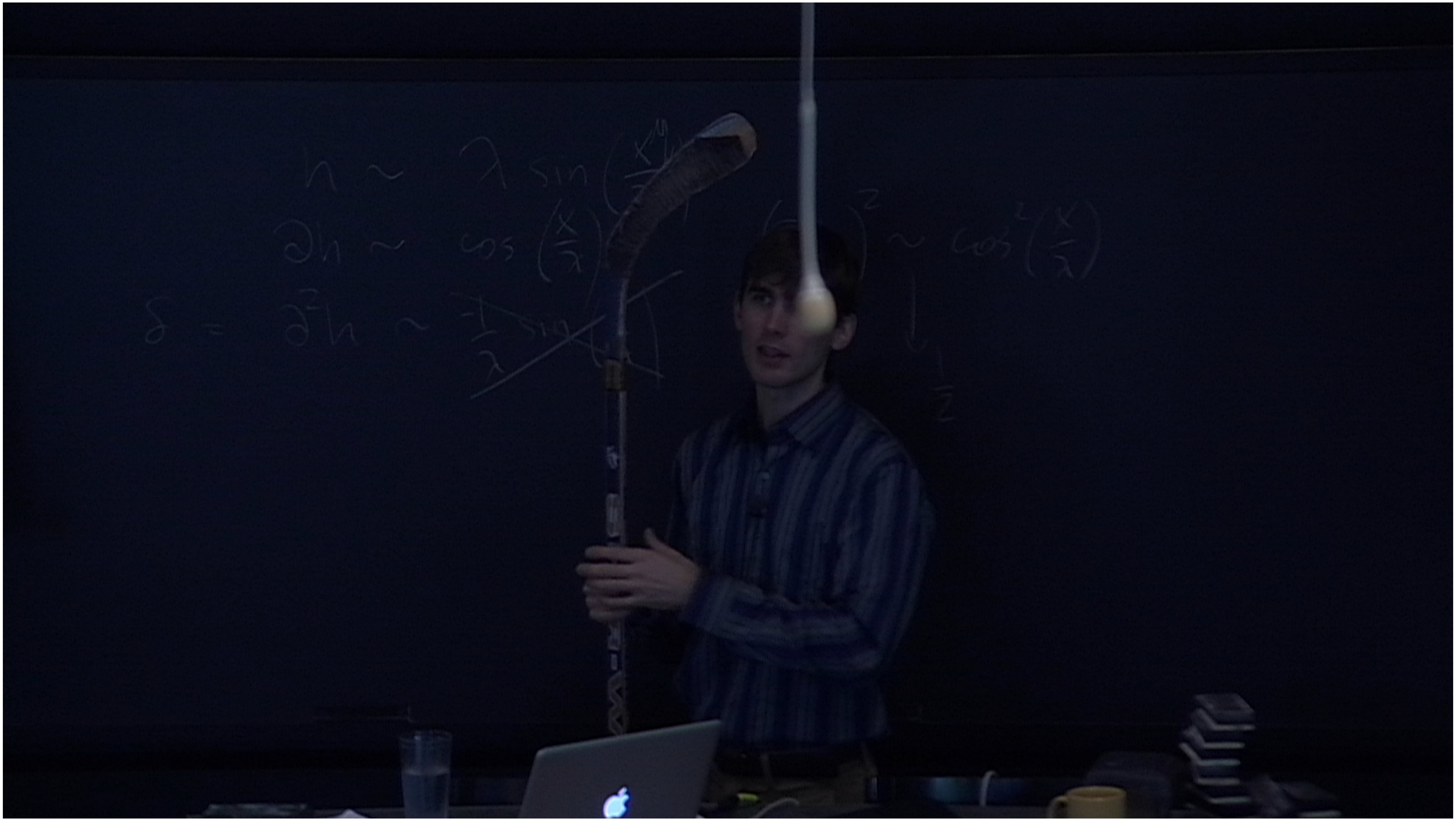

h ~ λ sin(x^μ k_μ / λ)
∂h ~ cos(x/λ)
∂²h ~ 1/λ
(∂h)² ~ cos²(x/λ)
1/2


$h \sim \lambda \sin\left(\frac{x^\mu k_\mu}{\lambda}\right)$
$\partial h \sim \cos\left(\frac{x}{\lambda}\right)$
$\delta = \partial^2 h \sim -\frac{1}{\lambda} \sin\left(\frac{x}{\lambda}\right)$
$\sim \cos^2\left(\frac{x}{\lambda}\right)$



General framework

- **Perturbations:** Large and small scales are treated separately, such that significant non-linearity is admitted on small scales, while large scale motion behaves linearly.

- Must assume there exists a homogeneity scale L, such that $|\delta_N| \ll 1$ beyond this scale. Then using a window function we can define local averages

$$\langle F\rangle(x) = \int d^3x'\, W_L(x,x')F(x')$$

and we split quantities $F = \langle F\rangle + (F - \langle F\rangle)$

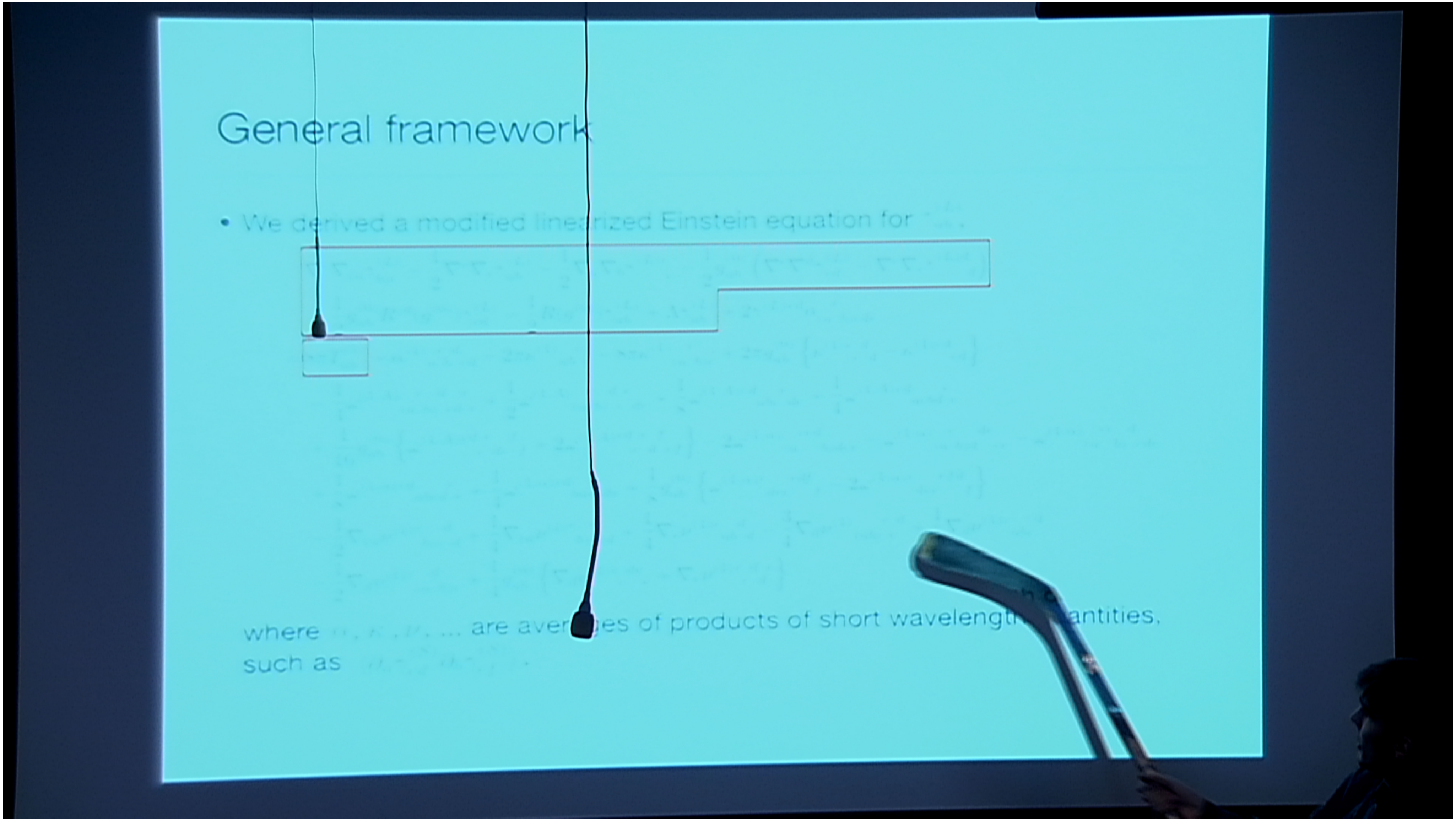
$$\equiv \underbrace{F^{(L)}}_{\text{Large scale part *behaves linearly*}} + \underbrace{F^{(S)}}_{\text{Small scale part *non-linear*}}$$

General framework

- We derived a modified linearized Einstein equation for $\gamma^{(L)}_{ab}$,

$$
\begin{aligned}
&\nabla^c\nabla_{(a}\gamma^{(L)}_{b)c} - \frac{1}{2}\nabla^c\nabla_c\gamma^{(L)}_{ab} - \frac{1}{2}\nabla_a\nabla_b\gamma^{(L)c}{}_c - \frac{1}{2}g^{(0)}_{ab}\left(\nabla^c\nabla^d\gamma^{(L)}_{cd} - \nabla^c\nabla_c\gamma^{(L)d}{}_d\right) \\
&+ \frac{1}{2}g^{(0)}_{ab}R^{cd}(g^{(0)})\gamma^{(L)}_{cd} - \frac{1}{2}R(g^{(0)})\gamma^{(L)}_{ab} + \Lambda\gamma^{(L)}_{ab} + 2\gamma^{(L)cd}\alpha_{(a}{}^e{}_{b)cde} \\
&= \boxed{8\pi T^{(1)}_{ab}} + \alpha^{(1)}{}_a{}^c{}_b{}^d{}_{cd} - 2\pi\kappa^{(1)}{}_{ab}{}^c{}_c - 8\pi\kappa^{(1)}{}_{(a}{}^c{}_{b)c} + 2\pi g^{(0)}_{ab}\left\{\kappa^{(1)c}{}_c{}^d{}_d - \kappa^{(1)cd}{}_{cd}\right\} \\
&- \frac{1}{4}\omega^{(1,A)}{}_{(a}{}^c{}_{b)}{}^d{}_{cd}{}^e{}_e + \frac{1}{2}\omega^{(1,A)}{}_{(a}{}^c{}_{b)}{}^d{}_c{}^e{}_{de} + \frac{1}{8}\omega^{(1,A)cd}{}_{abc}{}^e{}_{de} + \frac{1}{4}\omega^{(1,A)cd}{}_{acbd}{}^e{}_e \\
&+ \frac{1}{16}g^{(0)}_{ab}\left\{\omega^{(1,A)cd}{}_c{}^e{}_{de}{}^f{}_f + 2\omega^{(1,A)cd}{}_c{}^e{}_d{}^f{}_{ef}\right\} - 2\omega^{(1,\alpha)}{}_{(a}{}^{cd}{}_{b)dce} - \omega^{(1,\alpha)}{}_{(a}{}^c{}_{b)d}{}^{de}{}_{ce} - \omega^{(1,\alpha)}{}_{(a}{}^{ce}{}_{b)}{}^d{}_{cde} \\
&+ \frac{1}{8}\omega^{(1,\alpha)cd}{}_{abcd}{}^e{}_e + \frac{1}{4}\omega^{(1,\alpha)cd}{}_{bac}{}^e{}_{de} + \frac{1}{8}g^{(0)}_{ab}\left\{\omega^{(1,\alpha)c}{}_{dec}{}^{edf}{}_f - 2\omega^{(1,\alpha)c}{}_{dec}{}^{efd}{}_f\right\} \\
&- \frac{1}{2}\nabla_{(a}\nu^{(1)c}{}_{b)c}{}^d{}_d + \frac{1}{4}\nabla_{(a}\nu^{(1)c}{}_{b)}{}^d{}_{cd} + \frac{1}{4}\nabla_c\nu^{(1)c}{}_{ab}{}^d{}_d - \frac{3}{4}\nabla_d\nu^{(1)}{}_{(ab)}{}^{cd}{}_c + \frac{1}{4}\nabla_d\nu^{(1)c}{}_{abc}{}^d \\
&- \frac{1}{2}\nabla_d\nu^{(1)c}{}_{(a}{}^d{}_{b)c} + \frac{1}{4}g^{(0)}_{ab}\left\{\nabla_d\nu^{(1)c}{}_c{}^{de}{}_e + \nabla_c\nu^{(1)c}{}_e{}^d{}_d{}^e\right\}
\end{aligned}
$$

where α, κ, ν, ... are averages of products of short wavelength quantities, such as $\langle\partial_a\gamma^{(S)}_{cd}\partial_b\gamma^{(S)}_{ef}\rangle$.


General framework
• We derived a modified linearized Einstein equation for
where
are aver
ges of products of short wavelength
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Assumptions

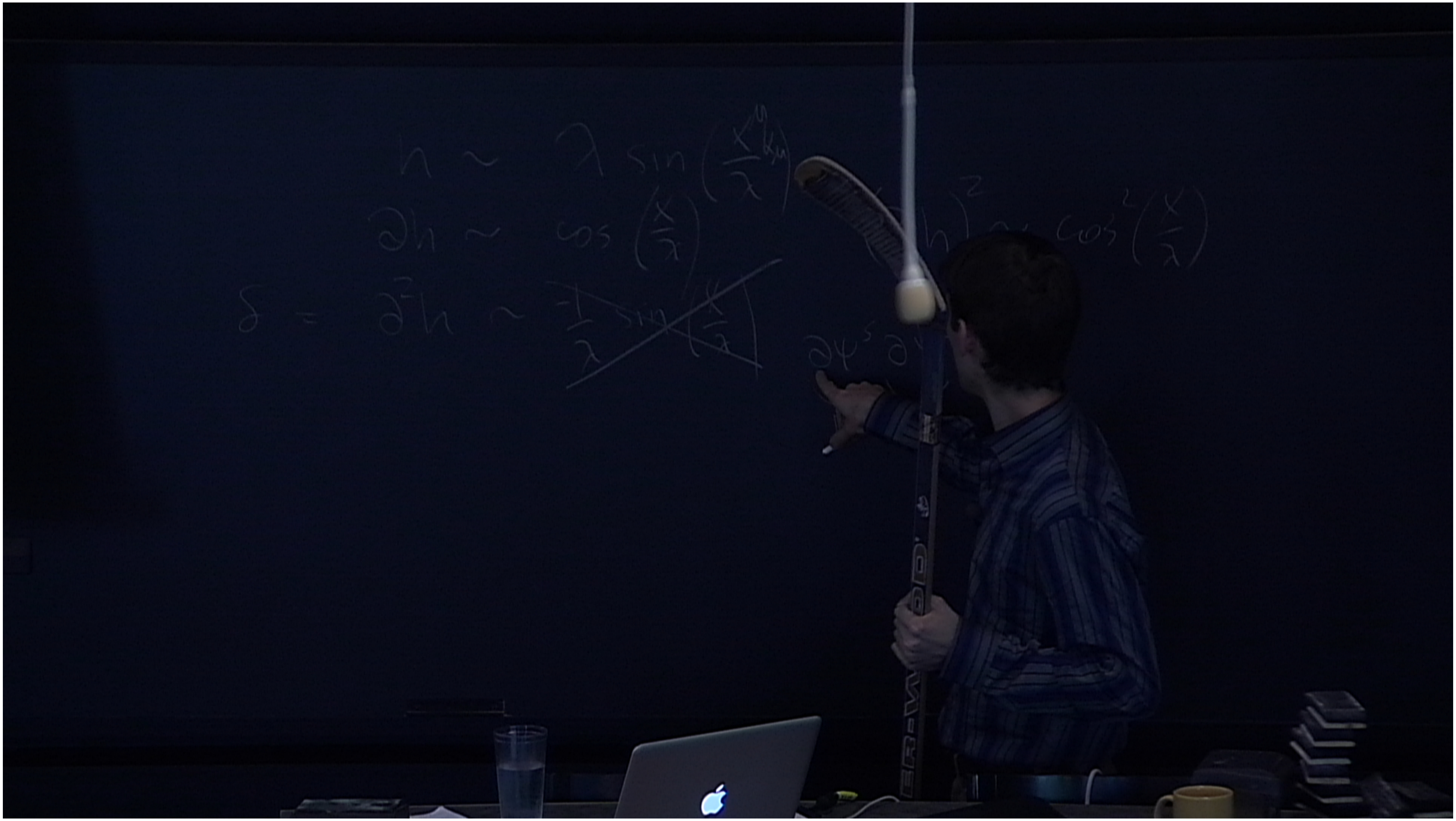
- Assume that for our Newtonian solution,

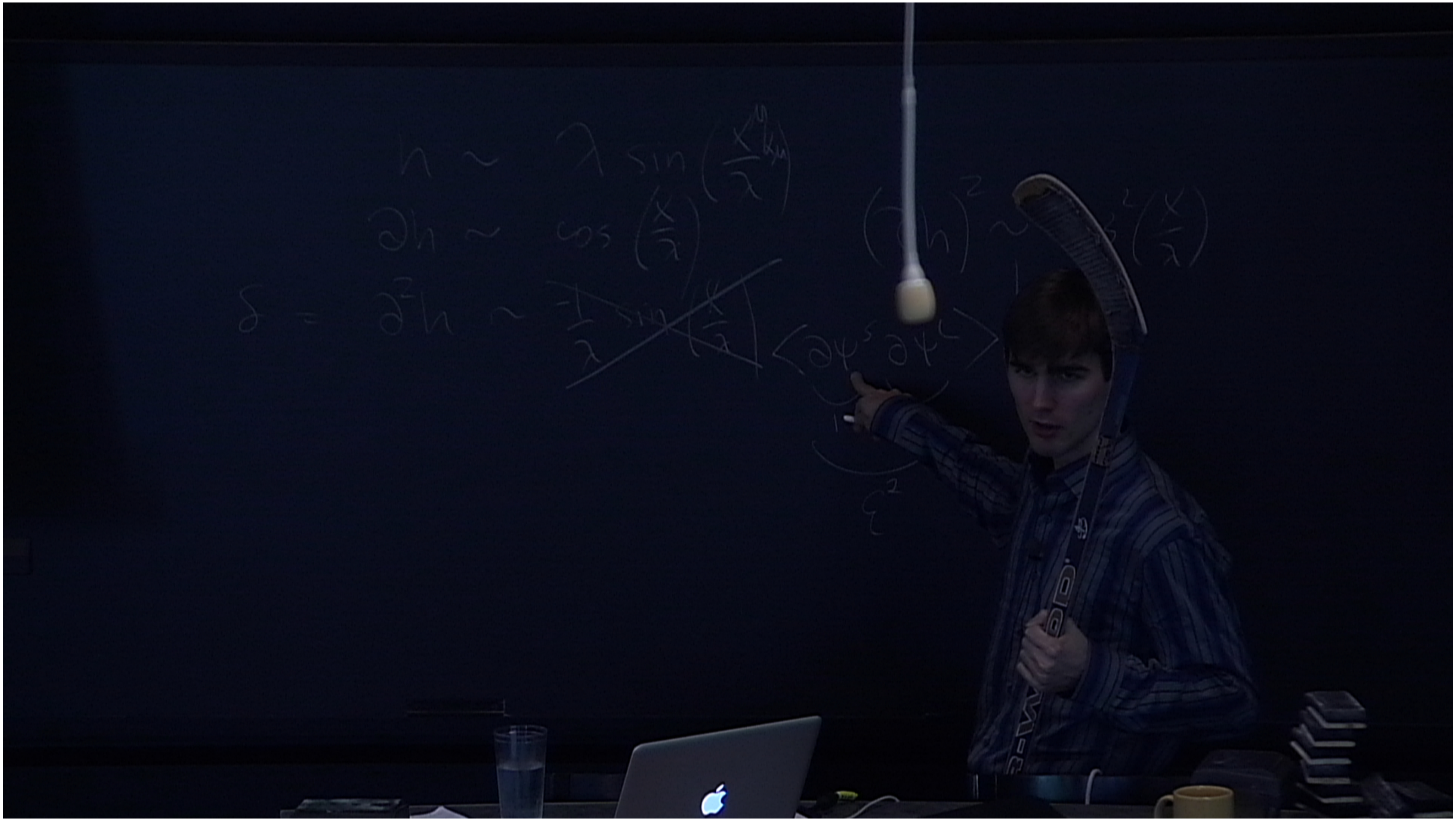
Quantity	Small scale order	Large scale order
ψ_N	ϵ	ϵ
v^i_N	$\epsilon^{1/2}$	ϵ
δ_N	ϵ^{-1}	ϵ
∂_0	$\epsilon^{-1/2}$	1
∂_i	ϵ^{-1}	1

so that, for composite quantities,

Quantity	Small scale order	Large scale order	
$\partial_i\psi_N$	1	ϵ	automatic under averaging
$\partial_i\psi_N\partial_j\psi_N$	1	ϵ	a priori $O(1)$, but proven to be $O(\epsilon)$
$\delta_N\psi_N$	1	ϵ	
$(1+\delta_N)v^i_N v^j_N$	1	ϵ	
$\delta_N v^i_N$	$\epsilon^{-1/2}$	ϵ	

- Key fact: $\langle\delta_N\psi_N\rangle = O(\epsilon)$ arises because $\delta_N > -1$ (positivity of mass).

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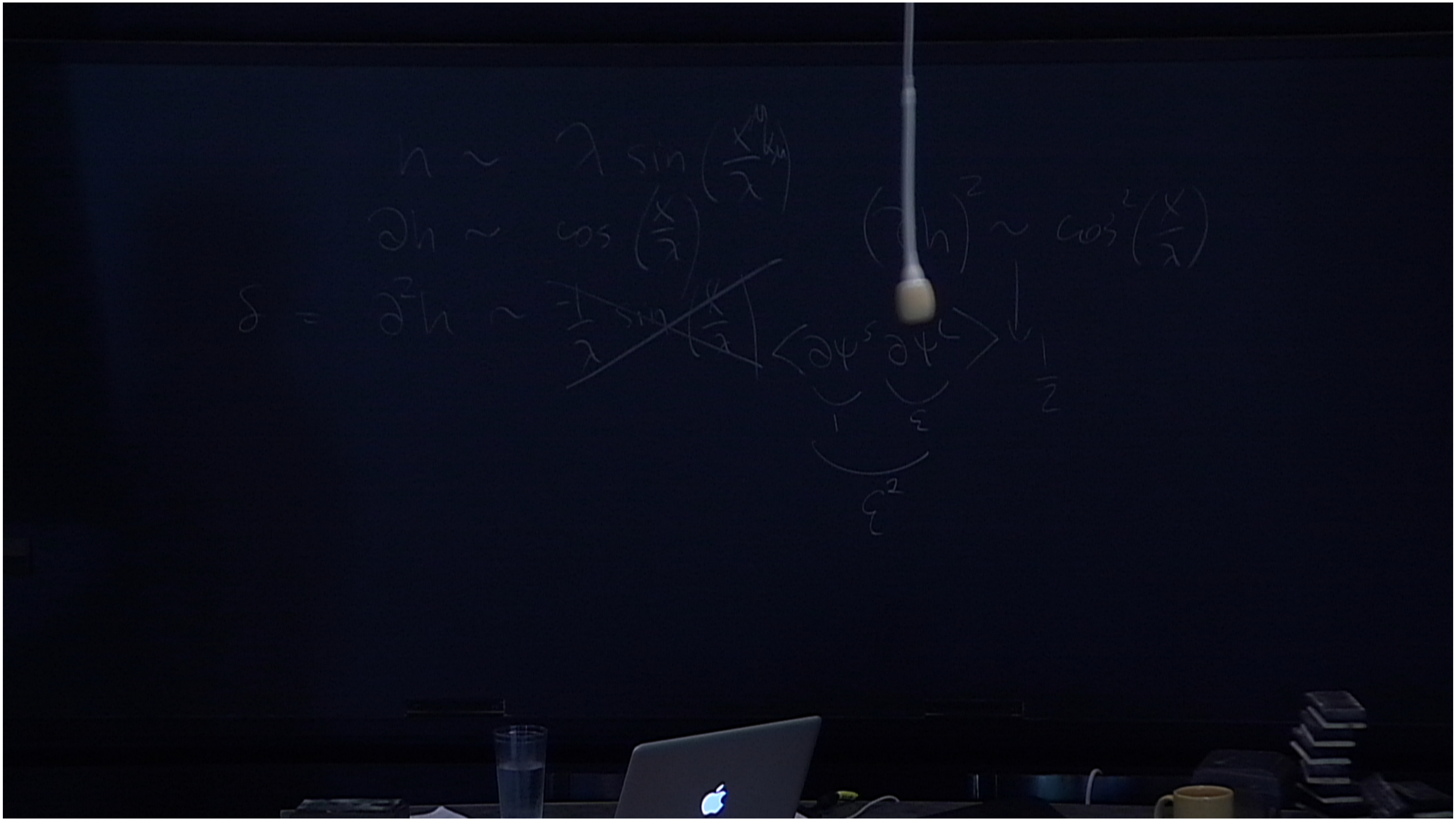
δ = ∂²h ~
⟨∂ψ ∂ψ⟩



$h \sim \lambda \sin\left(\frac{x^\mu k_\mu}{\lambda}\right)$
$\partial h \sim \cos\left(\frac{x}{\lambda}\right)$
$(\partial h)^2 \sim \cos^2\left(\frac{x}{\lambda}\right) \rightarrow \frac{1}{2}$
$\delta = \partial^2 h \sim \cancel{-\frac{1}{\lambda}\sin\left(\frac{x}{\lambda}\right)}$
$\underbrace{\langle \underbrace{\partial\psi^s}_{1} \underbrace{\partial\psi^L}_{\varepsilon} \rangle}_{\varepsilon^2}$

Tentative mapping

- Linearized correspondence motivates a "dictionary":

Chisari and Zaldarriaga dictionary

$$A = \psi_N$$

$$H_L = -\psi_N$$

$$v^i = v^i_N$$ [redacted]

$$\delta = \delta_N - \frac{3}{4\pi\rho_0 a^2}\left[\left(\frac{\dot{a}}{a}\right)^2 \psi_N + \frac{\dot{a}}{a}\dot{\psi}_N\right]$$

$$\partial^j \partial_j B^i =$$ [redacted]

$$h_{ij} = 0$$

- How well does this solve Einstein's equation within our approximation scheme? Recall that we wish to solve Einstein's equation to $O(1)$ at small scales and $O(\epsilon)$ at large scales.

How well does the tentative mapping work?

- Look at time-time component of Einstein's equation, under this dictionary

$$
\begin{aligned}
&G^0{}_0(g) + \Lambda - 8\pi T^0{}_0 \\
&= \frac{3}{a^2}\left\{\boxed{-2\frac{\dot a}{a}\dot H_L + \frac{2}{3}\partial^i\partial_i H_L + 2\left(\frac{\dot a}{a}\right)^2 A} - \frac{8}{3}H_L\partial^i\partial_i H_L - \partial_i H_L \partial^i H_L\right\} \\
&\quad - 8\pi\rho_0\left\{\boxed{-\delta} - (1+\delta)v^i(v_i - B_i)\right\} + o(1:\epsilon) \\
&= \frac{3}{a^2}\left\{-\frac{2}{3}\partial^i\partial_i \psi_N - \frac{8}{3}\psi_N\partial^i\partial_i\psi_N - \partial^i\psi_N\partial_i\psi_N\right\} - 8\pi\left\{-\delta_N - (1+\delta_N)v^i_N v_{N,i}\right\} + o(1:\epsilon) \\
&= \boxed{\frac{3}{a^2}\left\{-\frac{8}{3}\psi_N\partial^i\partial_i\psi_N - \partial^i\psi_N\partial_i\psi_N\right\} + 8\pi\rho_0(1+\delta_N)v^i_N v_{N,i}} + o(1:\epsilon)
\end{aligned}
$$

- Linear perturbation terms successfully dealt with, however important non-linear terms remain.

- Extra terms are $O(1)$ at small scales, and $O(\epsilon)$ at large scales, within our counting scheme. Must improve the dictionary.

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&\quad - 8\pi\rho_0\left\{\boxed{-\delta} - (1+\delta)v^i(v_i - B_i)\right\} + o(1:\epsilon) \\
&= \frac{3}{a^2}\left\{-\frac{2}{3}\partial^i\partial_i v_N - \frac{8}{3}v_N\partial^i\partial_i v_N - \partial^i v_N \partial_i v_N\right\} - 8\pi\left\{-\delta_N - (1+\delta_N)v^i_N v_{N,i}\right\} + o(1:\epsilon) \\
&= \boxed{\frac{3}{a^2}\left\{-\frac{8}{3}v_N\partial^i\partial_i v_N - \partial^i v_N \partial_i v_N\right\} + 8\pi\rho_0(1+\delta_N)v^i_N v_{N,i}} + o(1:\epsilon)
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Improving the dictionary

- To determine $\chi, \xi, \jmath_{ij}$, we substitute the dictionary into Einstein's equation, drop terms which are $o(1)$, and use the Newtonian equations to simplify.

- Be systematic:

$$E^0{}_0$$

$$E^0{}_i = Y_i + \partial_i Z$$

Scalar/Vector/Tensor decomposition

Einstein equation components

$$E_{ij} = U\delta_{ij} + \partial_i\partial_j V - \frac{1}{3}\partial^k\partial_k V + 2\partial_{(i}W_{j)} + X_{ij}$$

- For example, to isolate one of the scalars in the space-space components, take the double divergence of the traceless part:

$$\begin{aligned}
&-\frac{2}{3a^2}\partial^i\partial_i\partial^j\partial_j\xi + \frac{1}{a^2}\partial_i\partial^j\left\{4\psi_N\partial^i\partial_j\psi_N + 2\partial^i\psi_N\partial_j\psi_N\right\}\\
&-\frac{1}{3a^2}\partial^i\partial_i\left\{4\psi_N\partial^j\partial_j\psi_N + 2\partial^j\psi_N\partial_j\psi_N\right\}\\
&=8\pi\rho_0\partial_i\partial^j\left[(1+\delta_N)v^i_N v_{Nj}\right] - \frac{8\pi}{3}\rho_0\partial^i\partial_i\left[(1+\delta_N)v^k_N v_{Nk}\right] + o\left(\frac{1}{\epsilon^2}\right)
\end{aligned}$$

Improving the dictionary

- **Obtaining an $O(1)$ solution at small scales**

$$A = \psi_N + \boxed{\chi + \xi}$$
$$H_L = -\psi_N \boxed{- \chi}$$

Additional corrections TBD

$$(1+\delta_N)v_i = (1+\delta_N)(v_{N,i} + B_i) - \overline{(1+\delta_N)v_{N,i}}\Big|_V$$

$$\overline{F}(\tau) = \int_r d^3x\, F(x)$$

$$\delta = \delta_N - \frac{3}{4\pi\rho_0 a^2}\left[\left(\frac{\dot{a}}{a}\right)^2 \psi_N + \frac{\dot{a}}{a}\dot{\psi}_N\right]$$

$$\partial^j\partial_j B^i = -16\pi\rho_0 a^2 \left((1+\delta_N)v_N^i - \overline{(1+\delta_N)v_N^i}\right)\Big|_V$$

$$h_{ij} = \boxed{\chi_{ij}}$$

Required for consistency with nonlinear momentum constraint at small scales

- Since B_i satisfies Poisson equation with $O(1/\epsilon^{1/2})$ source, it is consistent within our scheme to assign $B_i^{(S)} = O(\epsilon^{3/2})$. Of course, $B_i^{(L)} = O(\epsilon)$.

- Assume other correction terms are all $O(\epsilon^2)$ at small scales; verify consistency later.

Improving the dictionary

- **Obtaining an $O(1)$ solution at small scales**

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Improving the dictionary

- Obtain $O(1)$ solution by setting

$$\begin{aligned}
\partial^i\partial_i\partial^j\partial_j\xi =& 3\partial_i\partial^j\left\{2\psi_N\partial^i\partial_j\psi_N + \partial^i\psi_N\partial_j\psi_N\right\} - \partial^i\partial_i\left\{2\psi_N\partial^j\partial_j\psi_N + \partial^j\psi_N\partial_j\psi_N\right\} \\
& - 12\pi\rho_0 a^2\partial_i\partial^j\left[(1+\delta_N)v_N^i v_{N,j}\right] + 4\pi\rho_0 a^2\partial^i\partial_i\left[(1+\delta_N)v_N^k v_{N,k}\right] \\
\partial^i\partial_i\chi =& -4\psi_N\partial^i\partial_i\psi_N - \frac{3}{2}\partial^i\psi_N\partial_i\psi_N + 4\pi a^2\rho_0(1+\delta_N)v_N^i v_{N,i} \\
& - \frac{5}{2}\overline{\partial^i\psi_N\partial_i\psi_N} - 4\pi a^2\rho_0\overline{(1+\delta_N)v_N^i v_{N,i}} \\
\partial^k\partial_k J^i{}_j =& \left\{8\pi\rho_0 a^2(1+\delta_N)v_N^i v_{N,j} + 4\psi_N\partial^i\partial_j\psi_N + 2\partial^i\psi_N\partial_j\psi_N\right. \\
& \left.-8\pi\rho_0 a^2\overline{(1+\delta_N)v_N^i v_{N,j}} + 2\overline{\partial^i\psi_N\partial_j\psi_N}\right\}\Big|_t
\end{aligned}$$

1. All solvable Poisson equations. Arises because time derivatives are small compared to space derivatives at small scales.

2. All of these corrections are $O(\epsilon^2)$. Important as a matter of principle, but negligible in practice. Post-Newtonian type corrections to the metric.

Improving the dictionary

- **Obtaining an $O(\epsilon)$ solution at large scales**

$$A = \psi_N + \chi + \xi + \boxed{X + \Xi}$$

$$H_L = -\psi_N - \chi - \boxed{X}$$

$$(1+\delta_N)v_i = (1+\delta_N)(v_{N,i} + B_i) - \boxed{\left.\overline{(1+\delta_N)v_{N,i}}\right|_V + P_i}$$

$$\delta = \delta_N - \frac{3}{4\pi\rho_0 a^2}\left[\left(\frac{\dot{a}}{a}\right)^2 \psi_N + \frac{\dot{a}}{a}\dot{\psi}_N\right] + \boxed{\Delta}$$

$$h_{ij} = \chi_{ij} + \boxed{J_{ij}}$$

Purely large scale corrections

To solve for these corrections:

1. Substitute dictionary

2. Apply averaging operator to Einstein equation

3. Impose Newtonian equations, and equations defining $(\chi, \xi, \chi_{ij}, B_i)$

Improving the dictionary

- Then we set

$$\Xi = -2\overline{X} \quad \text{(chosen so that we continue to use conformal time)}$$

$$P^i = -\frac{1}{4\pi\rho_0 a^2}\partial^i\left(\dot{X} + \langle\dot{\chi}\rangle + \frac{\dot{a}}{a}\left(X + \langle\chi\rangle + \langle\xi\rangle\right)\right)$$

$$\Delta = -\overline{(1+\delta_N)v_N^i v_{N,i}} - \frac{5}{8\pi a^2\rho_0}\overline{\partial^i v_N \partial_i v_N} - \frac{3}{8\pi\rho_0 a^2}\left\{-\frac{2}{3}\partial^i\partial_i X \right.$$
$$\left. + 2\frac{\dot{a}}{a}\left(\dot{X} + \langle\dot{\chi}\rangle\right) + 2\left(\frac{\dot{a}}{a}\right)^2\left(X - 2\overline{X} + \langle\chi\rangle + \langle\xi\rangle\right)\right\}$$

- **Initial conditions:** Could fix X by requiring, e.g., $\Delta = P_i = 0$ at some initial time.

The full dictionary

- Our “dictionary”:

Post-Newtonian corrections
Purely large scale corrections

$$A = \psi_N + \boxed{\chi + \xi} + \boxed{X + \Xi}$$

$$H_L = -\psi_N \boxed{-\chi} - \boxed{X}$$

$$(1+\delta_N)v_i = (1+\delta_N)(v_{N,i} + \boxed{B_i}) - \boxed{\overline{(1+\delta_N)v_{N,i}}\Big|_V + P_i}$$

$$\delta = \delta_N - \frac{3}{4\pi\rho_0 a^2}\left[\left(\frac{\dot a}{a}\right)^2 \psi_N + \frac{\dot a}{a}\dot\psi_N\right] + \boxed{\Delta}$$

$$h_{ij} = \boxed{j_{ij}} + \boxed{J_{ij}}$$

Quantity	Small scale order
B_i	$\epsilon^{3/2}$
ξ	ϵ^2
χ	ϵ^2
j_{ij}	ϵ^2

Infinite wavelength part

- We can absorb the infinite wavelength corrections $(\overline{X}, \overline{\Delta}, \overline{J}_{ij})$ into a modified background spacetime.

- $(\overline{X}, \overline{\Delta})$: Modified scale factor $\hat{a} = a(1 - \overline{X})$ obeys Friedmann equations with effective energy density and pressure,

$$\rho_{\rm eff} = \hat{\rho}\left(1 + \frac{1}{2}\overline{(1+\delta_N)v_N^i v_{Ni}} - \frac{1}{8\pi\hat{\rho}\hat{a}^2}\overline{\partial^i\psi_N\partial_i\psi_N}\right)$$
$$P_{\rm eff} = \hat{\rho}\left(\frac{1}{3}\overline{(1+\delta_N)v_N^i v_{Ni}} - \frac{1}{24\pi\hat{\rho}\hat{a}^2}\overline{\partial^i\psi_N\partial_i\psi_N}\right)$$

  corresponding to including gravitational potential and kinetic energy.

- $\overline{J}_{ij}$: Anisotropically expanding Bianchi model.

- This is the dominant effect of long wavelength corrections.

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Conclusions

- We expect that the corrections we computed will be small in realistic cosmologies, and it should therefore be acceptable to use an abridged dictionary (Chisari and Zaldarriaga, 2011):

$$A = \psi_N \qquad B_i = 0$$
$$H_L = -\psi_N \qquad h_{ij} = 0$$
$$v_i = v_{N,i}$$
$$\delta = \delta_N - \frac{3}{4\pi\rho_0 a^2}\left[\left(\frac{\dot{a}}{a}\right)^2 \psi_N + \frac{\dot{a}}{a}\dot{\psi}_N\right]$$

- Main purpose in obtaining complete dictionary is to evaluate accuracy of abridged dictionary. If corrections are small, we can have confidence that the Newtonian cosmology is accurately representing a general relativistic spacetime.