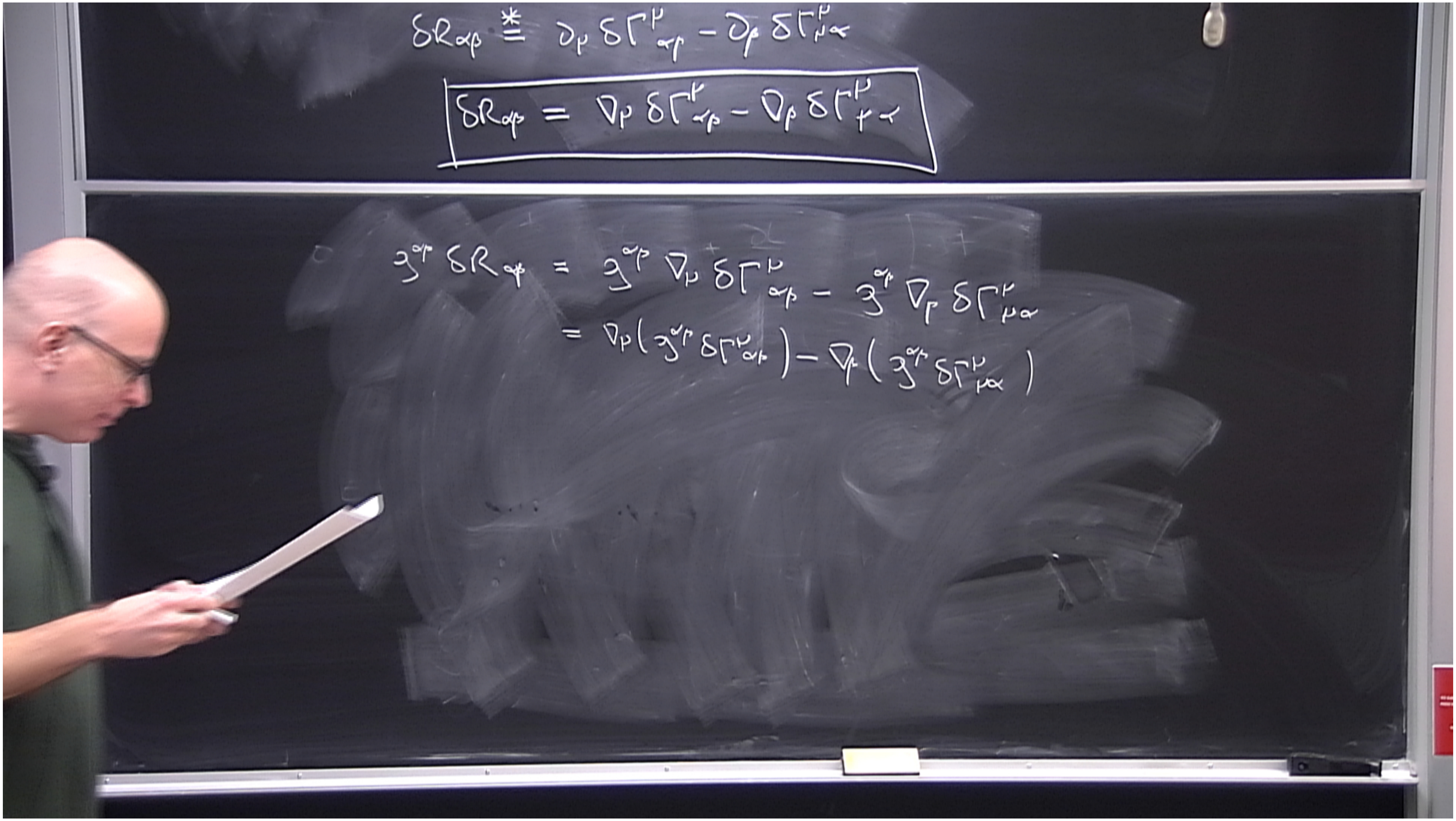
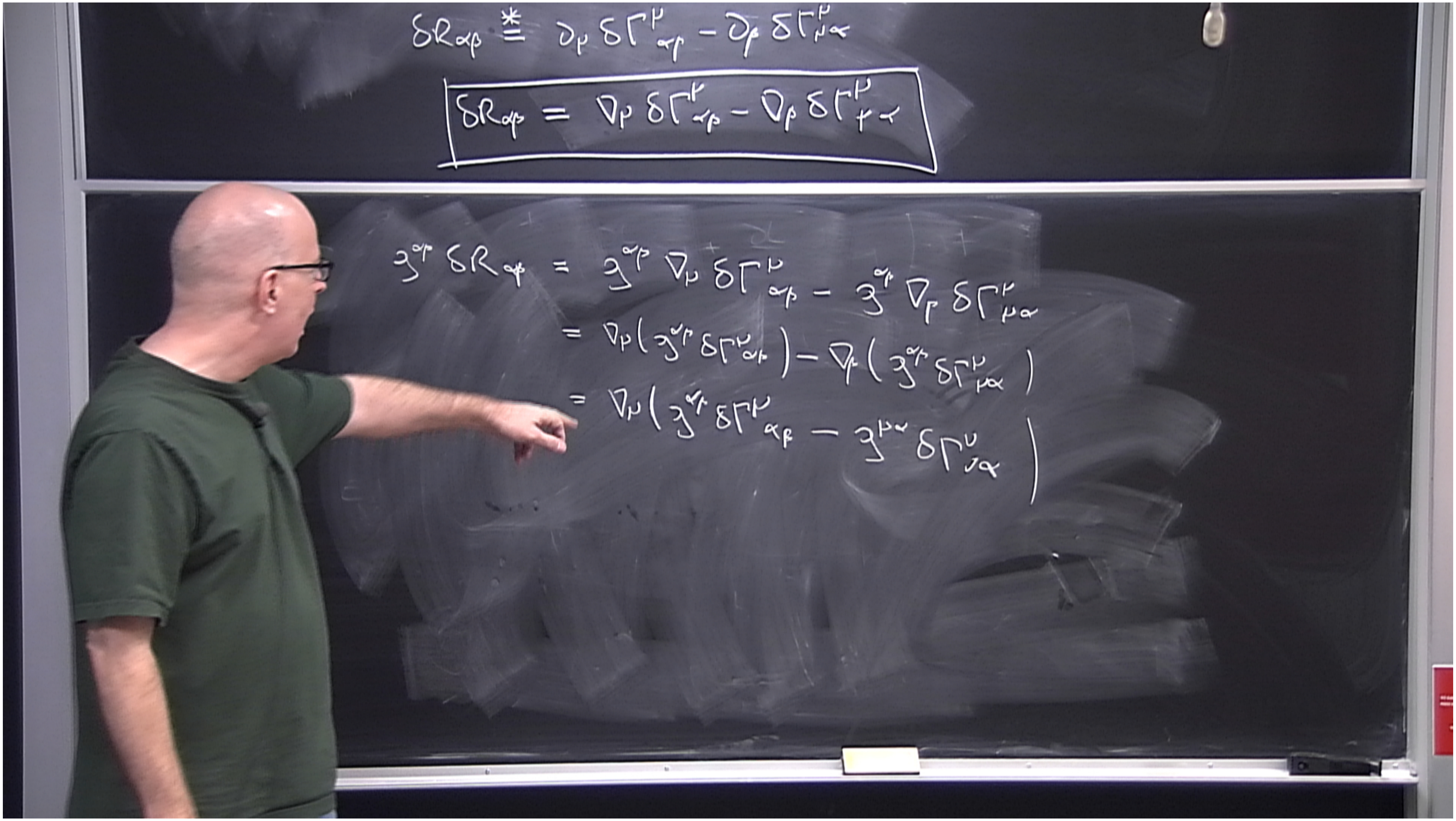
Title: Advanced General Relativity - Lecture 14

Date: Feb 29, 2012 03:30 PM

URL: http://pirsa.org/12020115

Abstract:


$\delta R_{\alpha\beta} \overset{*}{=} \partial_\mu \delta\Gamma^\mu_{\alpha\beta} - \partial_\beta \delta\Gamma^\mu_{\mu\alpha}$
$\delta R_{\alpha\beta} = \nabla_\mu \delta\Gamma^\mu_{\alpha\beta} - \nabla_\beta \delta\Gamma^\mu_{\mu\alpha}$
$g^{\alpha\beta} \delta R_{\alpha\beta} = g^{\alpha\beta} \nabla_\mu \delta\Gamma^\mu_{\alpha\beta} - g^{\alpha\beta} \nabla_\beta \delta\Gamma^\mu_{\mu\alpha}$
$= \nabla_\mu (g^{\alpha\beta} \delta\Gamma^\mu_{\alpha\beta}) - \nabla_\beta (g^{\alpha\beta} \delta\Gamma^\mu_{\mu\alpha})$


$\delta R_{\alpha\beta} \overset{*}{=} \partial_\mu \delta\Gamma^\mu_{\alpha\beta} - \partial_\beta \delta\Gamma^\mu_{\mu\alpha}$
$\delta R_{\alpha\beta} = \nabla_\mu \delta\Gamma^\mu_{\alpha\beta} - \nabla_\beta \delta\Gamma^\mu_{\mu\alpha}$
$g^{\alpha\beta}\delta R_{\alpha\beta} = g^{\alpha\beta}\nabla_\mu \delta\Gamma^\mu_{\alpha\beta} - g^{\alpha\beta}\nabla_\beta \delta\Gamma^\mu_{\mu\alpha}$
$= \nabla_\mu(g^{\alpha\beta}\delta\Gamma^\mu_{\alpha\beta}) - \nabla_\beta(g^{\alpha\beta}\delta\Gamma^\mu_{\mu\alpha})$
$= \nabla_\mu(g^{\alpha\beta}\delta\Gamma^\mu_{\alpha\beta} - g^{\mu\alpha}\delta\Gamma^\nu_{\nu\alpha})$

$$\delta R_{\alpha\beta} \overset{*}{=} \partial_\mu \delta\Gamma^\mu_{\alpha\beta} - \partial_\beta \delta\Gamma^\mu_{\mu\alpha}$$

$$\boxed{\delta R_{\alpha\beta} = \nabla_\mu \delta\Gamma^\mu_{\alpha\beta} - \nabla_\beta \delta\Gamma^\mu_{\mu\alpha}}$$

$$\begin{aligned}
g^{\alpha\beta}\delta R_{\alpha\beta} &= g^{\alpha\beta}\nabla_\mu \delta\Gamma^\mu_{\alpha\beta} - g^{\alpha\beta}\nabla_\beta \delta\Gamma^\mu_{\mu\alpha} \\
&= \nabla_\mu\left(g^{\alpha\beta}\delta\Gamma^\mu_{\alpha\beta}\right) - \nabla_\beta\left(g^{\alpha\beta}\delta\Gamma^\mu_{\mu\alpha}\right) \\
&= \nabla_\mu\left(g^{\alpha\beta}\delta\Gamma^\mu_{\alpha\beta} - g^{\mu\alpha}\delta\Gamma^\nu_{\nu\alpha}\right) \\
&= \nabla_\mu \delta v^\mu
\end{aligned}$$

$$\delta v^\mu = g^{\alpha\beta}\delta\Gamma^\mu_{\alpha\beta} - g^{\mu\alpha}\delta\Gamma^\nu_{\nu\alpha}$$

$$\delta R_{\alpha\beta} \stackrel{*}{=} \partial_\mu \delta\Gamma^\mu_{\alpha\beta} - \partial_\beta \delta\Gamma^\mu_{\mu\alpha}$$

$$\boxed{\delta R_{\alpha\beta} = \nabla_\mu \delta\Gamma^\mu_{\alpha\beta} - \nabla_\beta \delta\Gamma^\mu_{\mu\alpha}}$$

$$g^{\alpha\beta}\delta R_{\alpha\beta} = g^{\alpha\beta}\nabla_\mu \delta\Gamma^\mu_{\alpha\beta} - g^{\alpha\beta}\nabla_\beta \delta\Gamma^\mu_{\mu\alpha}$$

$$= \nabla_\mu\left(g^{\alpha\beta}\delta\Gamma^\mu_{\alpha\beta}\right) - \nabla_\beta\left(g^{\alpha\beta}\delta\Gamma^\mu_{\mu\alpha}\right)$$

$$= \nabla_\mu\left(g^{\alpha\beta}\delta\Gamma^\mu_{\alpha\beta} - g^{\mu\alpha}\delta\Gamma^\nu_{\nu\alpha}\right)$$

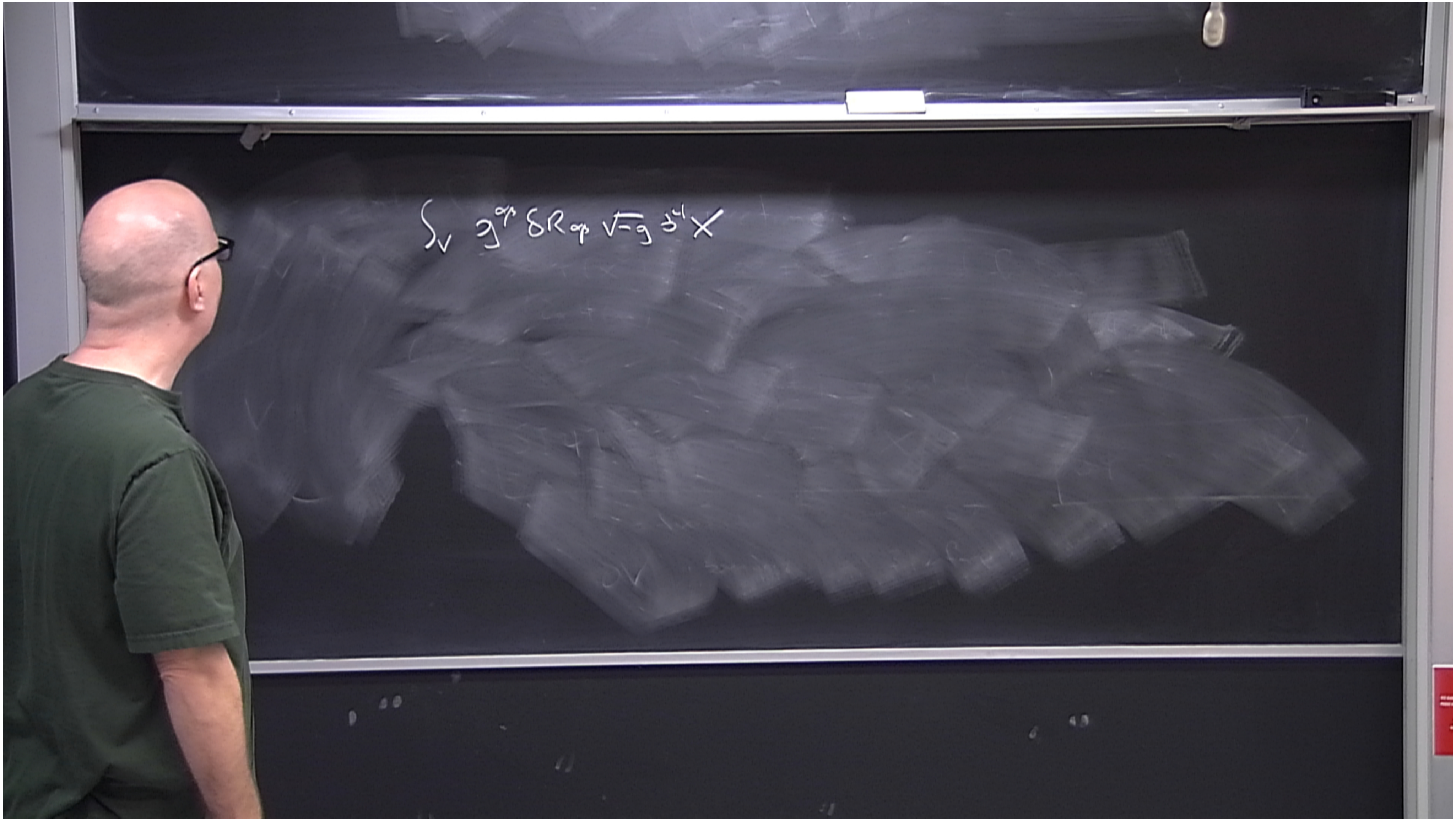
$$\boxed{\begin{aligned} g^{\alpha\beta}\delta R_{\alpha\beta} &= \nabla_\mu \delta v^\mu \\ \delta v^\mu &= g^{\alpha\beta}\delta\Gamma^\mu_{\alpha\beta} - g^{\mu\alpha}\delta\Gamma^\nu_{\nu\alpha} \end{aligned}}$$

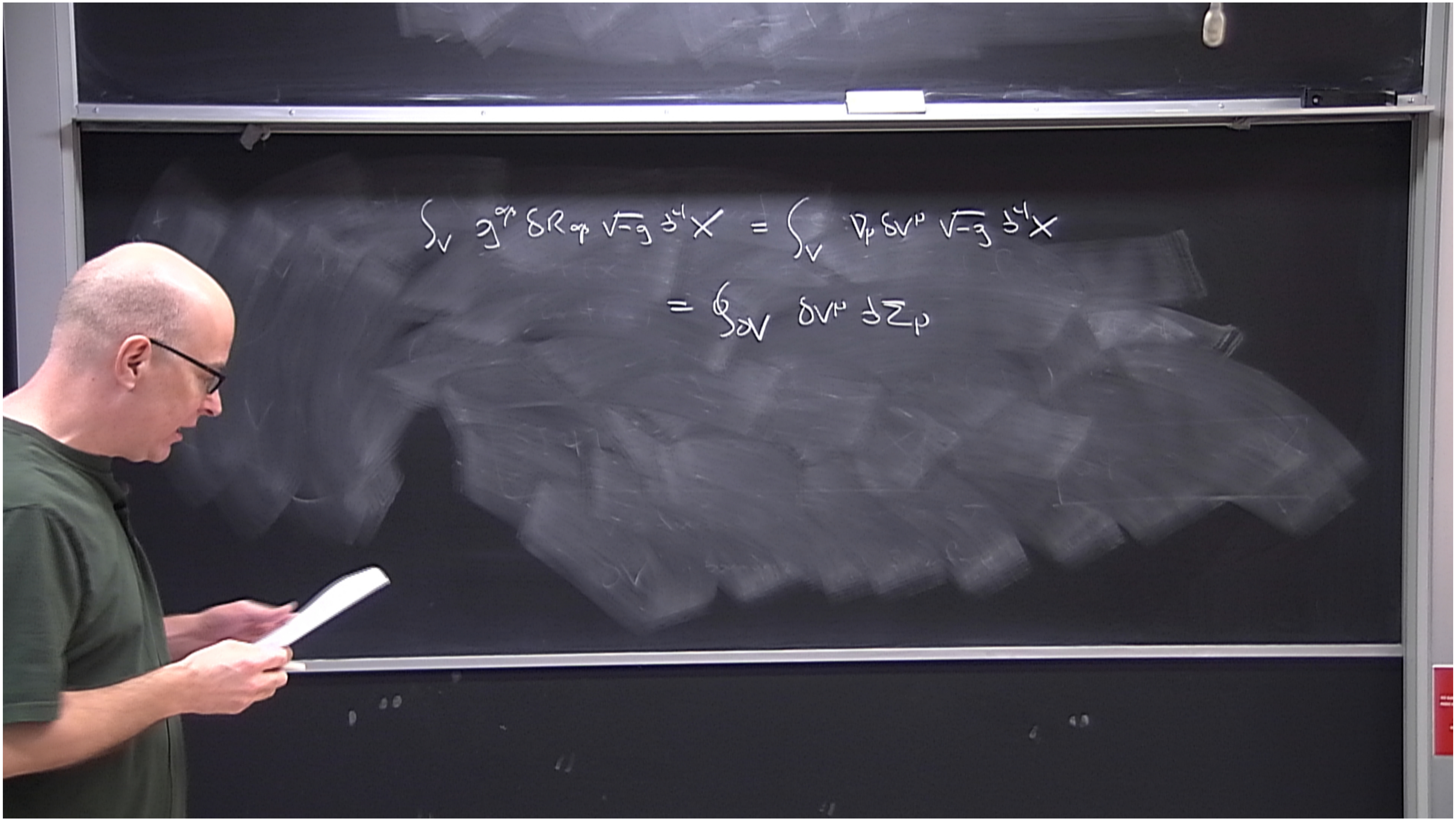
$$\delta R_{\alpha\beta} \overset{*}{=} \partial_\mu \delta\Gamma^\mu_{\alpha\beta} - \partial_\beta \delta\Gamma^\mu_{\mu\alpha}$$

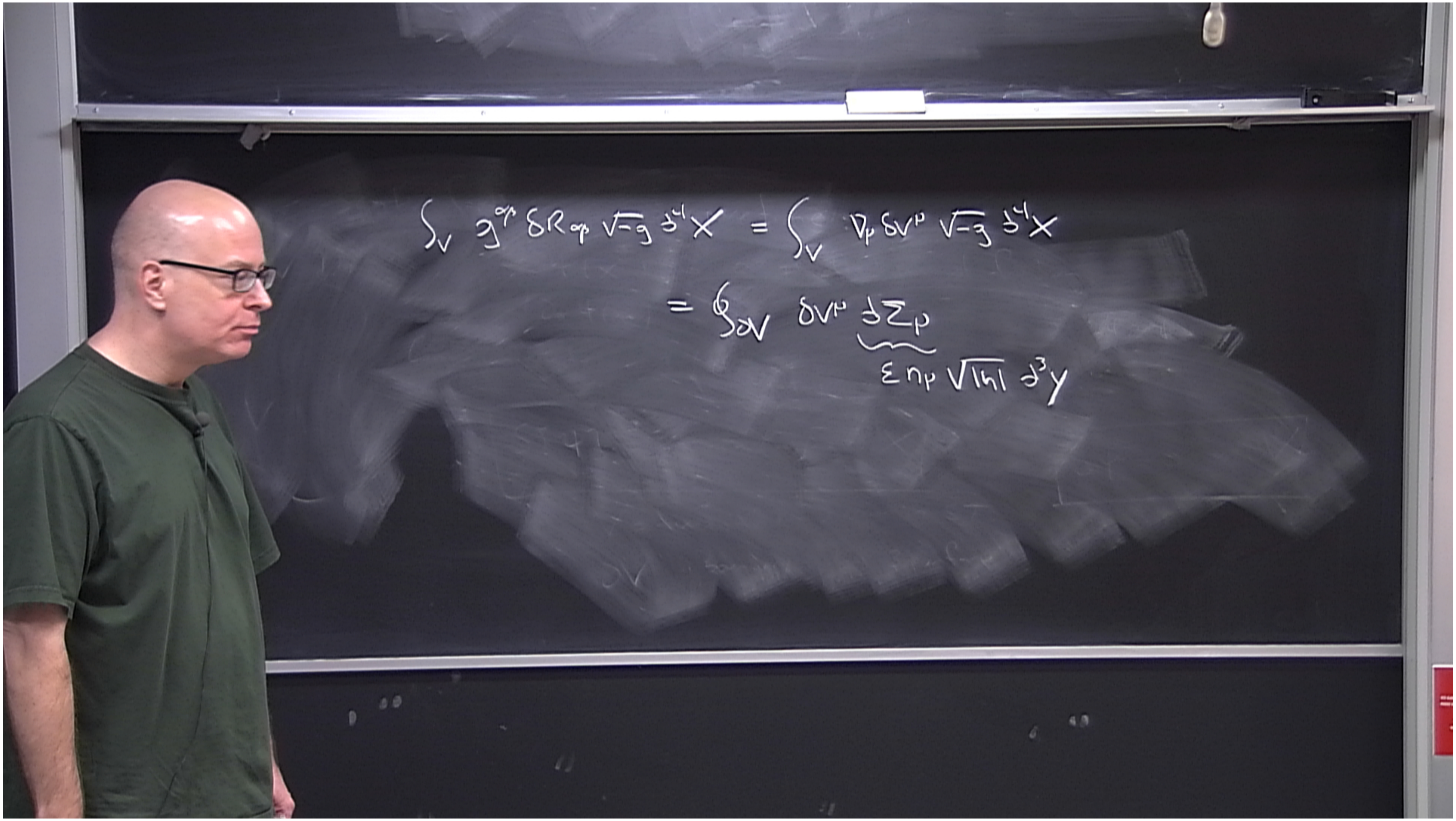
$$\boxed{\delta R_{\alpha\beta} = \nabla_\mu \delta\Gamma^\mu_{\alpha\beta} - \nabla_\beta \delta\Gamma^\mu_{\mu\alpha}}$$

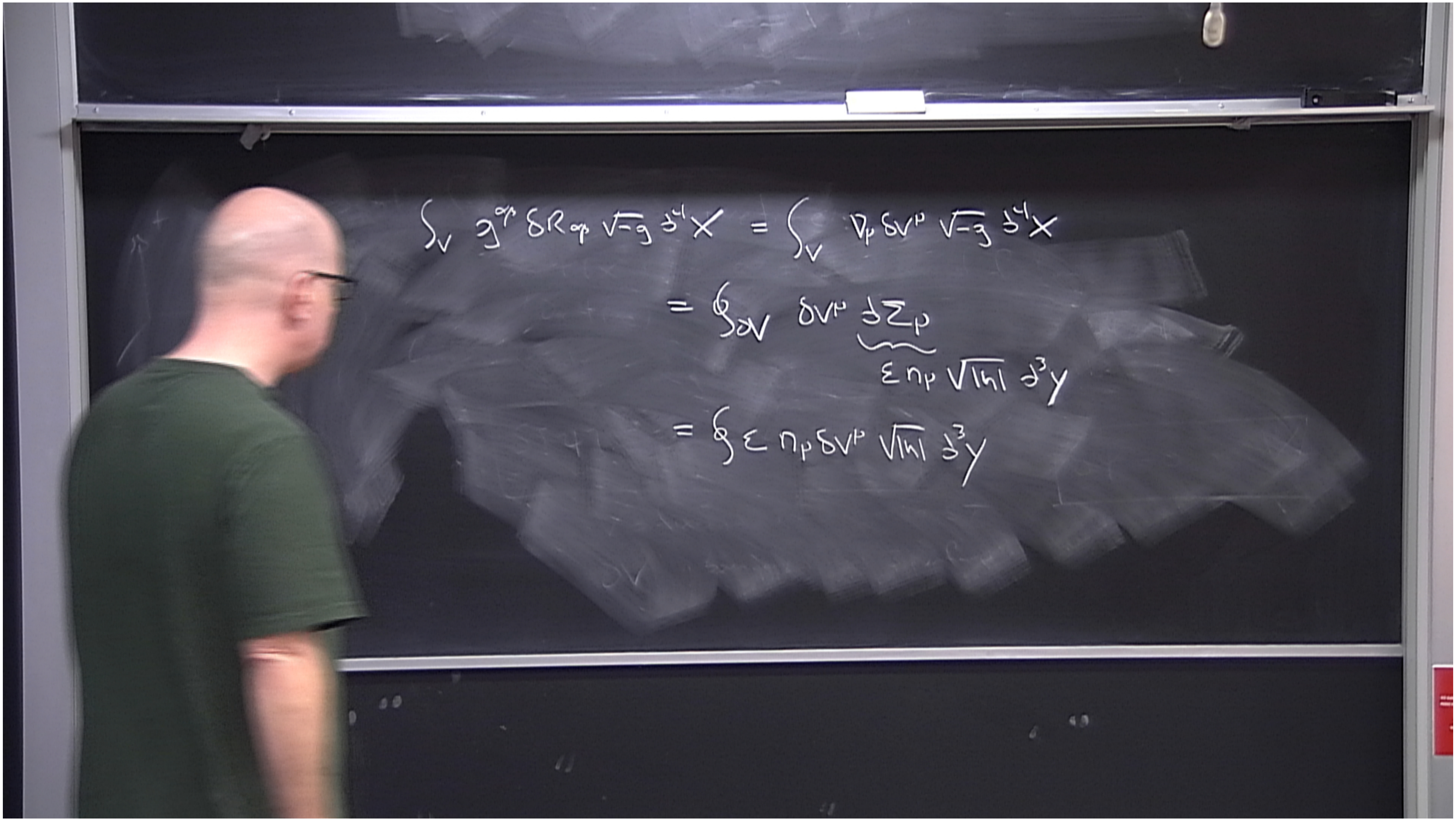
$$\begin{aligned} g^{\alpha\beta}\delta R_{\alpha\beta} &= g^{\alpha\beta}\nabla_\mu \delta\Gamma^\mu_{\alpha\beta} - g^{\alpha\beta}\nabla_\beta \delta\Gamma^\mu_{\mu\alpha} \\ &= \nabla_\mu(g^{\alpha\beta}\delta\Gamma^\mu_{\alpha\beta}) - \nabla_\beta(g^{\alpha\beta}\delta\Gamma^\mu_{\mu\alpha}) \\ &= \nabla_\mu(g^{\alpha\beta}\delta\Gamma^\mu_{\alpha\beta} - g^{\mu\alpha}\delta\Gamma^\nu_{\nu\alpha}) \end{aligned}$$

$$\boxed{\begin{aligned} g^{\alpha\beta}\delta R_{\alpha\beta} &= \nabla_\mu \delta v^\mu \\ \delta v^\mu &= g^{\alpha\beta}\delta\Gamma^\mu_{\alpha\beta} - g^{\mu\alpha}\delta\Gamma^\nu_{\nu\alpha} \end{aligned}}$$


$\int_V g^{\alpha\beta} \delta R_{\alpha\beta} \sqrt{-g}\, d^4x$


$\int_V g^{\alpha\beta}\,\delta R_{\alpha\beta}\sqrt{-g}\,d^4x = \int_V \nabla_\mu \delta V^\mu \sqrt{-g}\,d^4x$
$= \oint_{\partial V} \delta V^\mu \, d\Sigma_\mu$


$\int_V g^{\alpha\beta}\,\delta R_{\alpha\beta}\sqrt{-g}\,d^4x = \int_V \nabla_\mu \delta v^\mu \sqrt{-g}\,d^4x$
$= \oint_{\partial V} \delta v^\mu\, d\Sigma_\mu$
$\varepsilon\, n_\mu \sqrt{|h|}\, d^3y$


$\int_V g^{\alpha\beta}\,\delta R_{\alpha\beta}\sqrt{-g}\,d^4x = \int_V \nabla_\mu \delta v^\mu \sqrt{-g}\,d^4x$
$= \oint_{\partial V} \delta v^\mu \, d\Sigma_\mu$
$\varepsilon n_\mu \sqrt{|h|}\,d^3y$
$= \oint \varepsilon n_\mu \delta v^\mu \sqrt{|h|}\,d^3y$

$$\int_V g^{\alpha\beta}\,\delta R_{\alpha\beta}\,\sqrt{-g}\,d^4x = \int_V \nabla_\mu \delta v^\mu \sqrt{-g}\,d^4x$$

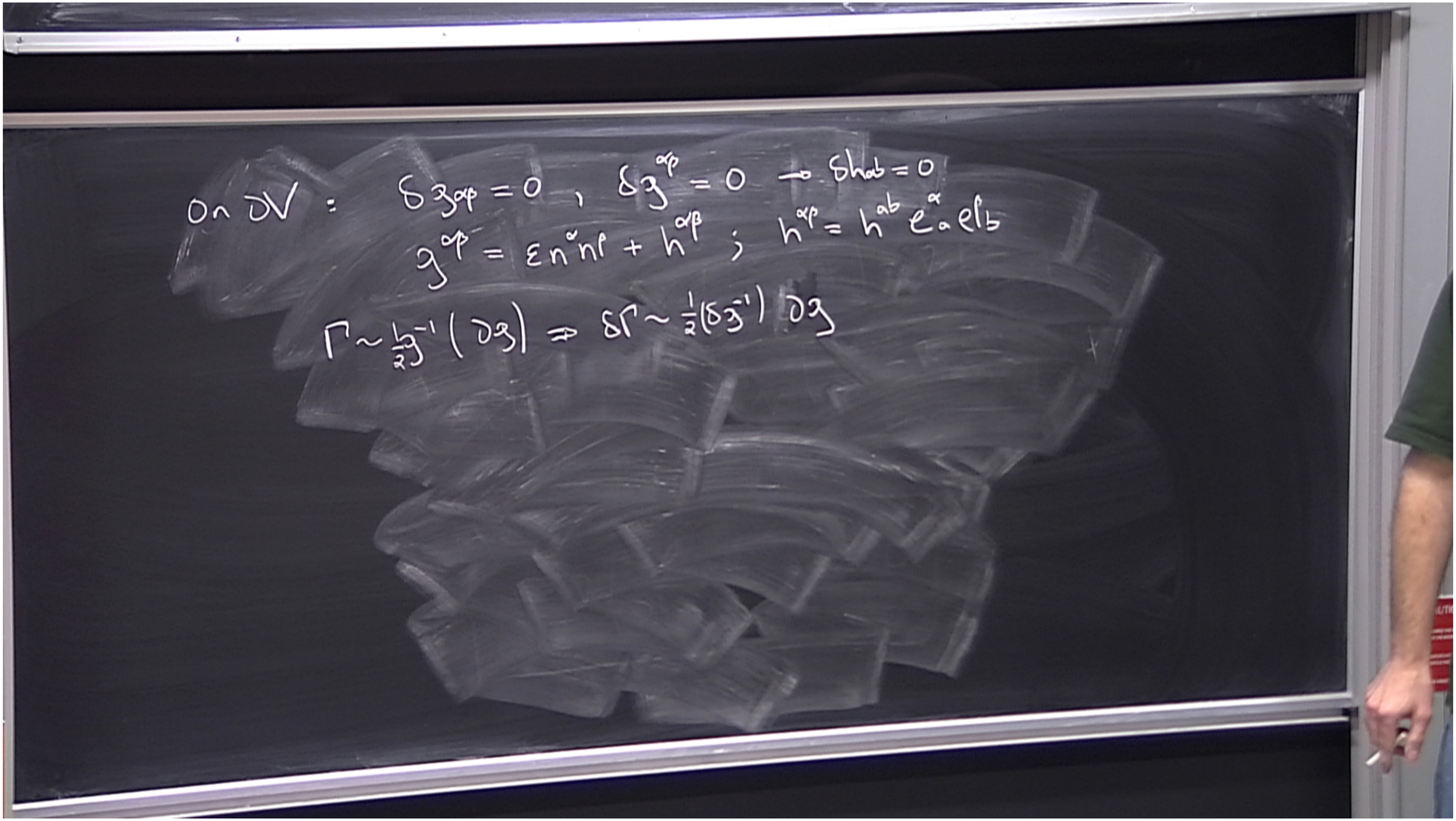
$$= \oint_{\partial V} \delta v^\mu \underbrace{d\Sigma_\mu}_{\varepsilon\, n_\mu \sqrt{|h|}\, d^3y}$$

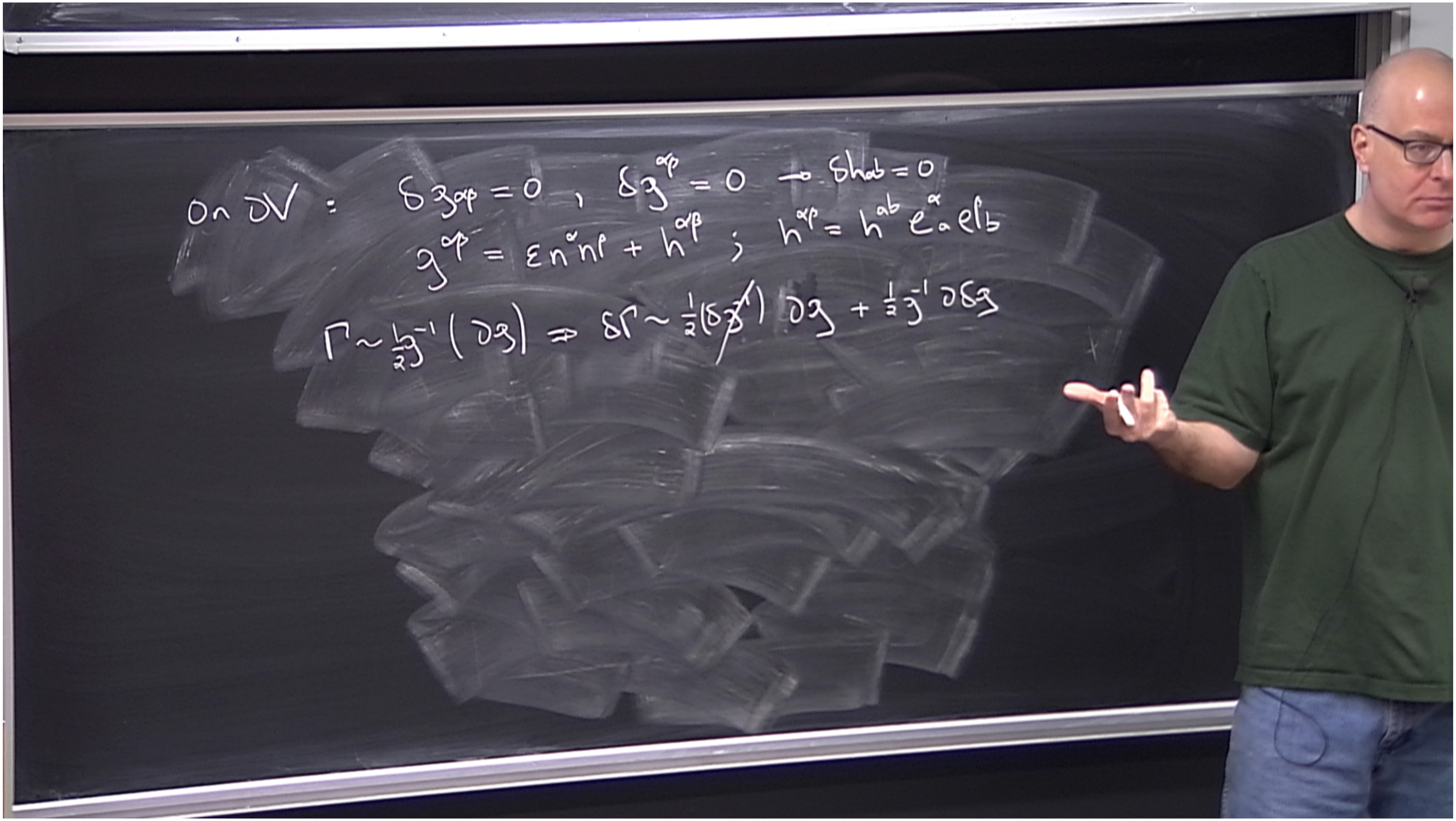
$$= \oint \varepsilon\, n_\mu\, \delta v^\mu \sqrt{|h|}\, d^3y$$

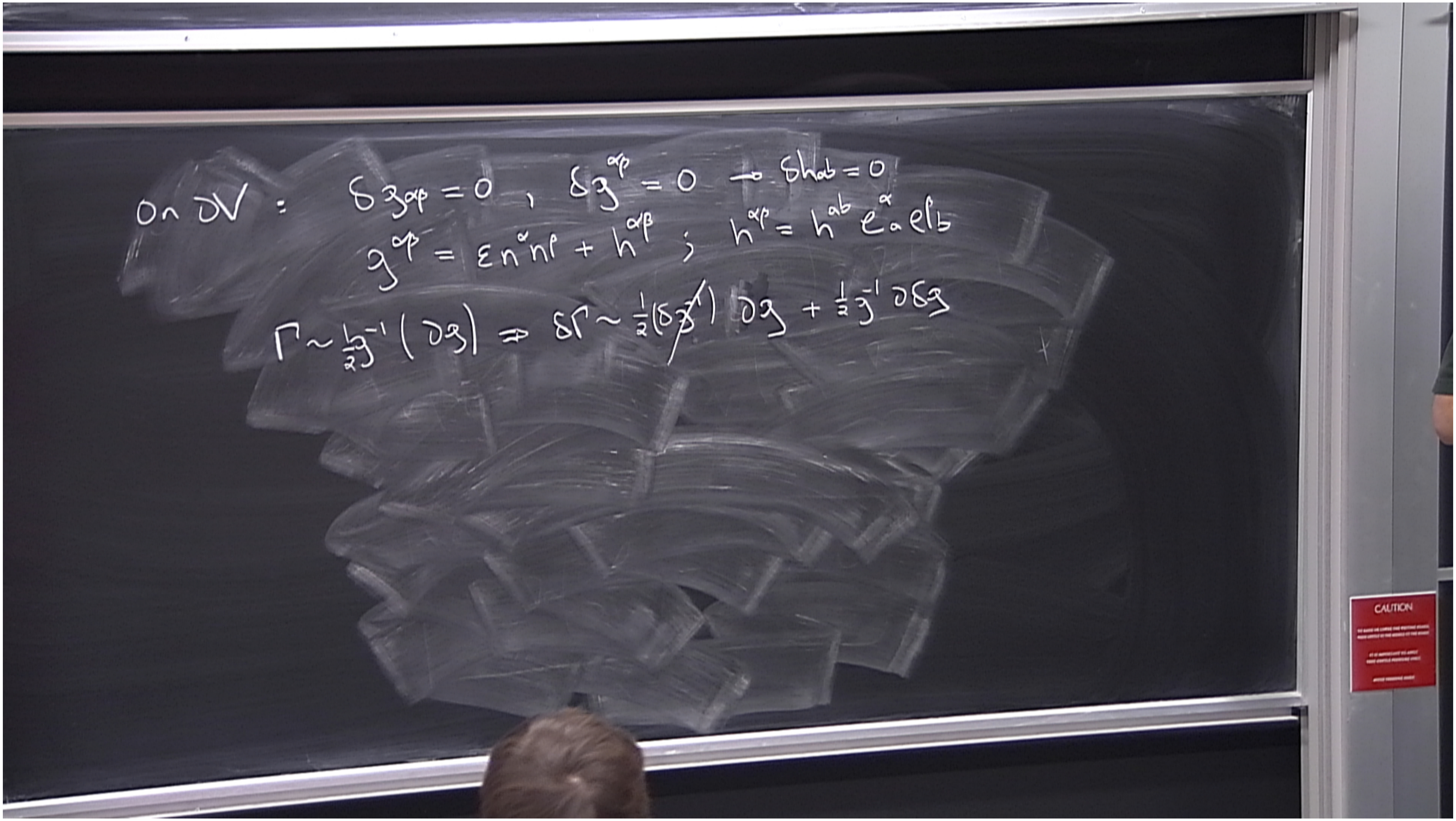
On ∂V : $\delta g_{\alpha\beta} = 0$, $\delta g^{\alpha\beta} = 0 \rightarrow \delta h_{ab} = 0$

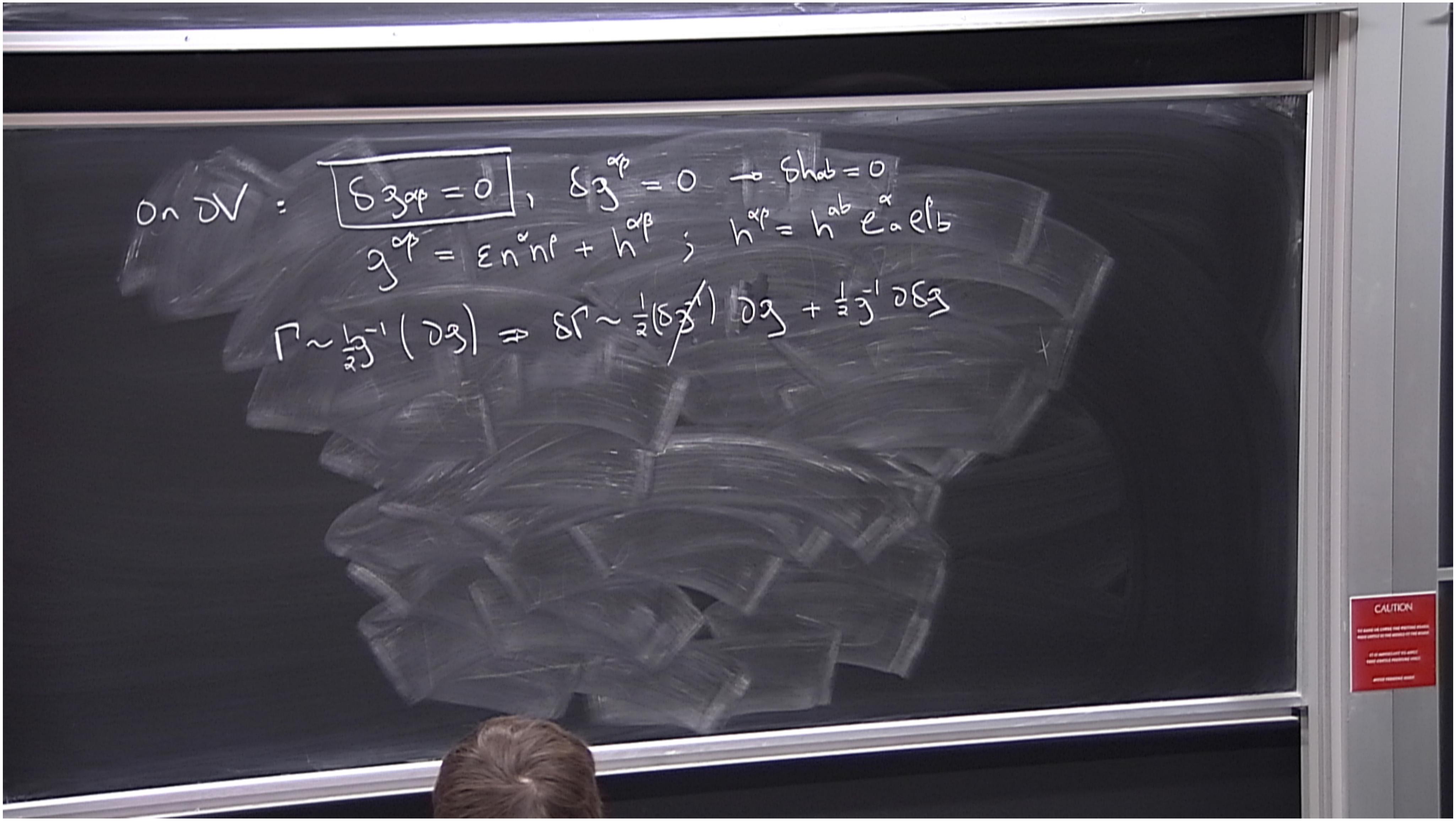
$$g^{\alpha\beta} = \varepsilon n^\alpha n^\beta + h^{\alpha\beta} \; ; \; h^{\alpha\beta} = h^{ab} e^\alpha_a e^\beta_b$$

$$\Gamma \sim \tfrac{1}{2} g^{-1} (\partial g)$$


On ∂V : δg_{αβ} = 0 , δg^{αβ} = 0 → δh_{ab} = 0
g^{αβ} = ε n^α n^β + h^{αβ} ; h^{αβ} = h^{ab} e^α_a e^β_b
Γ ~ ½ g^{-1} (∂g) ⇒ δΓ ~ ½(δg^{-1}) ∂g


On ∂V : δg_αβ = 0 , δg^αβ = 0 → δh_ab = 0
g^αβ = ε n^α n^β + h^αβ ; h^αβ = h^ab e^α_a e^β_b
Γ ~ ½ g^-1 (∂g) ⇒ δΓ ~ ½(δg^-1) ∂g + ½ g^-1 ∂δg


On ∂V : $\delta g_{\alpha\beta} = 0$, $\delta g^{\alpha\beta} = 0 \to \delta h_{ab} = 0$
$g^{\alpha\beta} = \varepsilon n^\alpha n^\beta + h^{\alpha\beta}$; $h^{\alpha\beta} = h^{ab} e^\alpha_a e^\beta_b$
$\Gamma \sim \frac{1}{2} g^{-1} (\partial g) \Rightarrow \delta\Gamma \sim \frac{1}{2} (\delta g^{-1}) \partial g + \frac{1}{2} g^{-1} \partial \delta g$


On ∂V : $\boxed{\delta g_{\alpha\beta} = 0}$, $\delta g^{\alpha\beta} = 0 \to \delta h_{ab} = 0$
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On ∂V : $\boxed{\delta g_{\alpha\beta} = 0}$, $\delta g^{\alpha\beta} = 0 \rightarrow \delta h_{ab} = 0$

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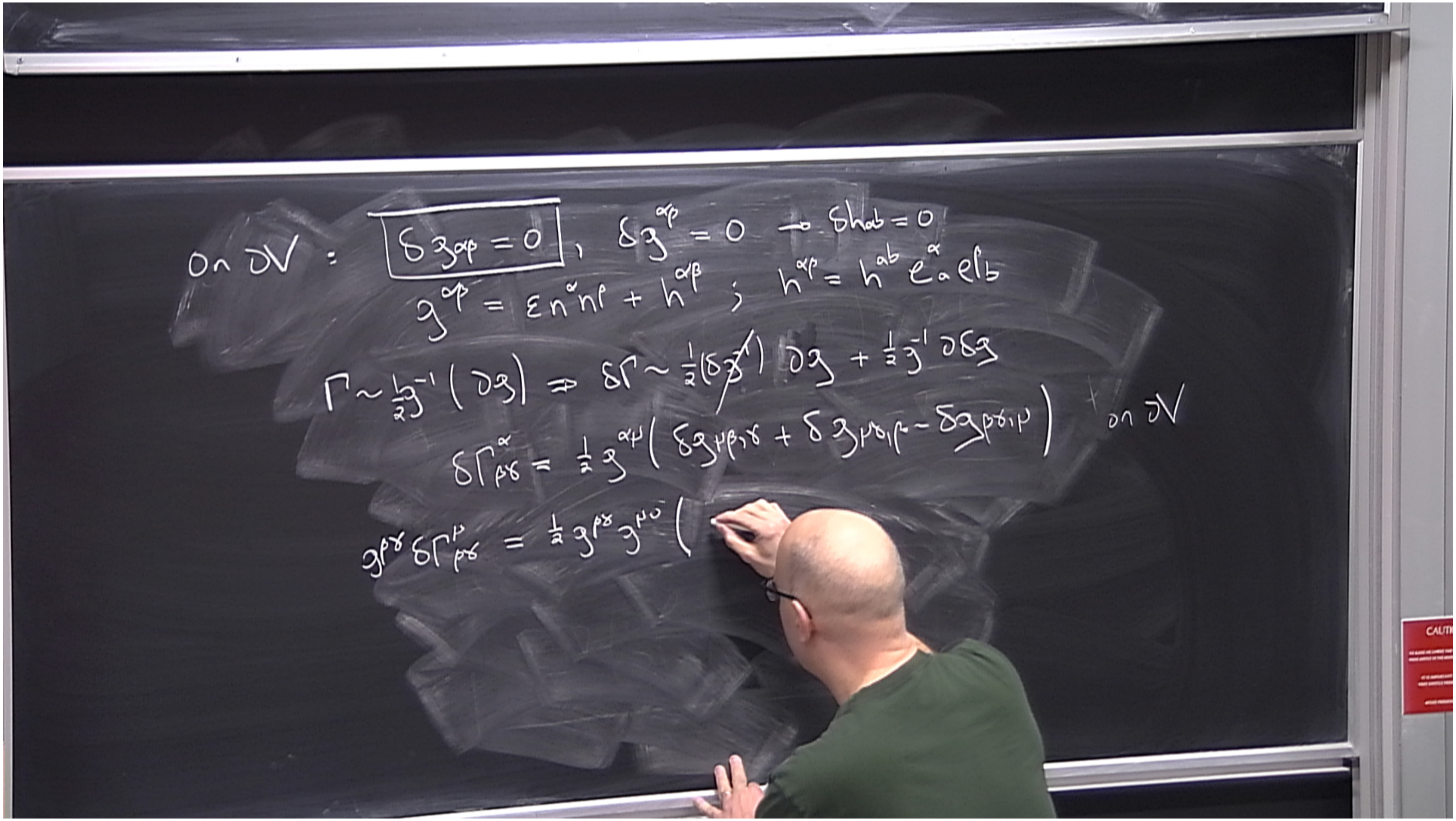
$$\Gamma \sim \tfrac{1}{2} g^{-1} (\partial g) \Rightarrow \delta\Gamma \sim \tfrac{1}{2}(\delta g^{-1})\, \partial g + \tfrac{1}{2} g^{-1} \partial \delta g$$

$$\delta\Gamma^\alpha_{\beta\gamma} = \tfrac{1}{2} g^{\alpha\mu} \left(\delta g_{\mu\beta,\gamma} + \delta g_{\mu\gamma,\beta} - \delta g_{\beta\gamma,\mu} \right)$$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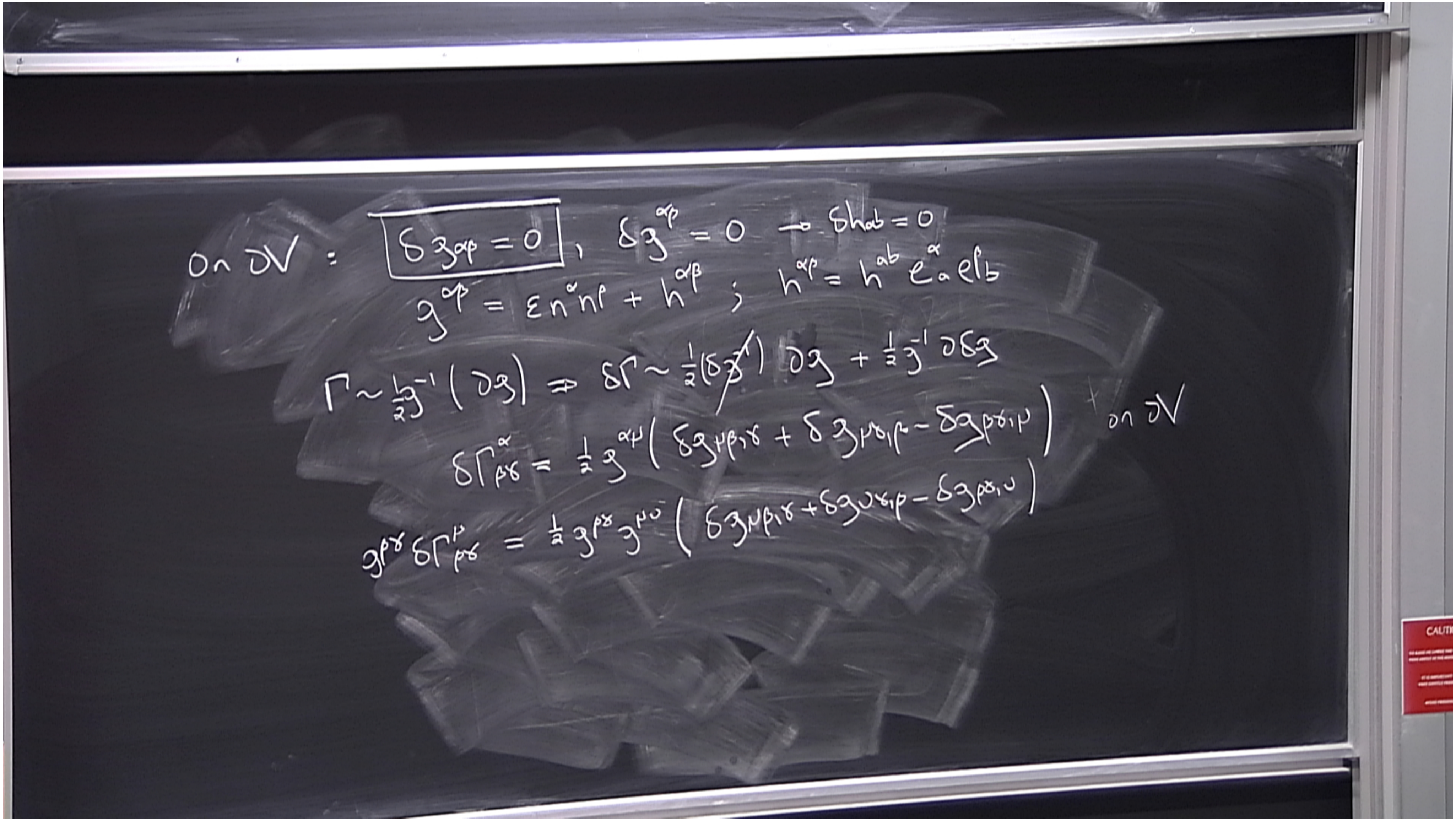
$\delta\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\mu} (\delta g_{\mu\beta,\gamma} + \delta g_{\mu\gamma,\beta} - \delta g_{\beta\gamma,\mu})$ on ∂V
$g^{\beta\gamma} \delta\Gamma^\mu_{\beta\gamma} = \frac{1}{2} g^{\beta\gamma} g^{\mu\sigma} ($


On ∂V : $\boxed{\delta g_{\alpha\beta} = 0}$, $\delta g^{\alpha\beta} = 0 \rightarrow \delta h_{ab} = 0$
$g^{\alpha\beta} = \varepsilon n^{\alpha} n^{\beta} + h^{\alpha\beta}$; $h^{\alpha\beta} = h^{ab} e^{\alpha}_{a} e^{\beta}_{b}$
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$g^{\beta\gamma} \delta \Gamma^{\mu}_{\beta\gamma} = \frac{1}{2} g^{\beta\gamma} g^{\mu\nu} (\delta g_{\nu\beta,\gamma} + \delta g_{\nu\gamma,\beta} - \delta g_{\beta\gamma,\nu})$

On ∂V : $\boxed{\delta g_{\alpha\beta} = 0}$, $\delta g^{\alpha\beta} = 0 \rightarrow \delta h_{ab} = 0$

$$g^{\alpha\beta} = \varepsilon n^\alpha n^\beta + h^{\alpha\beta} \; ; \; h^{\alpha\beta} = h^{ab} e^\alpha_a e^\beta_b$$

$$\Gamma \sim \tfrac{1}{2} g^{-1} (\partial g) \Rightarrow \delta\Gamma \sim \tfrac{1}{2}(\delta g^{-1})\, \partial g + \tfrac{1}{2} g^{-1} \partial \delta g$$

$$\delta\Gamma^\alpha_{\beta\gamma} = \tfrac{1}{2} g^{\alpha\mu} \left(\delta g_{\mu\beta,\gamma} + \delta g_{\mu\gamma,\beta} - \delta g_{\beta\gamma,\mu} \right) \quad \text{on } \partial V$$

$$g^{\beta\gamma} \delta\Gamma^\mu_{\beta\gamma} = \tfrac{1}{2} g^{\beta\gamma} g^{\mu\nu} \left(\delta g_{\nu\beta,\gamma} + \delta g_{\nu\gamma,\beta} - \delta g_{\beta\gamma,\nu} \right)$$

$$\delta\Gamma^\beta_{\beta\gamma} = \tfrac{1}{2} g^{\beta\mu} \left(\delta g_{\mu\beta,\gamma} + \delta g_{\mu\gamma,\beta} - \delta g_{\beta\gamma,\mu} \right)$$

$$\text{On } \partial V: \quad \boxed{\delta g_{\alpha\beta} = 0}, \quad \delta g^{\alpha\beta} = 0 \rightarrow \delta h_{ab} = 0$$

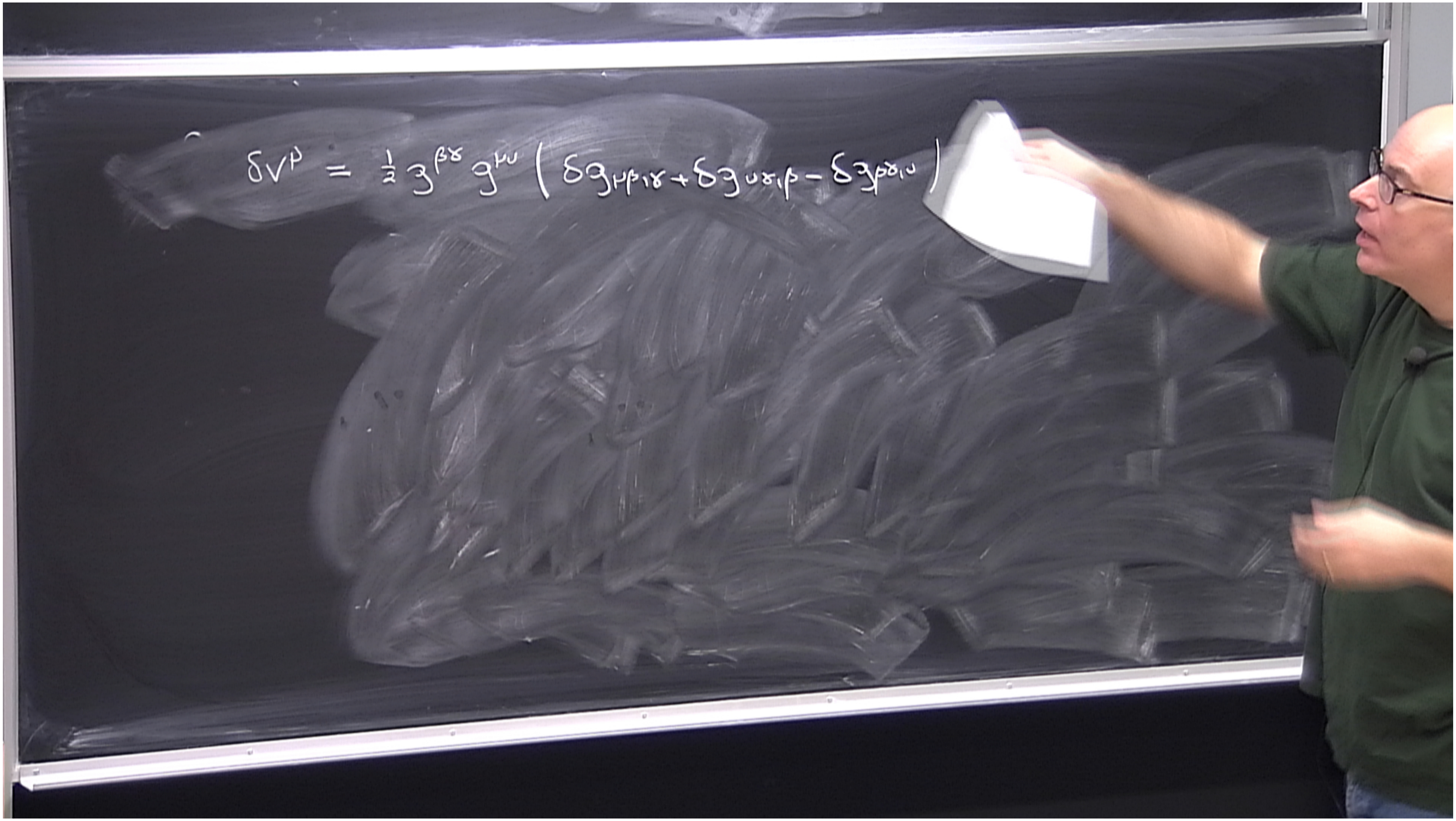
$$g^{\alpha\beta} = \varepsilon n^\alpha n^\beta + h^{\alpha\beta} \; ; \quad h^{\alpha\beta} = h^{ab} e^\alpha_a e^\beta_b$$

$$\Gamma \sim \tfrac{1}{2} g^{-1} (\partial g) \Rightarrow \delta\Gamma \sim \tfrac{1}{2} \cancel{(\delta g^{-1})} \, \partial g + \tfrac{1}{2} g^{-1} \partial \delta g$$

$$\delta\Gamma^\alpha_{\beta\gamma} = \tfrac{1}{2} g^{\alpha\mu} \left(\delta g_{\mu\beta,\gamma} + \delta g_{\mu\gamma,\beta} - \delta g_{\beta\gamma,\mu} \right) \quad \text{on } \partial V$$

$$g^{\beta\gamma} \delta\Gamma^\mu_{\beta\gamma} = \tfrac{1}{2} g^{\beta\gamma} g^{\mu\nu} \left(\delta g_{\nu\beta,\gamma} + \delta g_{\nu\gamma,\beta} - \delta g_{\beta\gamma,\nu} \right)$$

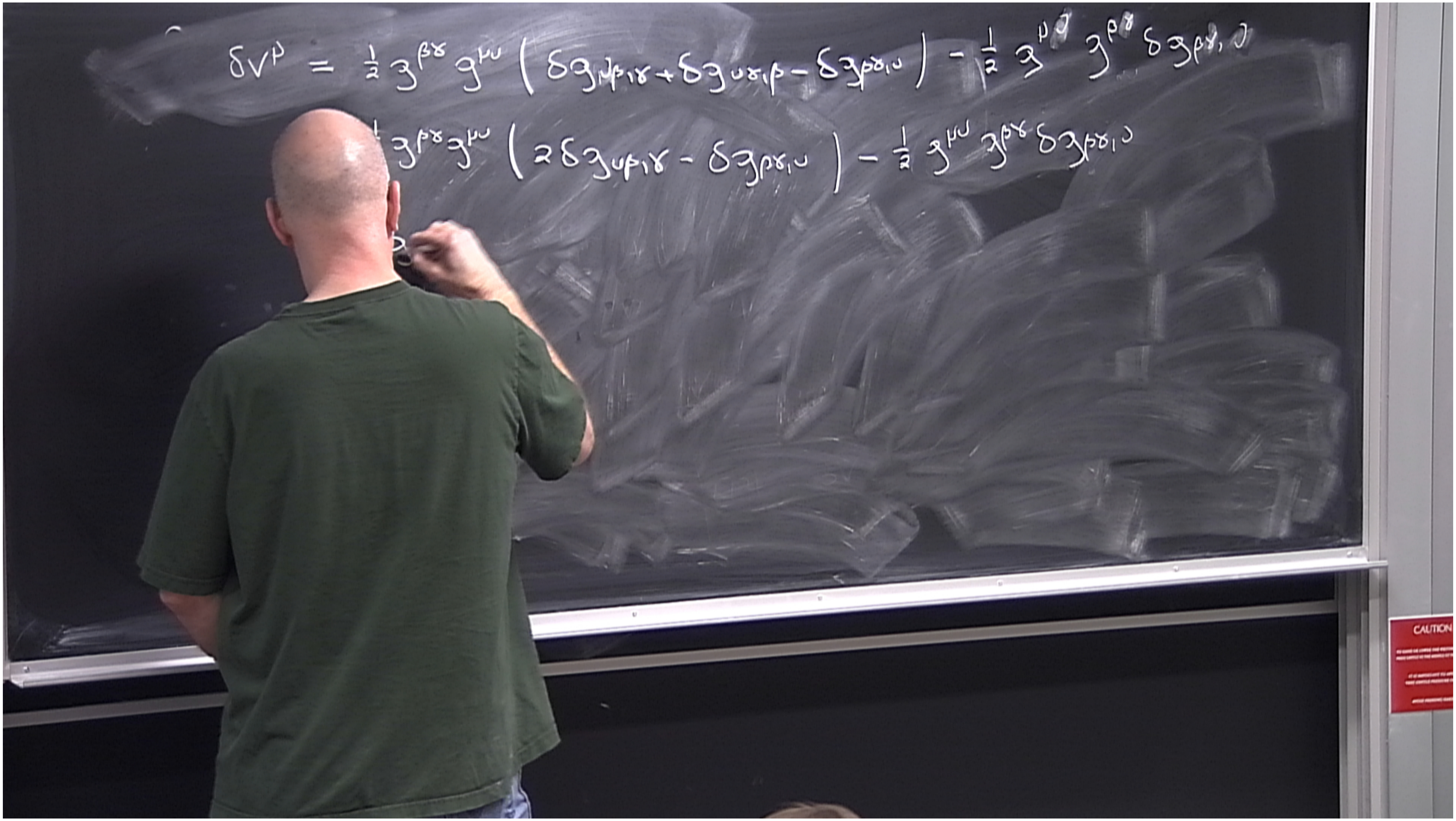
$$\delta\Gamma^\beta_{\beta\gamma} = \tfrac{1}{2} g^{\beta\mu} \left(\delta g_{\mu\beta,\gamma} + \cancel{\delta g_{\mu\gamma,\beta}} - \cancel{\delta g_{\beta\gamma,\mu}} \right) = \tfrac{1}{2} g^{\beta\mu} \delta g_{\beta\mu,\gamma}$$

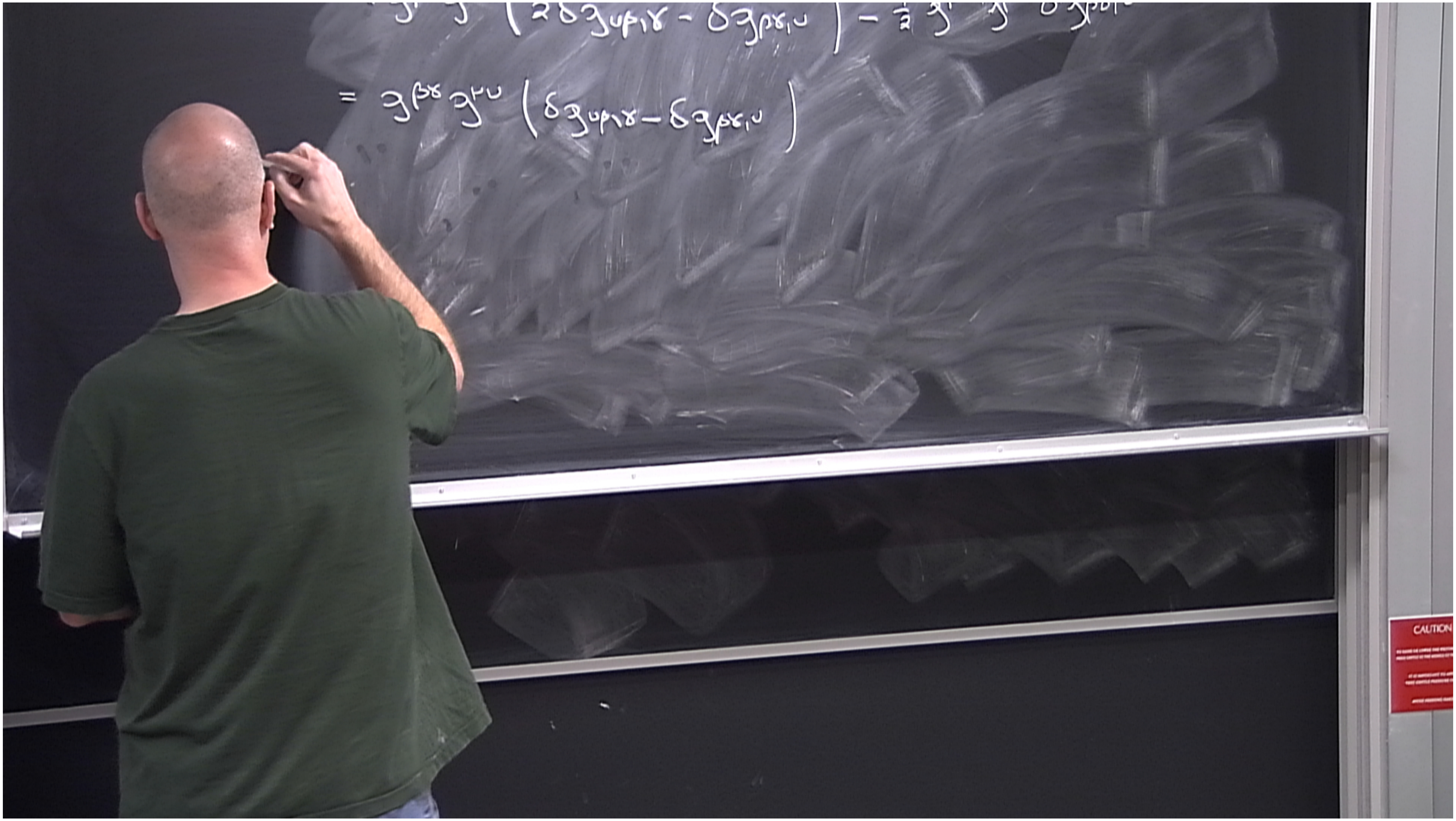

$\delta \Gamma^{\mu} = \frac{1}{2} g^{\beta\gamma} g^{\mu\nu} \left(\delta g_{\mu\beta,\gamma} + \delta g_{\nu\gamma,\beta} - \delta g_{\beta\gamma,\nu} \right)$

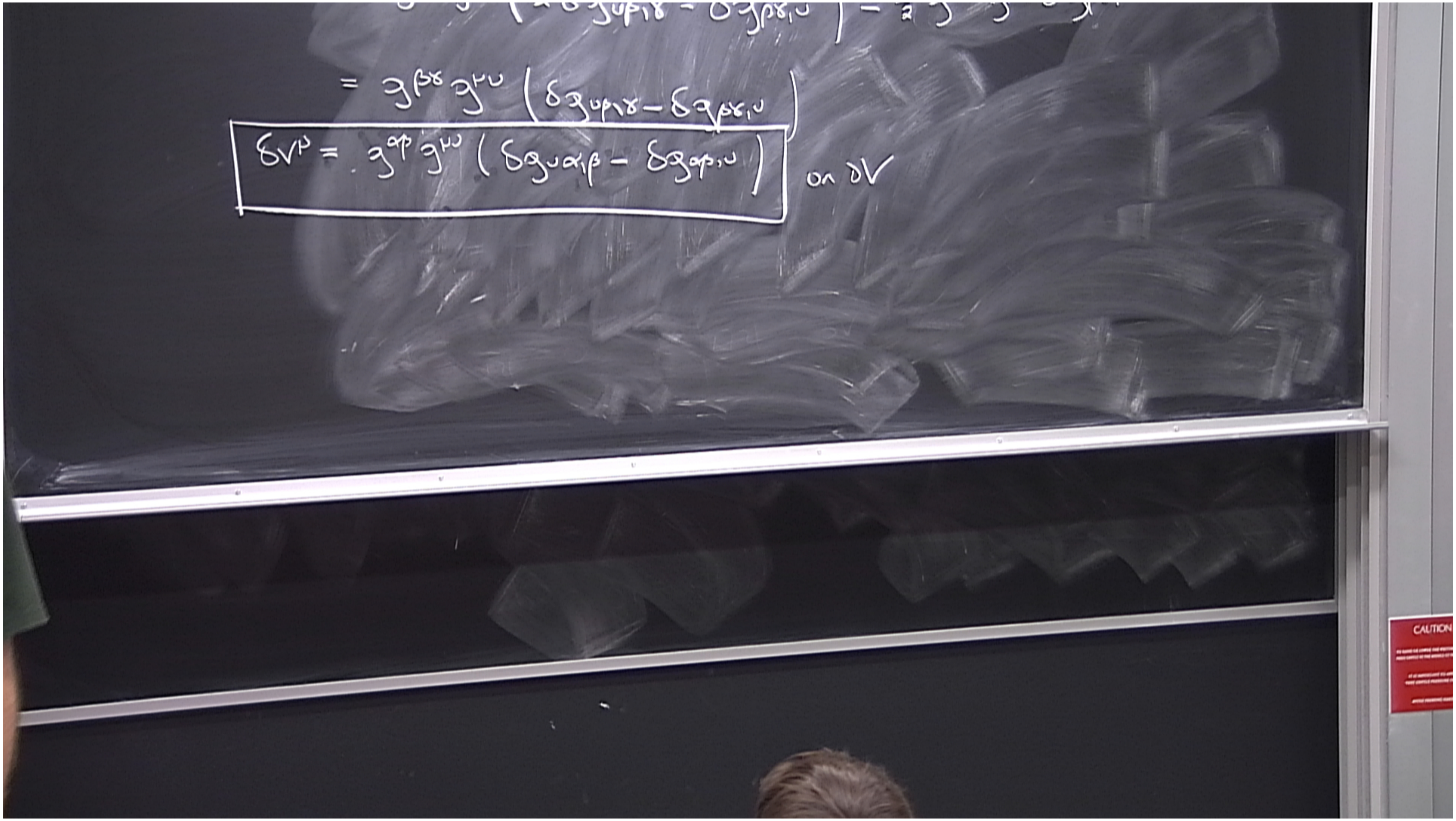
$$g^{\beta\gamma}\delta\Gamma^{\mu}_{\beta\gamma} = \frac{1}{2} g^{\beta\gamma} g^{\mu\nu}\left(\delta g_{\nu\beta,\gamma} + \delta g_{\nu\gamma,\beta} - \delta g_{\beta\gamma,\nu}\right)$$

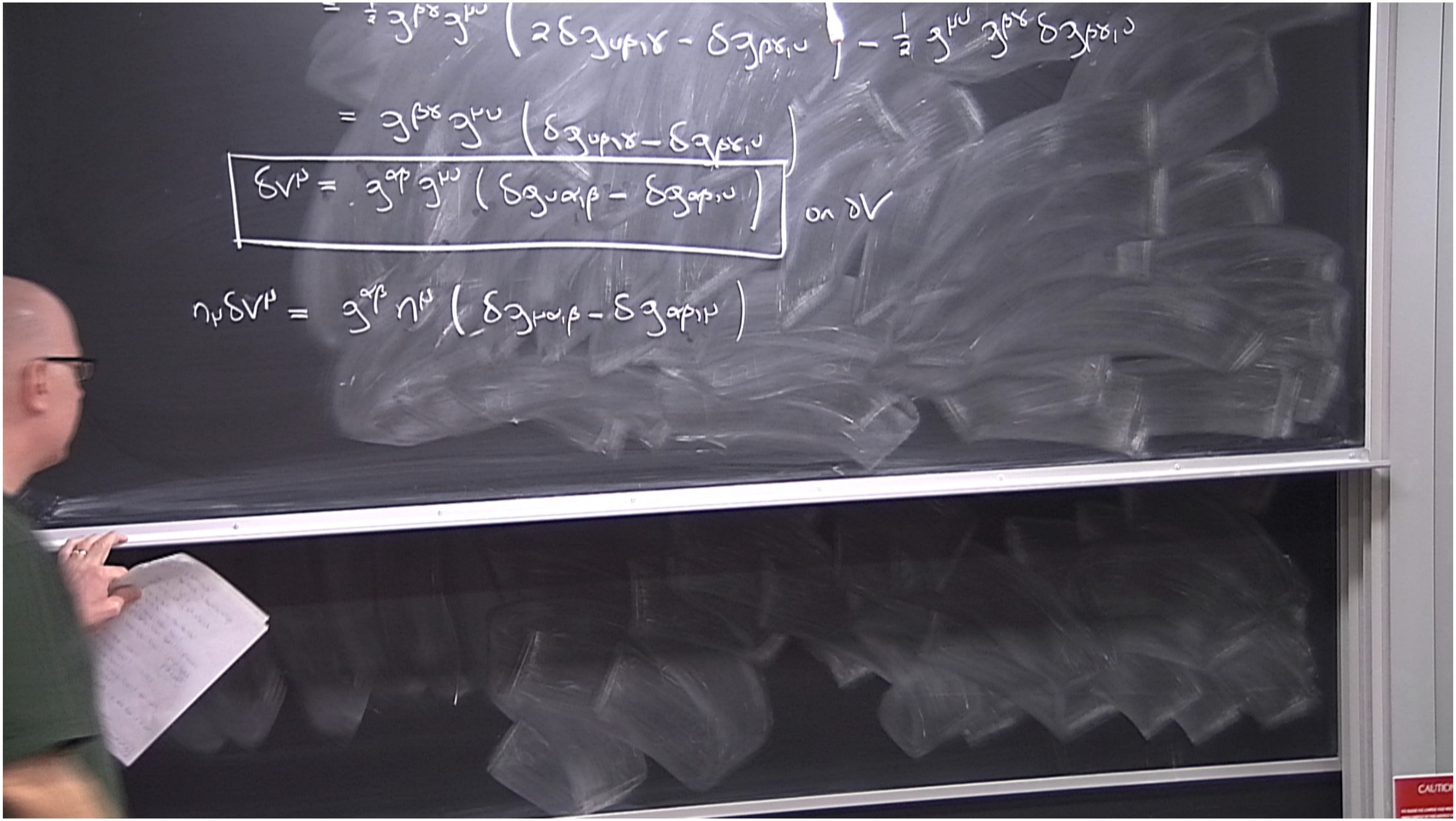
$$\delta\Gamma^{\beta}_{\beta\gamma} = \frac{1}{2} g^{\beta\mu}\left(\delta g_{\mu\beta,\gamma} + \cancel{\delta g_{\mu\gamma,\beta}} - \cancel{\delta g_{\beta\gamma,\mu}}\right) = \frac{1}{2} g^{\beta\nu}\delta g_{\beta\nu,\gamma}$$

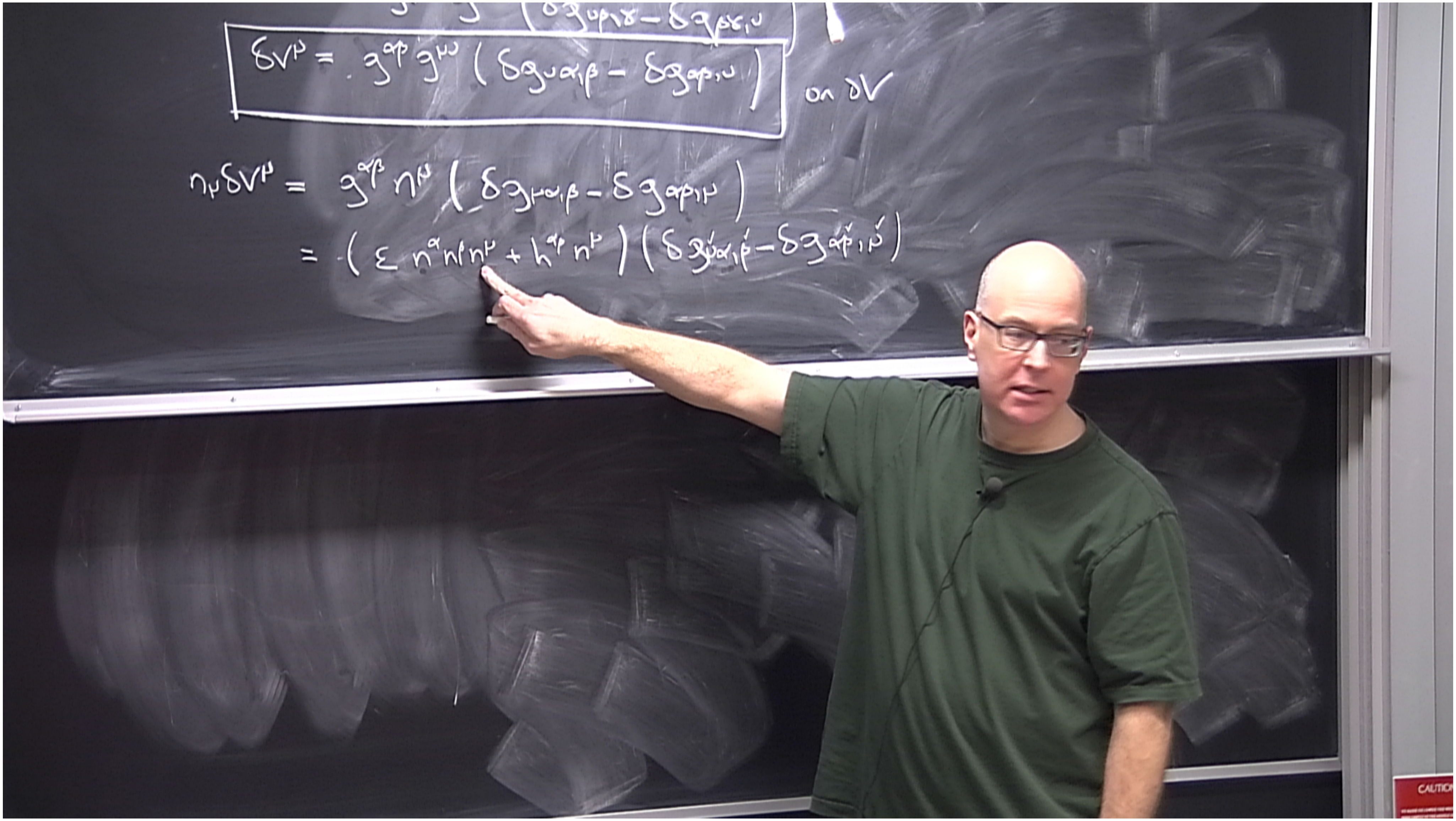
$$\delta V^{\mu} = \frac{1}{2} g^{\beta\gamma} g^{\mu\nu}\left(\delta g_{\nu\beta,\gamma} + \delta g_{\nu\gamma,\beta} - \delta g_{\beta\gamma,\nu}\right) - \frac{1}{2} g^{\mu\alpha} g^{\beta\nu}\delta g_{\beta\nu,\alpha}$$

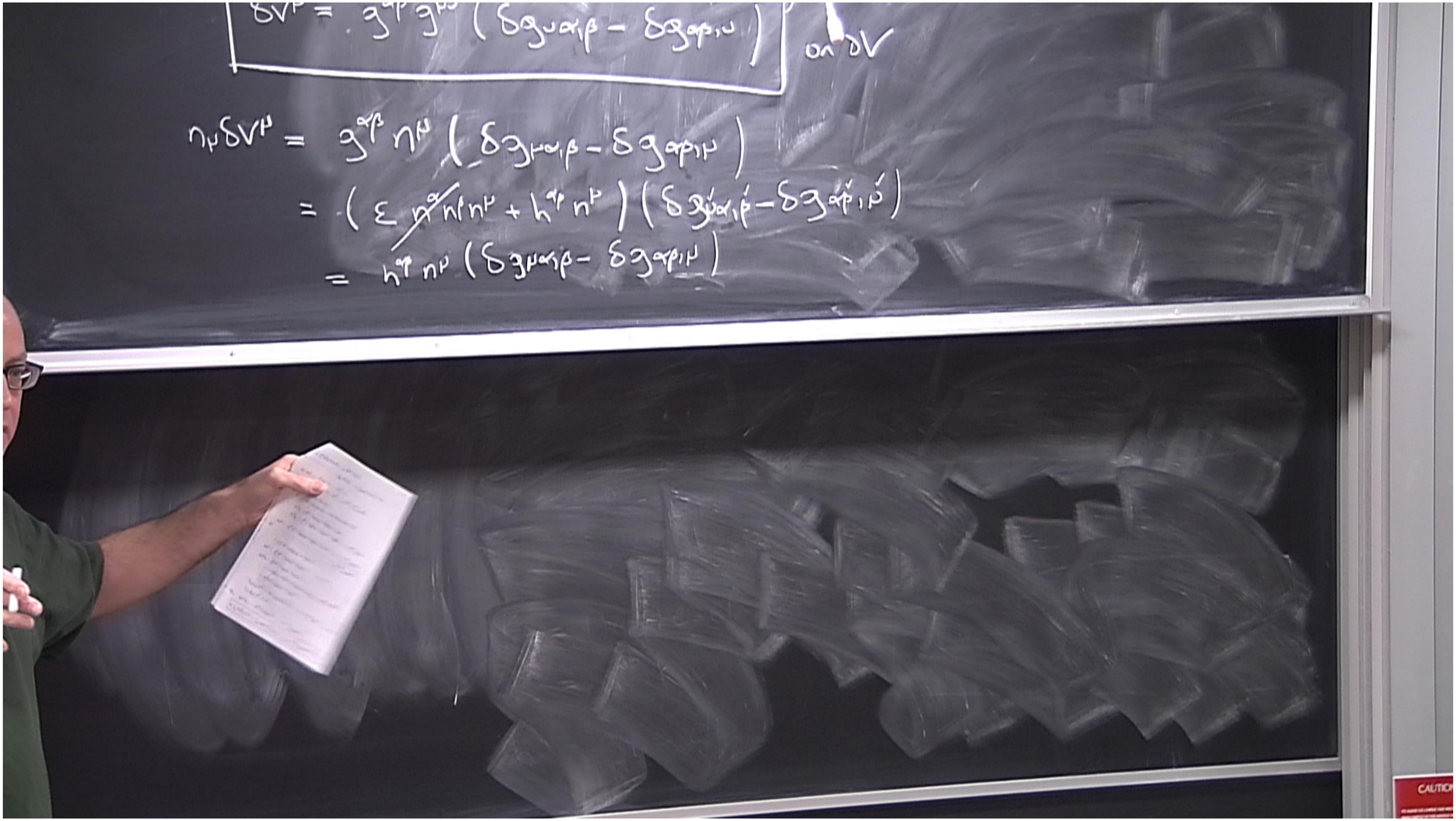

$\delta V^\mu = \frac{1}{2} g^{\beta\gamma} g^{\mu\nu} \left(\delta g_{\nu\beta,\gamma} + \delta g_{\nu\gamma,\beta} - \delta g_{\beta\gamma,\nu} \right) - \frac{1}{2} g^{\mu\nu} g^{\beta\gamma} \delta g_{\beta\gamma,\nu}$
$g^{\beta\gamma} g^{\mu\nu} \left(2\delta g_{\nu\beta,\gamma} - \delta g_{\beta\gamma,\nu} \right) - \frac{1}{2} g^{\mu\nu} g^{\beta\gamma} \delta g_{\beta\gamma,\nu}$

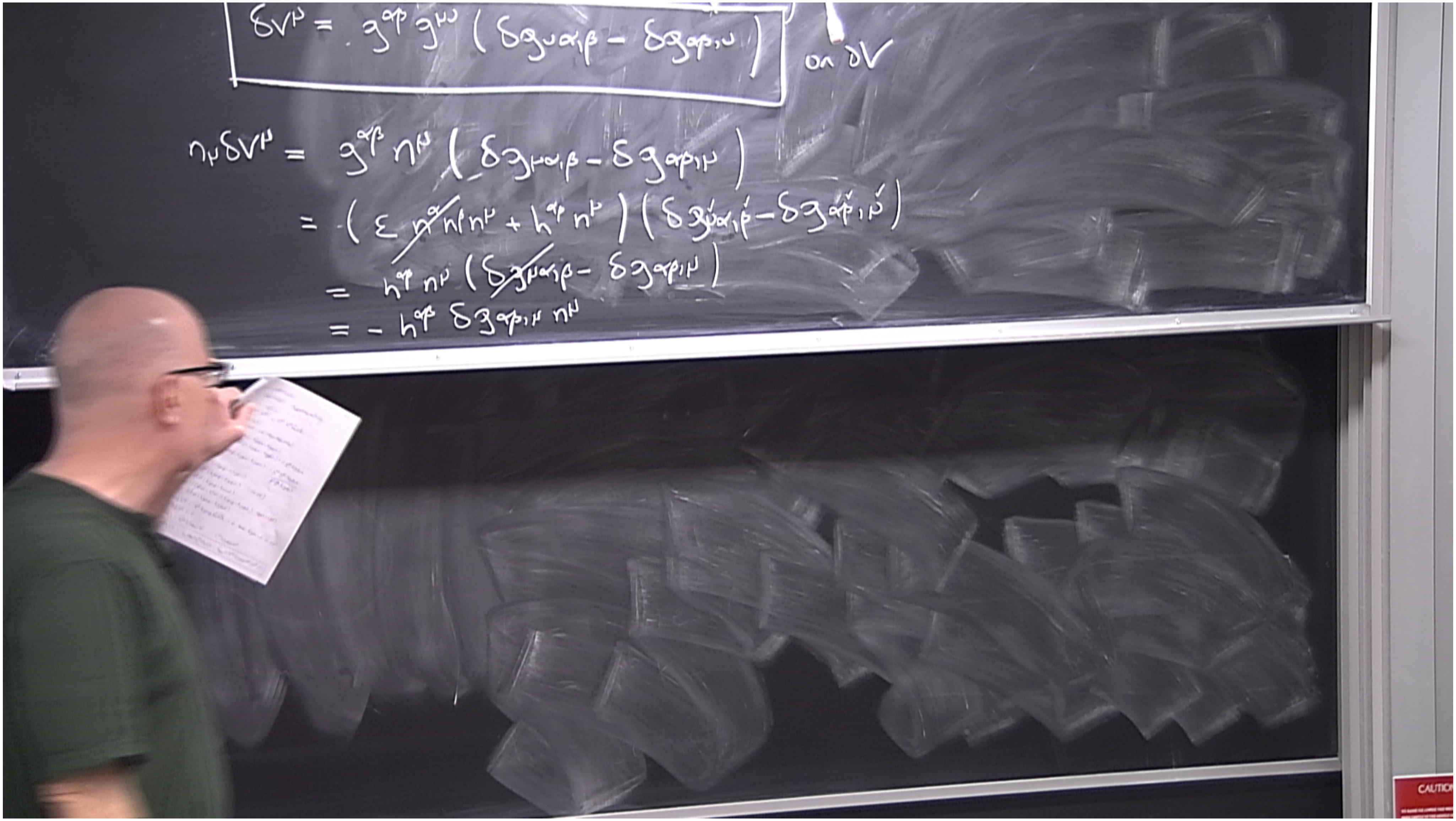

= $g^{\beta\gamma} g^{\mu\nu} (\delta g_{\nu\beta,\gamma} - \delta g_{\beta\gamma,\nu})$
CAUTION

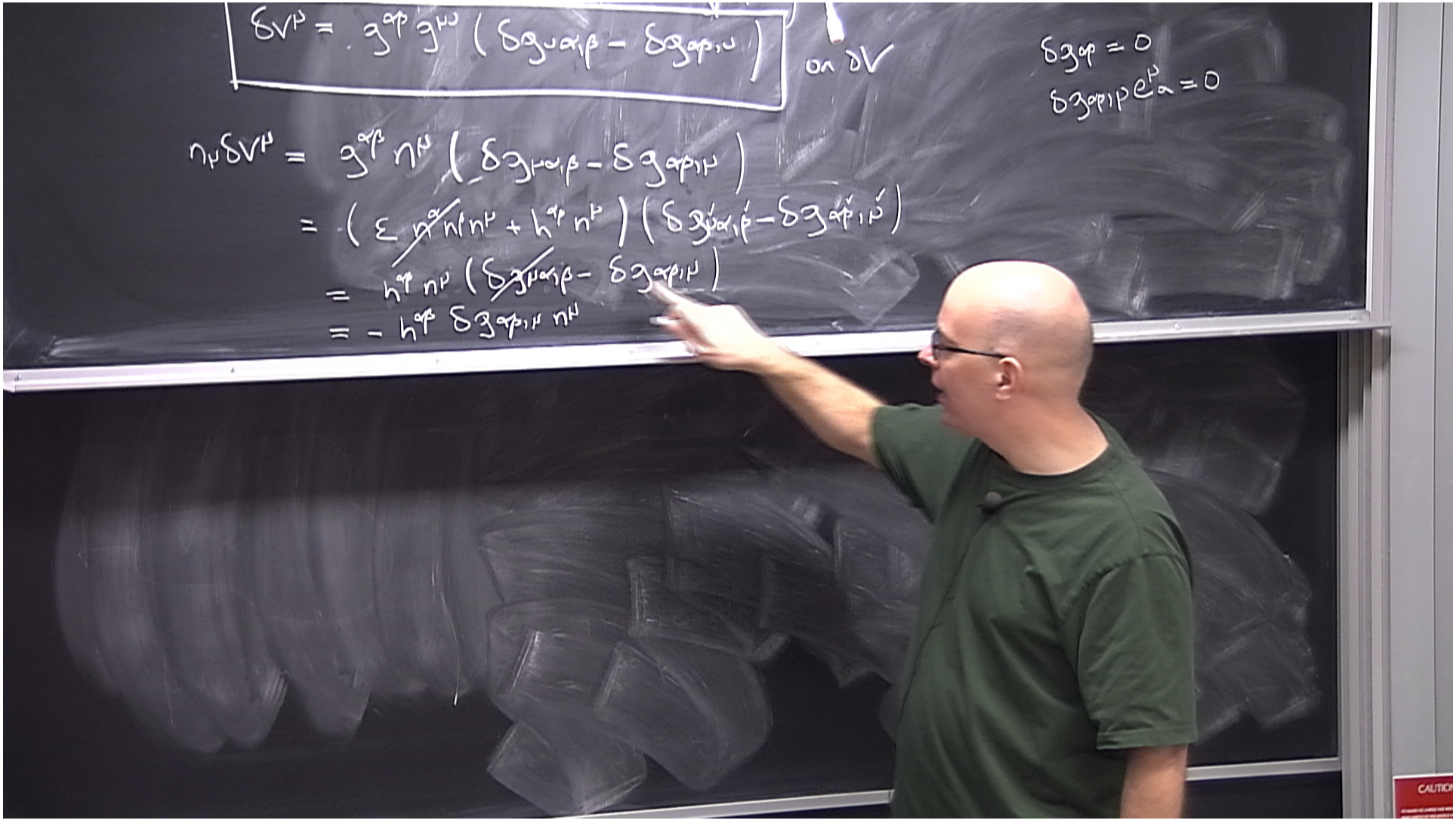

$= g^{\beta\gamma} g^{\mu\nu} \left(\delta g_{\nu\beta,\gamma} - \delta g_{\beta\gamma,\nu} \right)$
$\delta V^{\mu} = g^{\alpha\beta} g^{\mu\nu} \left(\delta g_{\nu\alpha,\beta} - \delta g_{\alpha\beta,\nu} \right)$ on ∂V


$= \frac{1}{2} g^{\beta\gamma} g^{\mu\nu} (2\delta g_{\nu\beta,\gamma} - \delta g_{\beta\gamma,\nu}) - \frac{1}{2} g^{\mu\nu} g^{\beta\gamma} \delta g_{\beta\gamma,\nu}$
$= g^{\beta\gamma} g^{\mu\nu} (\delta g_{\nu\beta,\gamma} - \delta g_{\beta\gamma,\nu})$
$\delta V^\mu = g^{\alpha\beta} g^{\mu\nu} (\delta g_{\nu\alpha,\beta} - \delta g_{\alpha\beta,\nu})$ on ∂V
$n_\mu \delta V^\mu = g^{\alpha\beta} n^\mu (\delta g_{\mu\alpha,\beta} - \delta g_{\alpha\beta,\mu})$
CAUTION

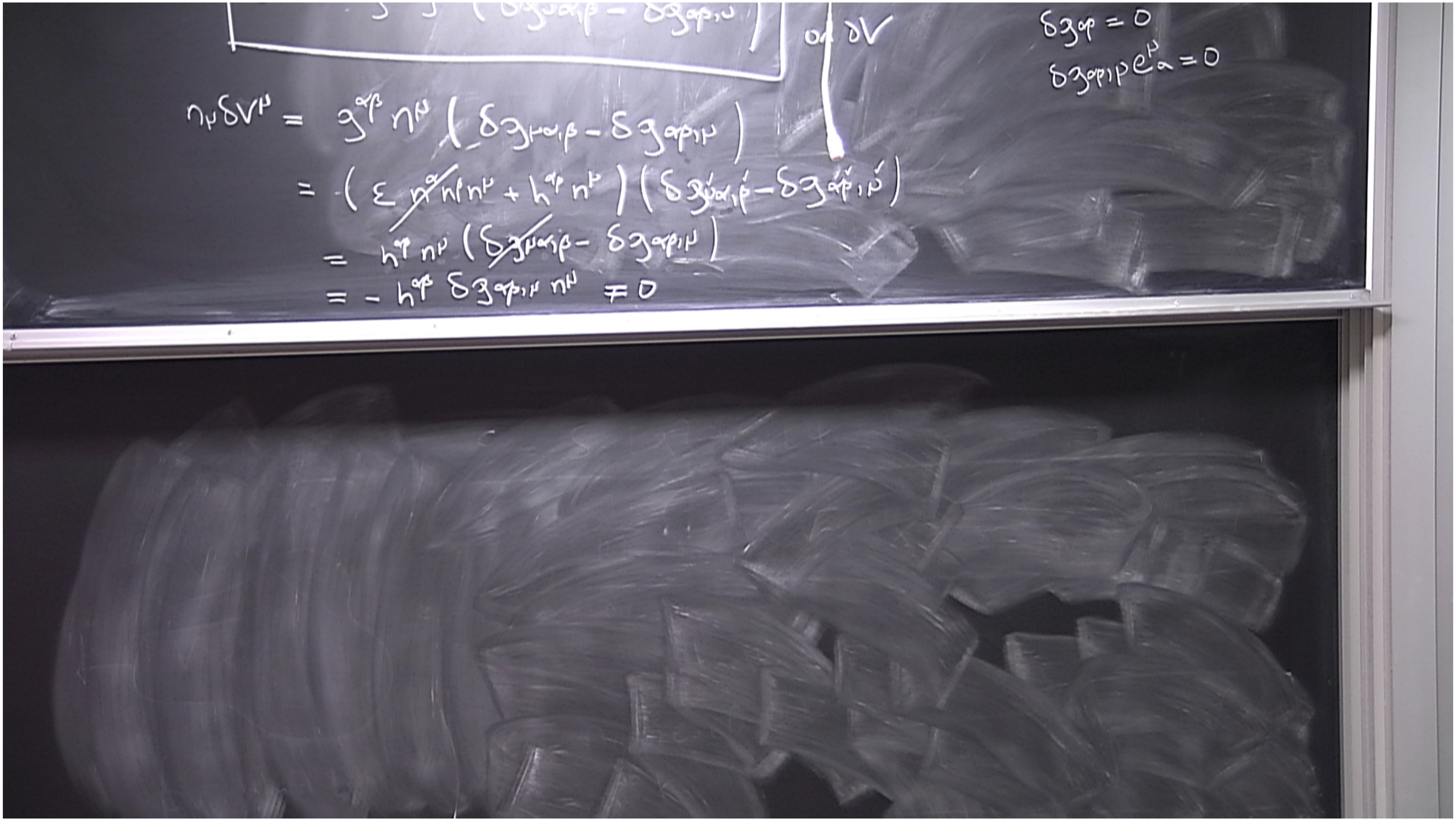

$\delta V^\mu = g^{\alpha\beta} g^{\mu\nu} (\delta g_{\nu\alpha,\beta} - \delta g_{\alpha\beta,\nu})$
on ∂V
$n_\mu \delta V^\mu = g^{\alpha\beta} n^\mu (\delta g_{\mu\alpha,\beta} - \delta g_{\alpha\beta,\mu})$
$= (\varepsilon n^\alpha n^\beta n^\mu + h^{\alpha\beta} n^\mu)(\delta g_{\mu\alpha,\beta} - \delta g_{\alpha\beta,\mu})$

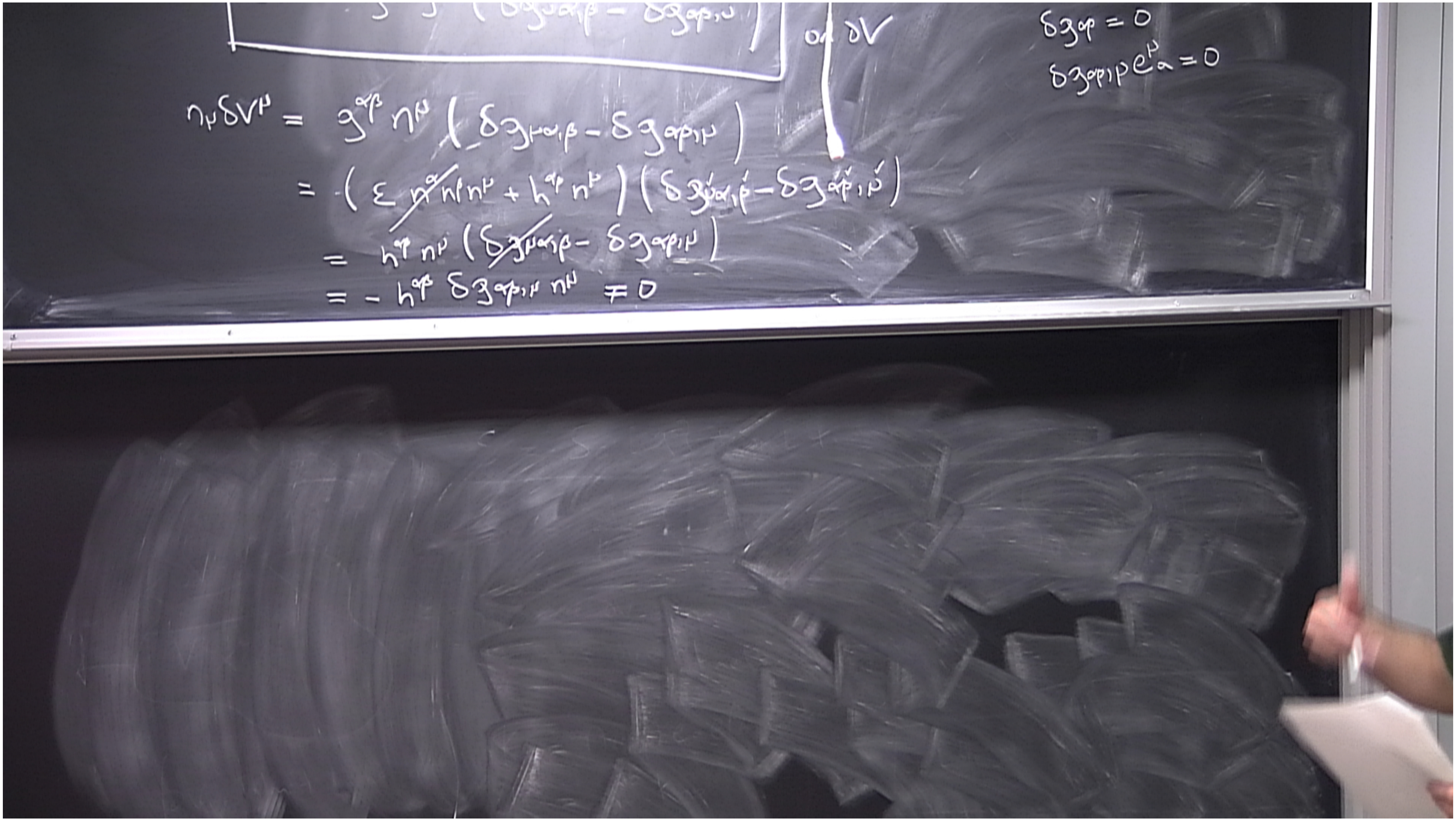

$\delta V^\mu = g^{\alpha\beta} g^{\mu\nu} (\delta g_{\nu\alpha;\beta} - \delta g_{\alpha\beta;\nu})$ on ∂V
$n_\mu \delta V^\mu = g^{\alpha\beta} n^\mu (\delta g_{\mu\alpha;\beta} - \delta g_{\alpha\beta;\mu})$
$= (\varepsilon n^\alpha n^\beta + h^{\alpha\beta}) n^\mu (\delta g_{\mu\alpha;\beta} - \delta g_{\alpha\beta;\mu})$
$= h^{\alpha\beta} n^\mu (\delta g_{\mu\alpha;\beta} - \delta g_{\alpha\beta;\mu})$

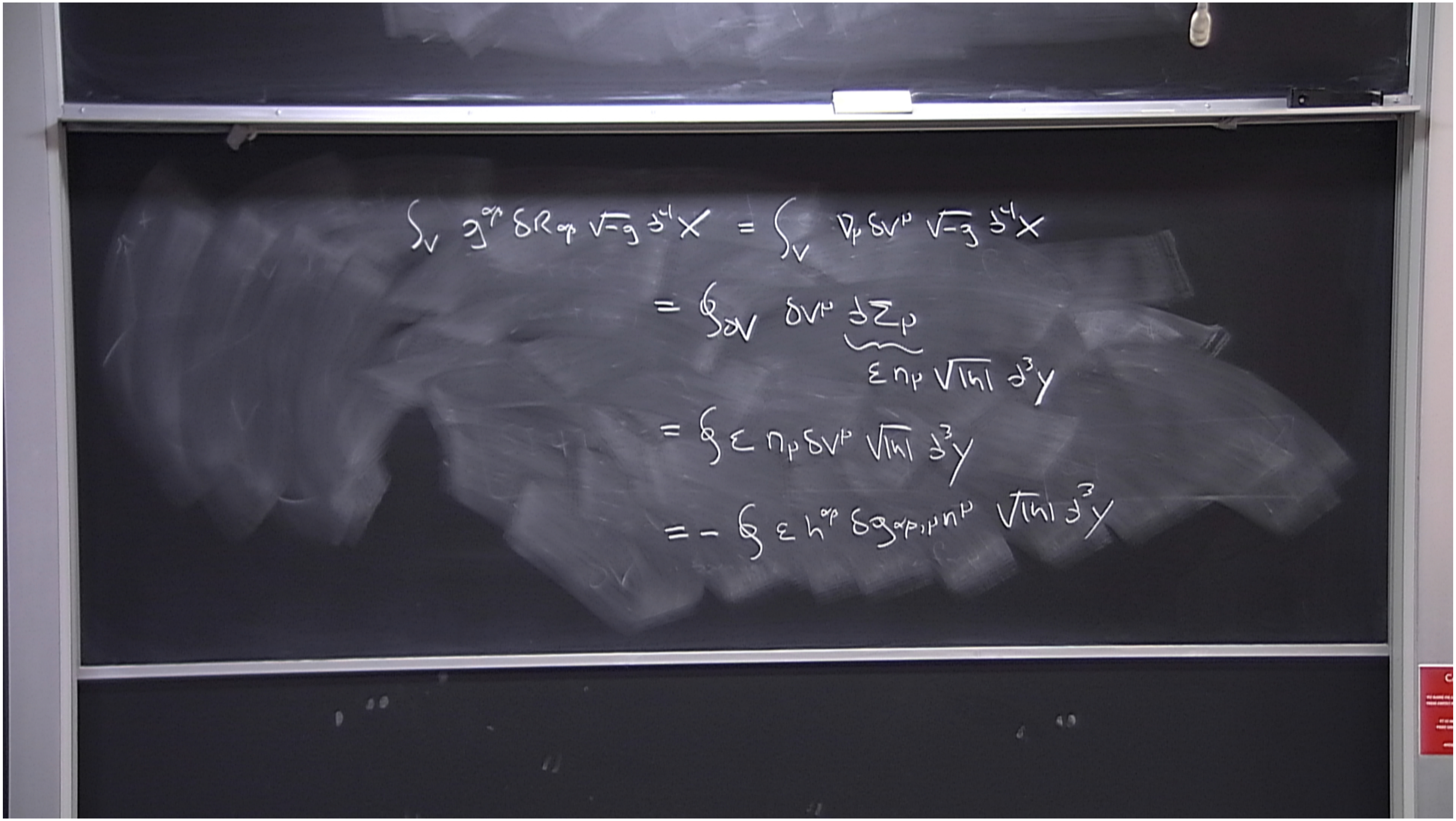

δV^μ = g^{αβ} g^{μν} (δg_{να,β} − δg_{αβ,ν})
on ∂V
n_μ δV^μ = g^{αβ} n^μ (δg_{μα,β} − δg_{αβ,μ})
= (ε n^α n^β + h^{αβ}) n^μ (δg_{μα,β} − δg_{αβ,μ})
= h^{αβ} n^μ (δg_{μα,β} − δg_{αβ,μ})
= − h^{αβ} δg_{αβ,μ} n^μ

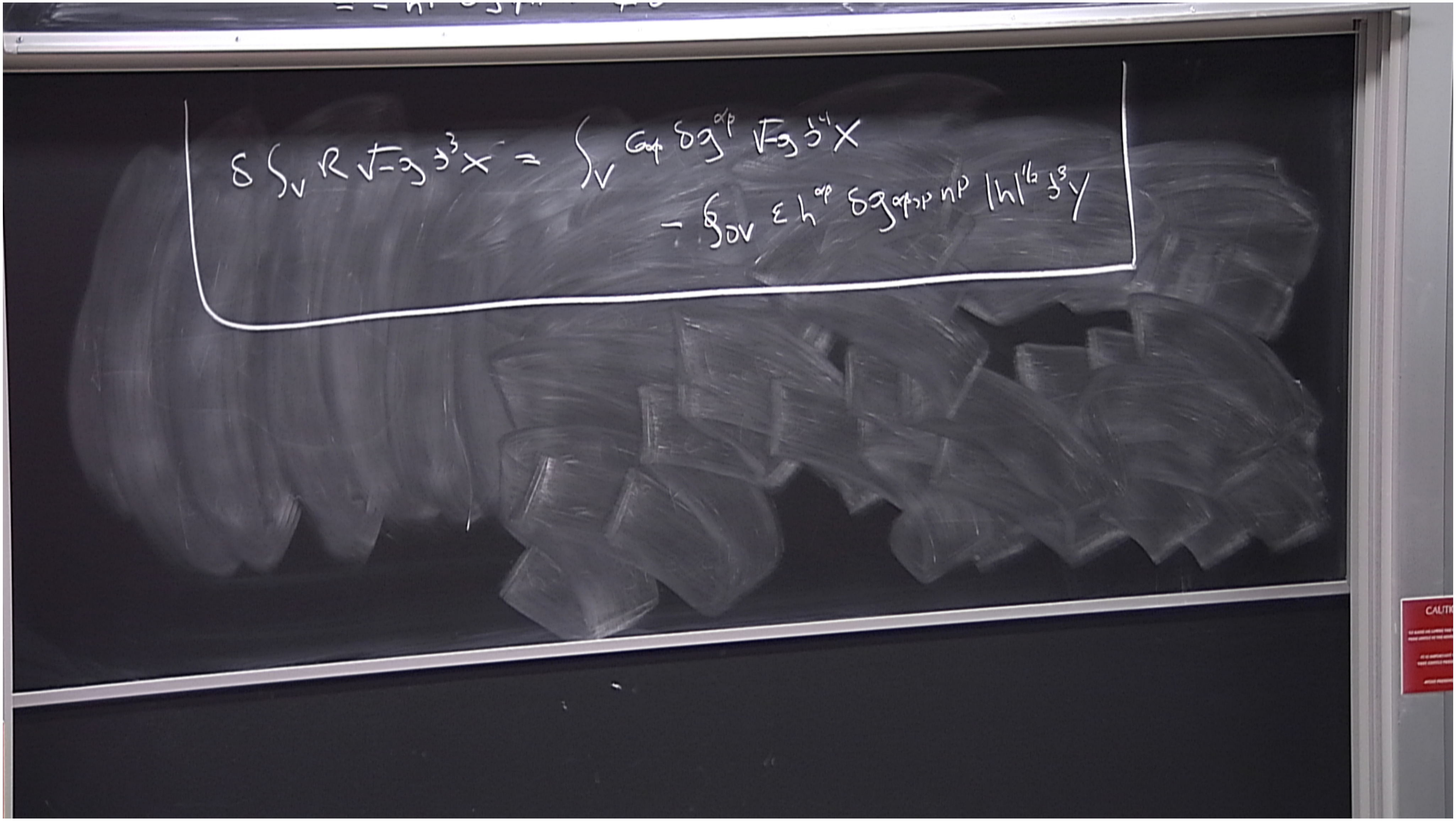


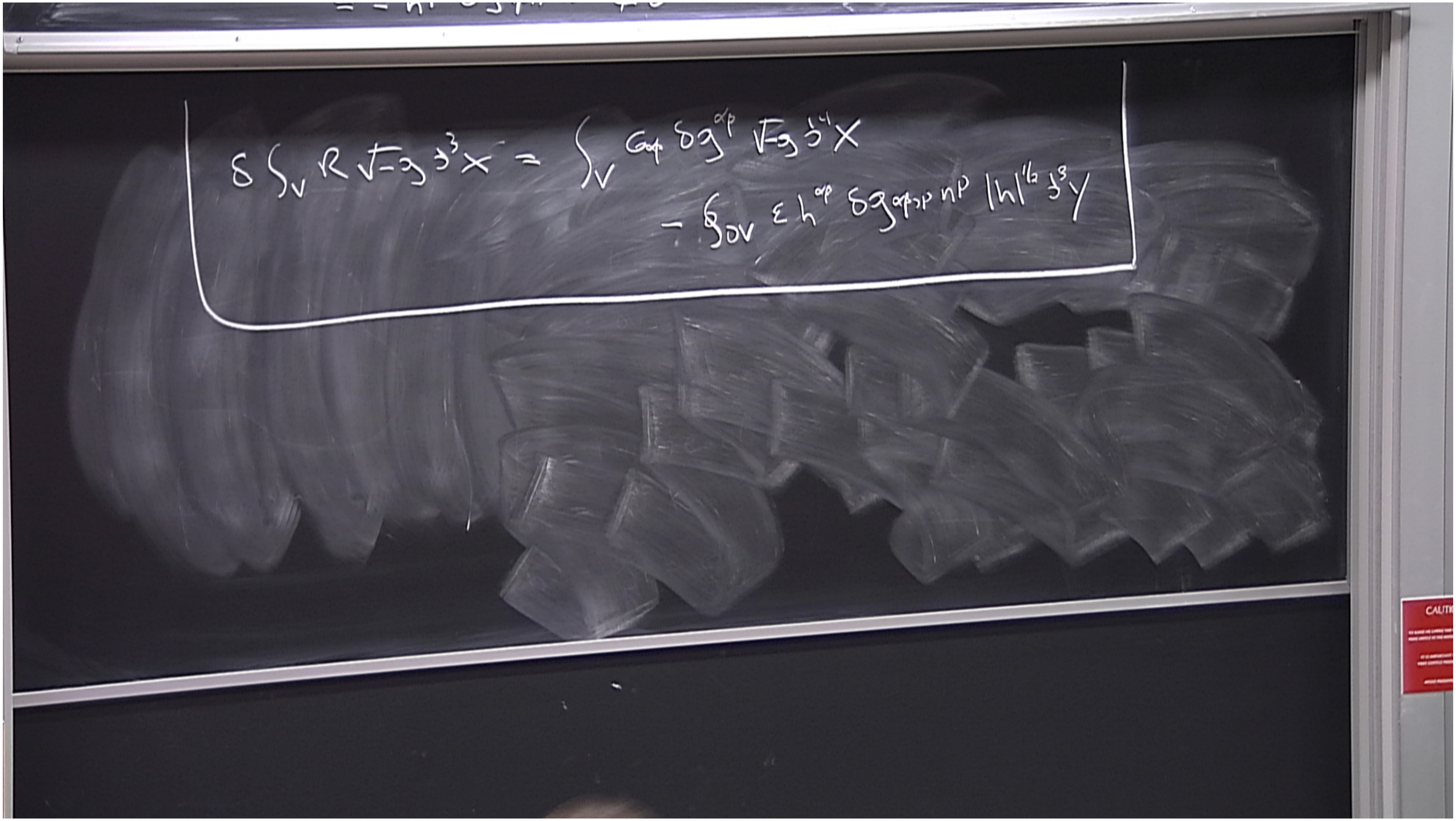
on ∂V
δg_{αβ} = 0

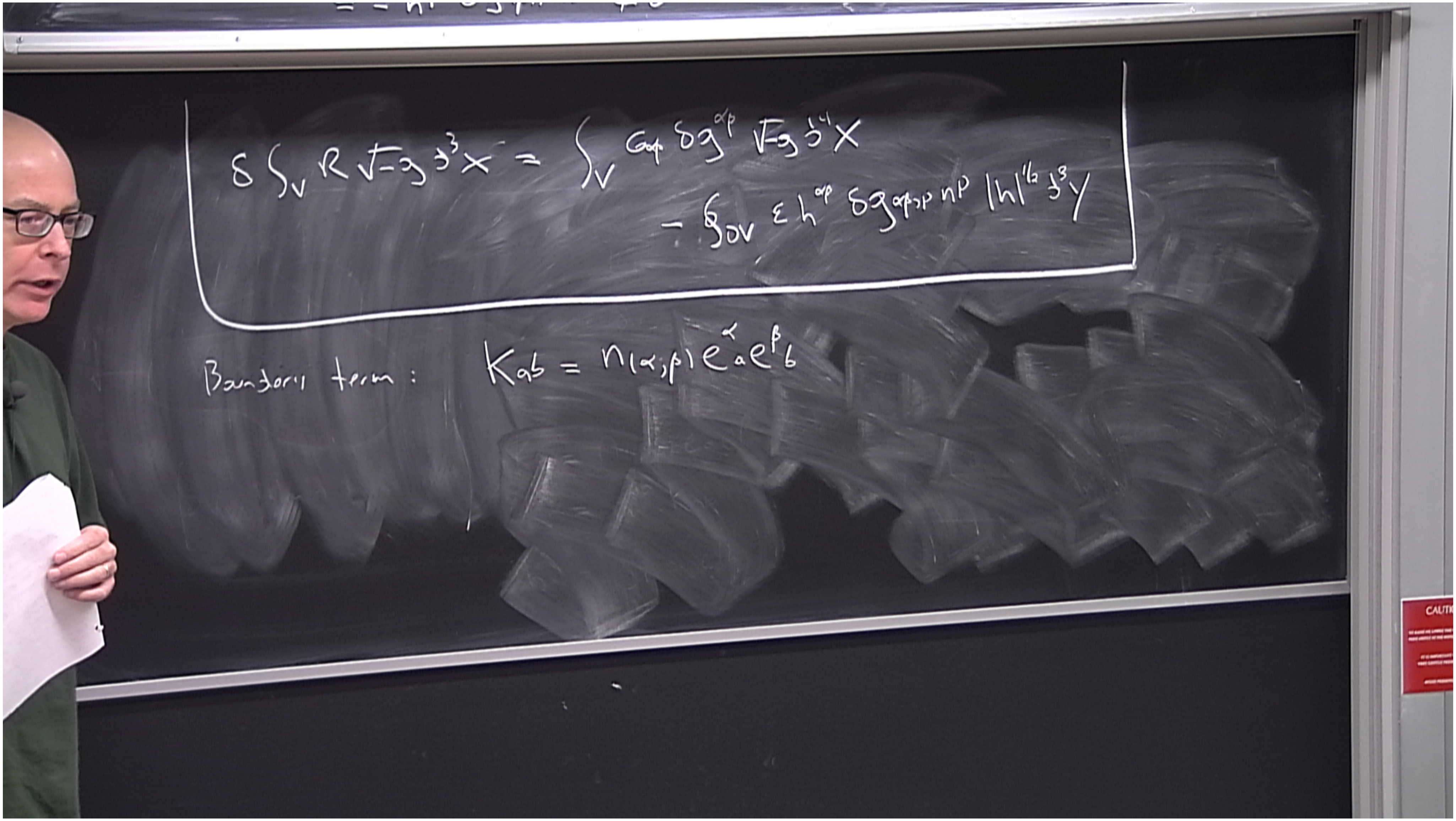

on ∂V
δg_{αβ} = 0
δg_{αβ,μ} e^μ_a = 0
n_μ δV^μ = g^{αβ} n^μ (δg_{μα,β} − δg_{αβ,μ})
= (ε n^α n^β + h^{αβ}) n^μ (δg_{μα,β} − δg_{αβ,μ})
= h^{αβ} n^μ (δg_{μα,β} − δg_{αβ,μ})
= − h^{αβ} δg_{αβ,μ} n^μ ≠ 0


on ∂V
δg_αβ = 0
δg_αβ,μ e^μ_a = 0
n_ν δV^ν = g^αβ n^ν (δg_να,β − δg_αβ,ν)
= (ε n^α n^β + h^αβ) n^ν (δg_να,β − δg_αβ,ν)
= h^αβ n^ν (δg_να,β − δg_αβ,ν)
= − h^αβ δg_αβ,ν n^ν ≠ 0

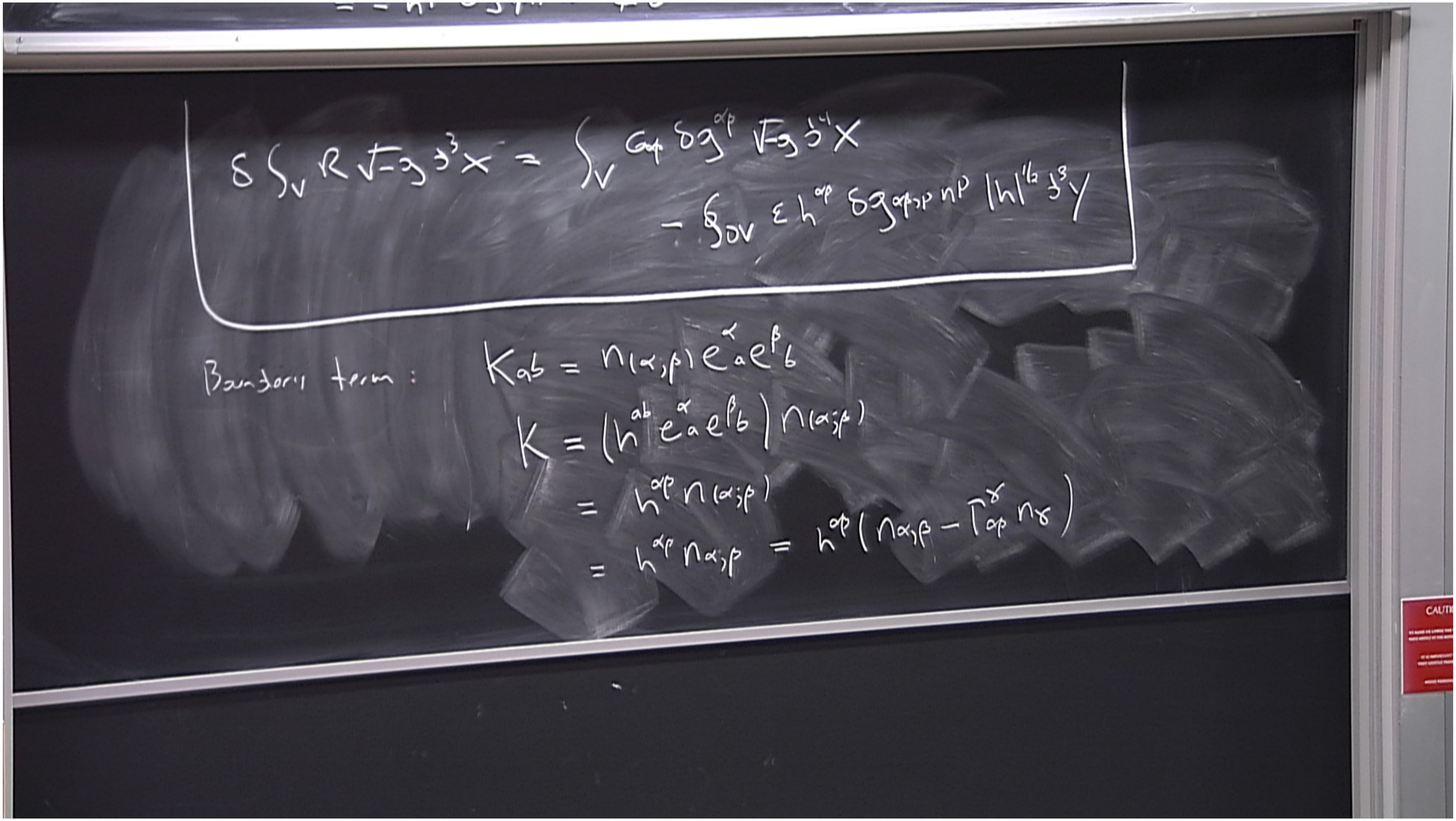

$\int_V g^{\alpha\beta} \delta R_{\alpha\beta} \sqrt{-g}\, d^4x = \int_V \nabla_\mu \delta v^\mu \sqrt{-g}\, d^4x$
$= \oint_{\partial V} \delta v^\mu\, d\Sigma_\mu$
$\varepsilon n_\mu \sqrt{|h|}\, d^3y$
$= \oint \varepsilon n_\mu \delta v^\mu \sqrt{|h|}\, d^3y$
$= -\oint \varepsilon h^{\alpha\beta} \delta g_{\alpha\beta,\mu} n^\mu \sqrt{|h|}\, d^3y$

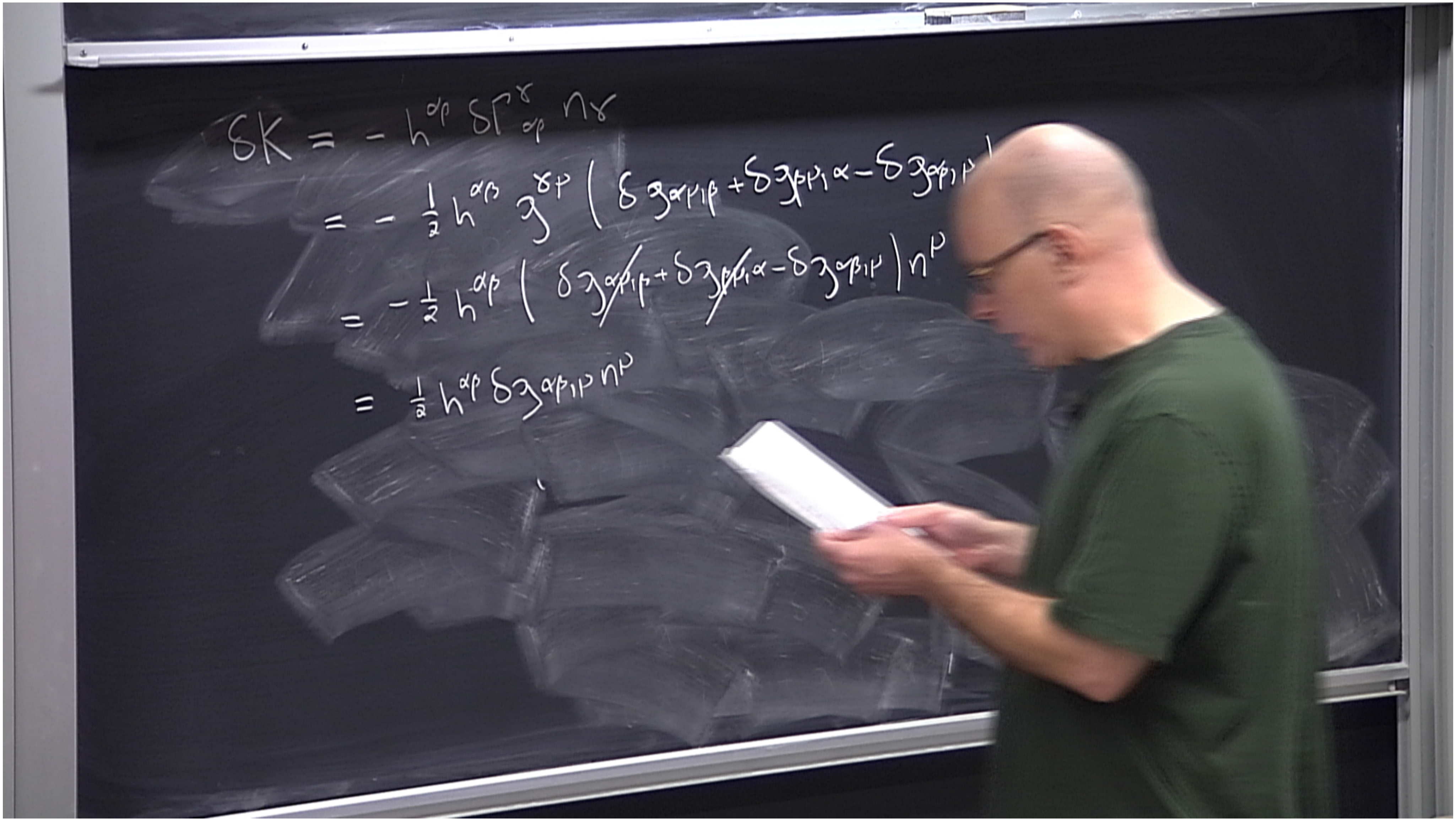

$\delta \int_V R \sqrt{-g}\, d^3x = \int_V G_{\alpha\beta}\, \delta g^{\alpha\beta} \sqrt{-g}\, d^4x$
$- \oint_{\partial V} \varepsilon\, h^{\alpha\beta}\, \delta g_{\alpha\beta;\mu}\, n^\mu\, |h|^{1/2}\, d^3y$


$\delta \int_V R \sqrt{-g}\, d^3x = \int_V G_{\alpha\beta}\, \delta g^{\alpha\beta} \sqrt{-g}\, d^4x$
$- \oint_{\partial V} \varepsilon\, h^{\alpha\beta}\, \delta g_{\alpha\beta,\mu}\, n^{\mu}\, |h|^{1/2}\, d^3y$



$\delta \int_V R \sqrt{-g}\, d^3x = \int_V G_{\alpha\beta}\, \delta g^{\alpha\beta} \sqrt{-g}\, d^4x$
$- \oint_{\partial V} \varepsilon h^{\alpha\beta}\, \delta g_{\alpha\beta;\mu} n^\mu |h|^{1/2} d^3y$
Boundary term: $K_{ab} = n_{(\alpha;\beta)} e^\alpha_a e^\beta_b$


$\delta \int_V R \sqrt{-g}\, d^3x = \int_V G_{\alpha\beta}\, \delta g^{\alpha\beta} \sqrt{-g}\, d^4x$
$- \oint_{\partial V} \varepsilon h^{\alpha\beta}\, \delta g_{\alpha\beta,\mu} n^{\mu} |h|^{1/2} d^3y$
Boundary term:
$K_{ab} = n_{(\alpha;\beta)} e^{\alpha}_{a} e^{\beta}_{b}$
$K = (h^{ab} e^{\alpha}_{a} e^{\beta}_{b})\, n_{(\alpha;\beta)}$
$= h^{\alpha\beta} n_{(\alpha;\beta)}$
$= h^{\alpha\beta} n_{\alpha;\beta} = h^{\alpha\beta}(n_{\alpha,\beta} - \Gamma^{\gamma}_{\alpha\beta} n_{\gamma})$


$\delta K = - h^{\alpha\beta} \delta\Gamma^{\gamma}_{\alpha\beta} n_\gamma$
$= -\frac{1}{2} h^{\alpha\beta} g^{\gamma\mu} (\delta g_{\alpha\mu,\beta} + \delta g_{\beta\mu,\alpha} - \delta g_{\alpha\beta,\mu}$
$= -\frac{1}{2} h^{\alpha\beta} (\delta g_{\alpha\mu,\beta} + \delta g_{\beta\mu,\alpha} - \delta g_{\alpha\beta,\mu}) n^\mu$
$= \frac{1}{2} h^{\alpha\beta} \delta g_{\alpha\beta,\mu} n^\mu$

$$\delta K = - h^{\alpha\beta}\,\delta\Gamma^{\gamma}_{\alpha\beta}\, n_\gamma$$

$$= -\tfrac{1}{2} h^{\alpha\beta} g^{\gamma\mu}\left(\delta g_{\alpha\mu;\beta} + \delta g_{\beta\mu;\alpha} - \delta g_{\alpha\beta;\mu}\right) n_\gamma$$

$$= -\tfrac{1}{2} h^{\alpha\beta}\left(\delta g_{\alpha\mu;\beta} + \delta g_{\beta\mu;\alpha} - \delta g_{\alpha\beta;\mu}\right) n^\mu$$

$$= \tfrac{1}{2} h^{\alpha\beta}\,\delta g_{\alpha\beta;\mu}\, n^\mu$$

$$\delta \oint_{\partial V} \varepsilon K\, |h|^{1/2} d^3y$$

$$\delta K = - h^{\alpha\beta}\,\delta\Gamma^{\gamma}_{\alpha\beta}\, n_\gamma$$

$$= -\tfrac{1}{2} h^{\alpha\beta} g^{\gamma\mu} \left(\delta g_{\alpha\mu;\beta} + \delta g_{\beta\mu;\alpha} - \delta g_{\alpha\beta;\mu} \right) n_\gamma$$

$$= -\tfrac{1}{2} h^{\alpha\beta} \left(\cancel{\delta g_{\alpha\mu;\beta}} + \cancel{\delta g_{\beta\mu;\alpha}} - \delta g_{\alpha\beta;\mu} \right) n^\mu$$

$$= \tfrac{1}{2} h^{\alpha\beta}\, \delta g_{\alpha\beta;\mu}\, n^\mu$$

$$\delta\, 2\oint_{\partial V} \varepsilon K |h|^{1/2} d^3y = 2\oint_{\partial V} \varepsilon\, \delta K\, |h|^{1/2} d^3y = \oint \varepsilon\, h^{\alpha\beta}\, \delta g_{\alpha\beta;\mu} n^\mu |h|^{1/2} d^3y$$

$$\delta K = -h^{\alpha\beta}\,\delta\Gamma^{\gamma}_{\alpha\beta}\,n_\gamma$$

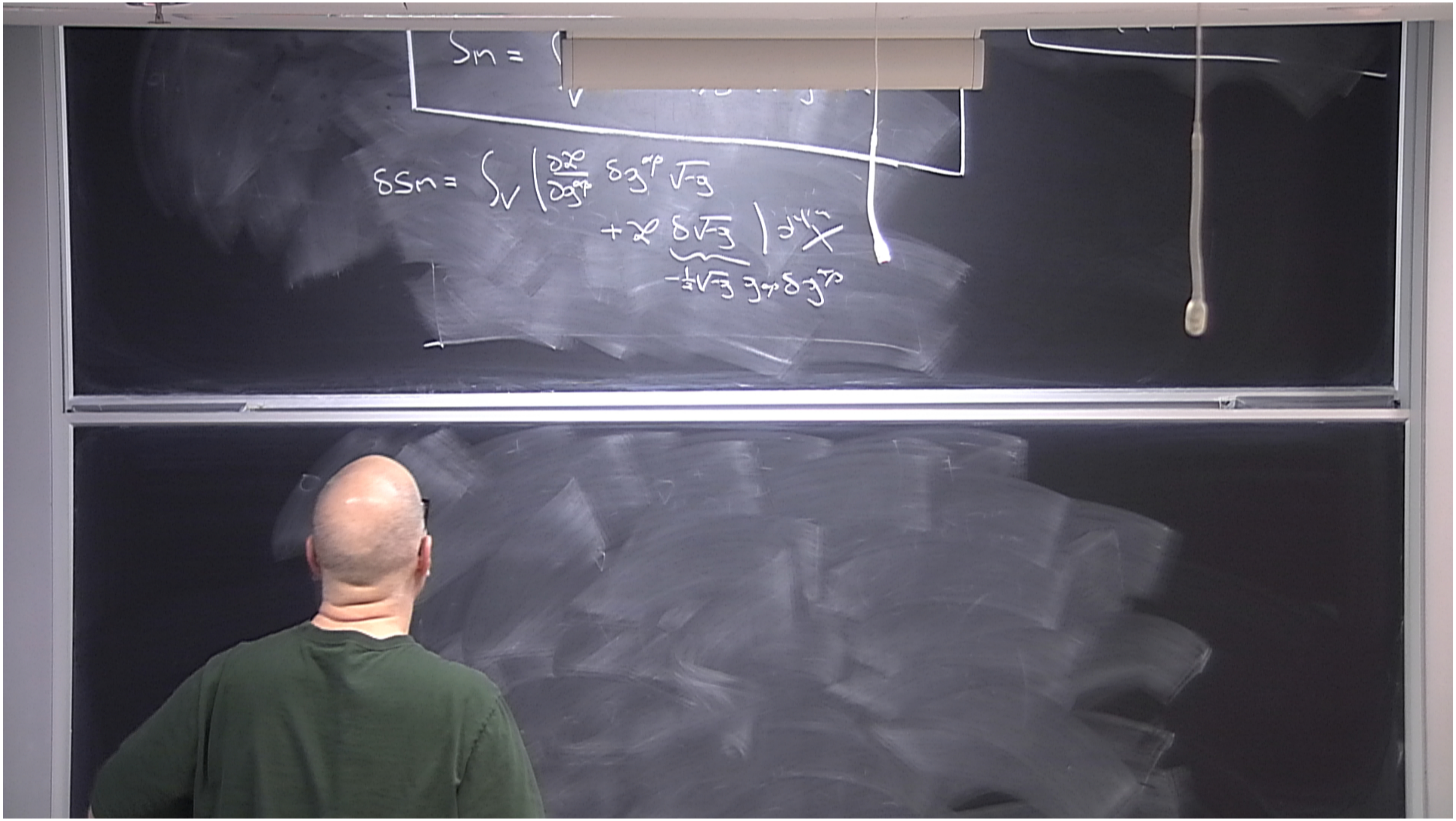
$$= -\tfrac{1}{2} h^{\alpha\beta} g^{\gamma\mu} \left(\delta g_{\alpha\mu;\beta} + \delta g_{\beta\mu;\alpha} - \delta g_{\alpha\beta;\mu} \right) n_\gamma$$

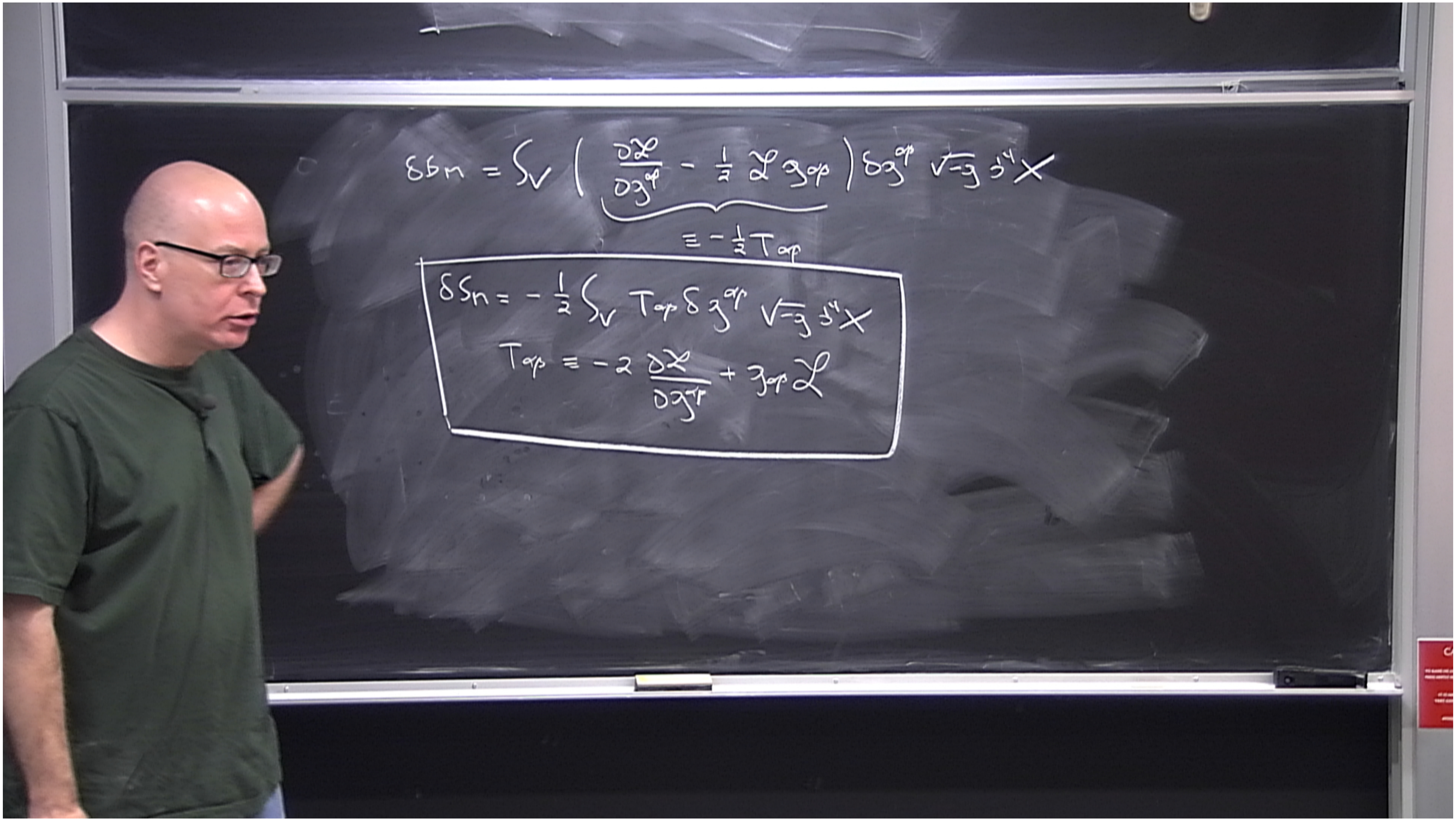
$$= -\tfrac{1}{2} h^{\alpha\beta} \left(\cancel{\delta g_{\alpha\mu;\beta}} + \cancel{\delta g_{\beta\mu;\alpha}} - \delta g_{\alpha\beta;\mu} \right) n^\mu$$

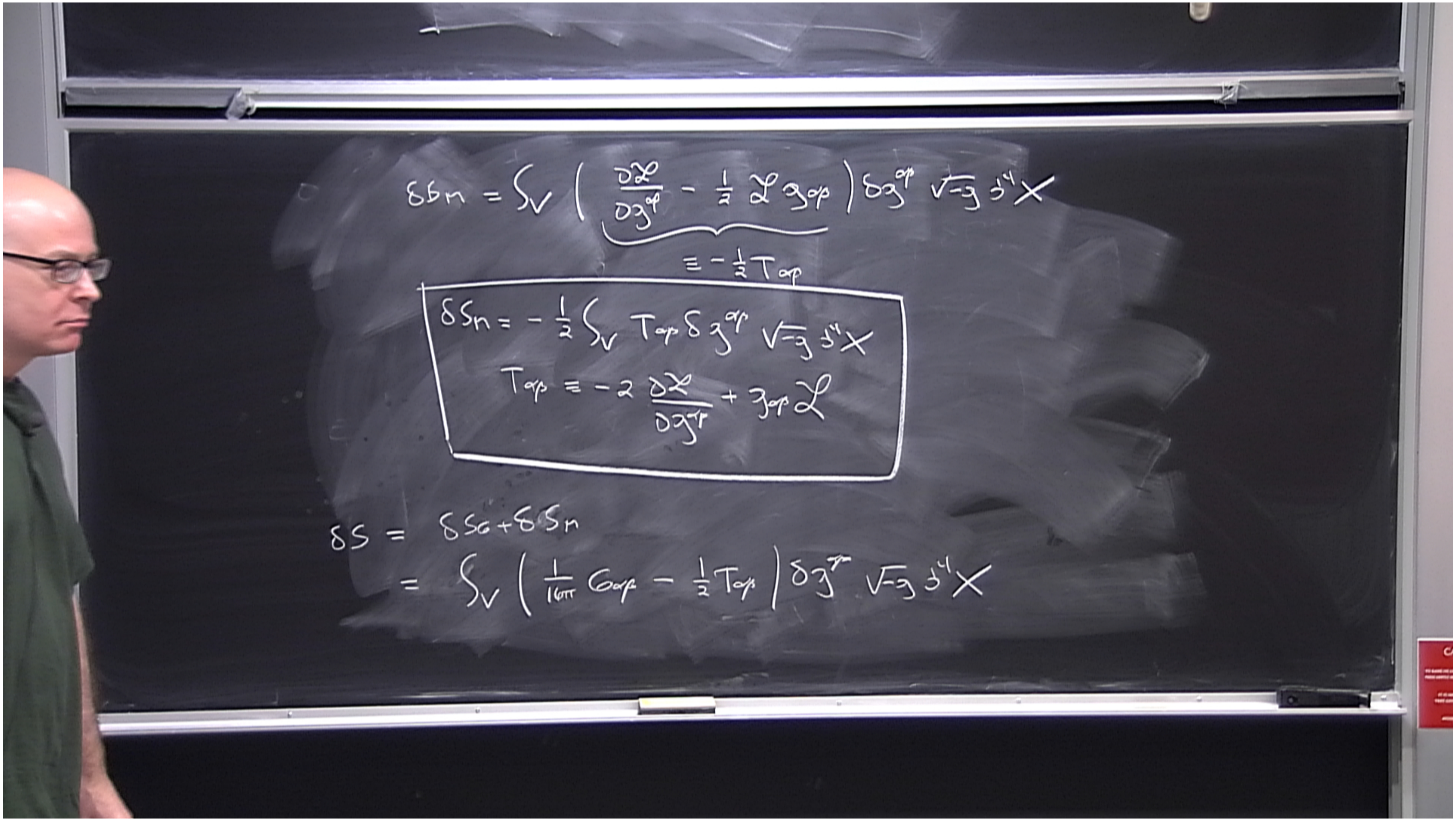
$$= \tfrac{1}{2} h^{\alpha\beta}\, \delta g_{\alpha\beta;\mu}\, n^\mu$$

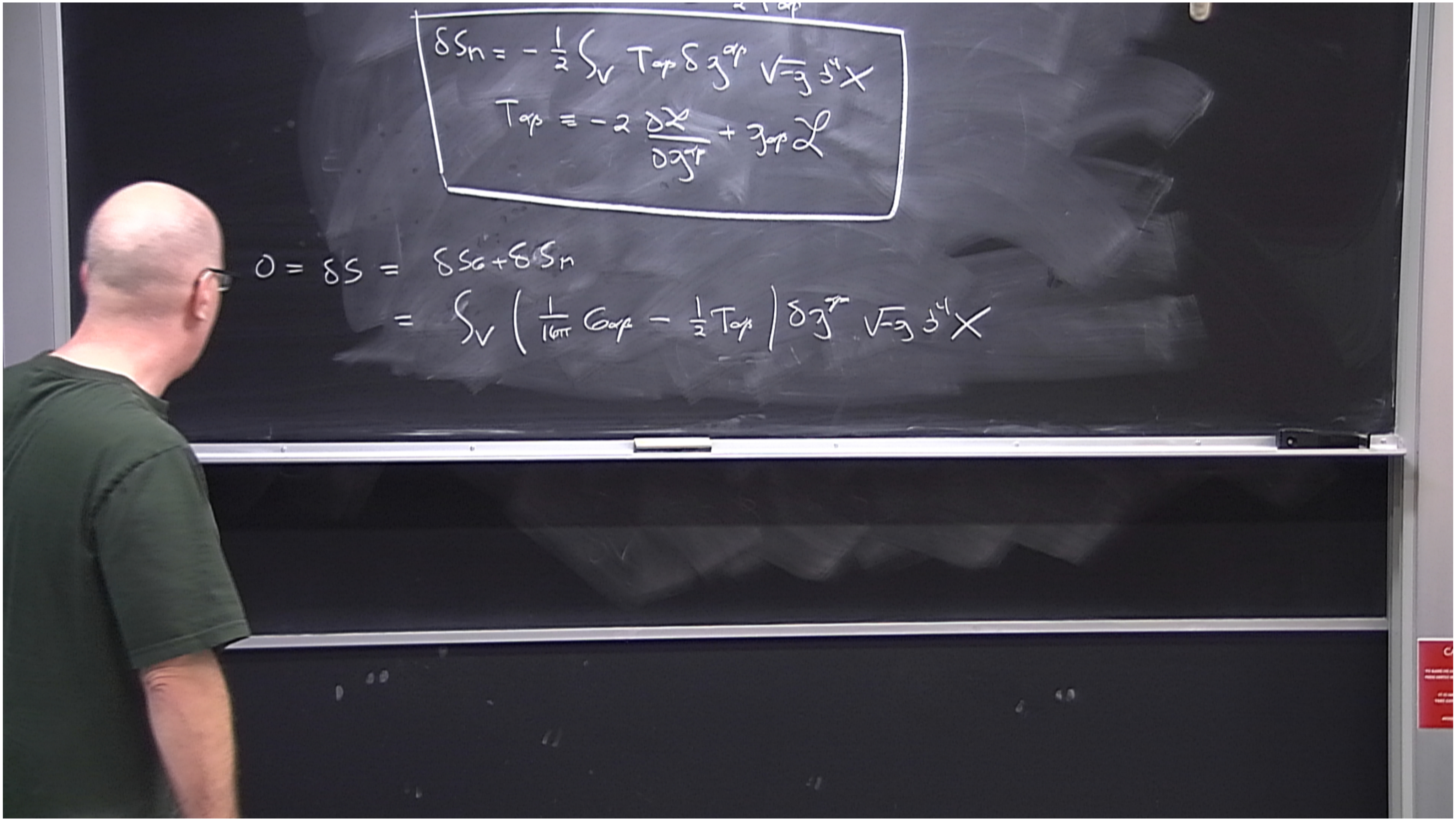
$$\delta\, 2\oint_{\partial V} \varepsilon K |h|^{1/2} d^3y = 2\oint_{\partial V} \varepsilon\, \delta K |h|^{1/2} d^3y = \oint \varepsilon\, h^{\alpha\beta} \delta g_{\alpha\beta;\mu} n^\mu |h|^{1/2} d^3y$$

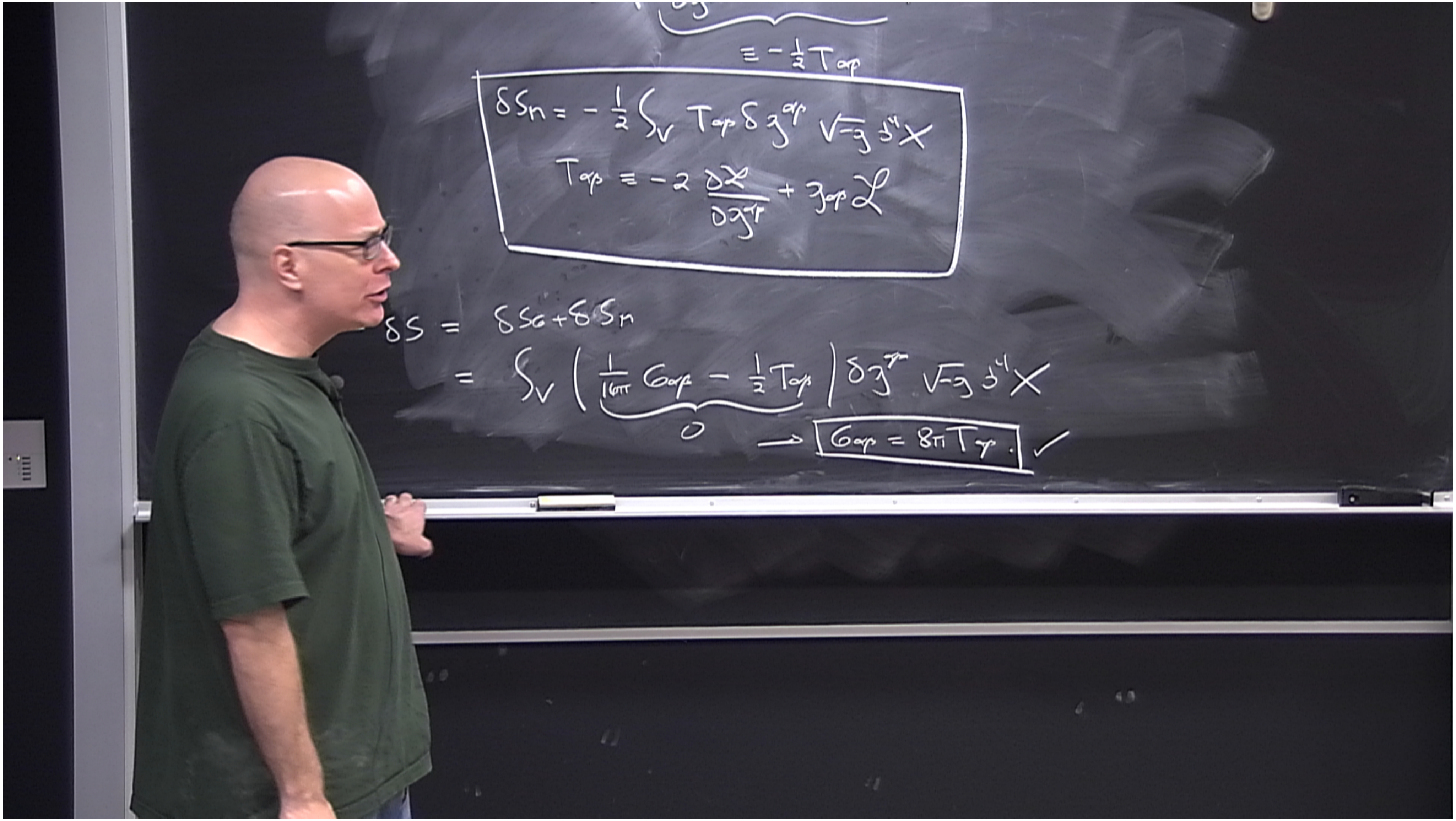
$$\delta \left\{ \int_V R \sqrt{-g}\, d^4x + 2\oint_{\partial V} \varepsilon K |h|^{1/2} d^3y \right\} = \int_V G_{\alpha\beta}\, \delta g^{\alpha\beta} \sqrt{-g}\, d^4x$$

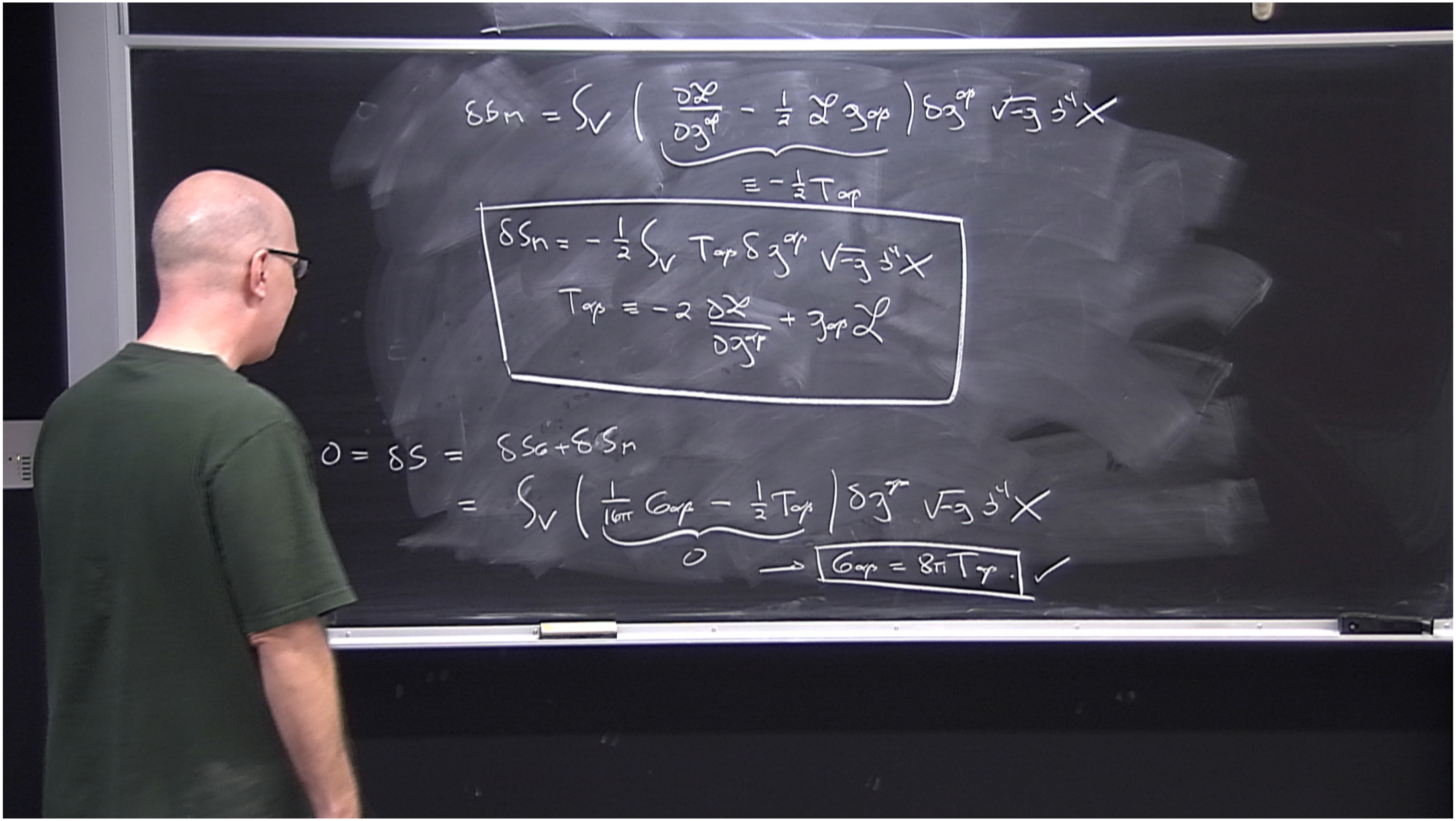

$\delta S_n = \int_V \left(\frac{\partial \mathcal{L}}{\partial g^{\alpha\beta}} \delta g^{\alpha\beta} \sqrt{-g} + \mathcal{L}\, \delta\sqrt{-g} \right) d^4x$
$-\frac{1}{2}\sqrt{-g}\, g_{\alpha\beta}\, \delta g^{\alpha\beta}$


$\delta S_M = \int_V \left(\frac{\partial \mathcal{L}}{\partial g^{\alpha\beta}} - \frac{1}{2} \mathcal{L} g_{\alpha\beta} \right) \delta g^{\alpha\beta} \sqrt{-g}\, d^4x$
$\equiv -\frac{1}{2} T_{\alpha\beta}$
$\delta S_M = -\frac{1}{2} \int_V T_{\alpha\beta} \delta g^{\alpha\beta} \sqrt{-g}\, d^4x$
$T_{\alpha\beta} \equiv -2 \frac{\partial \mathcal{L}}{\partial g^{\alpha\beta}} + g_{\alpha\beta} \mathcal{L}$


$\delta S_M = \int_V \left(\frac{\partial \mathcal{L}}{\partial g^{\alpha\beta}} - \frac{1}{2} \mathcal{L} g_{\alpha\beta} \right) \delta g^{\alpha\beta} \sqrt{-g}\, d^4X$
$\equiv -\frac{1}{2} T_{\alpha\beta}$
$\delta S_M = -\frac{1}{2} \int_V T_{\alpha\beta}\, \delta g^{\alpha\beta} \sqrt{-g}\, d^4X$
$T_{\alpha\beta} \equiv -2 \frac{\partial \mathcal{L}}{\partial g^{\alpha\beta}} + g_{\alpha\beta} \mathcal{L}$
$\delta S = \delta S_G + \delta S_M$
$= \int_V \left(\frac{1}{16\pi} G_{\alpha\beta} - \frac{1}{2} T_{\alpha\beta} \right) \delta g^{\alpha\beta} \sqrt{-g}\, d^4X$


δSm = − ½ ∫V Tαβ δg^αβ √−g d⁴x
Tαβ ≡ −2 ∂ℒ/∂g^αβ + gαβ ℒ
0 = δS = δSG + δSm
= ∫V (1/16π Gαβ − ½ Tαβ) δg^αβ √−g d⁴x


≡ − ½ Tαβ
δSn = − ½ ∫V Tαβ δgαβ √−g d⁴x
Tαβ ≡ −2 ∂ℒ/∂gαβ + gαβ ℒ
δS = δSG + δSn
= ∫V (1/16π Gαβ − ½ Tαβ) δgαβ √−g d⁴x
0
→ Gαβ = 8π Tαβ .


$\delta S_M = \int_V \left(\frac{\partial \mathcal{L}}{\partial g^{\alpha\beta}} - \frac{1}{2} \mathcal{L} g_{\alpha\beta} \right) \delta g^{\alpha\beta} \sqrt{-g}\, d^4x$
$\equiv -\frac{1}{2} T_{\alpha\beta}$
$\delta S_M = -\frac{1}{2} \int_V T_{\alpha\beta}\, \delta g^{\alpha\beta} \sqrt{-g}\, d^4x$
$T_{\alpha\beta} \equiv -2 \frac{\partial \mathcal{L}}{\partial g^{\alpha\beta}} + g_{\alpha\beta} \mathcal{L}$
$0 = \delta S = \delta S_G + \delta S_M$
$= \int_V \left(\frac{1}{16\pi} G_{\alpha\beta} - \frac{1}{2} T_{\alpha\beta} \right) \delta g^{\alpha\beta} \sqrt{-g}\, d^4x$
0
$\longrightarrow G_{\alpha\beta} = 8\pi T_{\alpha\beta}.$ ✓

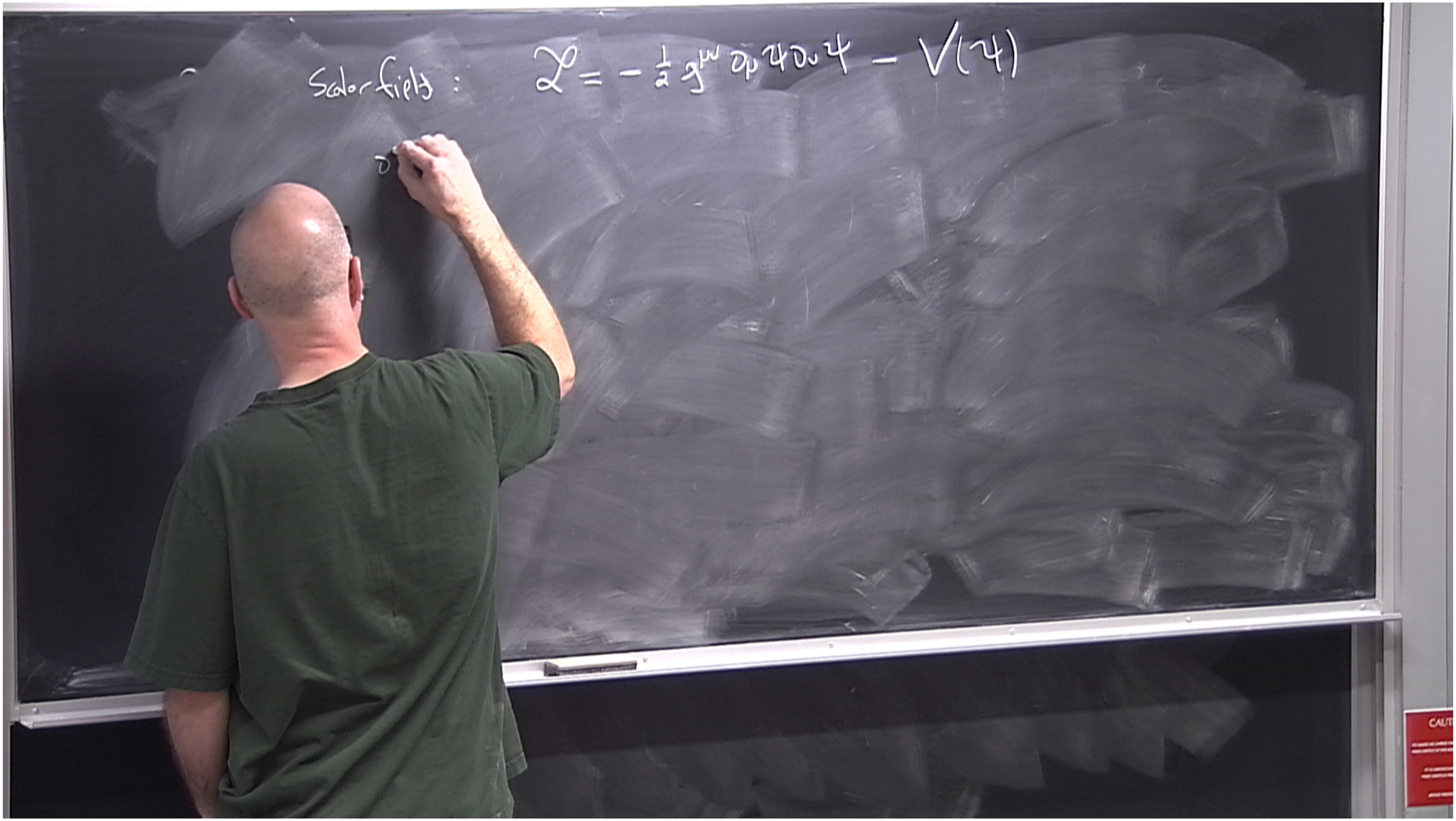
$$\delta S_M = \int_V \underbrace{\left(\frac{\partial \mathcal{L}}{\partial g^{\alpha\beta}} - \frac{1}{2} \mathcal{L} g_{\alpha\beta} \right)}_{\equiv -\frac{1}{2} T_{\alpha\beta}} \delta g^{\alpha\beta} \sqrt{-g}\, d^4x$$

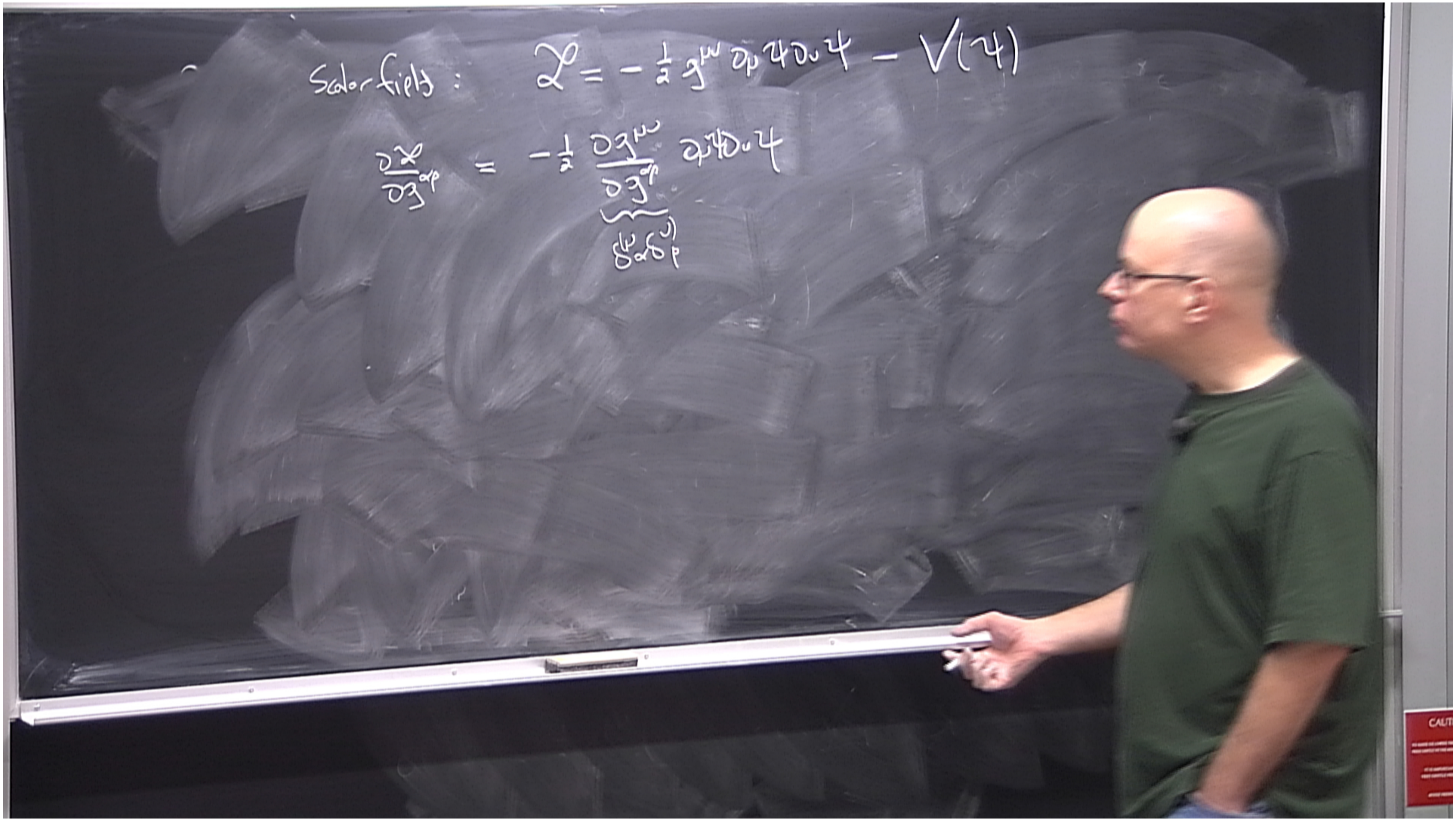
$$\boxed{\begin{aligned} \delta S_M &= -\frac{1}{2} \int_V T_{\alpha\beta}\, \delta g^{\alpha\beta} \sqrt{-g}\, d^4x \\ T_{\alpha\beta} &\equiv -2 \frac{\partial \mathcal{L}}{\partial g^{\alpha\beta}} + g_{\alpha\beta} \mathcal{L} \end{aligned}}$$

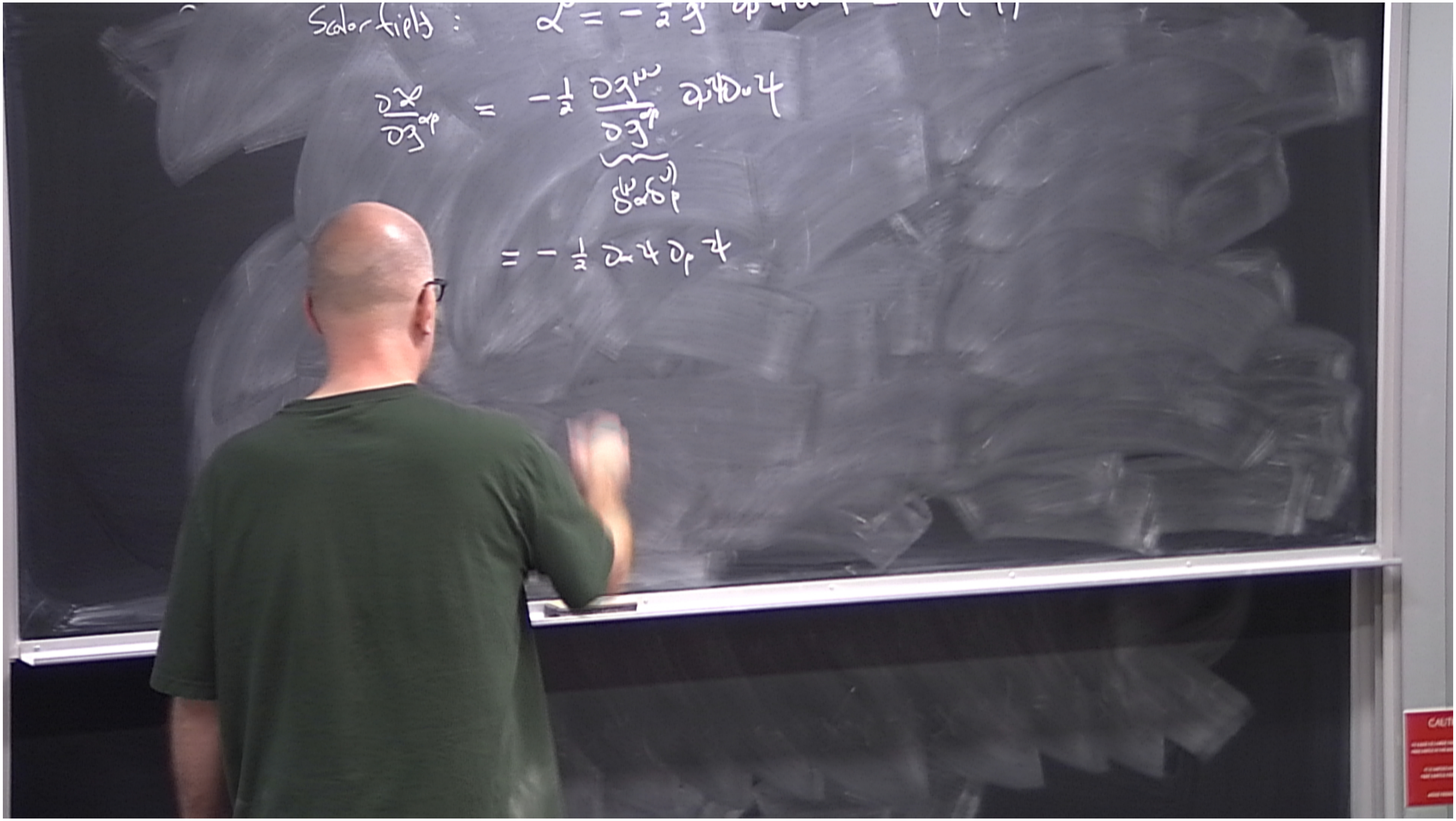
$$0 = \delta S = \delta S_G + \delta S_M$$

$$= \int_V \underbrace{\left(\frac{1}{16\pi} G_{\alpha\beta} - \frac{1}{2} T_{\alpha\beta} \right)}_{0} \delta g^{\alpha\beta} \sqrt{-g}\, d^4x$$

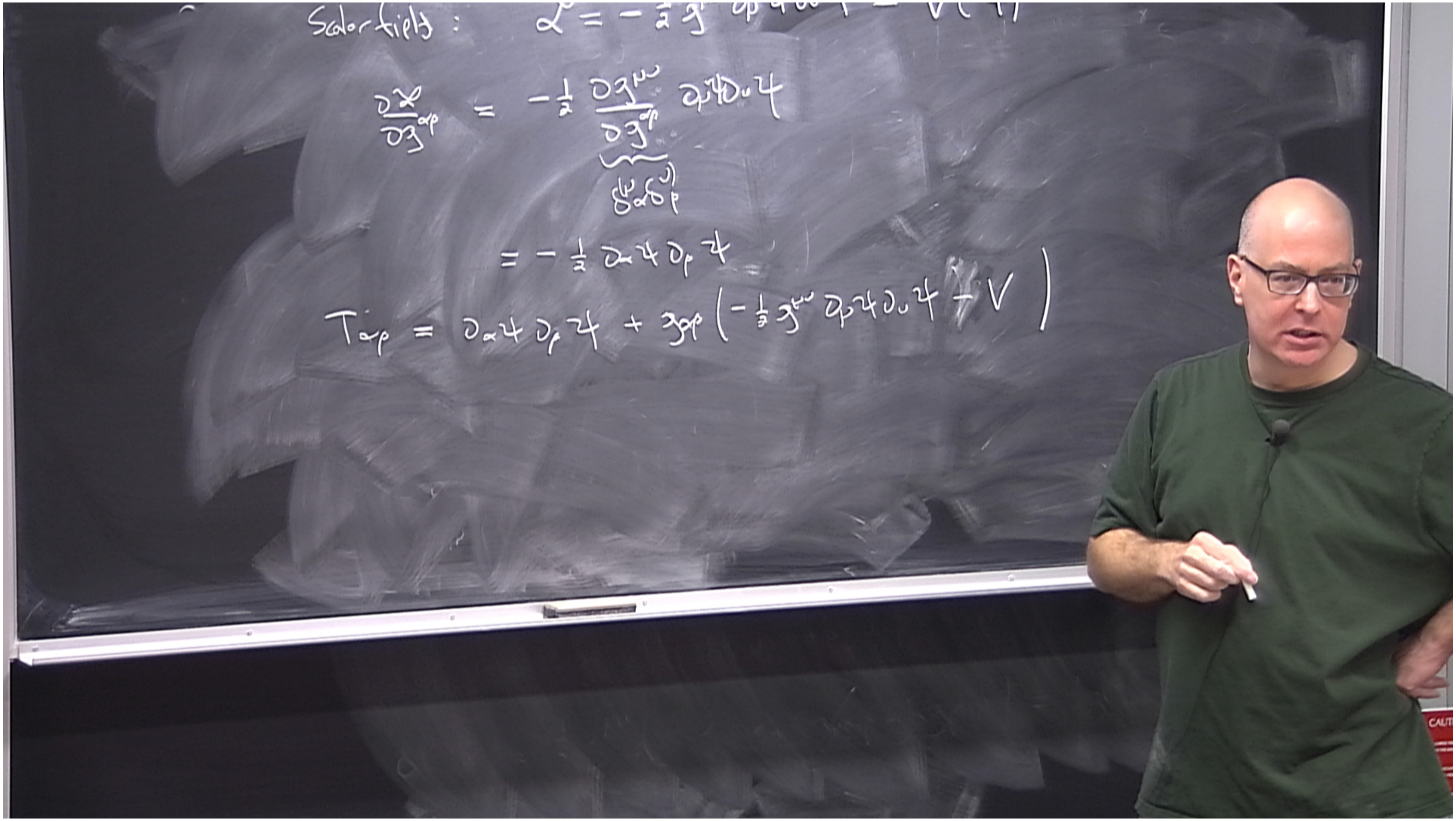
$$\longrightarrow \boxed{G_{\alpha\beta} = 8\pi T_{\alpha\beta}.} \checkmark$$

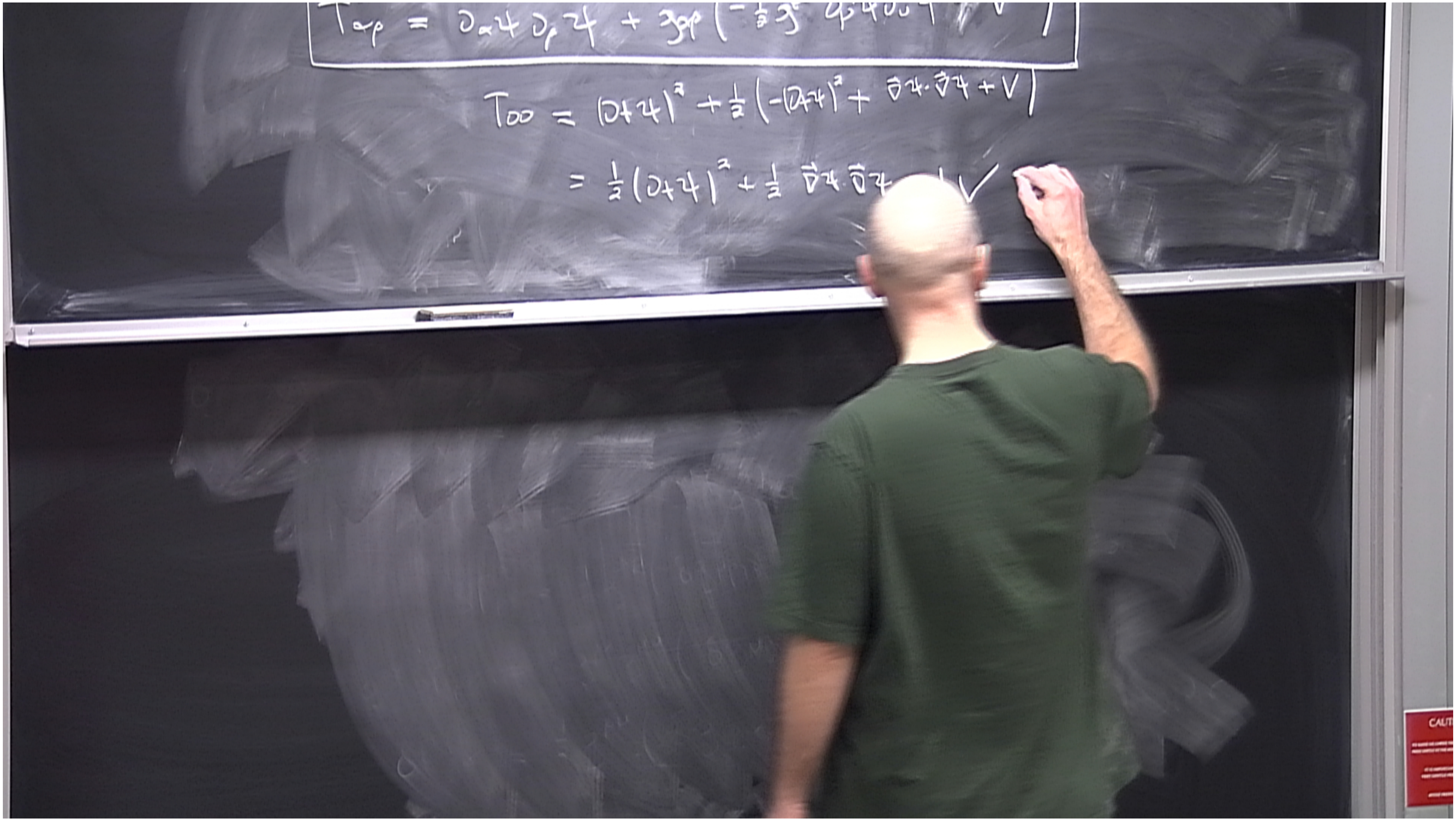

Scalar field :
ℒ = − ½ g^μν ∂_μ φ ∂_ν φ − V(φ)

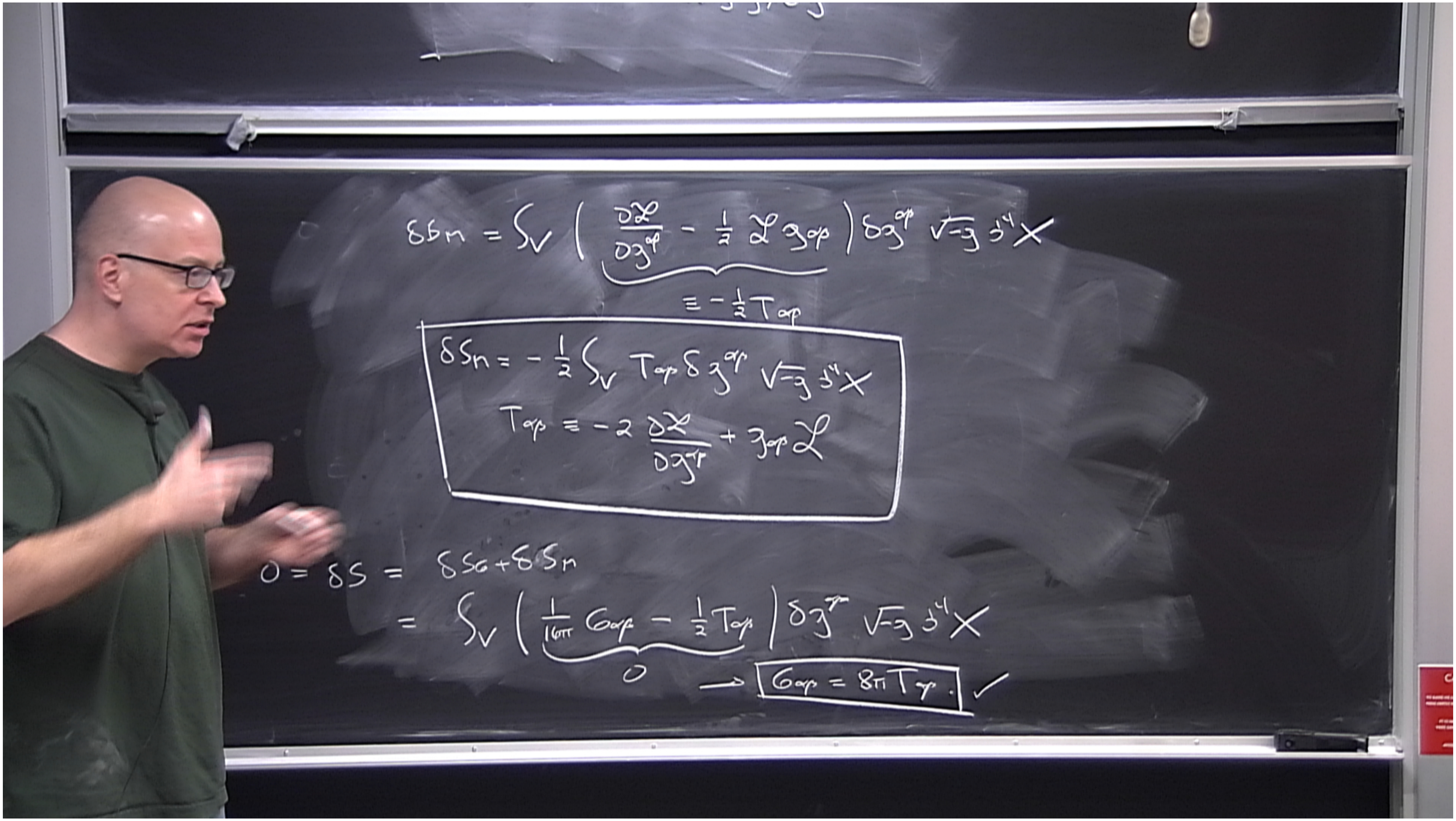

Scalar fields :
$\mathcal{L} = -\frac{1}{2} g^{\mu\nu} \partial_\mu \psi \partial_\nu \psi - V(\psi)$
$\frac{\partial \mathcal{L}}{\partial g^{\alpha\beta}} = -\frac{1}{2} \frac{\partial g^{\mu\nu}}{\partial g^{\alpha\beta}} \partial_\mu \psi \partial_\nu \psi$
$\delta^{(\mu}_{\alpha} \delta^{\nu)}_{\beta}$

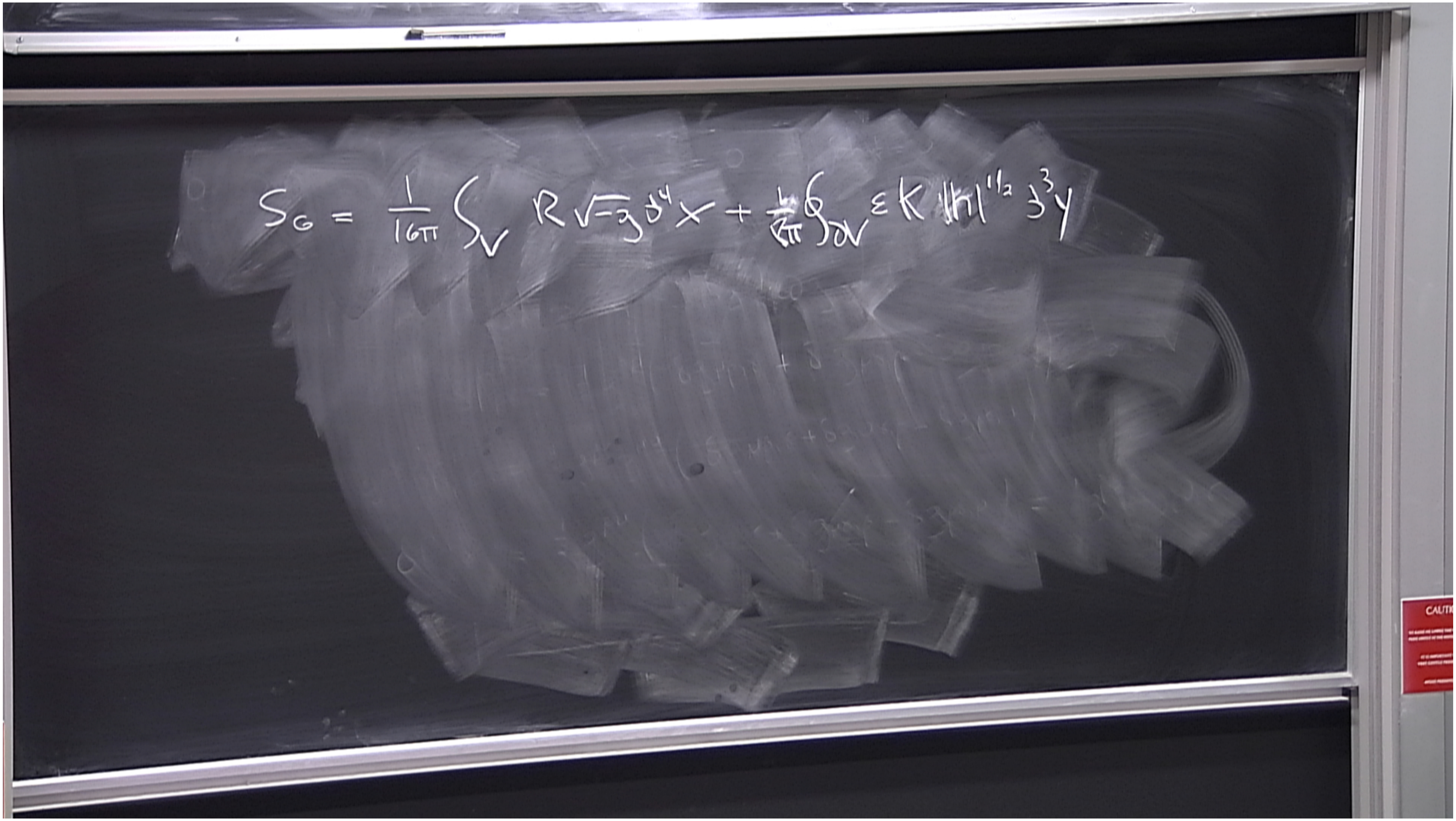


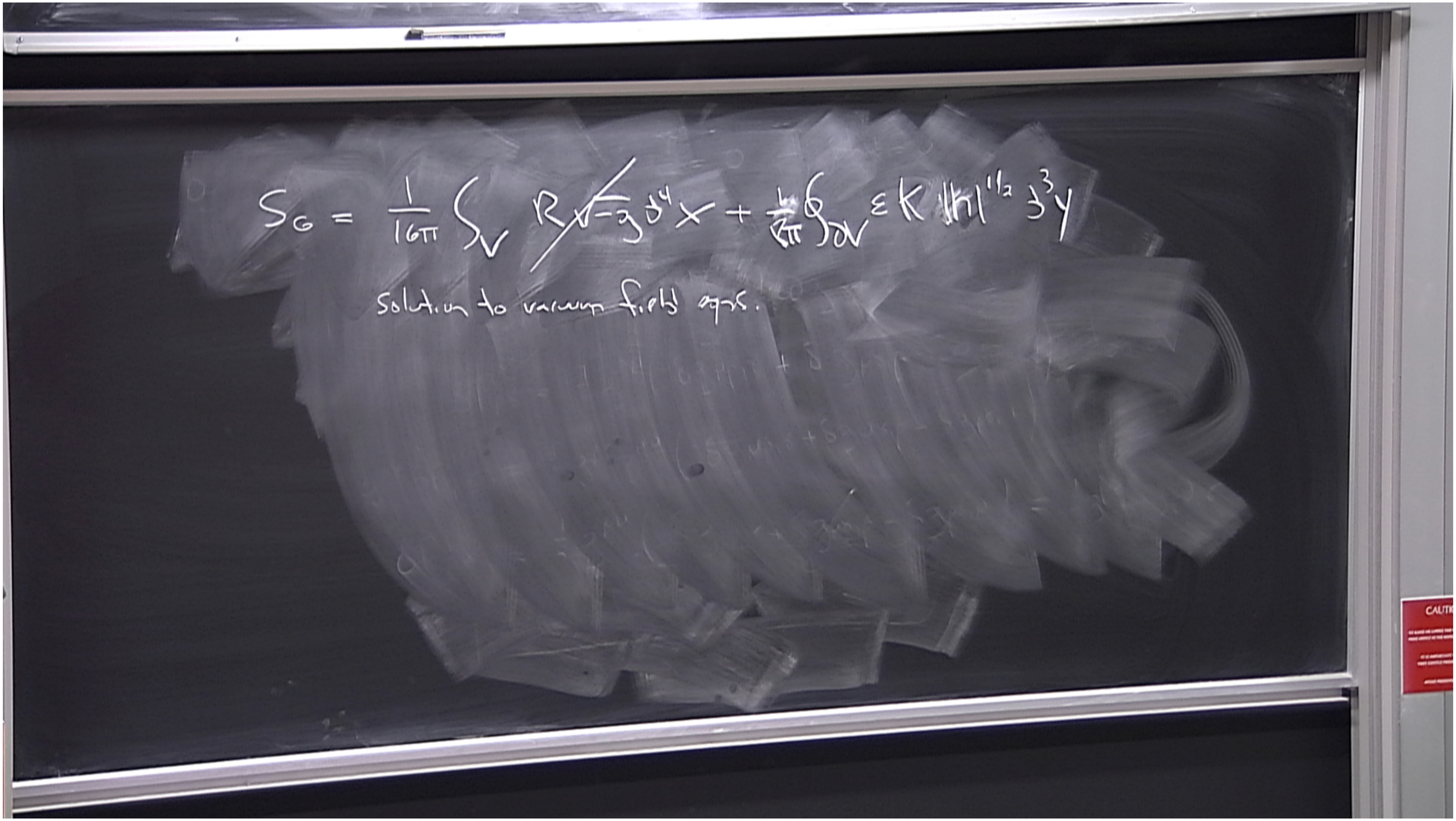
Scalar field :
$\frac{\partial \mathcal{L}}{\partial g^{\alpha\beta}} = -\frac{1}{2}\frac{\partial g^{\mu\nu}}{\partial g^{\alpha\beta}}\partial_\mu\phi\partial_\nu\phi$
$\delta^\mu_\alpha\delta^\nu_\beta$
$= -\frac{1}{2}\partial_\alpha\phi\partial_\beta\phi$

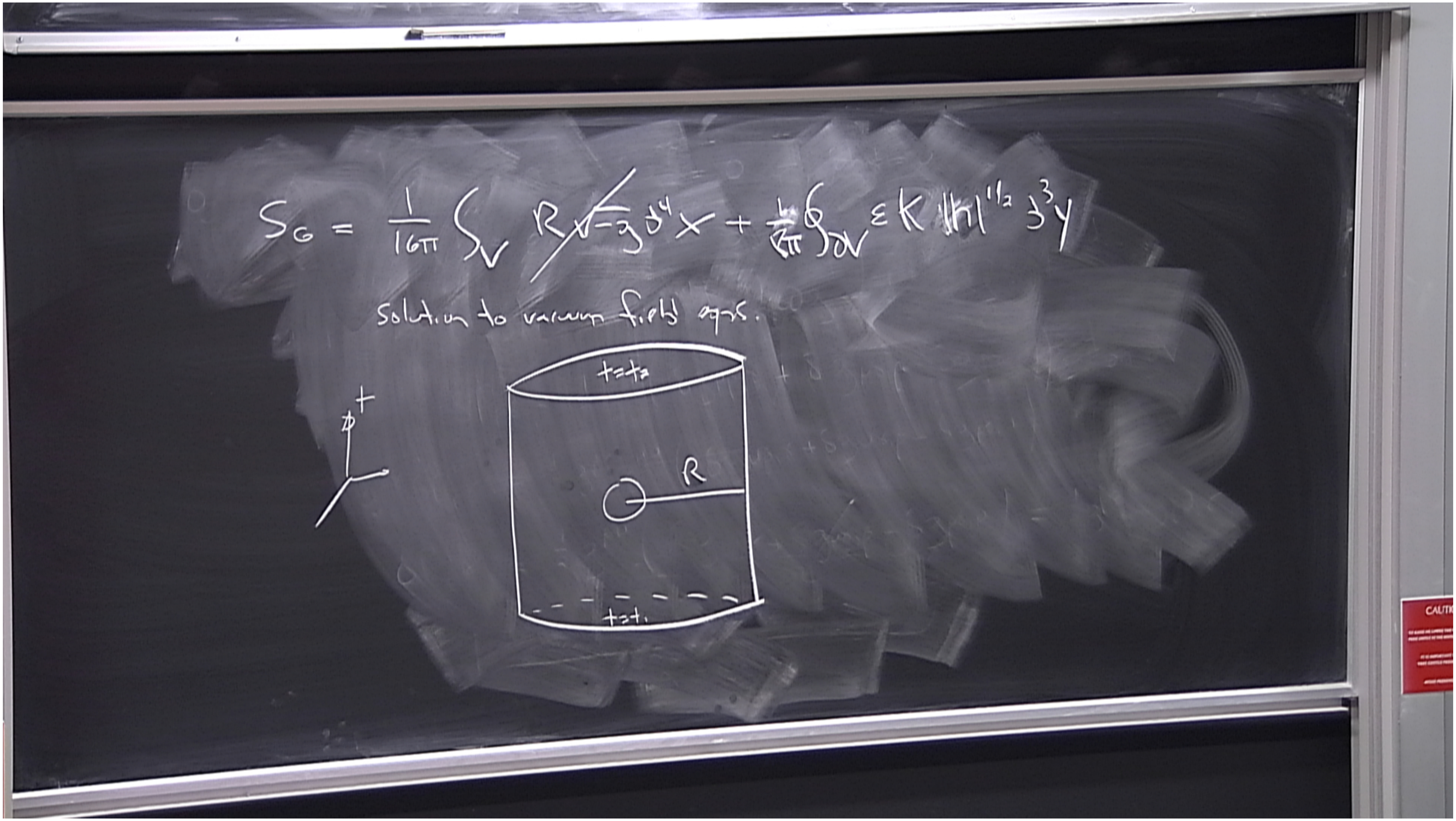

Scalar field :
$\frac{\partial \mathcal{L}}{\partial g^{\alpha\beta}} = -\frac{1}{2} \frac{\partial g^{\mu\nu}}{\partial g^{\alpha\beta}} \partial_\mu \phi \partial_\nu \phi$
$\delta^\mu_\alpha \delta^\nu_\beta$
$= -\frac{1}{2} \partial_\alpha \phi \partial_\beta \phi$
$T_{\alpha\beta} = \partial_\alpha \phi \partial_\beta \phi + g_{\alpha\beta} \left(-\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V \right)$

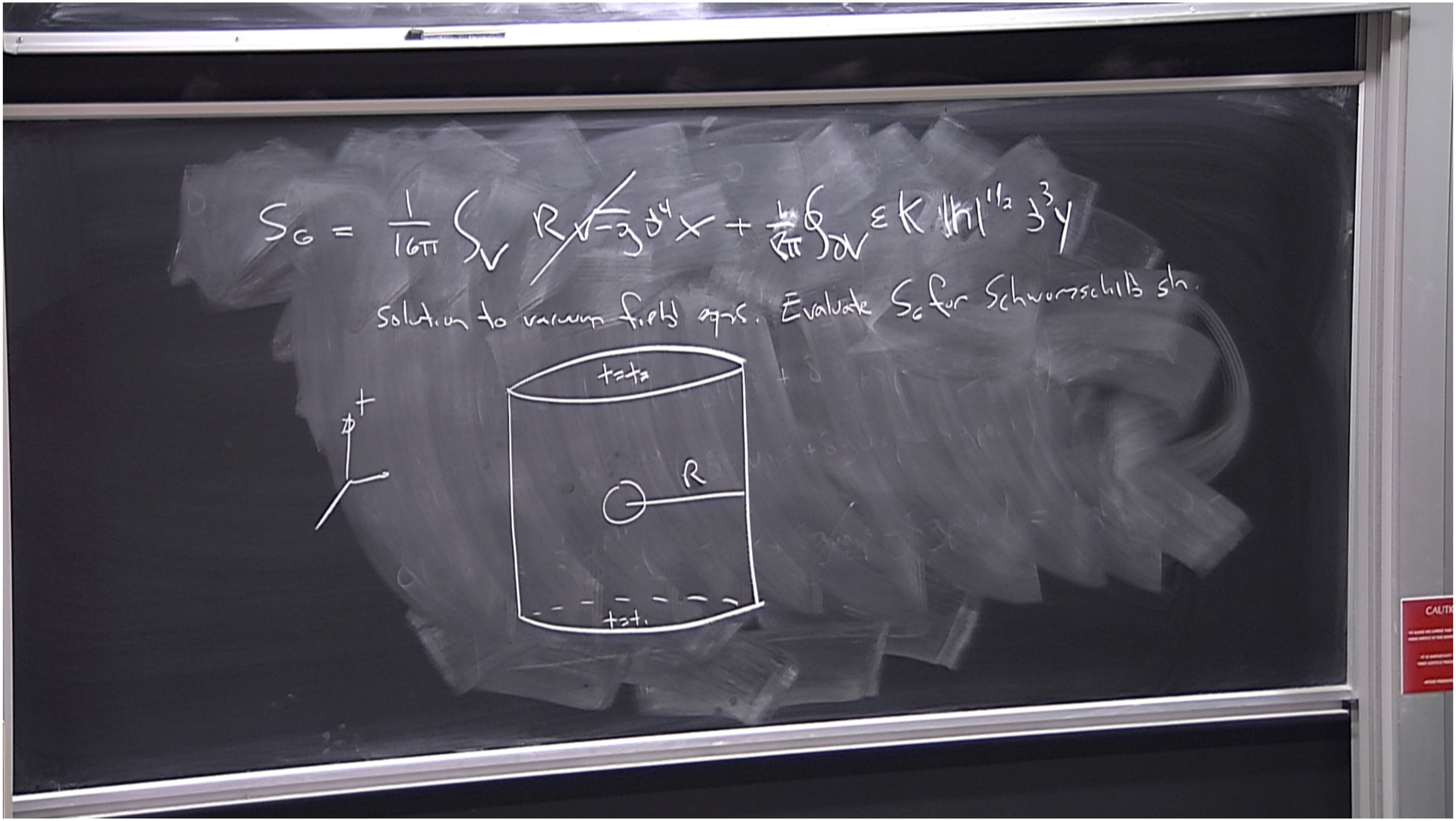

$T_{00} = (\partial_t \varphi)^2 + \frac{1}{2}\left(-(\partial_t \varphi)^2 + \vec{\partial}\varphi \cdot \vec{\partial}\varphi + V\right)$
$= \frac{1}{2}(\partial_t \varphi)^2 + \frac{1}{2}\,\vec{\partial}\varphi \cdot \vec{\partial}\varphi$

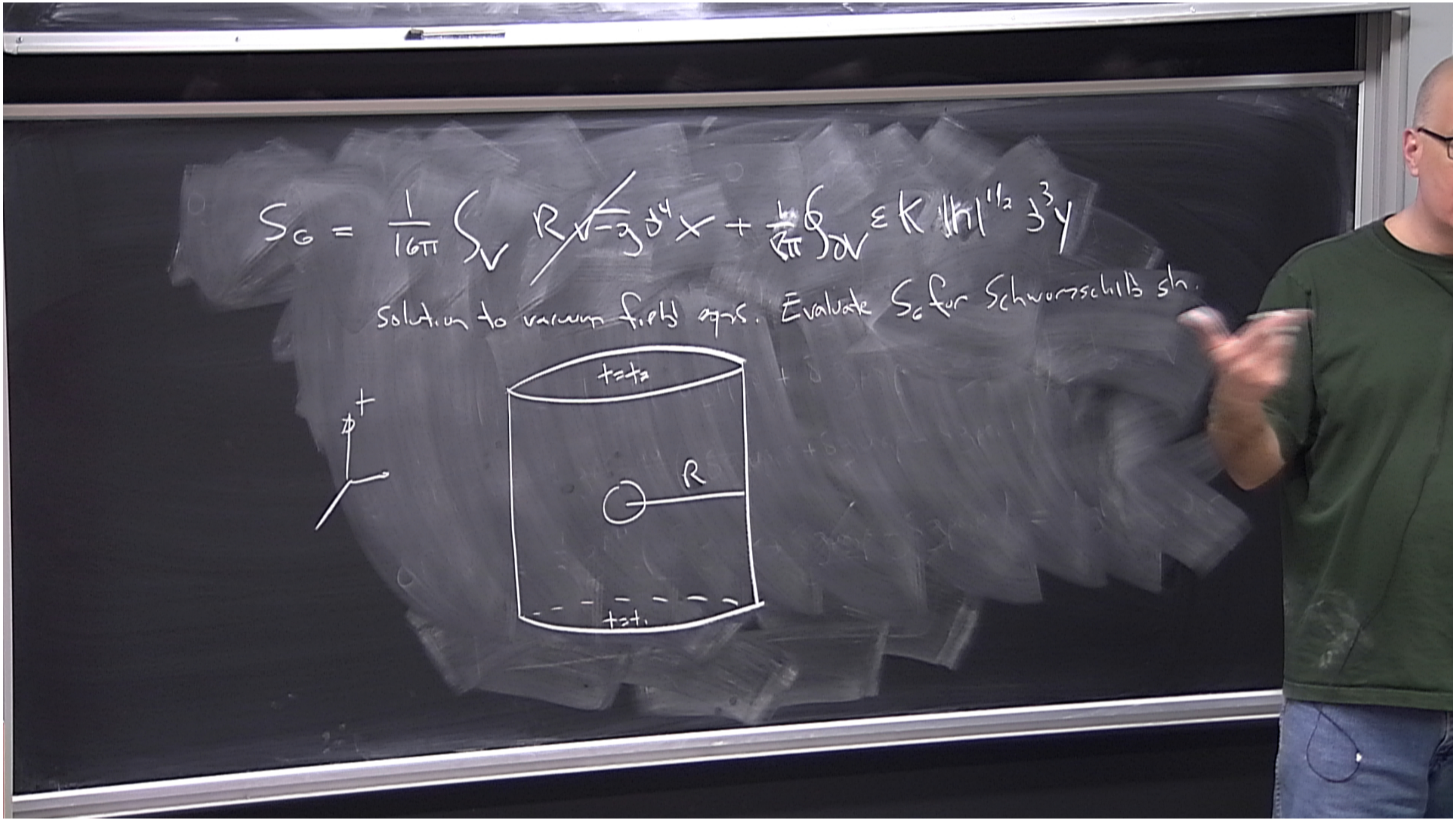

$\delta S_m = \int_V \left(\frac{\partial \mathcal{L}}{\partial g^{\alpha\beta}} - \frac{1}{2} \mathcal{L} g_{\alpha\beta} \right) \delta g^{\alpha\beta} \sqrt{-g}\, d^4x$
$\equiv -\frac{1}{2} T_{\alpha\beta}$
$\delta S_m = -\frac{1}{2} \int_V T_{\alpha\beta}\, \delta g^{\alpha\beta} \sqrt{-g}\, d^4x$
$T_{\alpha\beta} \equiv -2 \frac{\partial \mathcal{L}}{\partial g^{\alpha\beta}} + g_{\alpha\beta} \mathcal{L}$
$0 = \delta S = \delta S_G + \delta S_m$
$= \int_V \left(\frac{1}{16\pi} G_{\alpha\beta} - \frac{1}{2} T_{\alpha\beta} \right) \delta g^{\alpha\beta} \sqrt{-g}\, d^4x$
0
$\rightarrow G_{\alpha\beta} = 8\pi T_{\alpha\beta}$ ✓

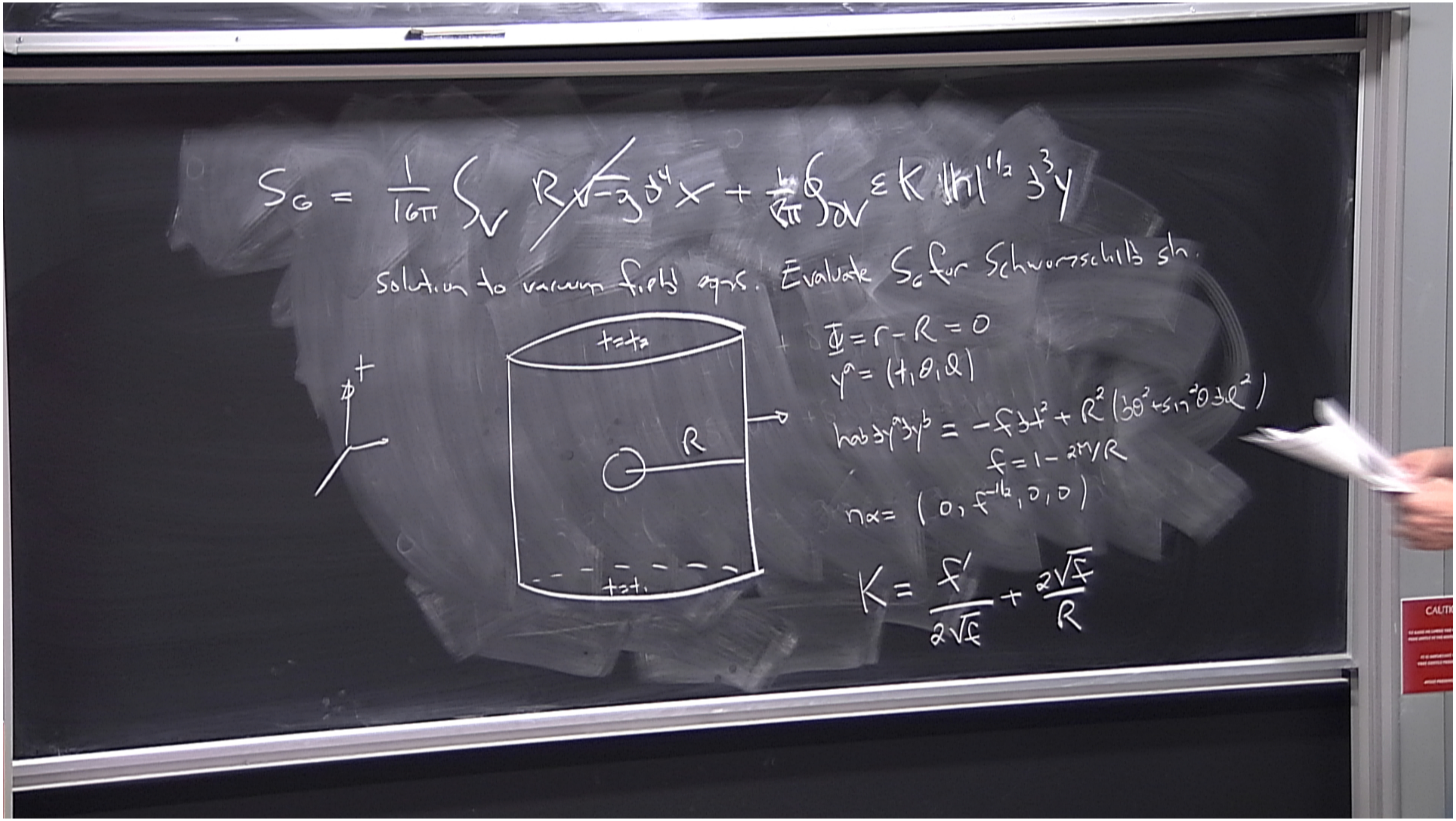

$S_G = \frac{1}{16\pi}\int_V R\sqrt{-g}\,d^4x + \frac{1}{8\pi}\oint_{\partial V}\varepsilon K |h|^{1/2}\, d^3y$

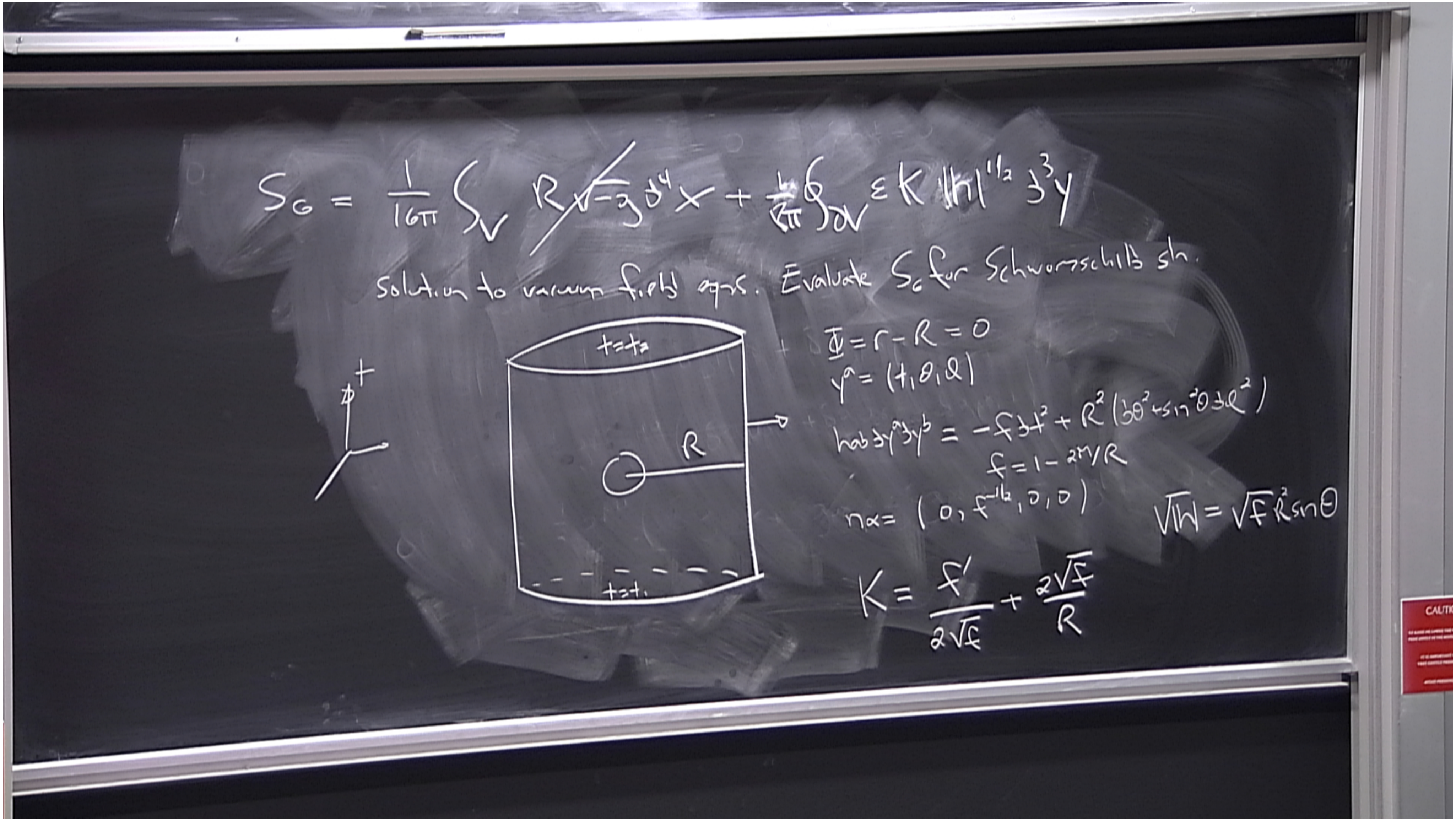

$S_G = \frac{1}{16\pi}\int_V R\sqrt{-g}\, d^4x + \frac{1}{8\pi}\oint_{\partial V} \varepsilon K |h|^{1/2} d^3y$
Solution to vacuum field eqns.

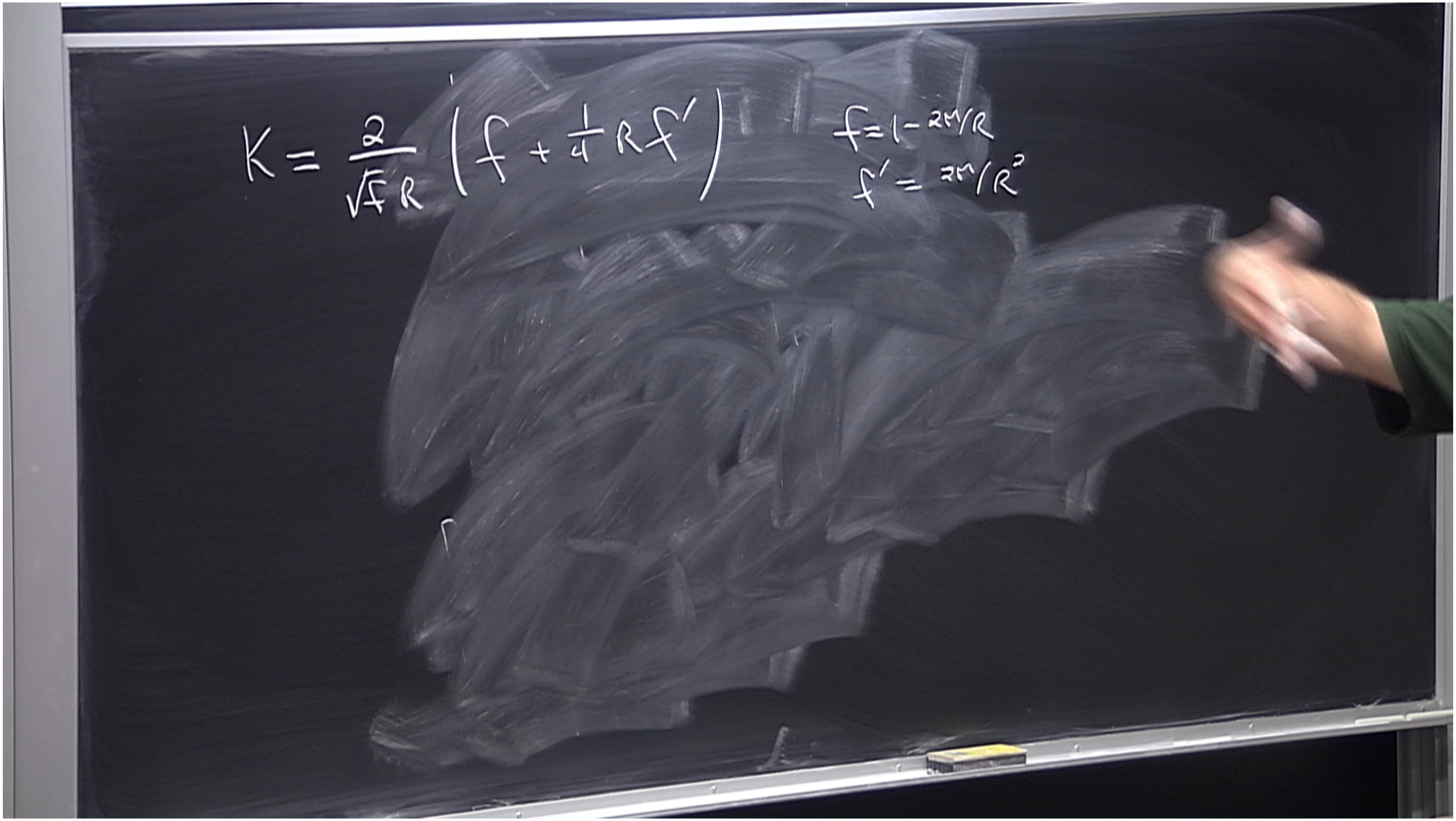

$S_G = \frac{1}{16\pi}\int_V R\sqrt{-g}\, d^4x + \frac{1}{8\pi}\oint_{\partial V} \varepsilon K |h|^{1/2} d^3y$
Solution to vacuum field eqns.
t=t2
t=t1
R
φ
t


$S_G = \frac{1}{16\pi} \int_V R\sqrt{-g}\, d^4x + \frac{1}{8\pi} \oint_{\partial V} \varepsilon K |h|^{1/2} d^3y$
Solution to vacuum field eqns. Evaluate S_G for Schwarzschild sln.
t=t2
t
R
t=t1


$S_G = \frac{1}{16\pi} \int_V R \sqrt{-g}\, d^4x + \frac{1}{8\pi} \oint_{\partial V} \varepsilon K |h|^{1/2} d^3y$
Solution to vacuum field eqns. Evaluate S_G for Schwarzschild sln.
t=t₂
R
t=t₁


S_G = 1/16π ∫_V R √-g d^4x + 1/8π ∮_∂V ε K |h|^1/2 d^3y
Solution to vacuum field eqns. Evaluate S_G for Schwarzschild sln.
t=t2
t=t1
R
Φ = r − R = 0
y^a = (t, θ, φ)
h_ab dy^a dy^b = −f dt^2 + R^2 (dθ^2 + sin^2θ dφ^2)
f = 1 − 2M/R
n_α = (0, f^-1/2, 0, 0)
K = f′/2√f + 2√f/R


S_G = 1/16π ∫_V R √-g d⁴x + 1/8π ∮_∂V ε K |h|^1/2 d³y
Solution to vacuum field eqns. Evaluate S_G for Schwarzschild sln.
t=t₂
t=t₁
R
Φ = r − R = 0
y^a = (t, θ, φ)
h_ab dy^a dy^b = −f dt² + R²(dθ² + sin²θ dφ²)
f = 1 − 2M/R
n_α = (0, f^-1/2, 0, 0)
√|h| = √f R² sin θ
K = f′/(2√f) + 2√f/R


$K = \frac{2}{\sqrt{f} R}\left(f + \frac{1}{4} R f'\right)$
$f = 1 - 2M/R$
$f' = 2M/R^2$