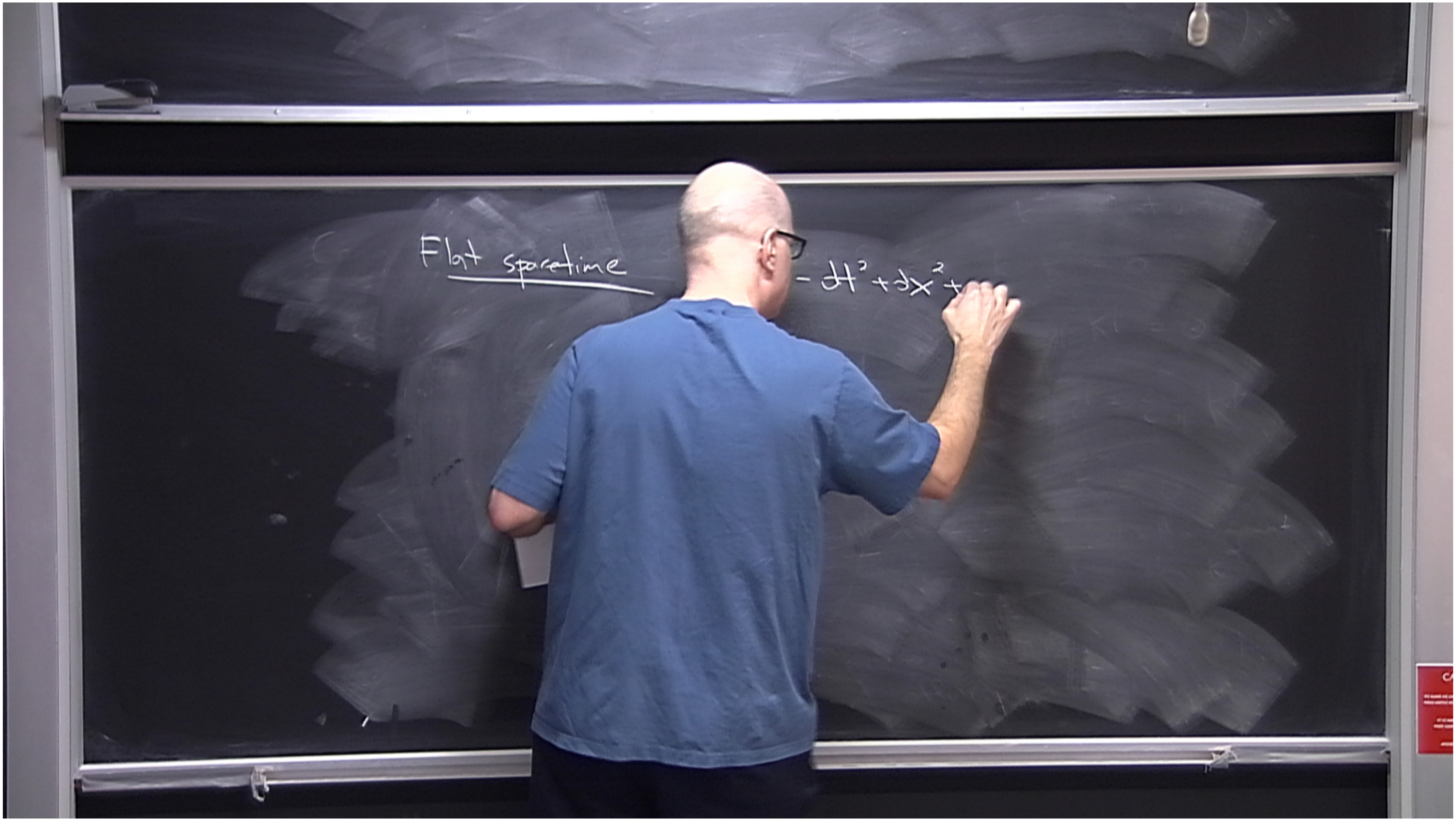
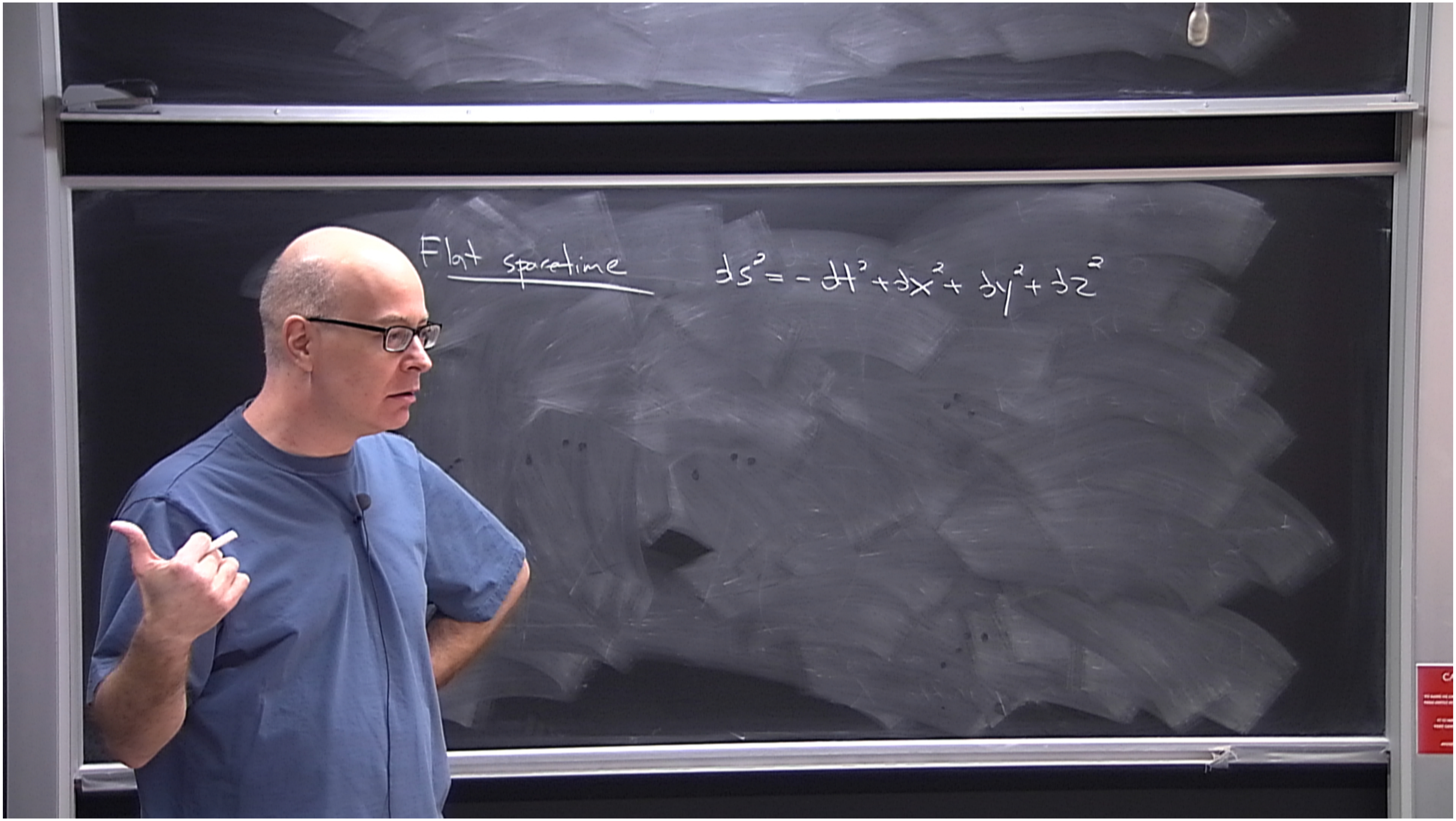
Title: Advanced General Relativity - Lecture 8

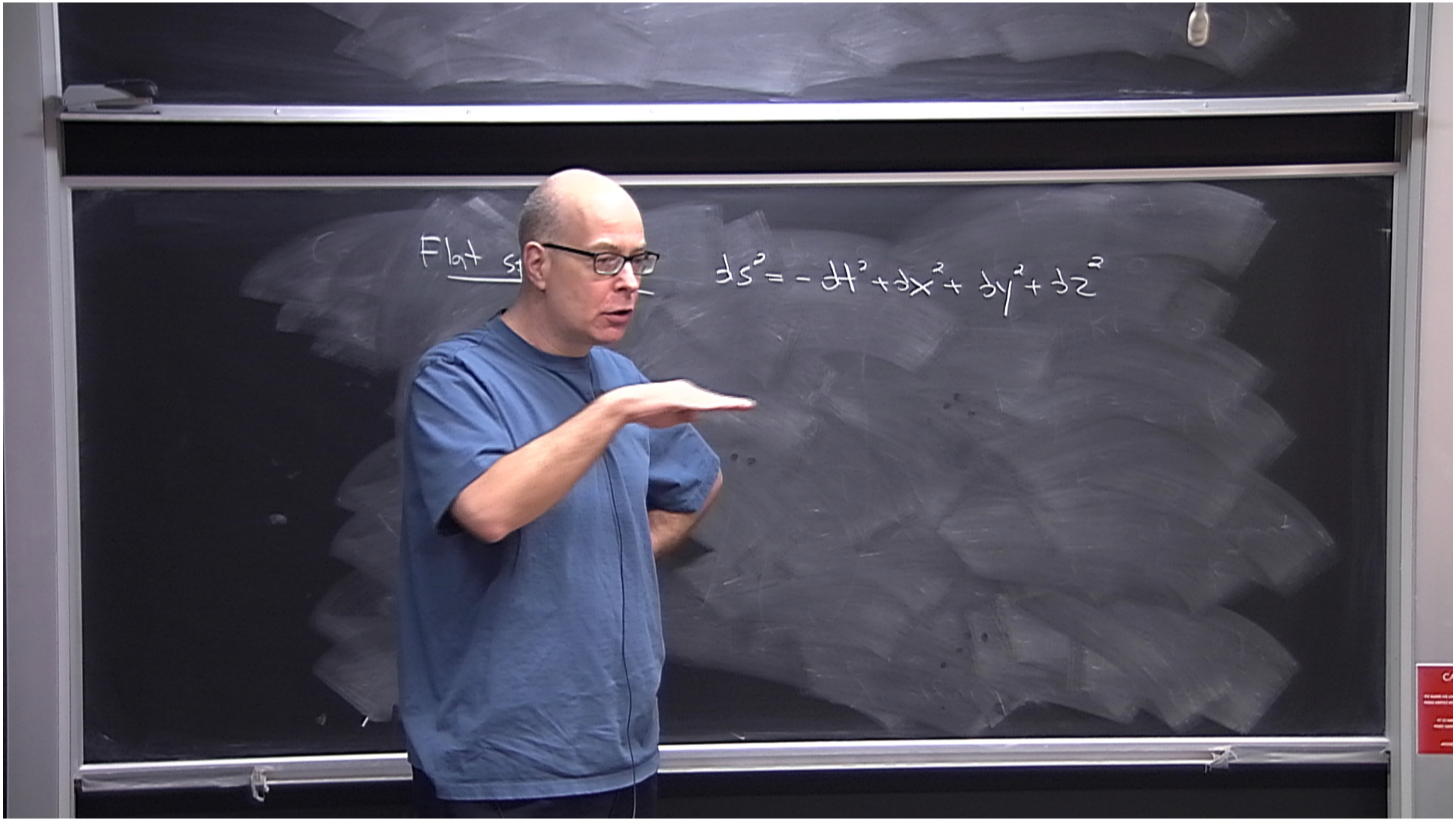
Date: Feb 01, 2012 03:30 PM

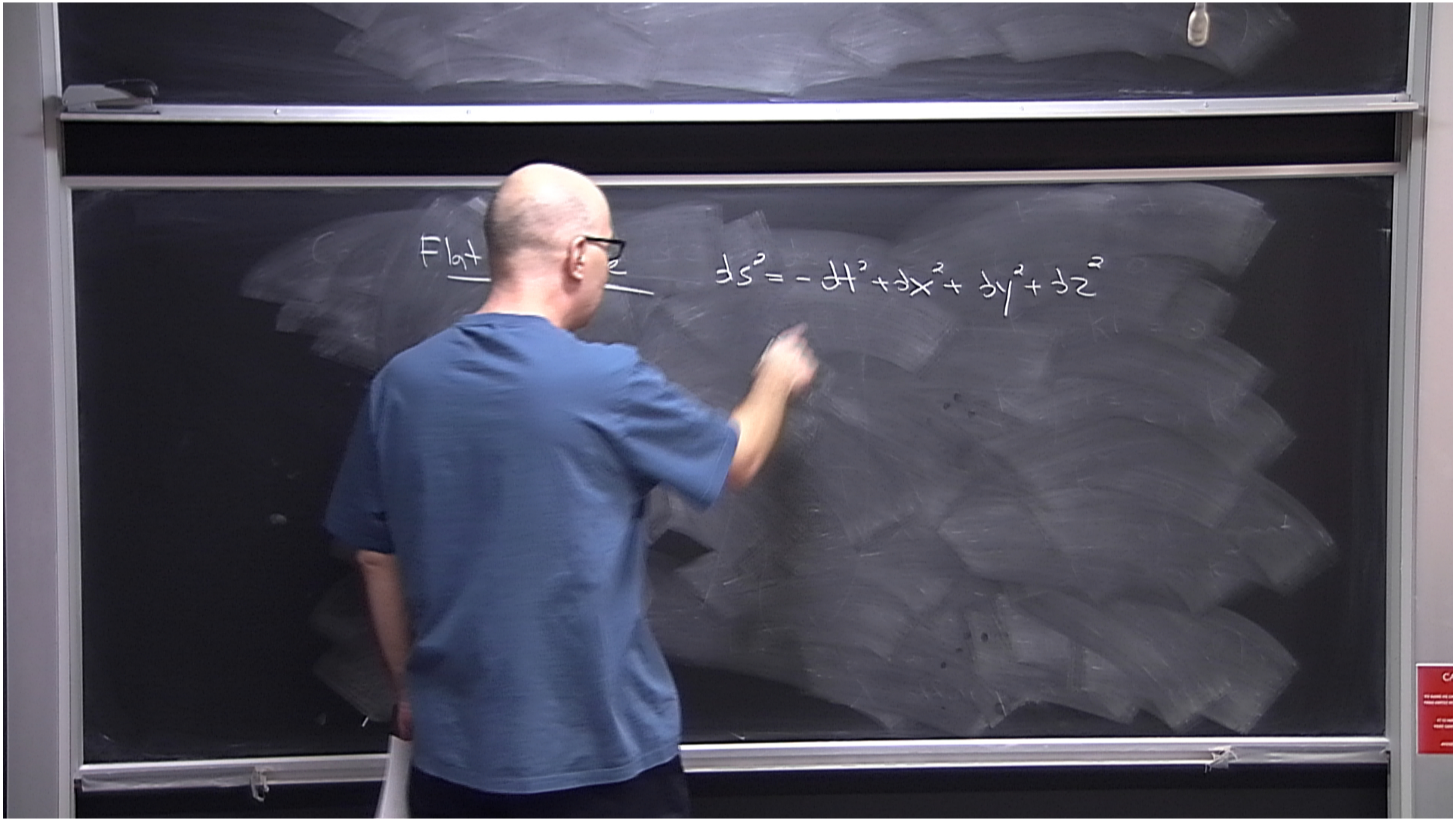
URL: http://pirsa.org/12020111

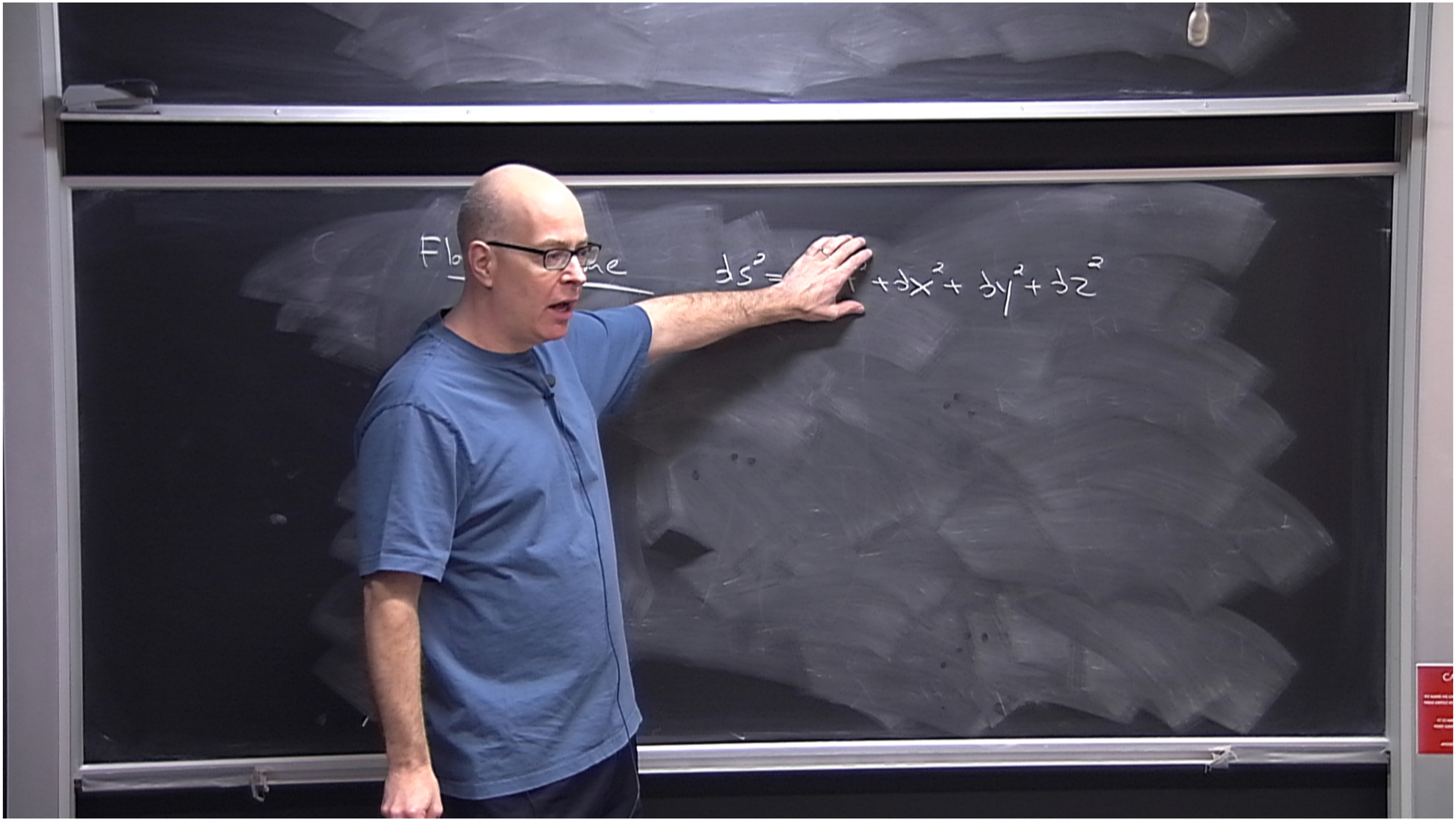
Abstract:

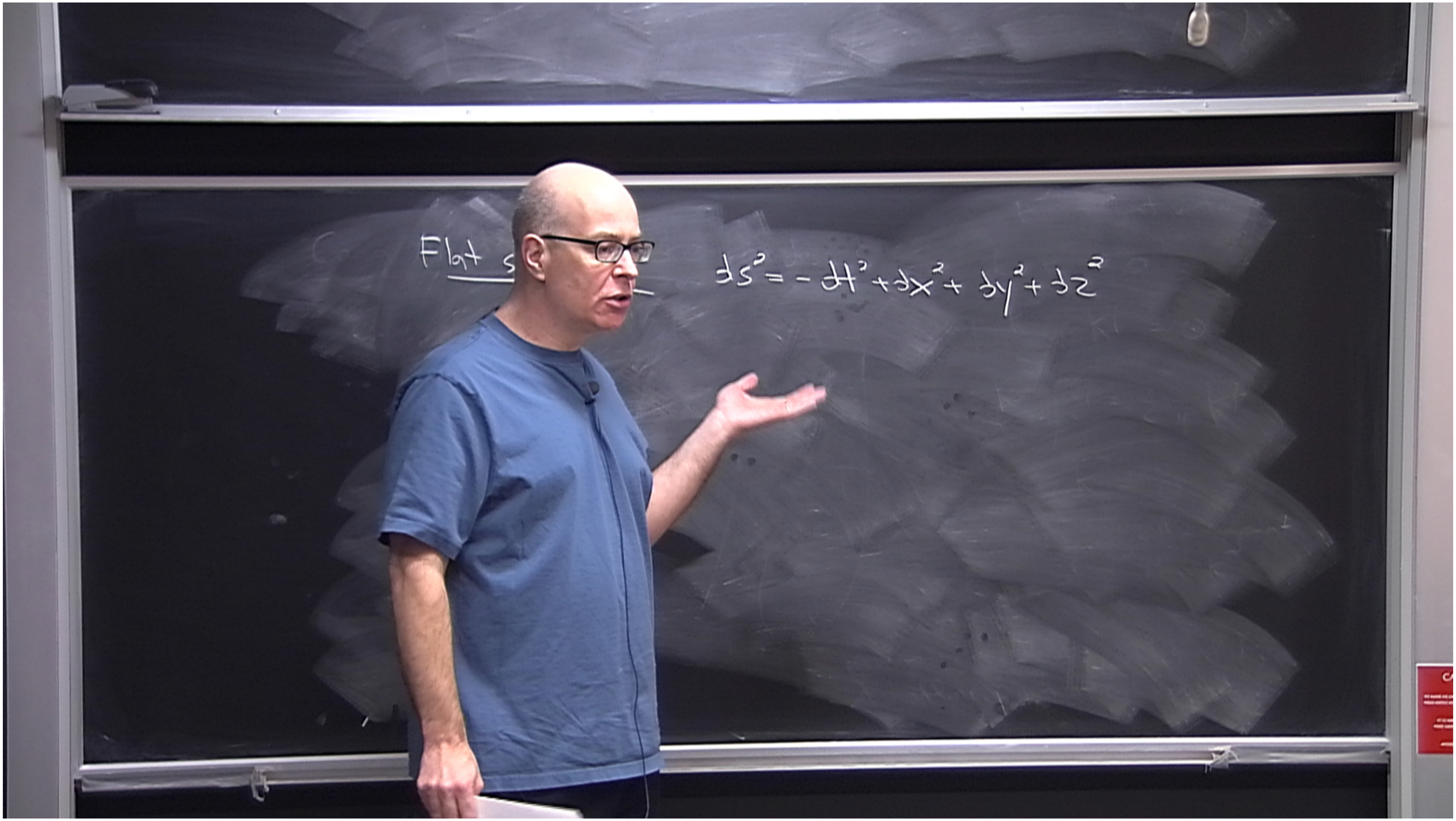

Flat spacetime
- dt² + dx² +

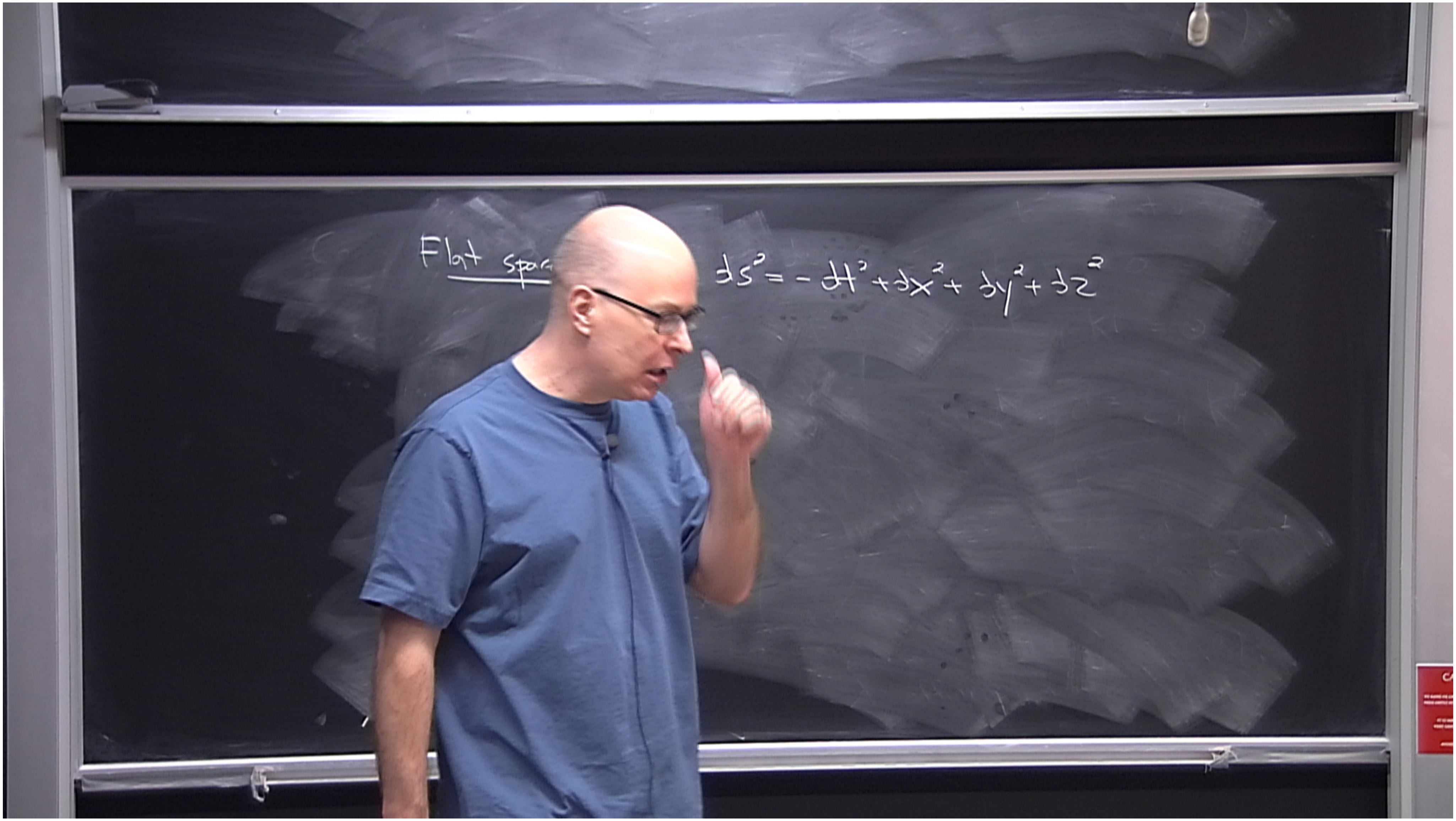

Flat spacetime
$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$

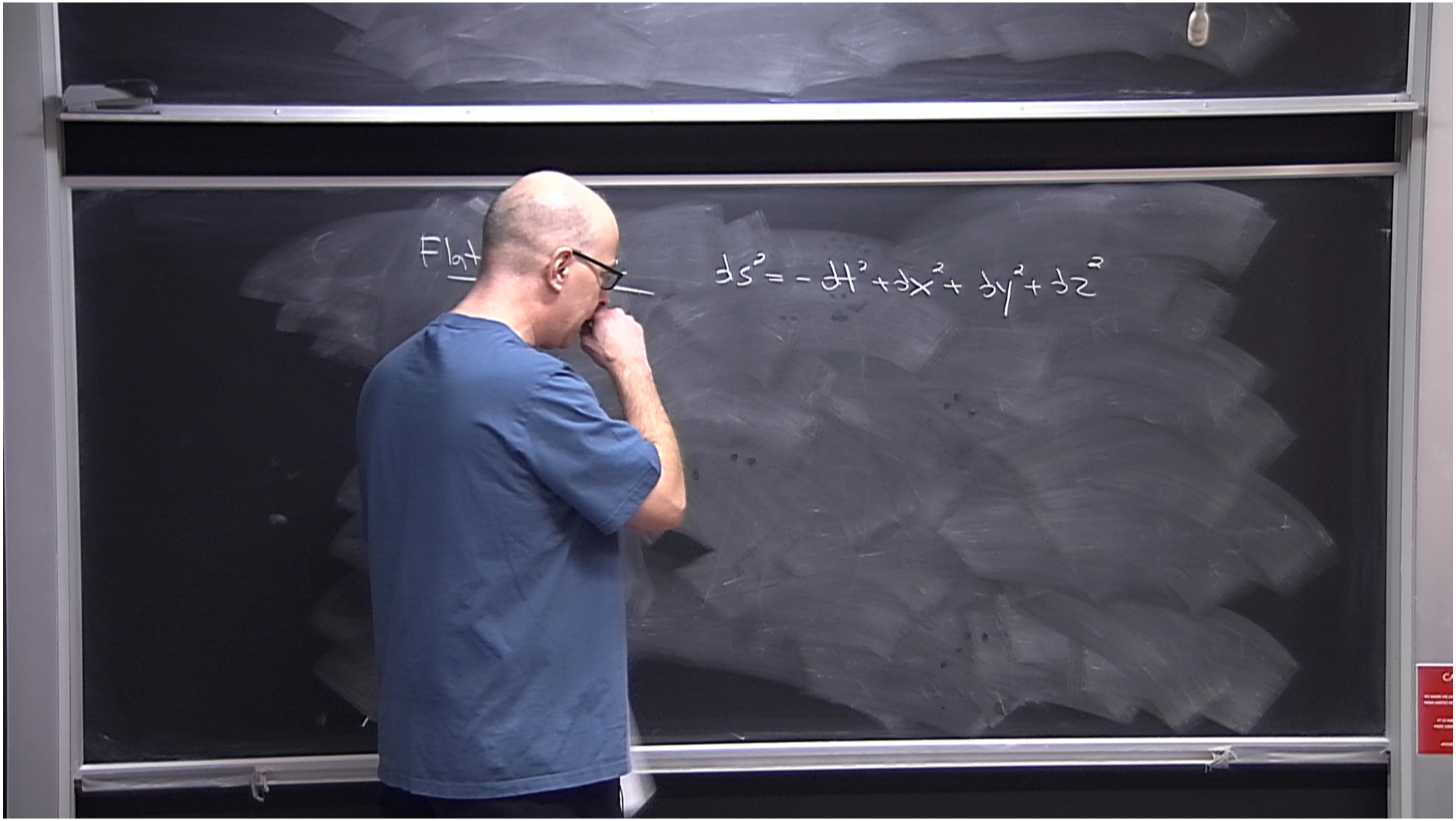

Flat
ds² = - dt² + dx² + dy² + dz²

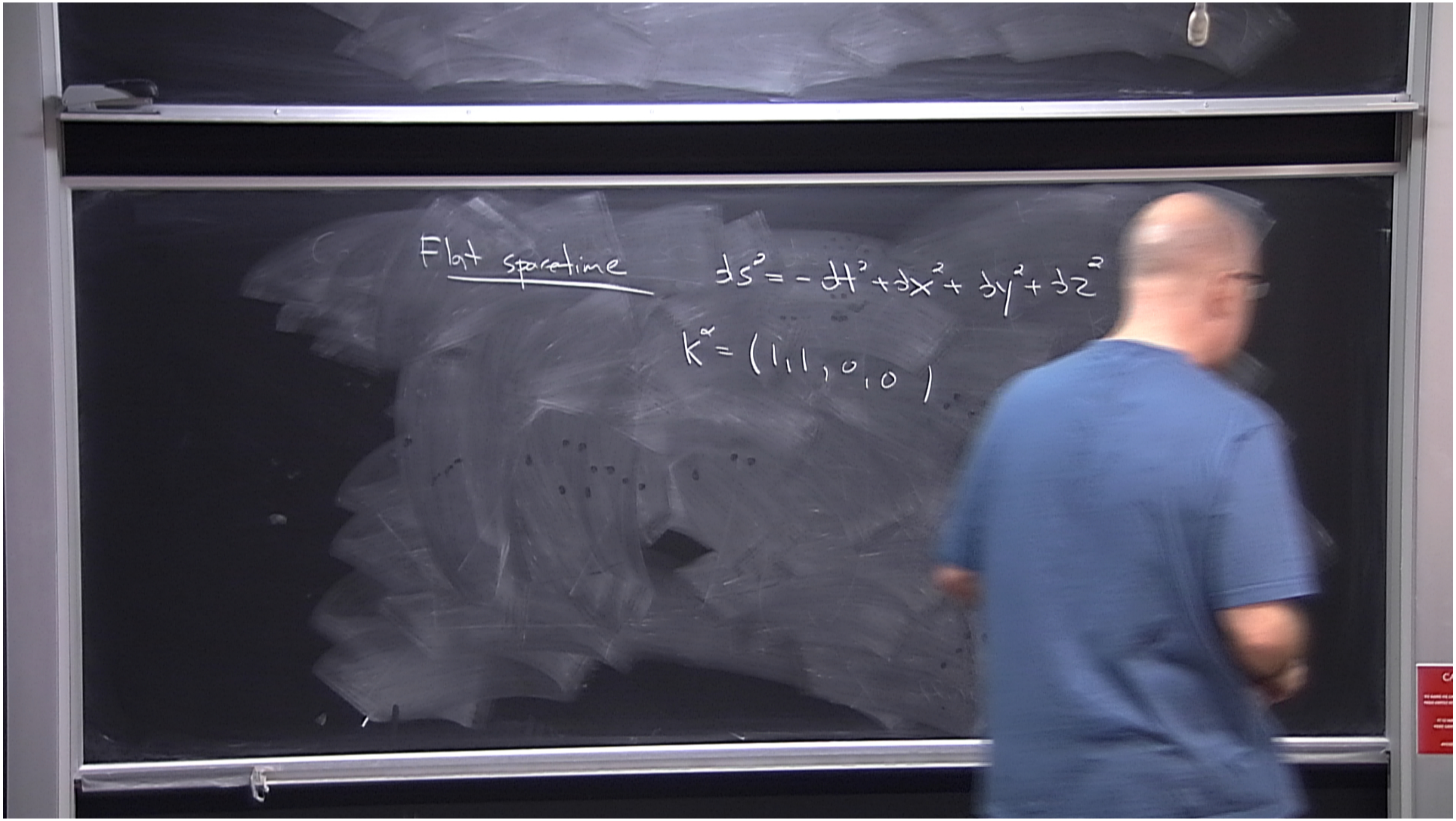

Flat
ds² = - dt² + dx² + dy² + dz²

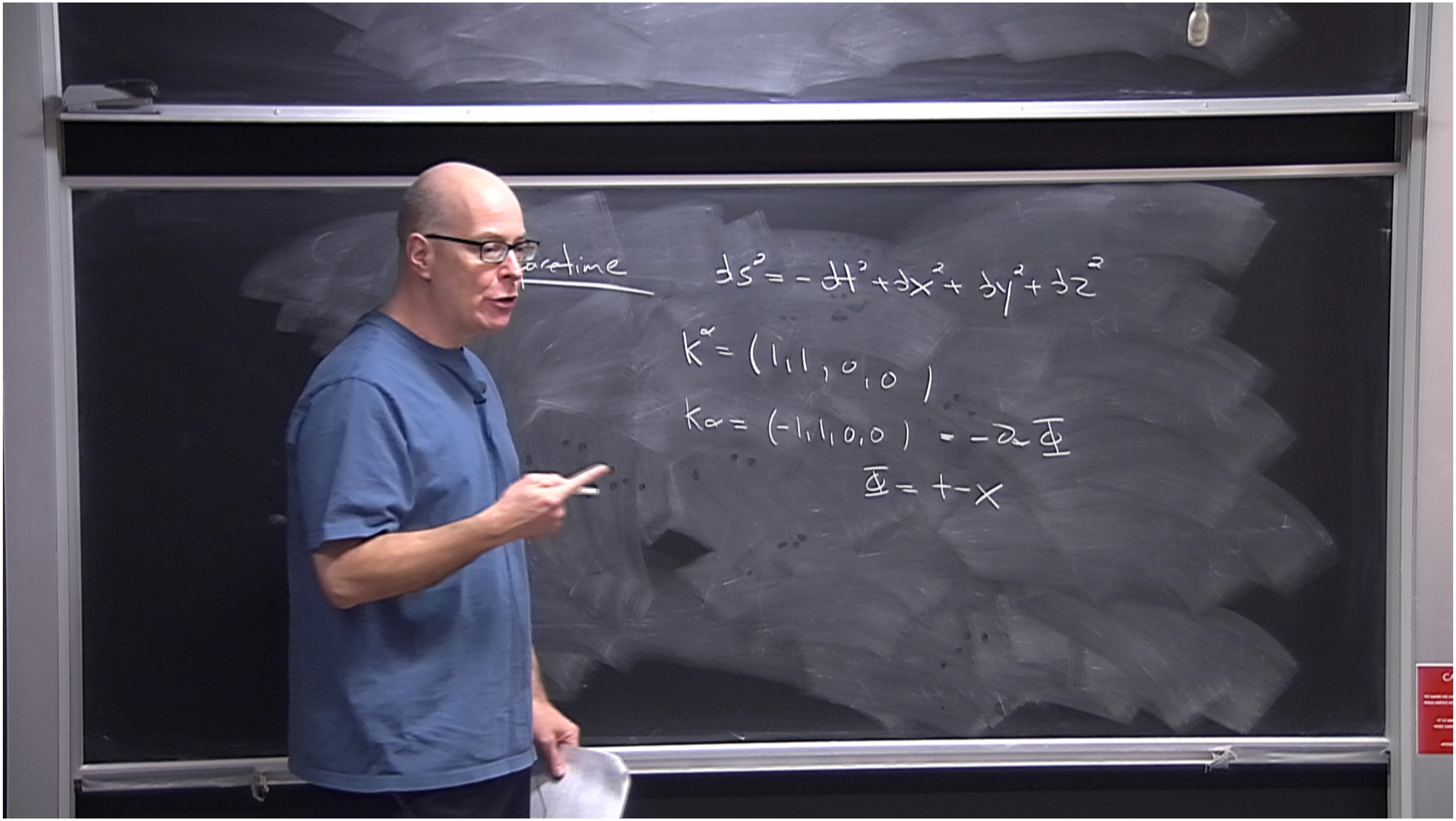

ds² = … + dx² + dy² + dz²

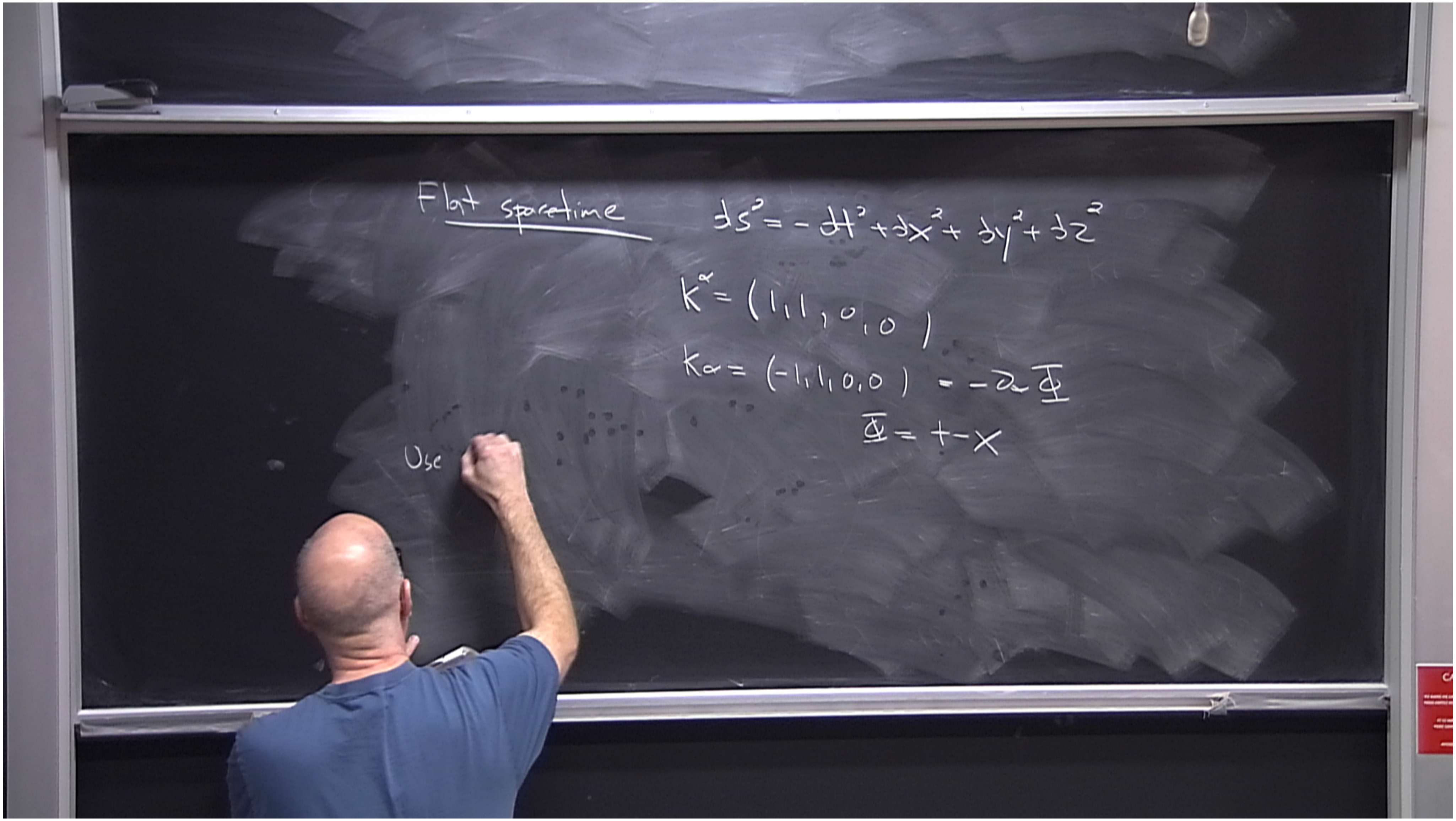

Flat
$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$


Flat spa
ds² = - dt² + dx² + dy² + dz²

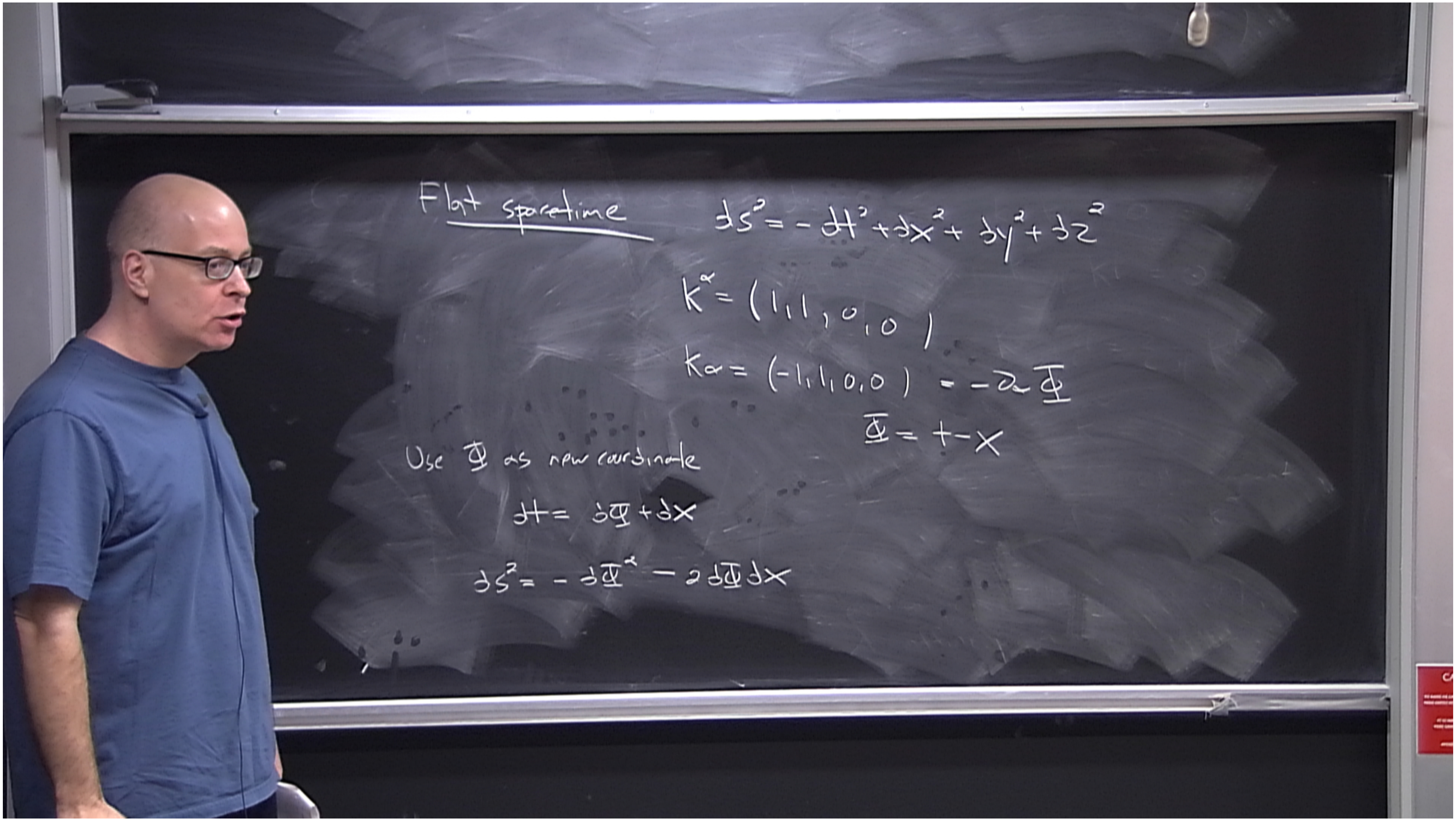

Flat
$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$

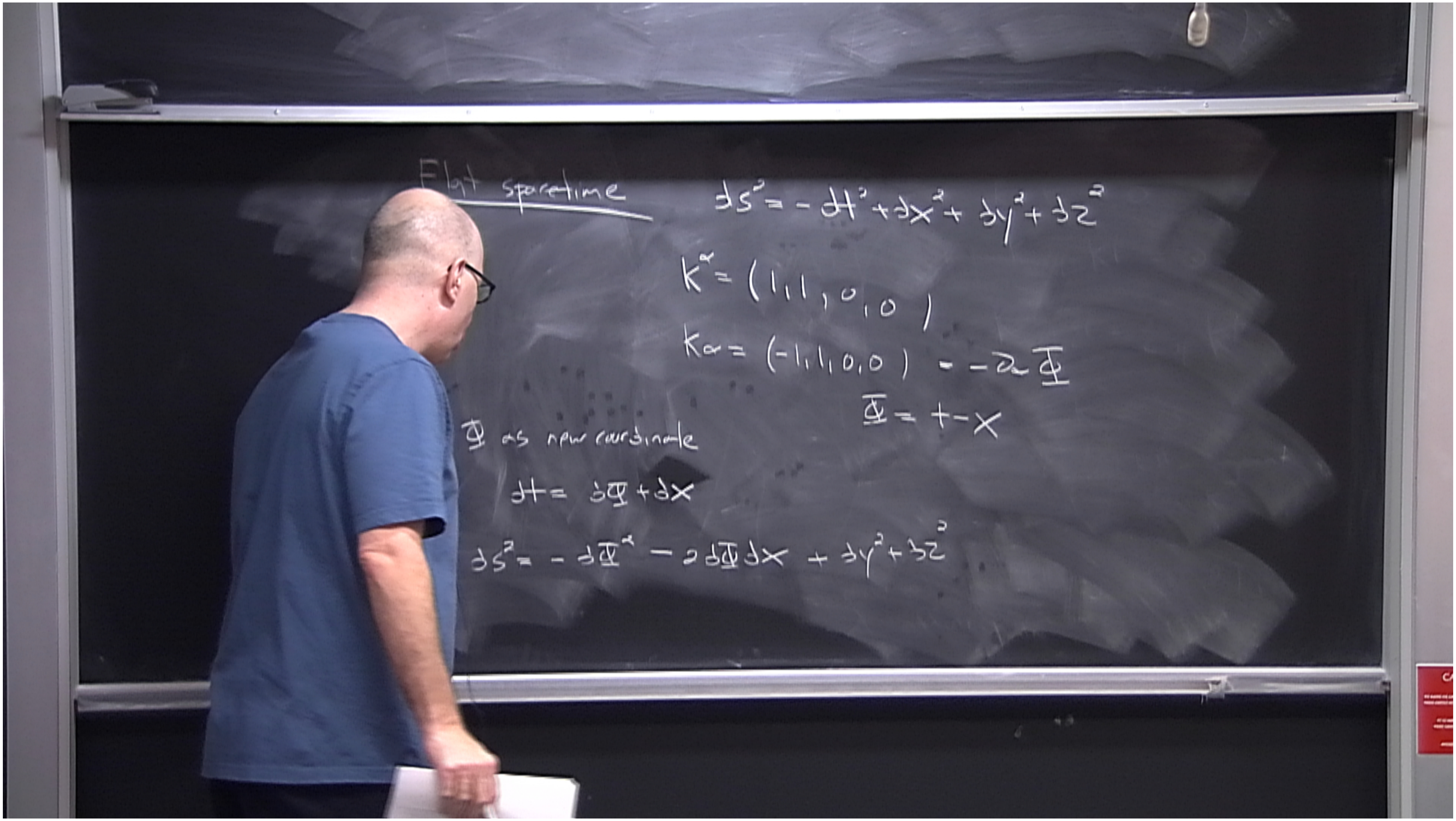

Flat spacetime
$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$
$k^\alpha = (1, 1, 0, 0)$

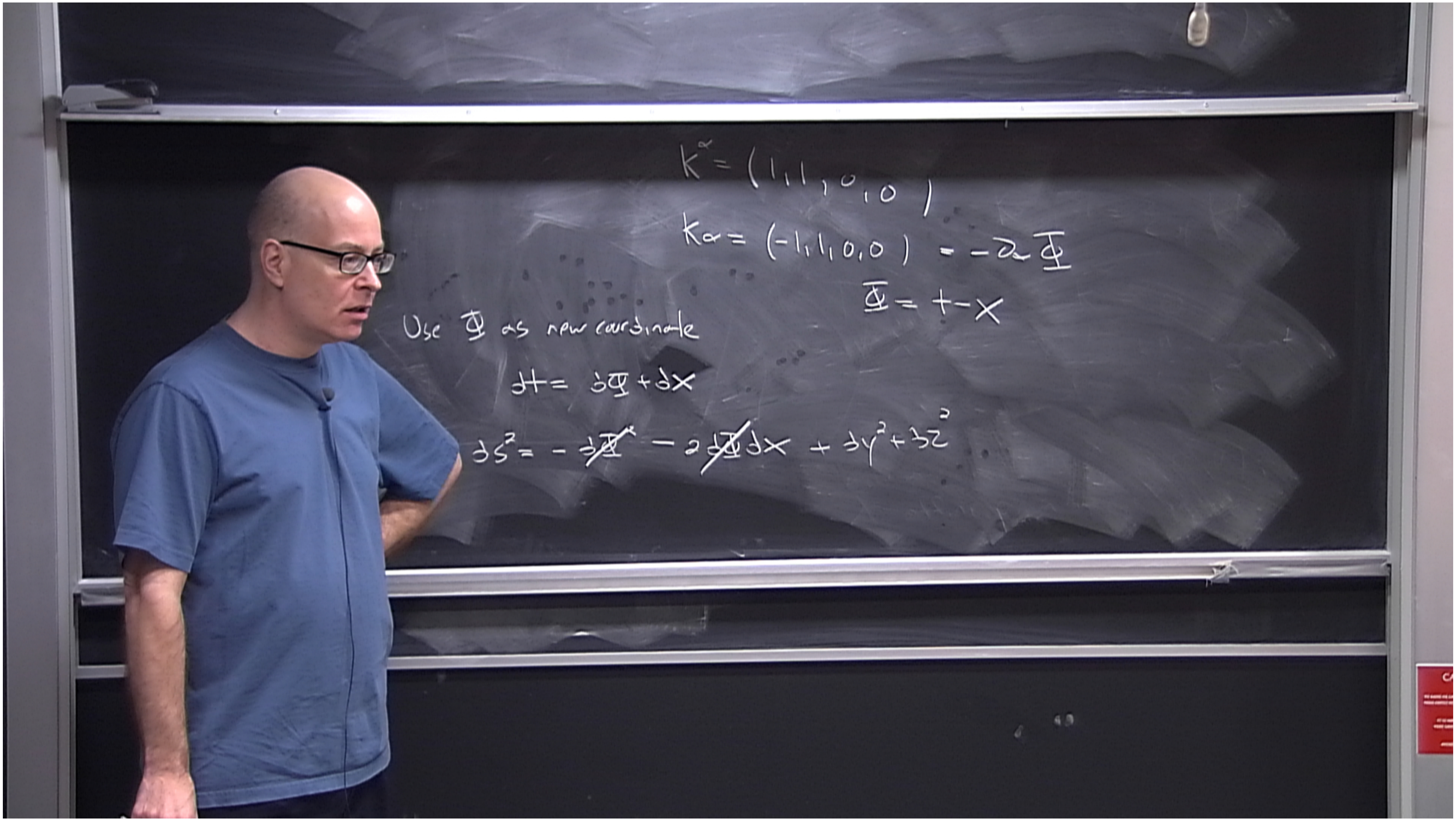

acetime
ds² = -dt² + dx² + dy² + dz²
k^α = (1,1,0,0)
k_α = (-1,1,0,0) = -∂_α Φ
Φ = t - x

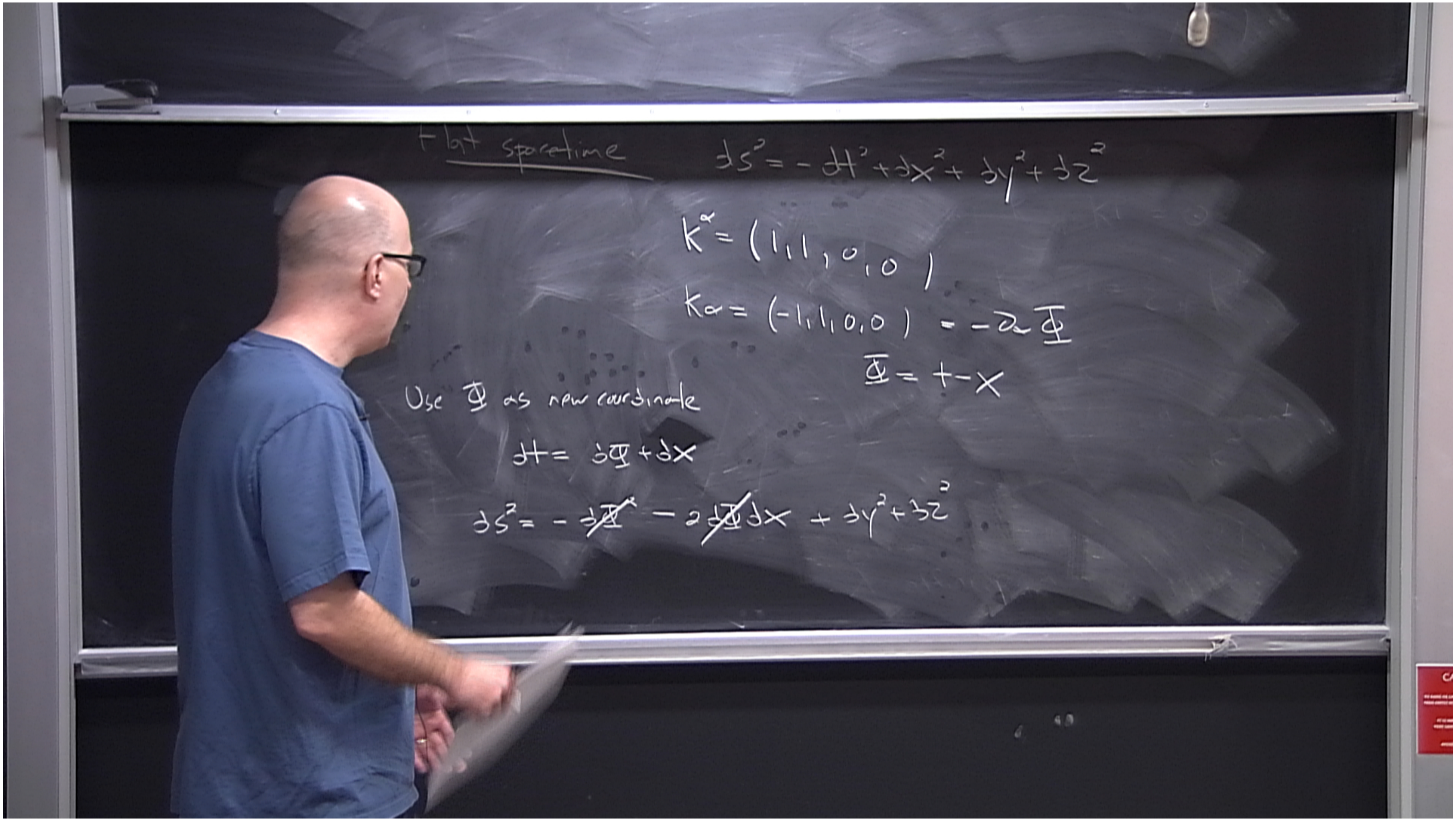


Flat spacetime
ds² = - dt² + dx² + dy² + dz²
k^α = (1, 1, 0, 0)
k_α = (-1, 1, 0, 0) = -∂_α Φ
Φ = t - x
Use


Flat spacetime
$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$
$k^\alpha = (1,1,0,0)$
$k_\alpha = (-1,1,0,0) = -\partial_\alpha \Phi$
$\Phi = t - x$
Use Φ as new coordinate
$dt = d\Phi + dx$
$ds^2 = -d\Phi^2 - 2\,d\Phi\,dx$


Flat spacetime
ds² = - dt² + dx² + dy² + dz²
k^α = (1,1,0,0)
k_α = (-1,1,0,0) = -∂_α Φ
Φ = t - x
Φ as new coordinate
dt = dΦ + dx
ds² = - dΦ² - 2dΦdx + dy² + dz²


kα = (1,1,0,0)
kα = (-1,1,0,0) = -∂α Φ
Φ = t-x
Use Φ as new coordinate
dt = dΦ + dx
ds2 = - dΦ2 - 2dΦdx + dy2 + dz2


Flat spacetime
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$k^\alpha = (1,1,0,0)$
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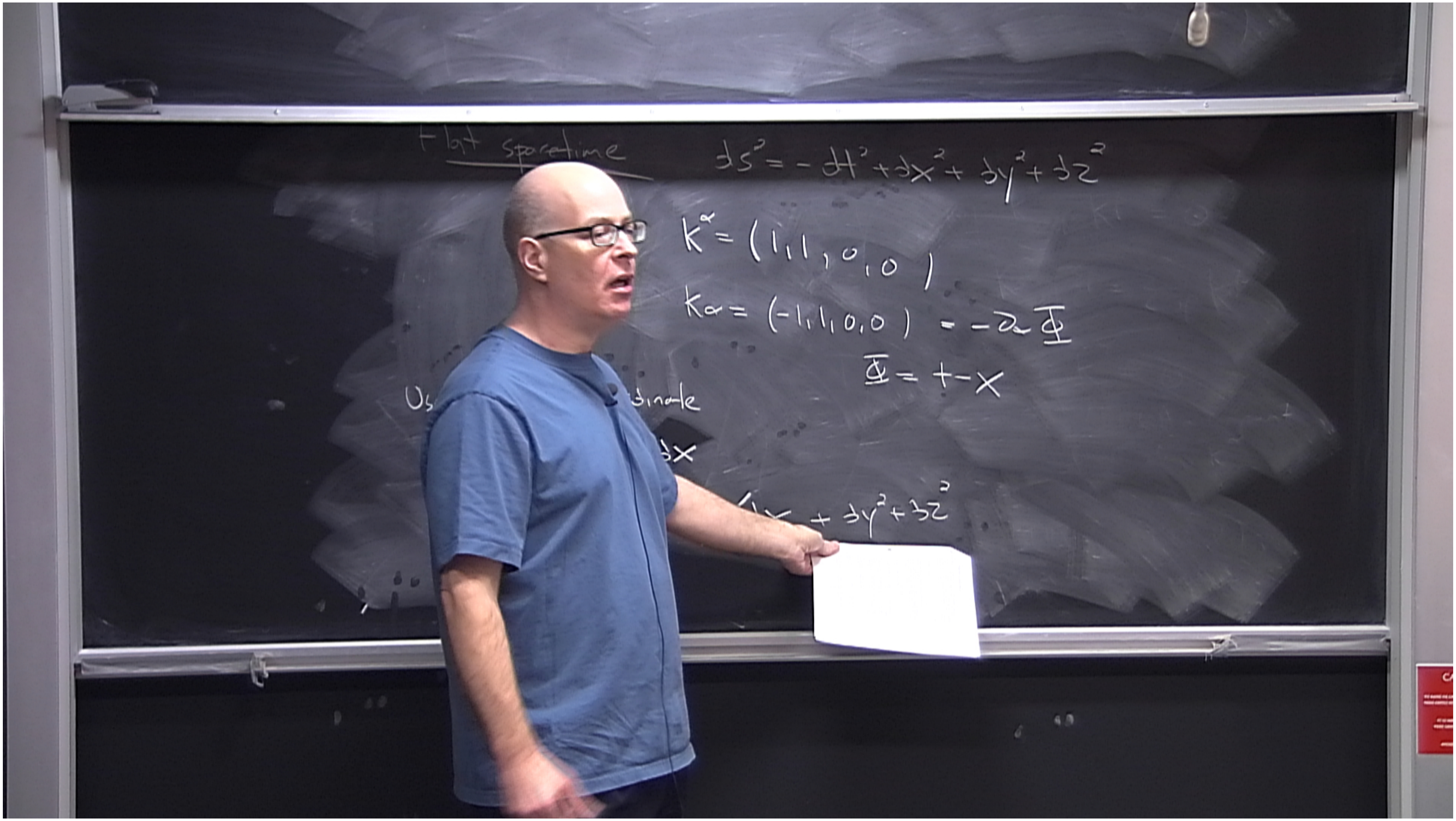
$ds^2 = -d\Phi^2 - 2\,d\Phi\,dx + dy^2 + dz^2$


Flat spacetime
ds² = - dt² + dx² + dy² + dz²
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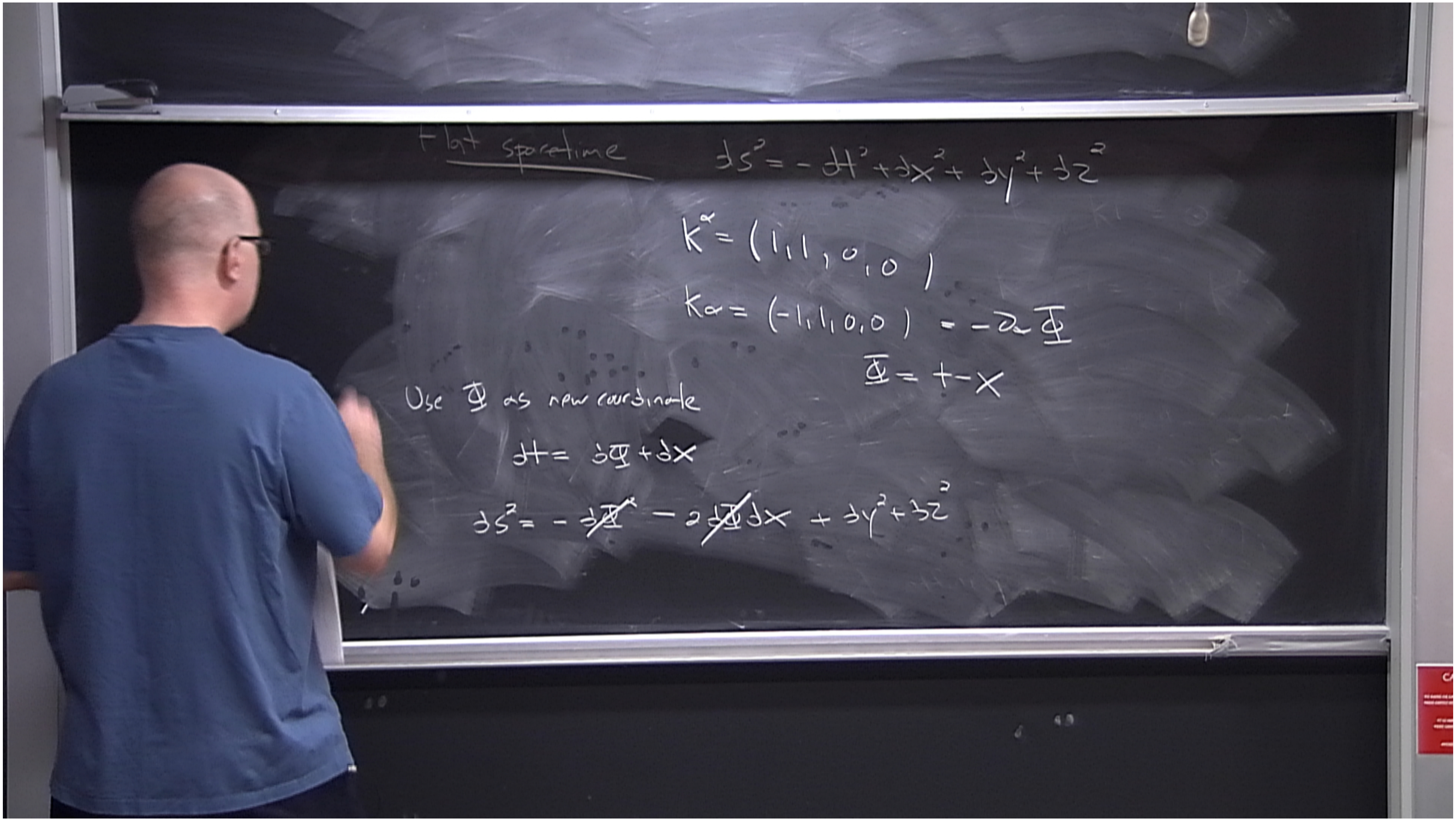
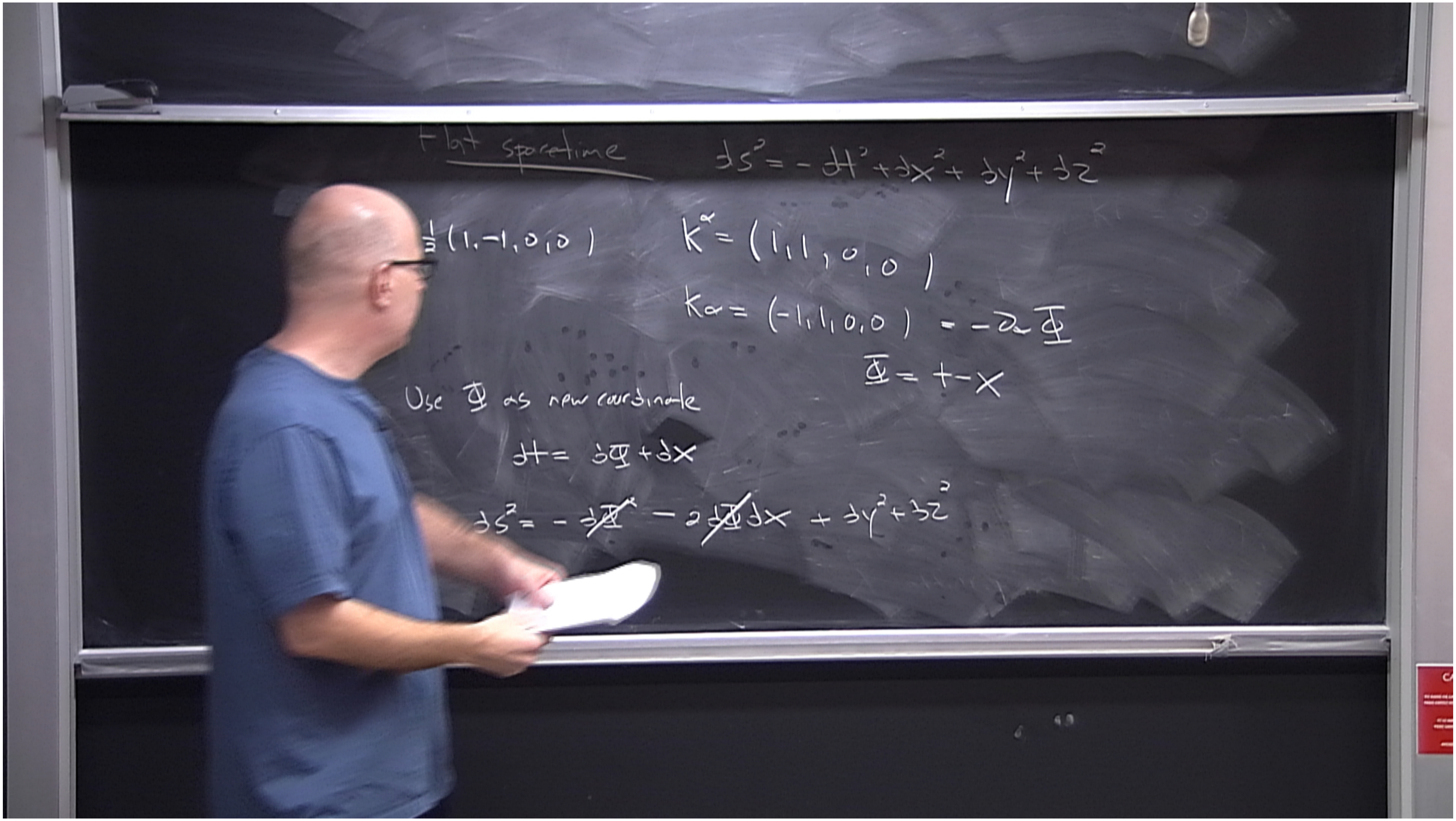
Φ = t-x


Flat spacetime
ds² = - dt² + dx² + dy² + dz²
k^α = (1,1,0,0)
k_α = (-1,1,0,0) = -∂_α Φ
Φ = t - x
Use Φ as new coordinate
dt = dΦ + dx
ds² = - dΦ² - 2dΦdx + dy² + dz²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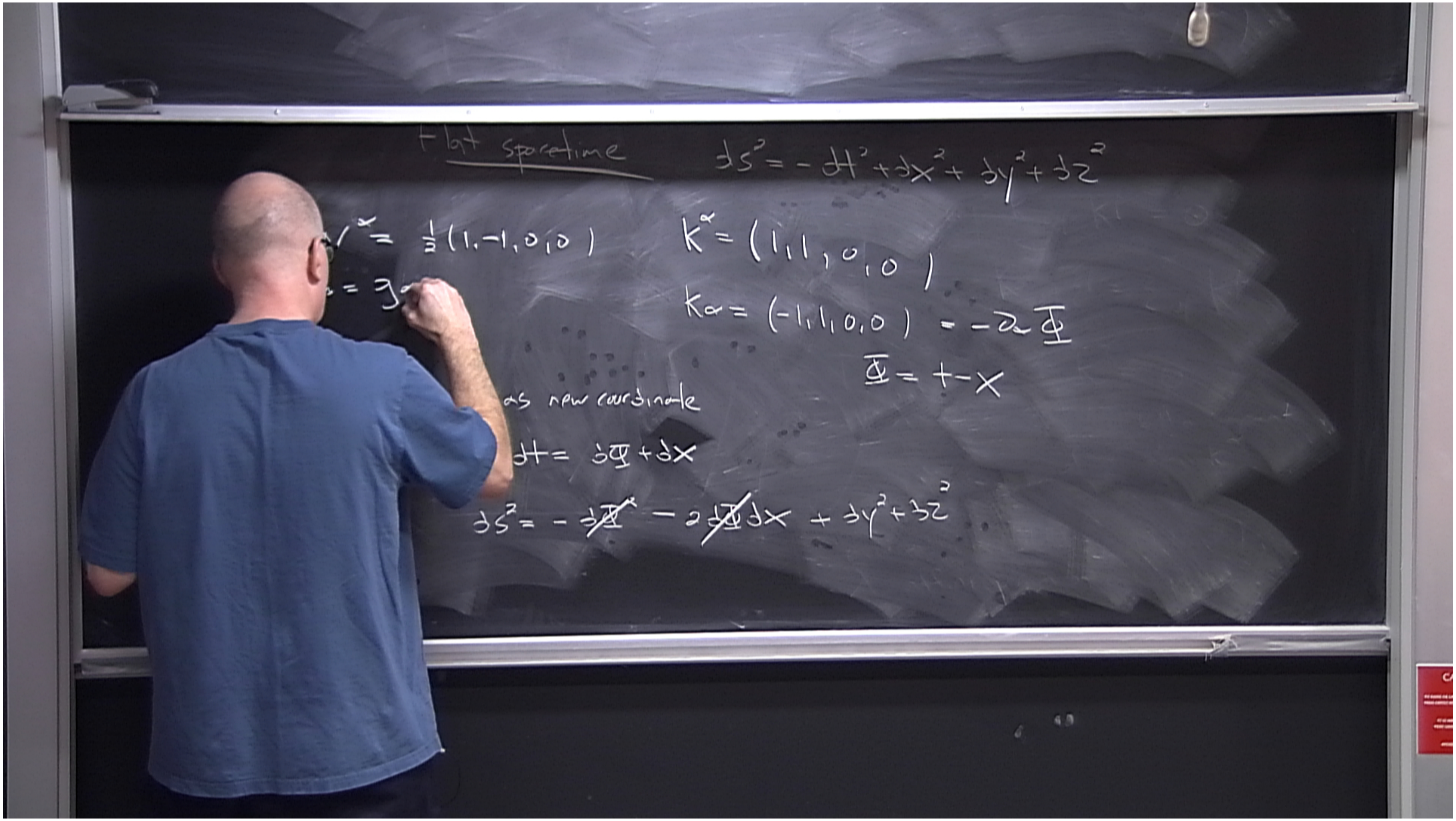
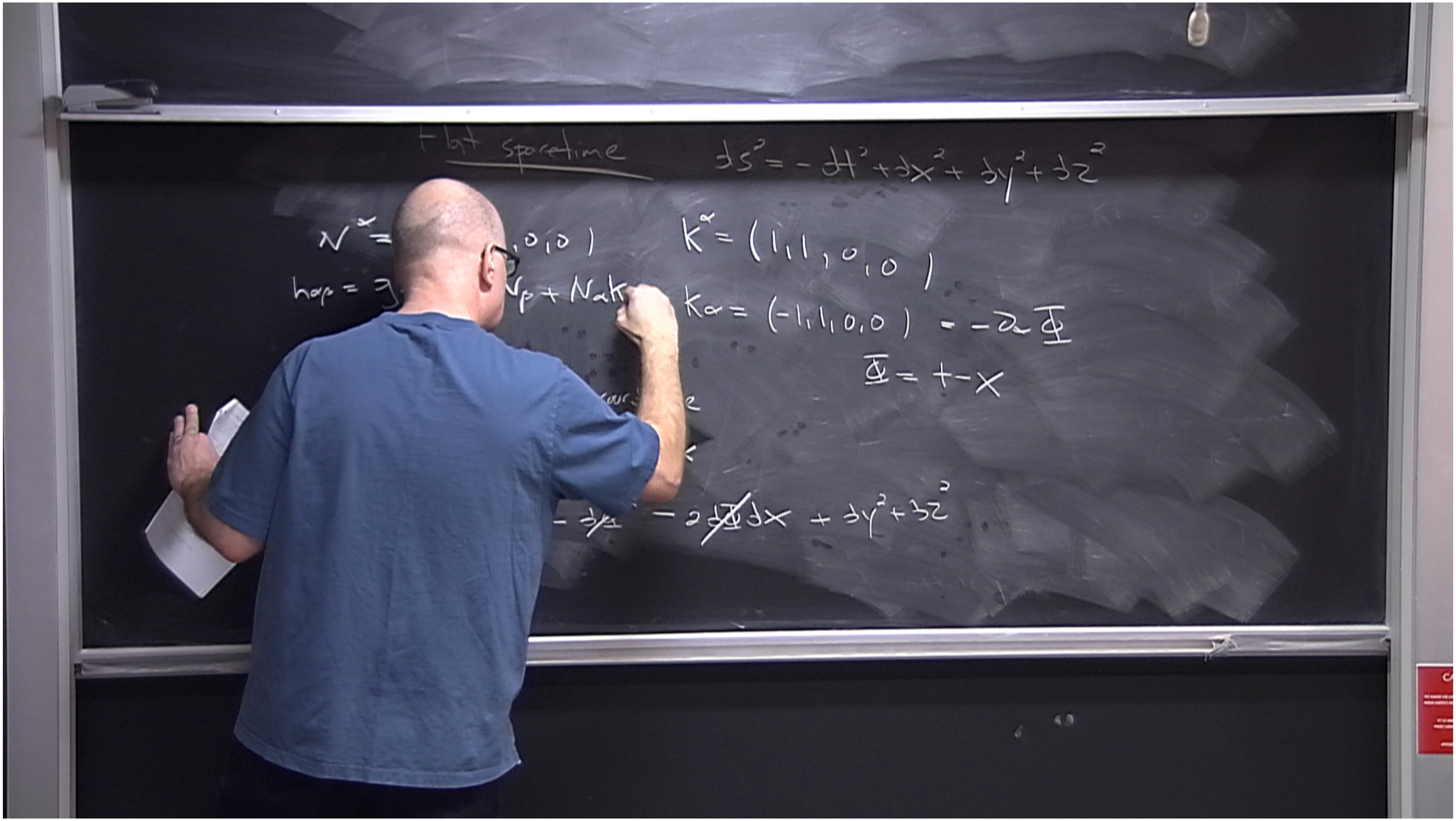
Φ = t - x
Use Φ as new coordinate
dt = dΦ + dx
ds² = - dΦ² - 2 dΦ dx + dy² + dz²


Flat spacetime
ds² = - dt² + dx² + dy² + dz²
k^α = (1, 1, 0, 0)
k_α = (-1, 1, 0, 0) = - ∂_α Φ
Φ = t - x
as new coordinate
dt = dΦ + dx
ds² = - dΦ² - 2 dΦ dx + dy² + dz²


Flat spacetime
ds² = - dt² + dx² + dy² + dz²
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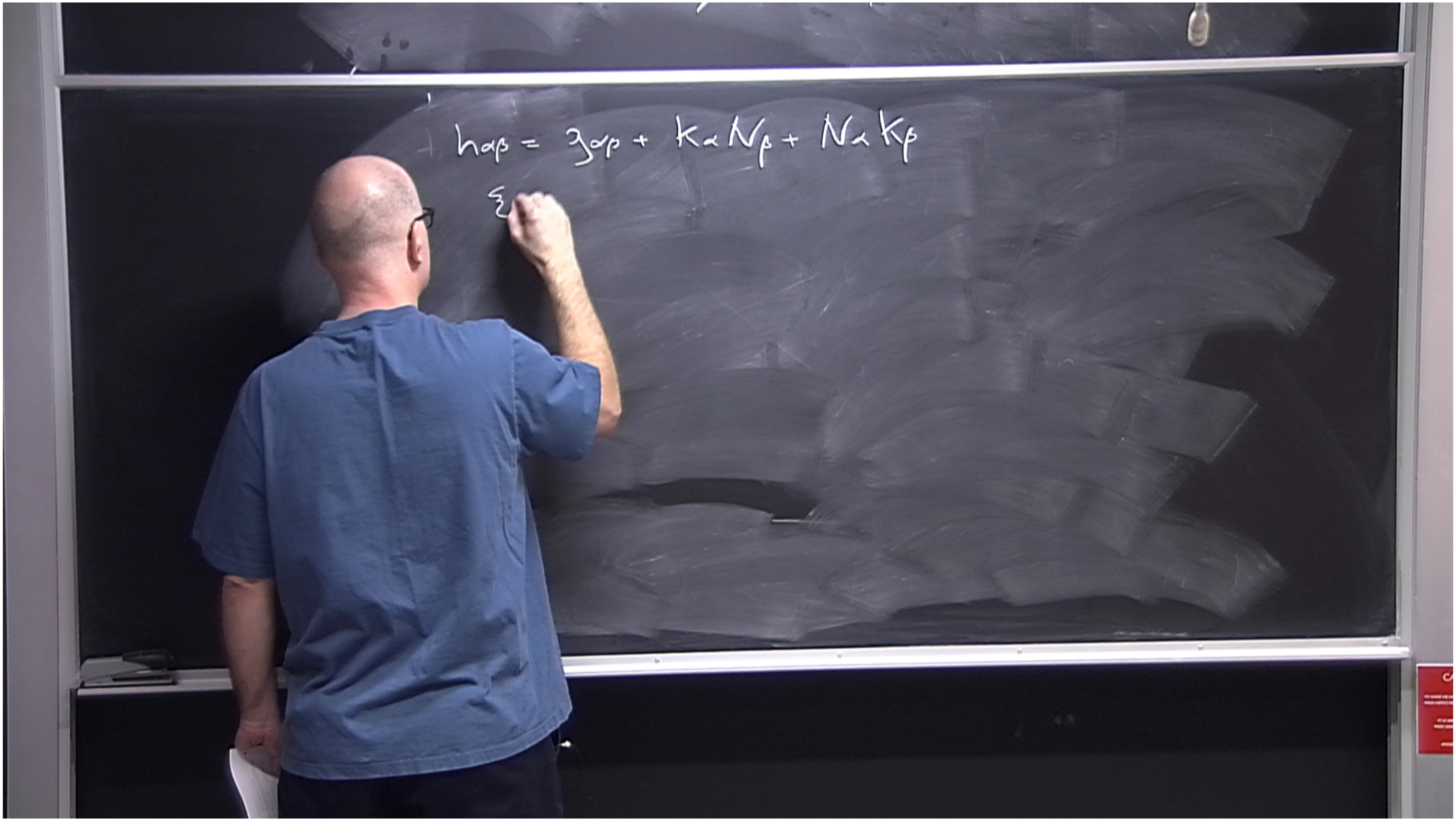
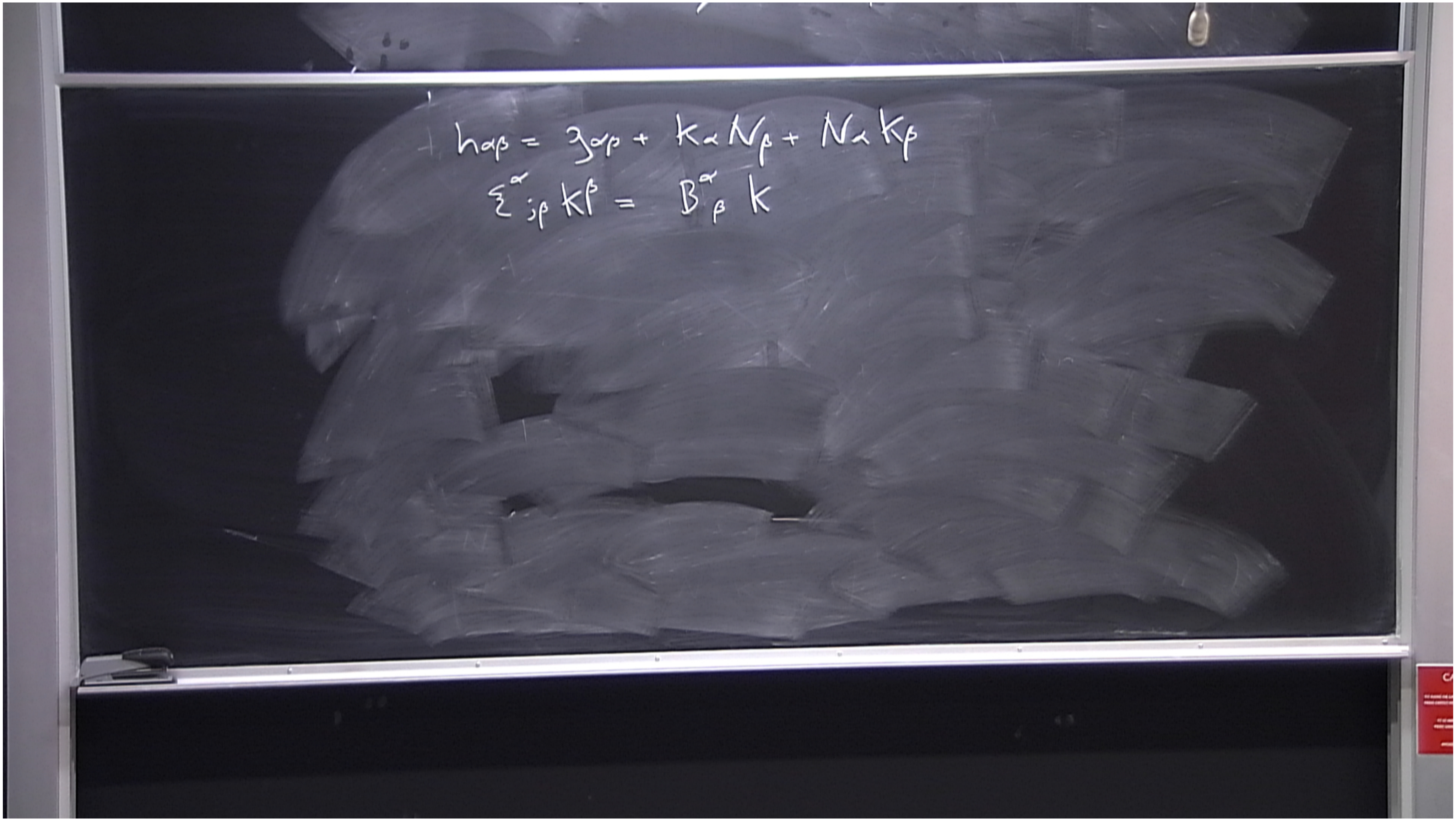
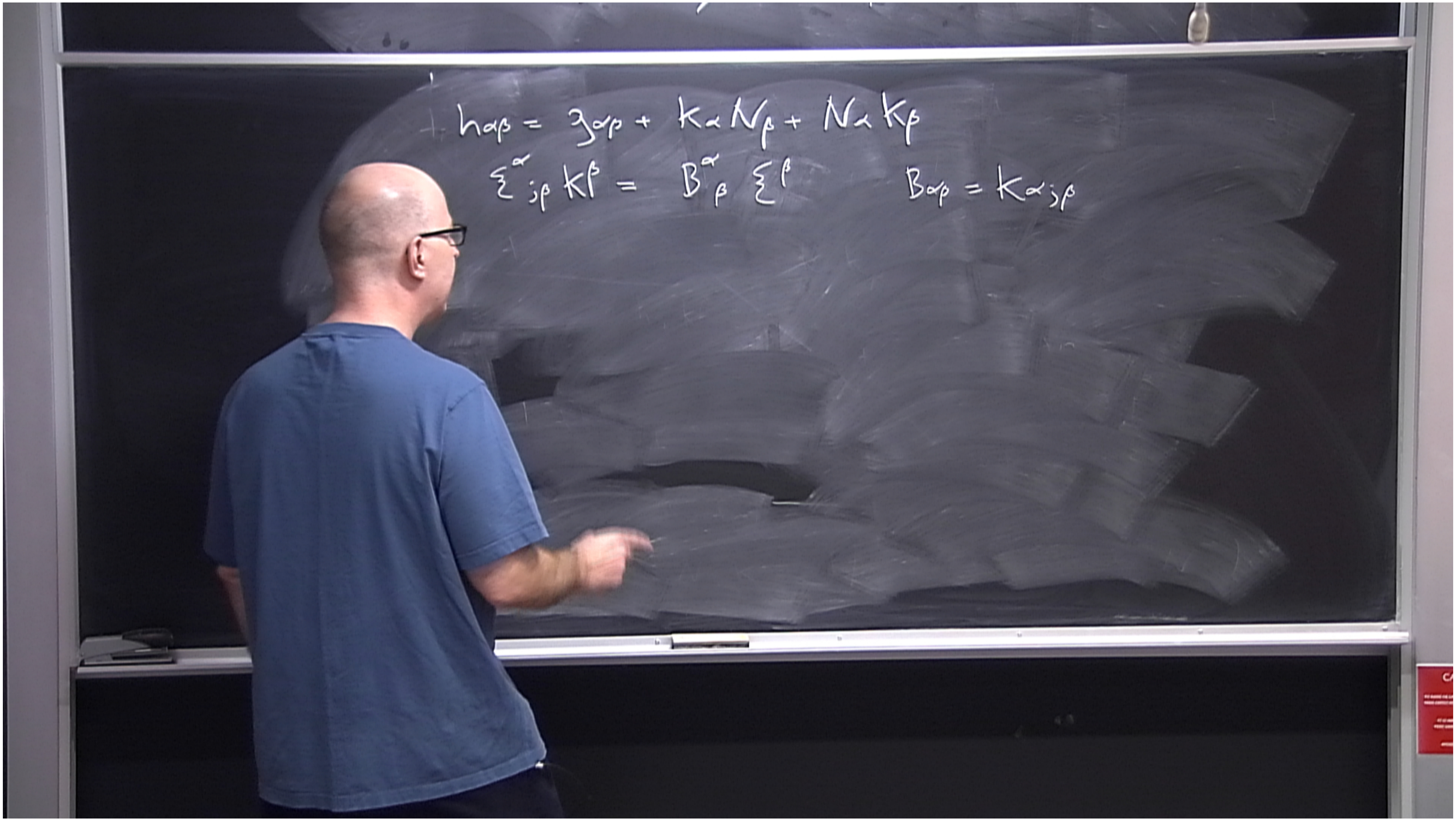
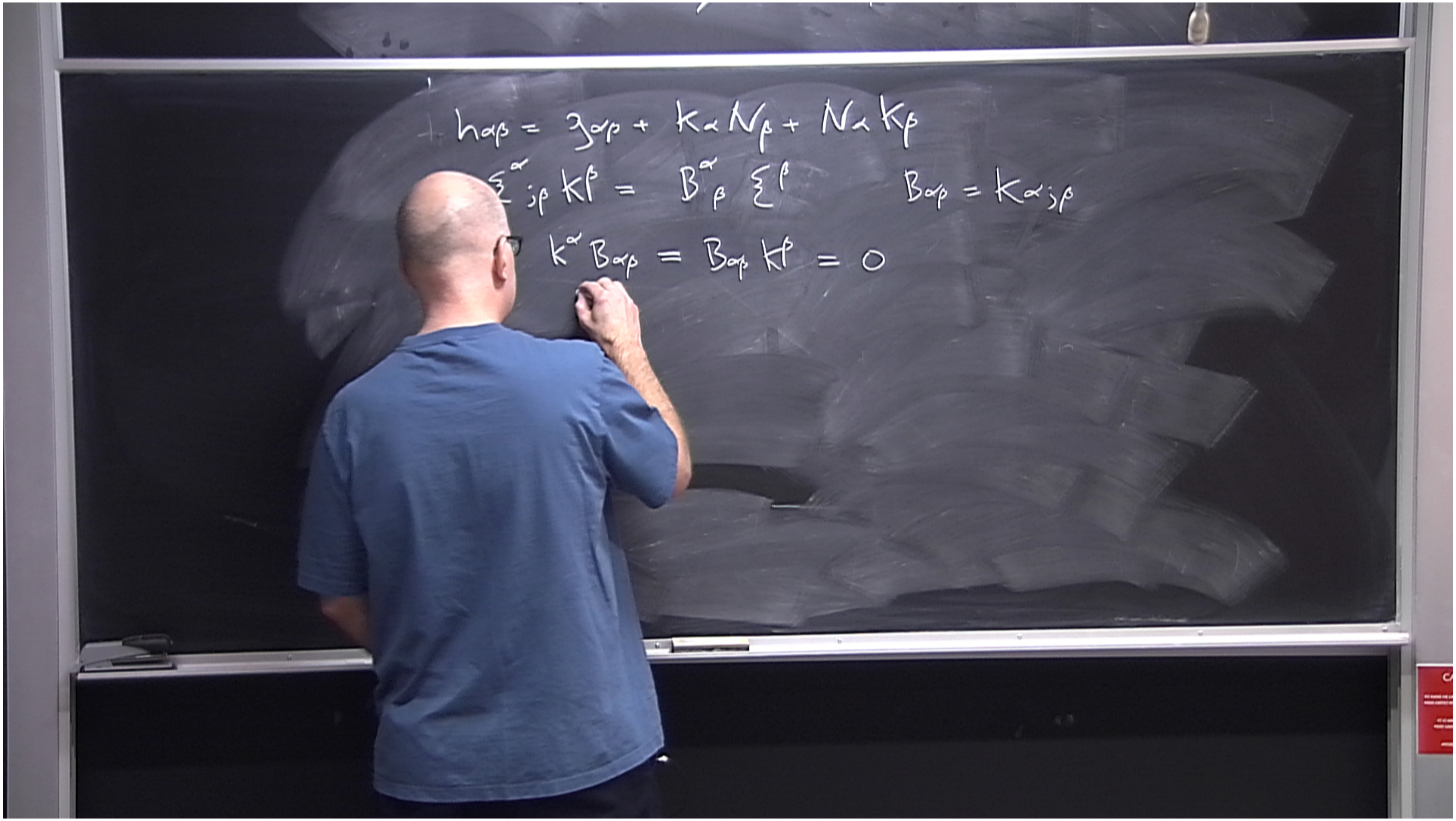
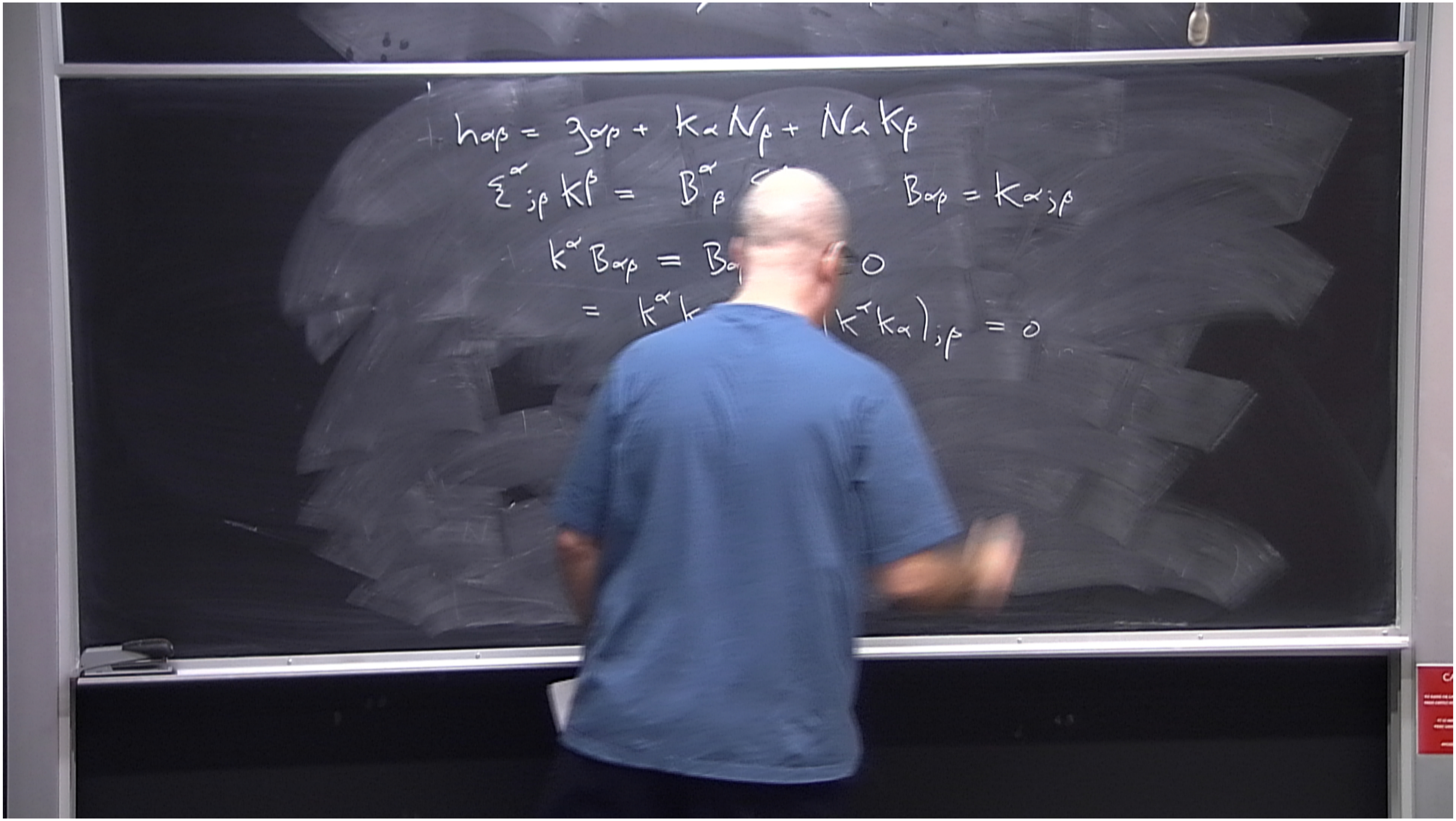
Φ = t - x

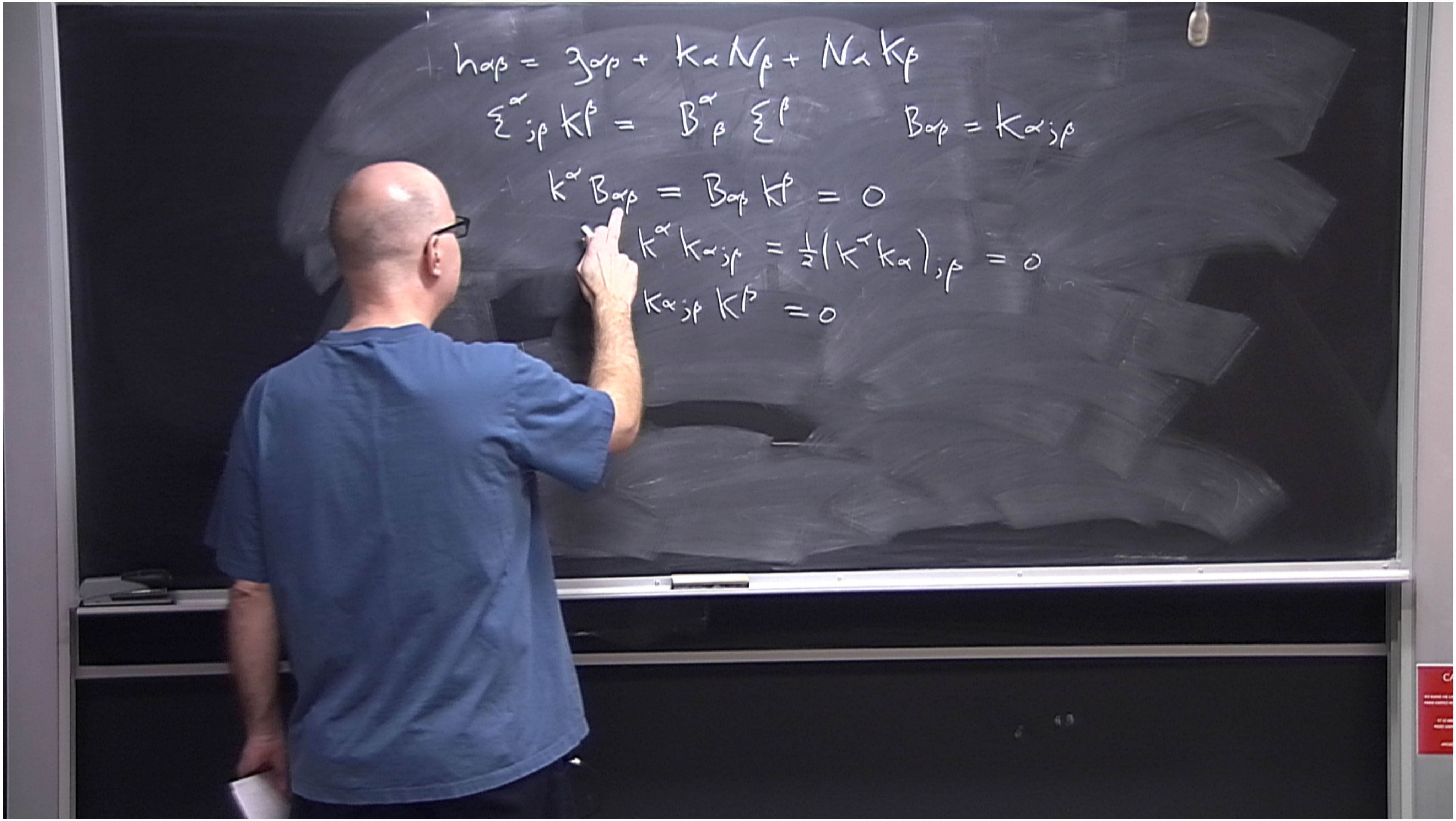

$h_{\alpha\beta} = g_{\alpha\beta} + k_\alpha N_\beta + N_\alpha k_\beta$
ξ

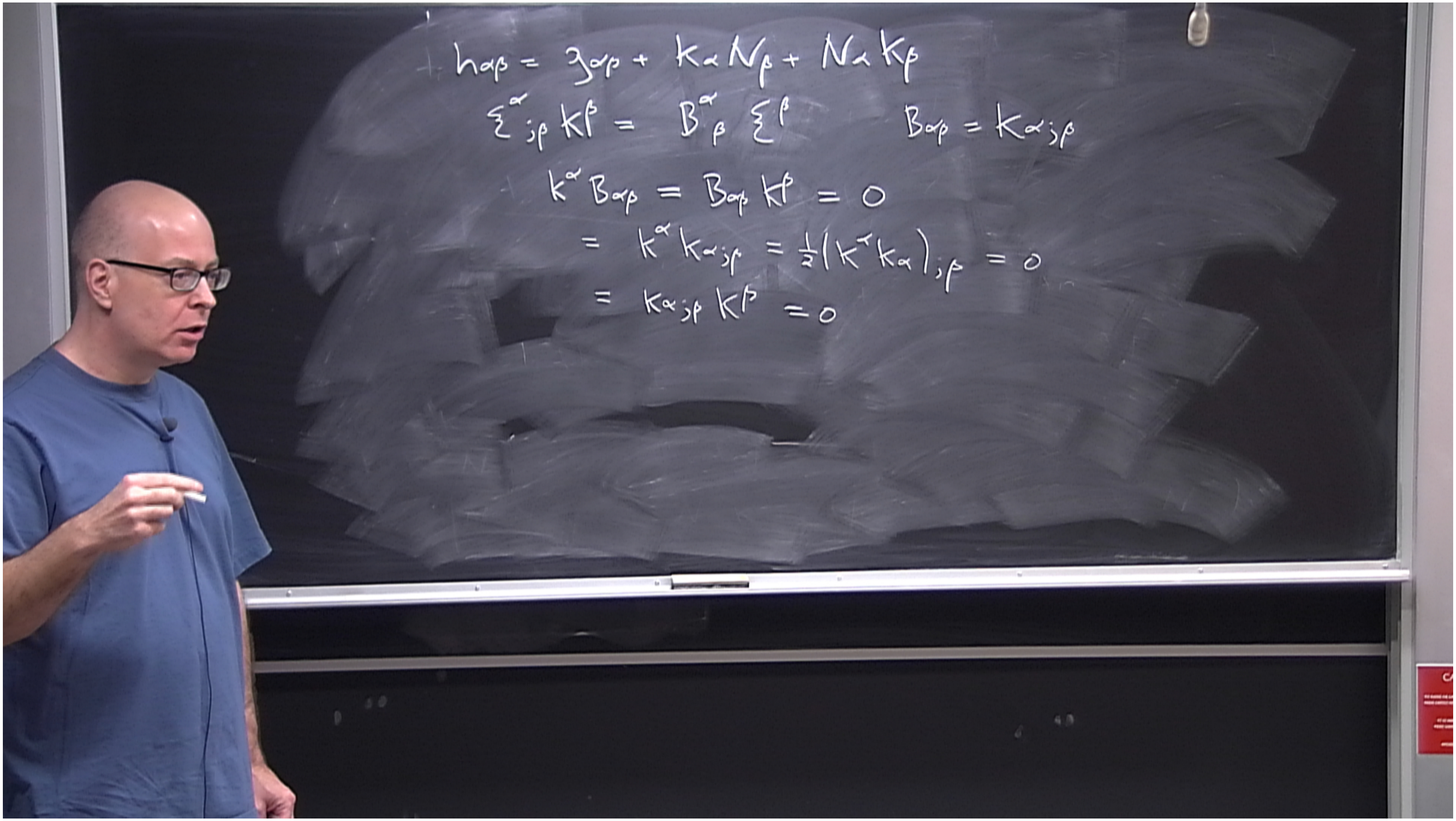

$h_{\alpha\beta} = g_{\alpha\beta} + k_\alpha N_\beta + N_\alpha k_\beta$
$\xi^\alpha{}_{;\beta} k^\beta = B^\alpha{}_\beta k$

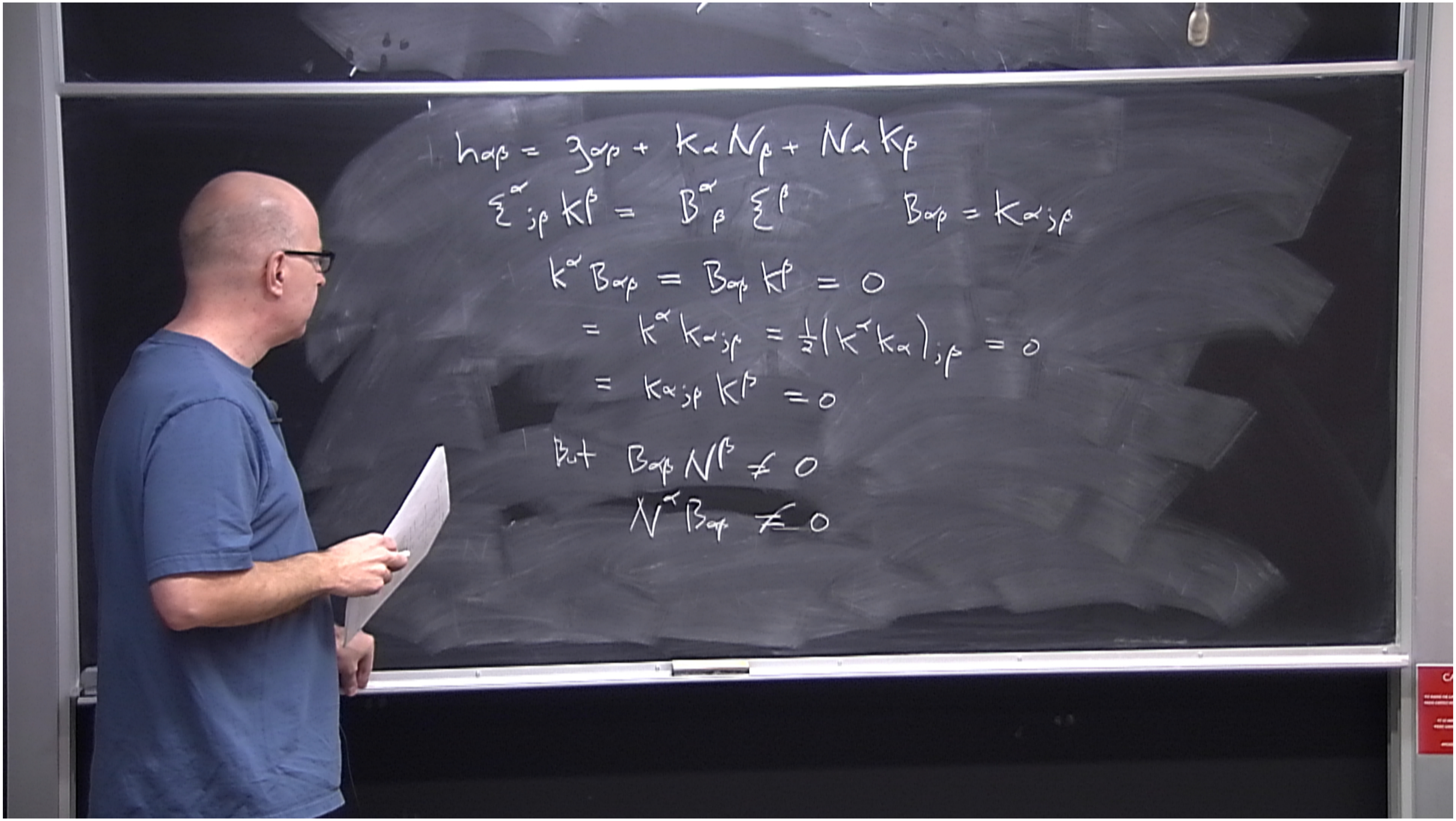

$h_{\alpha\beta} = g_{\alpha\beta} + k_\alpha N_\beta + N_\alpha k_\beta$
$\xi^\alpha{}_{;\beta} k^\beta = B^\alpha{}_\beta \xi^\beta$
$B_{\alpha\beta} = k_{\alpha;\beta}$

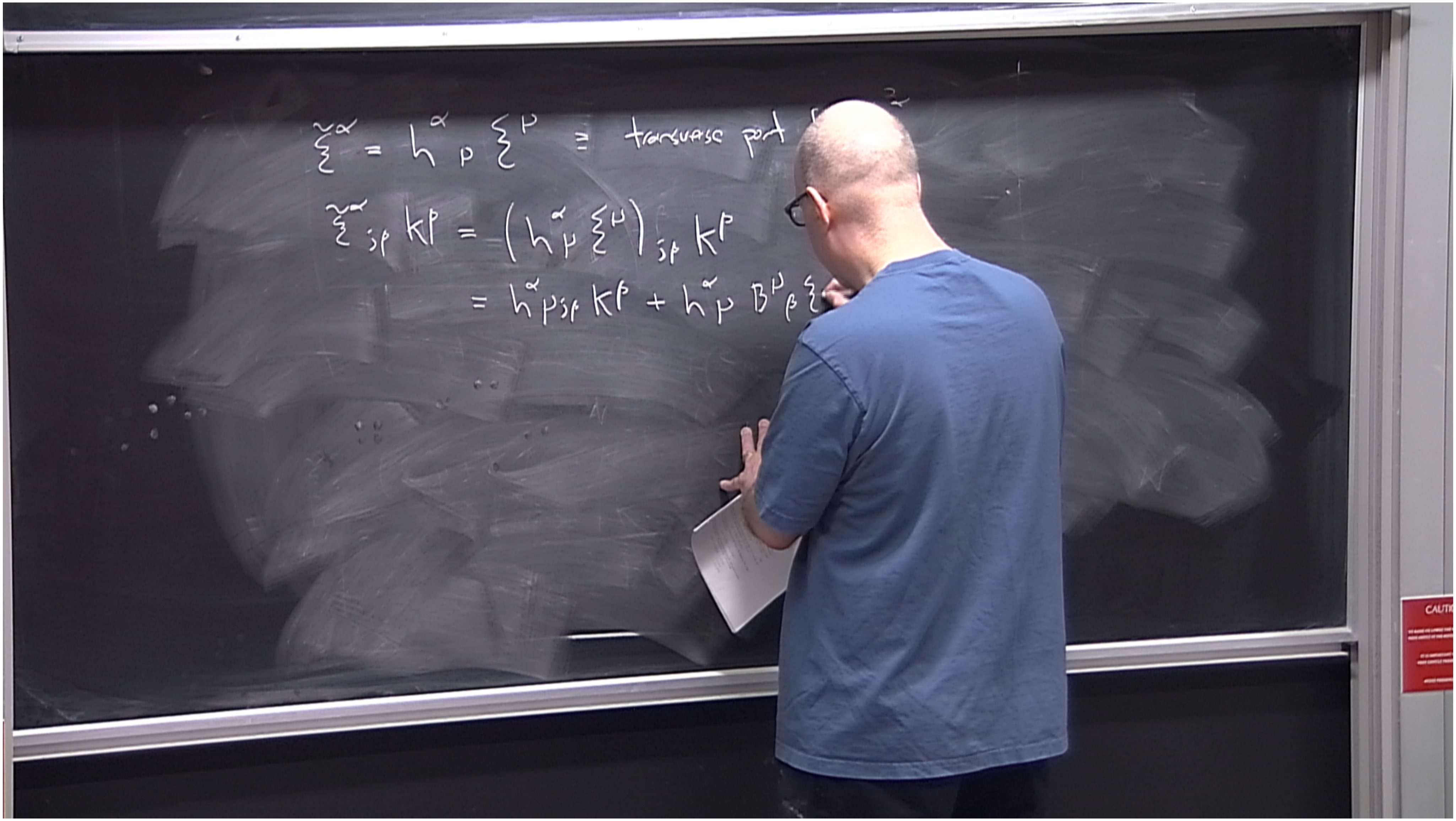

$h_{\alpha\beta} = g_{\alpha\beta} + k_\alpha N_\beta + N_\alpha k_\beta$
$\xi^\alpha{}_{;\beta} k^\beta = B^\alpha{}_\beta \xi^\beta$
$B_{\alpha\beta} = k_{\alpha;\beta}$
$k^\alpha B_{\alpha\beta} = B_{\alpha\beta} k^\beta = 0$


$h_{\alpha\beta} = g_{\alpha\beta} + k_\alpha N_\beta + N_\alpha k_\beta$
$\xi^\alpha{}_{;\beta} k^\beta = B^\alpha{}_\beta$
$B_{\alpha\beta} = k_{\alpha;\beta}$
$k^\alpha B_{\alpha\beta} = B$
$= 0$
$= k^\alpha k$
$(k^\alpha k_\alpha)_{;\beta} = 0$

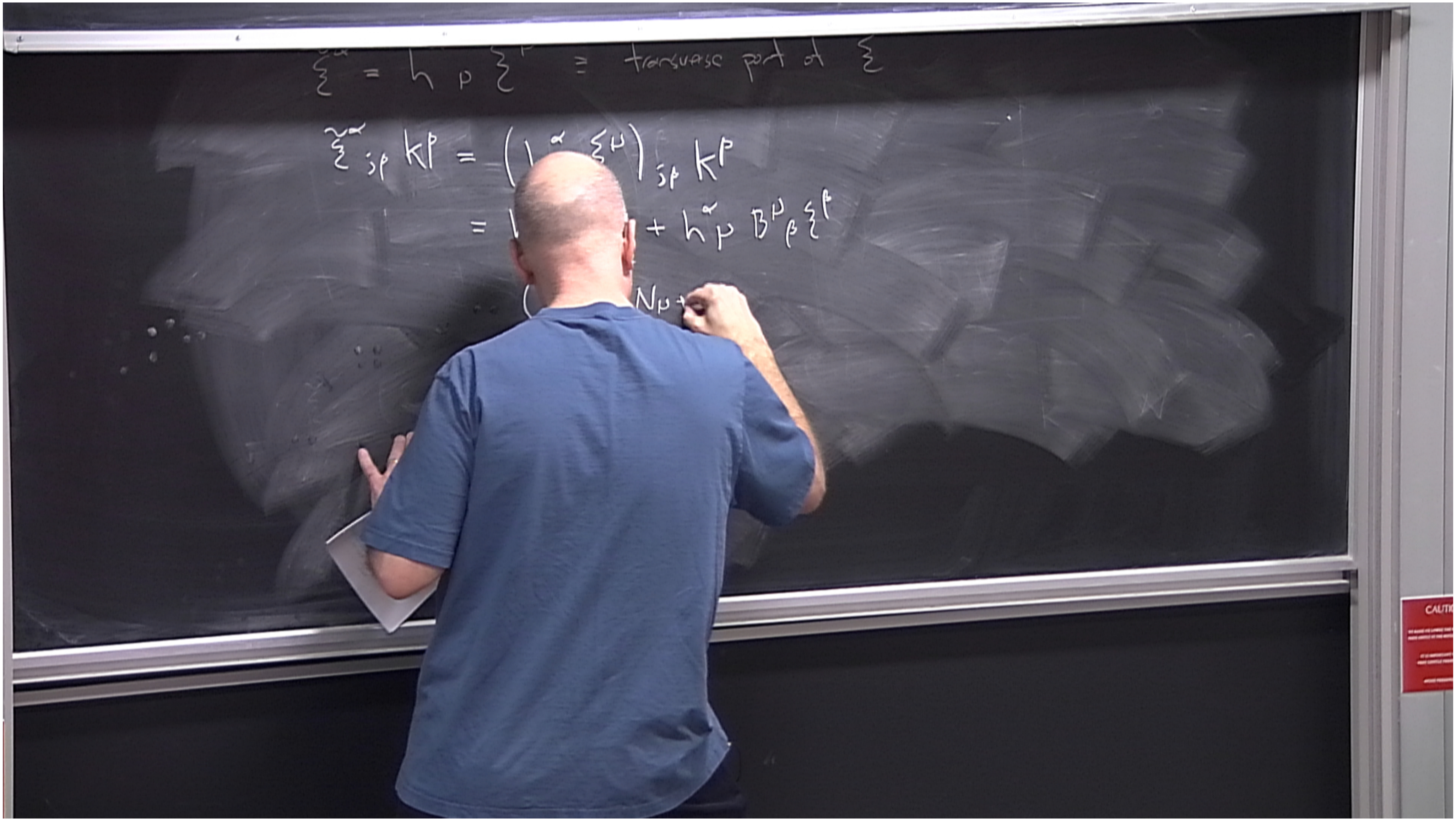

$h_{\alpha\beta} = g_{\alpha\beta} + k_\alpha N_\beta + N_\alpha k_\beta$
$\xi^\alpha{}_{;\beta} k^\beta = B^\alpha{}_\beta \xi^\beta$
$B_{\alpha\beta} = k_{\alpha;\beta}$
$k^\alpha B_{\alpha\beta} = B_{\alpha\beta} k^\beta = 0$
$k^\alpha k_{\alpha;\beta} = \frac{1}{2}(k^\alpha k_\alpha)_{;\beta} = 0$
$k_{\alpha;\beta} k^\beta = 0$

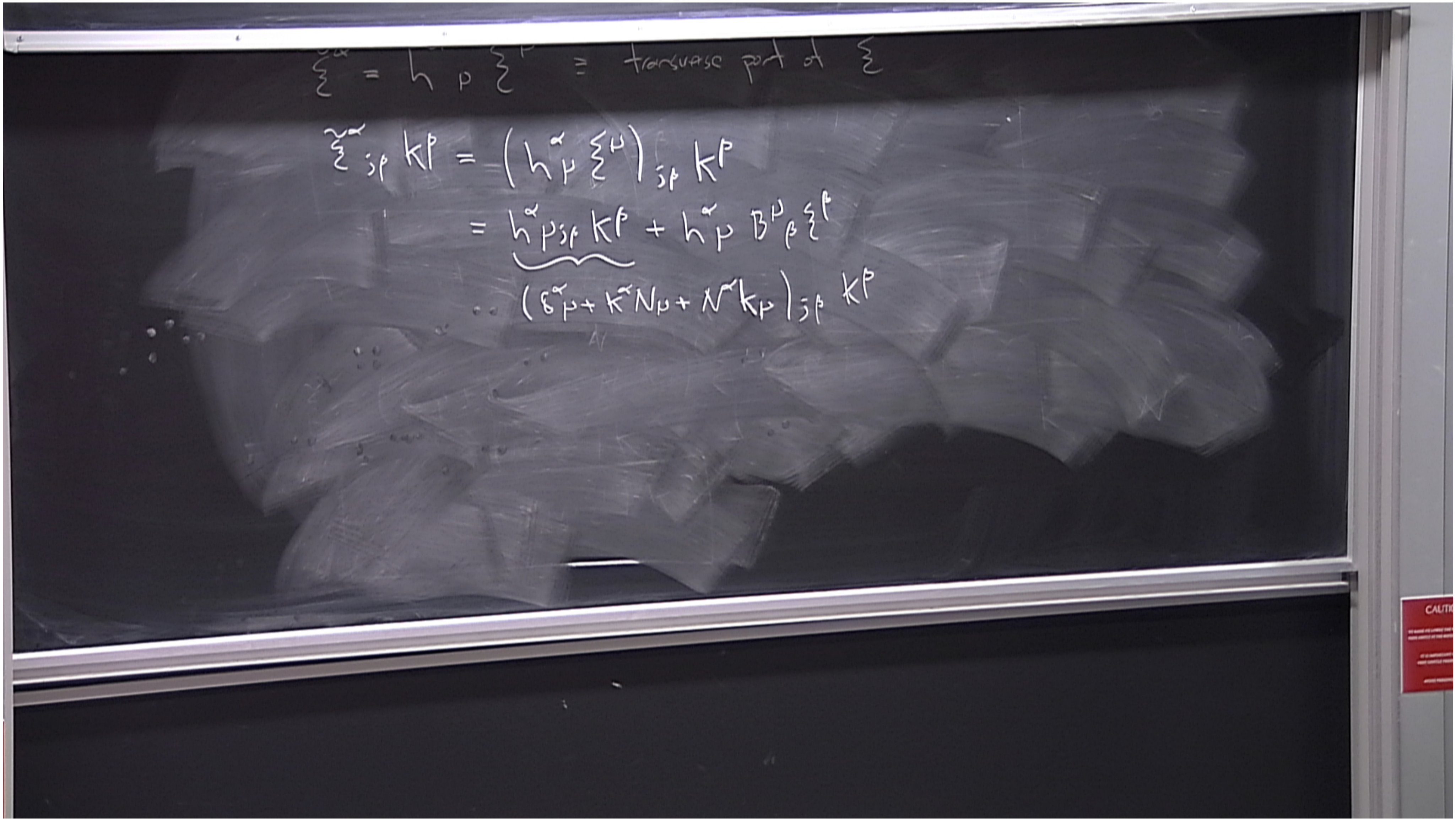

$h_{\alpha\beta} = g_{\alpha\beta} + k_\alpha N_\beta + N_\alpha k_\beta$
$\xi^\alpha{}_{;\beta} k^\beta = B^\alpha{}_\beta \xi^\beta$
$B_{\alpha\beta} = k_{\alpha;\beta}$
$k^\alpha B_{\alpha\beta} = B_{\alpha\beta} k^\beta = 0$
$= k^\alpha k_{\alpha;\beta} = \frac{1}{2}(k^\alpha k_\alpha)_{;\beta} = 0$
$= k_{\alpha;\beta} k^\beta = 0$

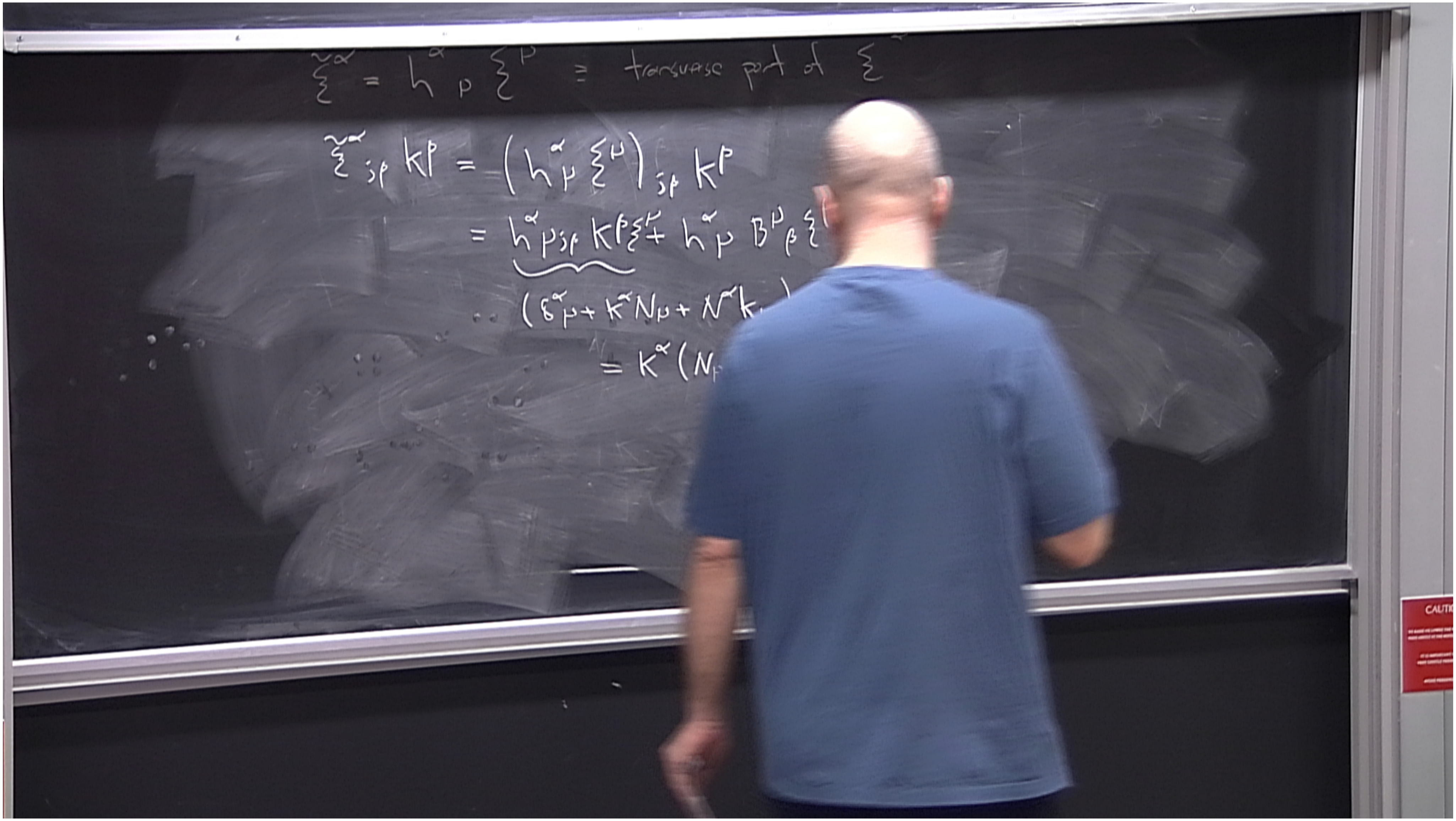

$h_{\alpha\beta} = \gamma_{\alpha\beta} + k_\alpha N_\beta + N_\alpha k_\beta$
$\xi^\alpha{}_{;\beta} k^\beta = B^\alpha{}_\beta \xi^\beta$
$B_{\alpha\beta} = k_{\alpha;\beta}$
$k^\alpha B_{\alpha\beta} = B_{\alpha\beta} k^\beta = 0$
$= k^\alpha k_{\alpha;\beta} = \frac{1}{2}(k^\alpha k_\alpha)_{;\beta} = 0$
$= k_{\alpha;\beta} k^\beta = 0$
But $B_{\alpha\beta} N^\beta \neq 0$
$N^\alpha B_{\alpha\beta} \neq 0$

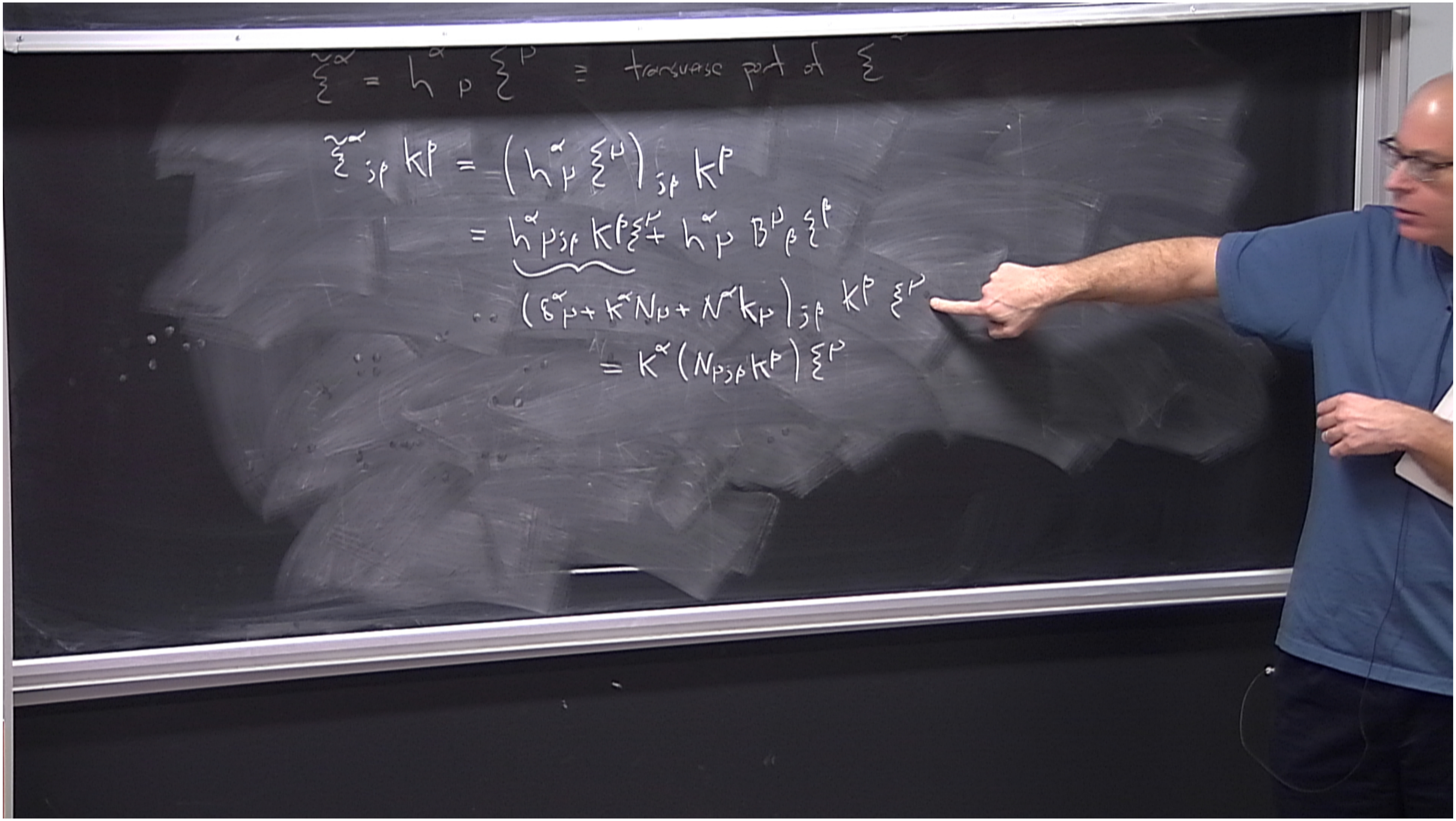


ξ̃^α = h^α_ρ ξ^ρ ≡ transverse part
ξ̃^α_;ρ k^ρ = (h^α_μ ξ^μ)_;ρ k^ρ
= h^α_μ;ρ k^ρ + h^α_μ B^μ_β


transverse part of ξ


$\tilde{\xi}^\alpha = h^\alpha{}_\beta \xi^\beta$ = transverse part of ξ
$\tilde{\xi}^\alpha{}_{;\rho} k^\rho = (h^\alpha{}_\mu \xi^\mu)_{;\rho} k^\rho$
$= h^\alpha{}_{\mu;\rho} k^\rho + h^\alpha{}_\mu B^\mu{}_\beta \xi^\beta$
$(\delta^\alpha{}_\mu + k^\alpha N_\mu + N^\alpha k_\mu)_{;\rho} k^\rho$


$\tilde{\xi}^\alpha = h^\alpha{}_\beta \xi^\beta$ = transverse part of ξ
$\tilde{\xi}^\alpha{}_{;\rho} k^\rho = (h^\alpha{}_\mu \xi^\mu)_{;\rho} k^\rho$
$= h^\alpha{}_{\mu;\rho} k^\rho \xi^\mu + h^\alpha{}_\mu B^\mu{}_\beta \xi^\beta$
$(\delta^\alpha{}_\mu + k^\alpha N_\mu + N^\alpha k_\mu)$
$= k^\alpha (N_\mu$


$\tilde{\xi}^\alpha = h^\alpha{}_\rho \xi^\rho$ = transverse part of ξ
$\tilde{\xi}^\alpha{}_{;\rho} k^\rho = (h^\alpha{}_\mu \xi^\mu)_{;\rho} k^\rho$
$= h^\alpha{}_{\mu;\rho} k^\rho \xi^\mu + h^\alpha{}_\mu B^\mu{}_\beta \xi^\beta$
$(\delta^\alpha{}_\mu + k^\alpha N_\mu + N^\alpha k_\mu)_{;\rho} k^\rho \xi^\mu$
$= k^\alpha (N_{\mu;\rho} k^\rho) \xi^\mu$

$\tilde{\xi}^{\alpha} = h^{\alpha}{}_{\rho}\, \xi^{\rho}$ = transverse part of ξ

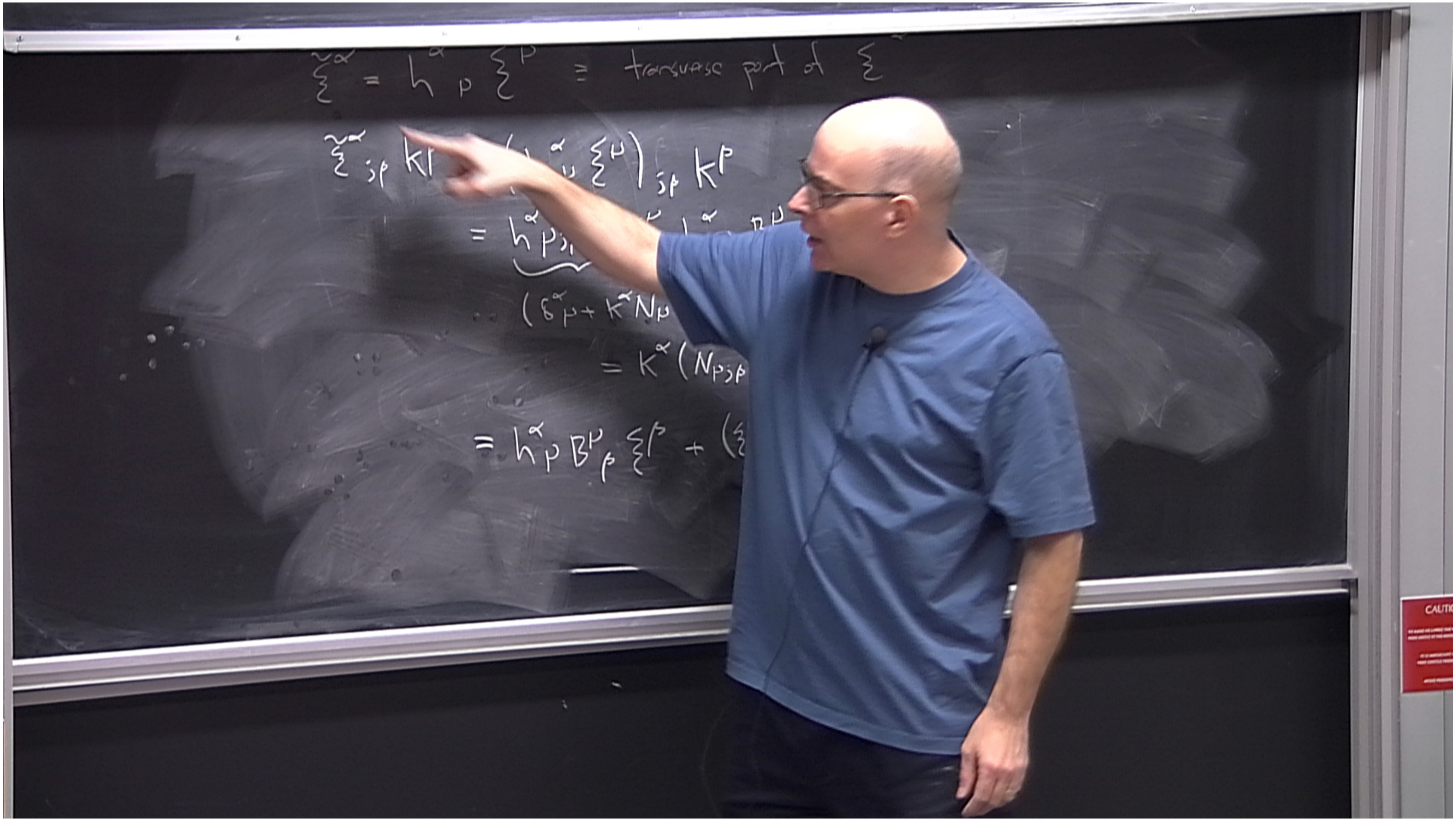
$$\tilde{\xi}^{\alpha}{}_{;\rho}\, k^{\rho} = \left(h^{\alpha}{}_{\mu}\, \xi^{\mu} \right)_{;\rho} k^{\rho}$$

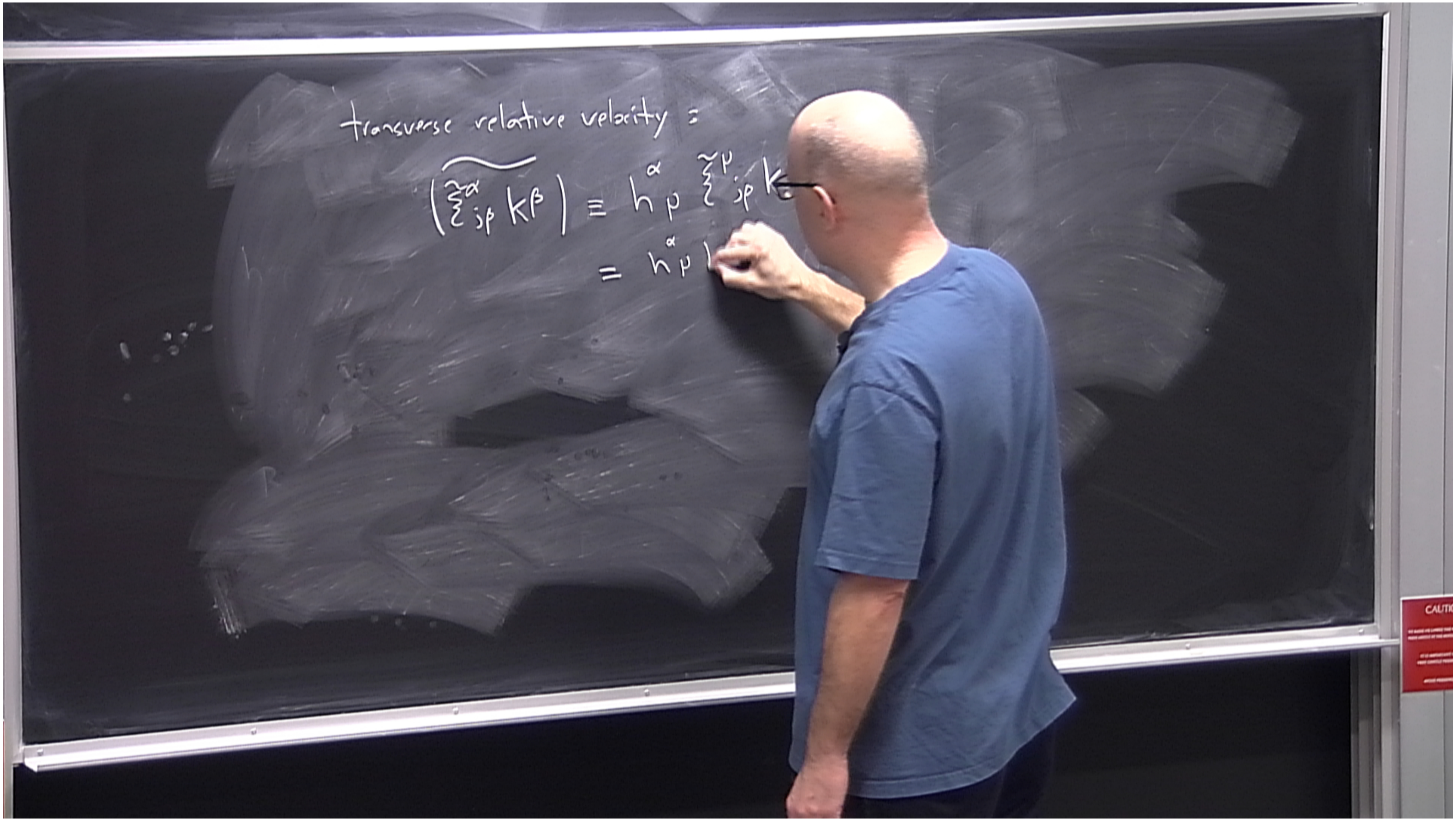
$$= \underbrace{h^{\alpha}{}_{\mu;\rho}\, k^{\rho}}\, \xi^{\mu} + h^{\alpha}{}_{\mu}\, B^{\mu}{}_{\beta}\, \xi^{\beta}$$

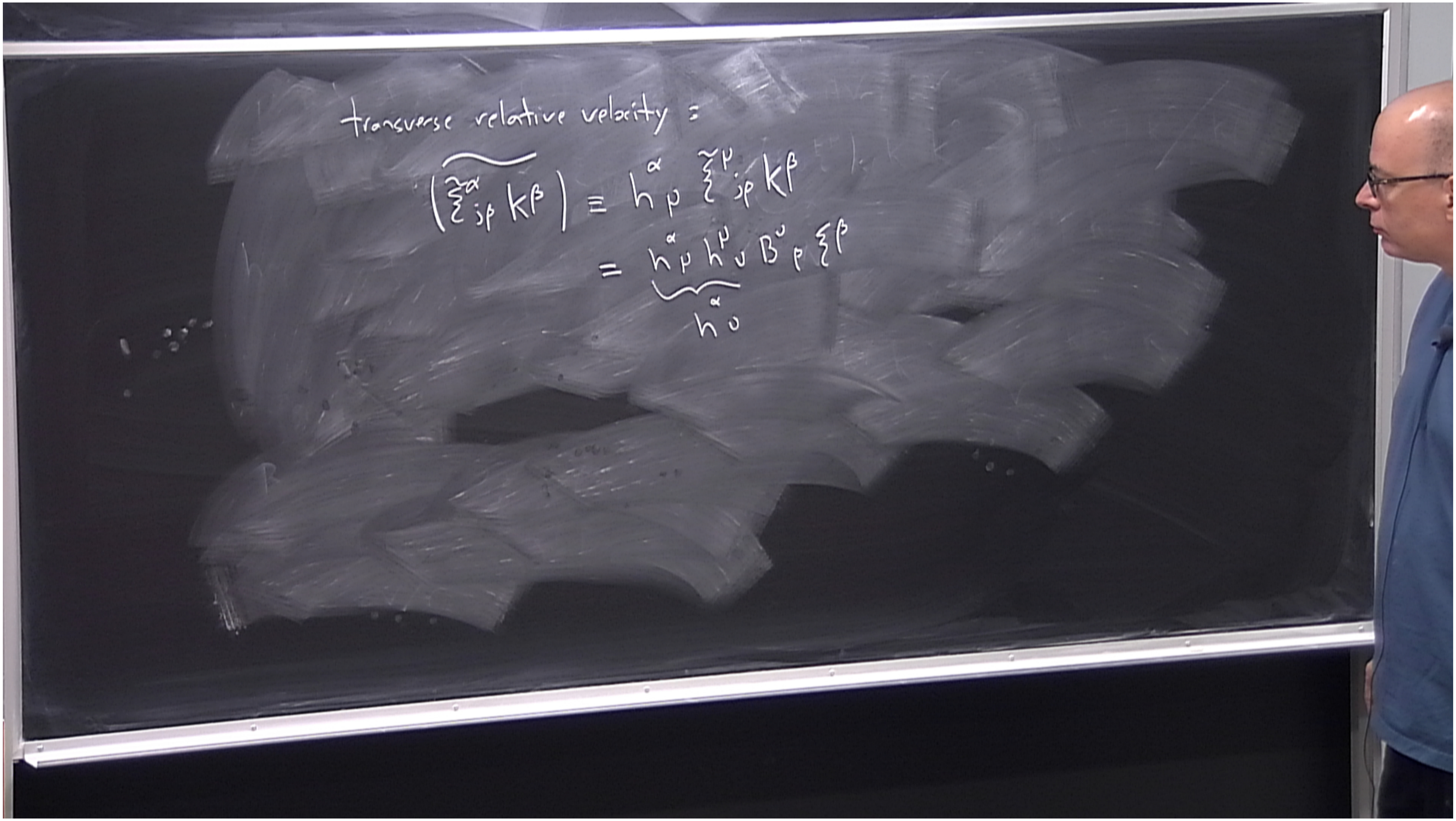
$$\left(\delta^{\alpha}{}_{\mu} + k^{\alpha} N_{\mu} + N^{\alpha} k_{\mu} \right)_{;\rho} k^{\rho}\, \xi^{\mu}$$

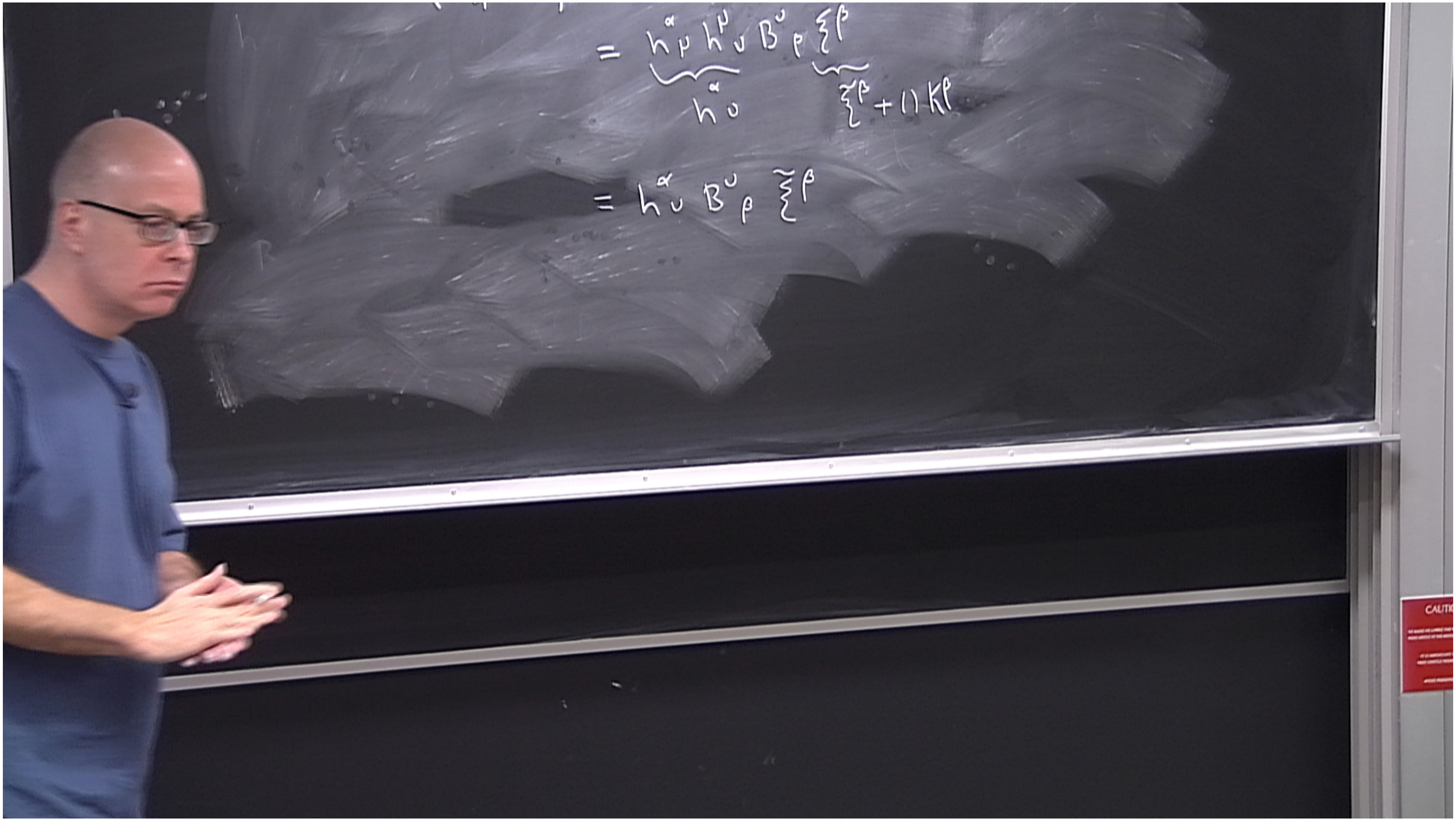
$$= k^{\alpha} \left(N_{\mu;\rho} k^{\rho} \right) \xi^{\mu}$$

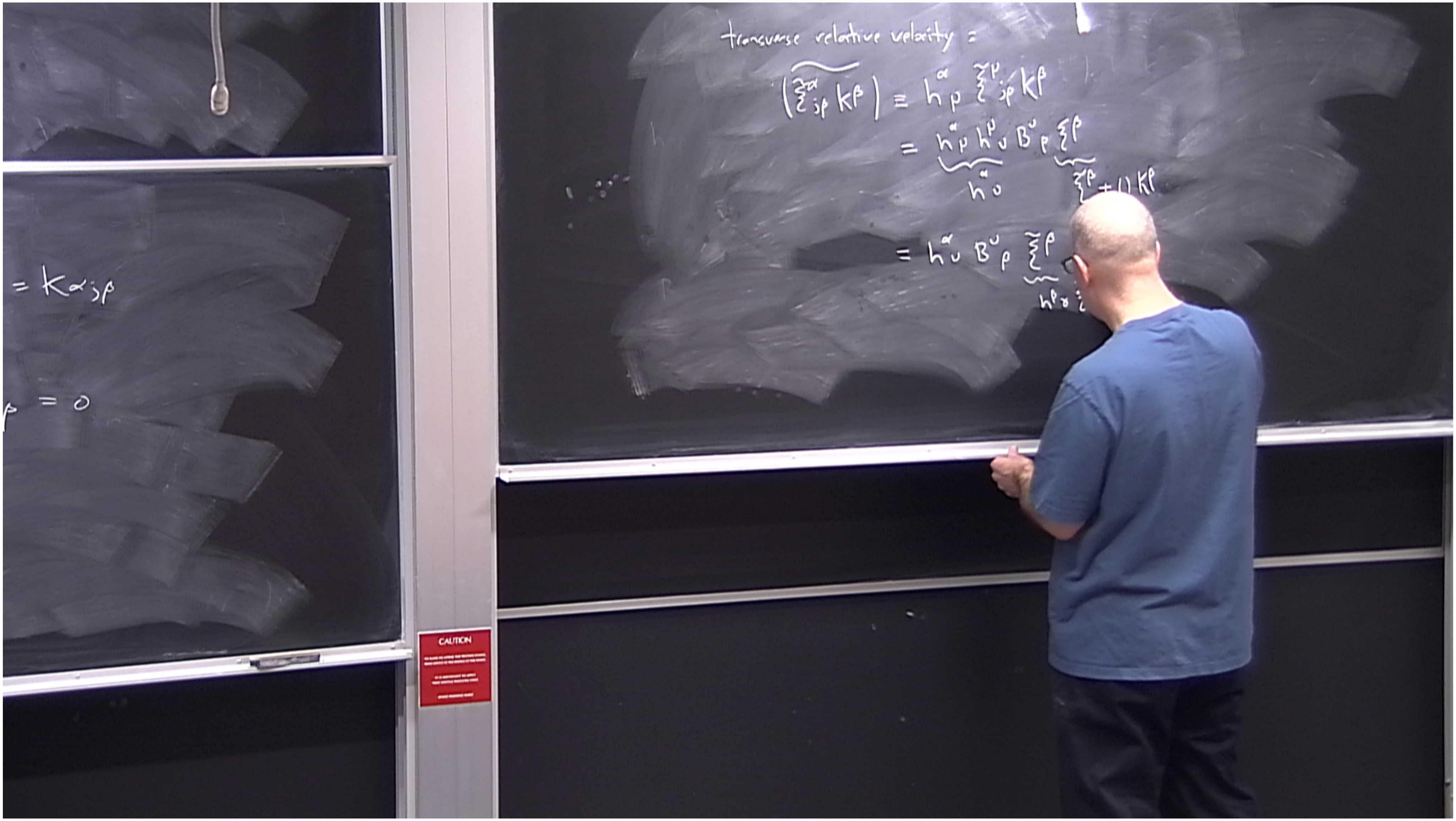
$$= h^{\alpha}{}_{\mu}\, B^{\mu}{}_{\beta}\, \xi^{\beta} + \left(\xi^{\mu} N_{\mu;\rho} k^{\beta} \right) k^{\alpha}$$

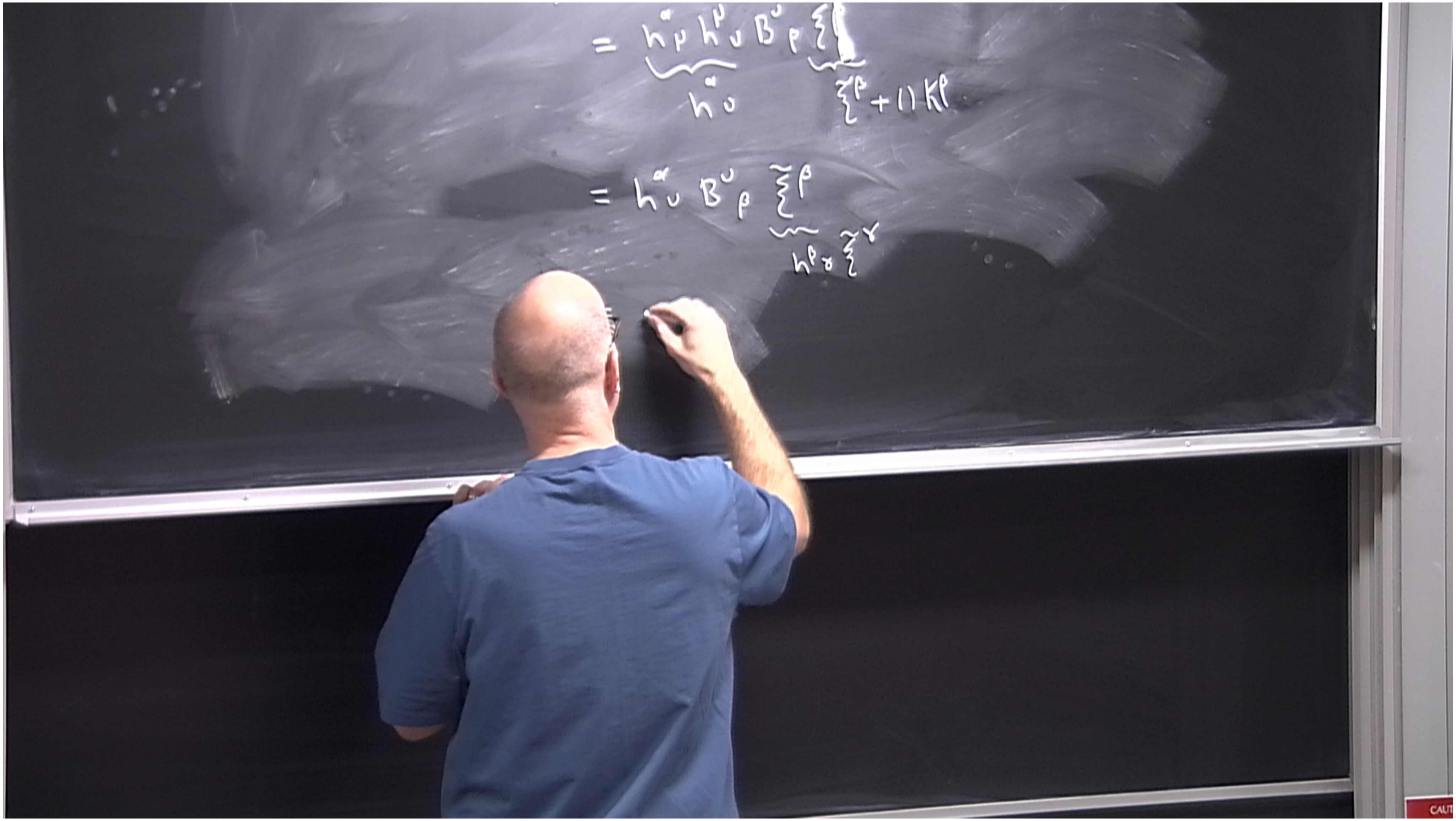

transverse part of

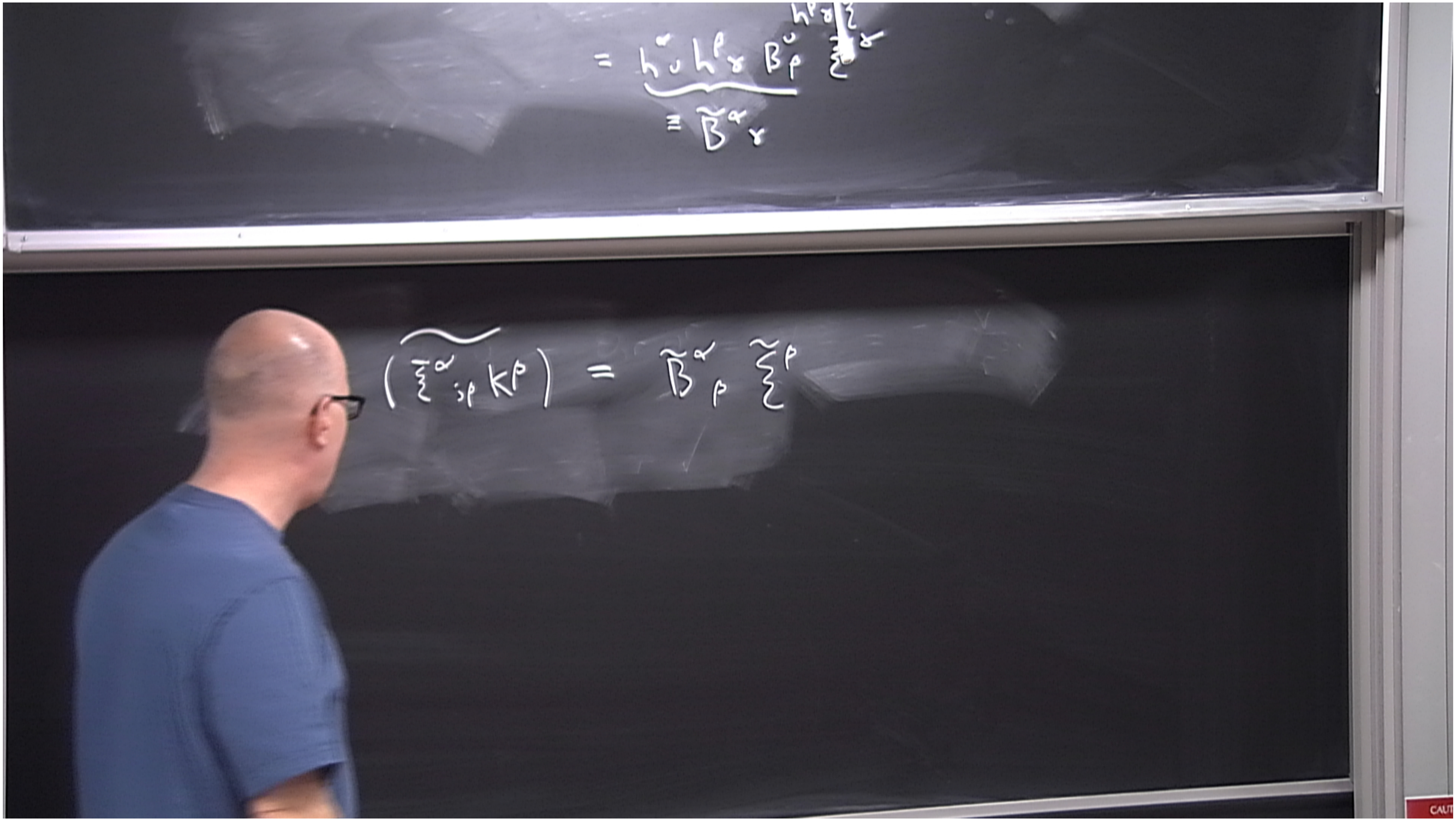

transverse relative velocity :
$\left(\widetilde{\tilde{\xi}^{\alpha}{}_{;\rho} k^{\beta}}\right) = h^{\alpha}{}_{\rho}\, \tilde{\xi}^{\rho}{}_{;\rho} k$
$= h^{\alpha}{}_{\rho}$

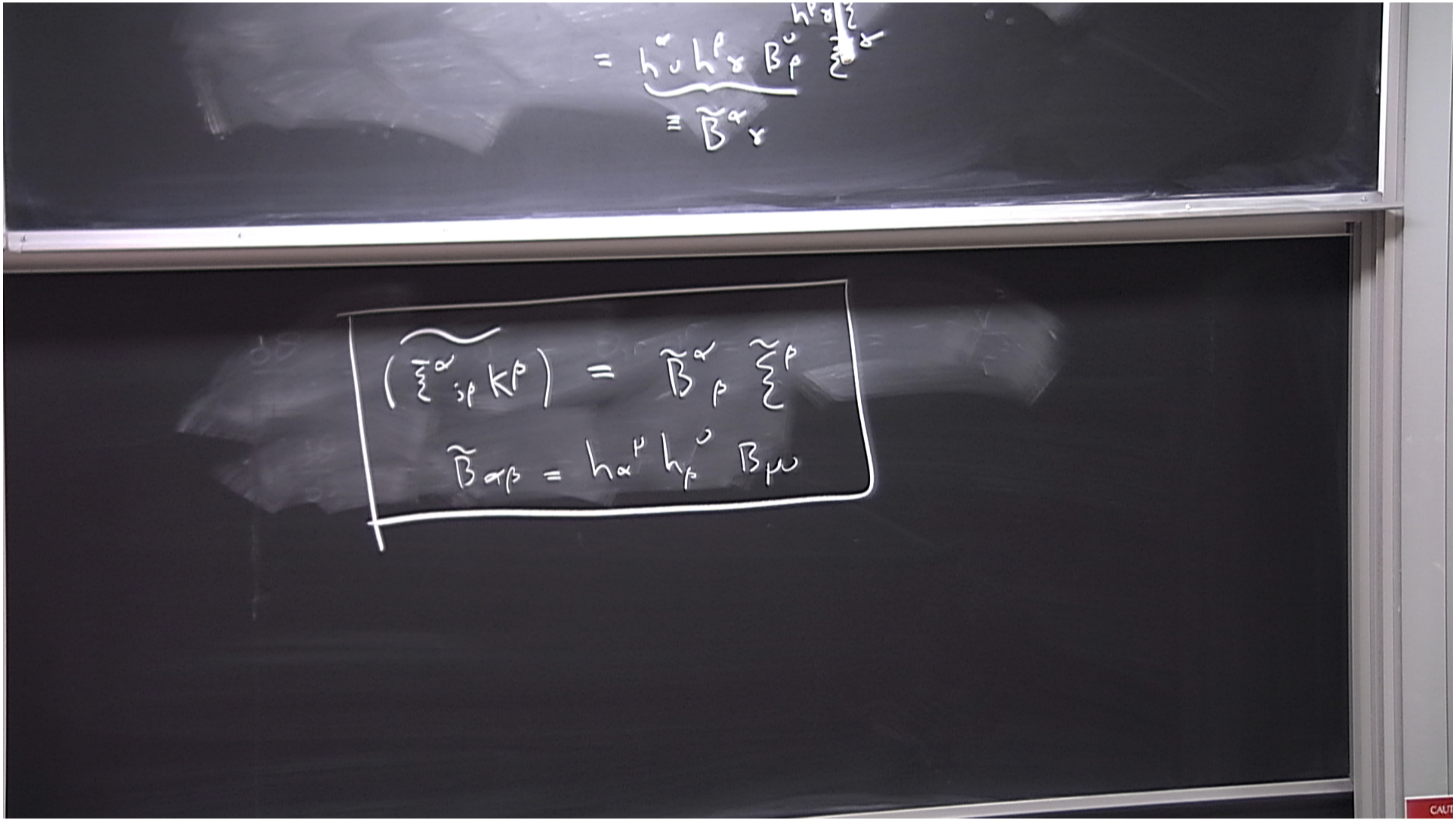

transverse relative velocity :
$\left(\widetilde{\tilde{\xi}^{\alpha}{}_{;\rho} k^{\rho}}\right) \equiv h^{\alpha}{}_{\mu} \tilde{\xi}^{\mu}{}_{;\rho} k^{\rho}$
$= \underbrace{h^{\alpha}{}_{\mu} h^{\mu}{}_{\nu}}_{h^{\alpha}{}_{\nu}} B^{\nu}{}_{\rho} \xi^{\rho}$

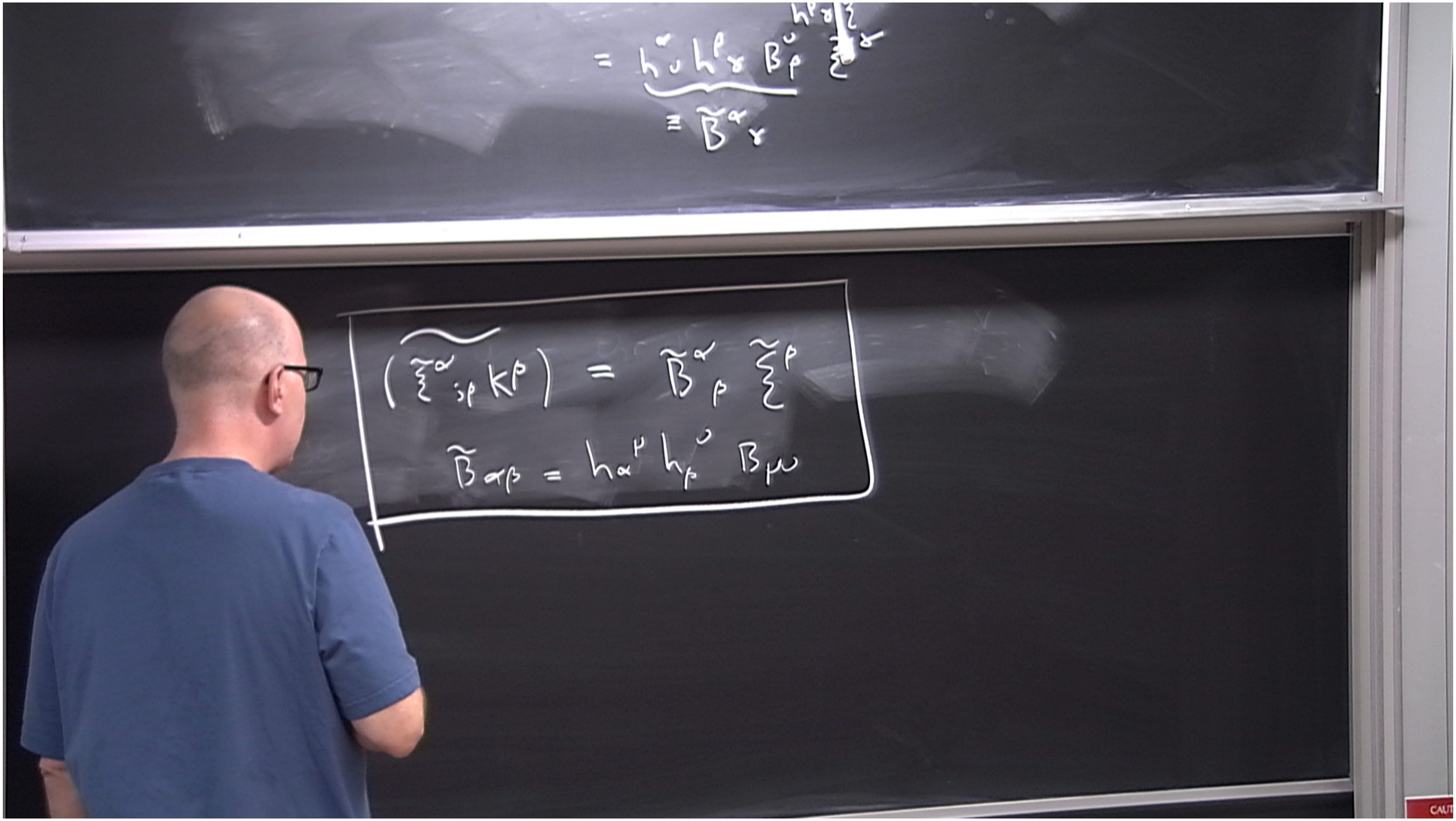

$= h^\alpha{}_\mu h^\mu{}_\nu B^\nu{}_\rho \xi^\rho$
$= h^\alpha{}_\nu B^\nu{}_\rho \tilde{\xi}^\rho$

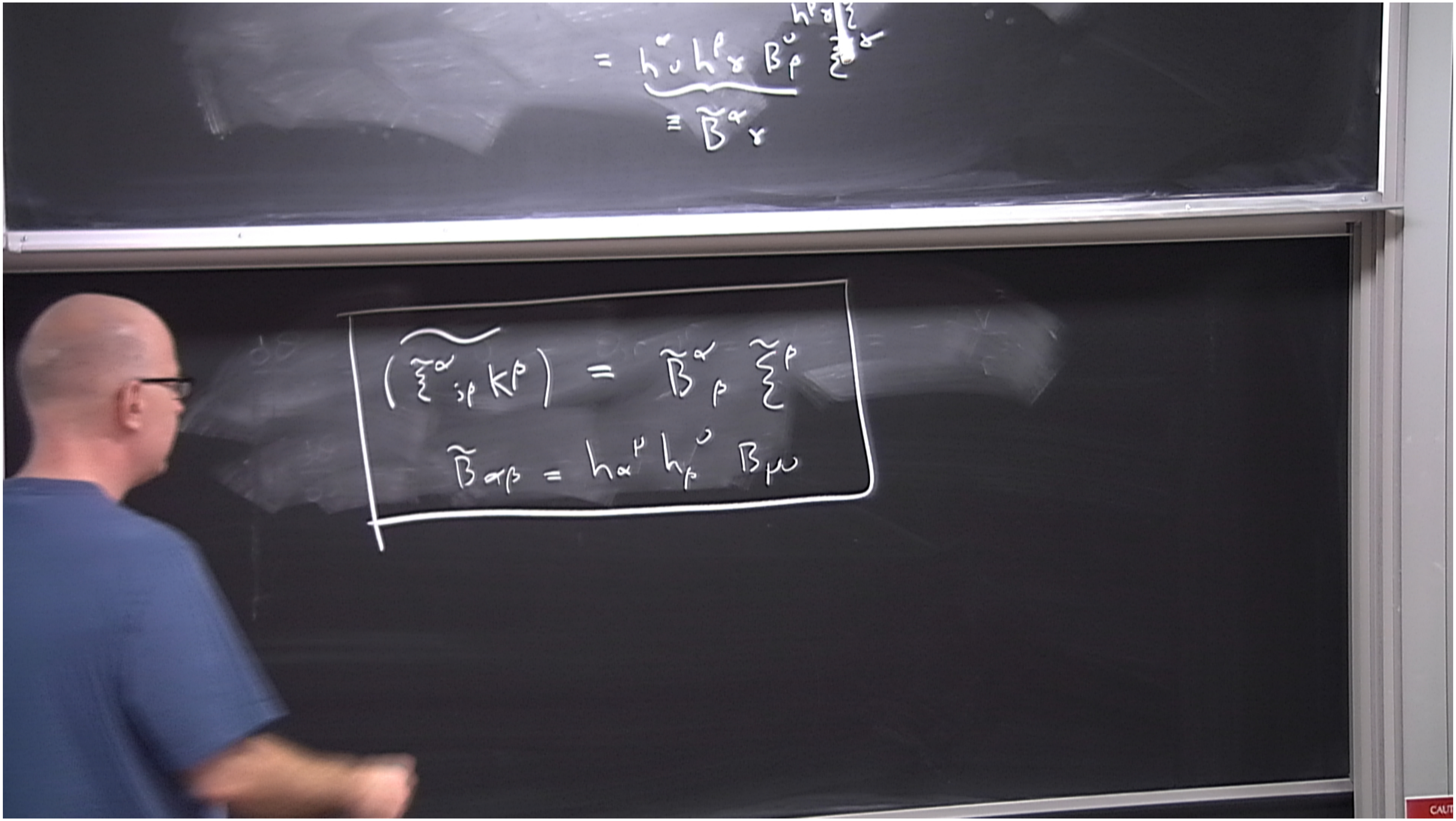

transverse relative velocity =
CAUTION




$= h^{\alpha}{}_{\nu} h^{\rho}{}_{\gamma} B^{\nu}{}_{\rho} \xi^{\gamma}$
$\equiv \tilde{B}^{\alpha}{}_{\gamma}$
$(\widetilde{\xi^{\alpha}{}_{;\rho} k^{\rho}}) = \tilde{B}^{\alpha}{}_{\rho} \tilde{\xi}^{\rho}$


$= h^{\alpha}{}_{\nu} h^{\rho}{}_{\gamma} B^{\nu}{}_{\rho} \xi^{\gamma}$
$\equiv \tilde{B}^{\alpha}{}_{\gamma}$
$(\widetilde{\xi^{\alpha}{}_{;\rho} k^{\rho}}) = \tilde{B}^{\alpha}{}_{\rho} \tilde{\xi}^{\rho}$
$\tilde{B}_{\alpha\beta} = h_{\alpha}{}^{\mu} h_{\beta}{}^{\nu} B_{\mu\nu}$


$= h^{\alpha}{}_{\nu} h^{\rho}{}_{\gamma} B^{\nu}{}_{\rho} \xi^{\gamma}$
$\equiv \tilde{B}^{\alpha}{}_{\gamma}$
$\widetilde{(\xi^{\alpha}{}_{;\rho} k^{\rho})} = \tilde{B}^{\alpha}{}_{\rho} \tilde{\xi}^{\rho}$
$\tilde{B}_{\alpha\beta} = h_{\alpha}{}^{\mu} h_{\beta}{}^{\nu} B_{\mu\nu}$

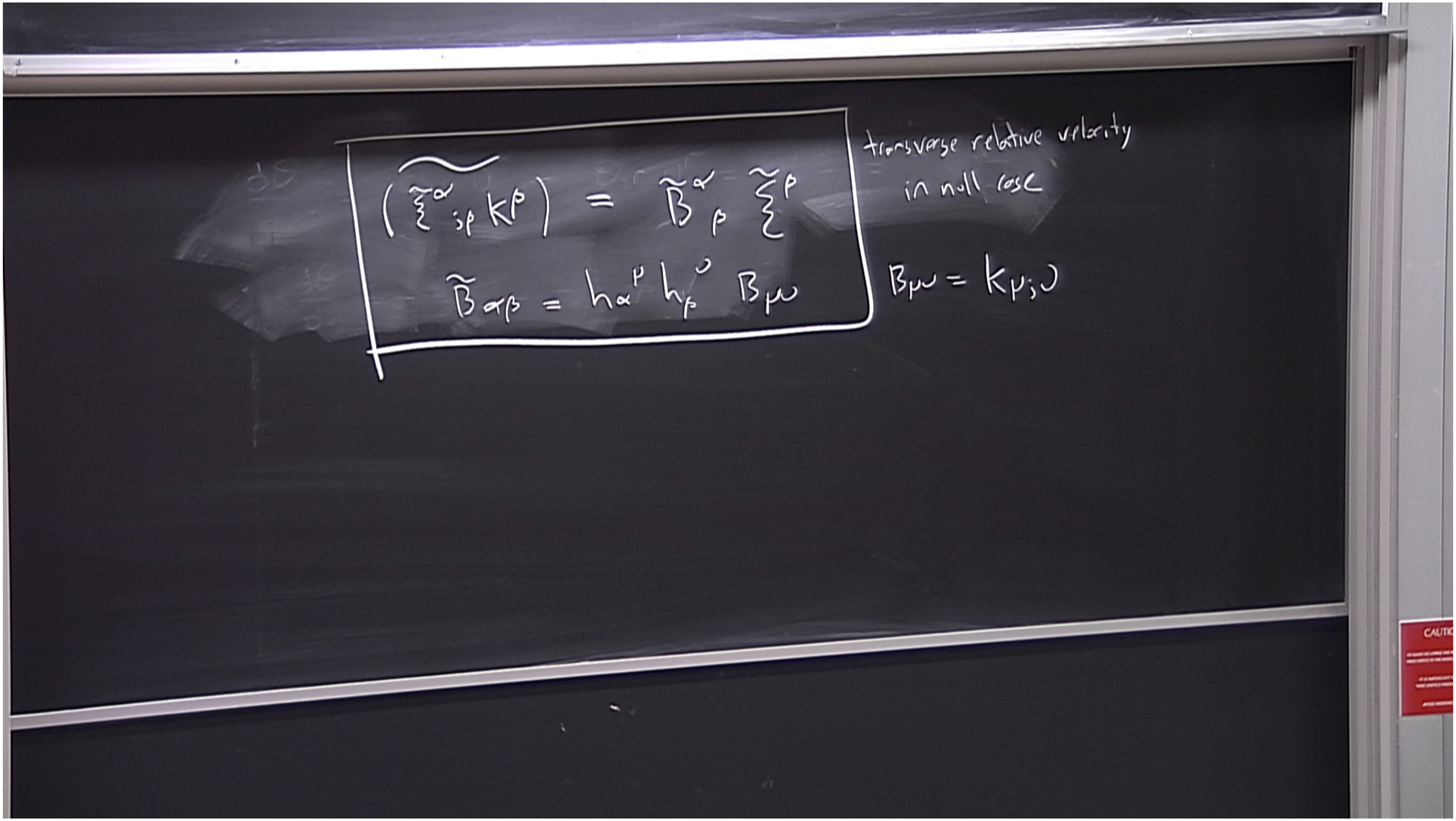

$= h^{\alpha}{}_{\nu} h^{\rho}{}_{\gamma} B^{\nu}{}_{\rho} \xi^{\gamma}$
$\equiv \tilde{B}^{\alpha}{}_{\gamma}$
$(\widetilde{\xi^{\alpha}{}_{;\rho} k^{\rho}}) = \tilde{B}^{\alpha}{}_{\rho} \tilde{\xi}^{\rho}$
$\tilde{B}_{\alpha\beta} = h_{\alpha}{}^{\mu} h_{\beta}{}^{\nu} B_{\mu\nu}$

$$= \underbrace{h^\alpha{}_\nu h^\rho{}_\gamma B^\nu{}_\rho}_{\equiv \tilde{B}^\alpha{}_\gamma} \xi^\gamma$$

$$\left(\widetilde{\xi^\alpha{}_{;\rho} k^\rho}\right) = \tilde{B}^\alpha{}_\rho \tilde{\xi}^\rho$$

$$\tilde{B}_{\alpha\beta} = h_\alpha{}^\mu h_\beta{}^\nu B_{\mu\nu}$$

$$B_{\mu\nu} = k_{\mu;\nu}$$


$\left(\widetilde{\xi^{\alpha}{}_{;\rho} k^{\rho}}\right) = \tilde{B}^{\alpha}{}_{\rho}\, \tilde{\xi}^{\rho}$
$\tilde{B}_{\alpha\beta} = h_{\alpha}{}^{\mu} h_{\beta}{}^{\nu} B_{\mu\nu}$
transverse relative velocity
in null case
$B_{\mu\nu} = k_{\mu;\nu}$

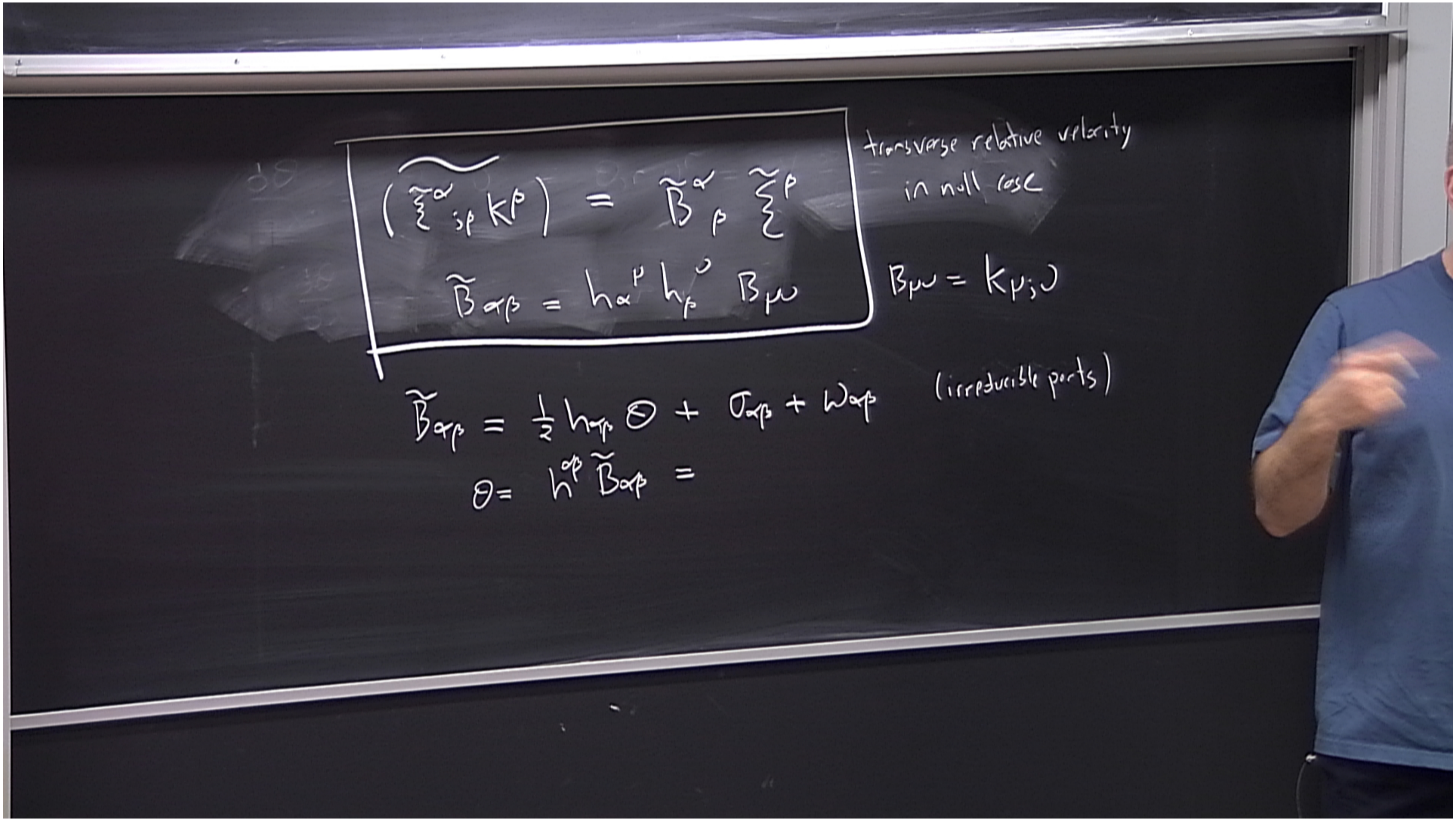
$$\left(\widetilde{\xi^{\alpha}{}_{;\rho}\, k^{\rho}}\right) = \tilde{B}^{\alpha}{}_{\rho}\, \tilde{\xi}^{\rho}$$

$$\tilde{B}_{\alpha\beta} = h_{\alpha}{}^{\mu}\, h_{\beta}{}^{\nu}\, B_{\mu\nu}$$

transverse relative velocity in null case

$$B_{\mu\nu} = k_{\mu;\nu}$$

$$\tilde{B}_{\alpha\beta} = \frac{1}{2} h_{\alpha\beta}\, \Theta + \sigma_{\alpha\beta} + \omega_{\alpha\beta}$$


$(\widetilde{\xi^\alpha{}_{;\beta} k^\beta}) = \tilde{B}^\alpha{}_\beta \tilde{\xi}^\beta$
$\tilde{B}_{\alpha\beta} = h_\alpha{}^\mu h_\beta{}^\nu B_{\mu\nu}$
transverse relative velocity
in null case
$B_{\mu\nu} = k_{\mu;\nu}$
$\tilde{B}_{\alpha\beta} = \frac{1}{2} h_{\alpha\beta} \Theta + \sigma_{\alpha\beta} + \omega_{\alpha\beta}$
(irreducible parts)
$\Theta = h^{\alpha\beta} \tilde{B}_{\alpha\beta} =$

$$\left(\widetilde{\tilde{\xi}^{\alpha}{}_{;\rho} k^{\rho}}\right) = \tilde{B}^{\alpha}{}_{\rho}\, \tilde{\xi}^{\rho}$$

$$\tilde{B}_{\alpha\beta} = h_{\alpha}{}^{\mu} h_{\beta}{}^{\nu} B_{\mu\nu}$$

transverse relative velocity in null case

$$B_{\mu\nu} = k_{\mu;\nu}$$

$$\tilde{B}_{\alpha\beta} = \frac{1}{2} h_{\alpha\beta}\Theta + \sigma_{\alpha\beta} + \omega_{\alpha\beta}$$ (irreducible parts)

$$\Theta = h^{\alpha\beta}\tilde{B}_{\alpha\beta} = g^{\alpha\beta}\tilde{B}_{\alpha\beta} = \dots = k^{\alpha}{}_{;\alpha}$$

$$\left(\widetilde{\xi^\alpha{}_{;\beta} k^\beta}\right) = \tilde{B}^\alpha{}_\beta \, \tilde{\xi}^\beta$$

transverse relative velocity in null case

$$\tilde{B}_{\alpha\beta} = h_\alpha{}^\mu h_\beta{}^\nu B_{\mu\nu}$$

$$B_{\mu\nu} = k_{\mu;\nu}$$

$$\tilde{B}_{\alpha\beta} = \frac{1}{2} h_{\alpha\beta}\,\theta + \sigma_{\alpha\beta} + \omega_{\alpha\beta}$$

(irreducible parts)

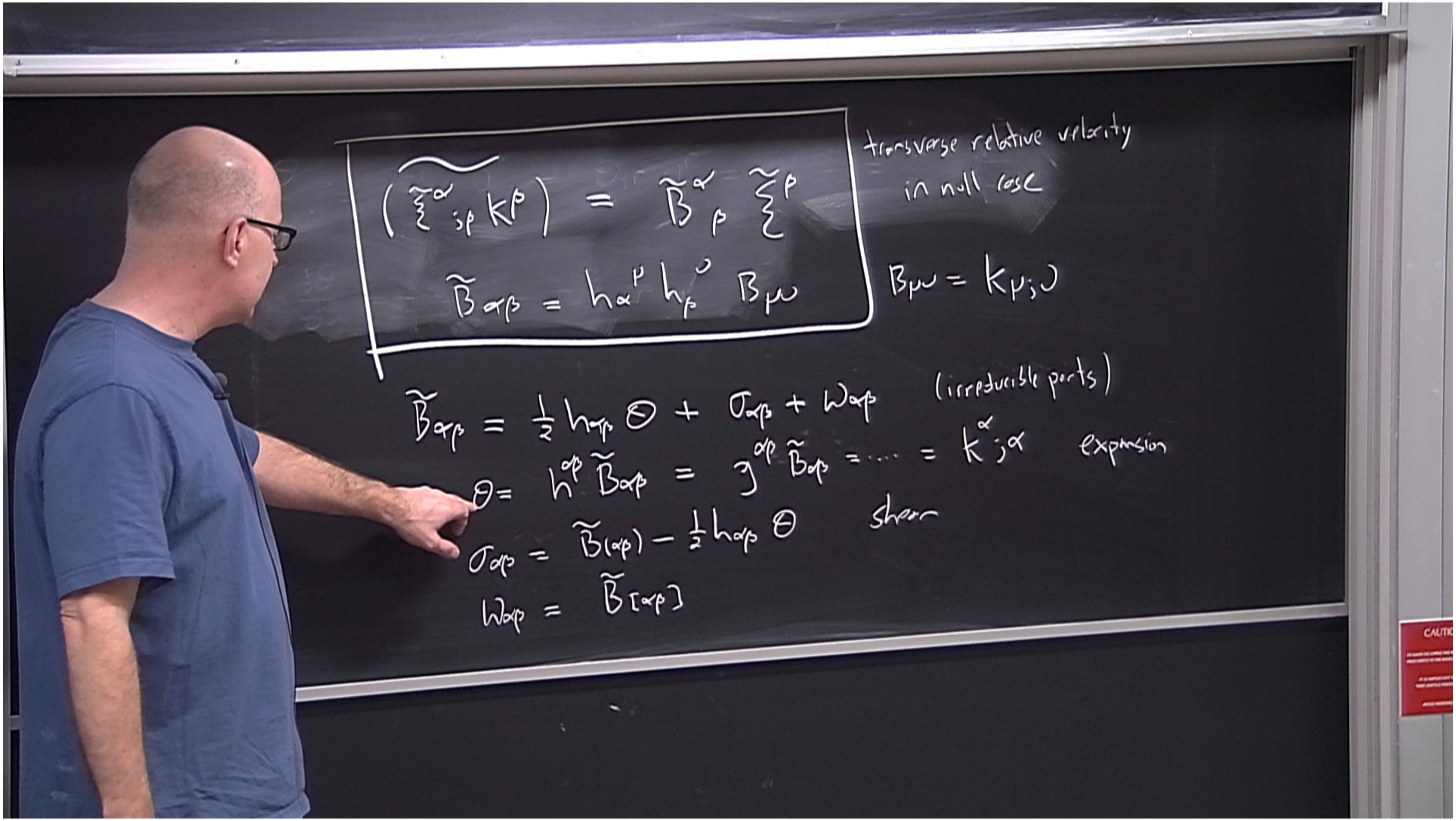
$$\theta = h^{\alpha\beta}\tilde{B}_{\alpha\beta} = g^{\alpha\beta}\tilde{B}_{\alpha\beta} = \dots = k^\alpha{}_{;\alpha}$$

expansion

$$\sigma_{\alpha\beta} = \tilde{B}_{(\alpha\beta)} - \frac{1}{2} h_{\alpha\beta}\,\theta$$

shear

$$\omega_{\alpha\beta} = \tilde{B}_{[\alpha\beta]}$$



$\widetilde{(\xi^\alpha{}_{;\rho} k^\rho)} = \tilde{B}^\alpha{}_\rho \tilde{\xi}^\rho$
transverse relative velocity in null case
$\tilde{B}_{\alpha\beta} = h_\alpha{}^\mu h_\beta{}^\nu B_{\mu\nu}$
$B_{\mu\nu} = k_{\mu;\nu}$
$\tilde{B}_{\alpha\beta} = \frac{1}{2} h_{\alpha\beta} \theta + \sigma_{\alpha\beta} + \omega_{\alpha\beta}$
(irreducible parts)
$\theta = h^{\alpha\beta} \tilde{B}_{\alpha\beta} = g^{\alpha\beta} \tilde{B}_{\alpha\beta} = \dots = k^\alpha{}_{;\alpha}$
expansion
$\sigma_{\alpha\beta} = \tilde{B}_{(\alpha\beta)} - \frac{1}{2} h_{\alpha\beta} \theta$
shear
$\omega_{\alpha\beta} = \tilde{B}_{[\alpha\beta]}$

$$\left(\widetilde{\xi^\alpha{}_{;\beta} k^\beta}\right) = \tilde{B}^\alpha{}_\beta \, \tilde{\xi}^\beta$$

$$\tilde{B}_{\alpha\beta} = h_\alpha{}^\mu h_\beta{}^\nu B_{\mu\nu}$$

transverse relative velocity in null case

$$B_{\mu\nu} = k_{\mu;\nu}$$

$$\Theta = \frac{1}{\delta A}\frac{d}{d\lambda}\delta A$$

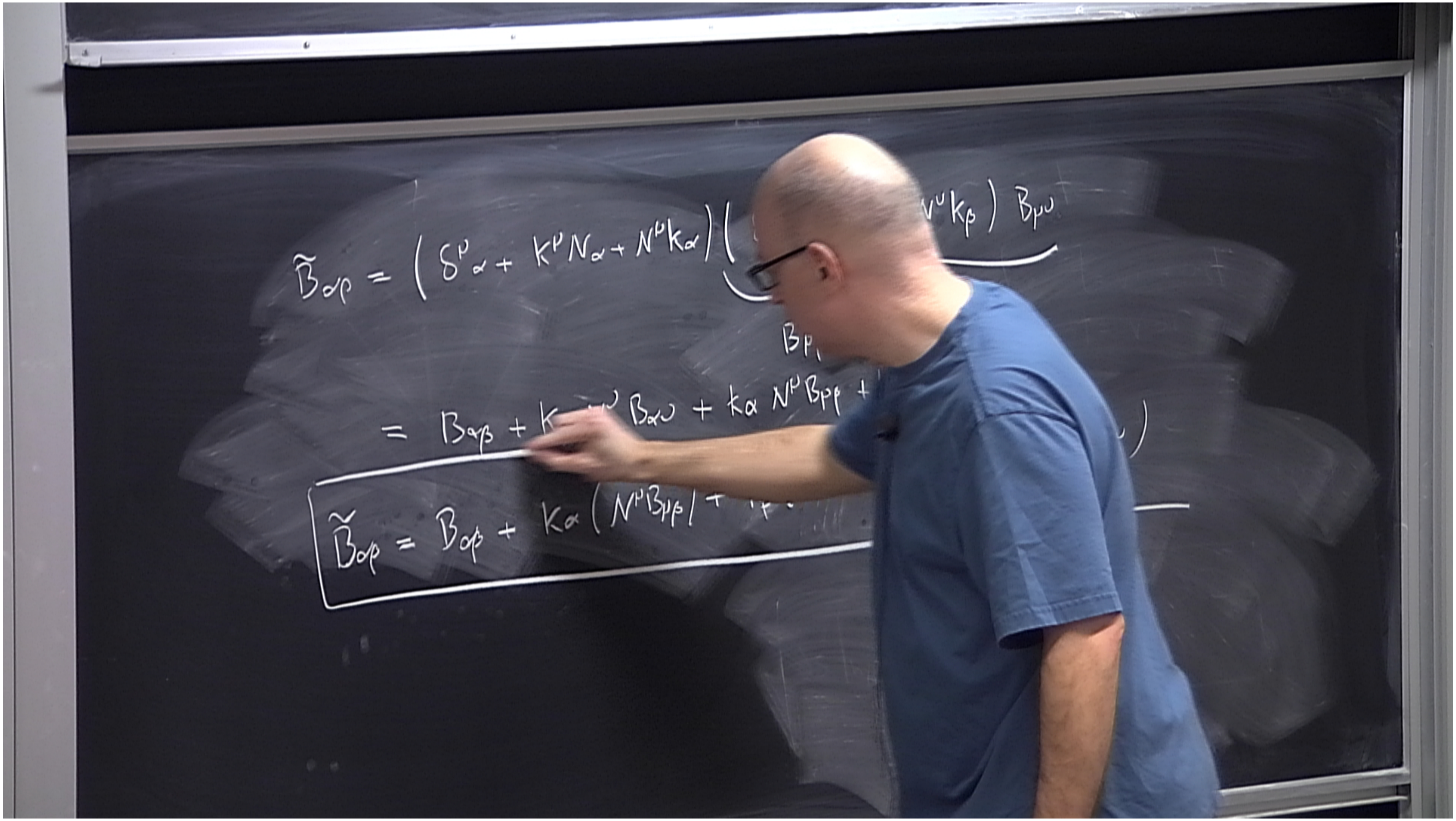
$\delta A \equiv$ cross-sectional areas.

$$\tilde{B}_{\alpha\beta} = \frac{1}{2} h_{\alpha\beta}\Theta + \sigma_{\alpha\beta} + \omega_{\alpha\beta}$$ (irreducible parts)

$$\Theta = h^{\alpha\beta}\tilde{B}_{\alpha\beta} = g^{\alpha\beta}\tilde{B}_{\alpha\beta} = \dots = k^\alpha{}_{;\alpha}$$ expansion

$$\sigma_{\alpha\beta} = \tilde{B}_{(\alpha\beta)} - \frac{1}{2} h_{\alpha\beta}\Theta$$ shear

$$\omega_{\alpha\beta} = \tilde{B}_{[\alpha\beta]}$$

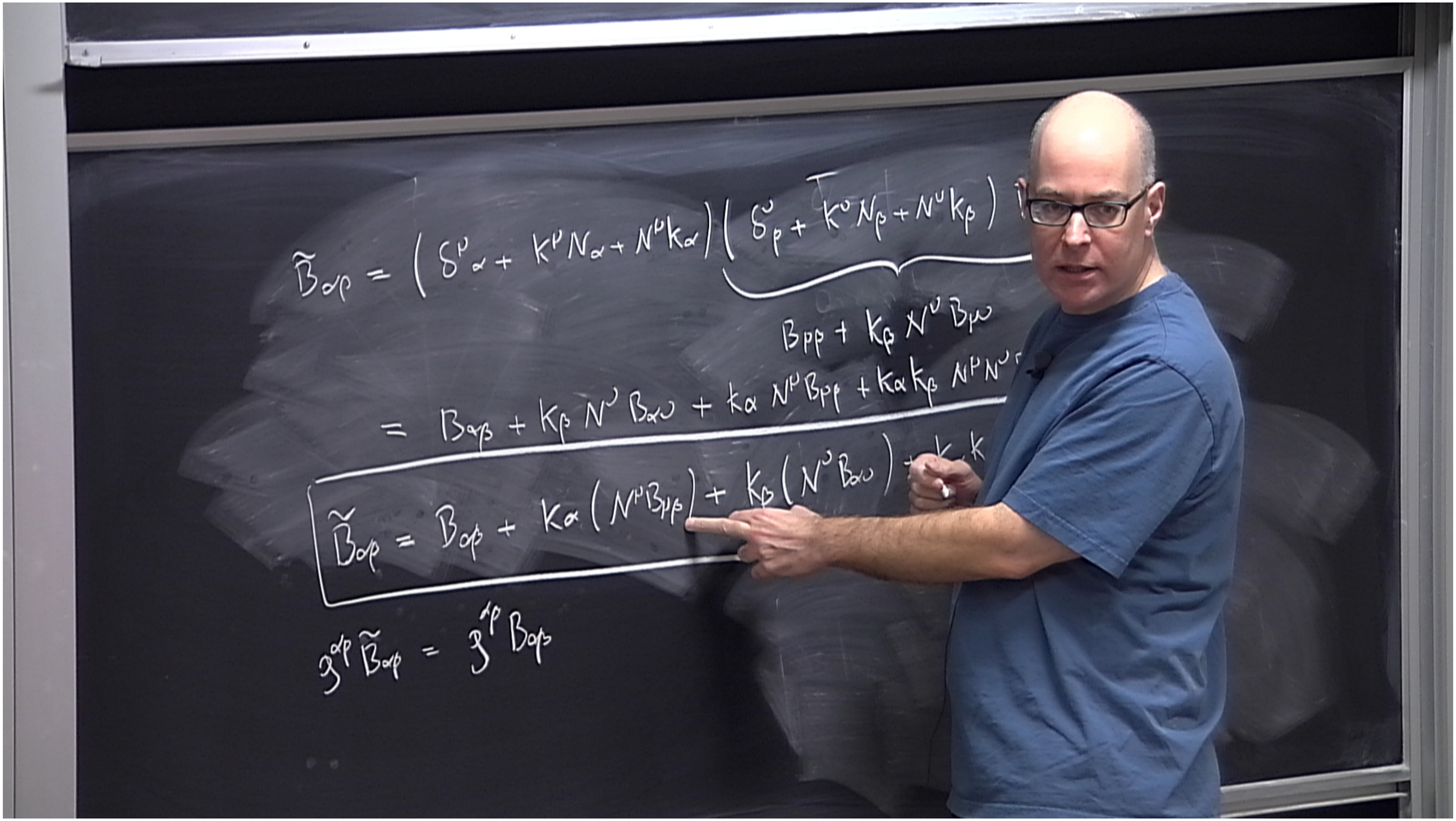


$$\tilde{B}_{\alpha\beta} = \left(\delta^{\mu}{}_{\alpha} + K^{\mu} N_{\alpha} + N^{\mu} K_{\alpha} \right) \underbrace{\left(\delta^{\nu}{}_{\beta} + K^{\nu} N_{\beta} + N^{\nu} K_{\beta} \right) B_{\mu\nu}}_{B_{\mu\beta} + K_{\beta} N^{\nu} B_{\mu\nu}}$$

$$= B_{\alpha\beta} + K_{\beta} N^{\nu} B_{\alpha\nu} + K_{\alpha} N^{\mu} B_{\mu\beta} + K_{\alpha} K_{\beta} N^{\mu} N^{\nu} B_{\mu\nu}$$

$$\boxed{\tilde{B}_{\alpha\beta} = B_{\alpha\beta} + K_{\alpha} \left(N^{\mu} B_{\mu\beta} \right) + K_{\beta} \left(N^{\nu} B_{\alpha\nu} \right) + K_{\alpha} K_{\beta} \left(N^{\mu} N^{\nu} B_{\mu\nu} \right)}$$

$$g^{\alpha\beta} \tilde{B}_{\alpha\beta} = g^{\alpha\beta} B_{\alpha\beta}$$

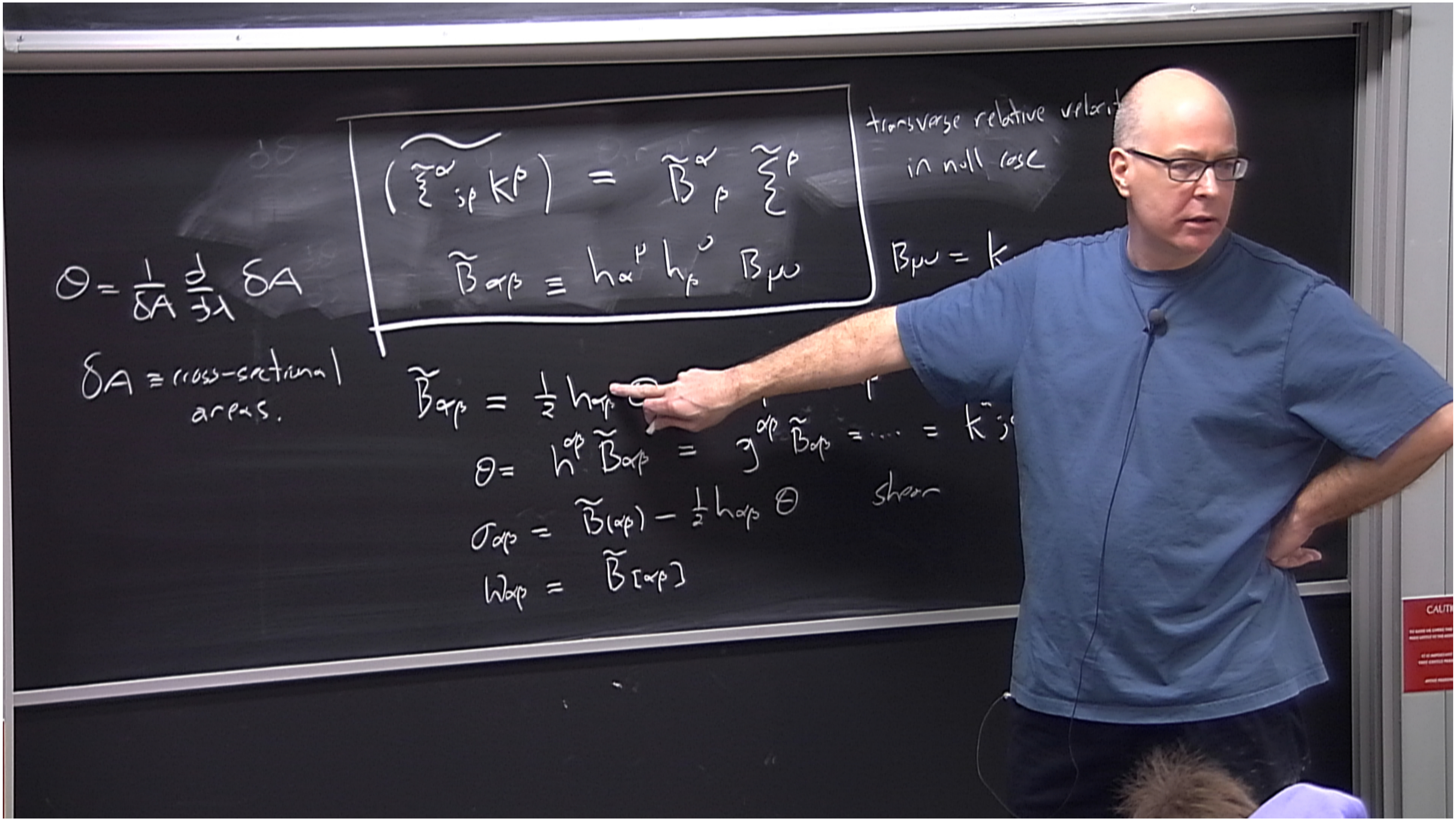

$\tilde{B}_{\alpha\beta} = (\delta^\mu{}_\alpha + K^\mu N_\alpha + N^\mu K_\alpha) (\delta^\nu{}_\beta + K^\nu N_\beta + N^\nu K_\beta)$
$B_{\mu\beta} + K_\beta N^\nu B_{\mu\nu}$
$= B_{\alpha\beta} + K_\beta N^\nu B_{\alpha\nu} + K_\alpha N^\mu B_{\mu\beta} + K_\alpha K_\beta N^\mu N^\nu$
$\tilde{B}_{\alpha\beta} = B_{\alpha\beta} + K_\alpha (N^\mu B_{\mu\beta}) + K_\beta (N^\nu B_{\alpha\nu}) + K_\alpha K$
$g^{\alpha\beta} \tilde{B}_{\alpha\beta} = g^{\alpha\beta} B_{\alpha\beta}$

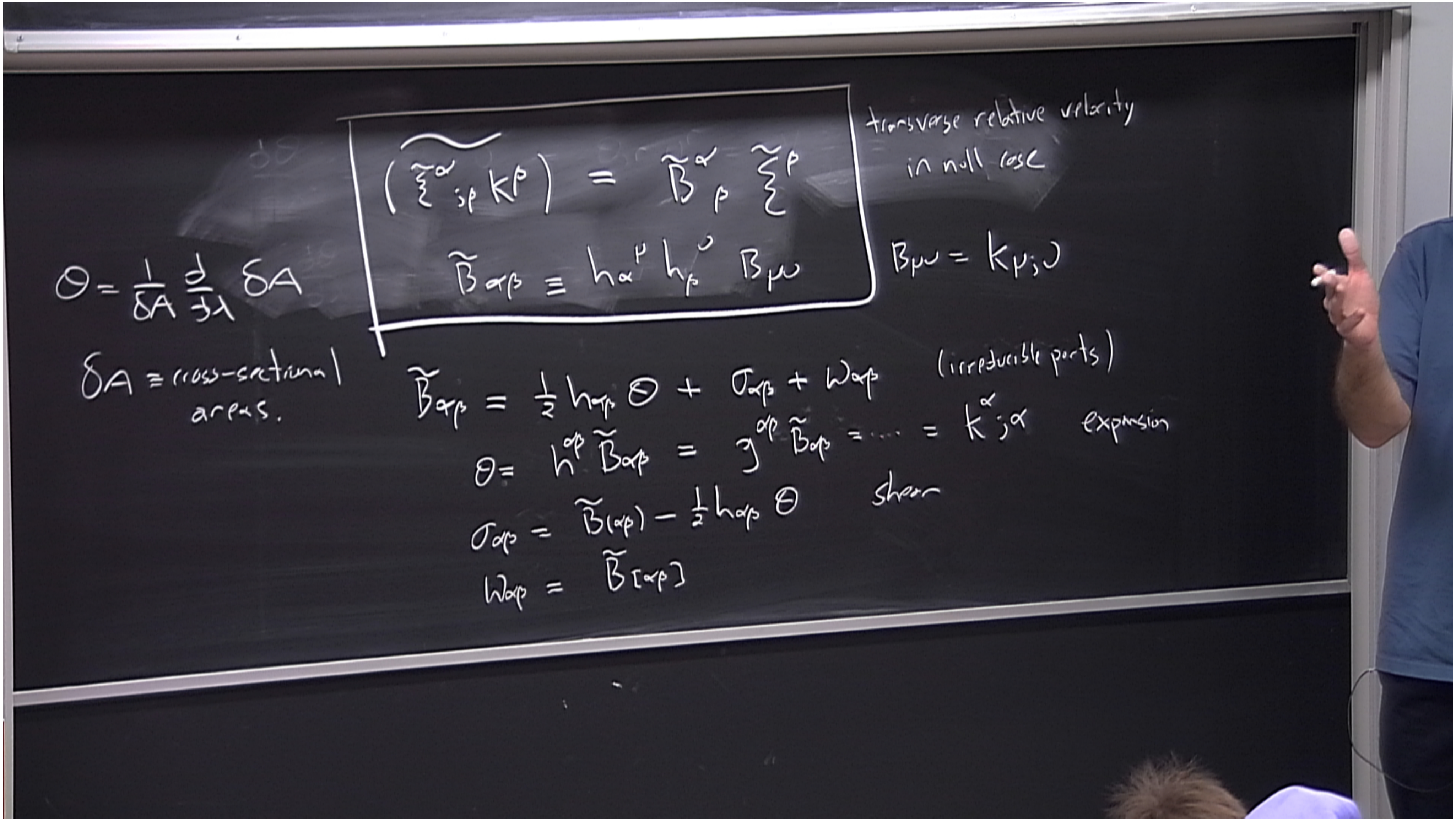
$$\tilde{B}_{\alpha\beta} = \left(\delta^{\mu}{}_{\alpha} + K^{\mu} N_{\alpha} + N^{\mu} K_{\alpha}\right)\underbrace{\left(\delta^{\nu}{}_{\beta} + K^{\nu} N_{\beta} + N^{\nu} K_{\beta}\right) B_{\mu\nu}}_{B_{\mu\beta} + K_{\beta} N^{\nu} B_{\mu\nu}}$$

$$= B_{\alpha\beta} + K_{\beta} N^{\nu} B_{\alpha\nu} + K_{\alpha} N^{\mu} B_{\mu\beta} + K_{\alpha} K_{\beta} N^{\mu} N^{\nu} B_{\mu\nu}$$

$$\boxed{\tilde{B}_{\alpha\beta} = B_{\alpha\beta} + K_{\alpha}\left(N^{\mu} B_{\mu\beta}\right) + K_{\beta}\left(N^{\nu} B_{\alpha\nu}\right) + K_{\alpha} K_{\beta}\left(N^{\mu} N^{\nu} B_{\mu\nu}\right)}$$

$$g^{\alpha\beta} \tilde{B}_{\alpha\beta} = g^{\alpha\beta} B_{\alpha\beta} = g^{\alpha\beta} K_{\alpha;\beta} = K^{\alpha}{}_{;\alpha}$$


in null case
δA ≡ cross-sectional
areas.
shear


$(\widetilde{\xi^\alpha{}_{;\rho} k^\rho}) = \tilde{B}^\alpha{}_\rho \tilde{\xi}^\rho$
$\tilde{B}_{\alpha\beta} \equiv h_\alpha{}^\mu h_\beta{}^\nu B_{\mu\nu}$
transverse relative velocity
in null case
$B_{\mu\nu} = k_{\mu;\nu}$
$\Theta = \frac{1}{\delta A} \frac{d}{d\lambda} \delta A$
$\delta A \equiv$ cross-sectional areas.
$\tilde{B}_{\alpha\beta} = \frac{1}{2} h_{\alpha\beta} \Theta + \sigma_{\alpha\beta} + \omega_{\alpha\beta}$
(irreducible parts)
$\Theta = h^{\alpha\beta} \tilde{B}_{\alpha\beta} = g^{\alpha\beta} \tilde{B}_{\alpha\beta} = \dots = k^\alpha{}_{;\alpha}$
expansion
$\sigma_{\alpha\beta} = \tilde{B}_{(\alpha\beta)} - \frac{1}{2} h_{\alpha\beta} \Theta$
shear
$\omega_{\alpha\beta} = \tilde{B}_{[\alpha\beta]}$

$$\tilde{\xi}^\alpha = h^\alpha{}_\mu \xi^\mu = \text{transverse part of } \xi^\alpha$$

$$\tilde{\xi}^\alpha{}_{;\beta} k^\beta = \left(h^\alpha{}_\mu \xi^\mu \right)_{;\beta} k^\beta$$

$$= \underbrace{h^\alpha{}_{\mu;\beta} k^\beta \xi^\mu} + h^\alpha{}_\mu B^\mu{}_\beta \xi^\beta$$

$$\left(\delta^\alpha{}_\mu + k^\alpha N_\mu + N^\alpha k_\mu \right)_{;\beta} k^\beta \xi^\mu$$

$$= k^\alpha \left(N_{\mu;\beta} k^\beta \right) \xi^\mu$$

$$= h^\alpha{}_\mu B^\mu{}_\beta \xi^\beta + \left(\xi^\mu N_{\mu;\beta} k^\beta \right) k^\alpha$$

$$\theta = \frac{1}{\delta A}\frac{d}{d\lambda}\delta A$$

$\delta A \equiv$ cross-sectional areas.

$$\left(\widetilde{\xi^\alpha{}_{;\beta} k^\beta}\right) = \tilde{B}^\alpha{}_\beta \, \tilde{\xi}^\beta$$

$$\tilde{B}_{\alpha\beta} \equiv h_\alpha{}^\mu h_\beta{}^\nu B_{\mu\nu}$$

transverse relative velocity in null case

$$B_{\mu\nu} = k_{\mu;\nu}$$

$$\tilde{B}_{\alpha\beta} = \frac{1}{2} h_{\alpha\beta}\,\theta + \sigma_{\alpha\beta} + \omega_{\alpha\beta} \quad \text{(irreducible parts)}$$

$$\theta = h^{\alpha\beta}\tilde{B}_{\alpha\beta} = g^{\alpha\beta}\tilde{B}_{\alpha\beta} = \cdots = k^\alpha{}_{;\alpha} \quad \text{expansion}$$

$$\sigma_{\alpha\beta} = \tilde{B}_{(\alpha\beta)} - \frac{1}{2} h_{\alpha\beta}\,\theta \quad \text{shear}$$

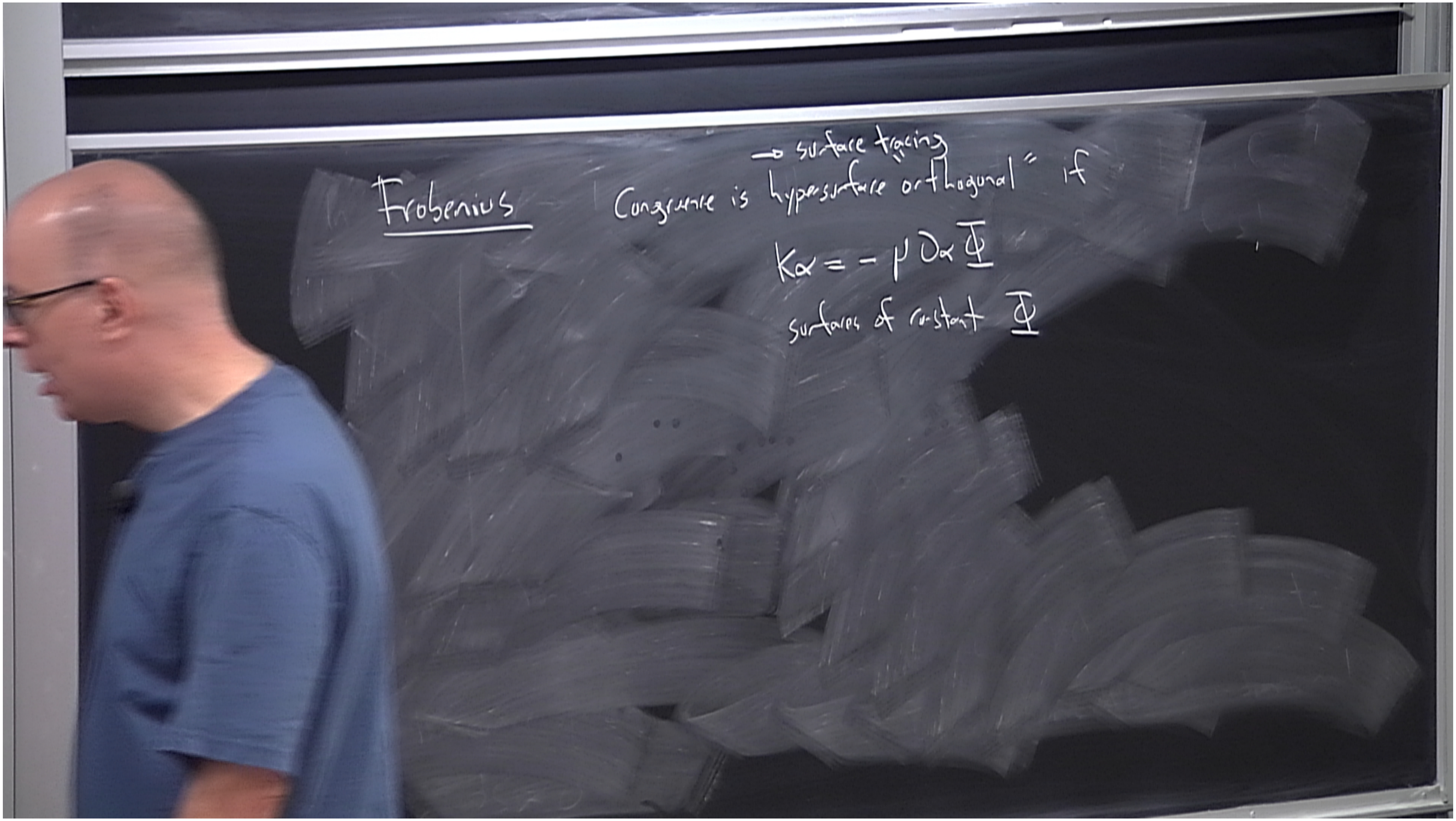
$$\omega_{\alpha\beta} = \tilde{B}_{[\alpha\beta]}$$

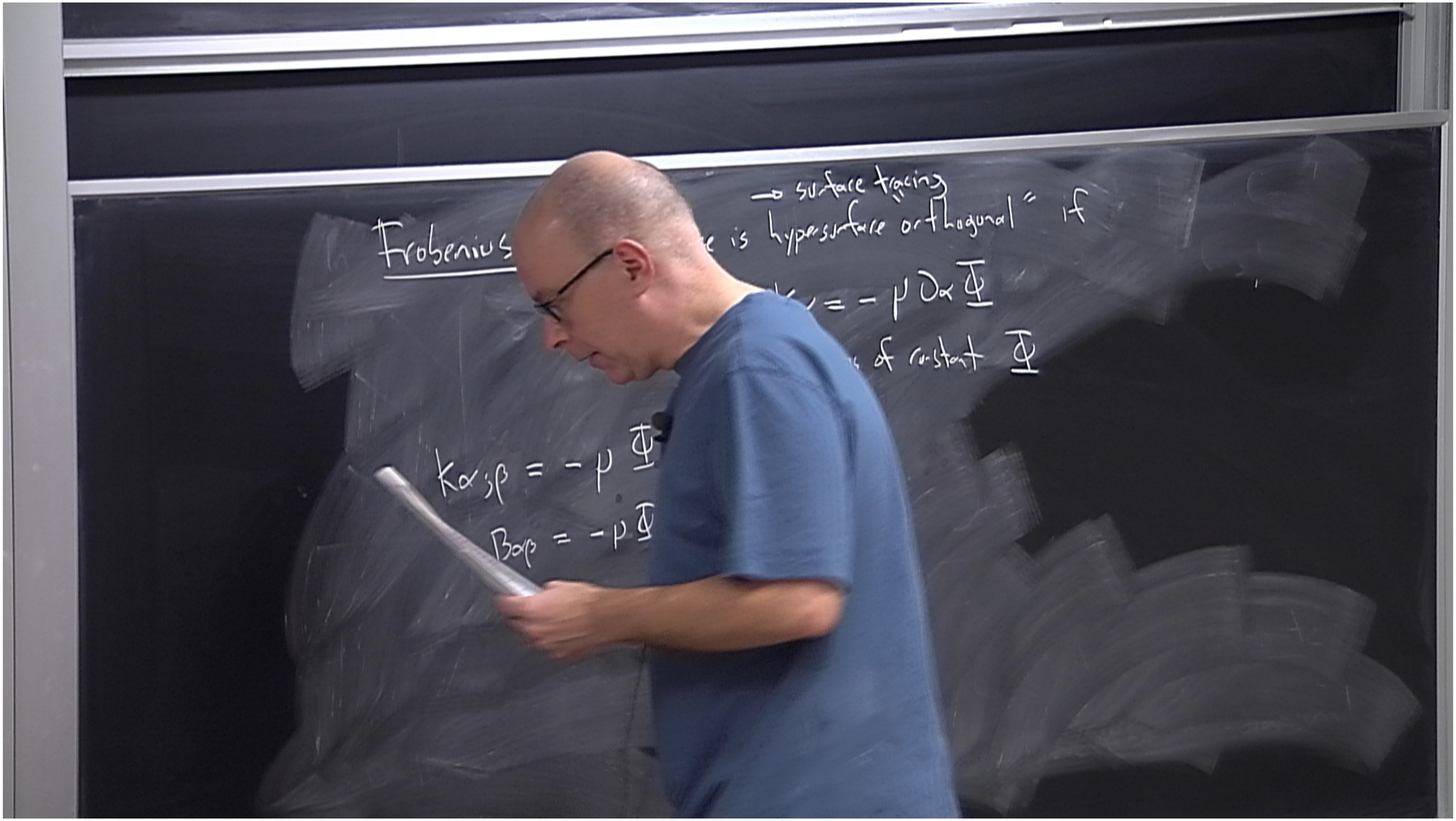
Frobenius

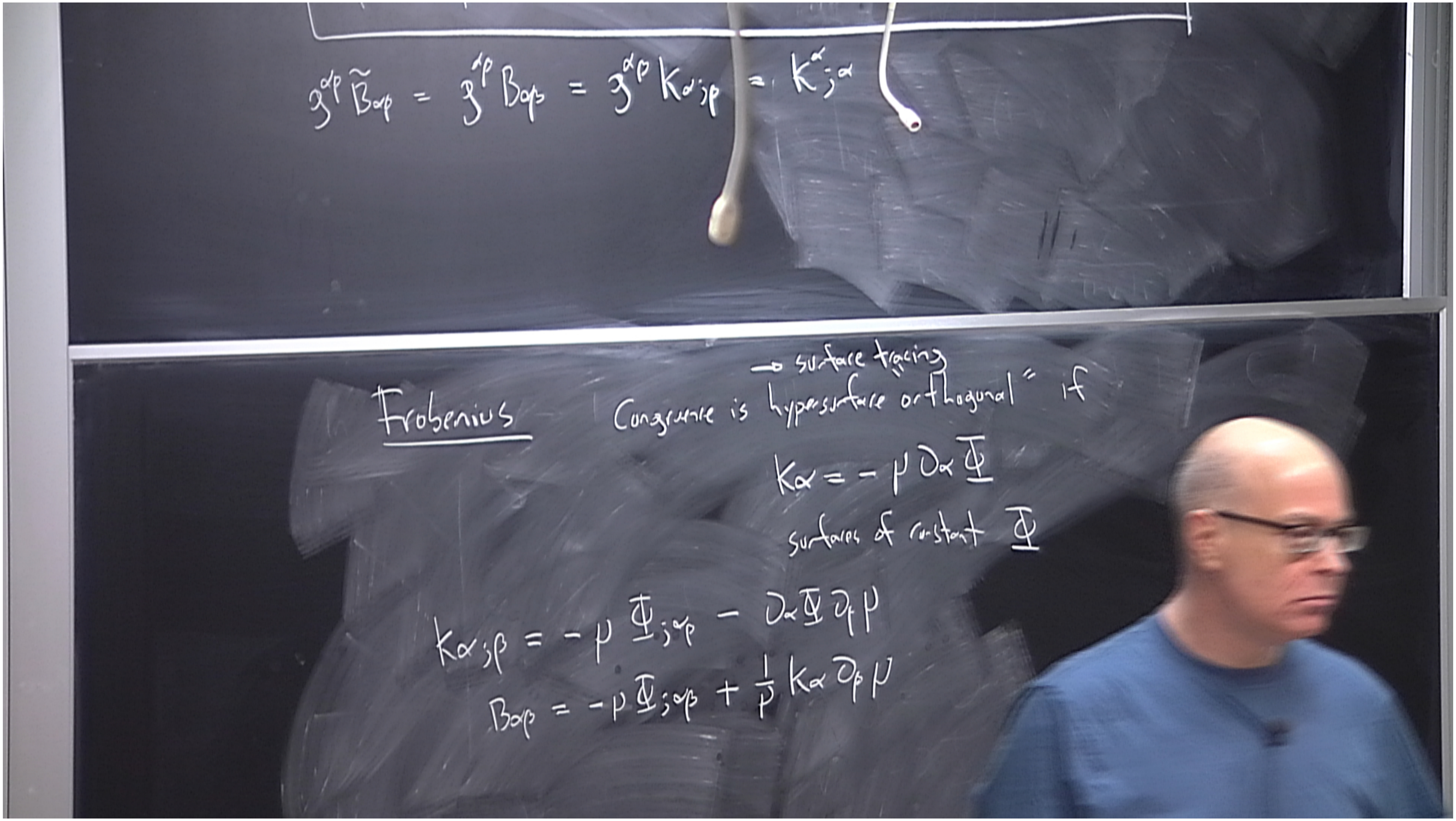
Congruence is hypersurface "orthogonal" if

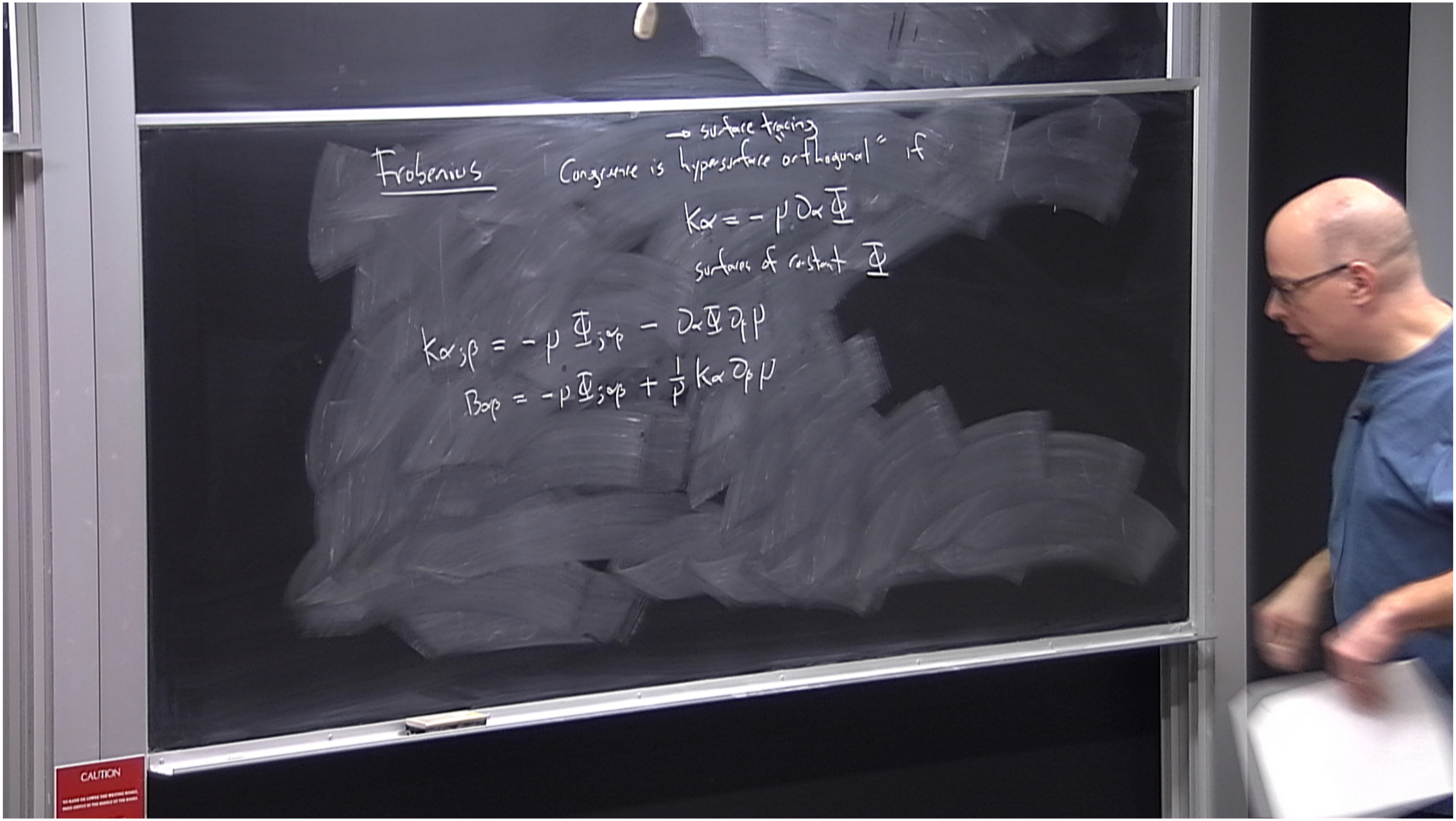
$$K_\alpha = -\mu \partial_\alpha \Phi$$

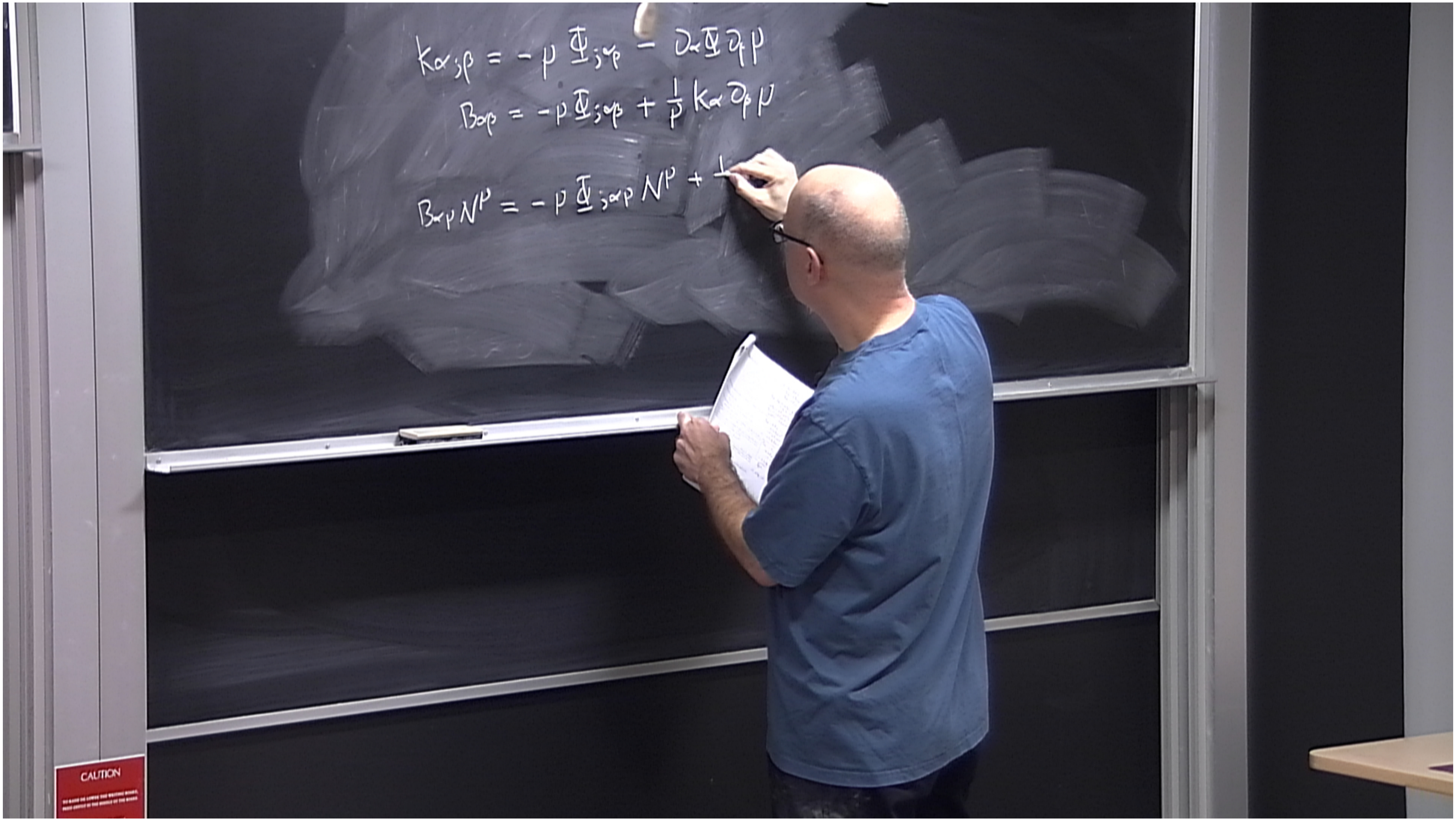
surfaces of constant Φ

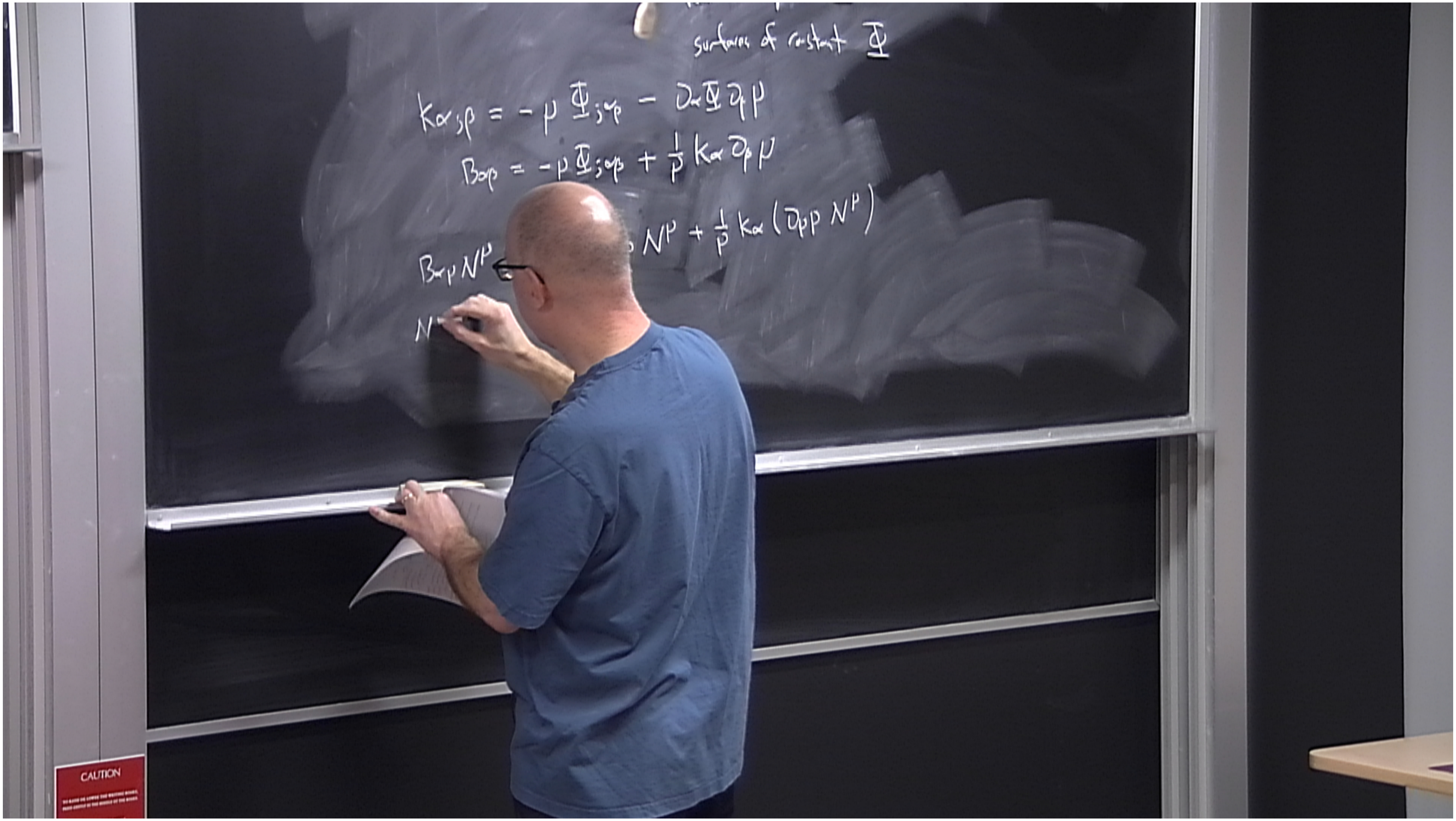

Frobenius
→ surface tracing
Congruence is "hypersurface orthogonal" if
Kα = - μ ∂α Φ
surfaces of constant Φ

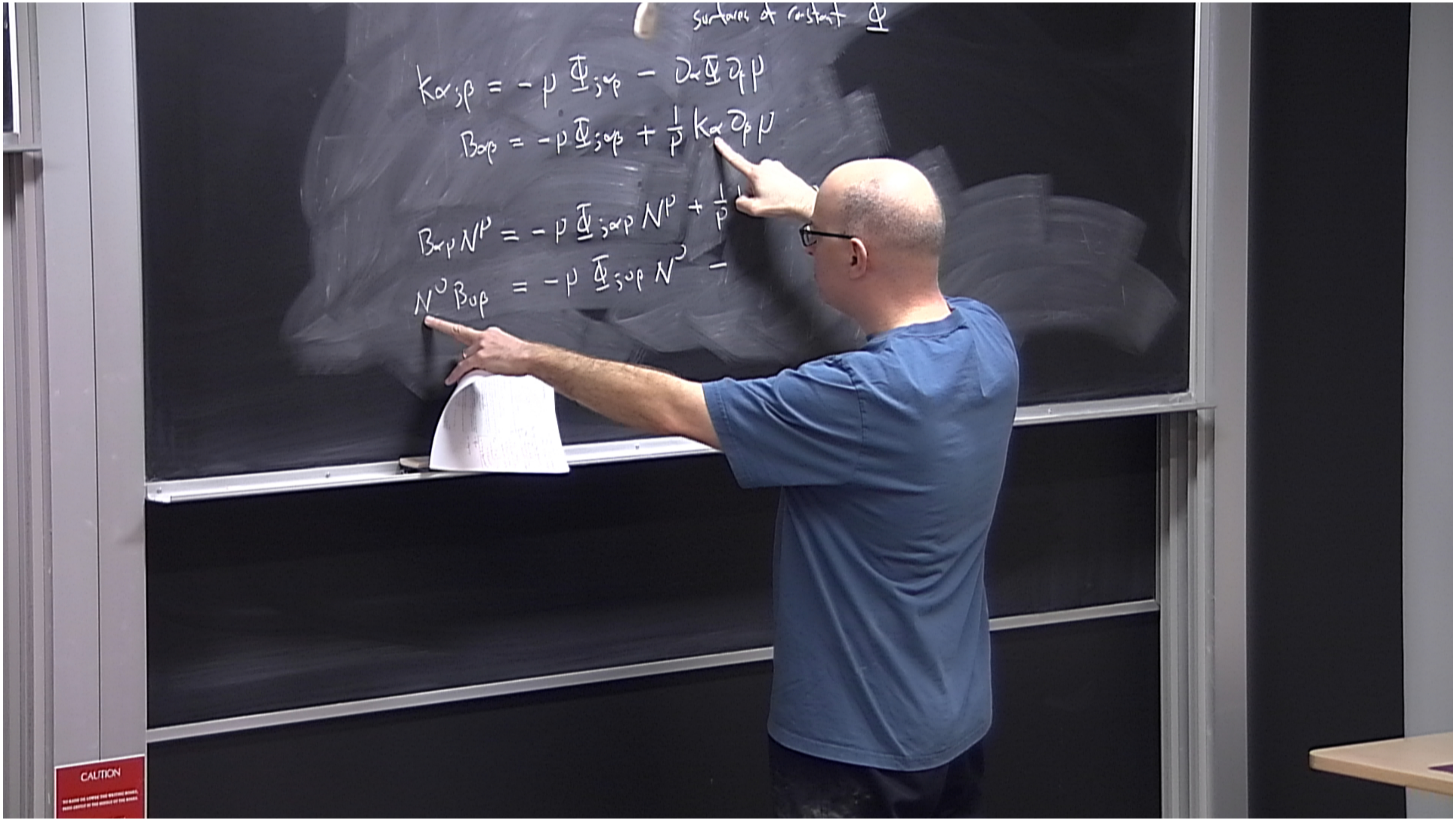

→ surface tracing
Frobenius
is hypersurface orthogonal" if
of constant Φ

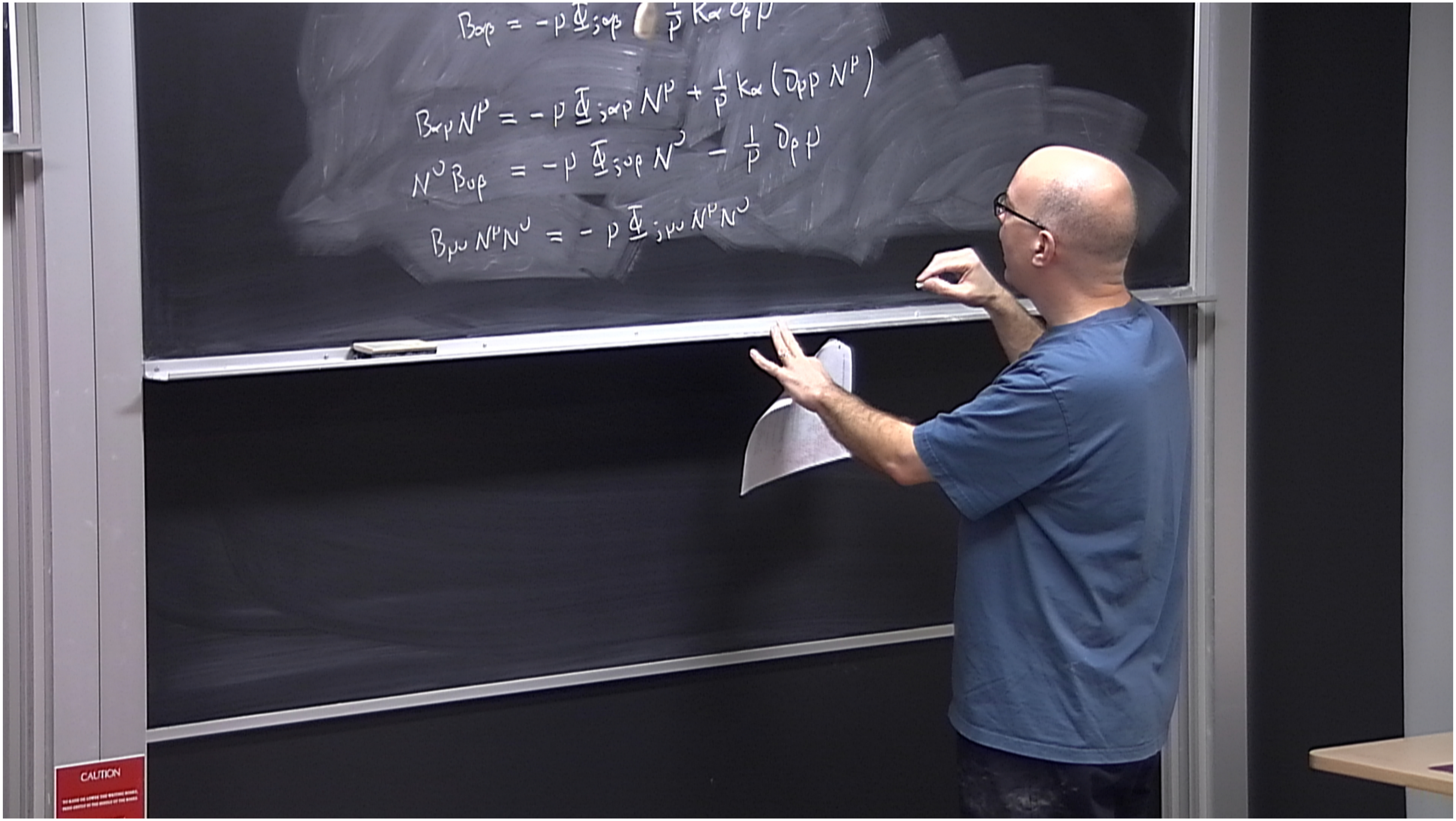

$g^{\alpha\beta}\tilde{B}_{\alpha\beta} = g^{\alpha\beta}B_{\alpha\beta} = g^{\alpha\beta}k_{\alpha;\beta} = k^{\alpha}{}_{;\alpha}$
Frobenius
→ surface tracing
Congruence is "hypersurface orthogonal" if
$k_\alpha = -\mu\,\partial_\alpha\Phi$
surfaces of constant Φ
$k_{\alpha;\beta} = -\mu\,\Phi_{;\alpha\beta} - \partial_\alpha\Phi\,\partial_\beta\mu$
$B_{\alpha\beta} = -\mu\,\Phi_{;\alpha\beta} + \frac{1}{\mu}k_\alpha\partial_\beta\mu$


Frobenius
→ surface tracing
Congruence is "hypersurface orthogonal" if
$k_\alpha = -\mu \partial_\alpha \Phi$
surfaces of constant Φ
$k_{\alpha;\beta} = -\mu \Phi_{;\alpha\beta} - \partial_\alpha \Phi \partial_\beta \mu$
$B_{\alpha\beta} = -\mu \Phi_{;\alpha\beta} + \frac{1}{\mu} k_\alpha \partial_\beta \mu$
CAUTION

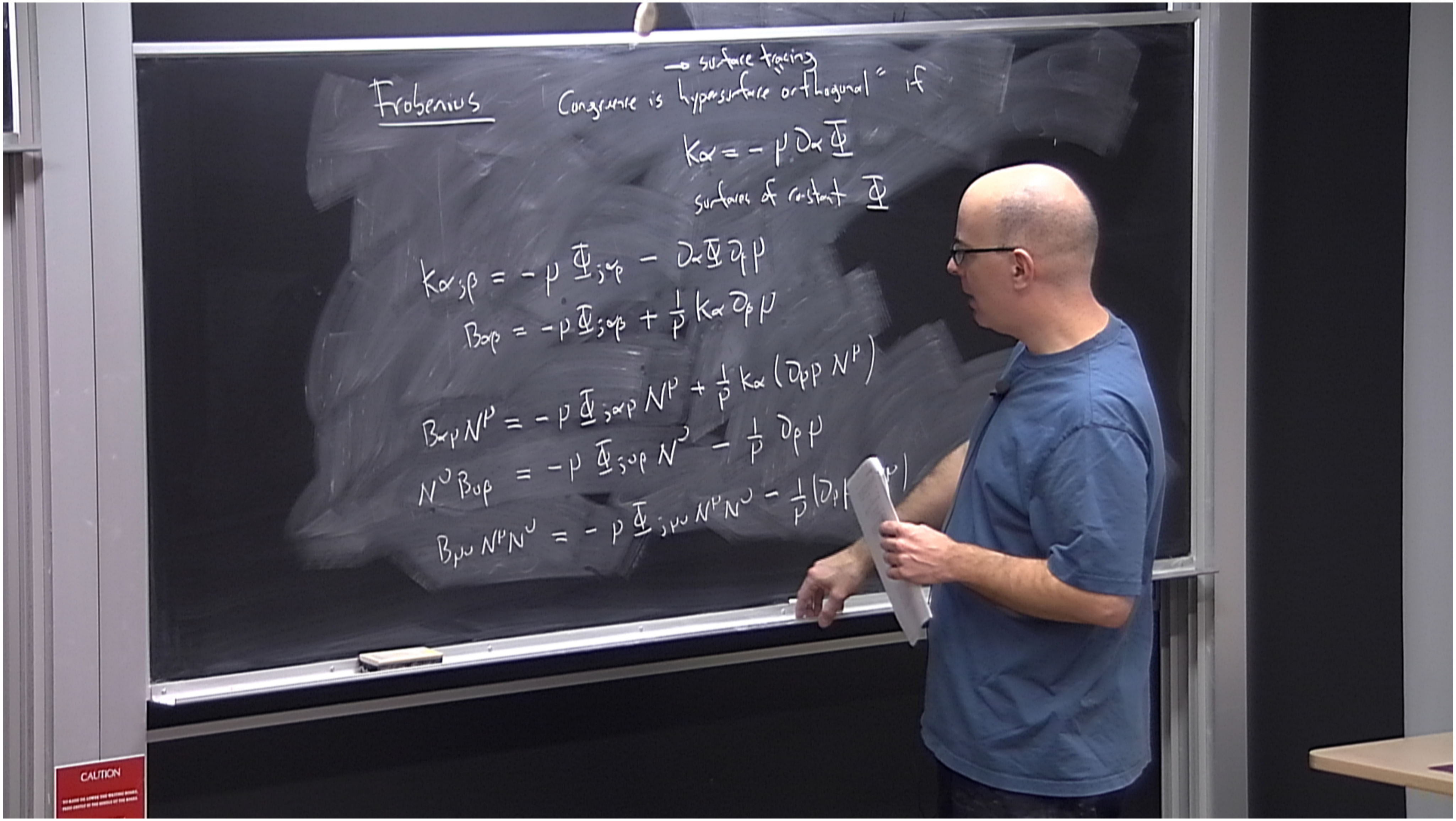

$k_{\alpha;\beta} = -\rho\, \Phi_{;\alpha\beta} - \partial_\alpha \Phi\, \partial_\beta \rho$
$B_{\alpha\beta} = -\rho\, \Phi_{;\alpha\beta} + \frac{1}{\rho} k_\alpha \partial_\beta \rho$
$B_{\alpha\rho} N^\rho = -\rho\, \Phi_{;\alpha\rho} N^\rho +$
CAUTION

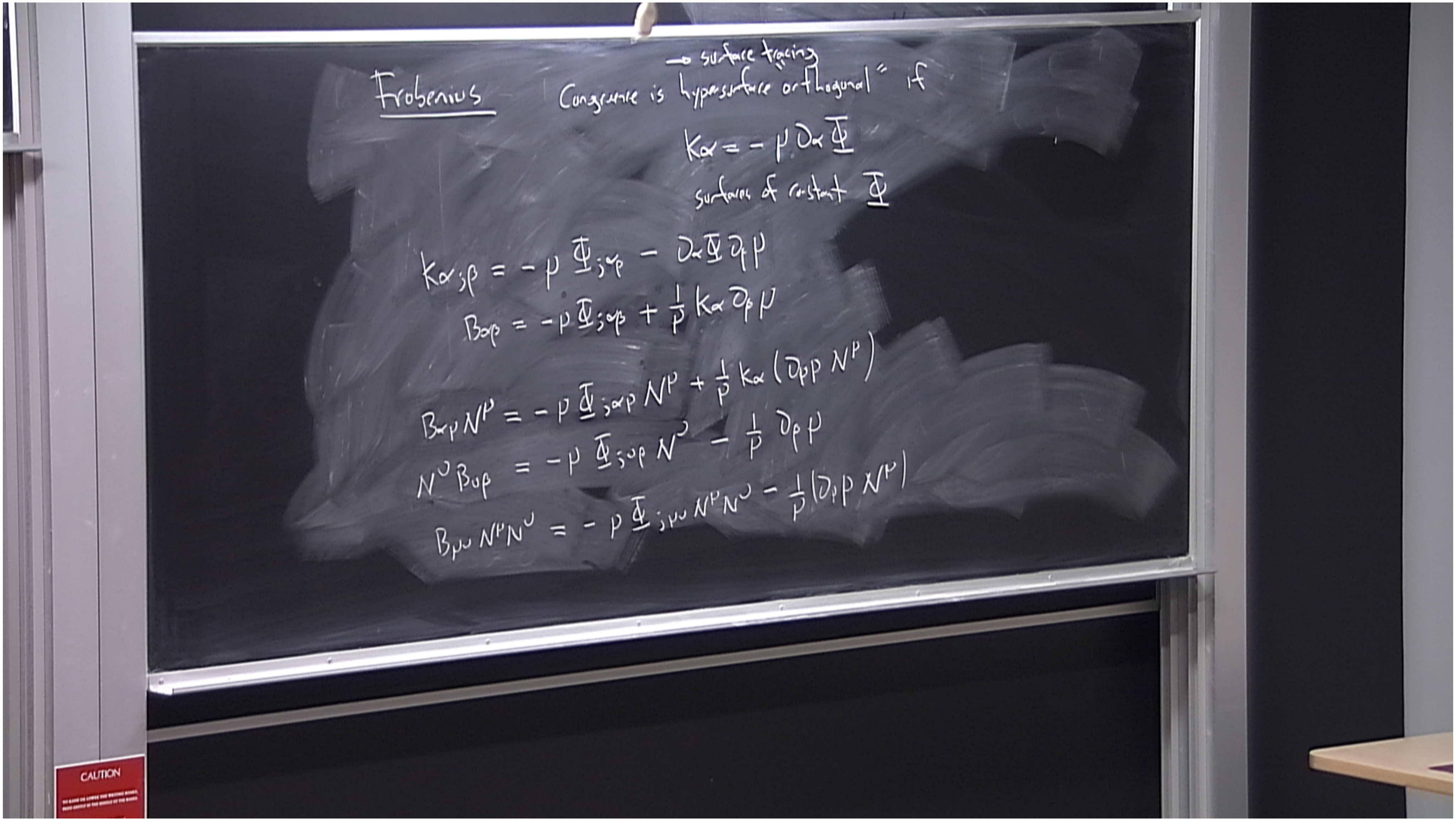

surfaces of constant Φ
kα;β = −ρ Φ;αβ − ∂αΦ ∂βρ
Bαβ = −ρ Φ;αβ + 1/ρ kα ∂βρ
NP + 1/ρ kα (∂βρ Nβ)
BαβNβ

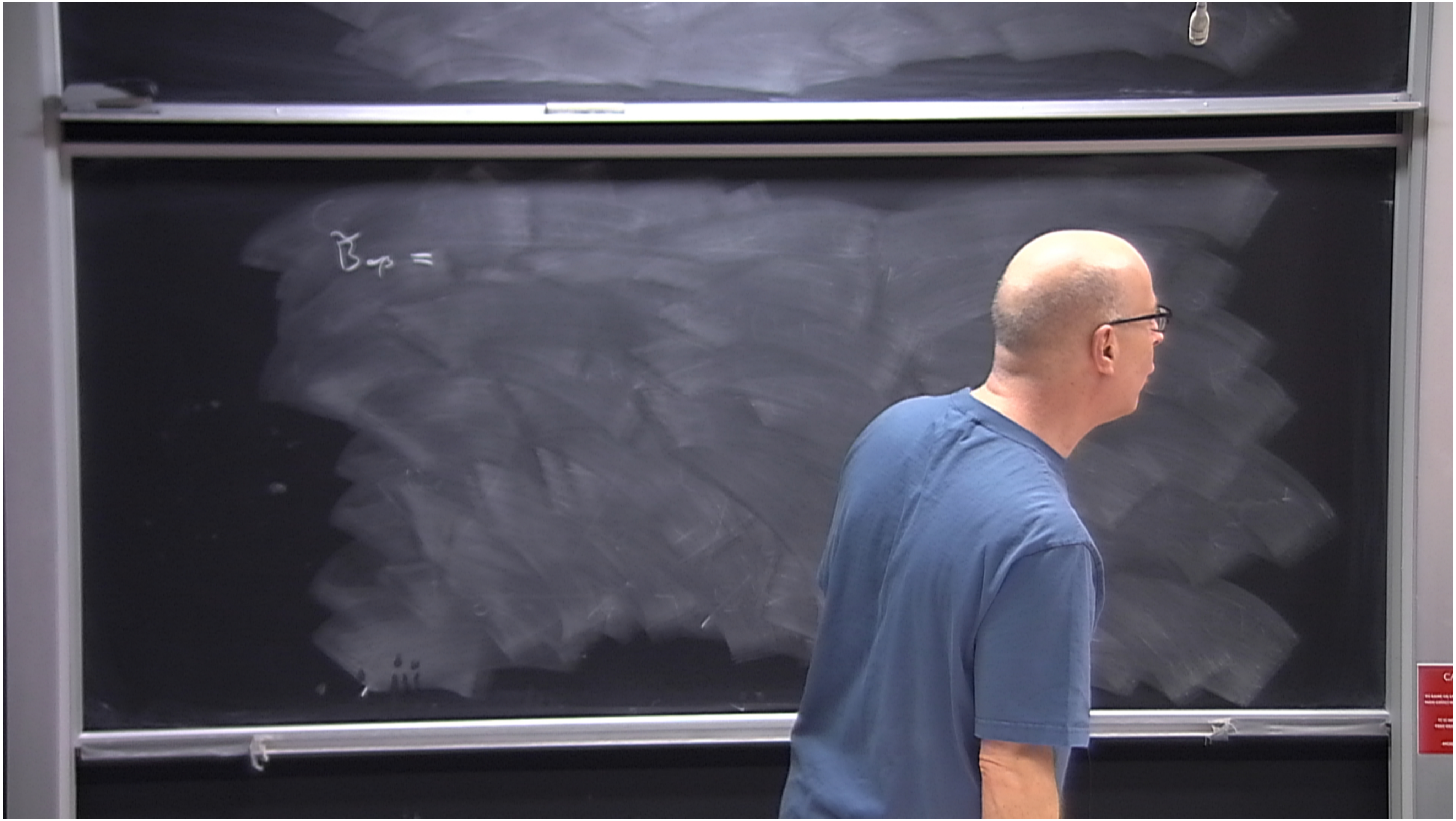

surfaces of constant Φ
Kα;β = −μ Φ;αβ − ∂αΦ ∂βμ
Bαβ = −μ Φ;αβ + 1/μ Kα ∂βμ
BαβN^β = −μ Φ;αβ N^β + 1/μ
N^α Bαβ = −μ Φ;αβ N^α −
CAUTION

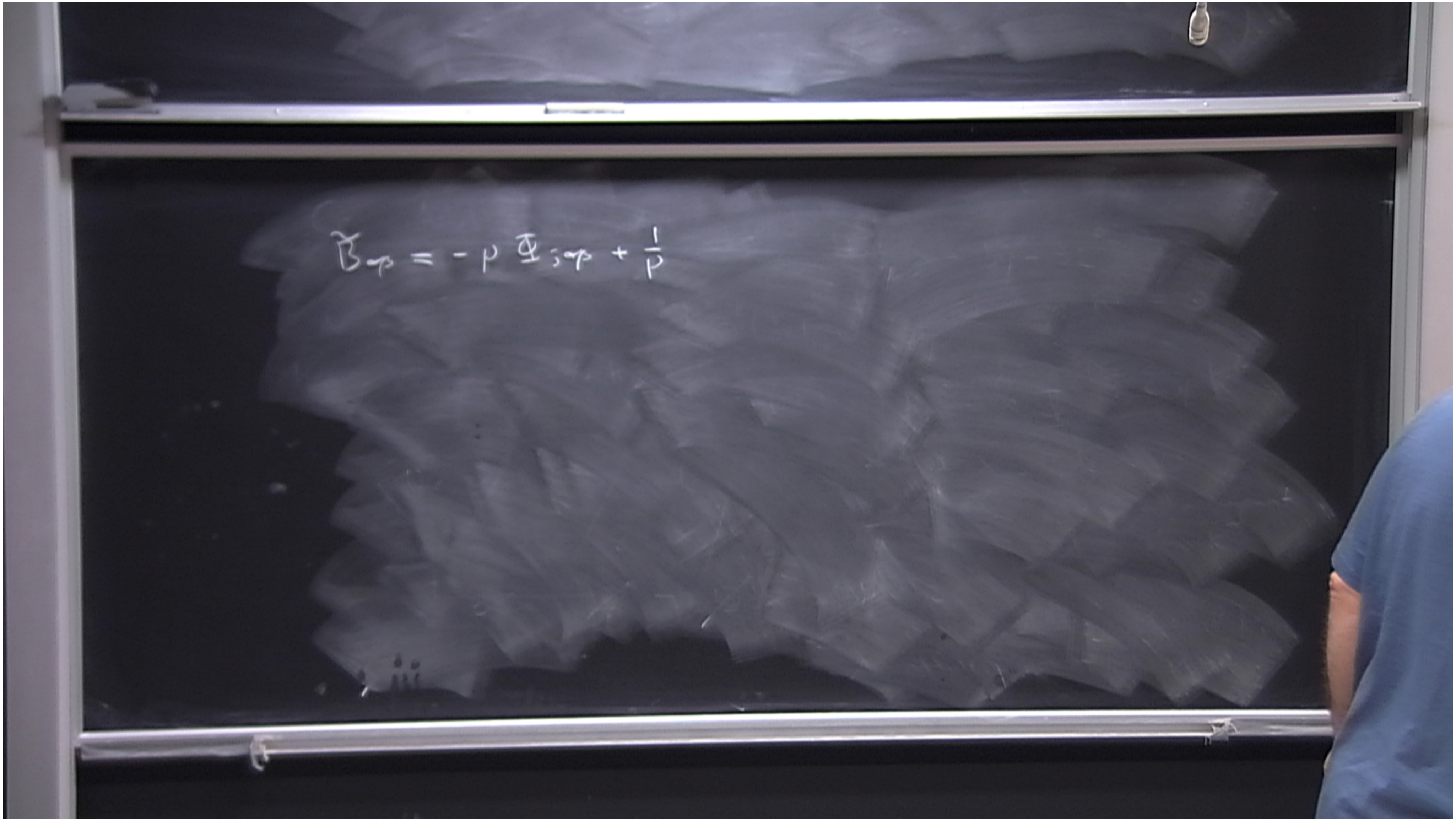


Bαρ N^ρ = - ρ Φ ;αρ N^ρ + 1/ρ Kα (∂ρρ N^ρ)
N^ν Bνβ = - ρ Φ ;νβ N^ν - 1/ρ ∂β ρ
Bμν N^μ N^ν = - ρ Φ ;μν N^μ N^ν
CAUTION

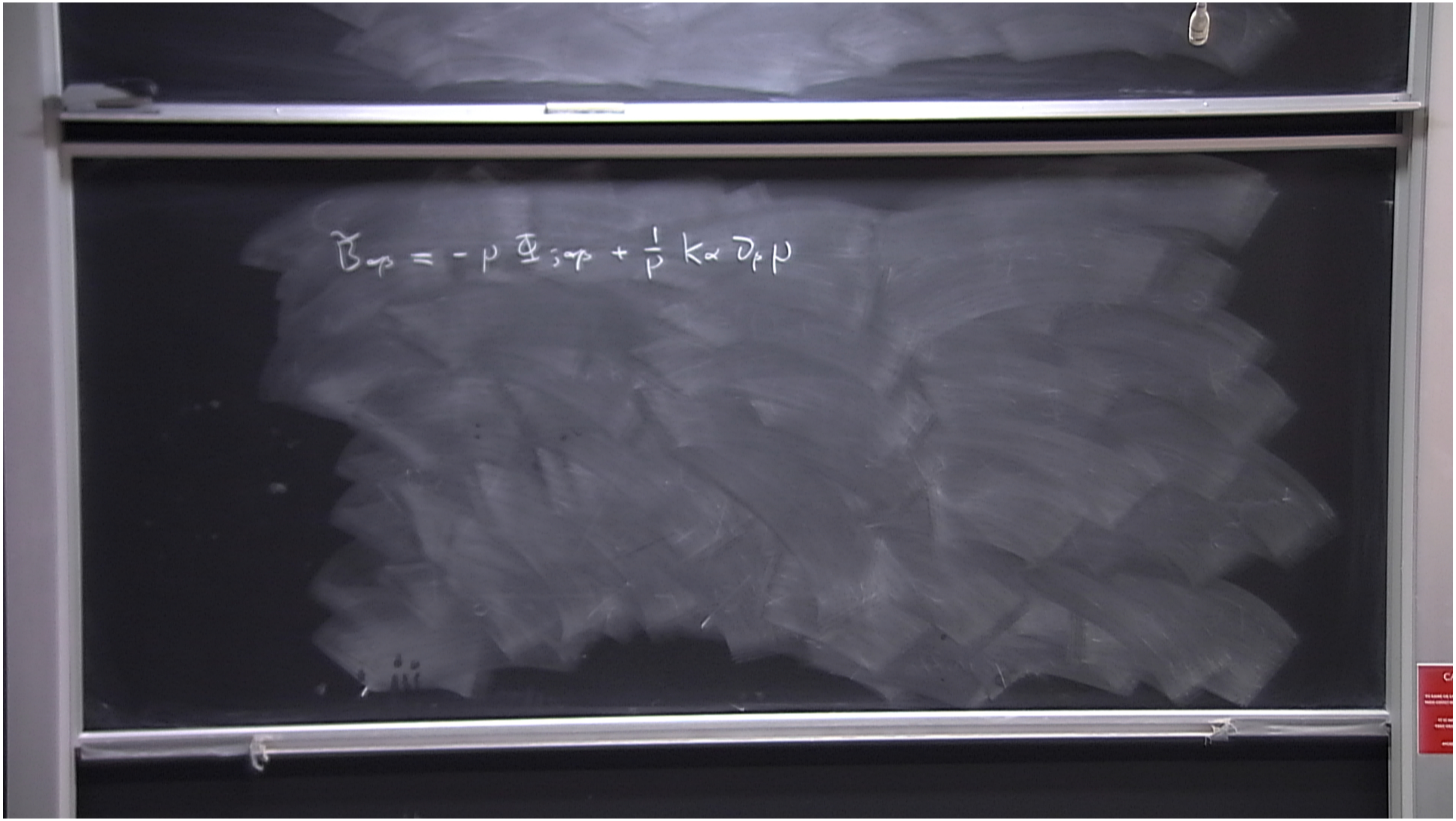

Frobenius
→ surface tracing
Congruence is "hypersurface orthogonal" if
$k_\alpha = -\rho\, \partial_\alpha \Phi$
surfaces of constant Φ
$k_{\alpha;\beta} = -\rho\, \Phi_{;\alpha\beta} - \partial_\alpha \Phi\, \partial_\beta \rho$
$B_{\alpha\beta} = -\rho\, \Phi_{;\alpha\beta} + \frac{1}{\rho} k_\alpha \partial_\beta \rho$
$B_{\alpha\mu} N^\mu = -\rho\, \Phi_{;\alpha\mu} N^\mu + \frac{1}{\rho} k_\alpha (\partial_\mu \rho\, N^\mu)$
$N^\nu B_{\nu\beta} = -\rho\, \Phi_{;\nu\beta} N^\nu - \frac{1}{\rho} \partial_\beta \rho$
$B_{\mu\nu} N^\mu N^\nu = -\rho\, \Phi_{;\mu\nu} N^\mu N^\nu - \frac{1}{\rho} (\partial_\mu$

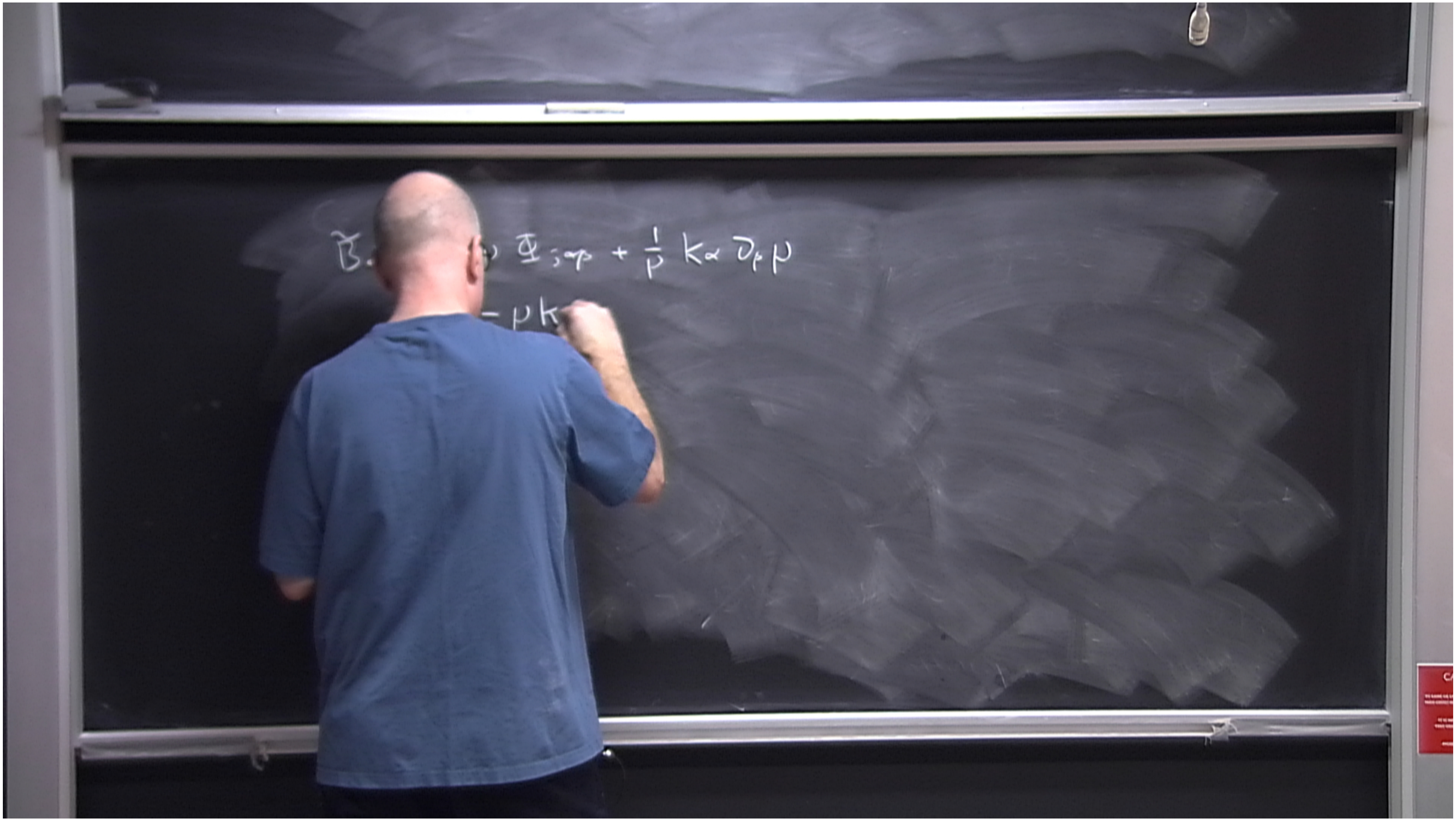

Frobenius
→ surface tracing
Congruence is "hypersurface orthogonal" if
$K_\alpha = -\mu\, \partial_\alpha \Phi$
surfaces of constant Φ
$K_{\alpha;\beta} = -\mu\, \Phi_{;\alpha\beta} - \partial_\alpha \Phi\, \partial_\beta \mu$
$B_{\alpha\beta} = -\mu\, \Phi_{;\alpha\beta} + \frac{1}{\mu} K_\alpha \partial_\beta \mu$
$B_{\alpha\rho} N^\rho = -\mu\, \Phi_{;\alpha\rho} N^\rho + \frac{1}{\mu} K_\alpha (\partial_\rho \mu\, N^\rho)$
$N^\nu B_{\nu\beta} = -\mu\, \Phi_{;\nu\beta} N^\nu - \frac{1}{\mu} \partial_\beta \mu$
$B_{\rho\nu} N^\rho N^\nu = -\mu\, \Phi_{;\rho\nu} N^\rho N^\nu - \frac{1}{\mu} (\partial_\rho \mu\, N^\rho)$
CAUTION

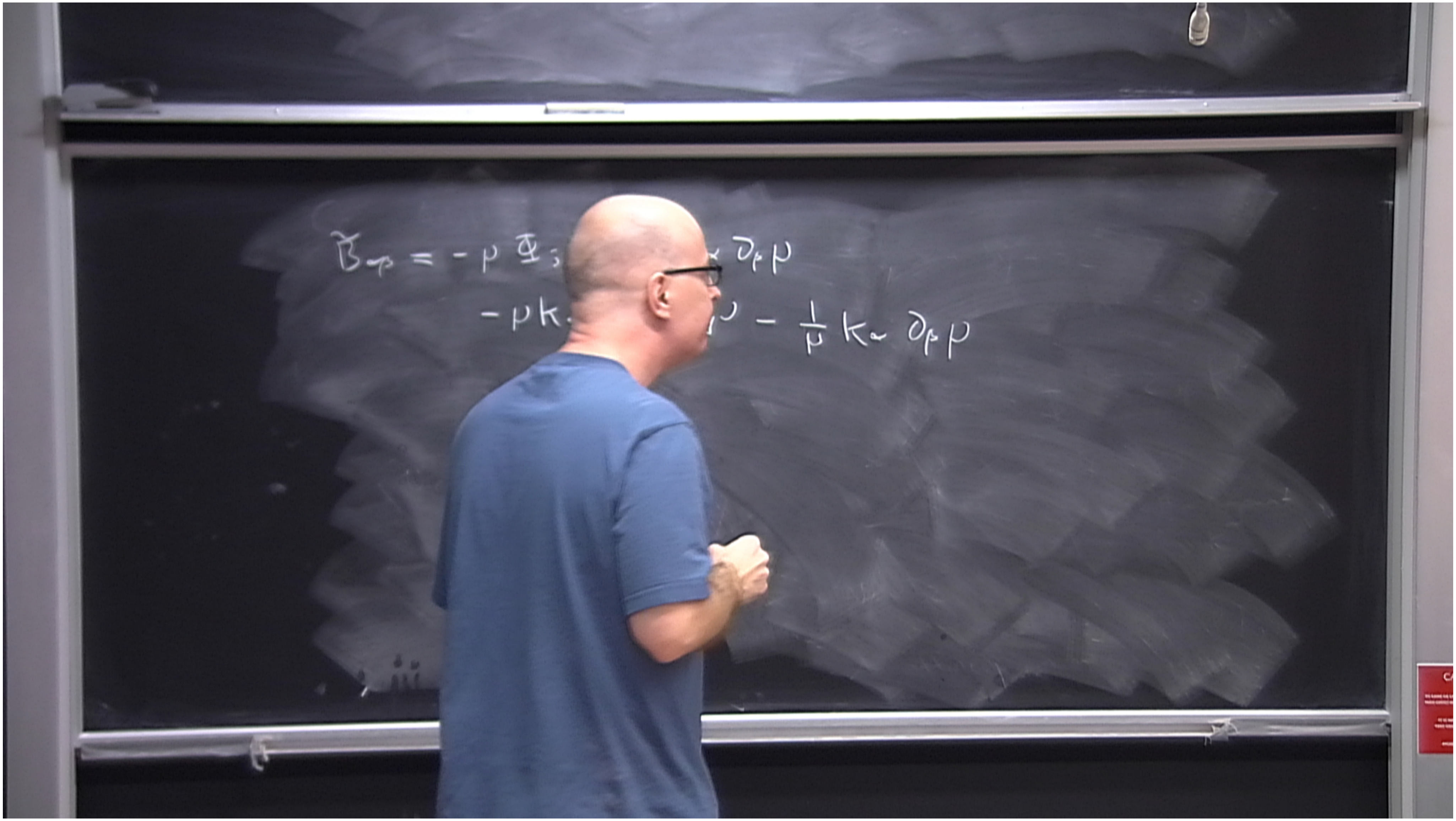

B_op =

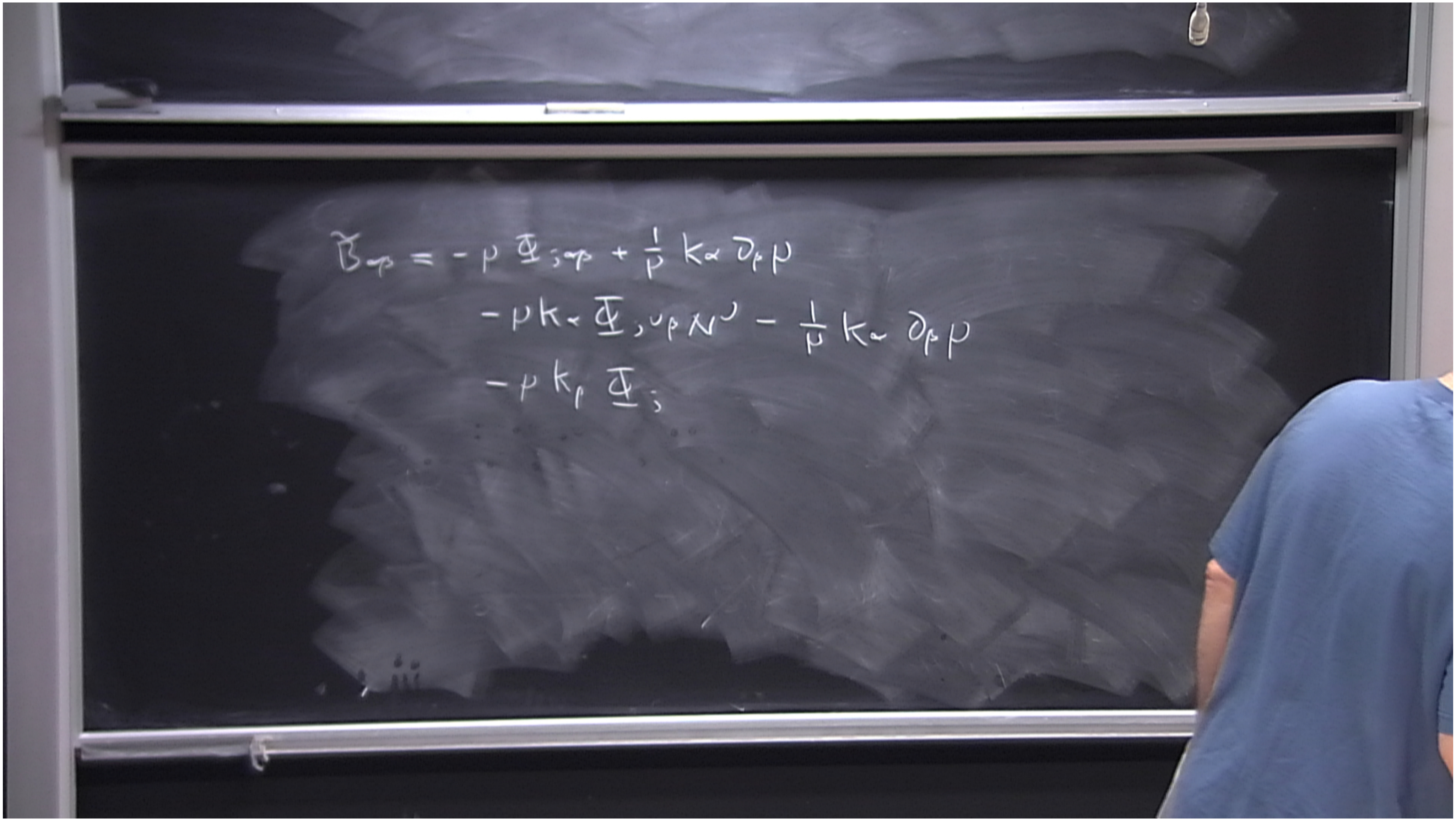


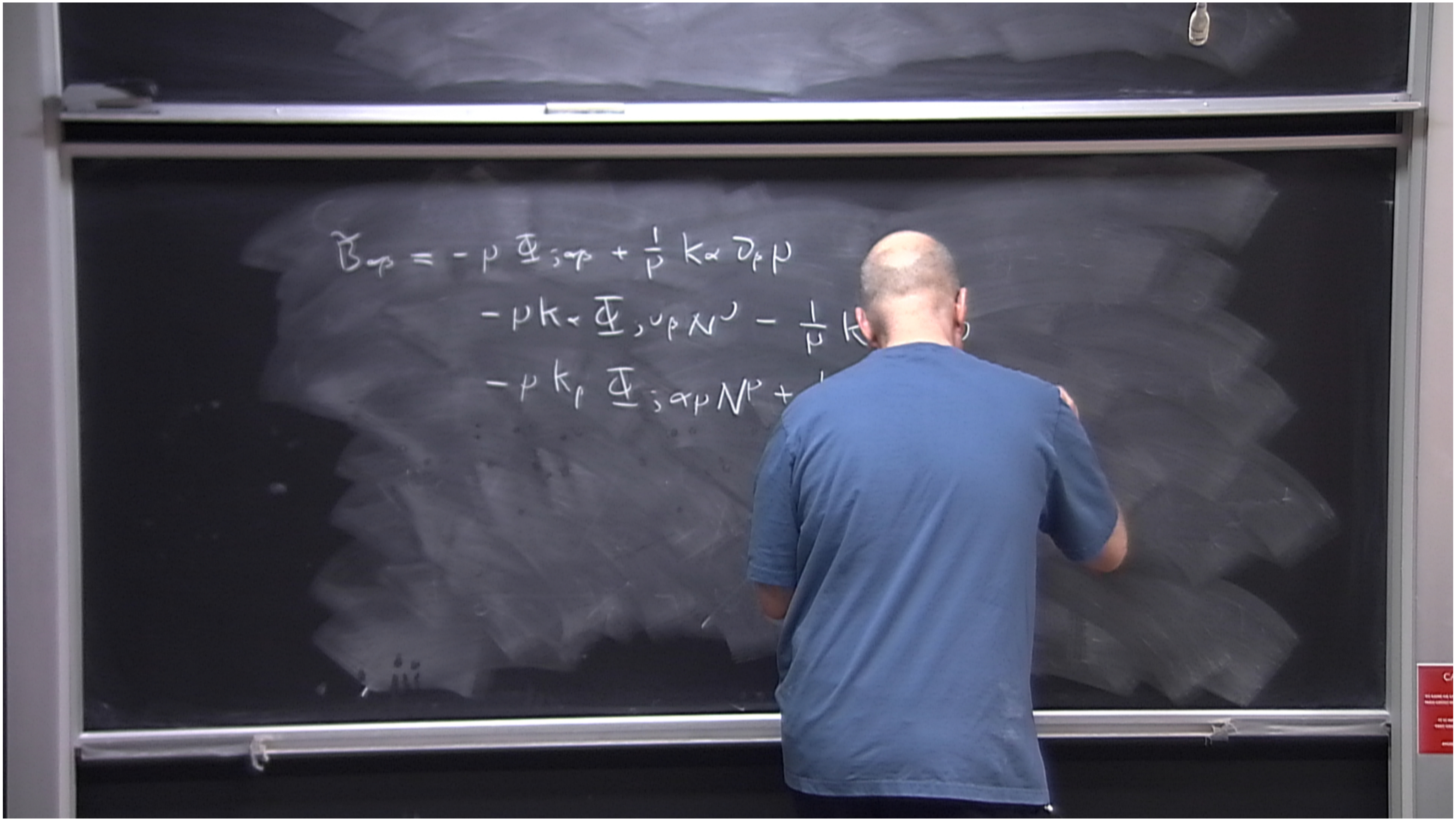
$\tilde{B}_{\alpha\beta} = -\rho\,\Phi_{;\alpha\beta} + \frac{1}{\rho}$


$\tilde{B}_{\alpha\beta} = -\rho\, \Phi_{;\alpha\beta} + \frac{1}{\rho} k_\alpha \partial_\beta \rho$






$\mathcal{B}_{\alpha\beta} = -\rho\, \Phi_{;\alpha\beta} + \frac{1}{\rho} k_\alpha \partial_\beta \rho$
$- \rho k_\alpha \Phi_{,\nu\beta} N^\nu - \frac{1}{\rho} k_\alpha \partial_\beta \rho$
$- \rho k_\beta \Phi_{;}$

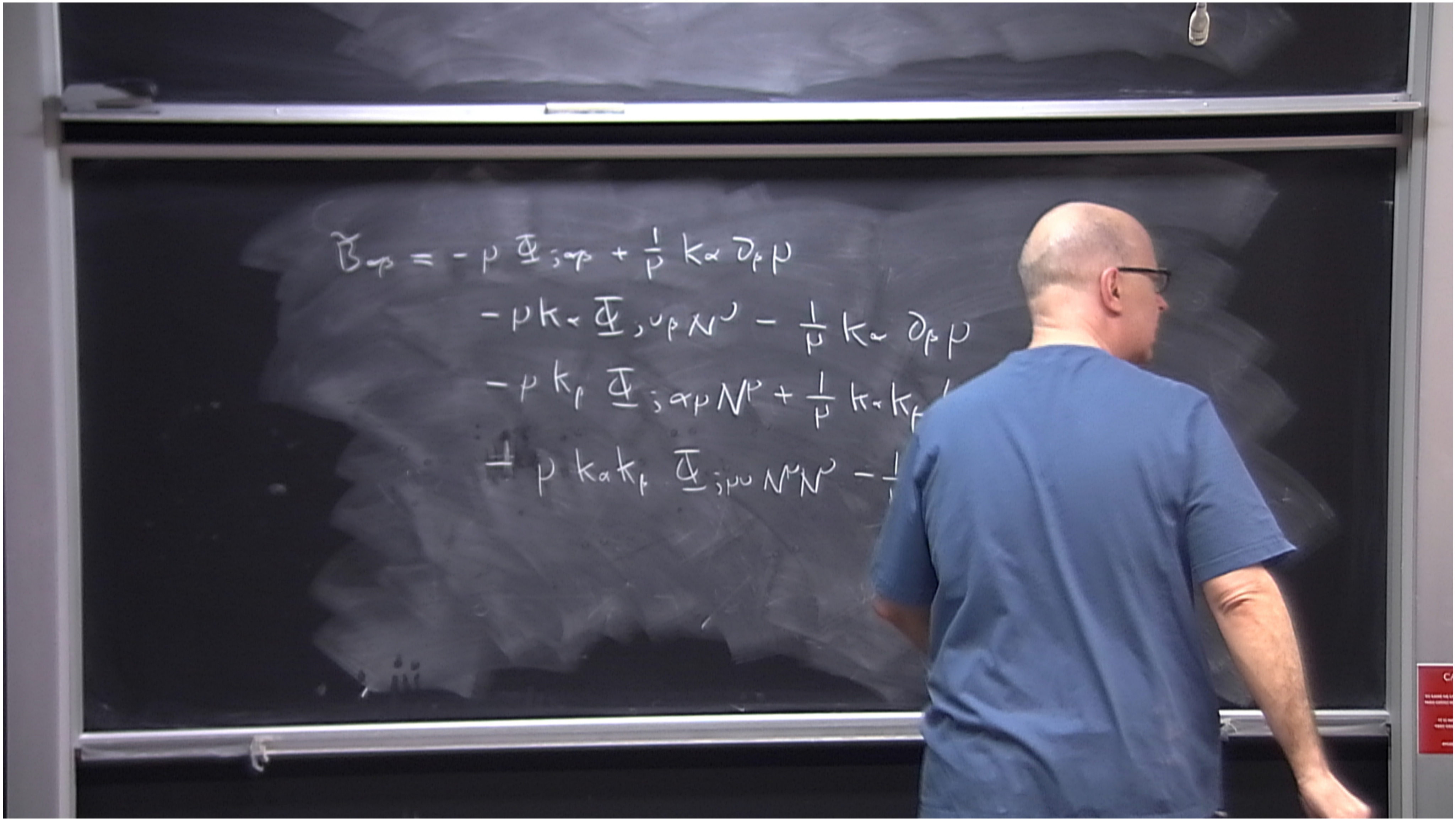

$B_{\alpha\beta} = -\rho\, \Phi_{;\alpha\beta} + \frac{1}{\rho} k_\alpha \partial_\beta \rho$
$-\rho k_\alpha \Phi_{,\nu\beta} N^\nu - \frac{1}{\rho}$
$-\rho k_\beta \Phi_{;\alpha\mu} N^\mu +$

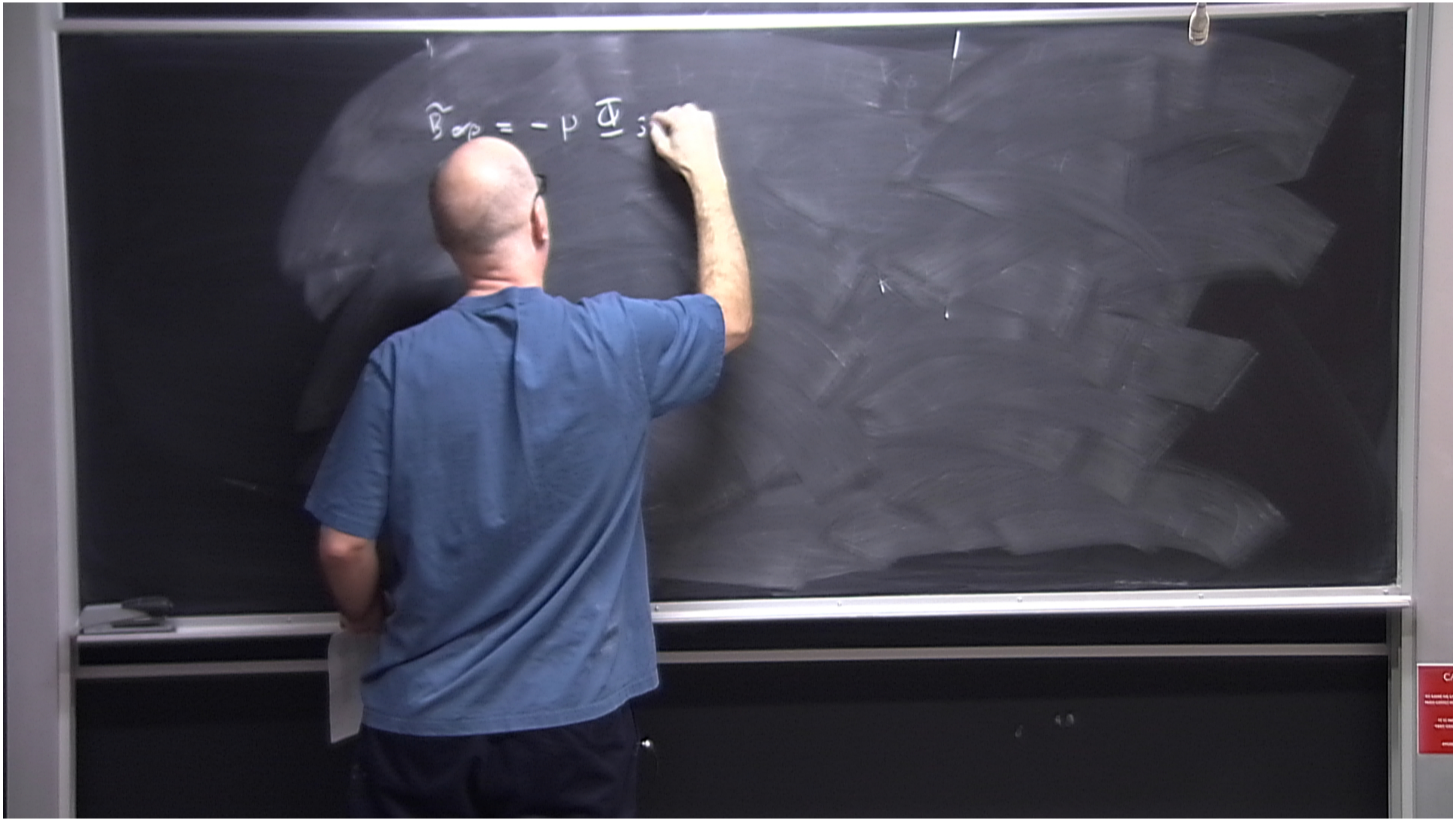
$$\tilde{B}_{\alpha\beta} = -\rho\, \Phi_{;\alpha\beta} + \frac{1}{\rho} k_\alpha \partial_\beta \rho$$

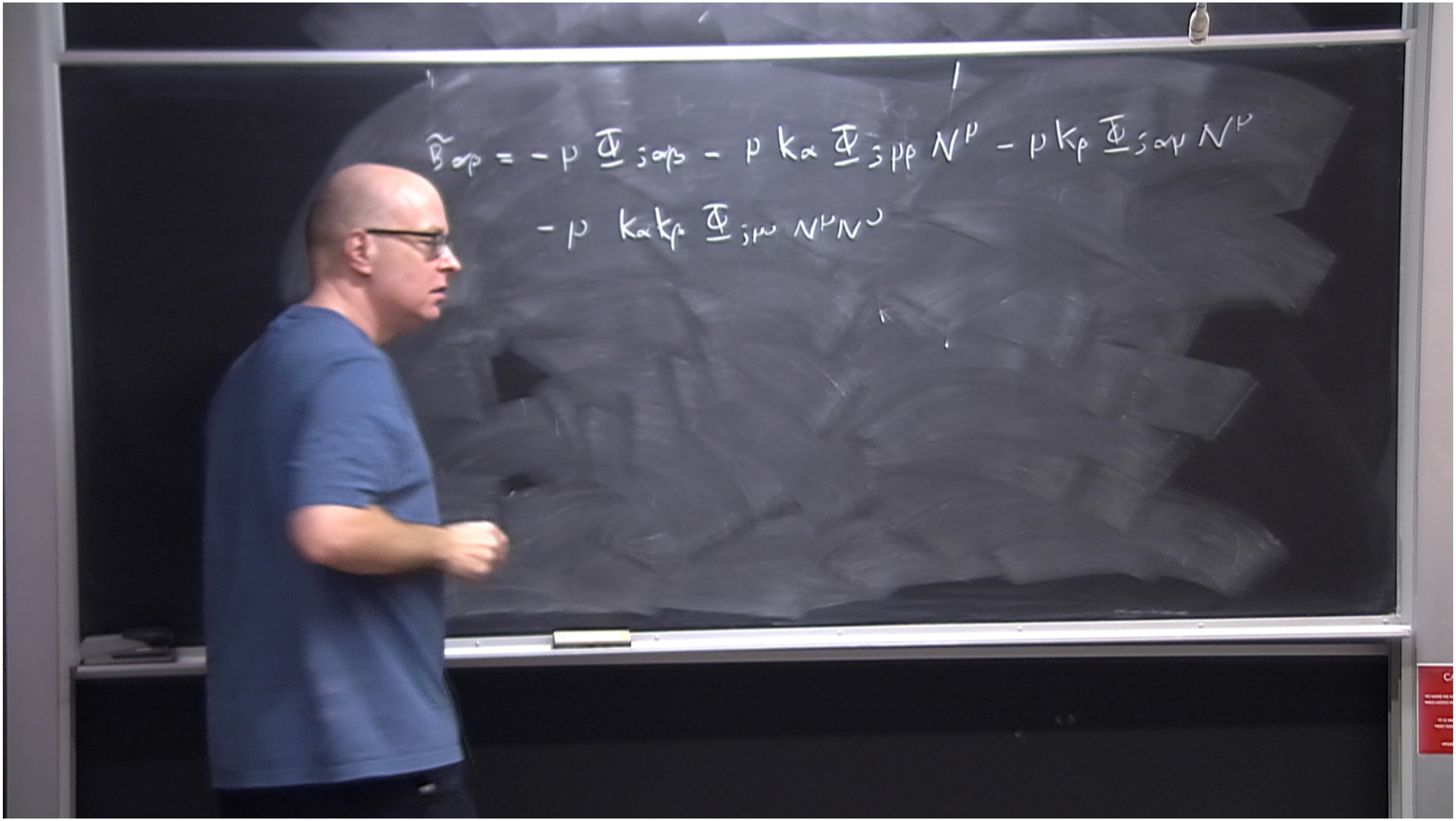
$$- \rho k_\alpha \Phi_{,\nu\beta} N^\nu - \frac{1}{\rho} k_\alpha \partial_\beta \rho$$

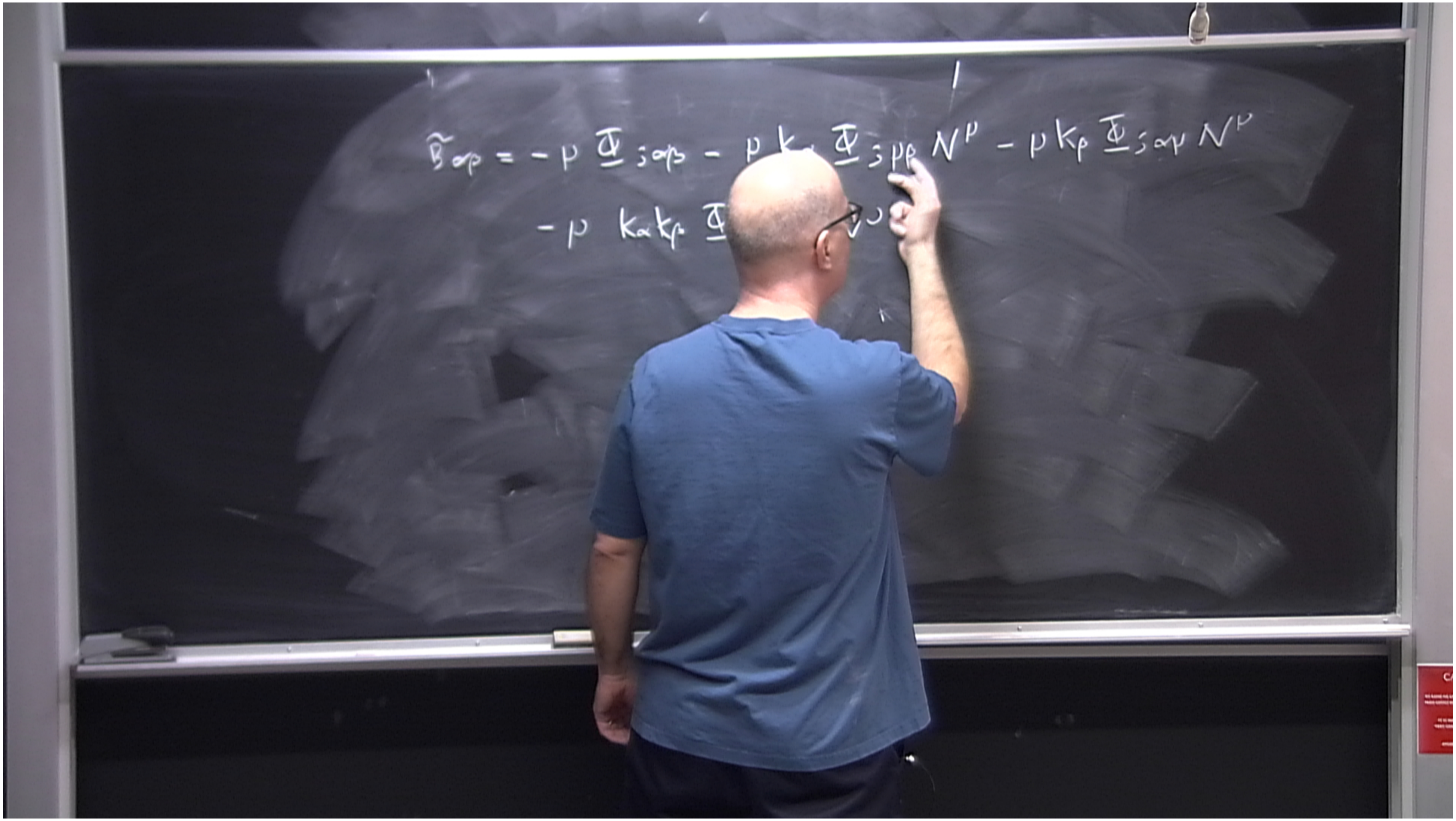
$$- \rho k_\beta \Phi_{;\alpha\mu} N^\mu + \frac{1}{\rho} k_\alpha k_\beta \left(\partial_\mu \rho N^\mu \right)$$

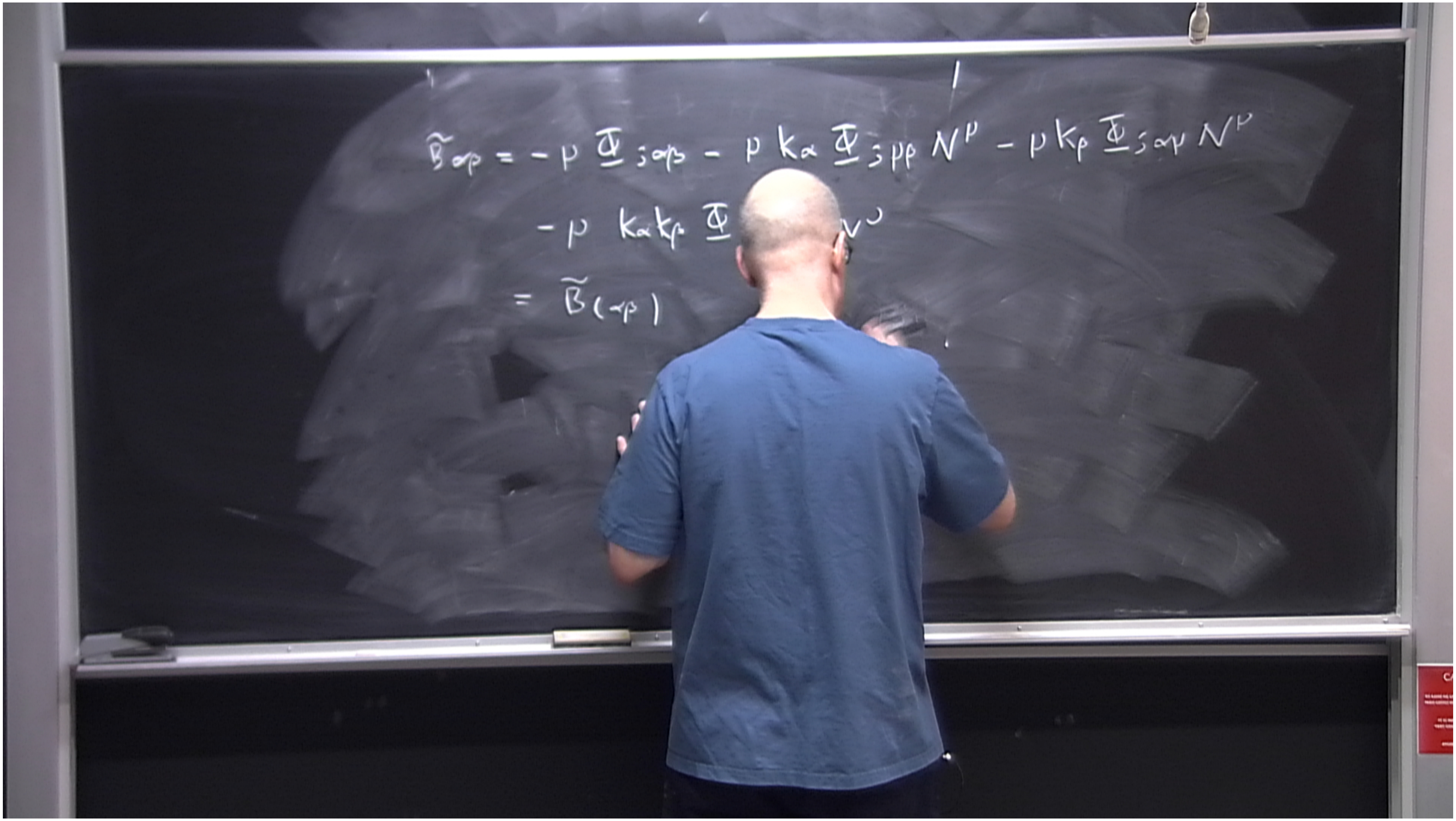
$$+ \rho k_\alpha k_\beta \Phi_{;\mu\nu} N^\mu N^\nu$$

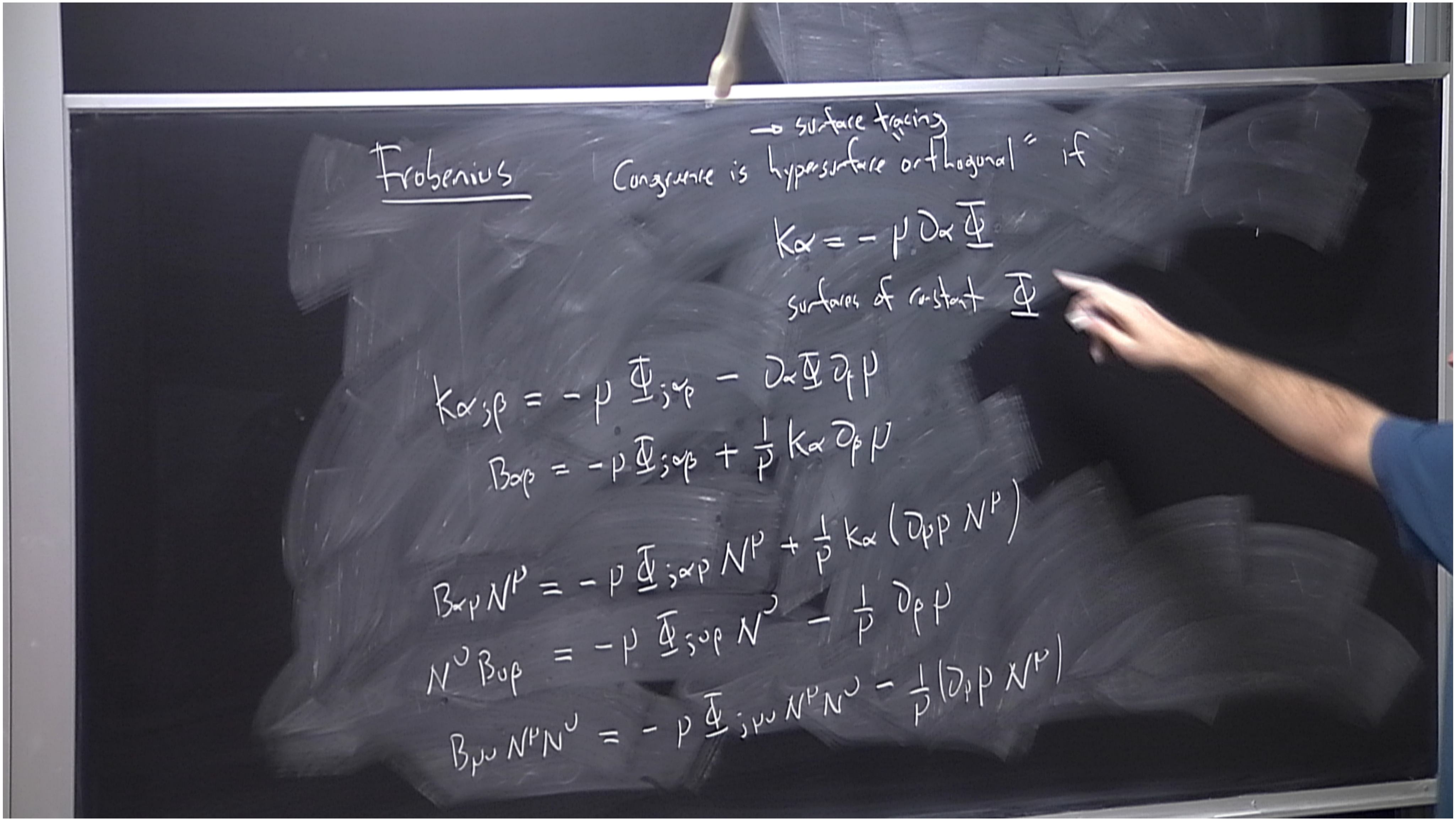

$B_{\alpha\beta} = -\rho\, \Phi_{;\alpha\beta} + \frac{1}{\rho} k_\alpha \partial_\beta \rho$
$- \rho k_\alpha \Phi_{;\nu\beta} N^\nu - \frac{1}{\rho} k_\alpha \partial_\beta \rho$
$- \rho k_\beta \Phi_{;\alpha\mu} N^\mu + \frac{1}{\rho} k_\alpha k_\beta$
$+ \rho k_\alpha k_\beta \Phi_{;\mu\nu} N^\mu N^\nu -$

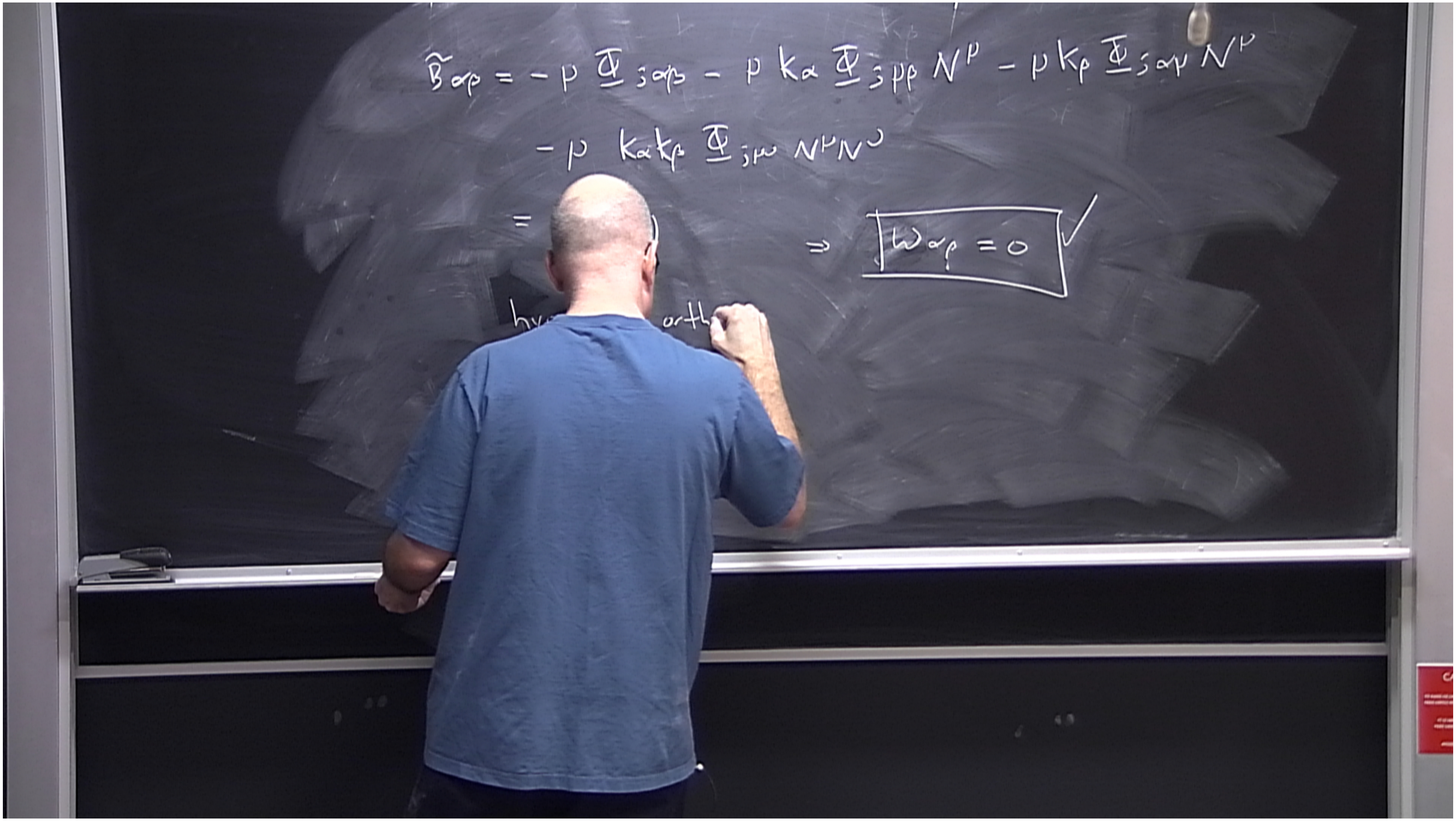

$\widehat{\tilde{B}}_{\alpha\beta} = -\mu \underline{\Phi}$

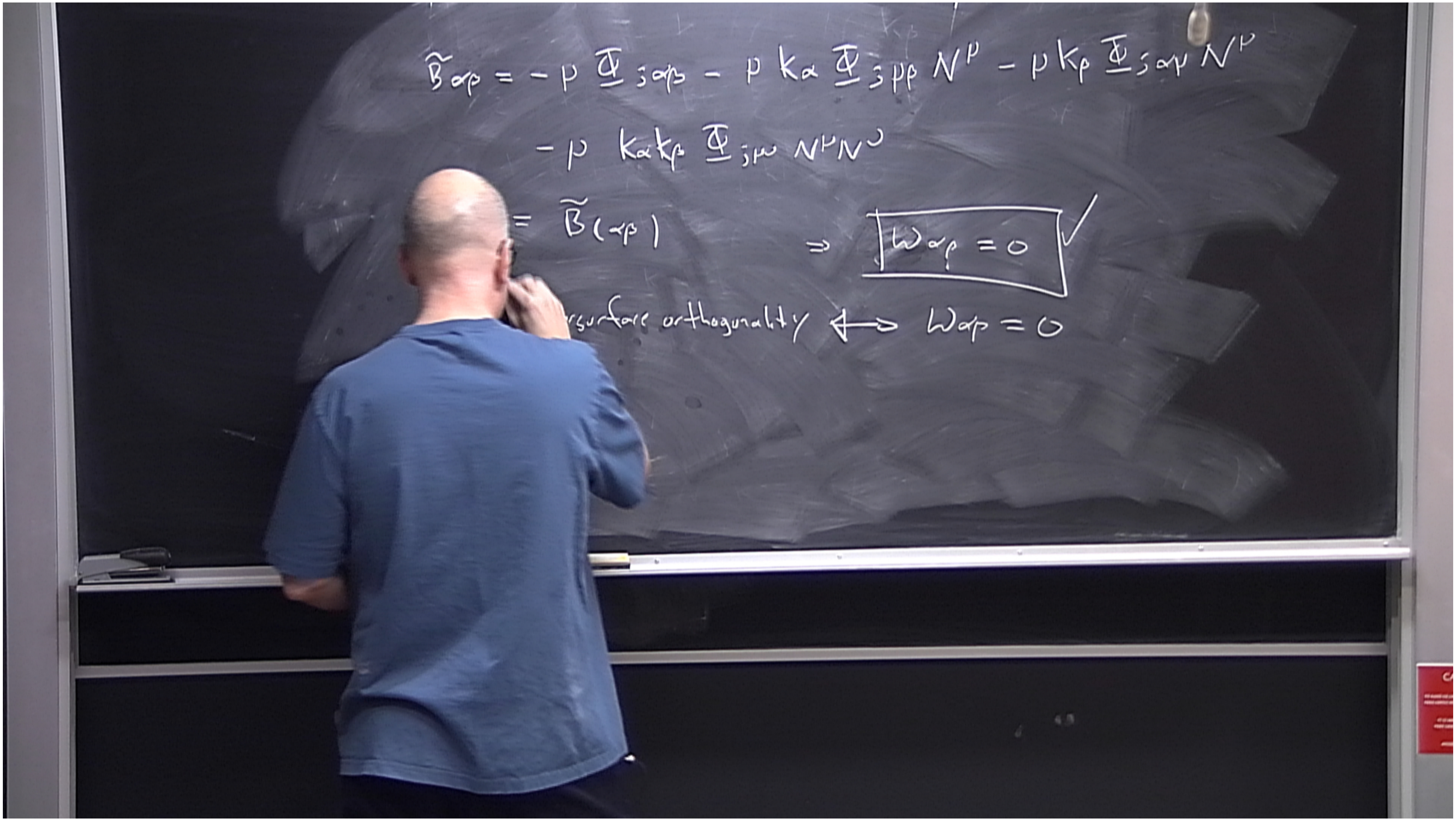

$\tilde{B}_{\alpha\beta} = -\rho\, \underline{\Phi}_{;\alpha\beta} - \rho\, k_\alpha \underline{\Phi}_{;\mu\beta} N^\mu - \rho\, k_\beta \underline{\Phi}_{;\alpha\mu} N^\mu$
$- \rho\, k_\alpha k_\beta \underline{\Phi}_{;\mu\nu} N^\mu N^\nu$

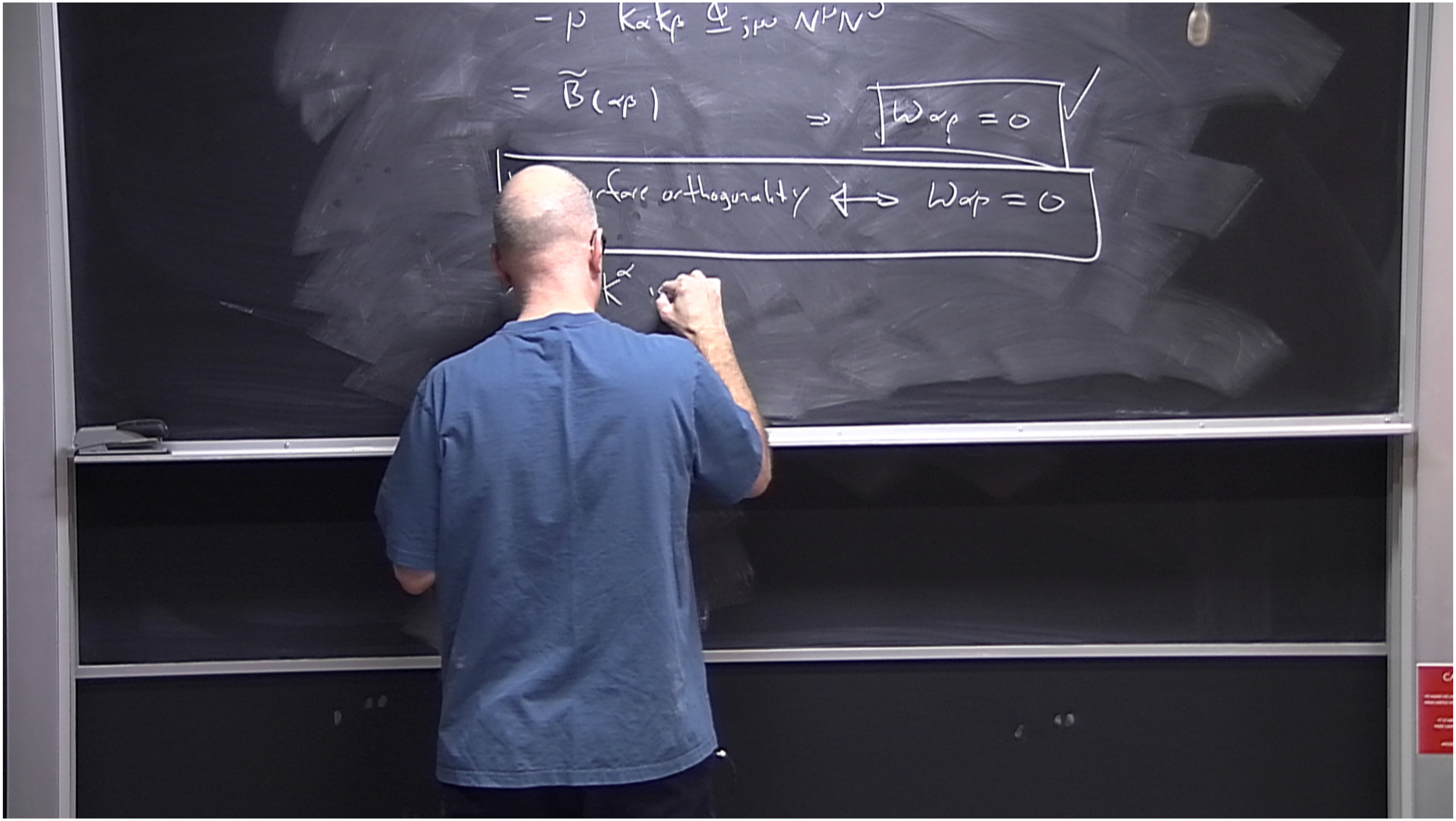

B̃αβ = −μ Φ;αβ − μ kα Φ;μβ N^μ − μ kβ Φ;αμ N^μ
−μ kα kβ Φ

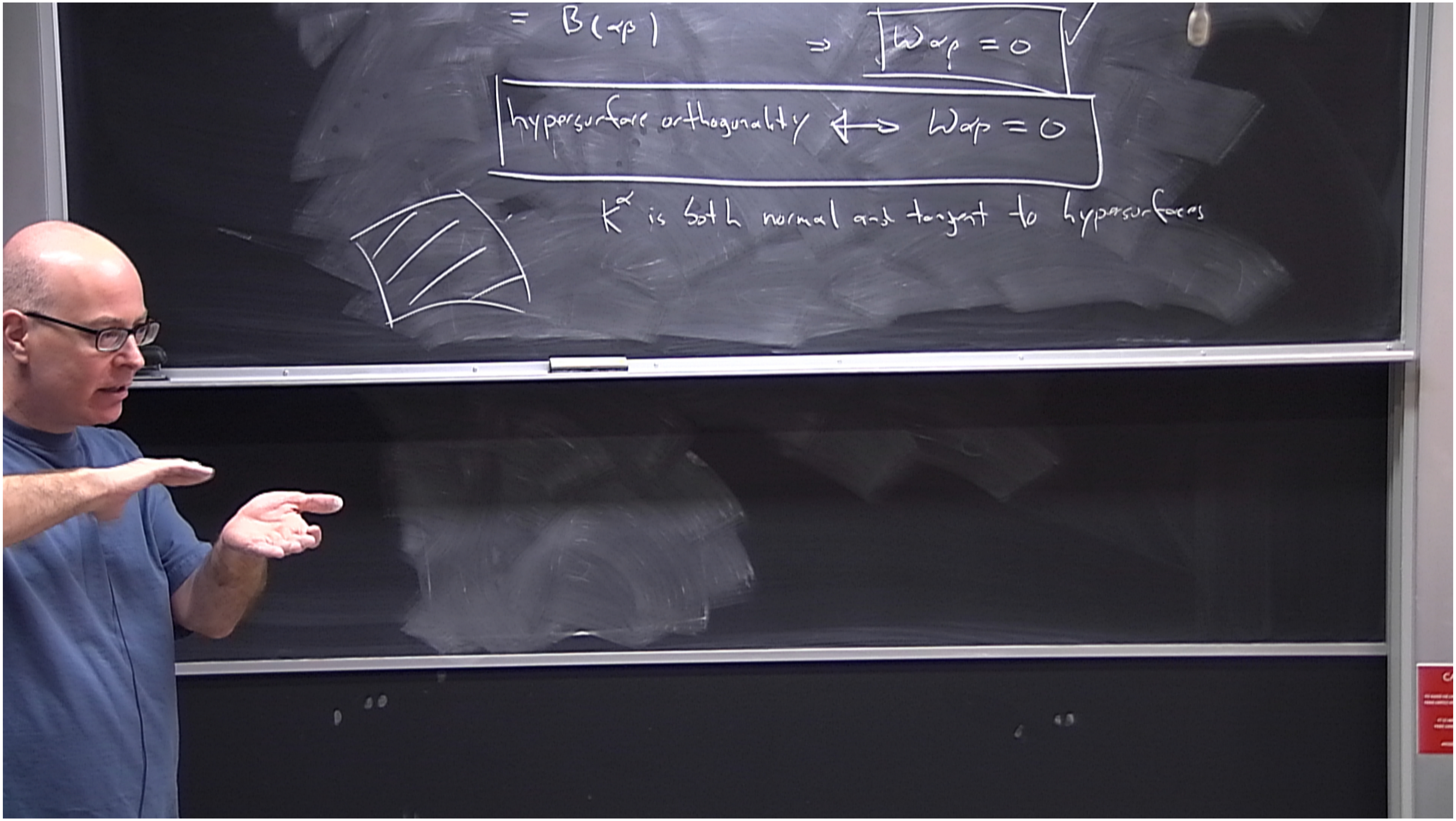

$\tilde{B}_{\alpha\beta} = -\mu\, \Phi_{;\alpha\beta} - \mu\, k_\alpha \Phi_{;\mu\beta} N^\mu - \mu\, k_\beta \Phi_{;\alpha\mu} N^\mu$
$-\mu\, k_\alpha k_\beta\, \Phi$
$= \tilde{B}_{(\alpha\beta)}$

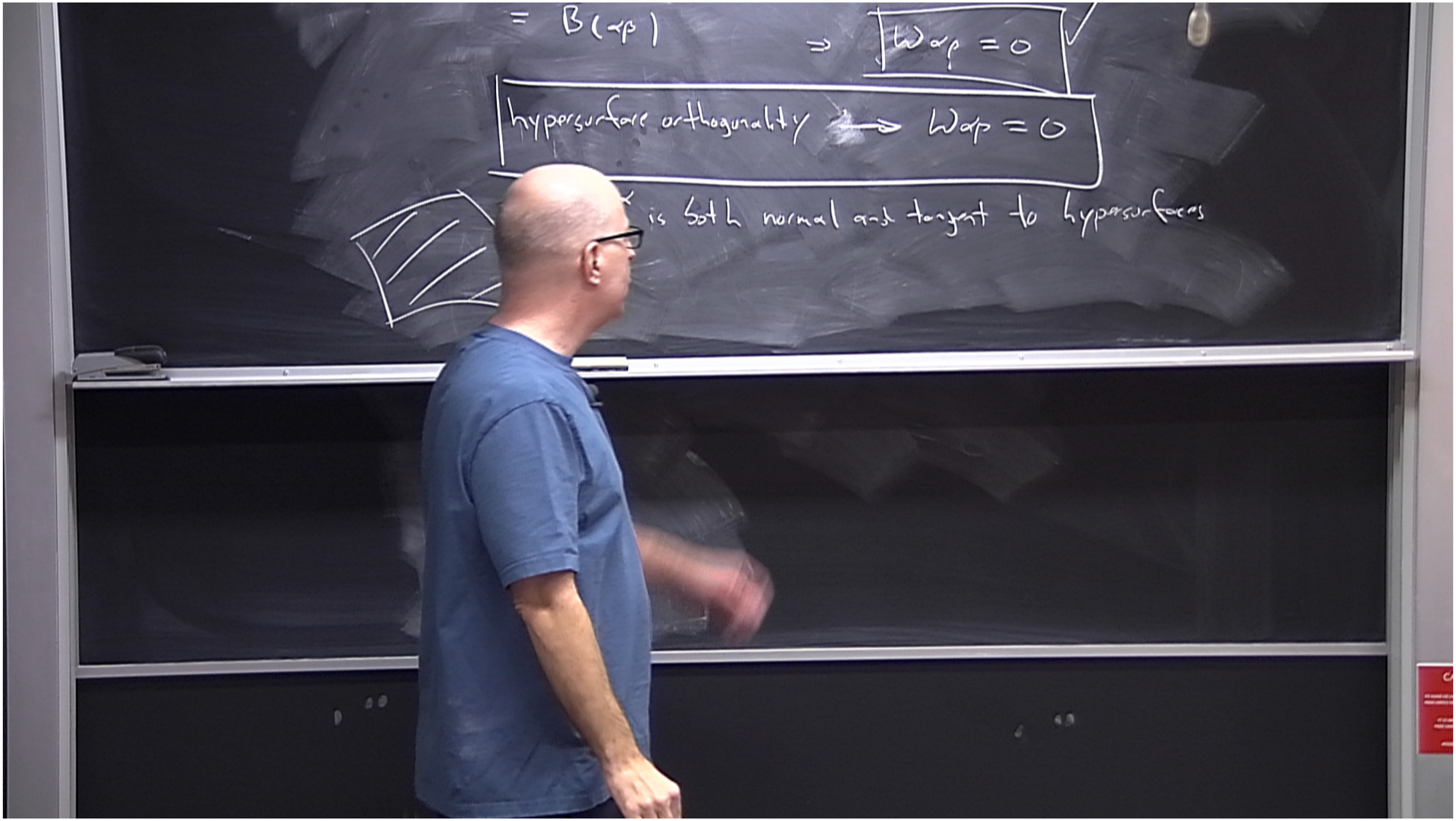

→ surface tracing
Frobenius
Congruence is "hypersurface orthogonal" if
Kα = - μ ∂α Φ
surfaces of constant Φ
Kα;β = - μ Φ;αβ - ∂α Φ ∂β μ
Bαβ = - μ Φ;αβ + 1/μ Kα ∂β μ
Bαβ N^β = - μ Φ;αβ N^β + 1/μ Kα (∂β μ N^β)
N^α Bαβ = - μ Φ;αβ N^α - 1/μ ∂β μ
Bμν N^μ N^ν = - μ Φ;μν N^μ N^ν - 1/μ (∂μ μ N^μ)

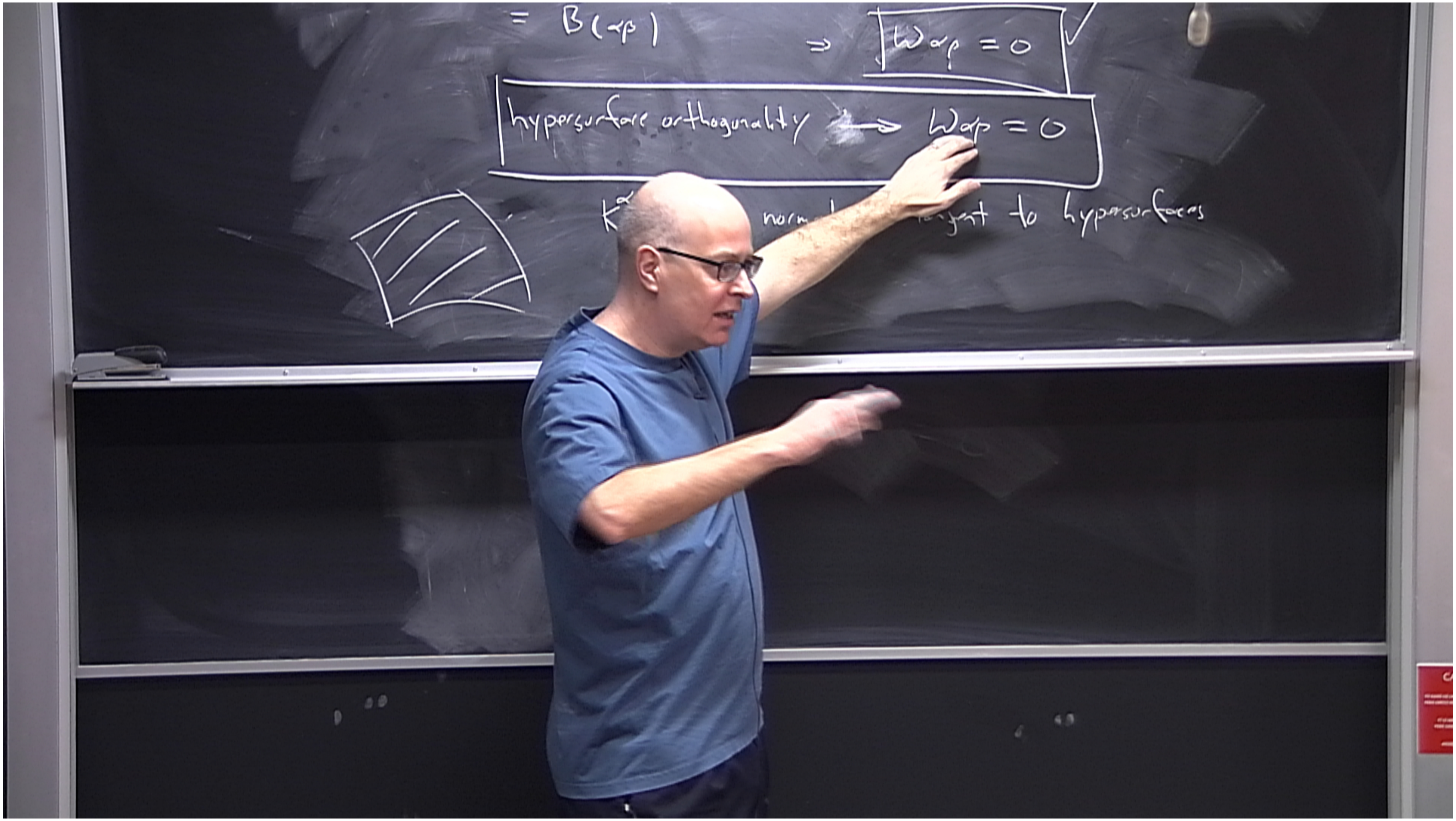

$\tilde{B}_{\alpha\beta} = -\mu \underline{\Phi}_{;\alpha\beta} - \mu k_\alpha \underline{\Phi}_{;\mu\beta} N^\mu - \mu k_\beta \underline{\Phi}_{;\alpha\mu} N^\mu$
$-\mu k_\alpha k_\beta \underline{\Phi}_{;\mu\nu} N^\mu N^\nu$
$\Rightarrow \omega_{\alpha\beta} = 0$

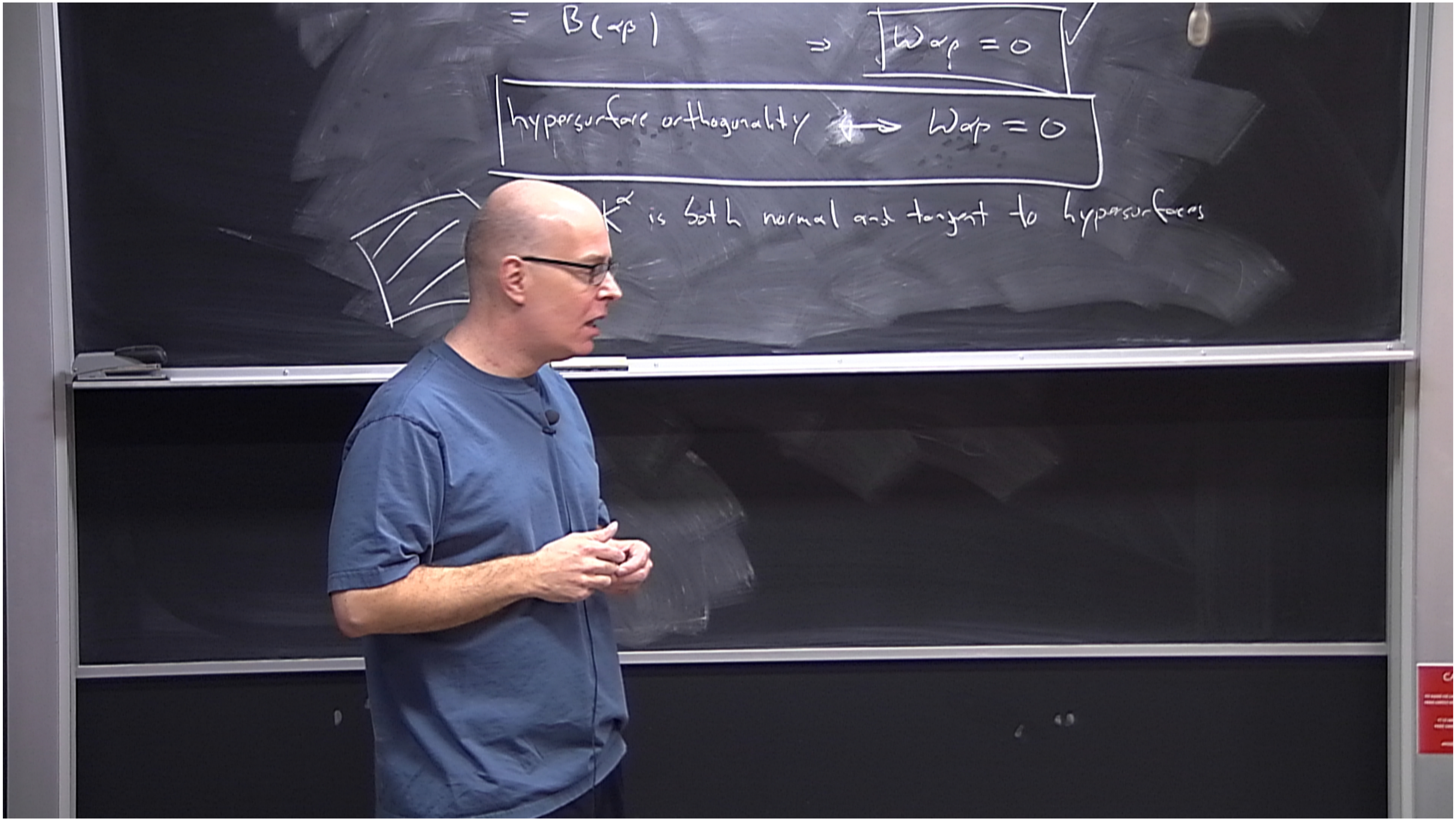

$\tilde{B}_{\alpha\beta} = -\mu \Phi_{;\alpha\beta} - \mu k_\alpha \Phi_{;\mu\beta} N^\mu - \mu k_\beta \Phi_{;\alpha\mu} N^\mu$
$- \mu k_\alpha k_\beta \Phi_{;\mu\nu} N^\mu N^\nu$
$= \tilde{B}_{(\alpha\beta)} \Rightarrow W_{\alpha\beta} = 0$
rsurface orthogonality $\Leftrightarrow W_{\alpha\beta} = 0$

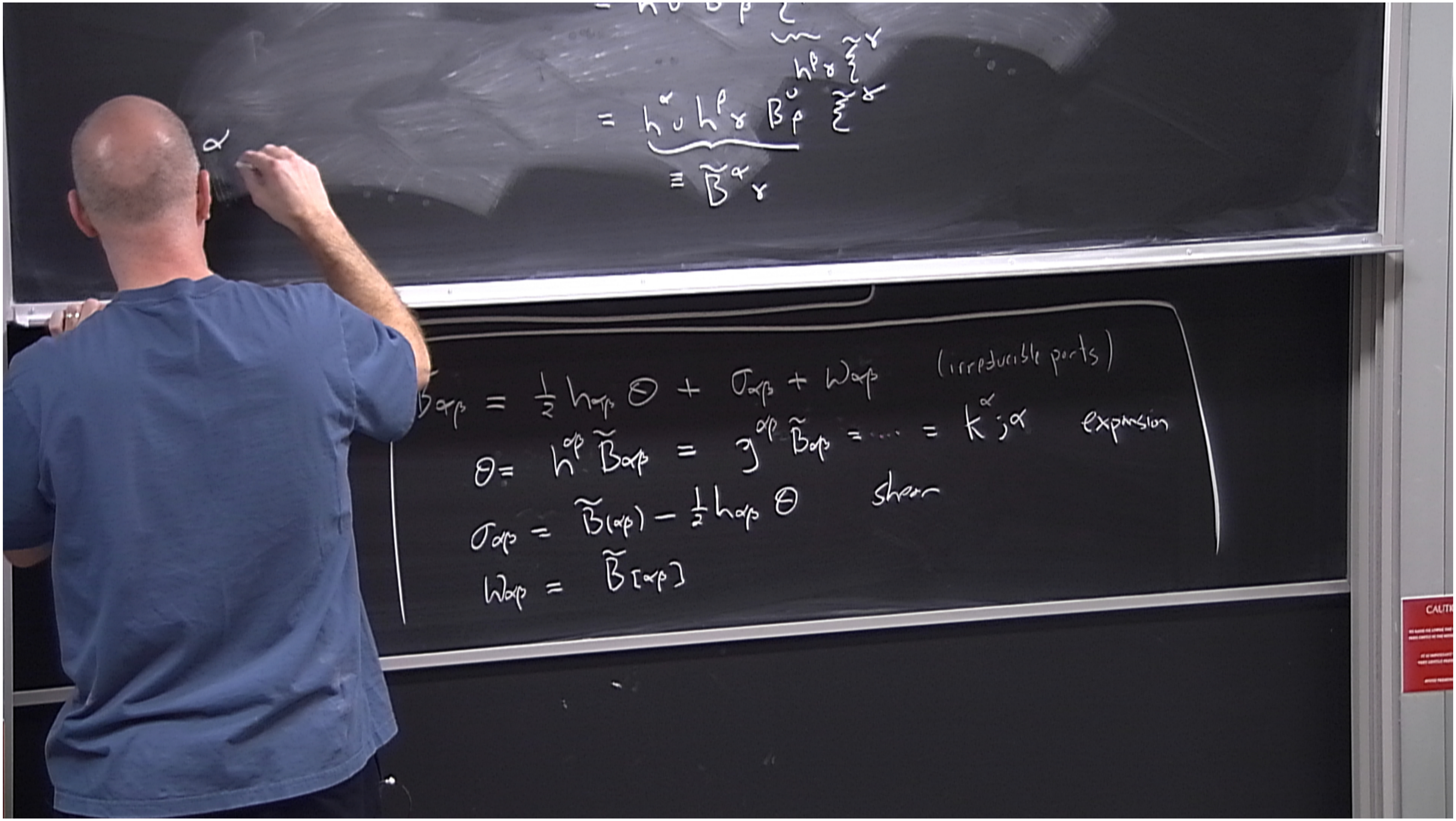

= B̃(αβ)
⇒ Wαβ = 0
orthogonality ⟺ Wαβ = 0

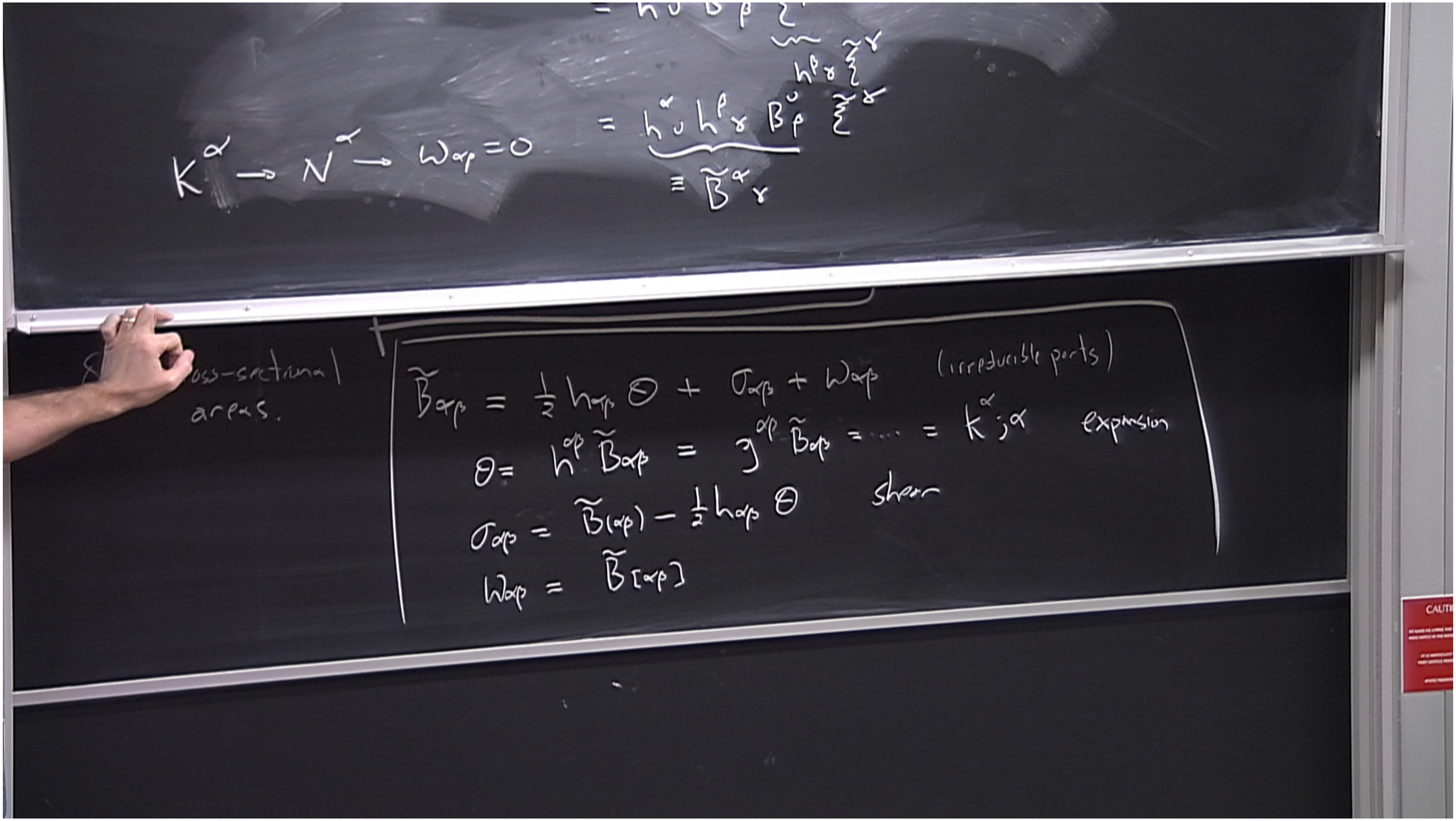

= B(αβ)
⇒ Wαβ = 0
hypersurface orthogonality ⟺ Wαβ = 0
K^α is both normal and tangent to hypersurfaces

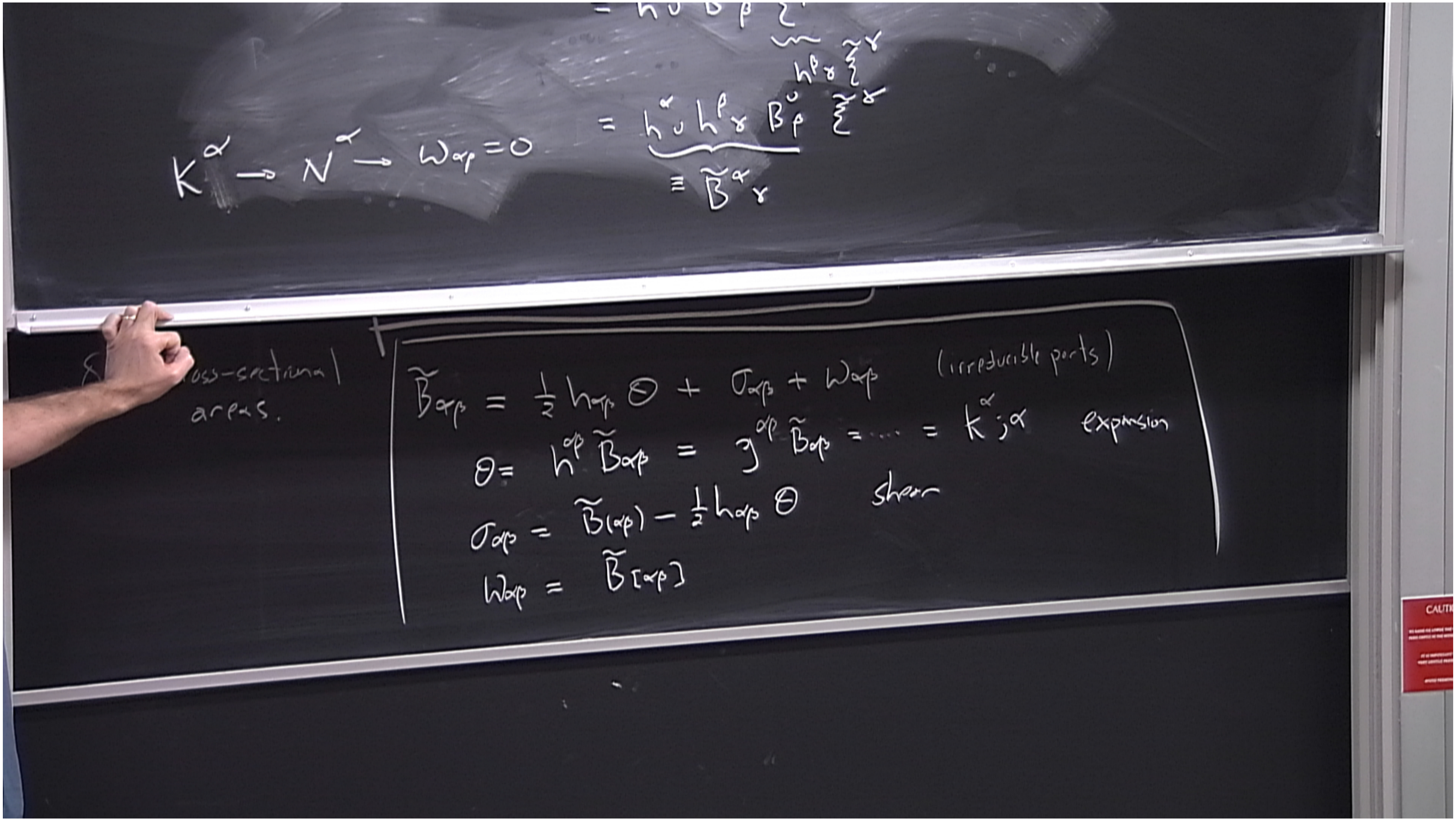

= B(αβ)
⇒ Wαβ = 0
hypersurface orthogonality ⟺ Wαβ = 0
is both normal and tangent to hypersurfaces

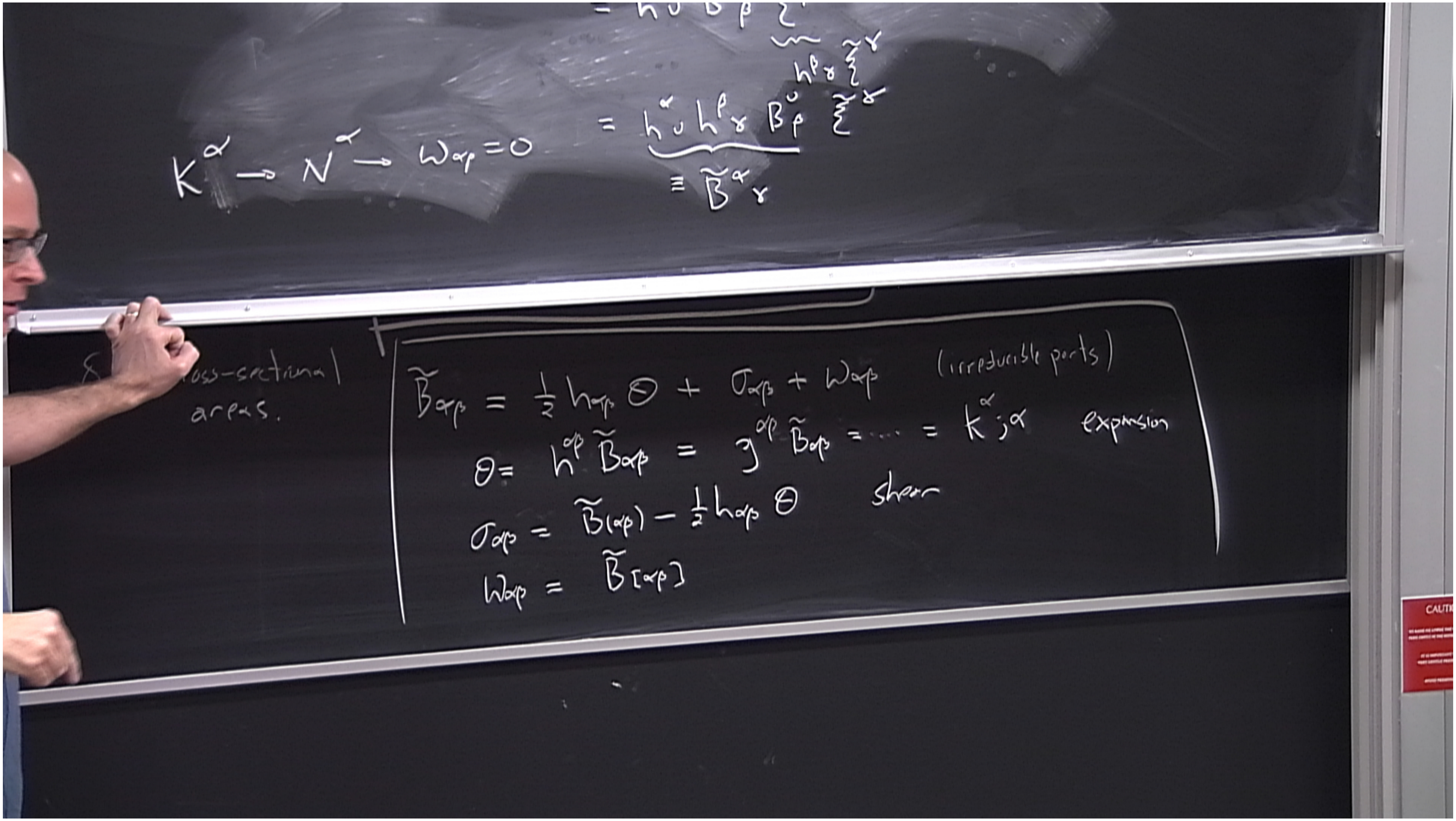

= B(αβ)
⇒ Wαβ = 0
hypersurface orthogonality → Wαβ = 0
tangent to hypersurfaces

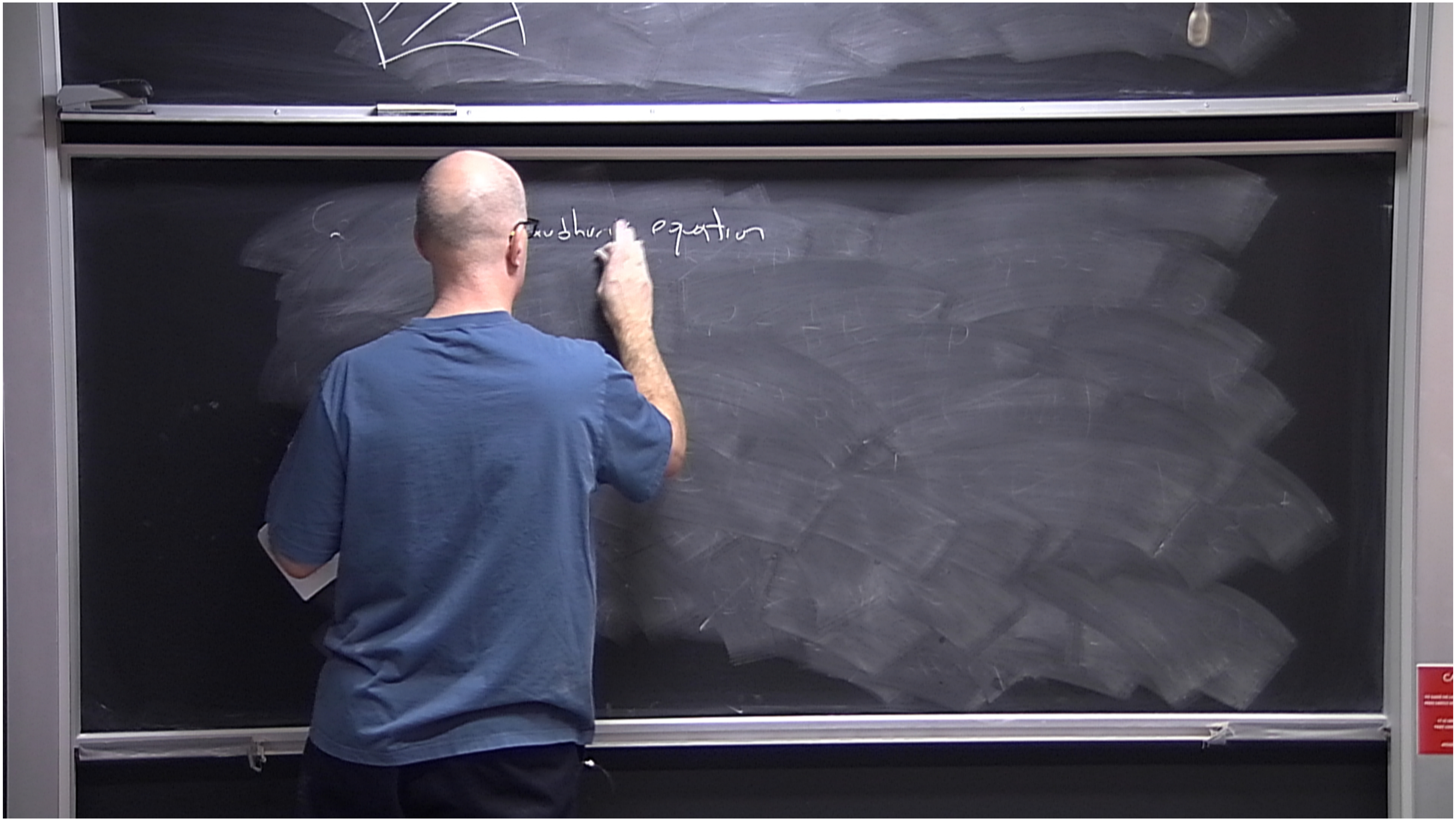

= B(αβ)
⇒ Wαβ = 0
hypersurface orthogonality ⟺ Wαβ = 0
Kα is both normal and tangent to hypersurfaces

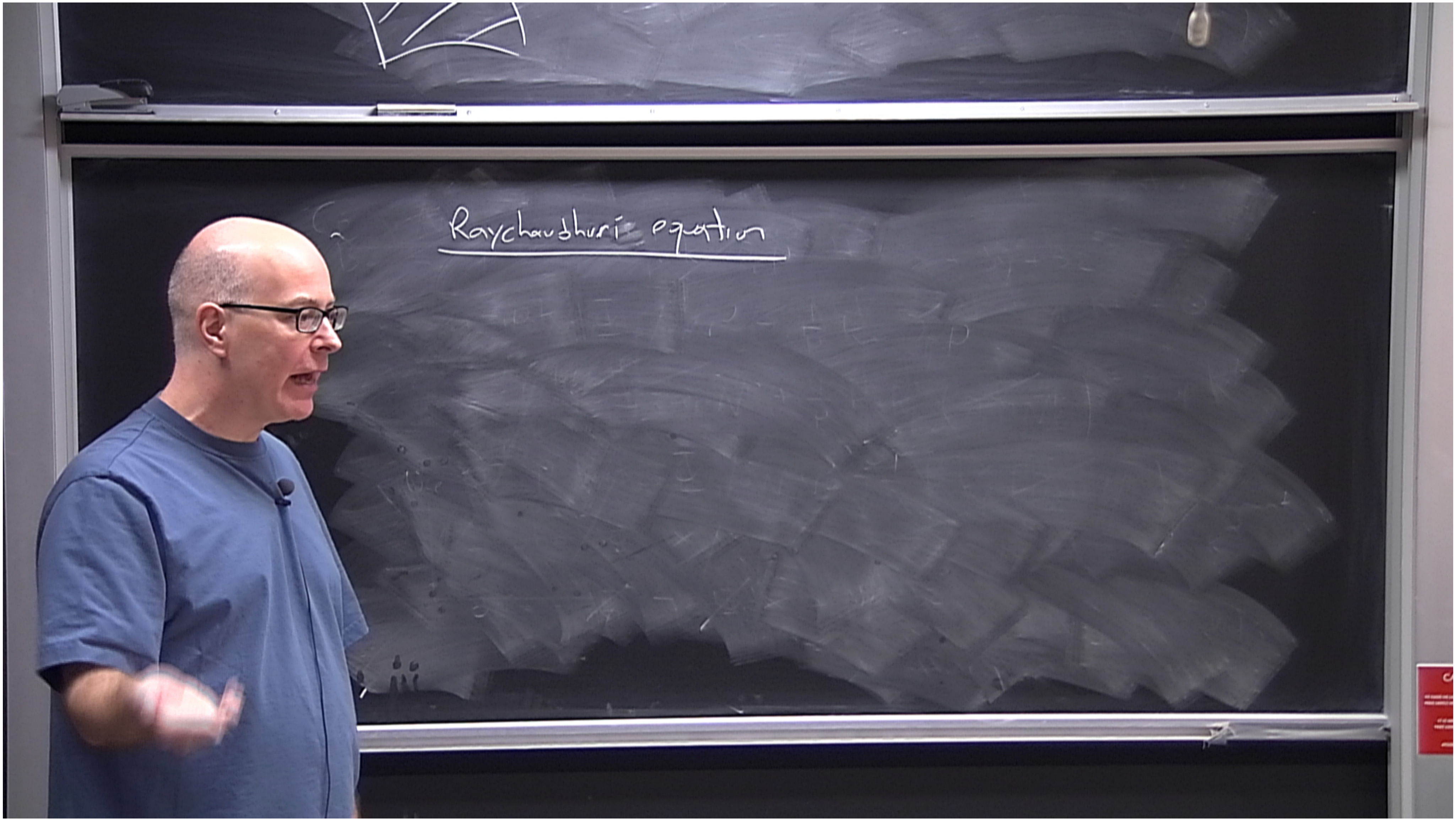

h^ρ_γ ξ̃^γ
= h^α_ν h^ρ_γ B^ν_ρ ξ̃^γ
≡ B̃^α_γ
(irreducible parts)
B_αβ = ½ h_αβ θ + σ_αβ + ω_αβ
θ = h^αβ B̃_αβ = g^αβ B̃_αβ = ... = k^α_;α expansion
σ_αβ = B̃_(αβ) − ½ h_αβ θ shear
ω_αβ = B̃_[αβ]

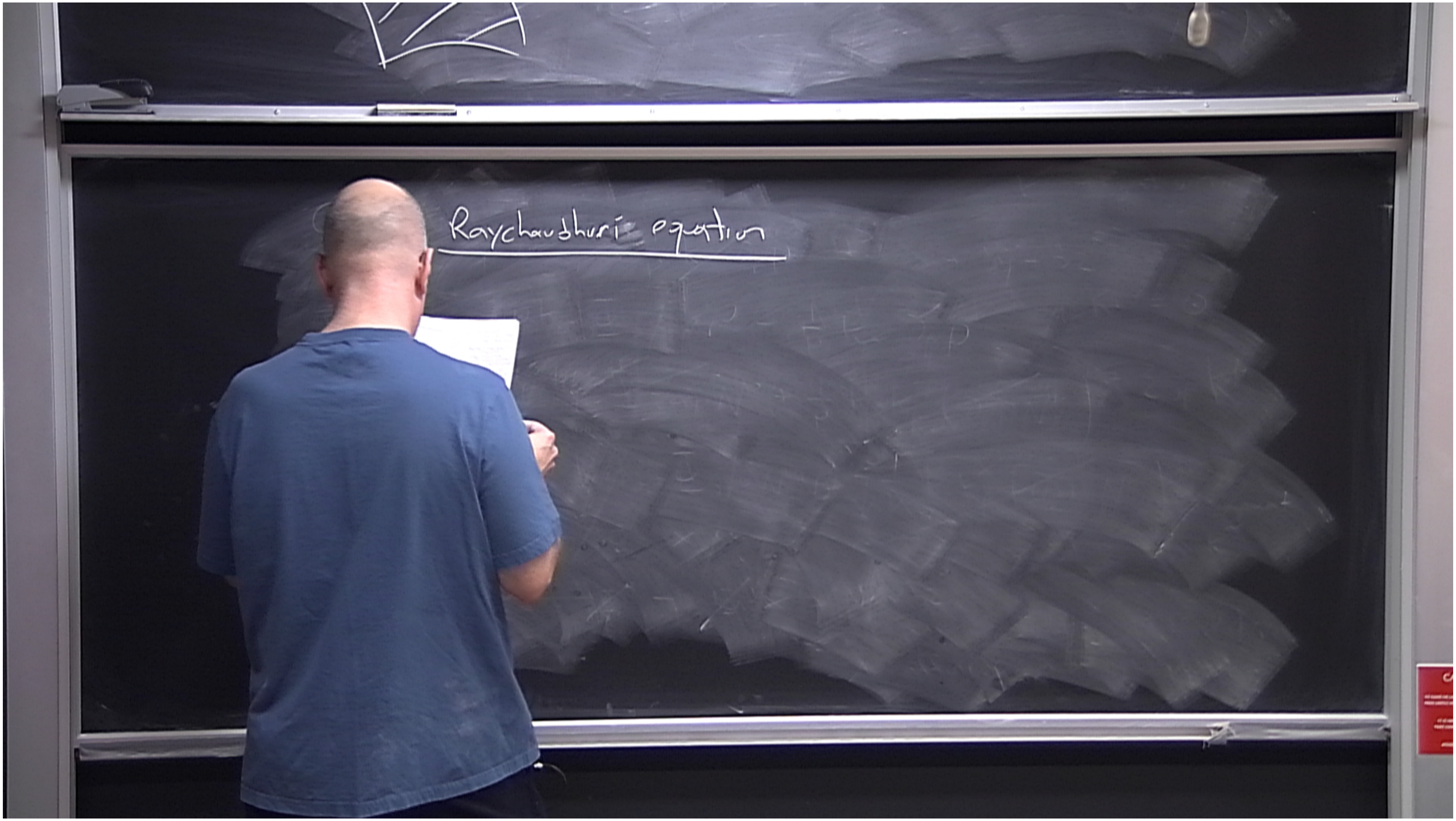

$K^\alpha \to N^\alpha \to \omega_{\alpha\beta} = 0$
$= h^\alpha{}_\nu h^\rho{}_\gamma B^\nu{}_\rho \xi^\gamma$
$\equiv \tilde{B}^\alpha{}_\gamma$
cross-sectional areas.
$\tilde{B}_{\alpha\beta} = \frac{1}{2} h_{\alpha\beta}\Theta + \sigma_{\alpha\beta} + \omega_{\alpha\beta}$ (irreducible parts)
$\Theta = h^{\alpha\beta}\tilde{B}_{\alpha\beta} = g^{\alpha\beta}\tilde{B}_{\alpha\beta} = \cdots = K^\alpha{}_{;\alpha}$ expansion
$\sigma_{\alpha\beta} = \tilde{B}_{(\alpha\beta)} - \frac{1}{2} h_{\alpha\beta}\Theta$ shear
$\omega_{\alpha\beta} = \tilde{B}_{[\alpha\beta]}$

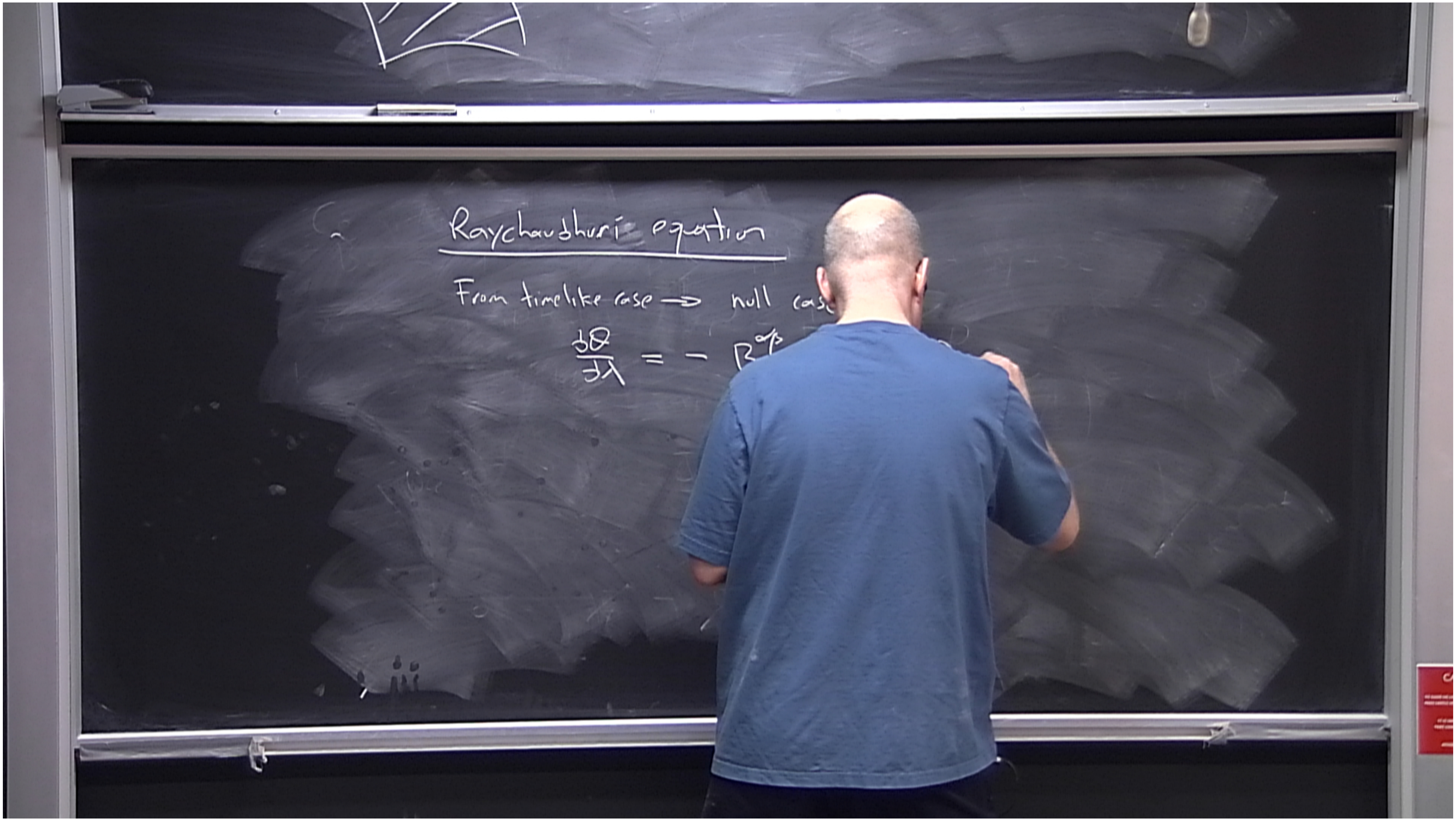

$h^{\rho}{}_{\gamma}\, \tilde{\xi}^{\gamma}$
$= h^{\alpha}{}_{\nu}\, h^{\rho}{}_{\gamma}\, B^{\nu}{}_{\rho}\, \tilde{\xi}^{\gamma}$
$\equiv \tilde{B}^{\alpha}{}_{\gamma}$
$K^{\alpha} \to N^{\alpha} \to \omega_{\alpha\beta} = 0$
cross-sectional areas.
$\tilde{B}_{\alpha\beta} = \frac{1}{2} h_{\alpha\beta}\, \theta + \sigma_{\alpha\beta} + \omega_{\alpha\beta}$ (irreducible parts)
$\theta = h^{\alpha\beta} \tilde{B}_{\alpha\beta} = g^{\alpha\beta} \tilde{B}_{\alpha\beta} = \cdots = K^{\alpha}{}_{;\alpha}$ expansion
$\sigma_{\alpha\beta} = \tilde{B}_{(\alpha\beta)} - \frac{1}{2} h_{\alpha\beta}\, \theta$ shear
$\omega_{\alpha\beta} = \tilde{B}_{[\alpha\beta]}$

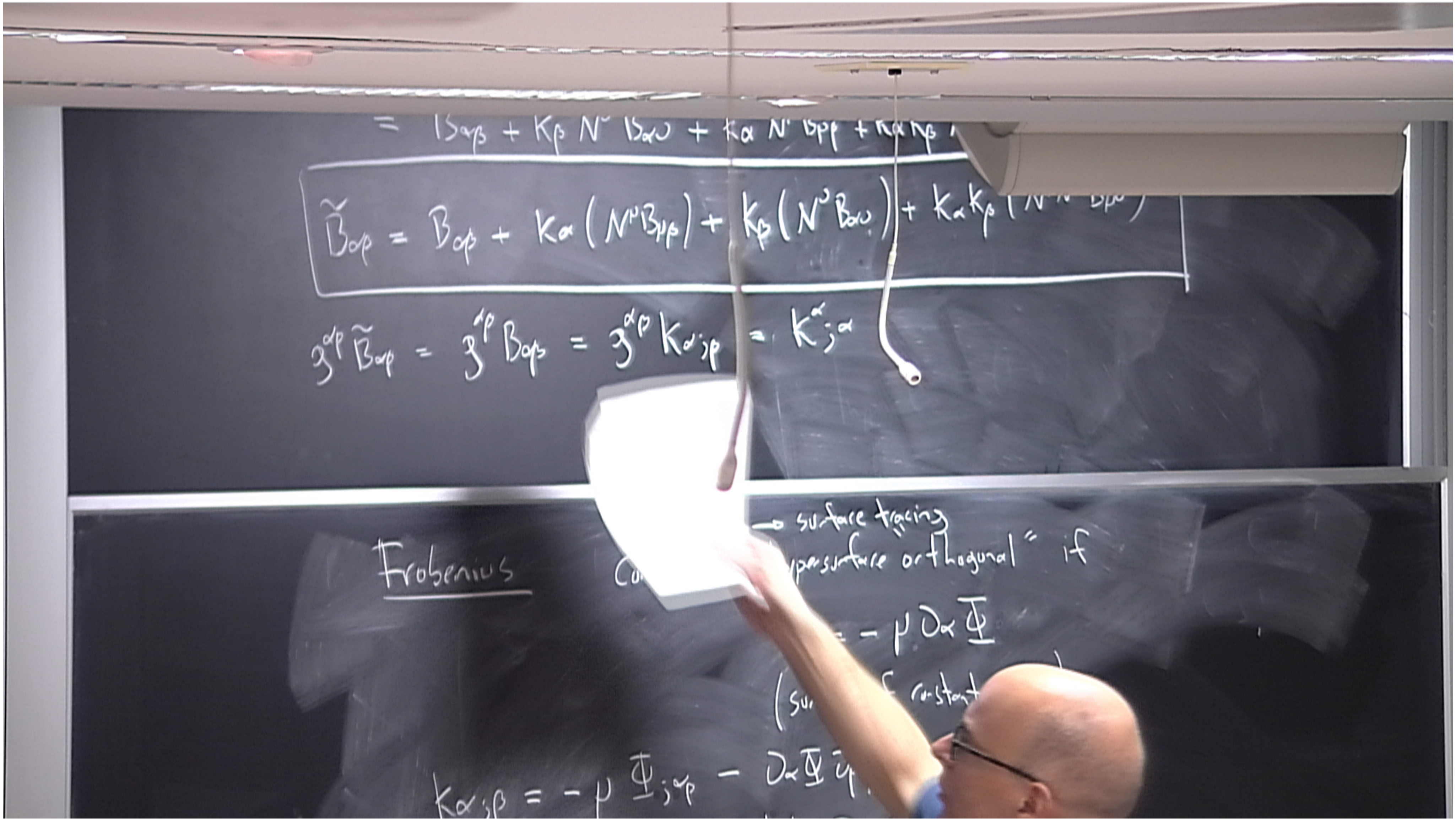

K^α → N^α → ω_αβ = 0
= h^α_υ h^ρ_γ B^υ_ρ ξ̃^γ
≡ B̃^α_γ
cross-sectional areas.
B̃_αβ = ½ h_αβ Θ + σ_αβ + ω_αβ (irreducible parts)
Θ = h^αβ B̃_αβ = g^αβ B̃_αβ = … = k^α_;α expansion
σ_αβ = B̃_(αβ) − ½ h_αβ Θ shear
ω_αβ = B̃_[αβ]

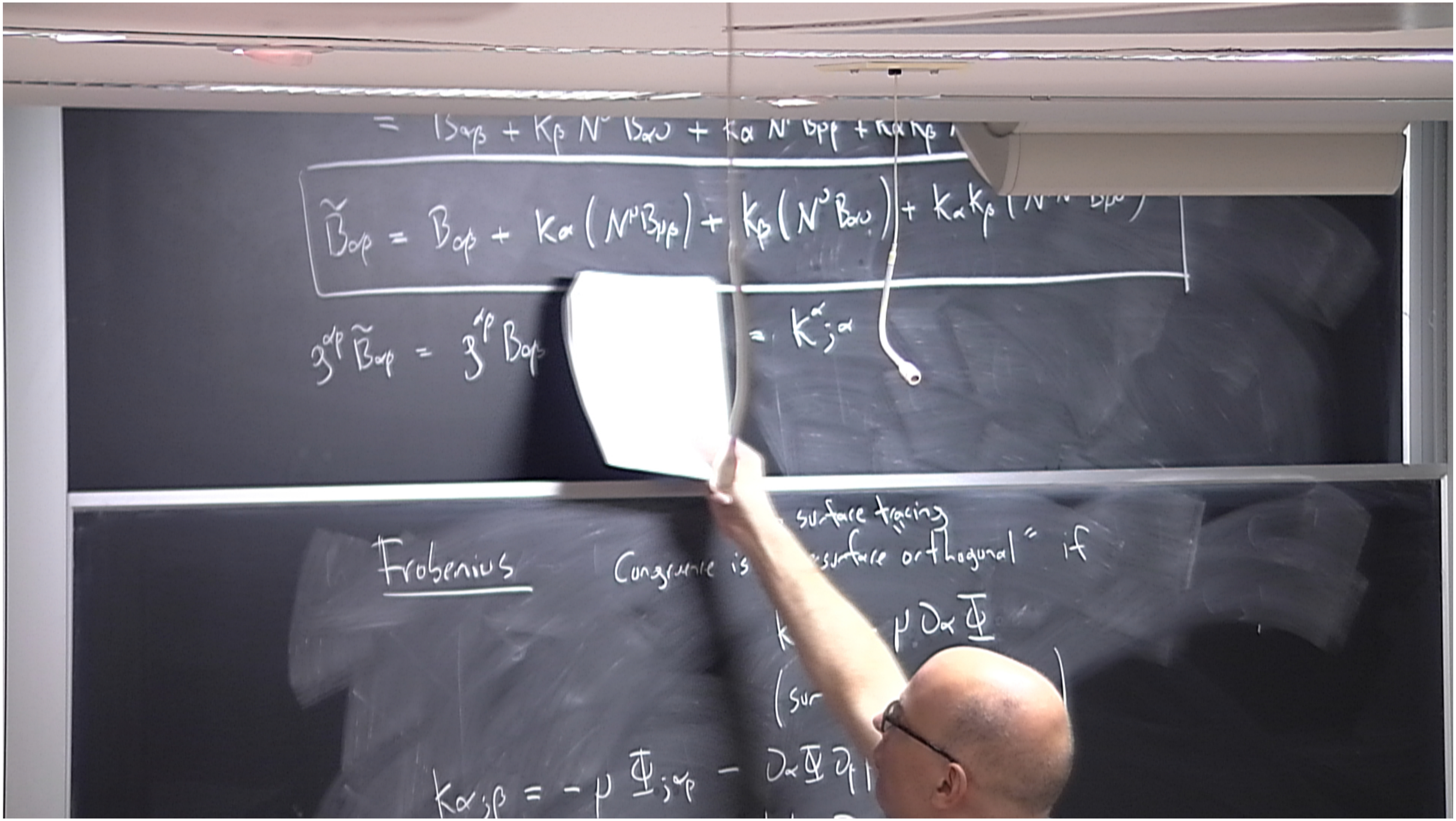

equation


Raychaudhuri equation


Raychaudhuri equation


Raychaudhuri equation
From timelike case → null case


surface tracing
Frobenius
orthogonal


$\tilde{B}_{\alpha\beta} = B_{\alpha\beta} + k_\alpha (N^\mu B_{\mu\beta}) + k_\beta (N^\nu B_{\alpha\nu}) + k_\alpha k_\beta$
$g^{\alpha\beta} \tilde{B}_{\alpha\beta} = g^{\alpha\beta} B_{\alpha\beta}$
$= k^\alpha{}_{;\alpha}$
Frobenius
surface tracing
Congruence is
surface orthogonal" if
$k_{\alpha;\beta} = -\mu \Phi_{;\alpha\beta} - \partial_\alpha \Phi \partial_\beta$

$$= B_{\alpha\beta} + K_\beta N^\nu B_{\alpha\nu} + K_\alpha N^\mu B_{\mu\beta} + K_\alpha K_\beta \ldots$$

$$\tilde{B}_{\alpha\beta} = B_{\alpha\beta} + K_\alpha \left(N^\mu B_{\mu\beta} \right) + K_\beta \left(N^\nu B_{\alpha\nu} \right) + K_\alpha K_\beta \left(N^\mu N^\nu B_{\mu\nu} \right)$$

$$g^{\alpha\beta} \tilde{B}_{\alpha\beta} = g^{\alpha\beta} B_{\alpha\beta} = g^{\alpha\beta} K_{\alpha;\beta} = K^{\alpha}{}_{;\alpha}$$

Frobenius

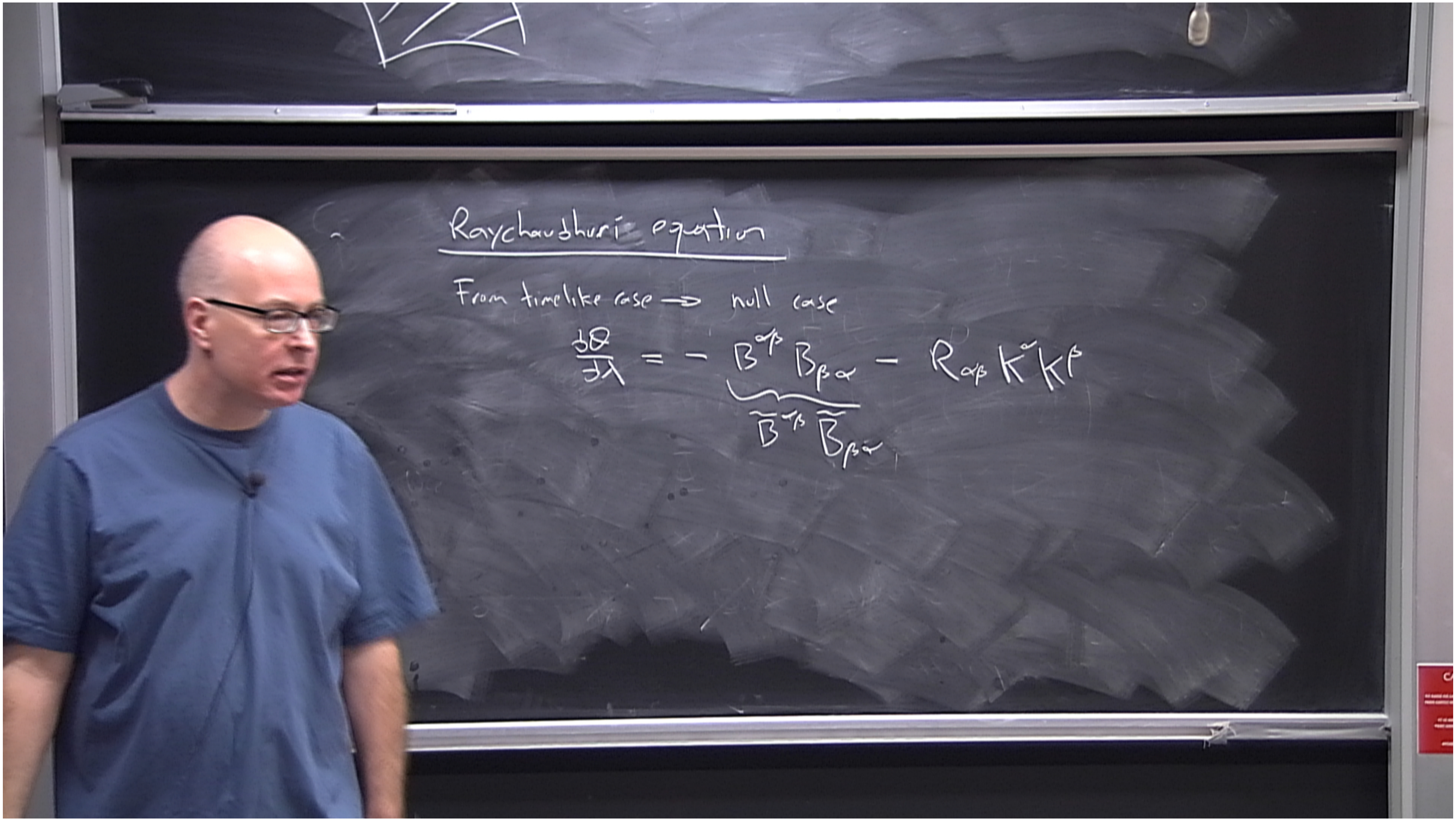
→ surface tracing

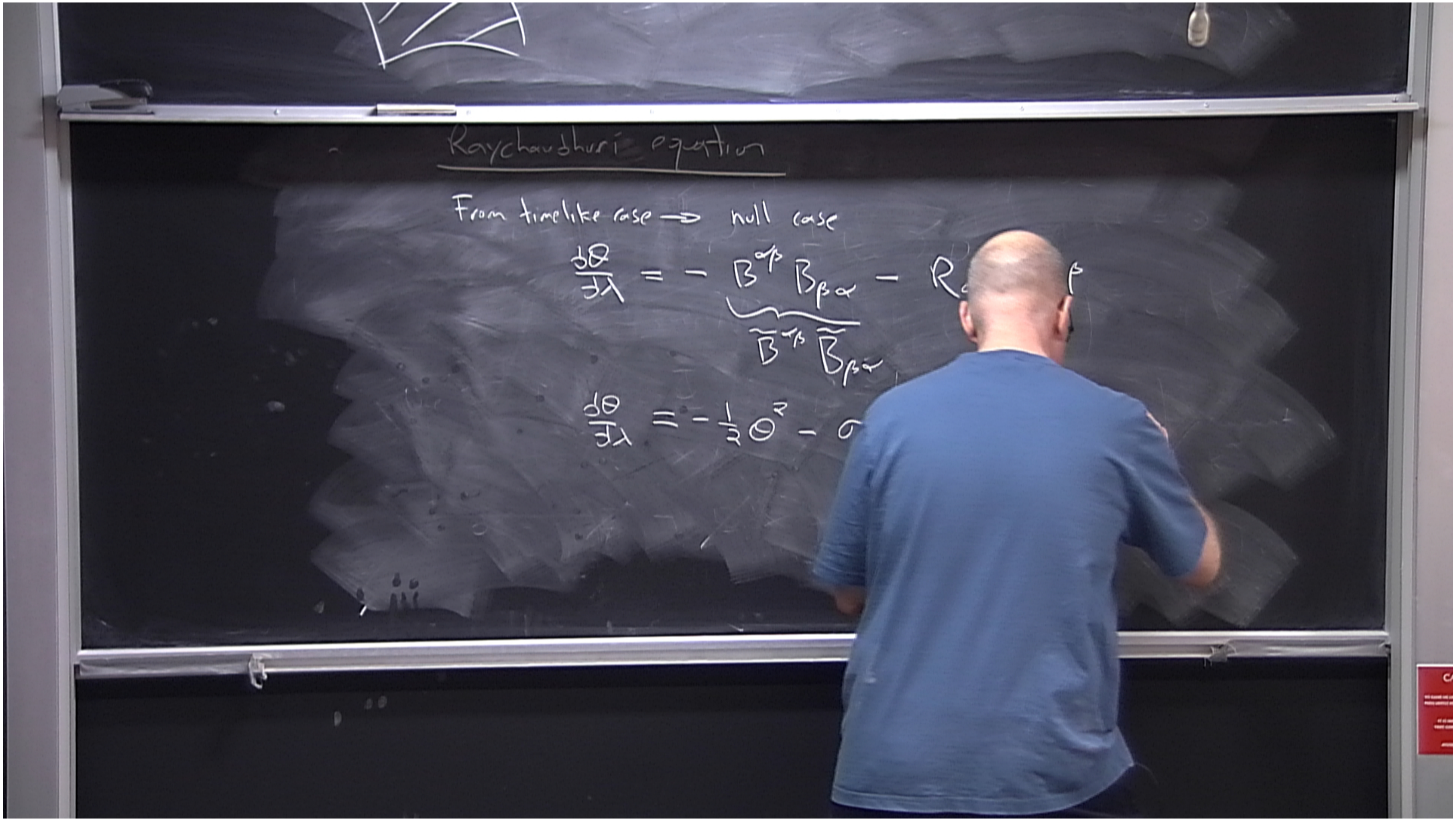
Congruence is "hypersurface orthogonal" if

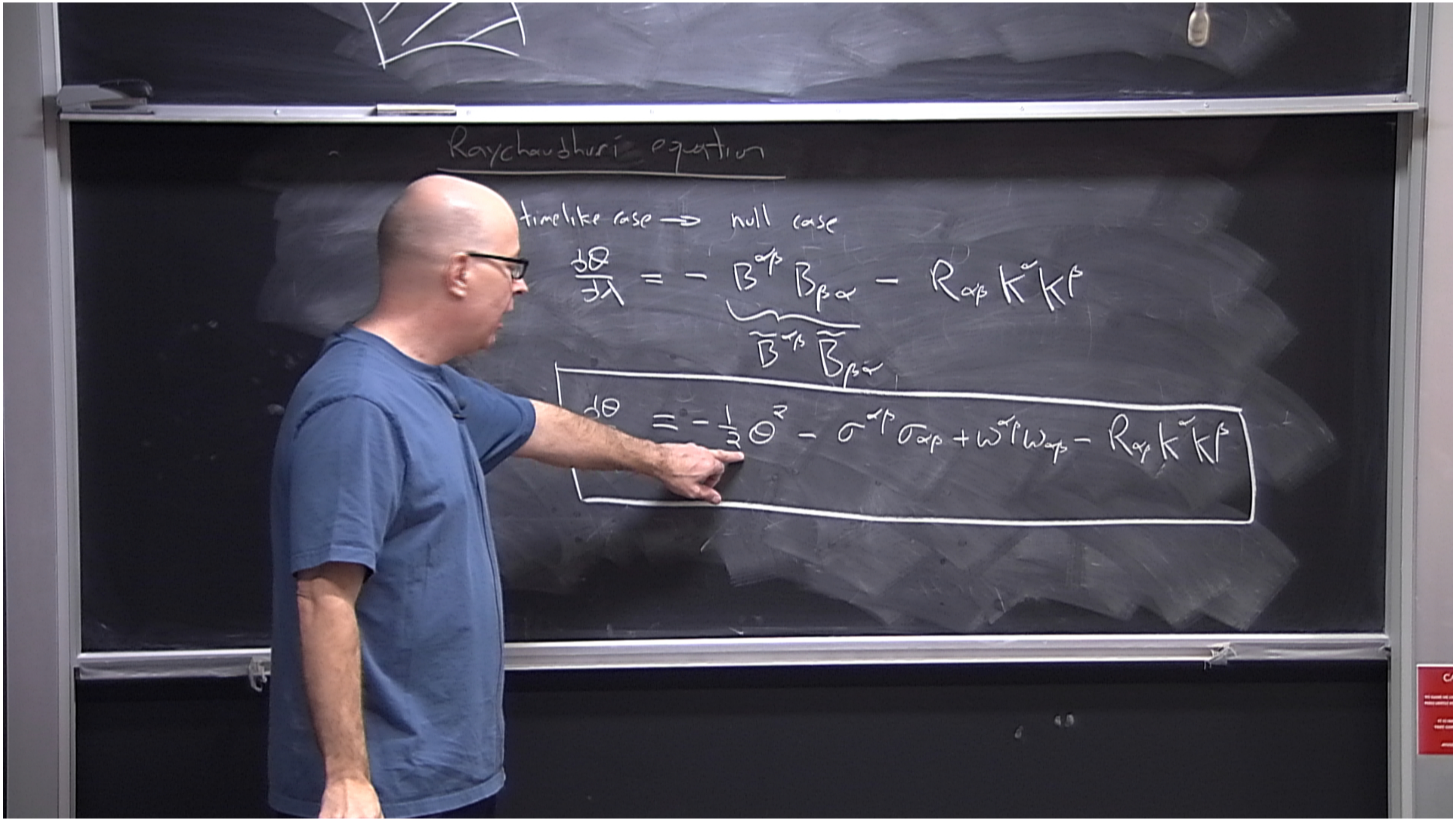
$$K_\alpha = -\mu \, \partial_\alpha \Phi$$

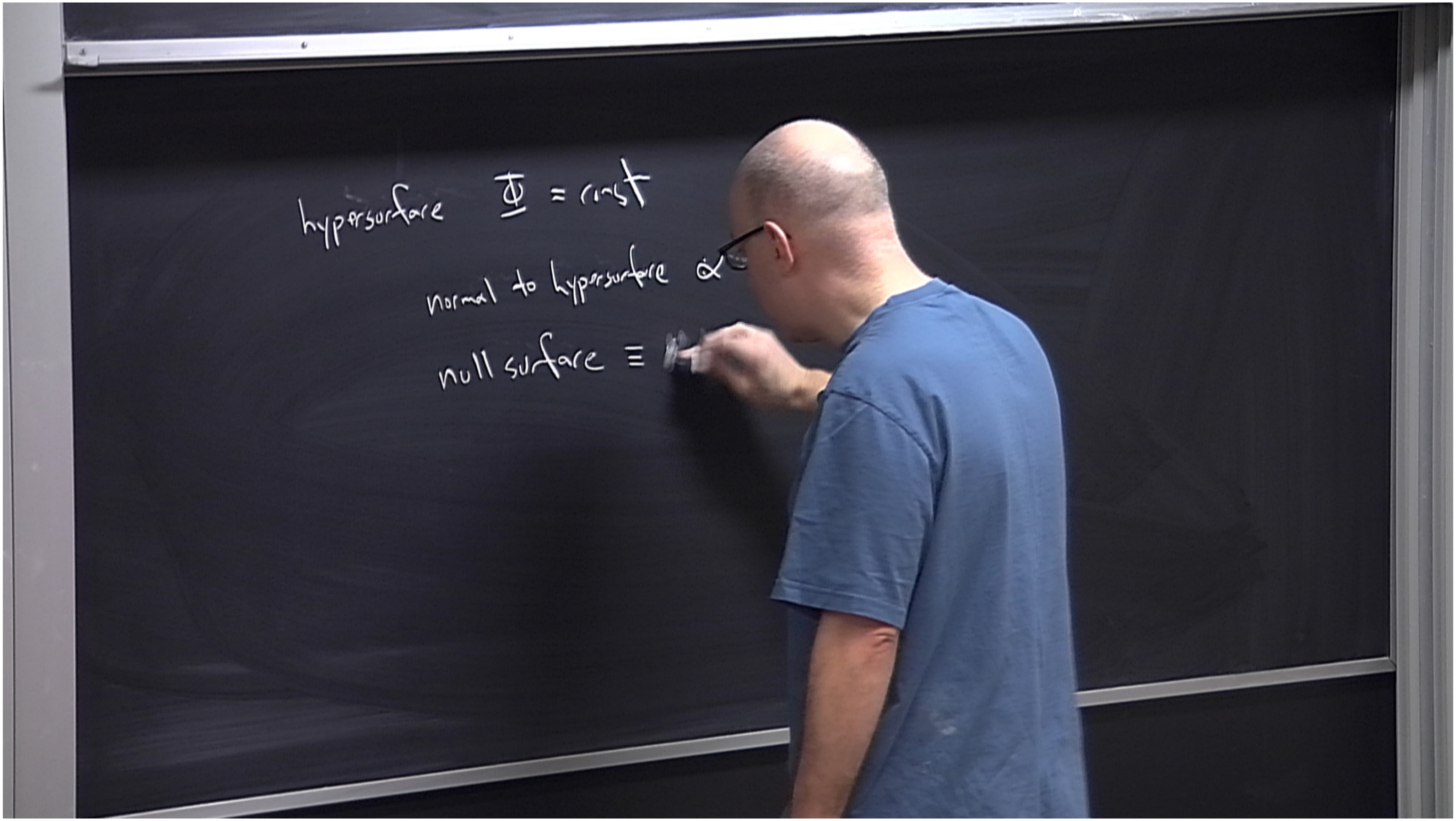
(surfaces of constant Φ)

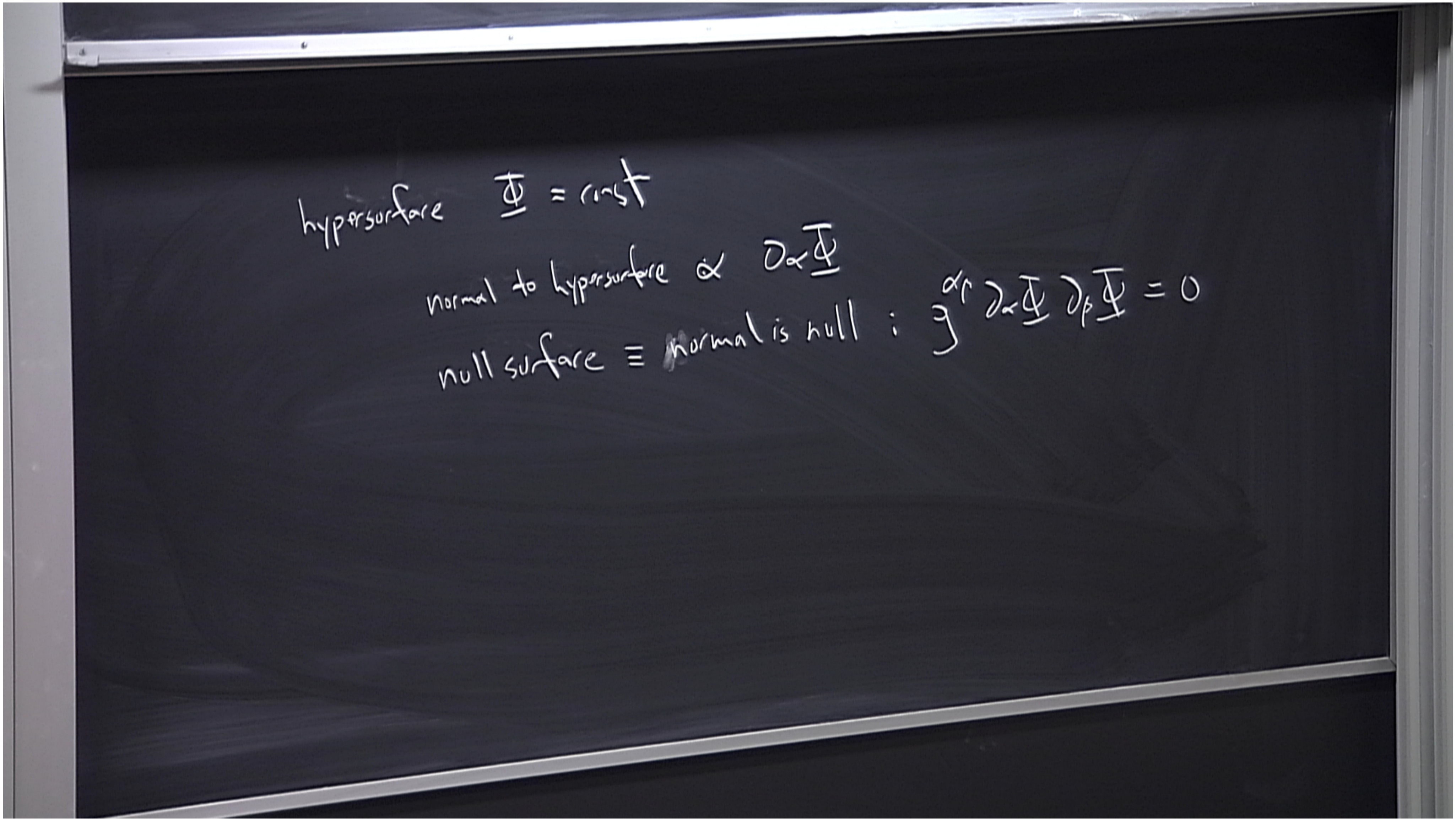
$$-\mu \, \Phi_{;\alpha\beta} - \partial_\alpha \Phi \, \partial_\beta \mu$$

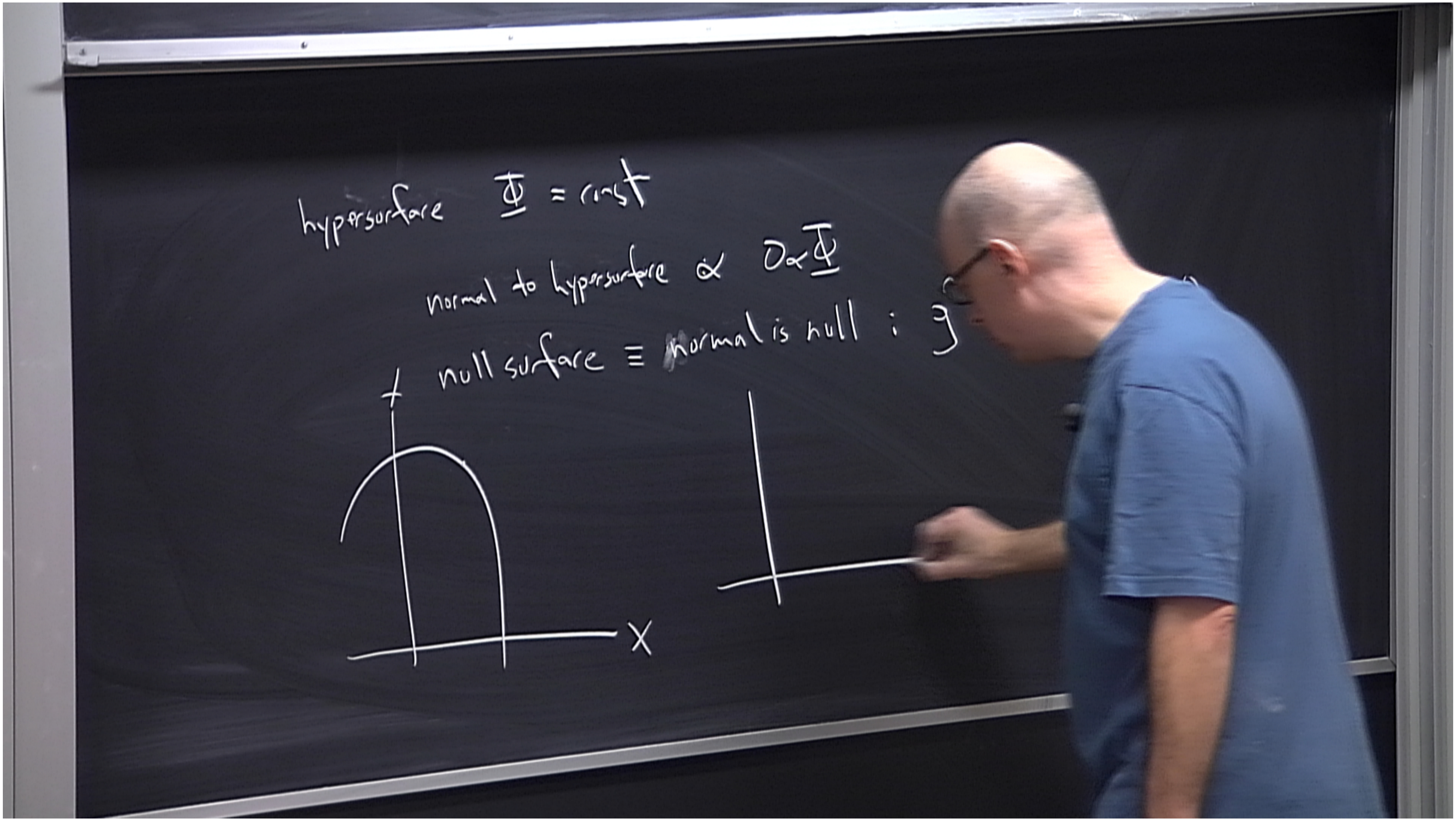

Raychaudhuri equation
From timelike case → null case
$\frac{d\theta}{d\lambda} = - B^{\alpha\beta} B_{\beta\alpha} - R_{\alpha\beta} k^{\alpha} k^{\beta}$
$\tilde{B}^{\alpha\beta} \tilde{B}_{\beta\alpha}$

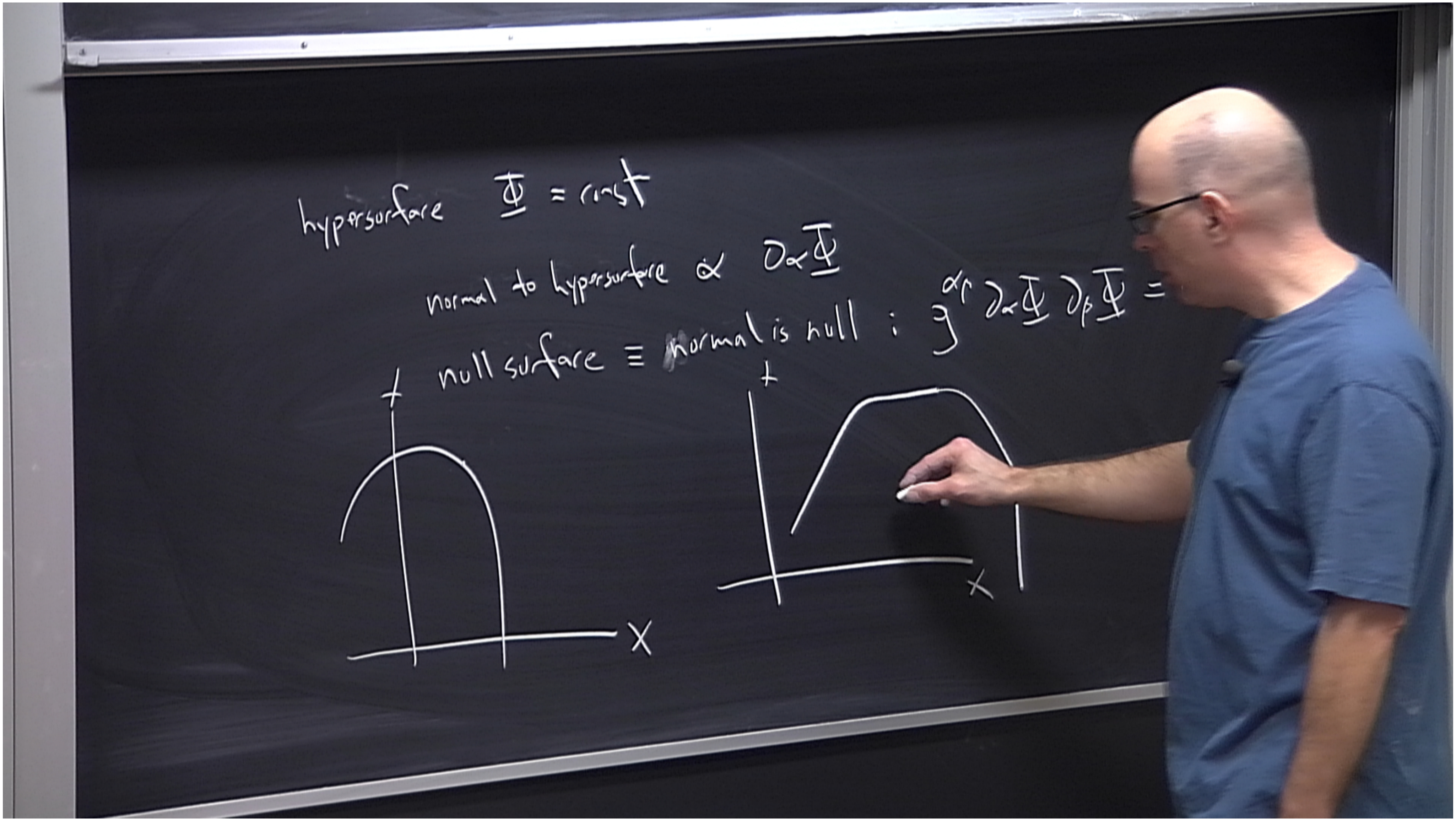

Raychaudhuri equation
From timelike case → null case
dθ/dλ = − B^{αβ} B_{βα} − R
B̃^{αβ} B̃_{βα}
dθ/dλ = − ½ θ² − σ

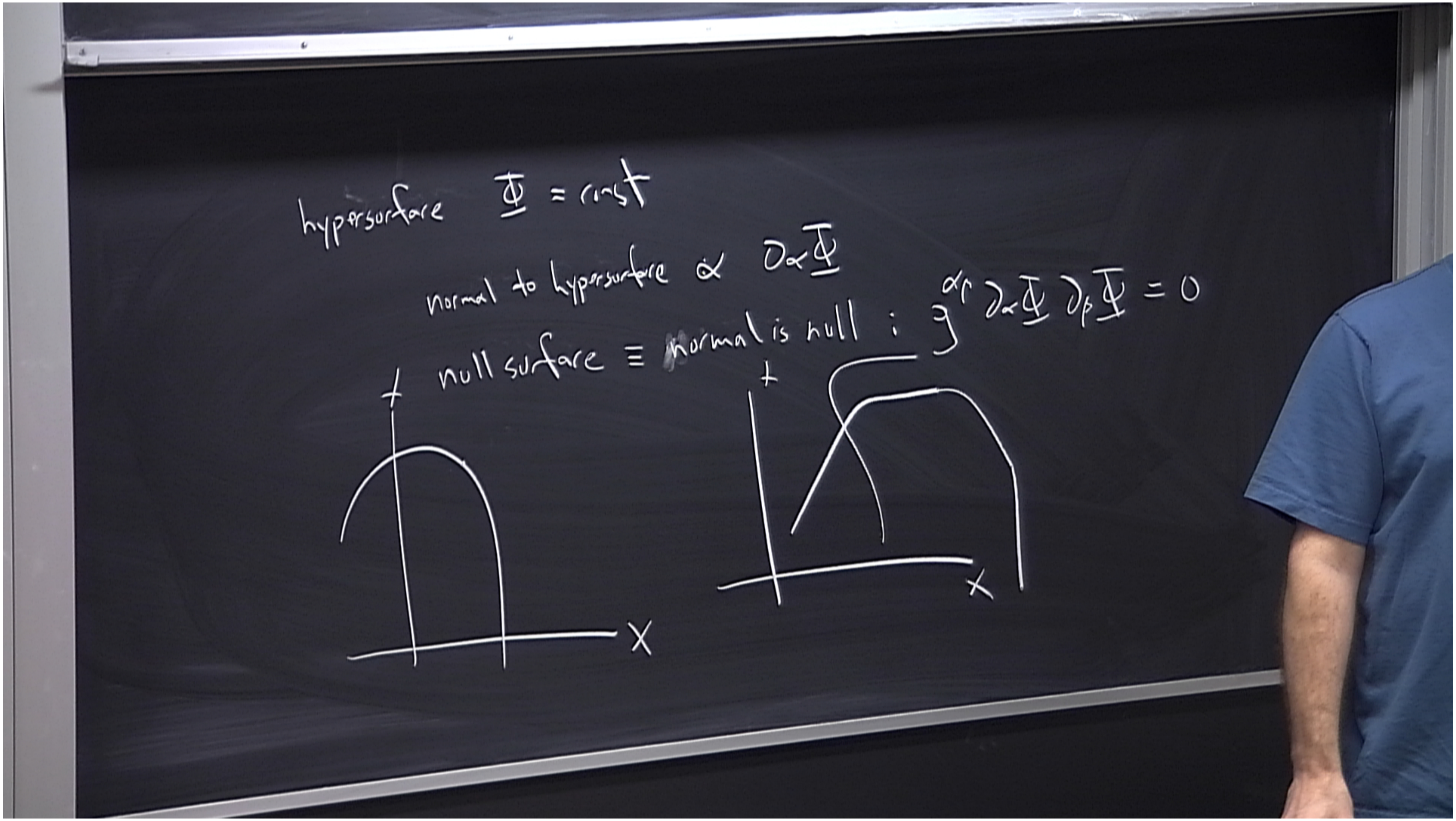

Raychaudhuri equation
timelike case → null case
$\frac{d\theta}{d\lambda} = - B^{\alpha\beta} B_{\beta\alpha} - R_{\alpha\beta} k^{\alpha} k^{\beta}$
$\tilde{B}^{\alpha\beta} \tilde{B}_{\beta\alpha}$
$\frac{d\theta}{d\lambda} = - \frac{1}{2}\theta^2 - \sigma^{\alpha\beta}\sigma_{\alpha\beta} + \omega^{\alpha\beta}\omega_{\alpha\beta} - R_{\alpha\beta} k^{\alpha} k^{\beta}$

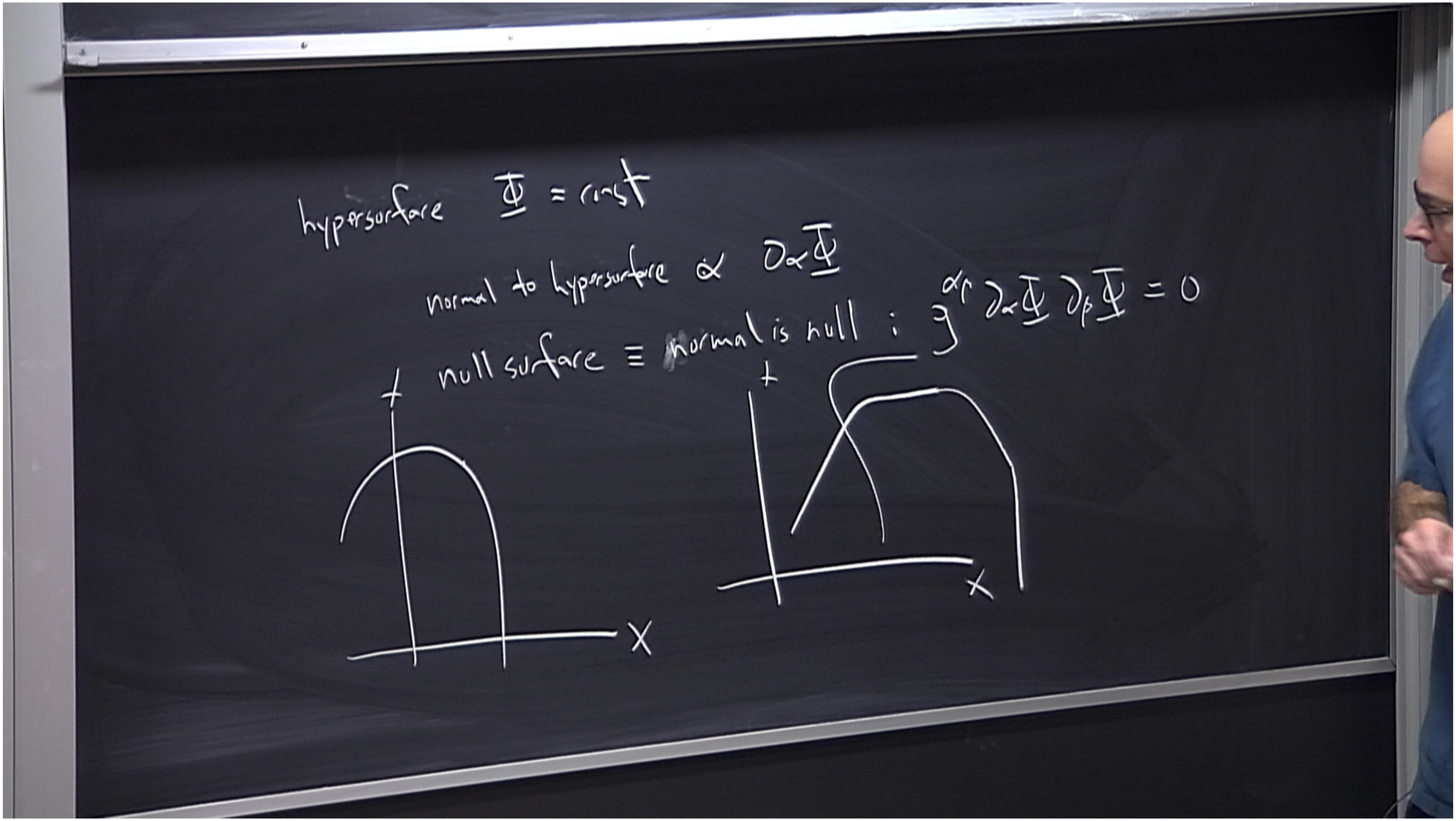

hypersurface Φ = const
normal to hypersurface ∝
null surface ≡


hypersurface $\Phi = \text{const}$
normal to hypersurface $\propto \partial_\alpha \Phi$
null surface $\equiv$ normal is null : $g^{\alpha\beta} \partial_\alpha \Phi \, \partial_\beta \Phi = 0$


hypersurface Φ = const
normal to hypersurface ∝ ∂αΦ
null surface ≡ normal is null : g
t
x


hypersurface Φ = const
normal to hypersurface $\propto$ $\partial_\alpha \Phi$
null surface $\equiv$ normal is null : $g^{\alpha\beta} \partial_\alpha \Phi \, \partial_\beta \Phi =$
t
x
t
x


hypersurface $\Phi = \text{const}$
normal to hypersurface $\propto \partial_\alpha \Phi$
null surface $\equiv$ normal is null : $g^{\alpha\beta} \partial_\alpha \Phi \, \partial_\beta \Phi = 0$
t
x
t
x

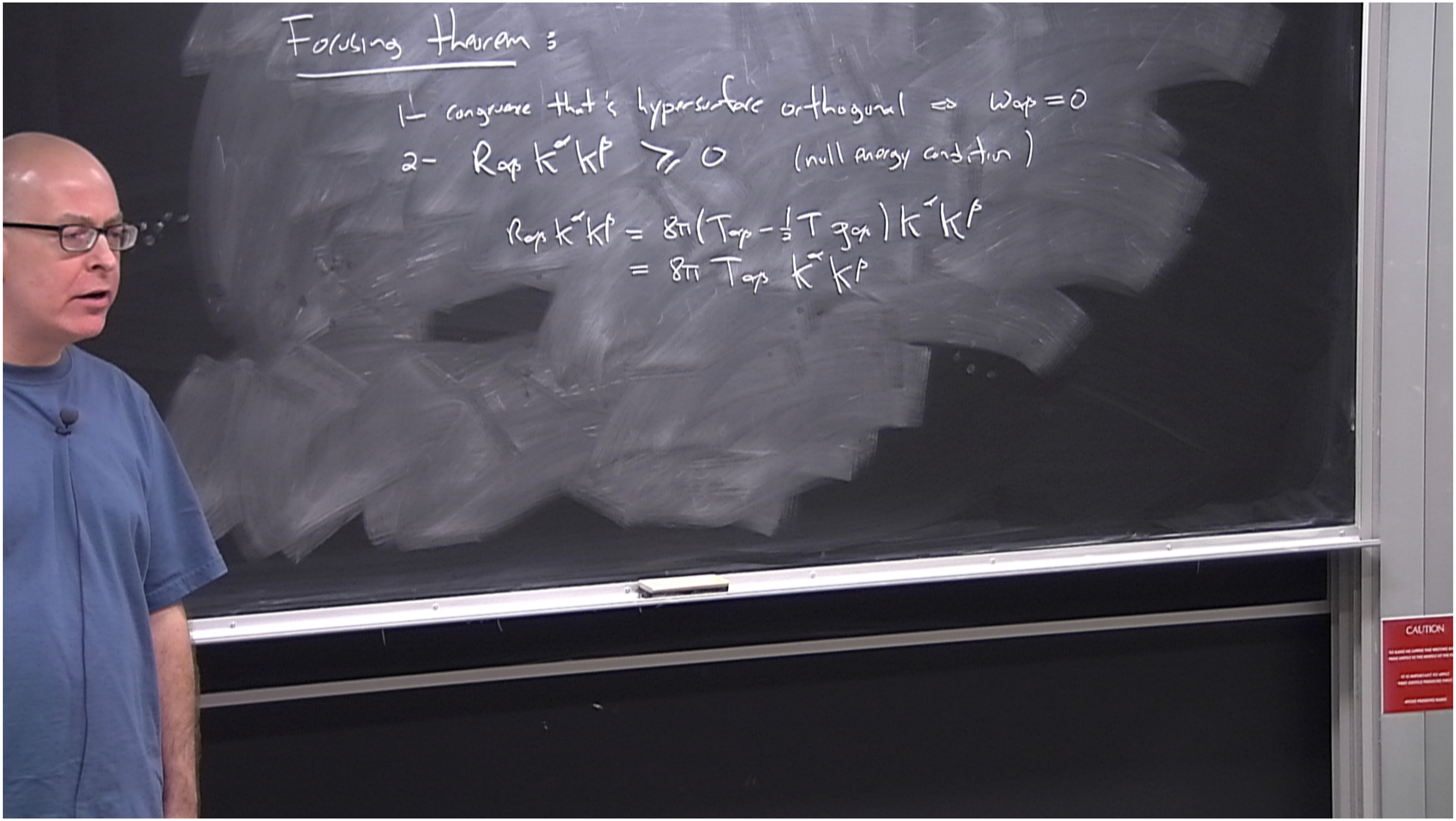


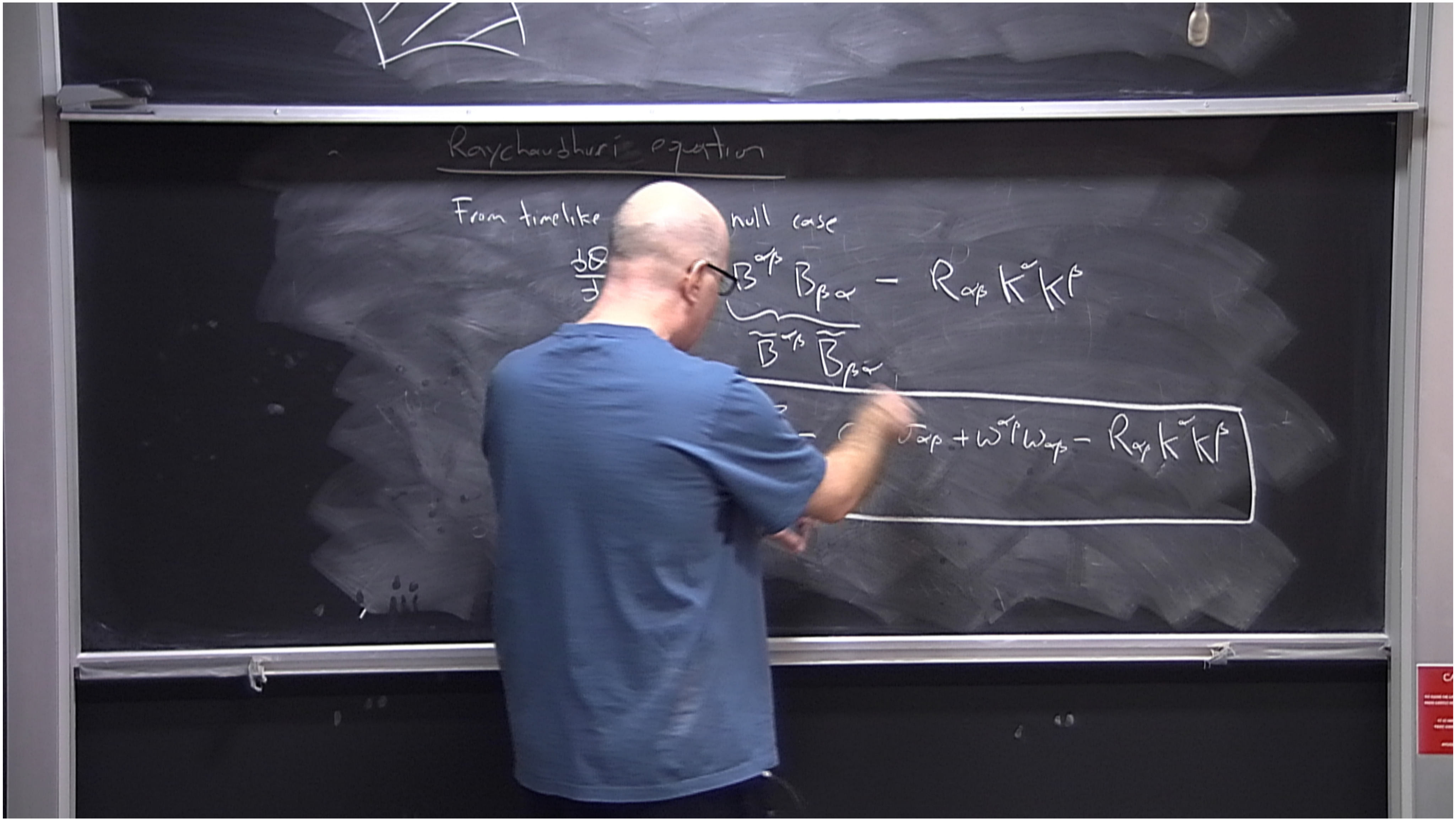
hypersurface $\Phi = \text{const}$
normal to hypersurface $\propto \partial_\alpha \Phi$
null surface $\equiv$ normal is null : $g^{\alpha\beta} \partial_\alpha \Phi \partial_\beta \Phi = 0$
t
x
t
x

Focusing theorem:

1- congruence that's hypersurface orthogonal $\Rightarrow$ $\omega_{\alpha\beta} = 0$

2- $R_{\alpha\beta} k^{\alpha} k^{\beta} \geq 0$ (null energy condition)


Focusing theorem:
1- congruence that's hypersurface orthogonal ⇒ $\omega_{\alpha\beta} = 0$
2- $R_{\alpha\beta} k^\alpha k^\beta \geq 0$ (null energy condition)
$R_{\alpha\beta} k^\alpha k^\beta = 8\pi (T_{\alpha\beta} - \frac{1}{2} T g_{\alpha\beta}) k^\alpha k^\beta$
$= 8\pi T_{\alpha\beta} k^\alpha k^\beta$
CAUTION


Raychaudhuri equation
From timelike
null case

$$R_{\alpha\beta} k^\alpha k^\beta = 8\pi \left(T_{\alpha\beta} - \tfrac{1}{2} T g_{\alpha\beta}\right) k^\alpha k^\beta$$
$$= 8\pi T_{\alpha\beta} k^\alpha k^\beta$$

$$\Rightarrow \frac{d\theta}{d\lambda} \leq 0$$

$\theta = h^{\alpha\beta} \tilde{B}_{\alpha\beta} = g^{\alpha\beta} \tilde{B}_{\alpha\beta} = \dots = k^\alpha{}_{;\alpha}$ expansion

$\sigma_{\alpha\beta} = \tilde{B}_{(\alpha\beta)} - \frac{1}{2} h_{\alpha\beta} \theta$ shear

$\omega_{\alpha\beta} = \tilde{B}_{[\alpha\beta]}$

Focusing theorem:

1- congruence that's hypersurface orthogonal $\Leftrightarrow$ $\omega_{\alpha\beta} = 0$

2- $R_{\alpha\beta} k^\alpha k^\beta \geq 0$ (null energy condition)

$G = -R = 8\pi T$

$G_{\alpha\beta} = 8\pi T_{\alpha\beta}$

$R_{\alpha\beta} = 8\pi \left(T_{\alpha\beta} - \frac{1}{2} T g_{\alpha\beta} \right)$

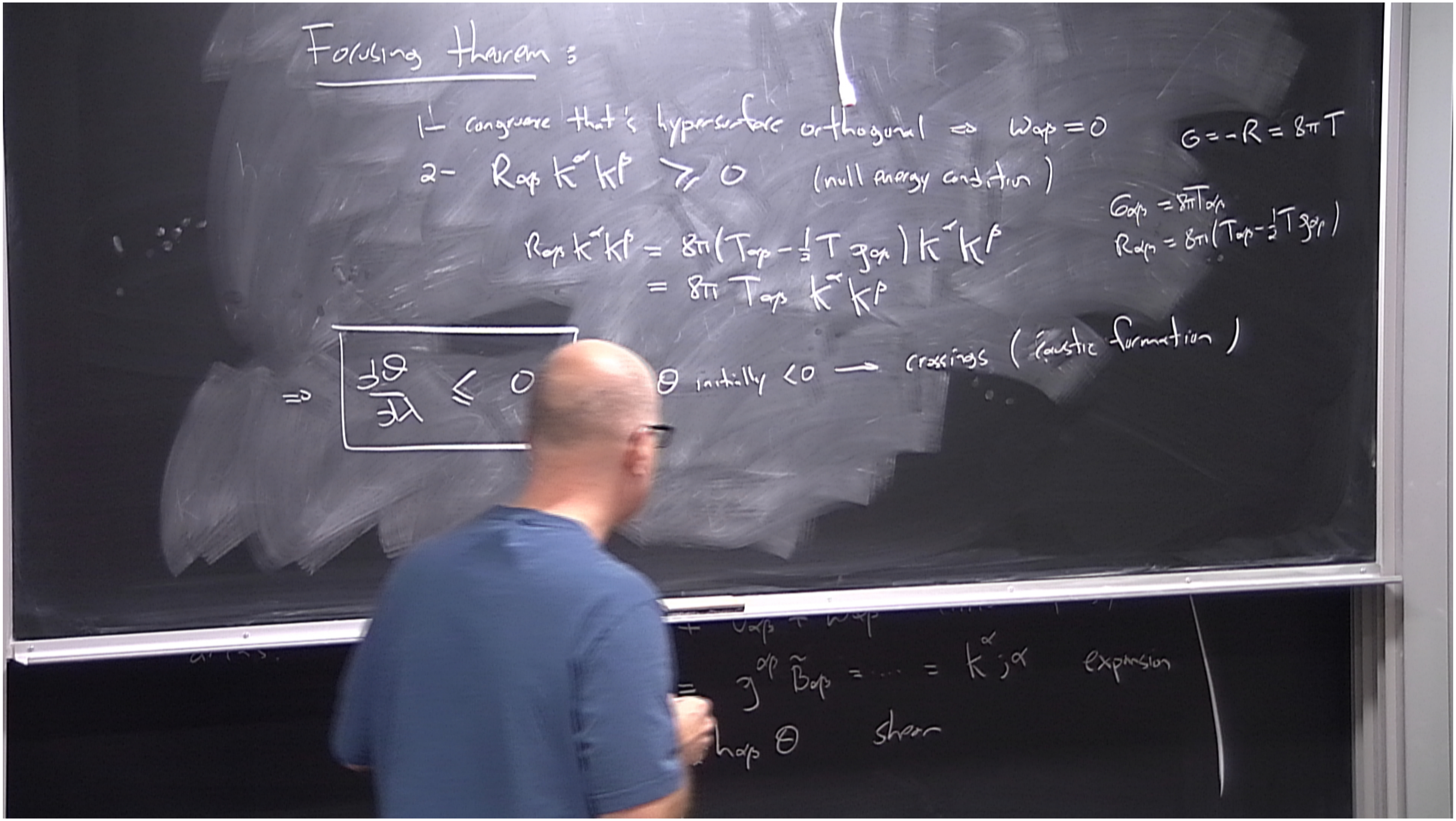
$$R_{\alpha\beta} k^\alpha k^\beta = 8\pi \left(T_{\alpha\beta} - \frac{1}{2} T g_{\alpha\beta} \right) k^\alpha k^\beta$$
$$= 8\pi T_{\alpha\beta} k^\alpha k^\beta$$

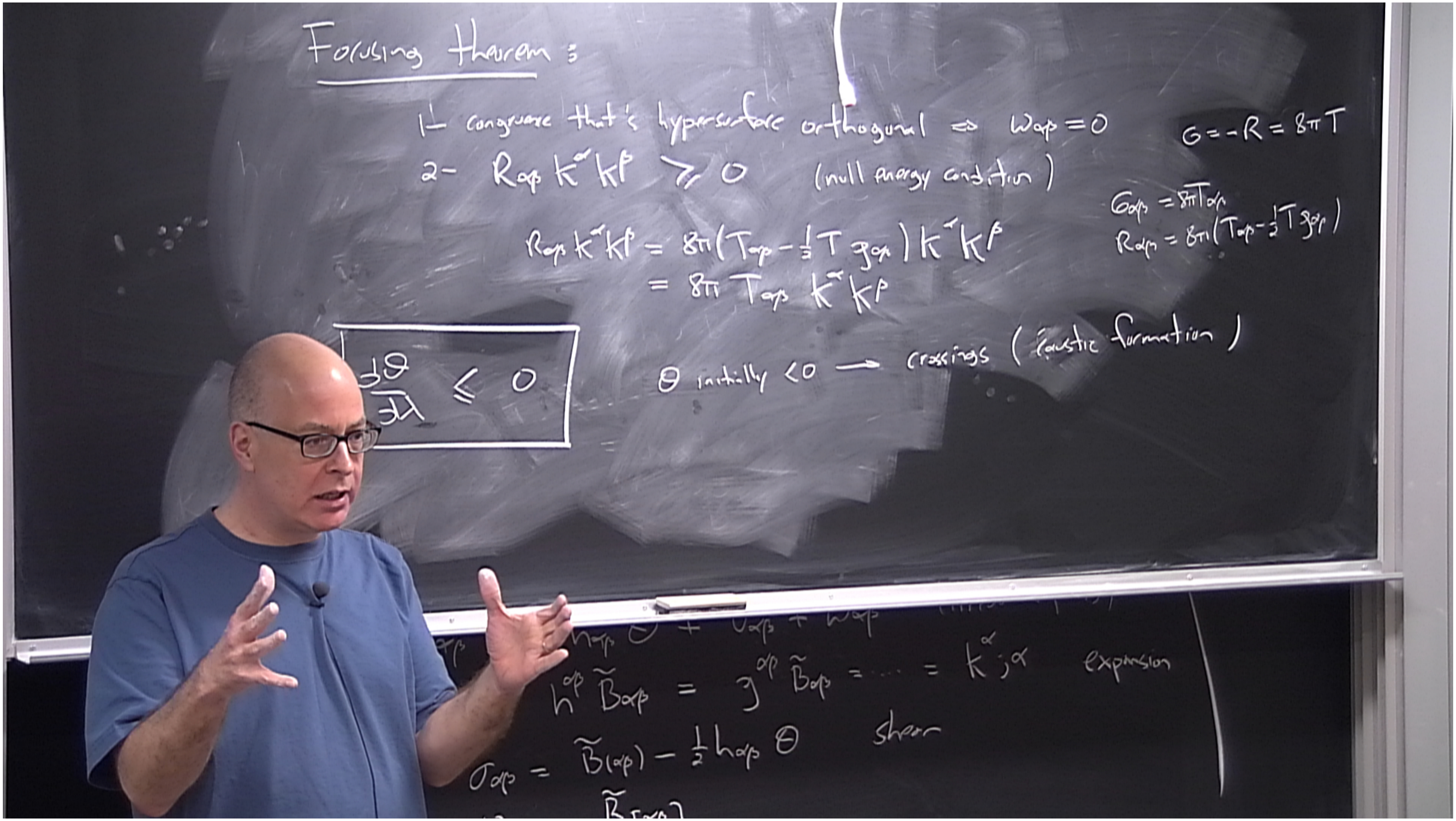
$$\Rightarrow \frac{d\theta}{d\lambda} \leq 0$$

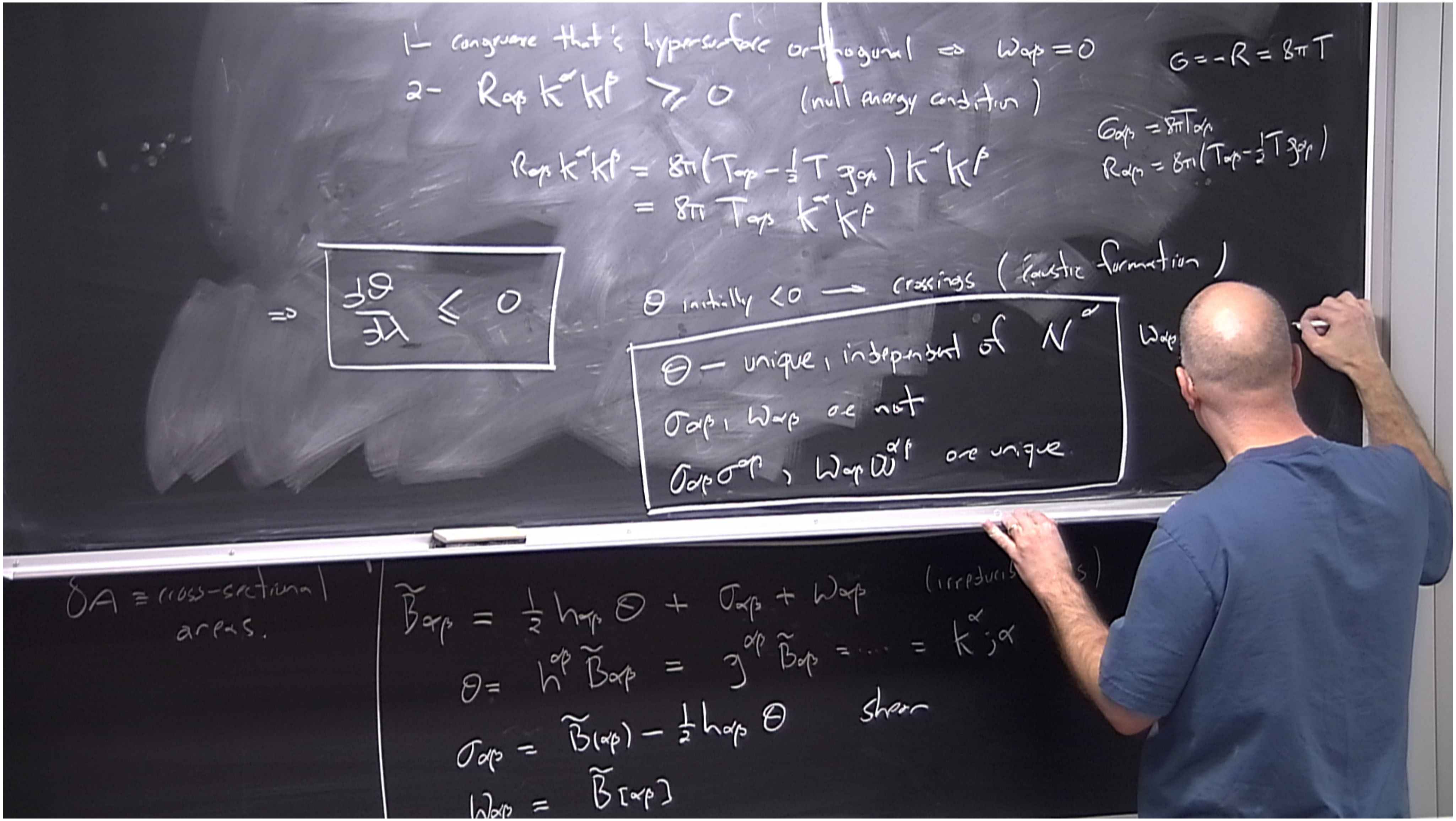
$\theta = h^{\alpha\beta} \tilde{B}_{\alpha\beta} = g^{\alpha\beta} \tilde{B}_{\alpha\beta} = \dots = k^\alpha{}_{;\alpha}$ expansion

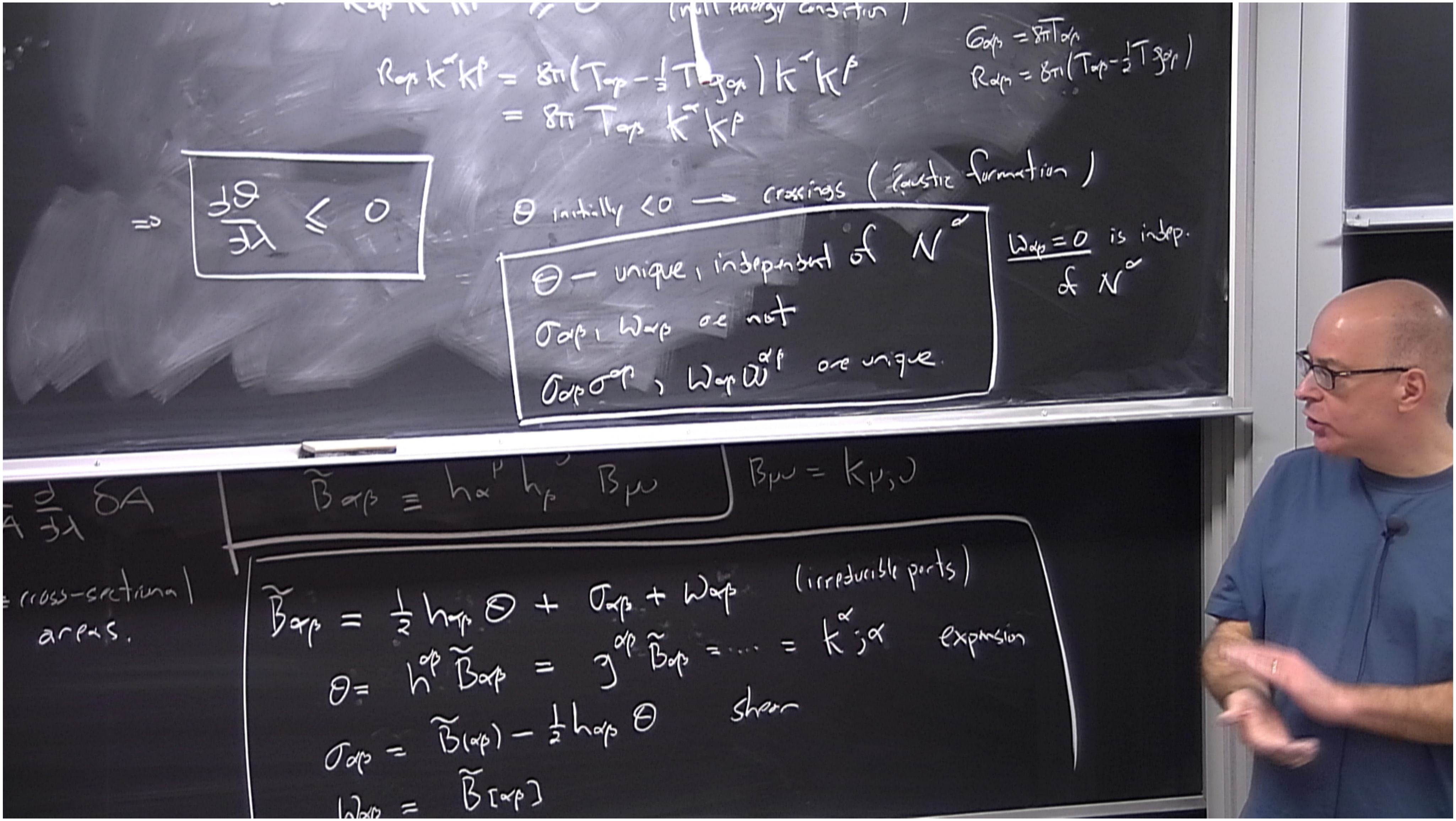
$\sigma_{\alpha\beta} = \tilde{B}_{(\alpha\beta)} - \frac{1}{2} h_{\alpha\beta} \theta$ shear

$\tilde{B}_{[\alpha\beta]}$


Focusing theorem:
1- congruence that's hypersurface orthogonal ⇒ $\omega_{\alpha\beta} = 0$
2- $R_{\alpha\beta} k^\alpha k^\beta \geq 0$ (null energy condition)
$G = -R = 8\pi T$
$G_{\alpha\beta} = 8\pi T_{\alpha\beta}$
$R_{\alpha\beta} = 8\pi(T_{\alpha\beta} - \frac{1}{2} T g_{\alpha\beta})$
$R_{\alpha\beta} k^\alpha k^\beta = 8\pi(T_{\alpha\beta} - \frac{1}{2} T g_{\alpha\beta}) k^\alpha k^\beta$
$= 8\pi T_{\alpha\beta} k^\alpha k^\beta$
$\Rightarrow \frac{d\theta}{d\lambda} \leq 0$
θ initially < 0 → crossings (caustic formation)
$= g^{\alpha\beta} \tilde{B}_{\alpha\beta} = \dots = k^\alpha{}_{;\alpha}$ expansion
shear


Focusing theorem :
1- congruence that's hypersurface orthogonal ⇒ Wαβ = 0
2- Rαβ k^α k^β ≥ 0 (null energy condition)
G = -R = 8π T
Gαβ = 8π Tαβ
Rαβ = 8π (Tαβ - ½ T gαβ)
Rαβ k^α k^β = 8π (Tαβ - ½ T gαβ) k^α k^β
= 8π Tαβ k^α k^β
dθ/dλ ≤ 0
θ initially <0 → crossings (caustic formation)
h^αβ B̃αβ = g^αβ B̃αβ = ... = k^α ;α expansion
σαβ = B̃(αβ) - ½ hαβ θ shear


1- congruence that's hypersurface orthogonal ⇔ ωαβ = 0
2- Rαβ k^α k^β ≥ 0 (null energy condition)
G = -R = 8π T
Gαβ = 8π Tαβ
Rαβ = 8π (Tαβ - ½ T gαβ)
Rαβ k^α k^β = 8π (Tαβ - ½ T gαβ) k^α k^β
= 8π Tαβ k^α k^β
⇒ dθ/dλ ≤ 0
θ initially <0 → crossings (caustic formation)
θ - unique, independent of N^α
σαβ, ωαβ are not
σαβ σ^αβ, ωαβ ω^αβ are unique
δA ≡ cross-sectional areas.
B̃αβ = ½ hαβ θ + σαβ + ωαβ
θ = h^αβ B̃αβ = g^αβ B̃αβ = ... = k^α;α
σαβ = B̃(αβ) - ½ hαβ θ shear
ωαβ = B̃[αβ]


$R_{\alpha\beta}k^\alpha k^\beta = 8\pi(T_{\alpha\beta} - \frac{1}{2}T g_{\alpha\beta})k^\alpha k^\beta$
$= 8\pi T_{\alpha\beta}k^\alpha k^\beta$
$G_{\alpha\beta} = 8\pi T_{\alpha\beta}$
$R_{\alpha\beta} = 8\pi(T_{\alpha\beta} - \frac{1}{2}T g_{\alpha\beta})$
$\Rightarrow \frac{d\theta}{d\lambda} \leq 0$
θ initially $< 0 \longrightarrow$ crossings (caustic formation)
θ — unique, independent of N^α
$\sigma_{\alpha\beta}, \omega_{\alpha\beta}$ are not
$\sigma_{\alpha\beta}\sigma^{\alpha\beta}, \omega_{\alpha\beta}\omega^{\alpha\beta}$ are unique.
$\omega_{\alpha\beta} = 0$ is indep. of N^α
$\tilde{B}_{\alpha\beta} \equiv h_\alpha{}^\mu h_\beta{}^\nu B_{\mu\nu}$
$B_{\mu\nu} = k_{\mu;\nu}$
cross-sectional areas.
$\tilde{B}_{\alpha\beta} = \frac{1}{2}h_{\alpha\beta}\theta + \sigma_{\alpha\beta} + \omega_{\alpha\beta}$ (irreducible parts)
$\theta = h^{\alpha\beta}\tilde{B}_{\alpha\beta} = g^{\alpha\beta}\tilde{B}_{\alpha\beta} = \dots = k^\alpha{}_{;\alpha}$ expansion
$\sigma_{\alpha\beta} = \tilde{B}_{(\alpha\beta)} - \frac{1}{2}h_{\alpha\beta}\theta$ shear
$\omega_{\alpha\beta} = \tilde{B}_{[\alpha\beta]}$