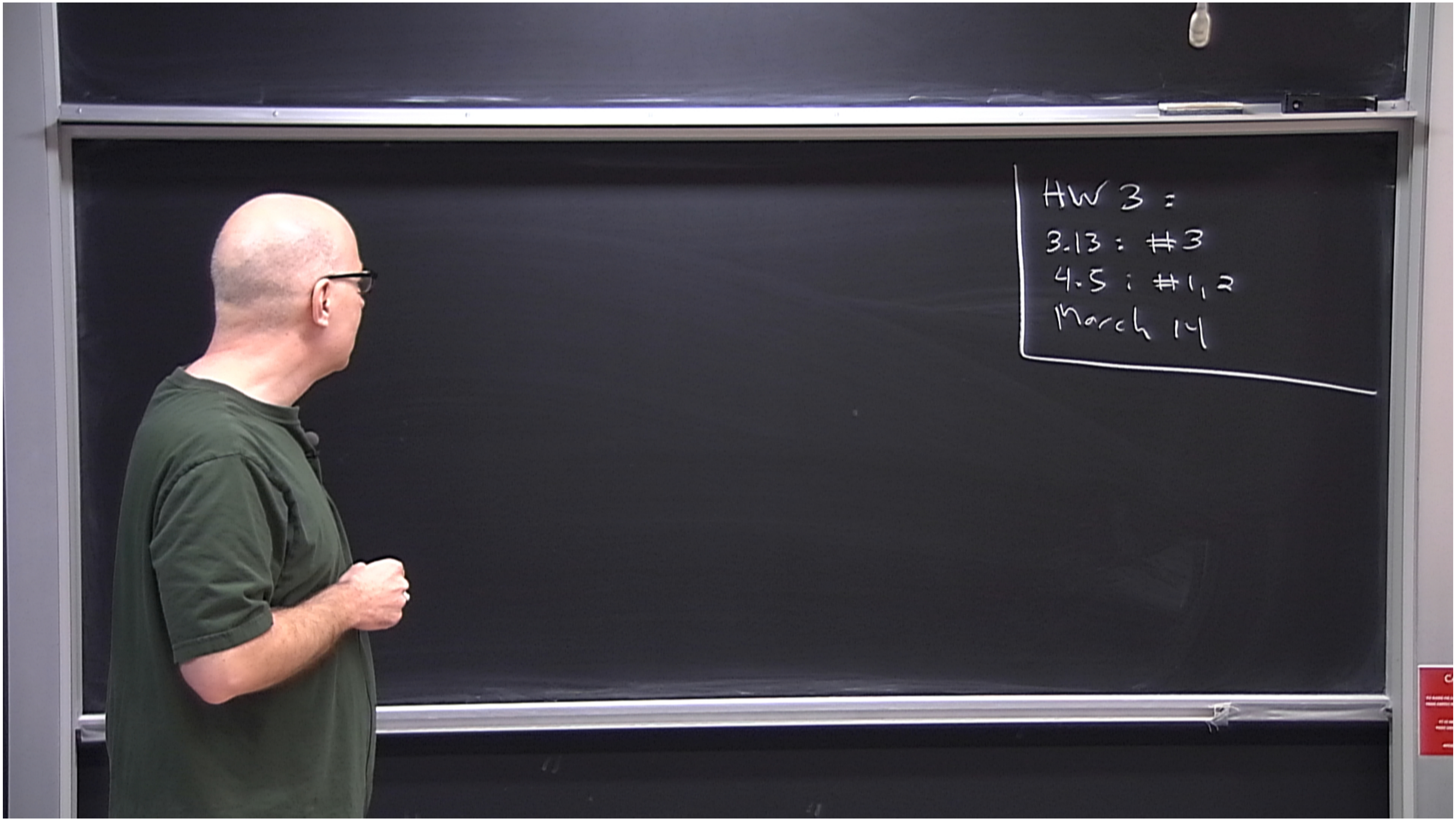
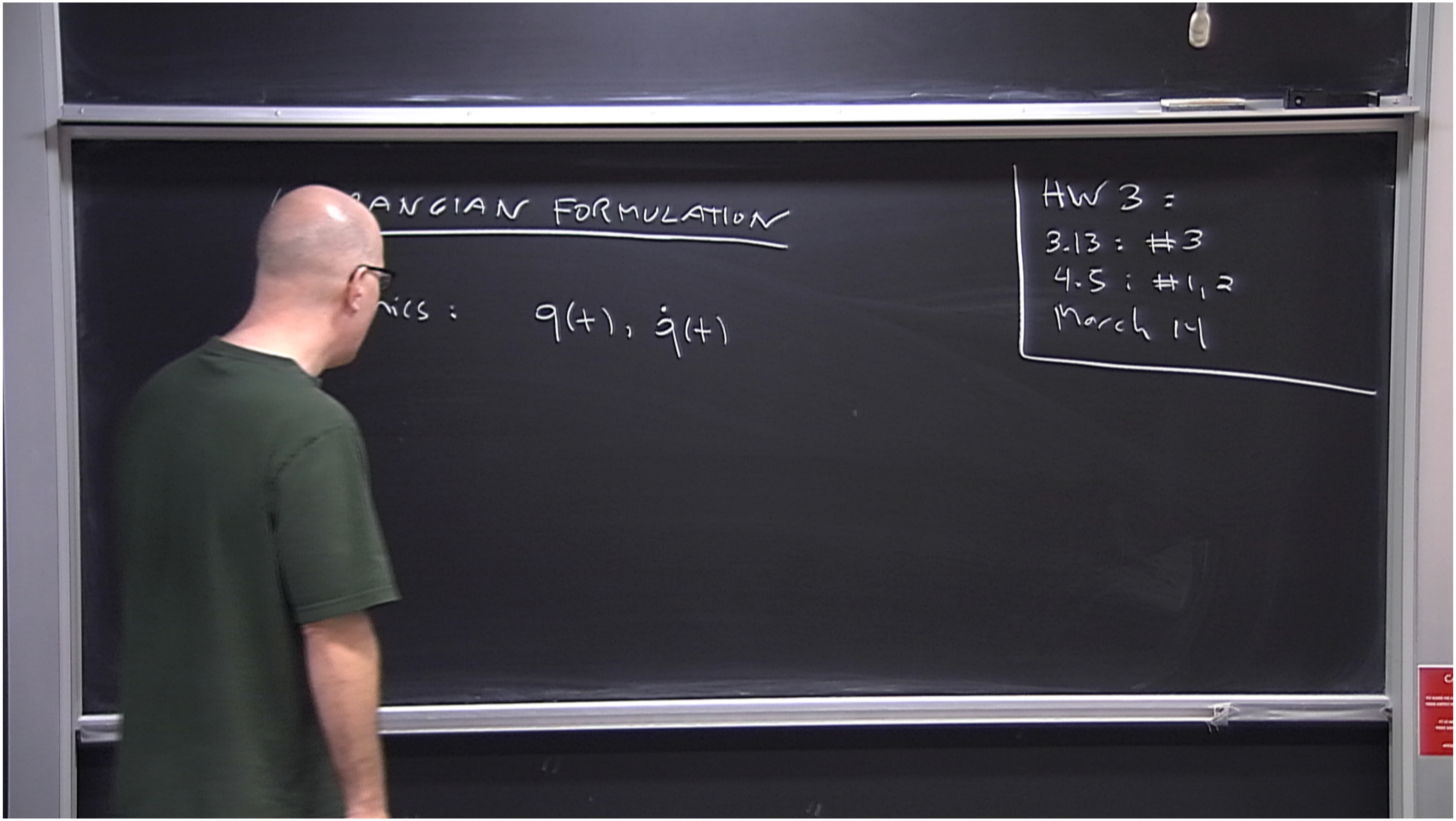
Title: Advanced General Relativity - Lecture 13

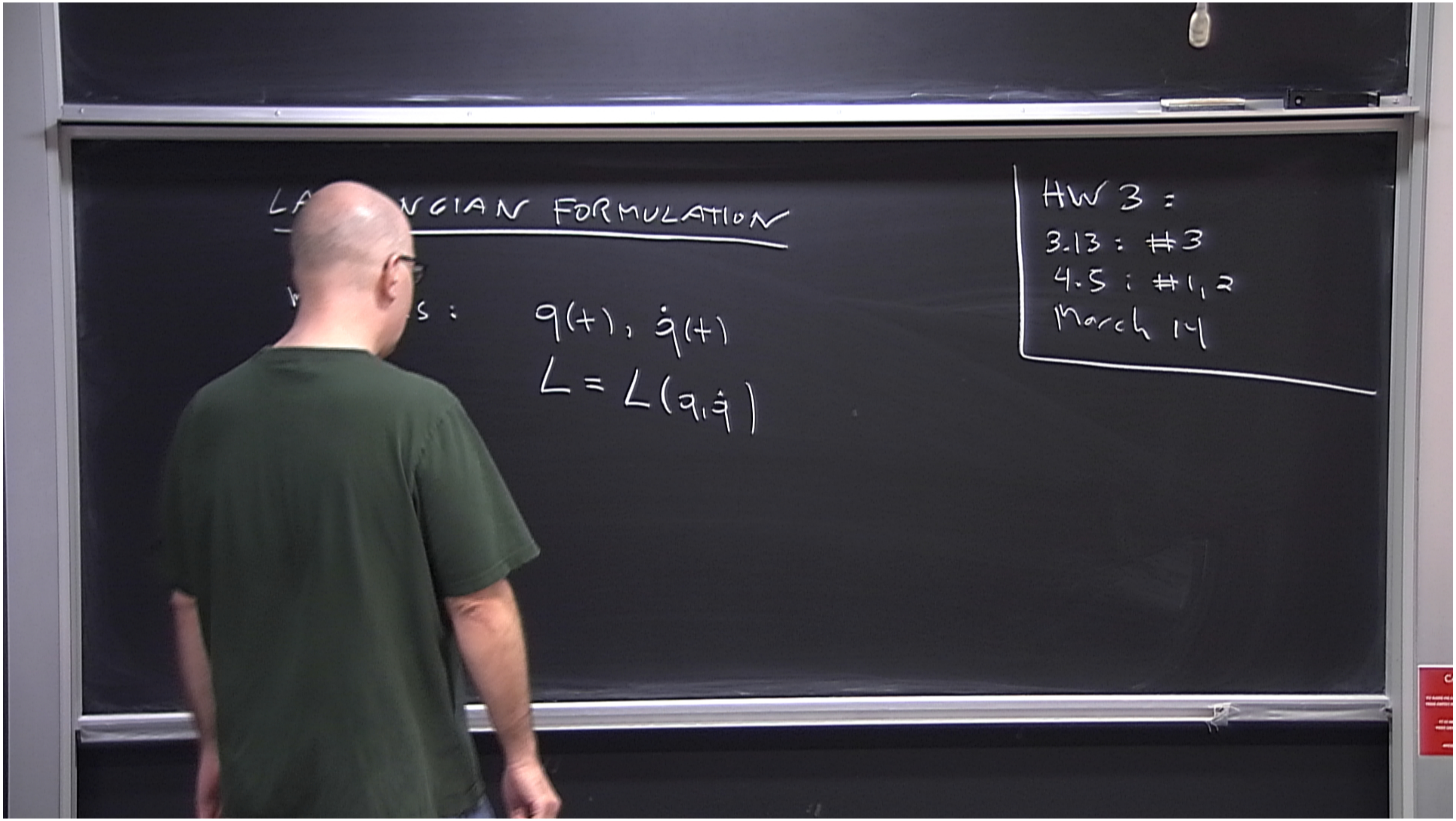
Date: Feb 29, 2012 10:00 AM

URL: http://www.pirsa.org/12020110

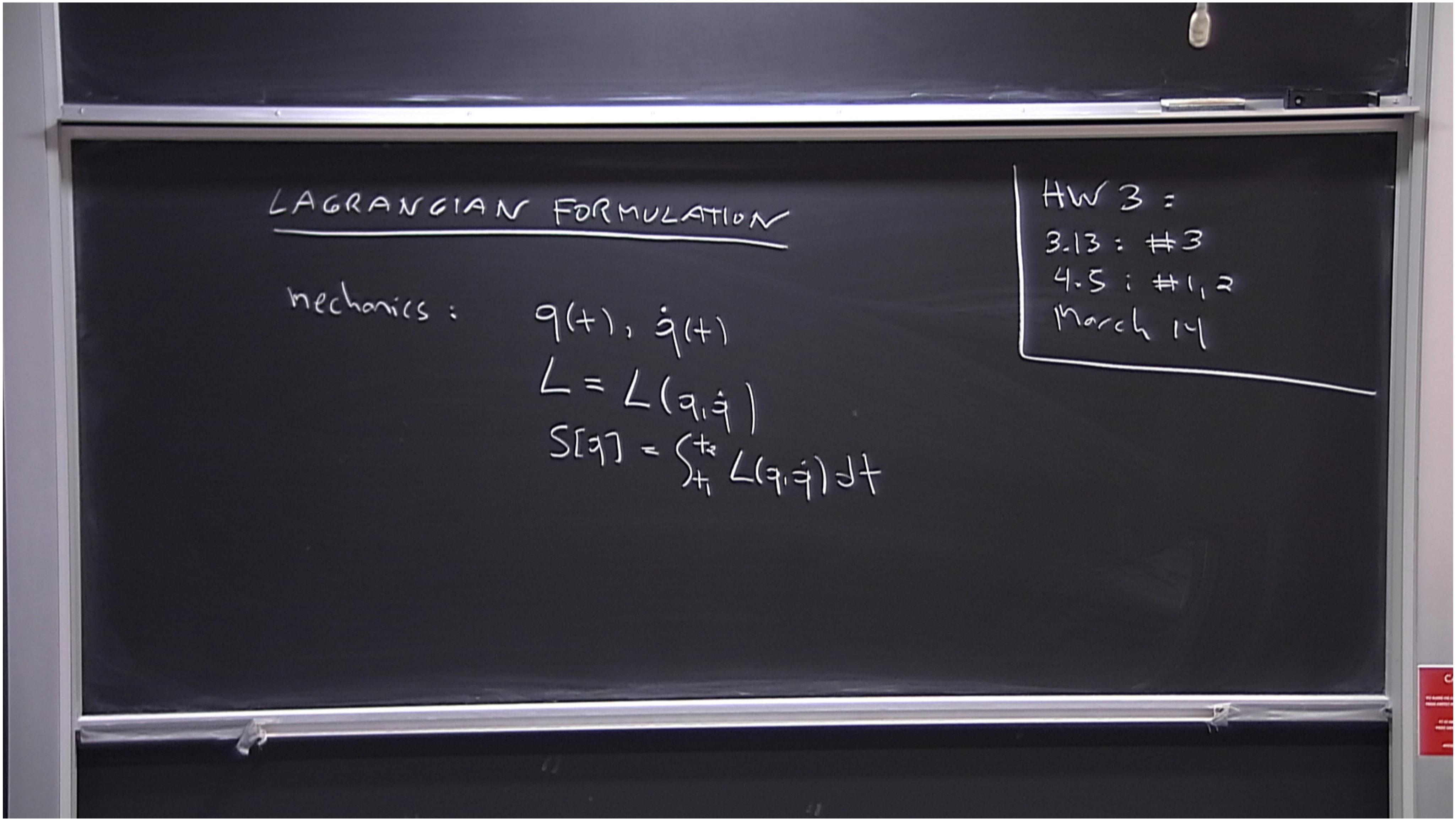
Abstract:

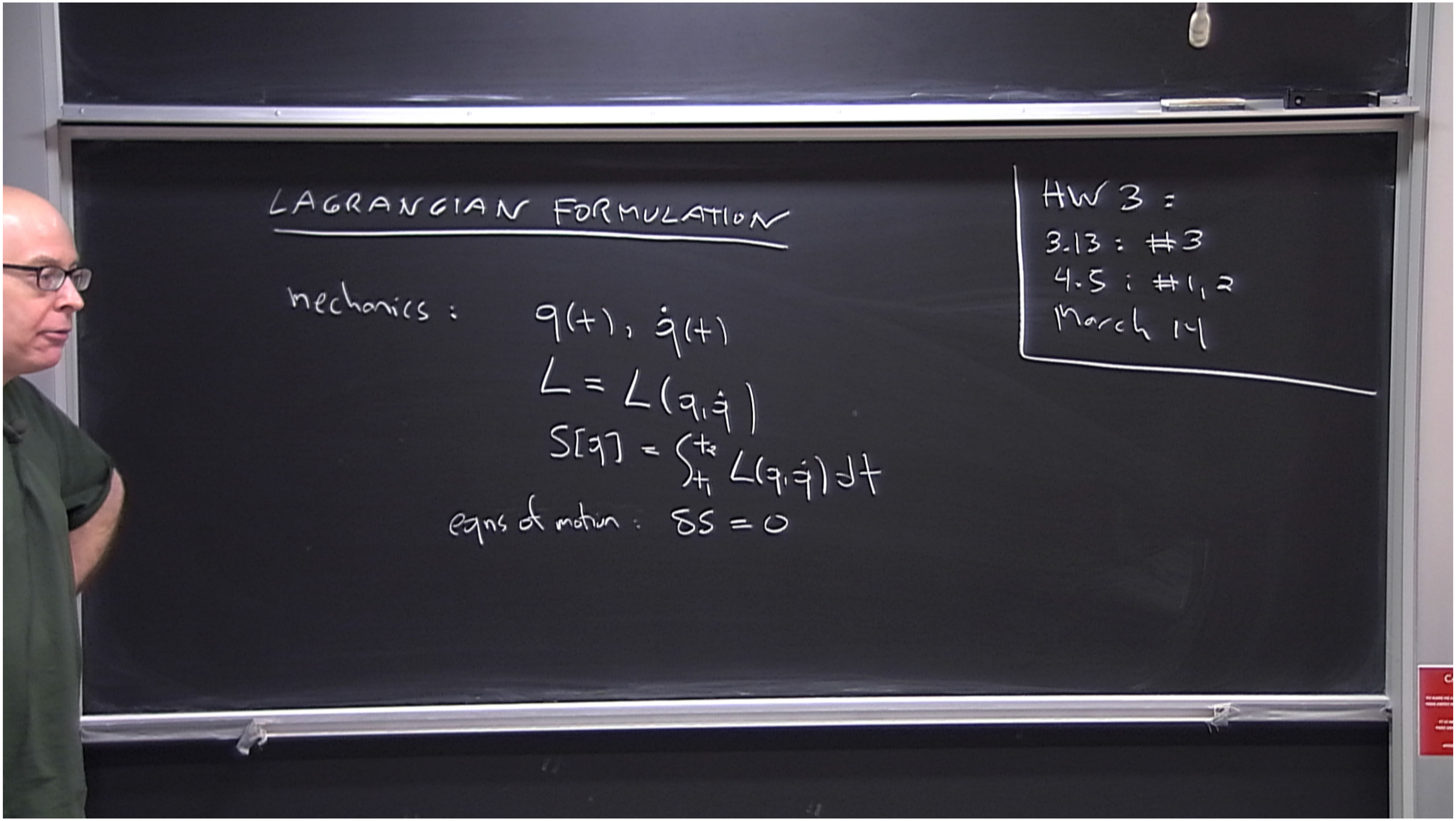

HW 3 :
3.13 : #3
4.5 : #1, 2
March 14

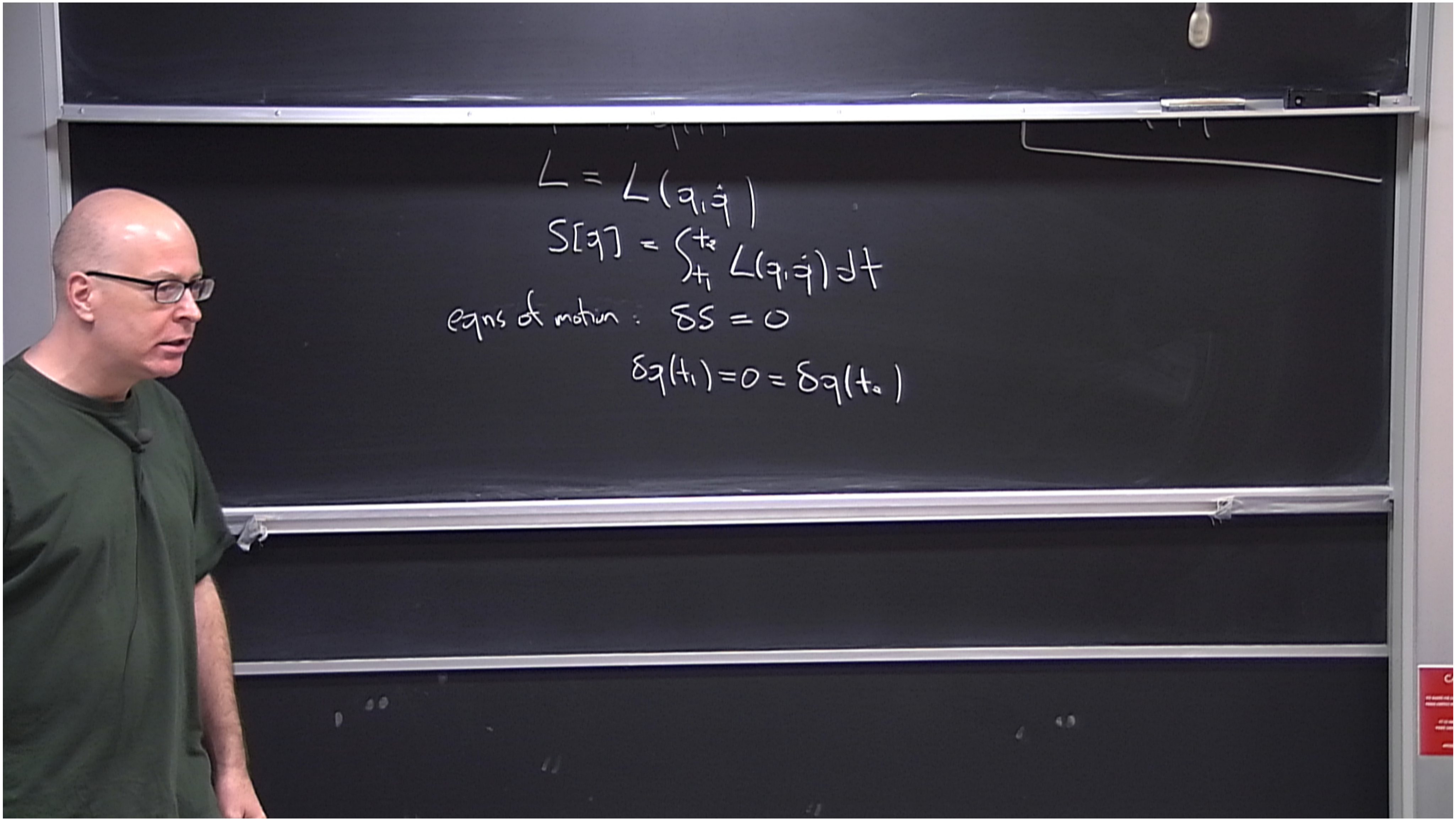

ANGIAN FORMULATION
q(t), q̇(t)
HW 3 :
3.13 : #3
4.5 : #1, 2
March 14

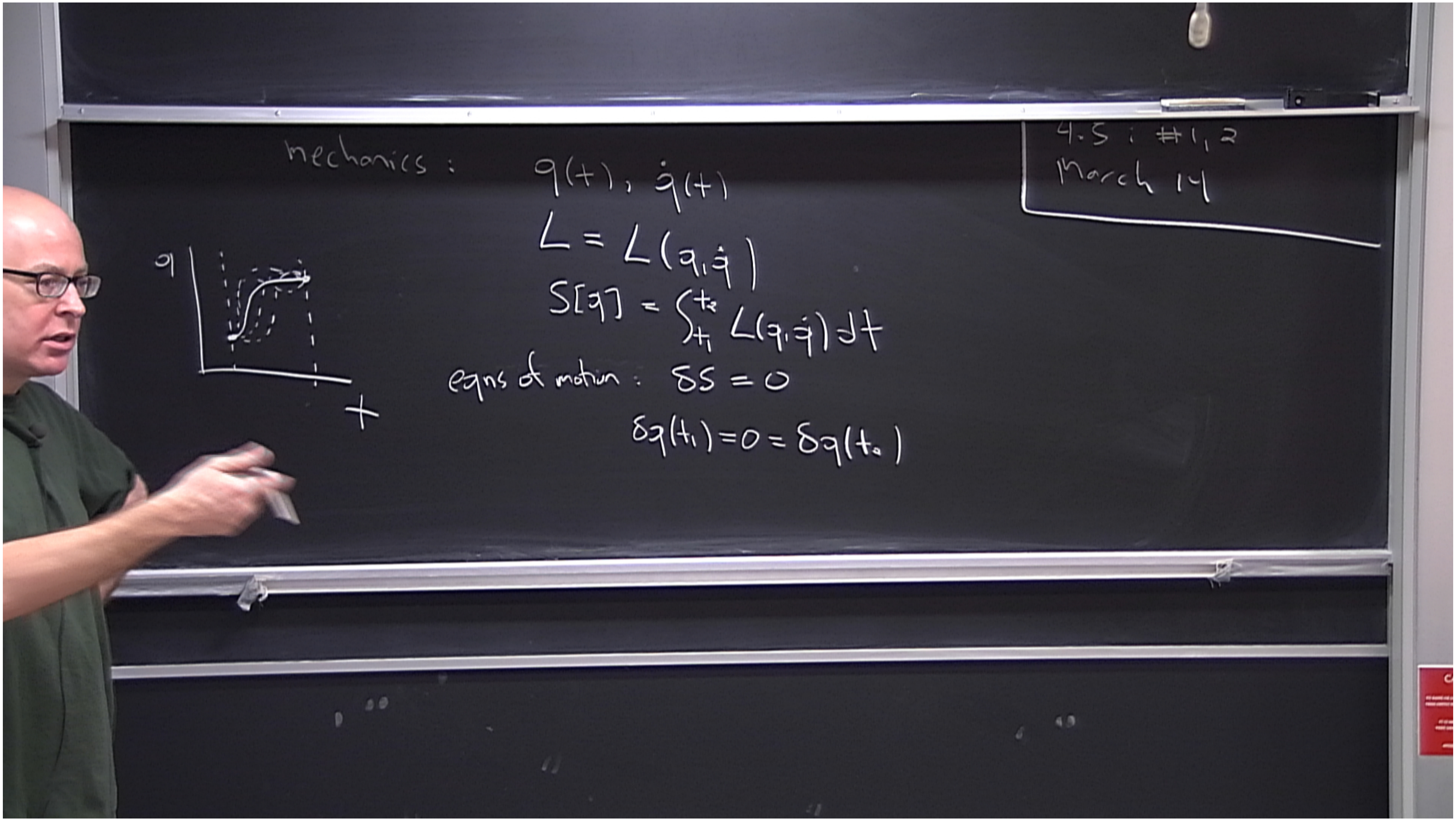


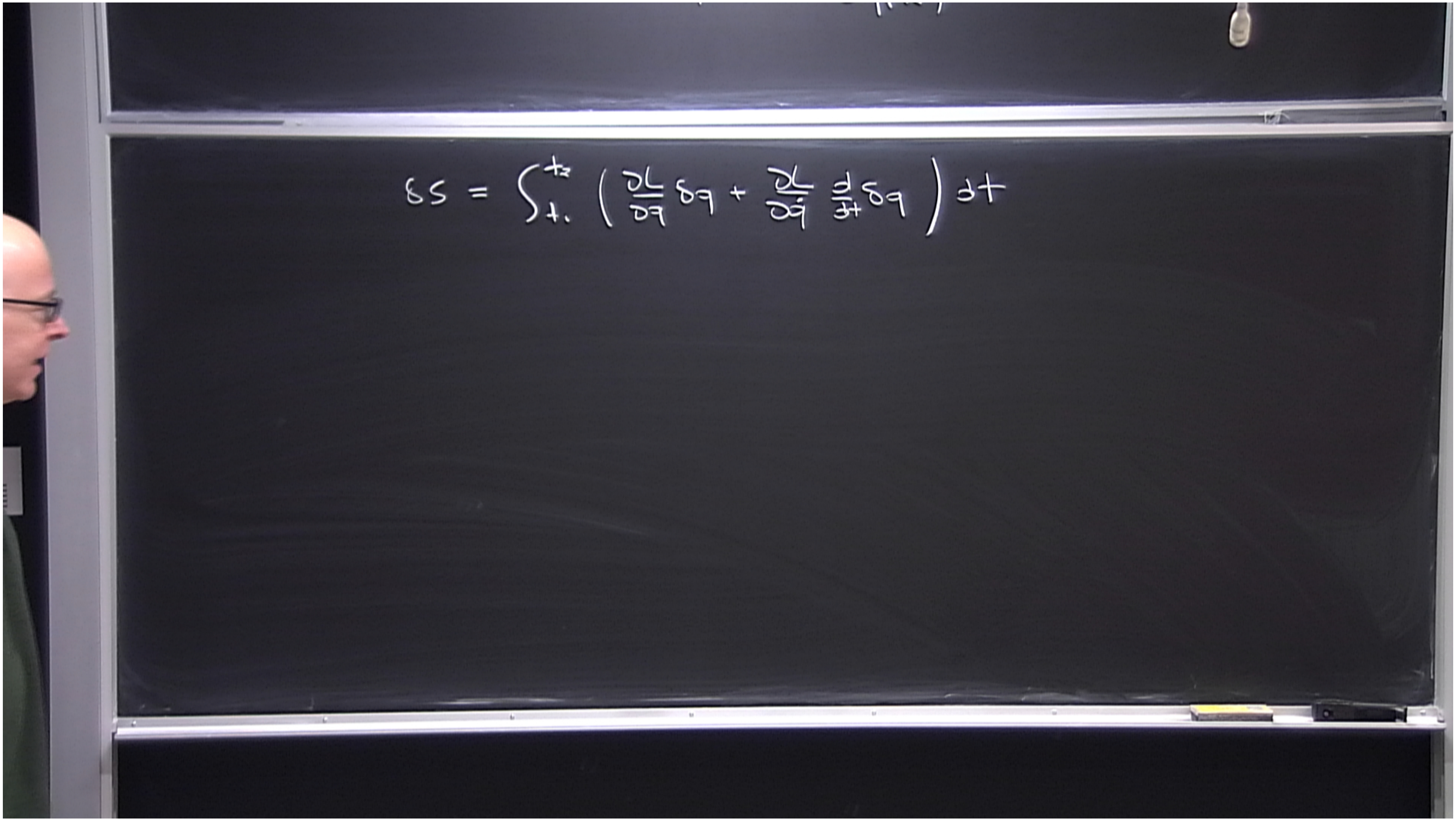
FORMULATION
q(t), q̇(t)
L = L(q, q̇)
HW 3 :
3.13 : #3
4.5 : #1, 2
March 14

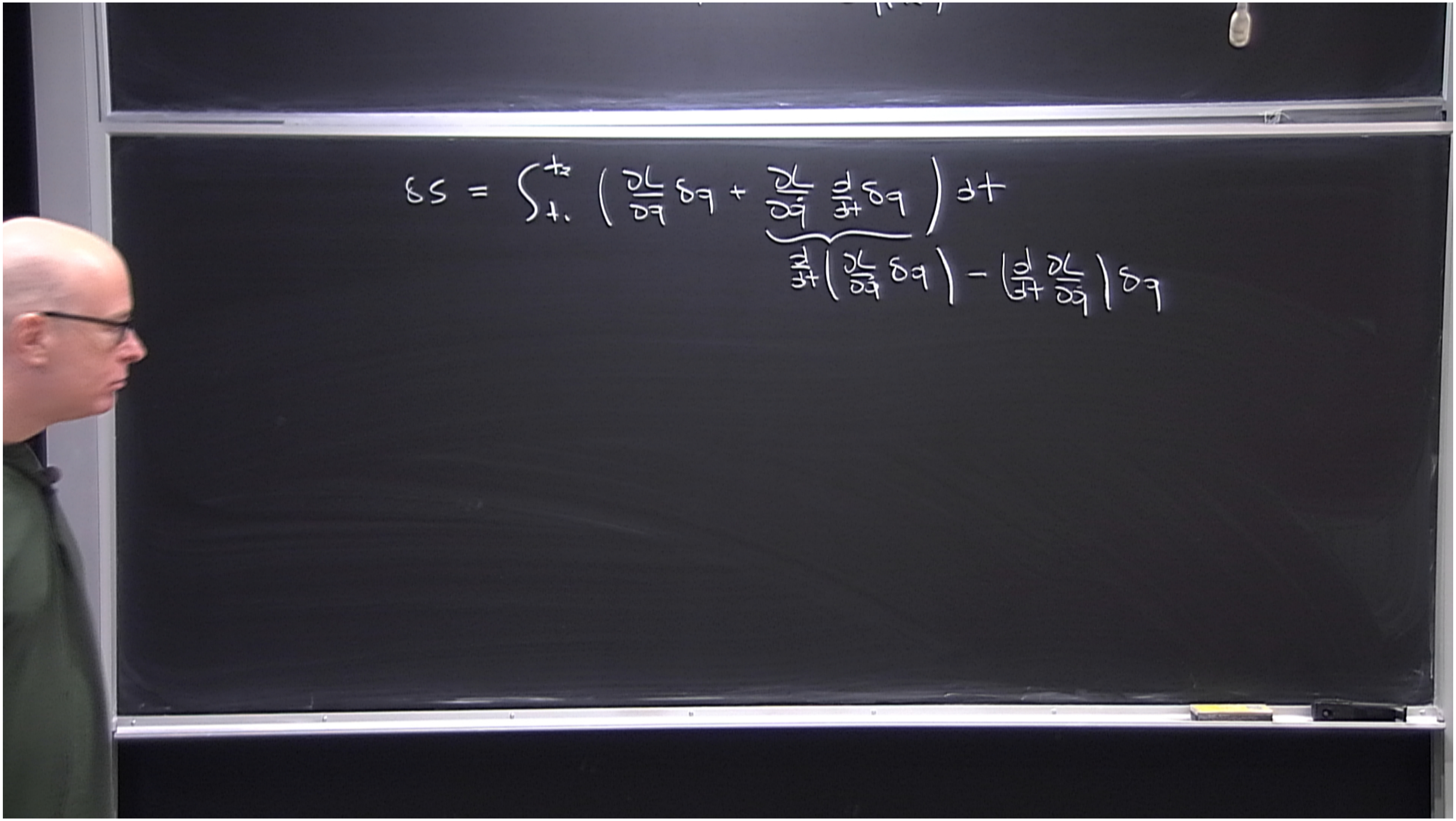

LAGRANGIAN FORMULATION
mechanics :
$q(t)$, $\dot{q}(t)$
$L = L(q, \dot{q})$
$S[q] = \int_{t_1}^{t_2} L(q, \dot{q})\, dt$
HW 3 :
3.13 : #3
4.5 : #1, 2
March 14

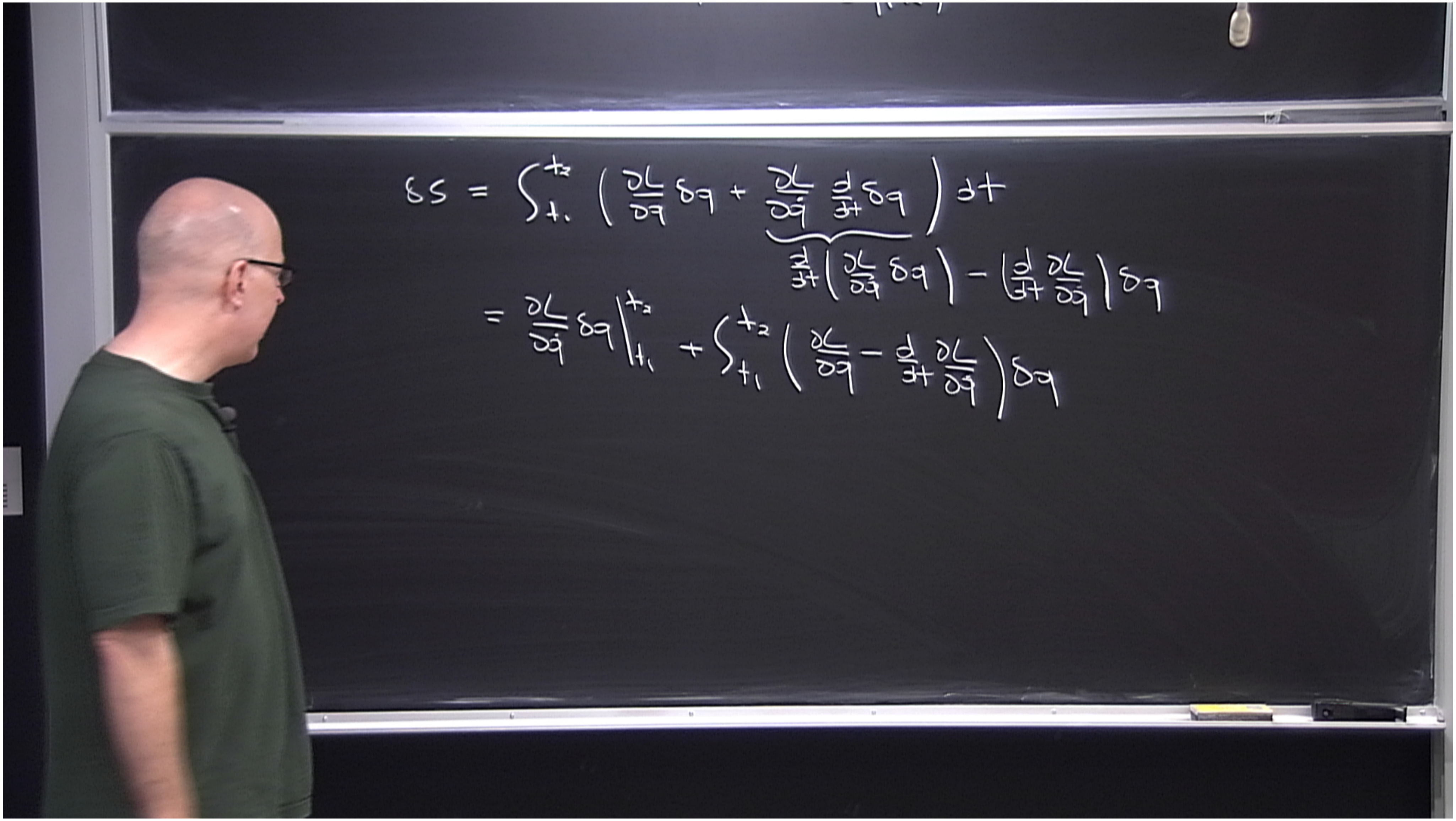

LAGRANGIAN FORMULATION
mechanics : $q(t), \dot{q}(t)$
$L = L(q, \dot{q})$
$S[q] = \int_{t_1}^{t_2} L(q, \dot{q})\, dt$
eqns of motion : $\delta S = 0$
HW 3 :
3.13 : #3
4.5 : #1, 2
March 14


$L = L(q, \dot{q})$
$S[q] = \int_{t_1}^{t_2} L(q, \dot{q})\, dt$
eqns of motion: $\delta S = 0$
$\delta q(t_1) = 0 = \delta q(t_2)$


mechanics :
$q(t)$, $\dot{q}(t)$
$L = L(q, \dot{q})$
$S[q] = \int_{t_1}^{t_2} L(q, \dot{q})\, dt$
eqns of motion : $\delta S = 0$
$\delta q(t_1) = 0 = \delta q(t_2)$
4.5 : #1, 2
March 14
q
t


$\delta S = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \frac{d}{dt} \delta q \right) dt$


$\delta S = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \frac{d}{dt} \delta q \right) dt$
$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}} \delta q \right) - \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \right) \delta q$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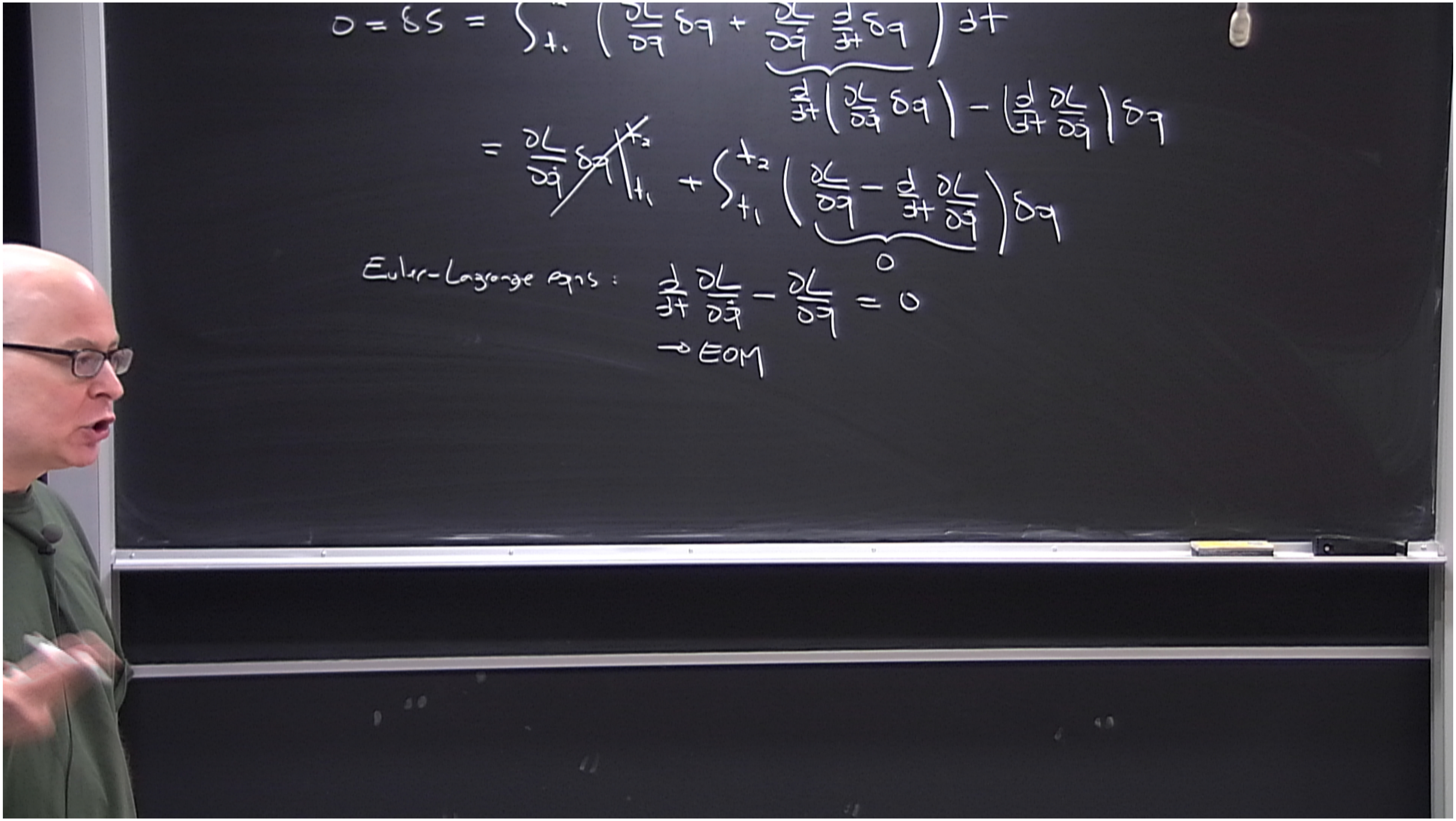
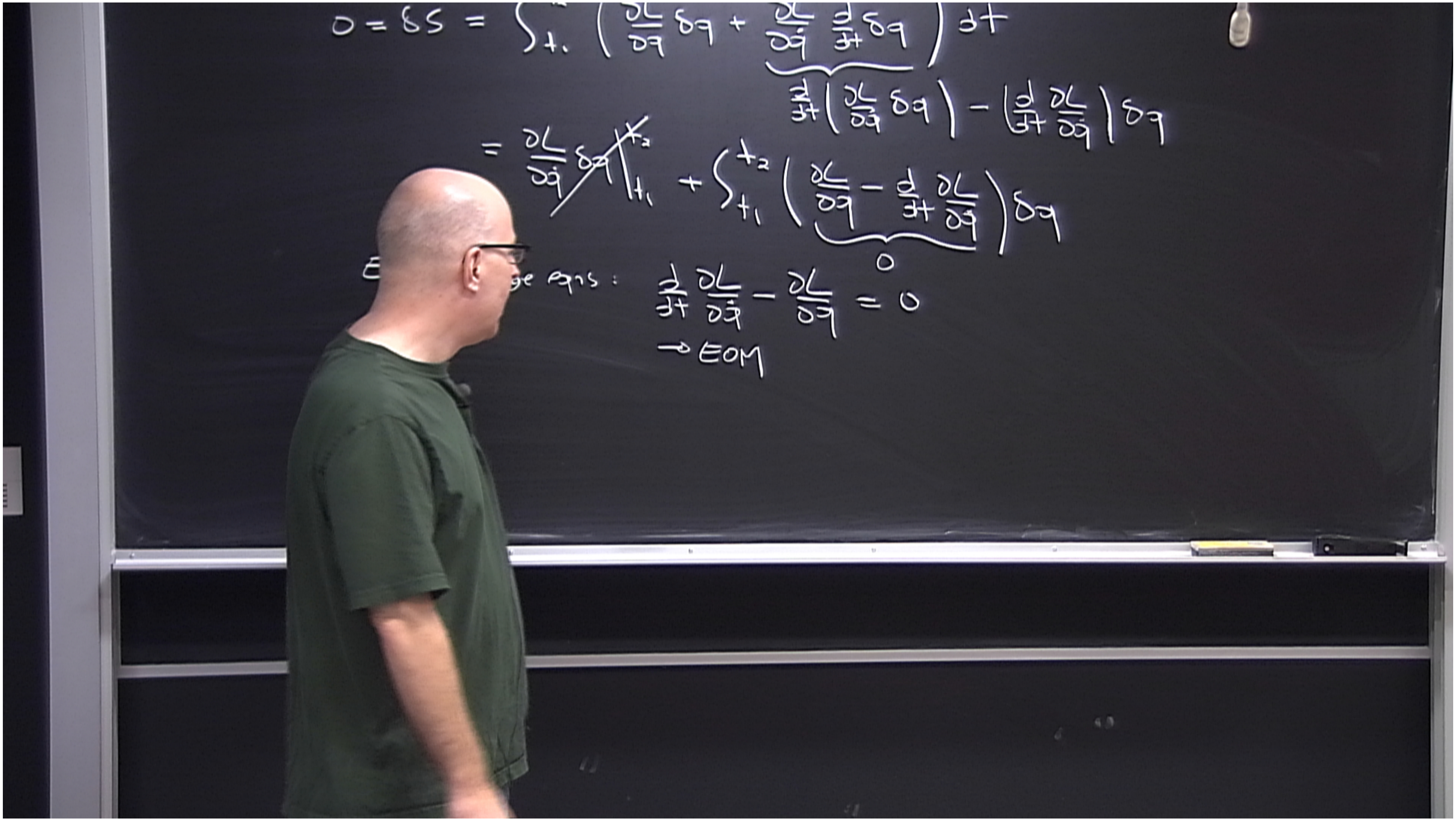
$= \frac{\partial L}{\partial \dot q} \delta q \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot q} \right) \delta q$



$0 = \delta S = \int_{t_1} \left(\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot q}\frac{d}{dt}\delta q \right) dt$
$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\delta q\right) - \left(\frac{d}{dt}\frac{\partial L}{\partial \dot q}\right)\delta q$
$= \frac{\partial L}{\partial \dot q}\delta q \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot q} \right)\delta q$
0
Euler-Lagrange eqns : $\frac{d}{dt}\frac{\partial L}{\partial \dot q} - \frac{\partial L}{\partial q} = 0$
→ EOM


$0 = \delta S = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot q} \frac{d}{dt} \delta q \right) dt$
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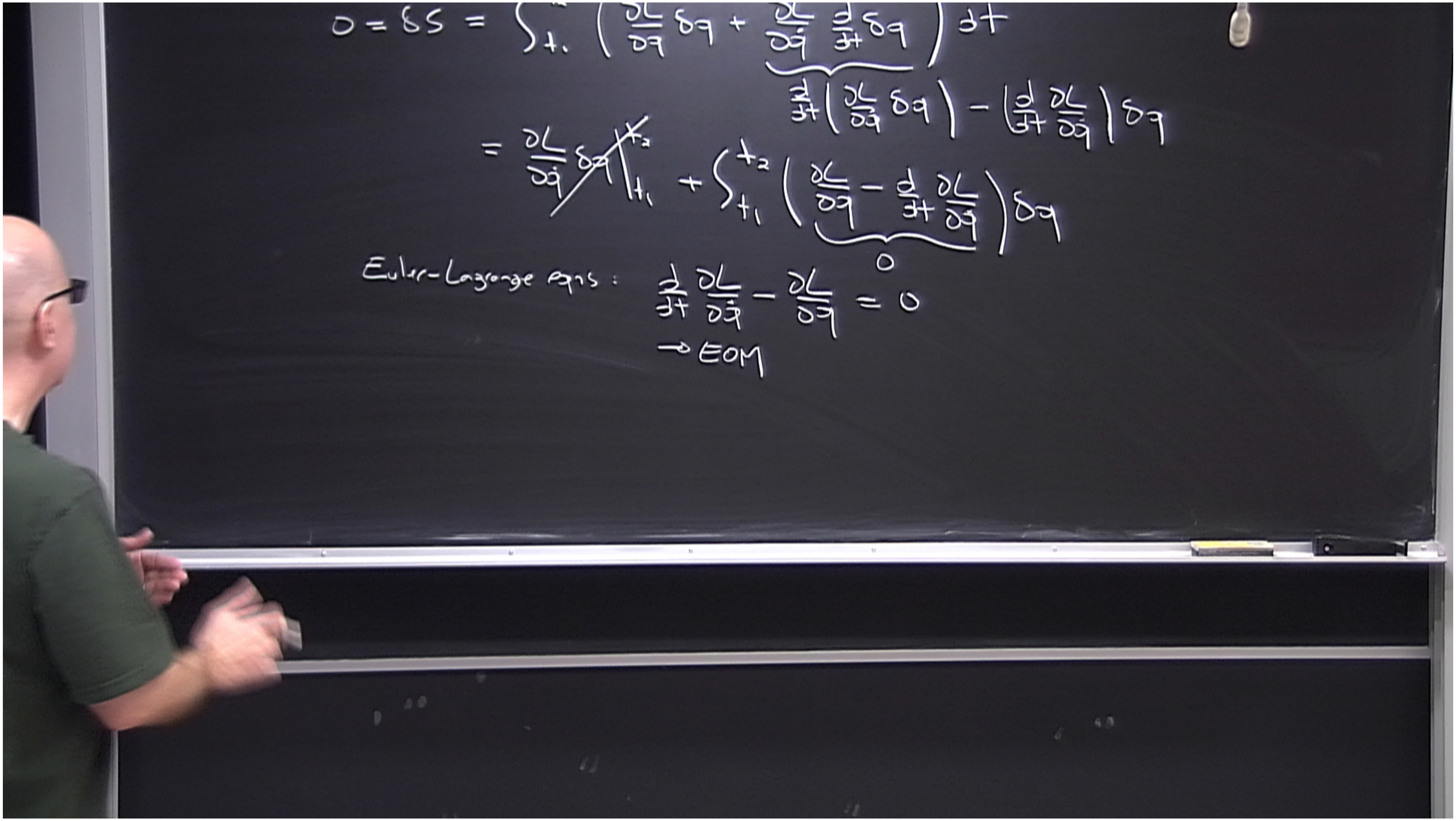
eqns: $\frac{d}{dt} \frac{\partial L}{\partial \dot q} - \frac{\partial L}{\partial q} = 0$
→ EOM


$0 = \delta S = \int_{t_1} \left(\frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot q} \frac{d}{dt} \delta q \right) dt$
$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q} \delta q \right) - \left(\frac{d}{dt} \frac{\partial L}{\partial \dot q} \right) \delta q$
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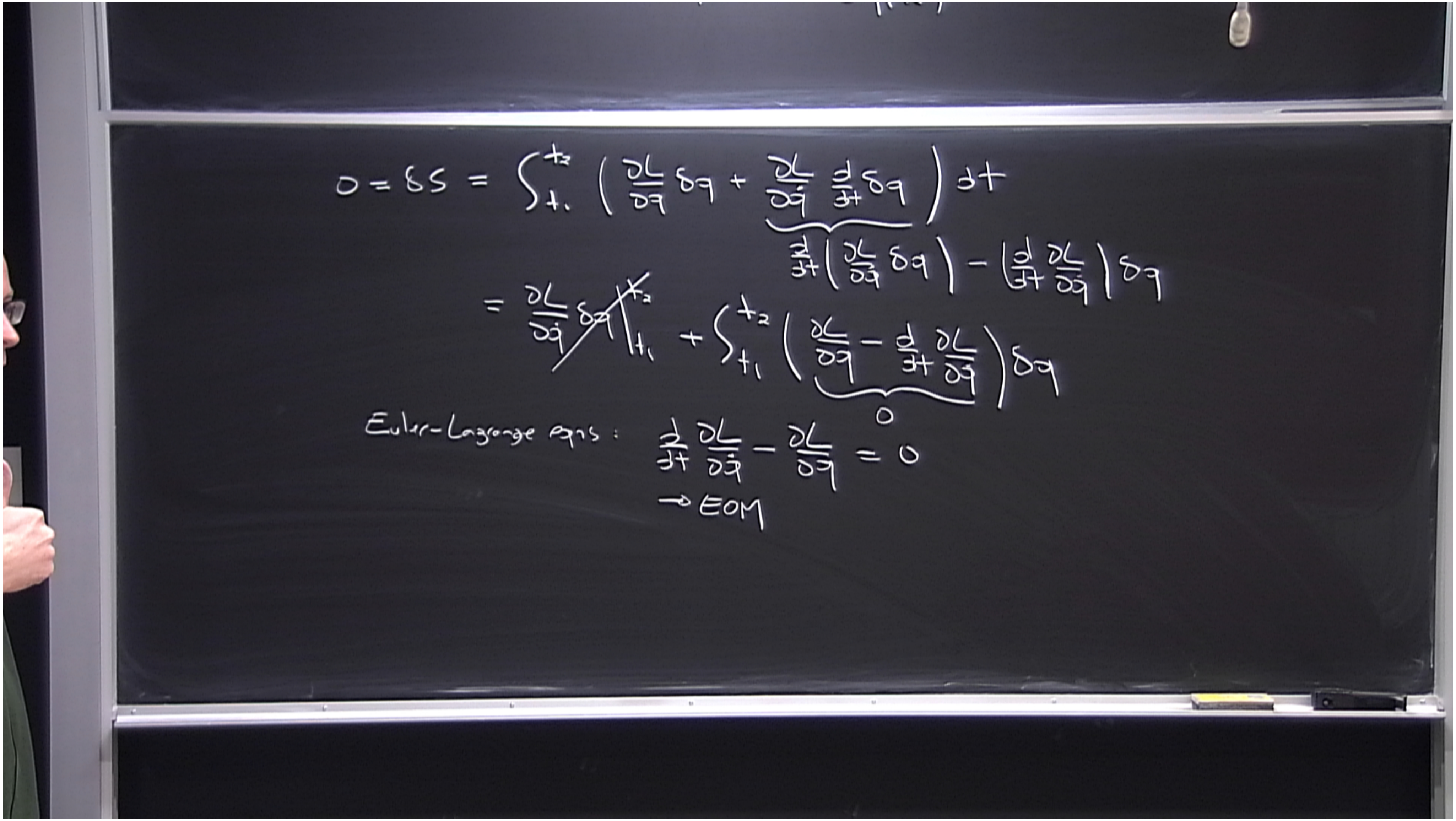
Euler-Lagrange eqns: $\frac{d}{dt} \frac{\partial L}{\partial \dot q} - \frac{\partial L}{\partial q} = 0$
→ EOM


$0 = \delta S = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \frac{d}{dt} \delta q \right) dt$
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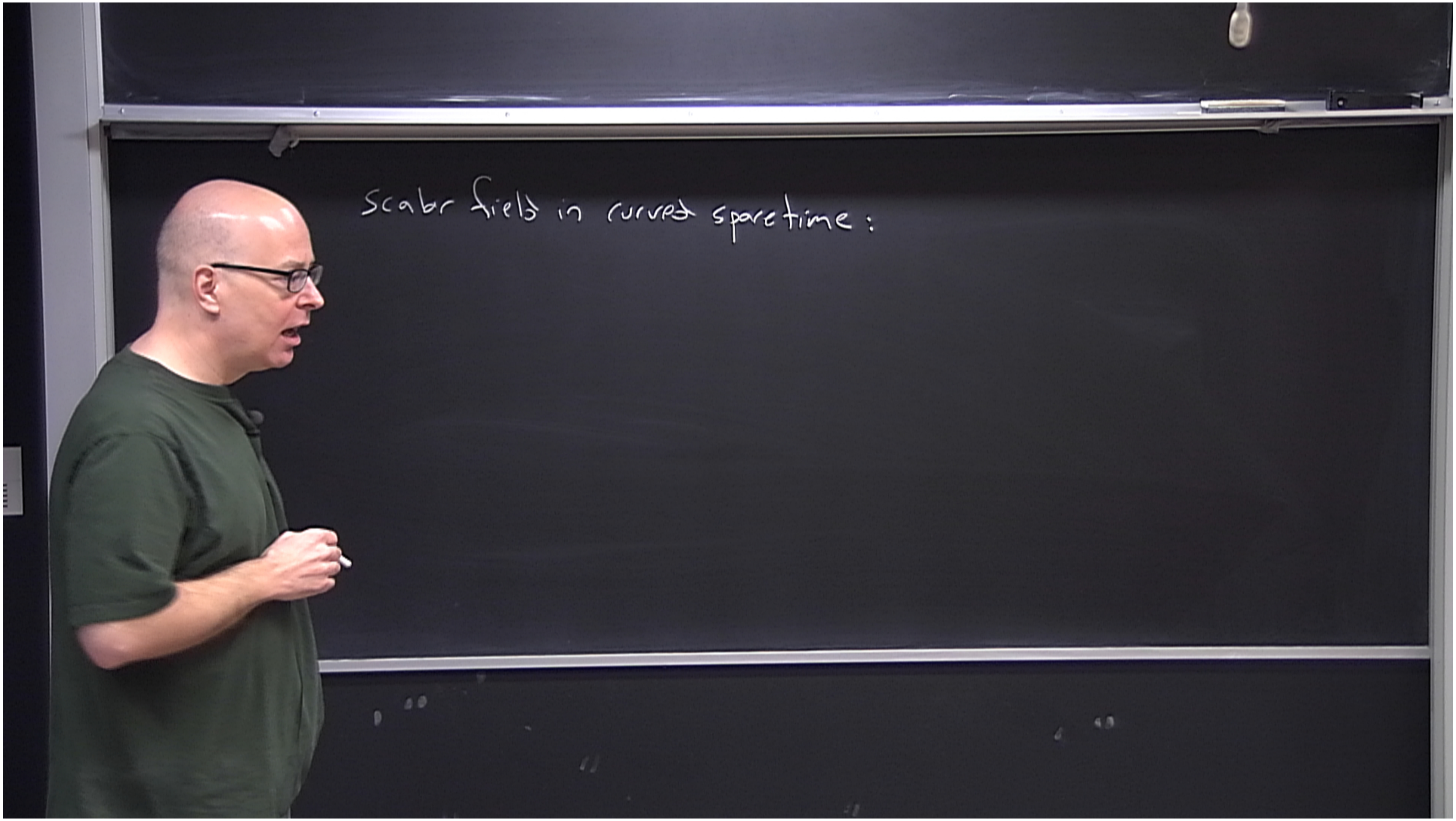
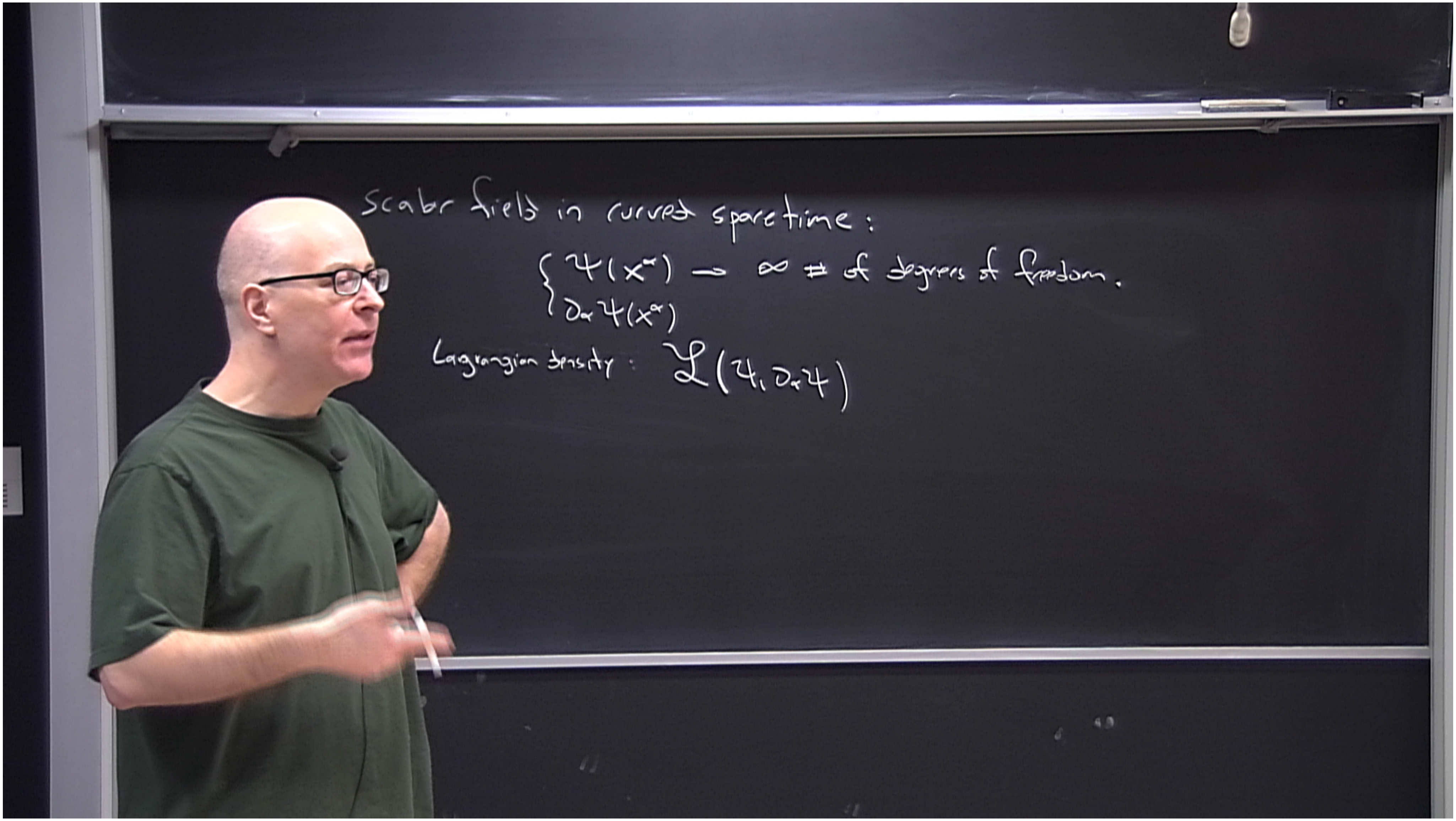
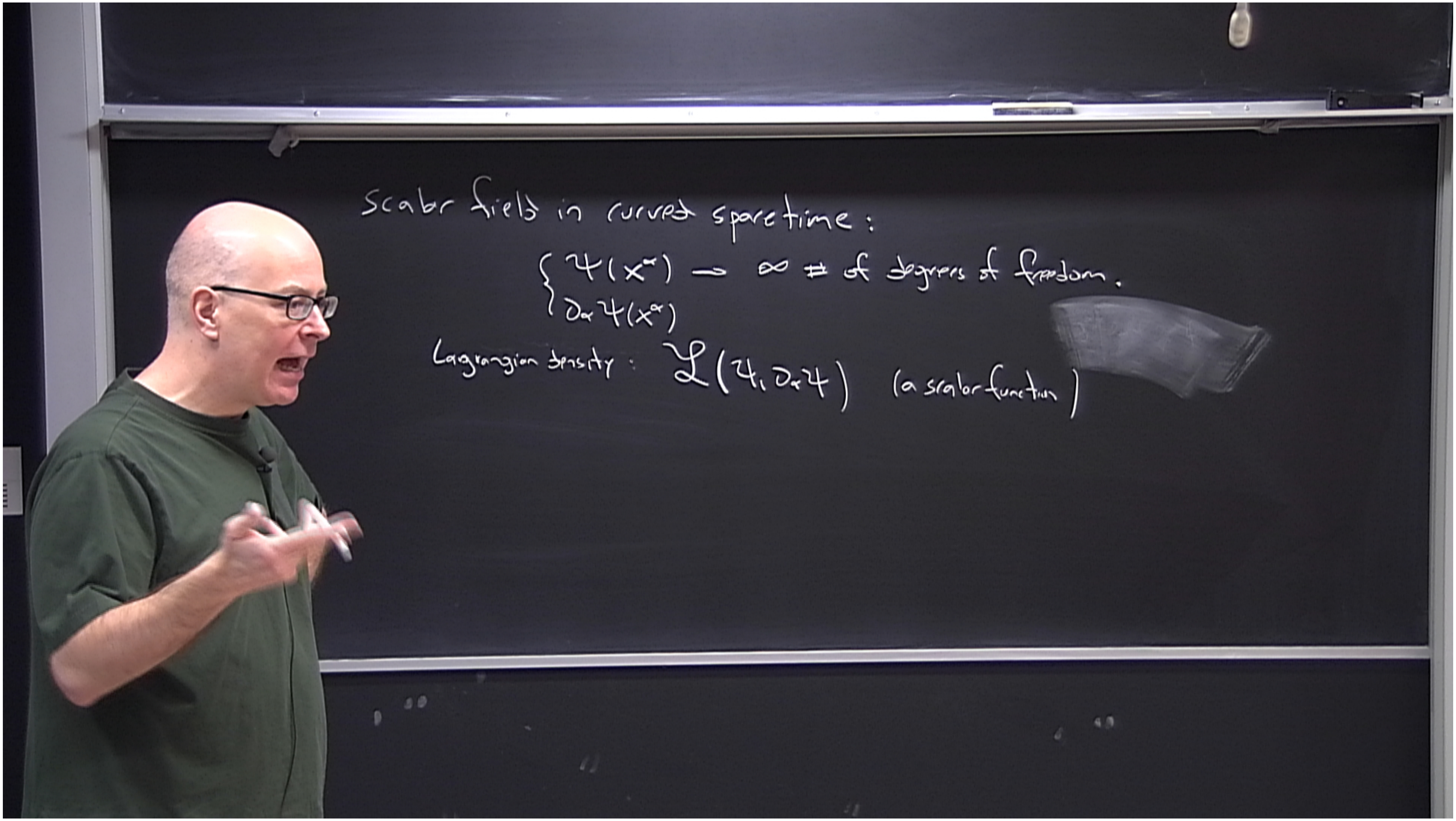
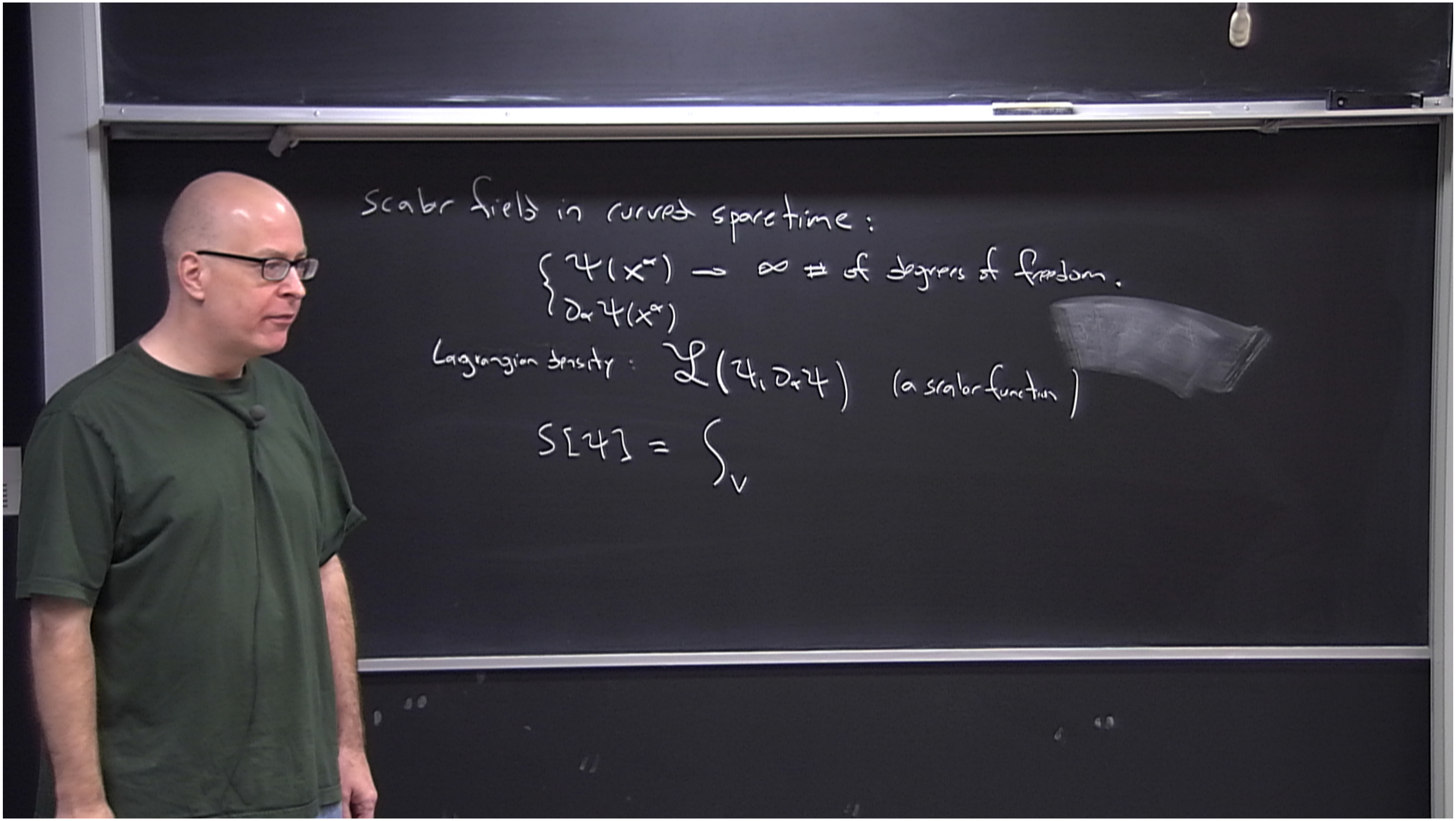
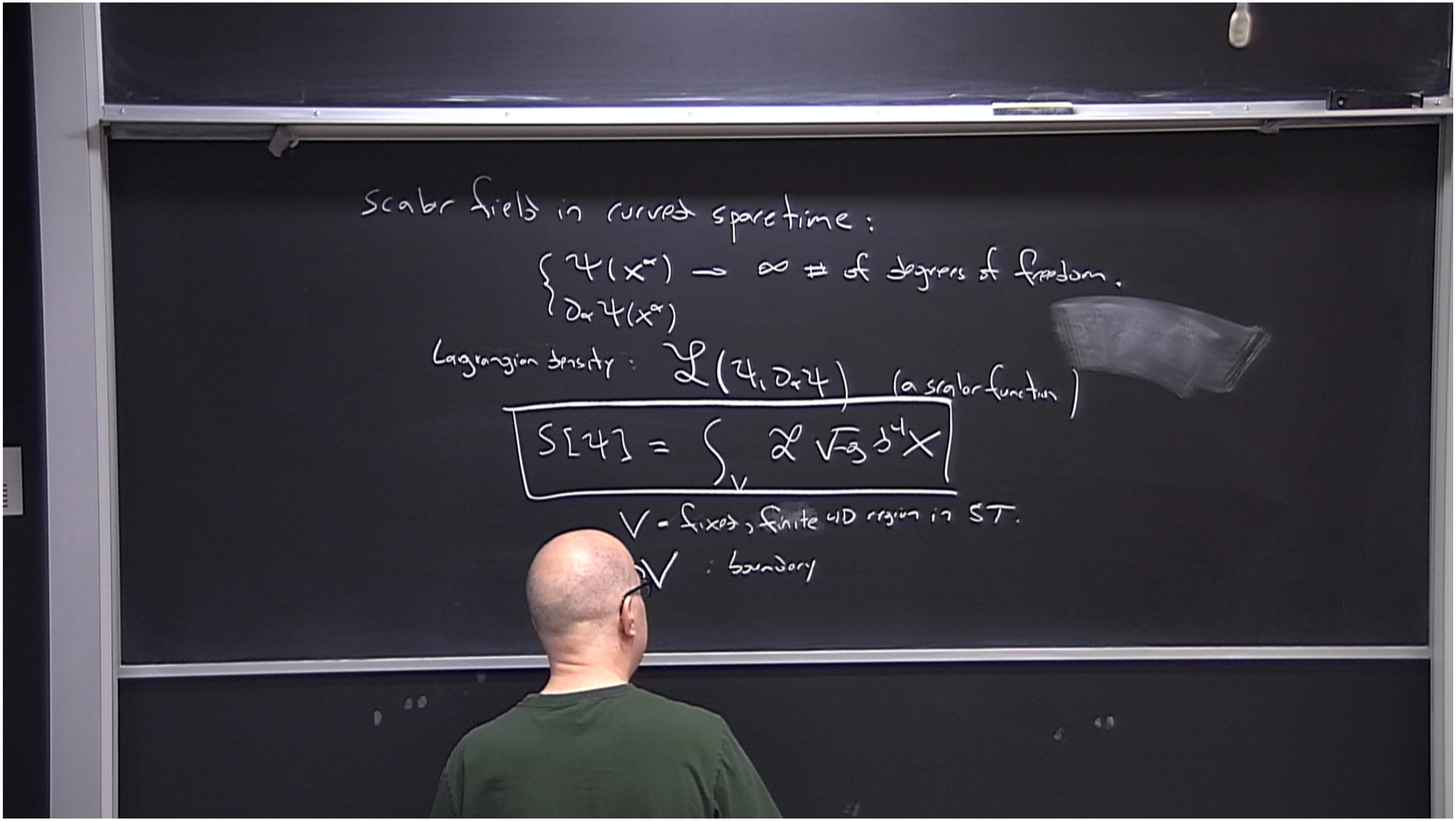
0
Euler-Lagrange eqns: $\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$
$\rightarrow$ EOM


scalar field in curved spacetime:


scalar field in curved spacetime:
$\{ \psi(x^\alpha) \to \infty$ # of degrees of freedom.
$\partial_\alpha \psi(x^\alpha)$
Lagrangian density: $\mathcal{L}(\psi, \partial_\alpha \psi)$


scalar field in curved spacetime :
$\psi(x^\alpha)$ → ∞ # of degrees of freedom.
$\partial_\alpha \psi(x^\alpha)$
Lagrangian density : $\mathcal{L}(\psi, \partial_\alpha \psi)$ (a scalar function)


Scalar field in curved spacetime:
{ ψ(x^α) → ∞ # of degrees of freedom.
∂_α ψ(x^α)
Lagrangian density: ℒ(ψ, ∂_α ψ) (a scalar function)
S[ψ] = ∫_V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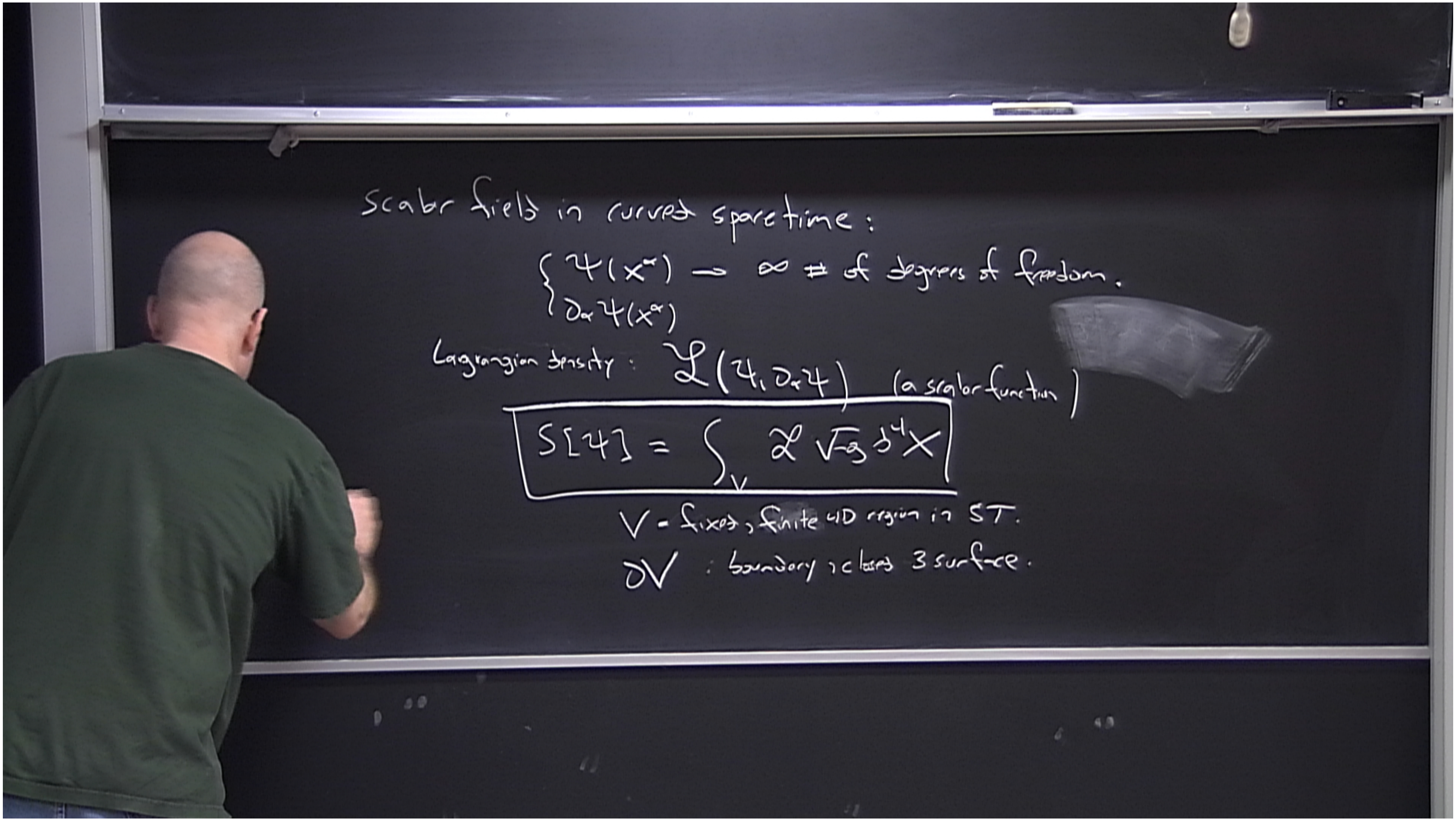
ψ(x^α) → ∞ # of degrees of freedom.
∂_α ψ(x^α)
Lagrangian density: ℒ(ψ, ∂_α ψ) (a scalar function)
S[ψ] = ∫_V ℒ √-g d^4x
V - fixed, finite 4D region in ST.
∂V : boundary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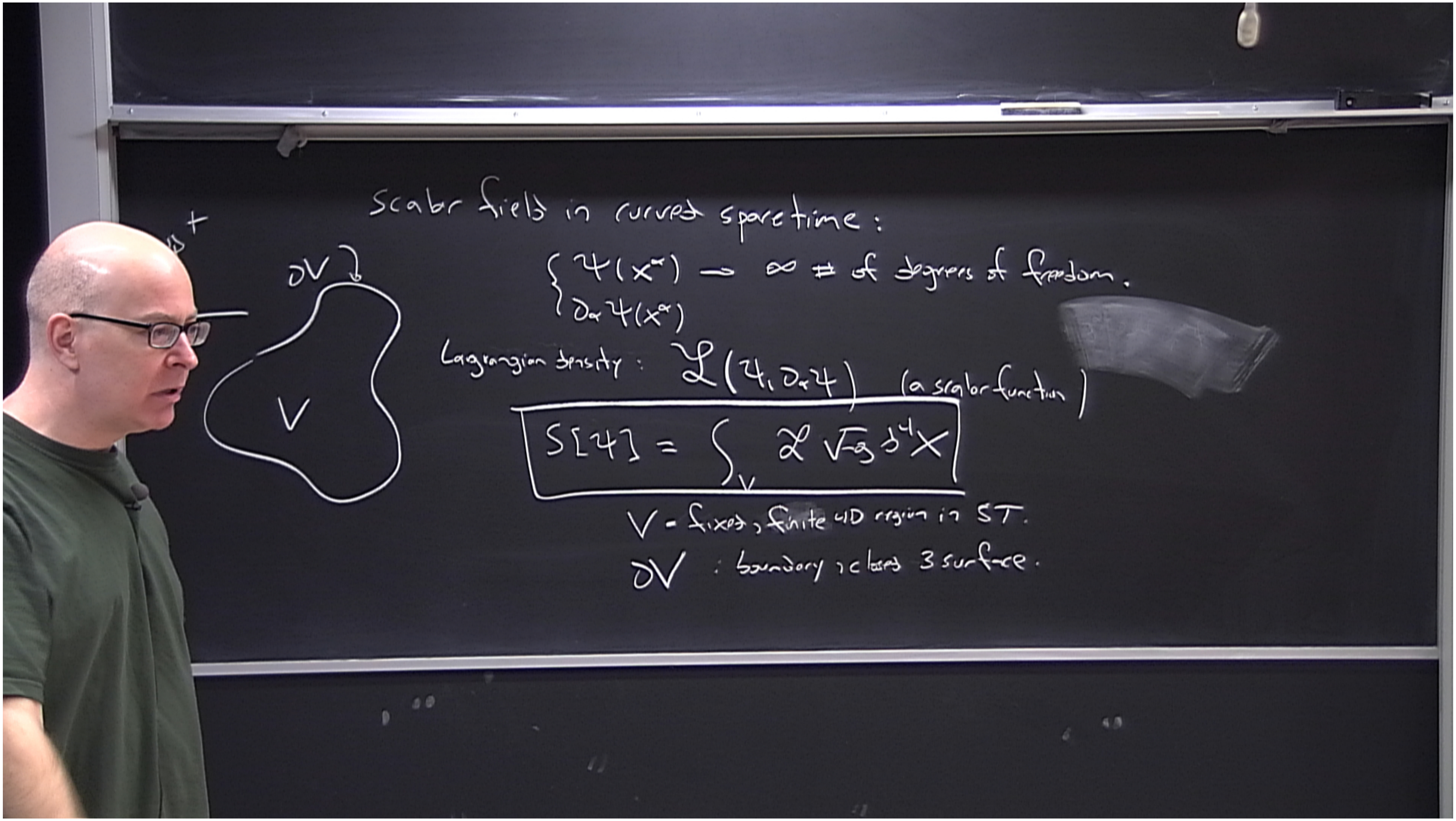
∂_α ψ(x^α)
Lagrangian density: ℒ(ψ, ∂_α ψ) (a scalar function)
S[ψ] = ∫_V ℒ √-g d^4x
V = fixed, finite 4D region in ST.
∂V : boundary is closed 3 surface.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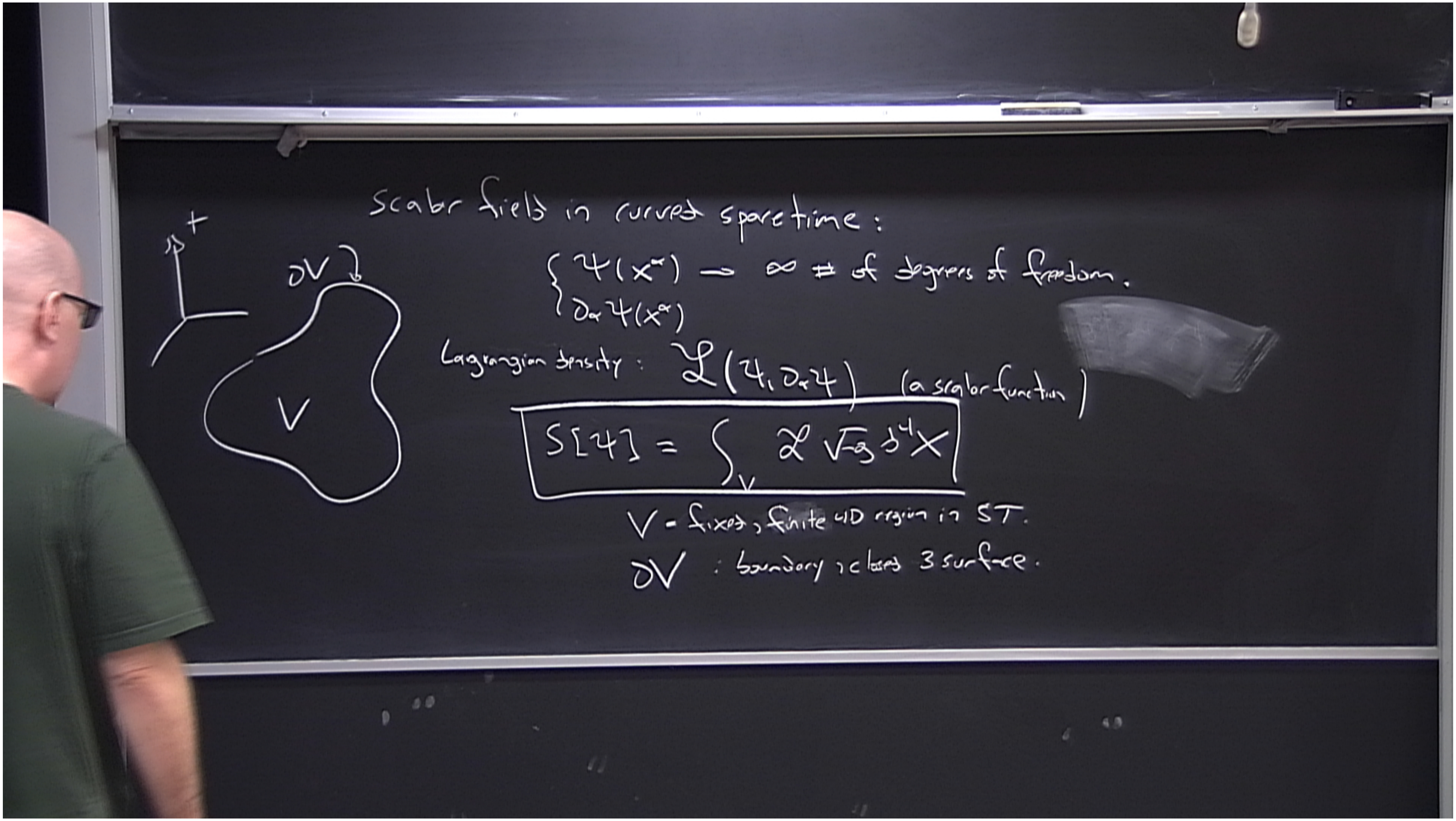
∂_α ψ(x^α)
Lagrangian density: 𝓛(ψ, ∂_α ψ) (a scalar function)
S[ψ] = ∫_V 𝓛 √-g d⁴x
V = fixed, finite 4D region in ST.
∂V : boundary, closed 3 surface.
∂V
V


Scalar fields in curved spacetime:
$\psi(x^\alpha)$ → ∞ # of degrees of freedom.
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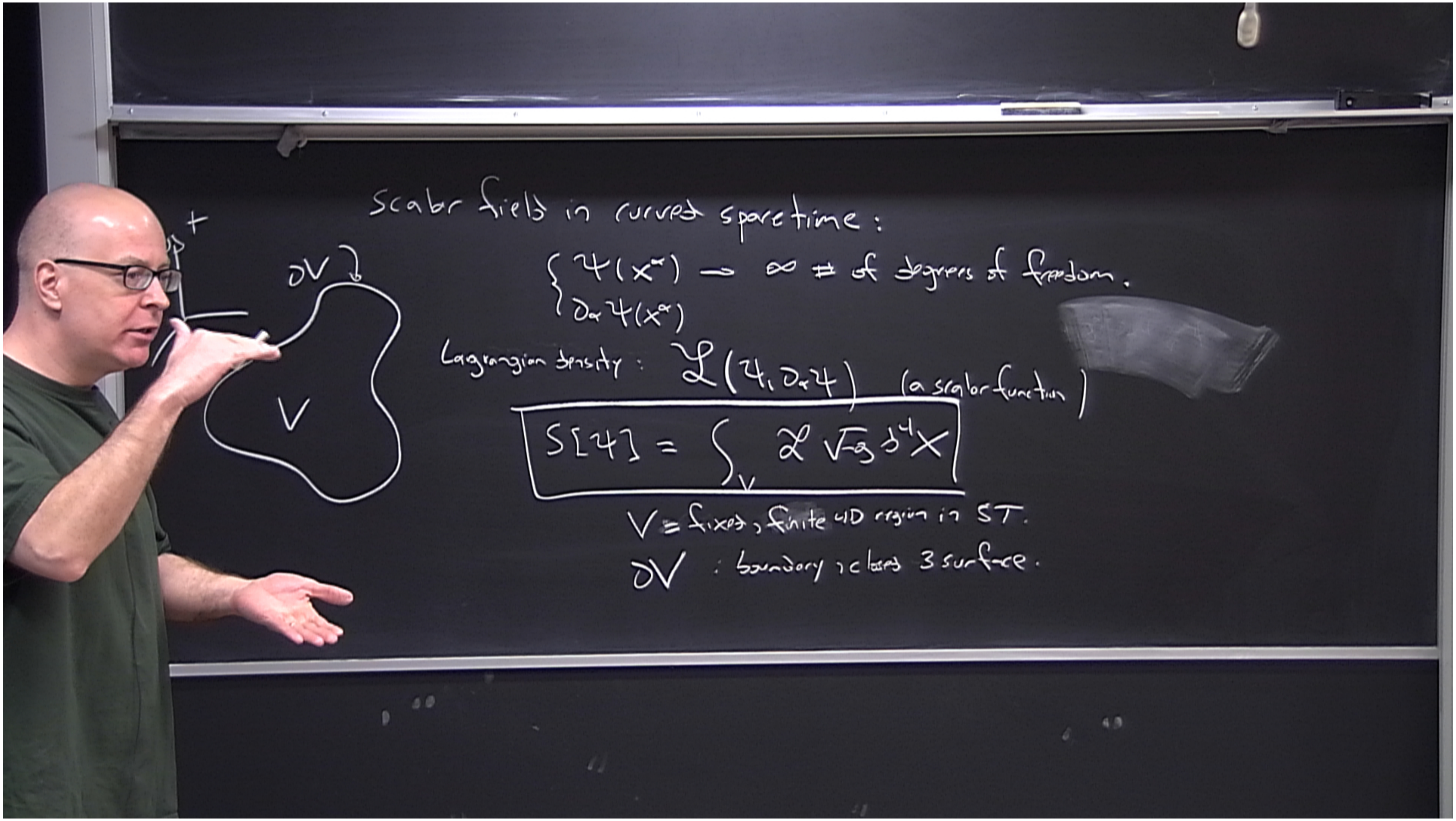
Lagrangian density: $\mathcal{L}(\psi, \partial_\alpha \psi)$ (a scalar function)
$S[\psi] = \int_V \mathcal{L} \sqrt{-g}\, d^4x$
V = fixed, finite 4D region in ST.
∂V : boundary, closed 3 surface.
t
∂V
V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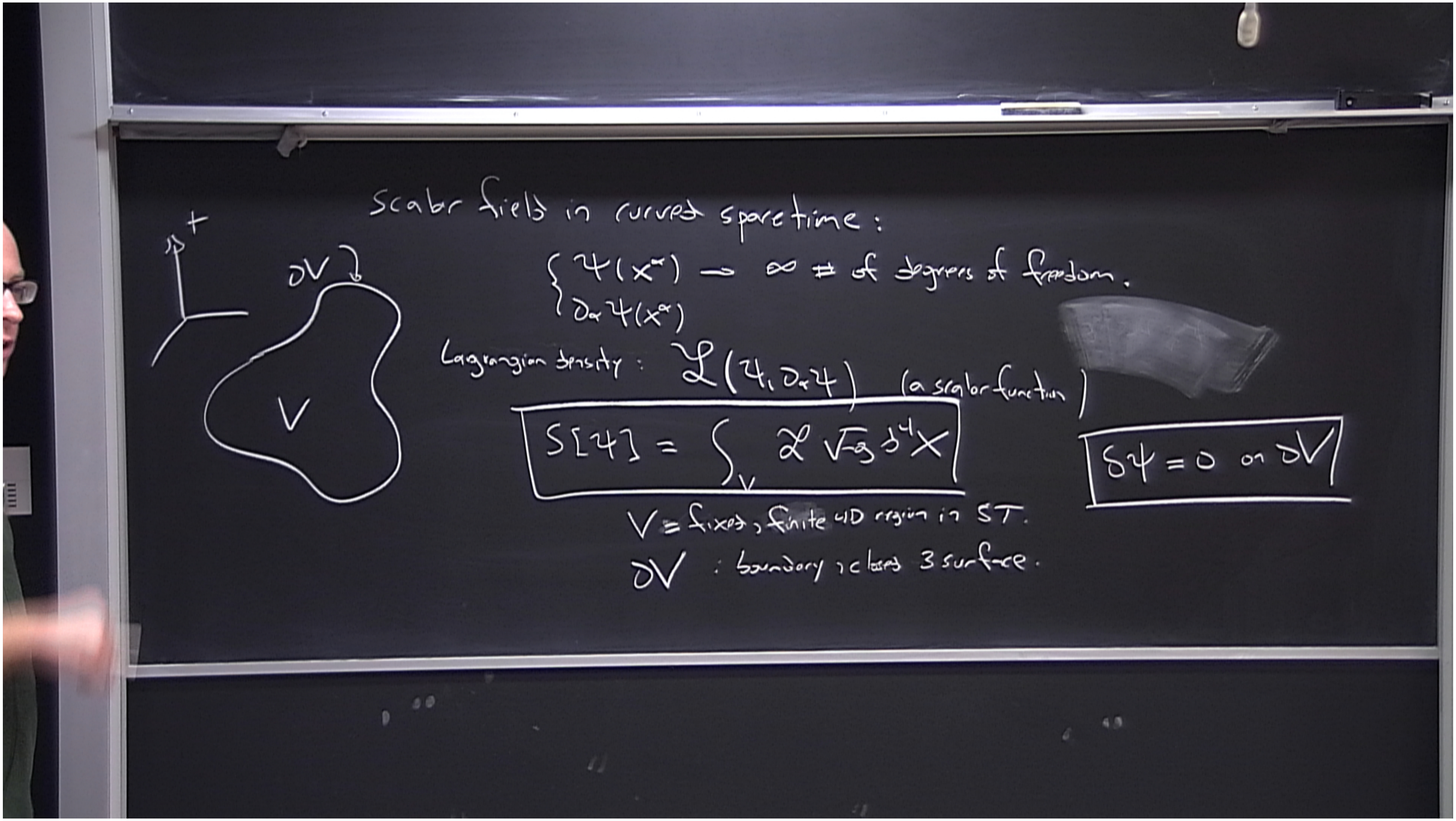
∂V
V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$\delta\psi = 0$ on ∂V
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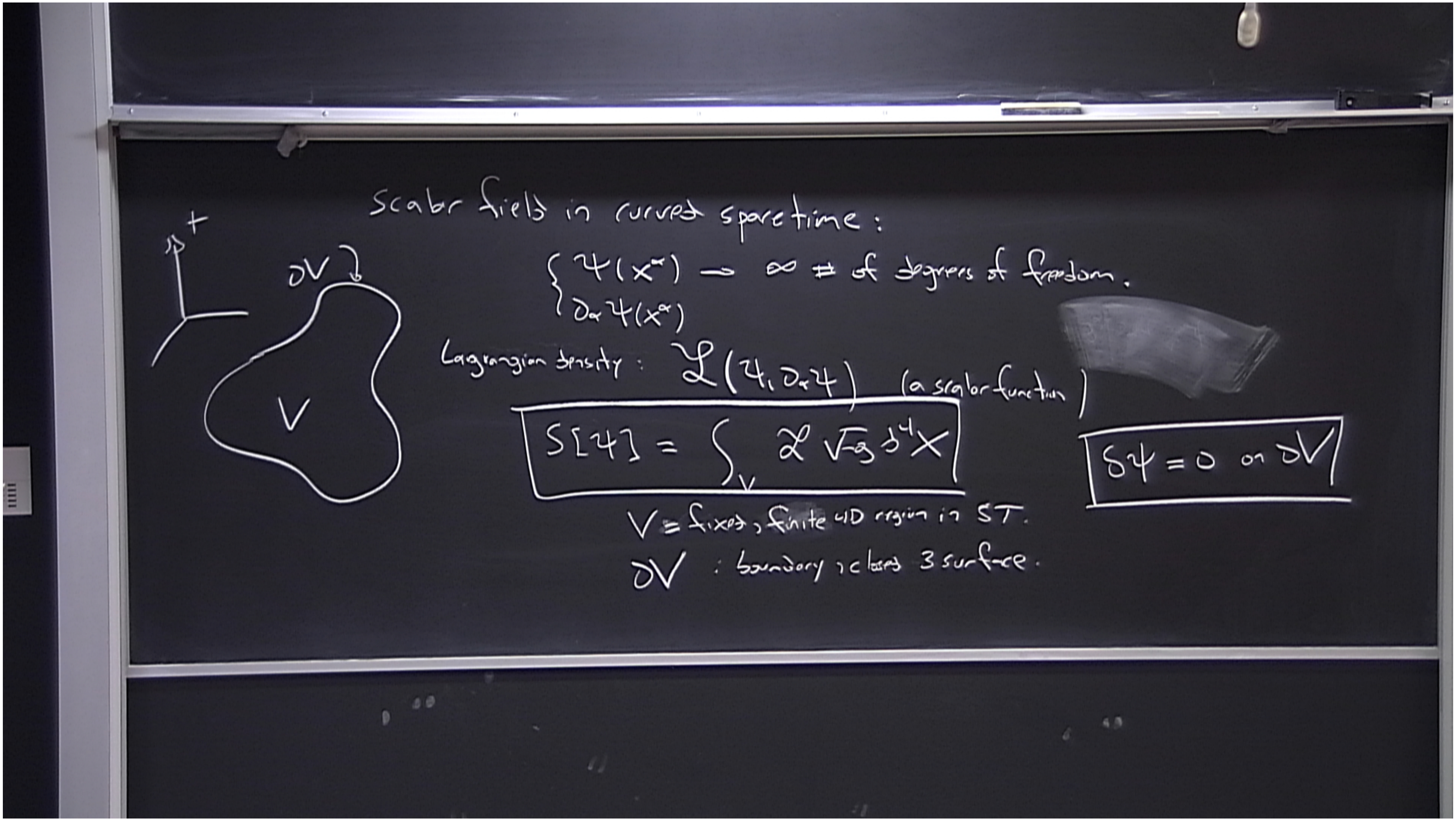
t
∂V
V


Scalar field in curved spacetime:
t
∂V
V
{ $\psi(x^\alpha)$ → ∞ # of degrees of freedom.
{ $\partial_\alpha \psi(x^\alpha)$
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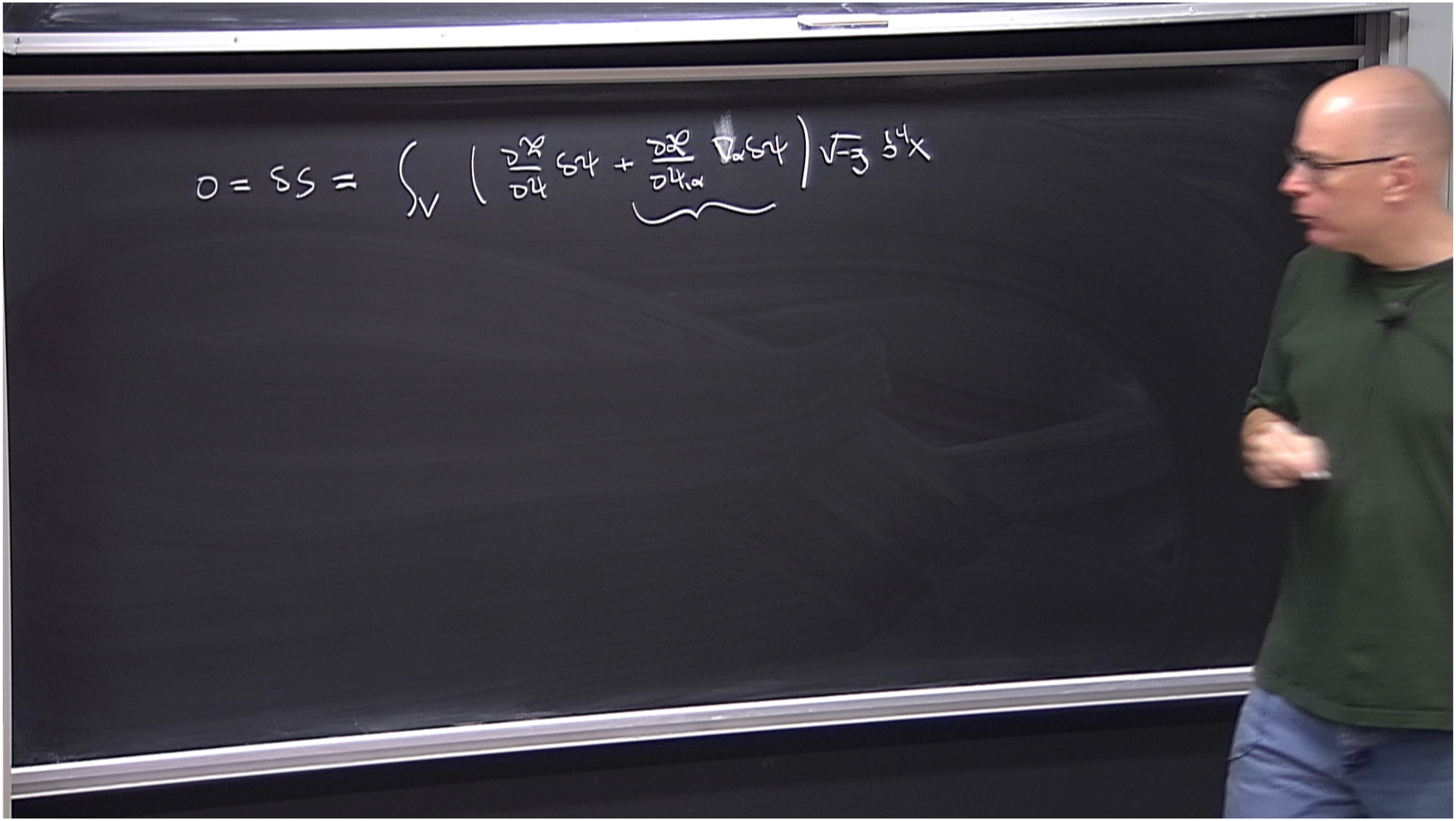
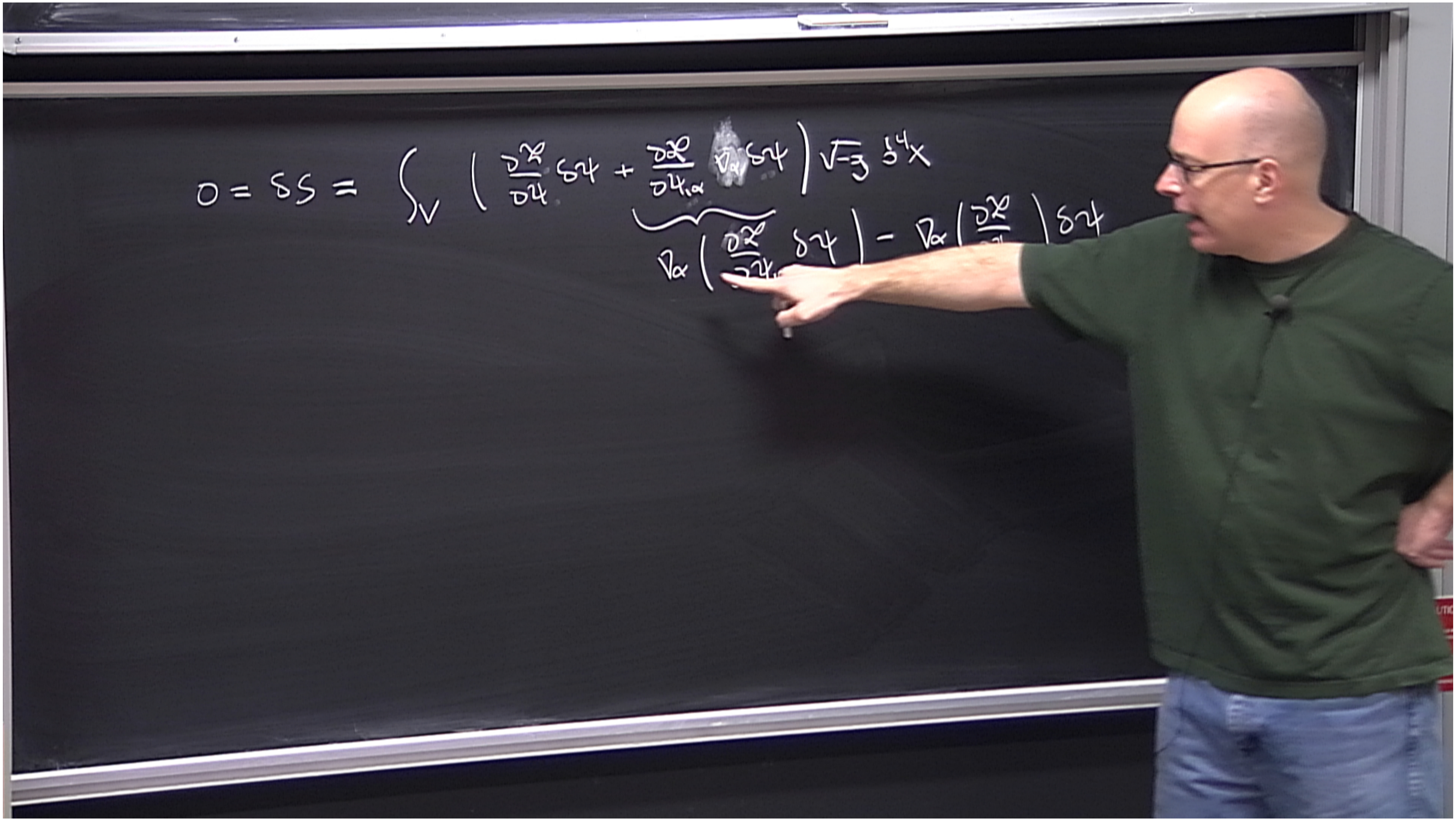
$S[\psi] = \int_V \mathcal{L} \sqrt{-g}\, d^4x$
$\delta\psi = 0$ on ∂V
V = fixed, finite 4D region in ST.
∂V : boundary, closed 3 surface.

$$0 = \delta S = \int_V \left(\frac{\partial \mathcal{L}}{\partial \psi} \delta\psi + \frac{\partial \mathcal{L}}{\partial \psi_{,\alpha}} \nabla_\alpha \delta\psi \right) \sqrt{-g}\, d^4x$$


$0 = \delta S = \int_V \left(\frac{\partial \mathcal{L}}{\partial \psi} \delta \psi + \frac{\partial \mathcal{L}}{\partial \psi_{,\alpha}} \nabla_\alpha \delta \psi \right) \sqrt{-g}\, d^4x$

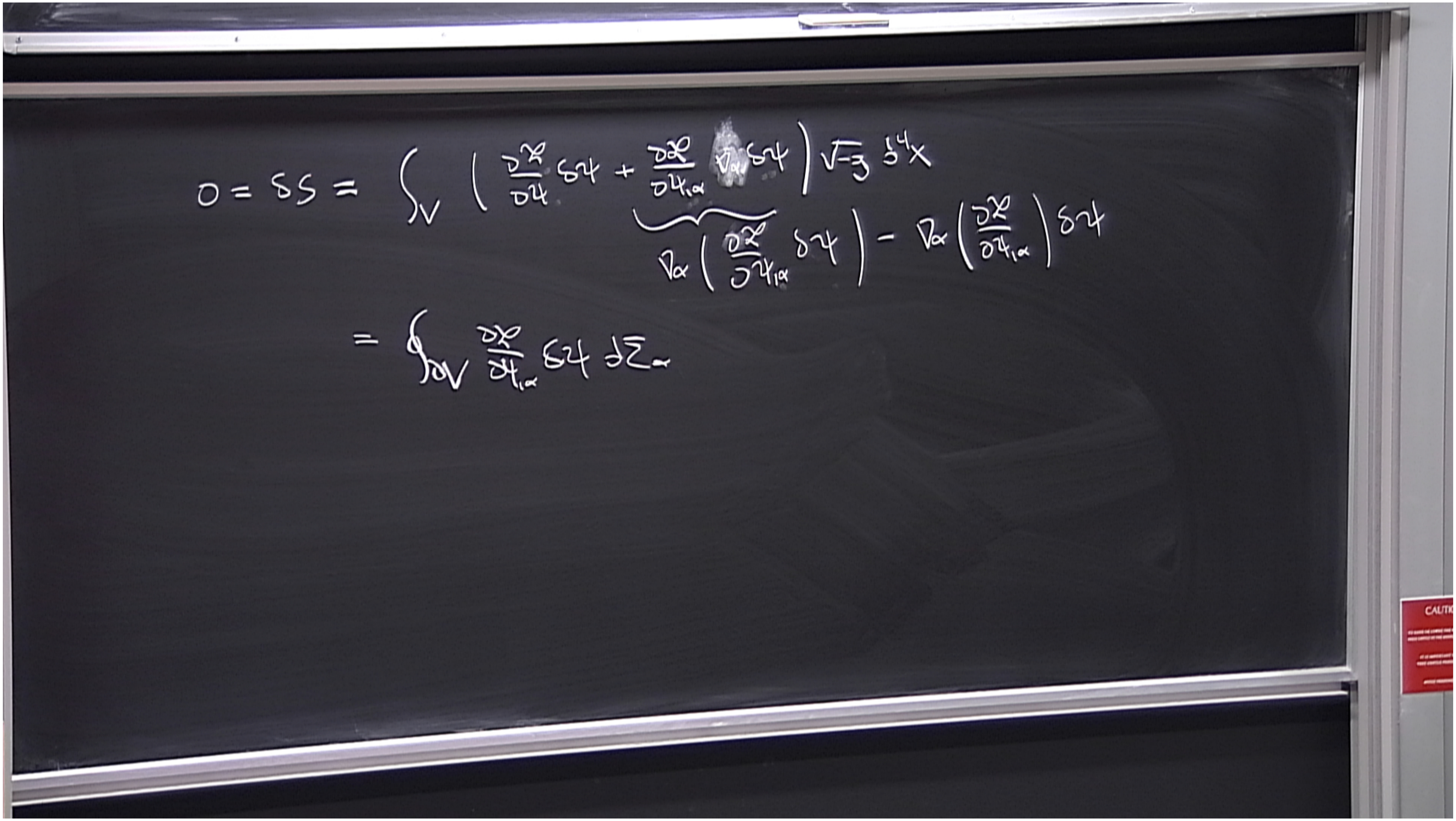
$$0 = \delta S = \int_V \left(\frac{\partial \mathcal{L}}{\partial \psi}\delta\psi + \underbrace{\frac{\partial \mathcal{L}}{\partial \psi_{,\alpha}} \nabla_\alpha \delta\psi}_{\nabla_\alpha\left(\frac{\partial \mathcal{L}}{\partial \psi_{,\alpha}}\delta\psi\right) - \nabla_\alpha\left(\frac{\partial \mathcal{L}}{\partial \psi_{,\alpha}}\right)} \right)\sqrt{-g}\, d^4x$$

$$0 = \delta S = \int_V \left(\frac{\partial \mathcal{L}}{\partial \psi}\,\delta\psi + \underbrace{\frac{\partial \mathcal{L}}{\partial \psi_{,\alpha}}\,\nabla_\alpha \delta\psi}_{\nabla_\alpha\left(\frac{\partial \mathcal{L}}{\partial \psi_{,\alpha}}\delta\psi\right) - \nabla_\alpha\left(\frac{\partial \mathcal{L}}{\partial \psi_{,\alpha}}\right)\delta\psi} \right)\sqrt{-g}\, d^4x$$



$0 = \delta S = \int_V \left(\frac{\partial \mathcal{L}}{\partial \psi} \delta\psi + \frac{\partial \mathcal{L}}{\partial \psi_{,\alpha}} \nabla_\alpha \delta\psi \right) \sqrt{-g}\, d^4x$
$\nabla_\alpha \left(\frac{\partial \mathcal{L}}{\partial \psi_{,\alpha}} \delta\psi \right) - \nabla_\alpha \left(\frac{\partial \mathcal{L}}{} \right) \delta\psi$

$$0 = \delta S = \int_V \left(\frac{\partial \mathcal{L}}{\partial \psi} \delta\psi + \underbrace{\frac{\partial \mathcal{L}}{\partial \psi_{,\alpha}} \nabla_\alpha \delta\psi}_{\nabla_\alpha \left(\frac{\partial \mathcal{L}}{\partial \psi_{,\alpha}} \delta\psi \right) - \nabla_\alpha \left(\frac{\partial \mathcal{L}}{\partial \psi_{,\alpha}} \right) \delta\psi} \right) \sqrt{-g}\, d^4x$$


$0 = \delta S = \int_V \left(\frac{\partial \mathcal{L}}{\partial \psi} \delta\psi + \frac{\partial \mathcal{L}}{\partial \psi_{,\alpha}} \nabla_\alpha \delta\psi \right) \sqrt{-g}\, d^4x$
$\nabla_\alpha \left(\frac{\partial \mathcal{L}}{\partial \psi_{,\alpha}} \delta\psi \right) - \nabla_\alpha \left(\frac{\partial \mathcal{L}}{\partial \psi_{,\alpha}} \right) \delta\psi$
$= \oint_{\partial V} \frac{\partial \mathcal{L}}{\partial \psi_{,\alpha}} \delta\psi \, d\Sigma_\alpha$
CAUTI

$$0 = \delta S = \int_V \left(\frac{\partial \mathcal{L}}{\partial \psi}\delta\psi + \underbrace{\frac{\partial \mathcal{L}}{\partial \psi_{,\alpha}} \nabla_\alpha \delta\psi}_{\nabla_\alpha\left(\frac{\partial\mathcal{L}}{\partial\psi_{,\alpha}}\delta\psi\right) - \nabla_\alpha\left(\frac{\partial \mathcal{L}}{\partial \psi_{,\alpha}}\right)\delta\psi} \right)\sqrt{-g}\, d^4x$$

$$= \oint_{\partial V}\left(\frac{\partial\mathcal{L}}{\partial\psi_{,\alpha}}\delta\psi\right) d\Sigma_\alpha + \int_V \left(\frac{\partial\mathcal{L}}{\partial\psi} - \nabla_\alpha \frac{\partial\mathcal{L}}{\partial\psi_{,\alpha}}\right)\delta\psi$$

$$0 = \delta S = \int_V \left(\frac{\partial \mathcal{L}}{\partial \psi}\delta\psi + \underbrace{\frac{\partial \mathcal{L}}{\partial \psi_{,\alpha}} \nabla_\alpha \delta\psi}_{\nabla_\alpha\left(\frac{\partial \mathcal{L}}{\partial \psi_{,\alpha}}\delta\psi\right) - \nabla_\alpha\left(\frac{\partial \mathcal{L}}{\partial \psi_{,\alpha}}\right)\delta\psi} \right)\sqrt{-g}\, d^4x$$

$$= \cancel{\oint_{\partial V}\left(\frac{\partial\mathcal{L}}{\partial\psi_{,\alpha}}\delta\psi\right) d\Sigma_\alpha} + \int_V \underbrace{\left(\frac{\partial\mathcal{L}}{\partial\psi} - \nabla_\alpha \frac{\partial\mathcal{L}}{\partial\psi_{,\alpha}}\right)}_{0}\delta\psi$$

field eqn:

$$\nabla_\alpha\left(\frac{\partial\mathcal{L}}{\partial\psi_{,\alpha}}\right) - \frac{\partial\mathcal{L}}{\partial\psi} = 0$$

$$\mathcal{L} = -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \, \partial_\nu \phi - V(\phi)$$

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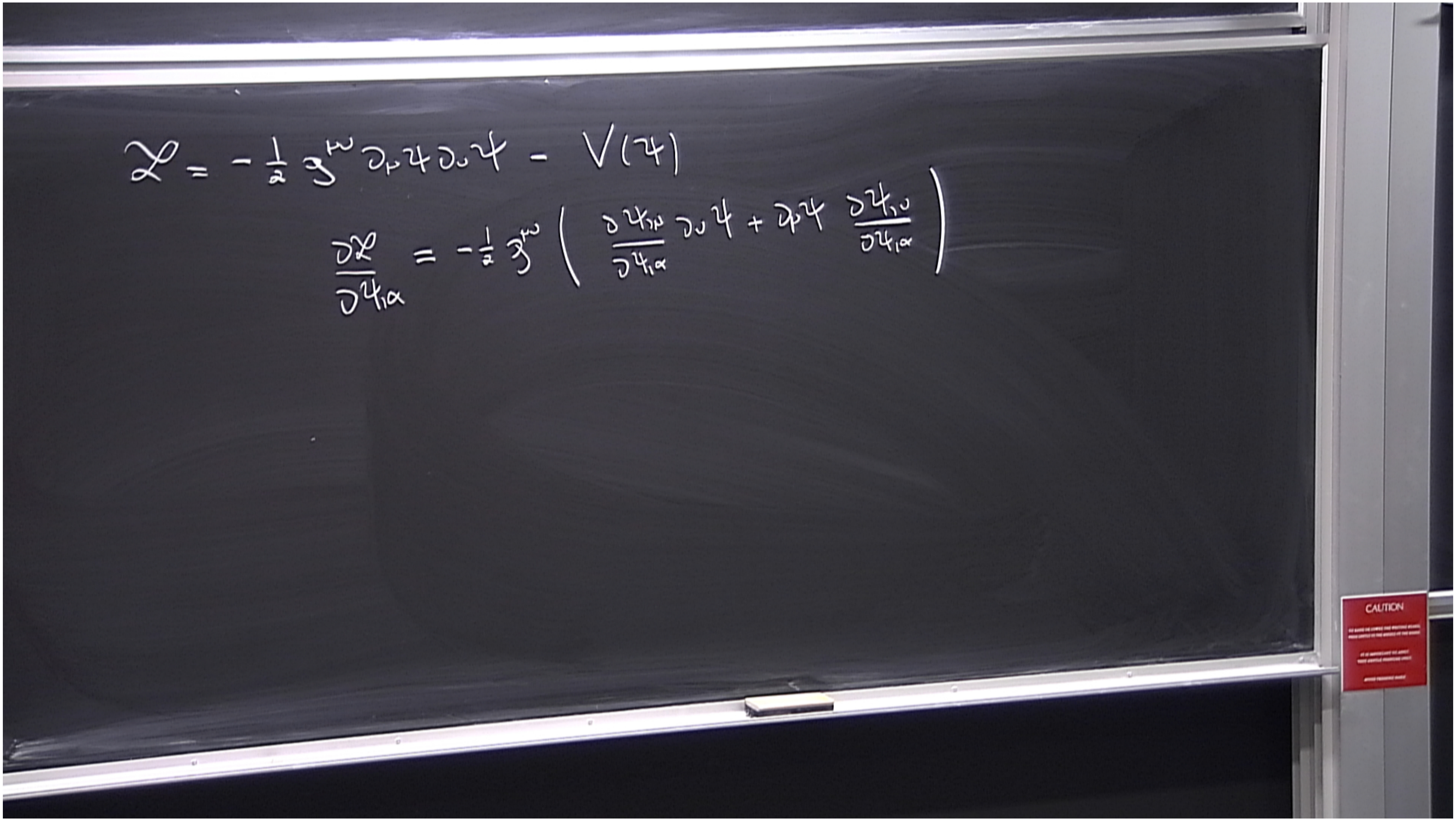
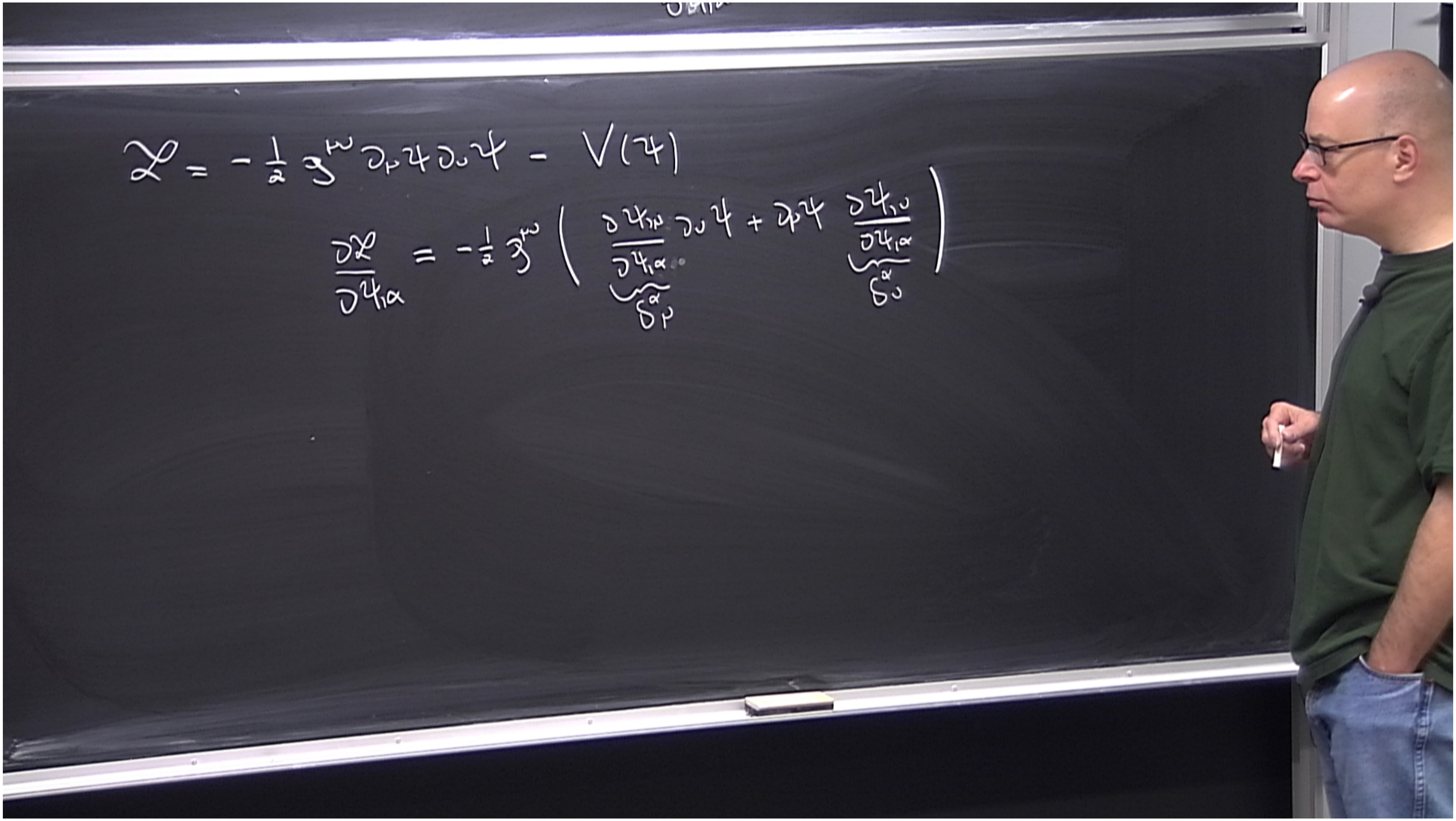
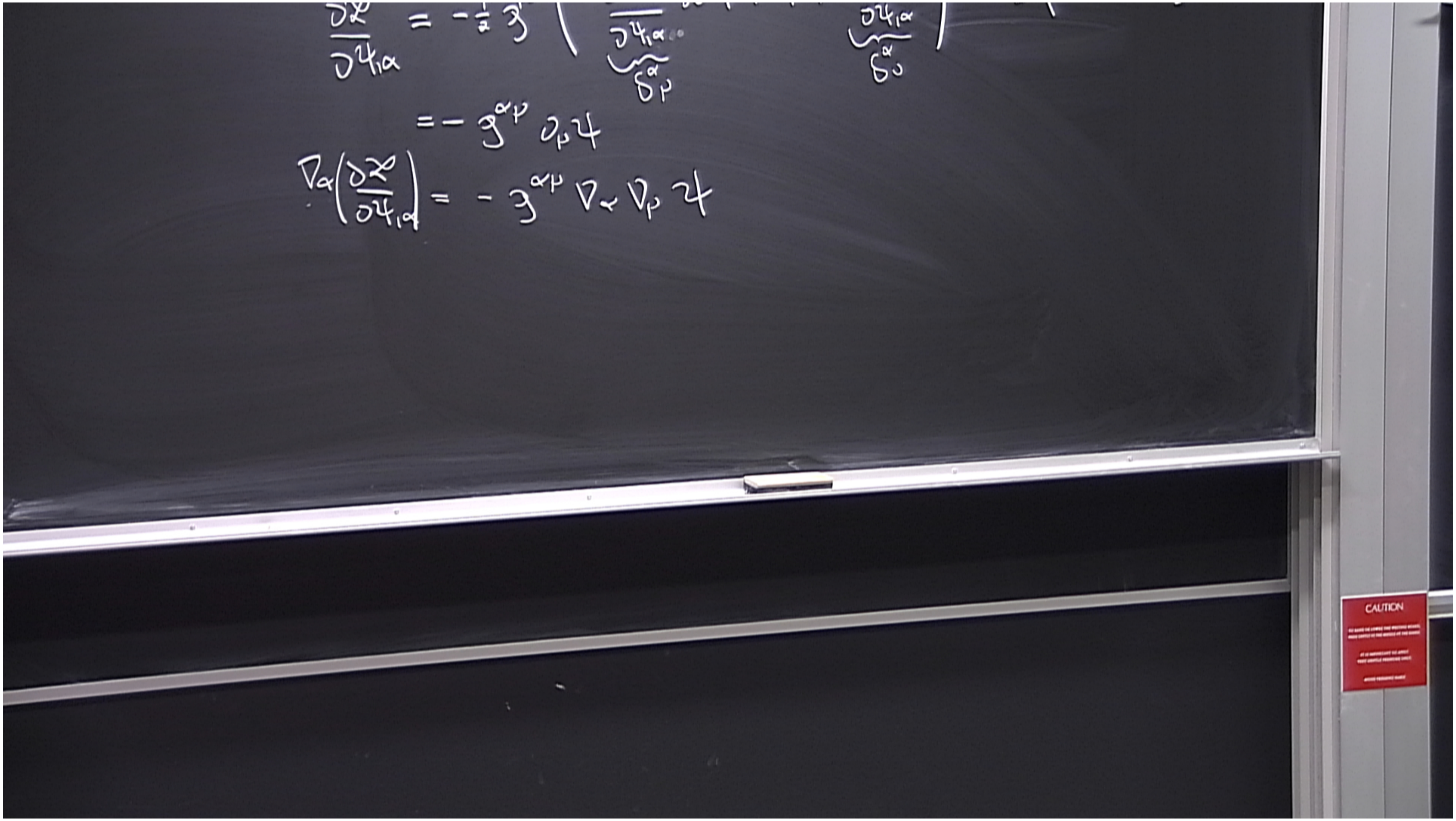

$\mathcal{L} = -\frac{1}{2} g^{\mu\nu} \partial_\mu \psi \partial_\nu \psi - V(\psi)$
$\frac{\partial \mathcal{L}}{\partial \psi_{,\alpha}} = -\frac{1}{2} g^{\mu\nu} \left(\frac{\partial \psi_{,\mu}}{\partial \psi_{,\alpha}} \partial_\nu \psi + \partial_\mu \psi \frac{\partial \psi_{,\nu}}{\partial \psi_{,\alpha}} \right)$

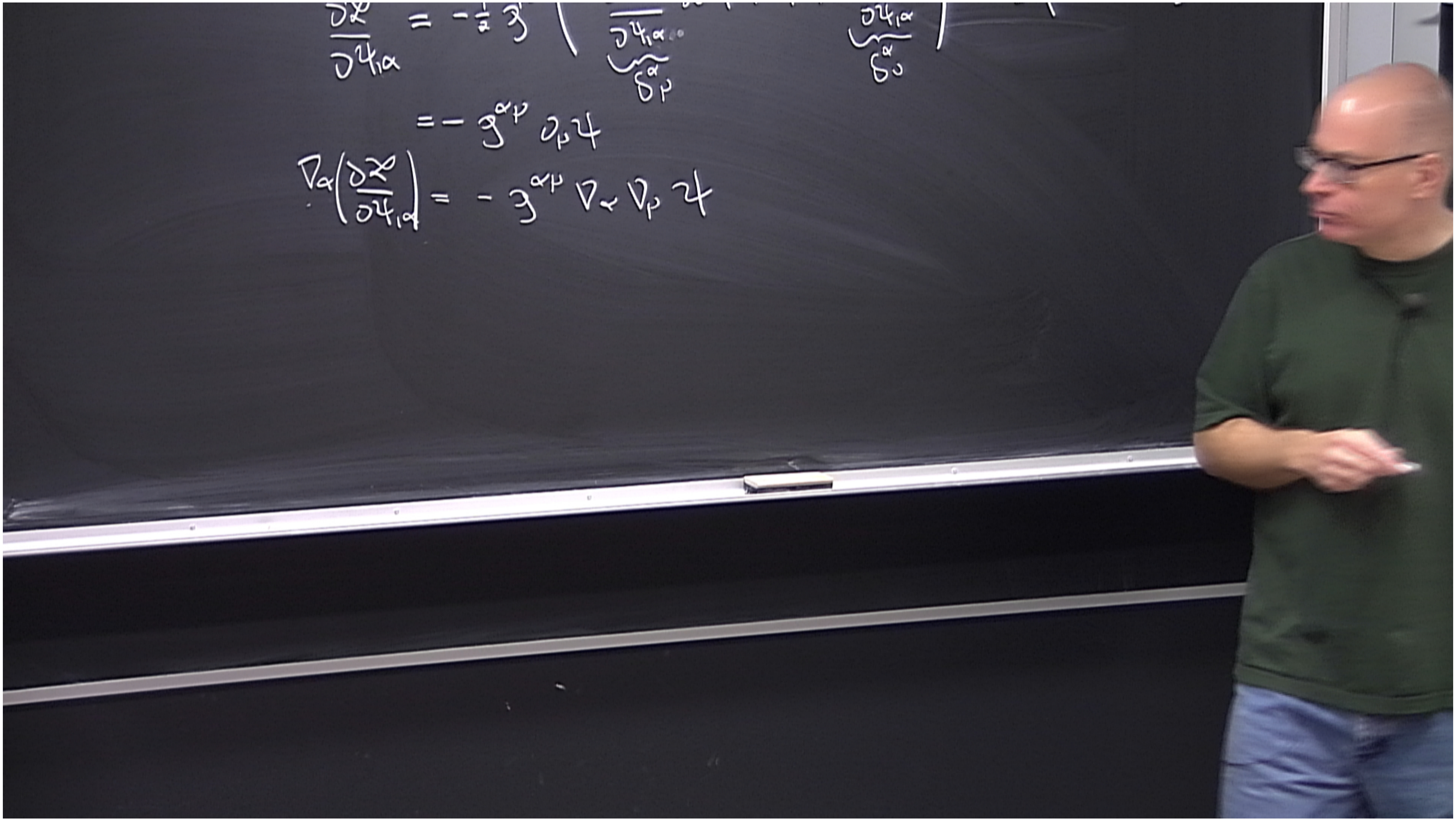

$\mathcal{L} = -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi)$
$\frac{\partial \mathcal{L}}{\partial \phi_{,\alpha}} = -\frac{1}{2} g^{\mu\nu} \left(\underbrace{\frac{\partial \phi_{,\mu}}{\partial \phi_{,\alpha}}}_{\delta^\alpha_\mu} \partial_\nu \phi + \partial_\mu \phi \underbrace{\frac{\partial \phi_{,\nu}}{\partial \phi_{,\alpha}}}_{\delta^\alpha_\nu} \right)$

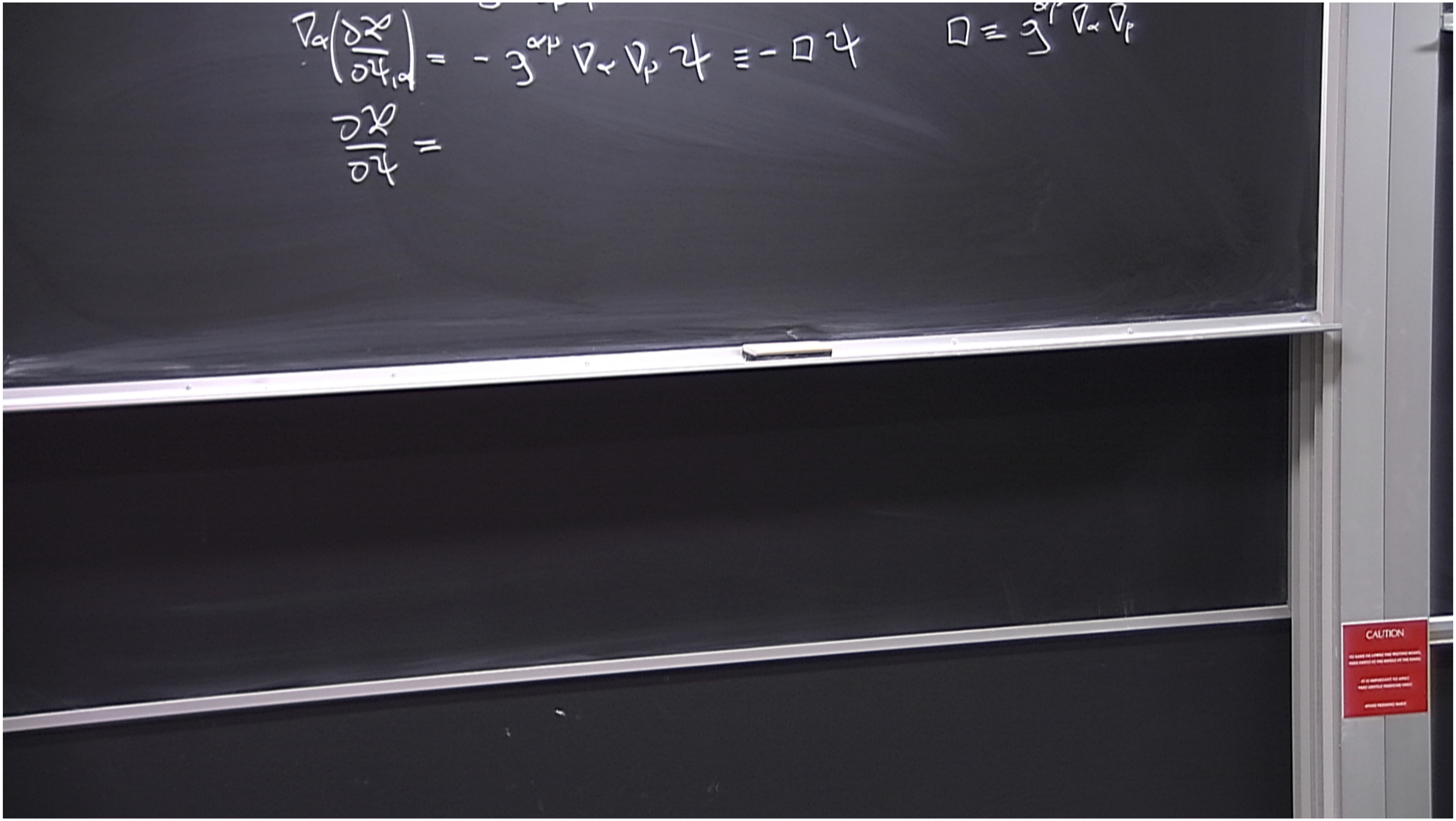
$$\mathcal{L} = -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \, \partial_\nu \phi - V(\phi)$$

$$\frac{\partial \mathcal{L}}{\partial \phi_{,\alpha}} = -\frac{1}{2} g^{\mu\nu} \left(\underbrace{\frac{\partial \phi_{,\mu}}{\partial \phi_{,\alpha}}}_{\delta^{\alpha}_{\mu}} \partial_\nu \phi + \partial_\mu \phi \underbrace{\frac{\partial \phi_{,\nu}}{\partial \phi_{,\alpha}}}_{\delta^{\alpha}_{\nu}} \right) = -\frac{1}{2} \left(g^{\alpha\nu} \partial_\nu \phi + g^{\mu\alpha} \partial_\mu \phi \right)$$

$$= - g^{\alpha\mu} \partial_\mu \phi$$


$\frac{\partial \mathcal{L}}{\partial \psi_{,\alpha}} = -\frac{1}{2} g \Big(\frac{\partial}{\partial \psi_{,\alpha}} \ldots \frac{\partial}{\partial \psi_{,\alpha}} \Big)$
δ^{α}_{μ}
δ^{α}_{ν}
$= - g^{\alpha\mu} \partial_\mu \psi$
$\nabla_\alpha \left(\frac{\partial \mathcal{L}}{\partial \psi_{,\alpha}} \right) = - g^{\alpha\mu} \nabla_\alpha \nabla_\mu \psi$
CAUTION


$= -g^{\alpha\mu}\partial_\mu\psi$
$\nabla_\alpha\left(\frac{\partial\mathcal{L}}{\partial\psi_{,\alpha}}\right) = -g^{\alpha\mu}\nabla_\alpha\nabla_\mu\psi$

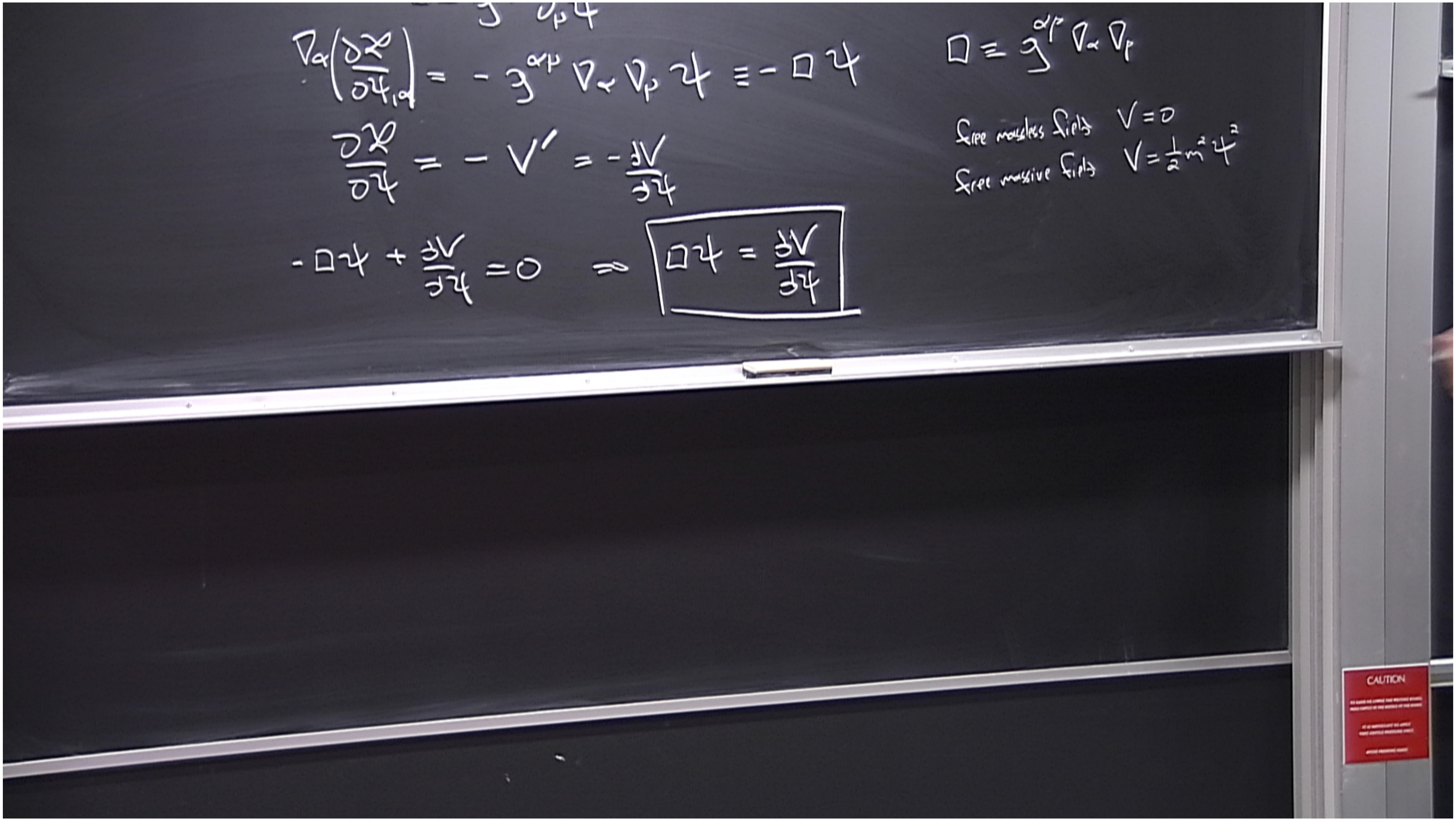

$\nabla_\alpha\left(\frac{\partial \mathcal{L}}{\partial \psi_{,\alpha}}\right) = - g^{\alpha\mu}\nabla_\alpha\nabla_\mu \psi \equiv -\Box\psi$
$\Box \equiv g^{\alpha\mu}\nabla_\alpha\nabla_\mu$
$\frac{\partial\mathcal{L}}{\partial\psi} =$
CAUTION

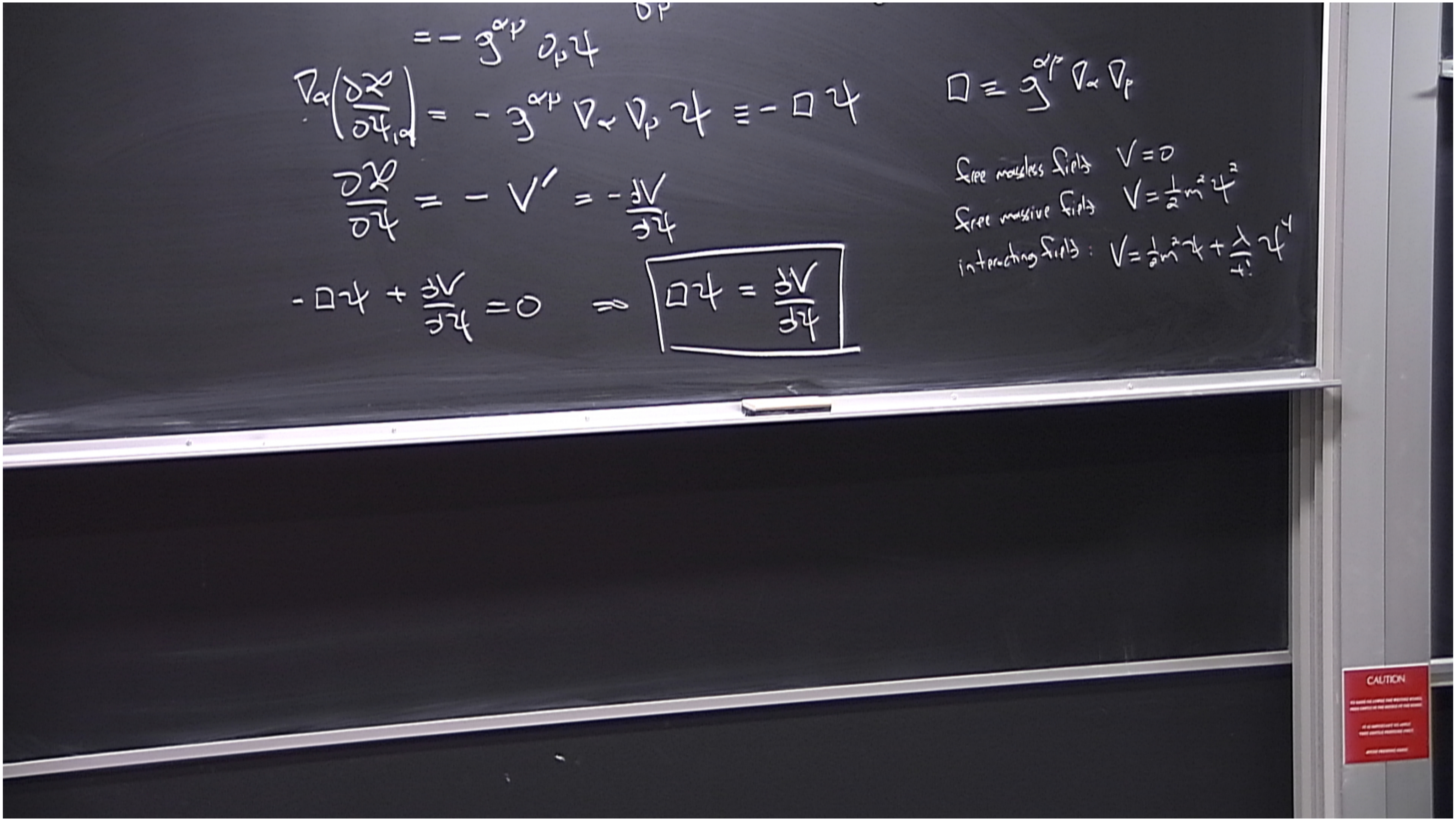
$$= - g^{\alpha\mu} \partial_\mu \psi$$

$$\nabla_\alpha \left(\frac{\partial \mathcal{L}}{\partial \psi_{,\alpha}} \right) = - g^{\alpha\mu} \nabla_\alpha \nabla_\mu \psi \equiv - \Box \psi \qquad \Box \equiv g^{\alpha\mu} \nabla_\alpha \nabla_\mu$$

$$\frac{\partial \mathcal{L}}{\partial \psi} = - V' = - \frac{\partial V}{\partial \psi}$$

$$- \Box \psi + \frac{\partial V}{\partial \psi} = 0 \quad \Rightarrow \quad \boxed{\Box \psi = \frac{\partial V}{\partial \psi}}$$


$\nabla_\alpha\left(\frac{\partial \mathcal{L}}{\partial \phi_{,\alpha}}\right) = -g^{\alpha\beta}\nabla_\alpha\nabla_\beta\phi \equiv -\Box\phi$
$\frac{\partial \mathcal{L}}{\partial \phi} = -V' = -\frac{\partial V}{\partial \phi}$
$-\Box\phi + \frac{\partial V}{\partial \phi} = 0 \Rightarrow \Box\phi = \frac{\partial V}{\partial \phi}$
$\Box \equiv g^{\alpha\beta}\nabla_\alpha\nabla_\beta$
free massless field $V = 0$
free massive field $V = \frac{1}{2}m^2\phi^2$
CAUTION


$= - g^{\alpha\beta} \partial_\beta \phi$
$\nabla_\alpha \left(\frac{\partial \mathcal{L}}{\partial \phi_{,\alpha}} \right) = - g^{\alpha\beta} \nabla_\alpha \nabla_\beta \phi \equiv - \Box \phi$
$\frac{\partial \mathcal{L}}{\partial \phi} = - V' = - \frac{\partial V}{\partial \phi}$
$- \Box \phi + \frac{\partial V}{\partial \phi} = 0 \Rightarrow \Box \phi = \frac{\partial V}{\partial \phi}$
$\Box \equiv g^{\alpha\beta} \nabla_\alpha \nabla_\beta$
free massless fields $V = 0$
free massive fields $V = \frac{1}{2} m^2 \phi^2$
interacting fields: $V = \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!} \phi^4$
CAUTION

field eqn:

$$\partial_\alpha\left(\frac{\partial \mathcal{L}}{\partial \psi_{,\alpha}}\right) - \frac{\partial \mathcal{L}}{\partial \psi} = 0$$

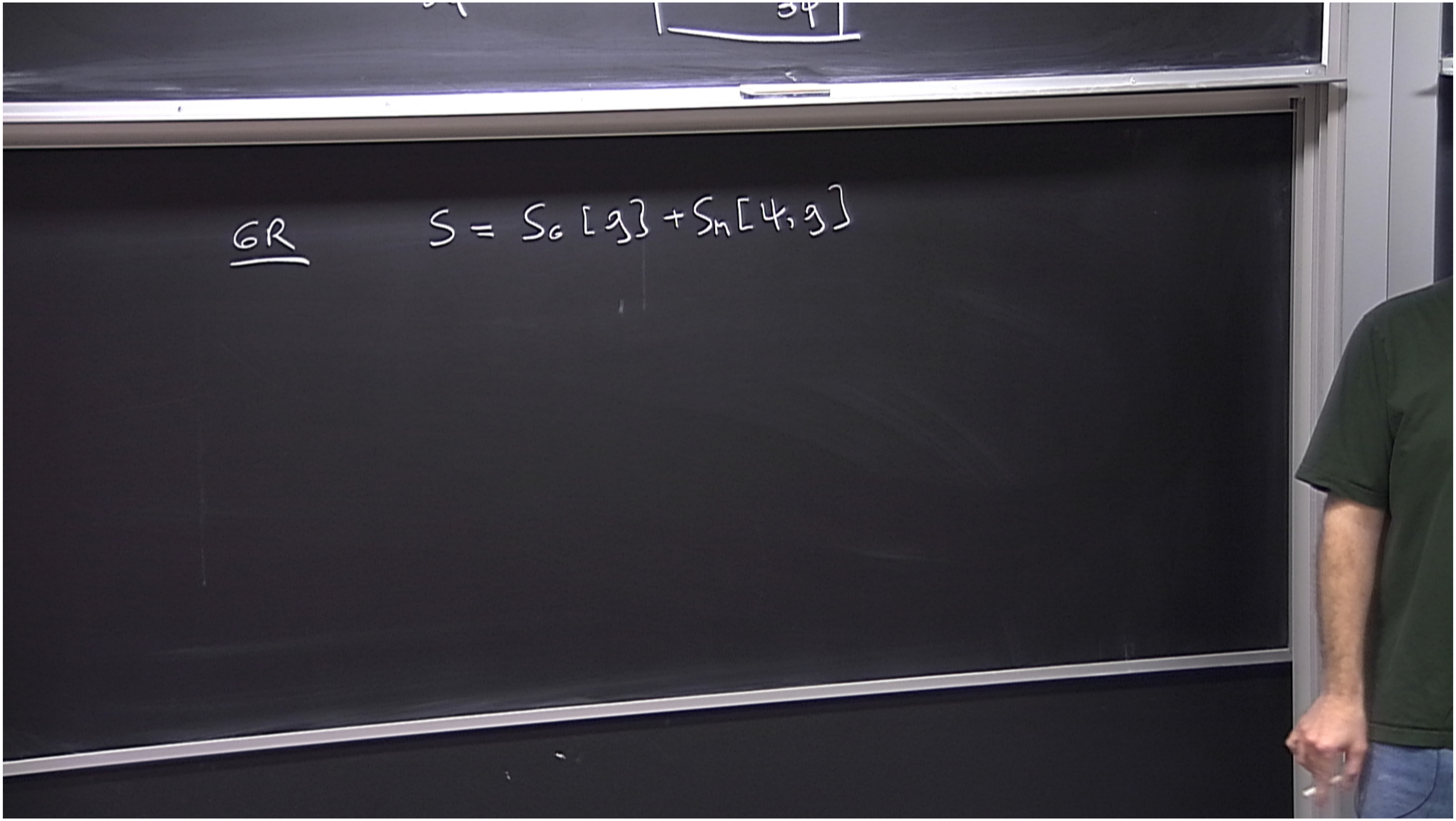
$$A^\alpha = \frac{\partial \mathcal{L}}{\partial \psi_{,\alpha}}$$

$$\mathcal{L} = -\frac{1}{2} g^{\mu\nu}\partial_\mu\psi\,\partial_\nu\psi - V(\psi)$$

$$\frac{\partial \mathcal{L}}{\partial \psi_{,\alpha}} = -\frac{1}{2} g^{\mu\nu}\left(\underbrace{\frac{\partial \psi_{,\mu}}{\partial \psi_{,\alpha}}}_{\delta^\alpha_\mu}\partial_\nu\psi + \partial_\mu\psi\underbrace{\frac{\partial \psi_{,\nu}}{\partial \psi_{,\alpha}}}_{\delta^\alpha_\nu}\right) = -\frac{1}{2}\left(g^{\alpha\nu}\partial_\nu\psi + g^{\mu\alpha}\partial_\mu\psi\right)$$

$$= -g^{\alpha\mu}\partial_\mu\psi$$

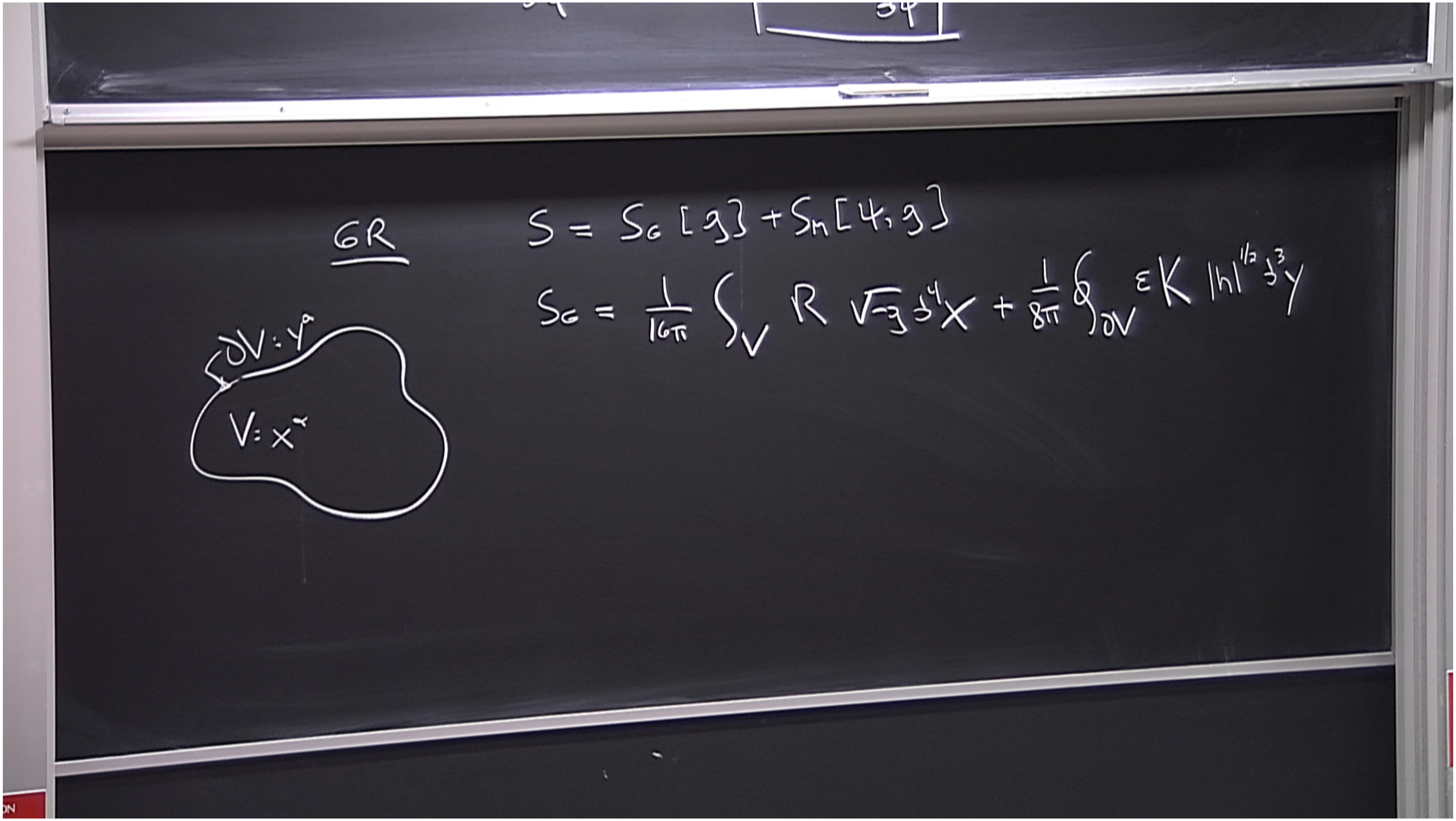
$$\partial_\alpha\left(\frac{\partial \mathcal{L}}{\ldots}\right) \ldots \partial_\alpha\partial_\mu\psi = -\Box\psi \qquad \Box \equiv g^{\alpha\beta}\partial_\alpha\partial_\beta$$

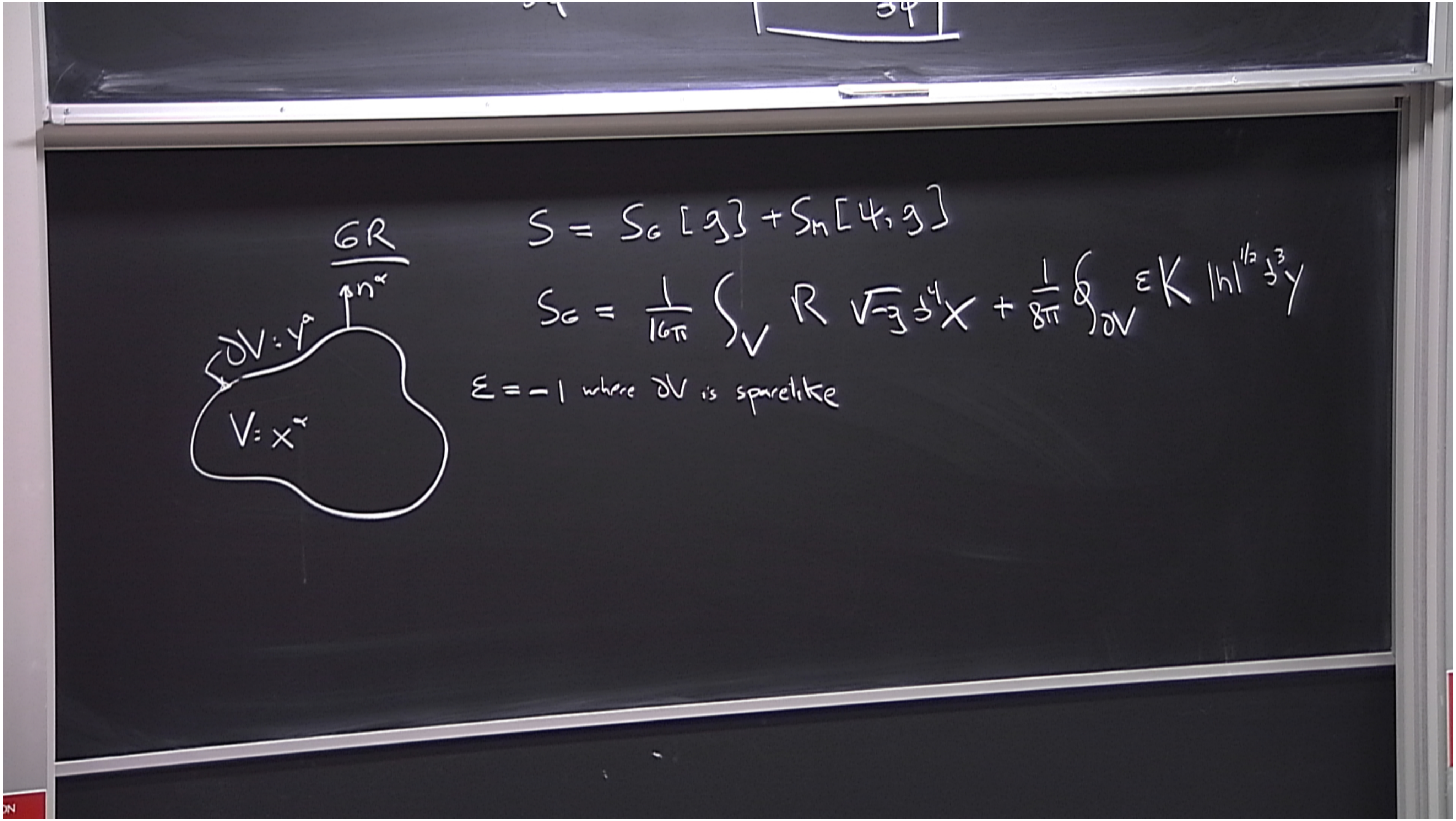

GR
S = S_G [g] + S_M [ψ, g]

GR

$$S = S_G[g] + S_M[\psi, g]$$

$$S_G = \frac{1}{16\pi} \int_V R \sqrt{-g}\, d^4x$$


GR
$S = S_G[g] + S_M[\psi, g]$
$S_G = \frac{1}{16\pi}\int_V R\sqrt{-g}\,d^4x + \frac{1}{8\pi}\oint_{\partial V}\varepsilon K |h|^{1/2} d^3y$
$\partial V : y^a$
$V : x^\alpha$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$\varepsilon = -1$ where ∂V is spacelike
n^α
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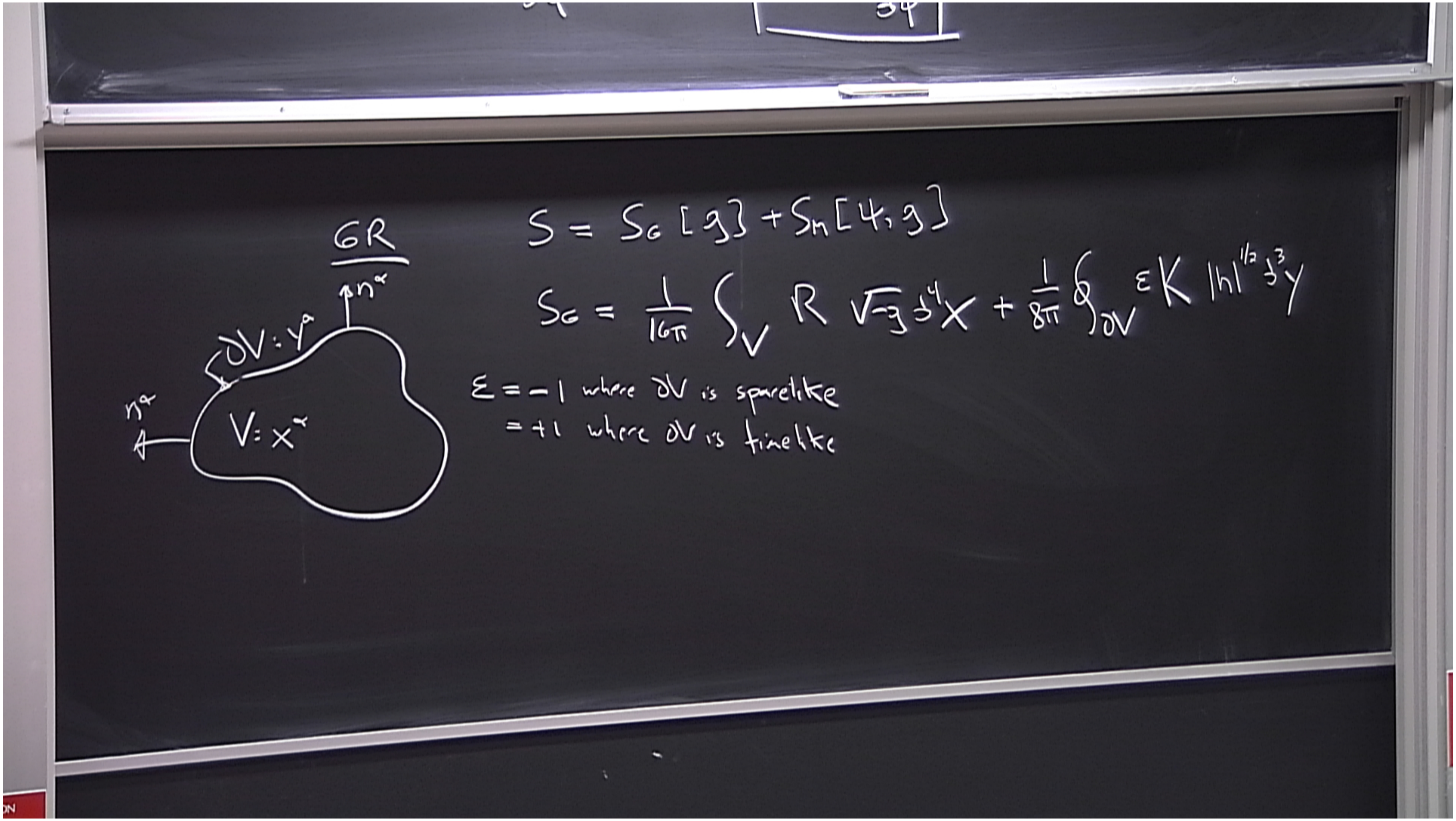
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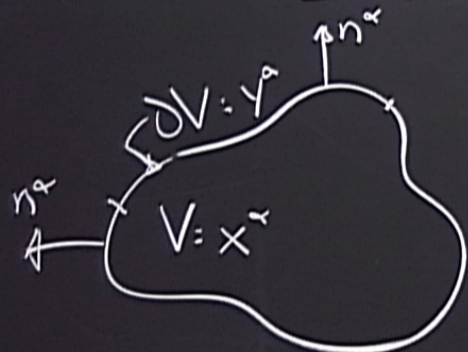

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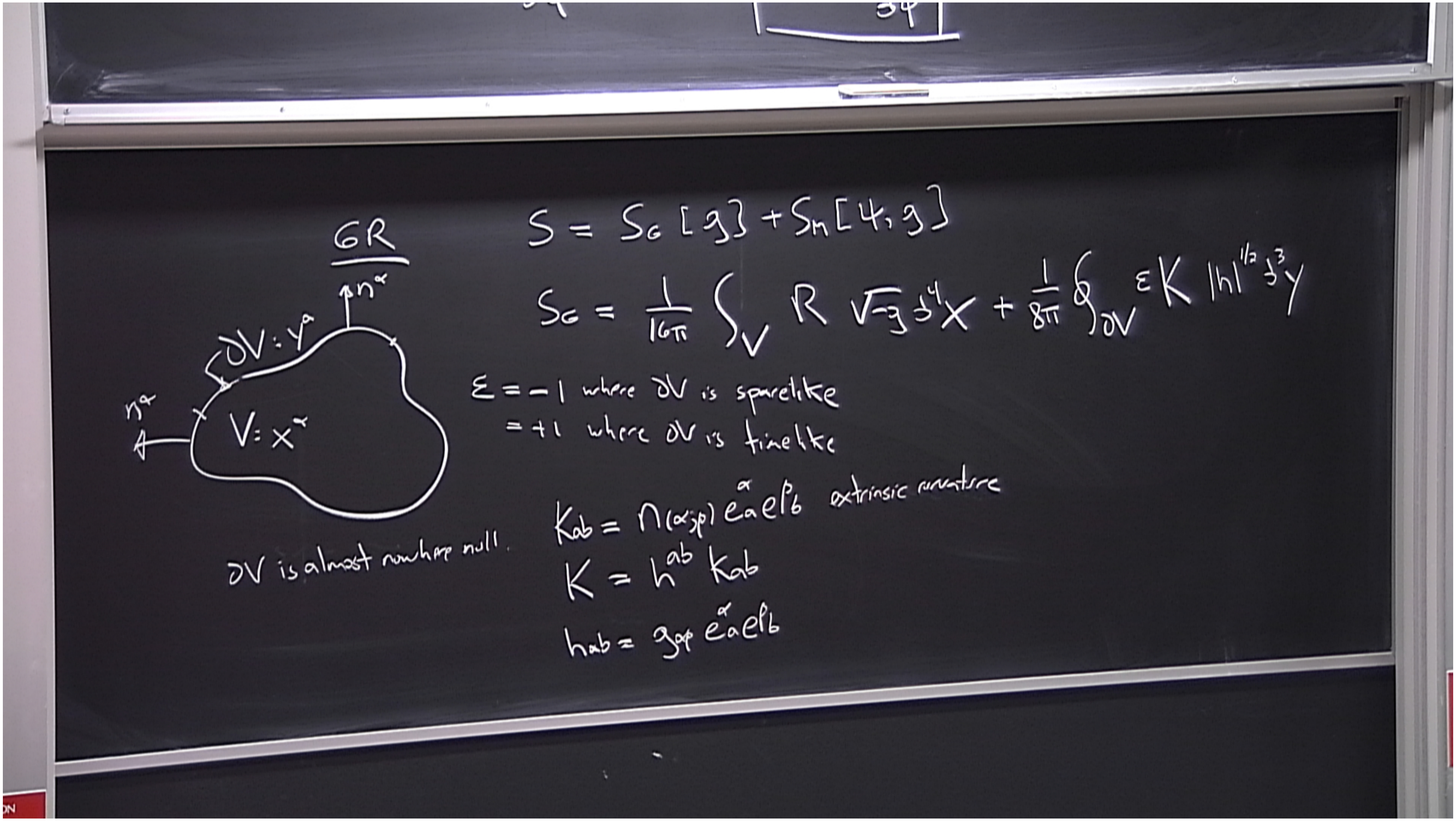
GR
n^α
$\partial V : y^a$
n^α
$V : x^\alpha$
∂V is almost nowhere null
$S = S_G[g] + S_M[\psi, g]$
$S_G = \frac{1}{16\pi}\int_V R\sqrt{-g}\,d^4x + \frac{1}{8\pi}\oint_{\partial V} \varepsilon K |h|^{1/2} d^3y$
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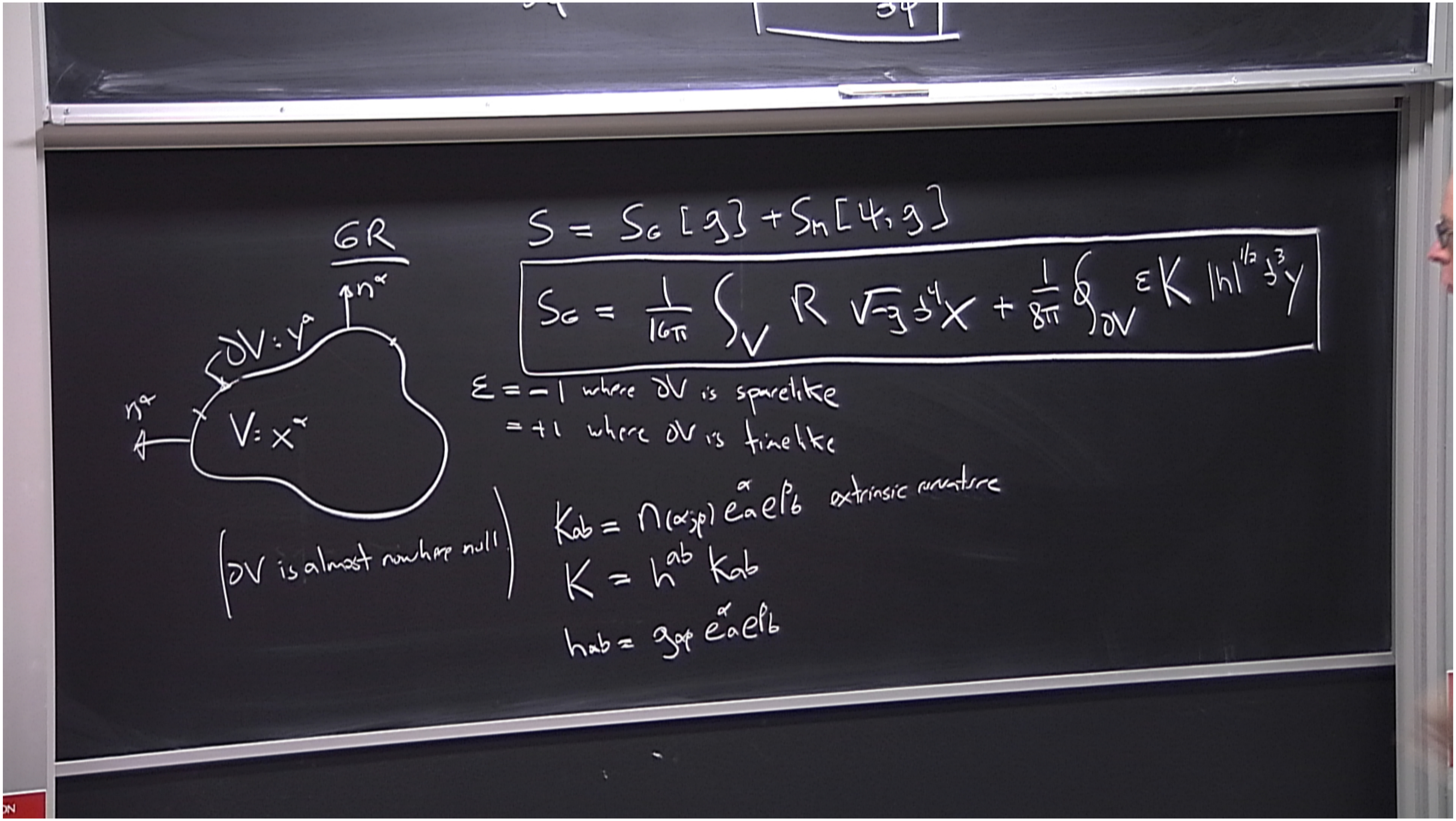
$K = h^{ab} K_{ab}$
$h_{ab} = g_{\alpha\beta} e^\alpha_a e^\beta_b$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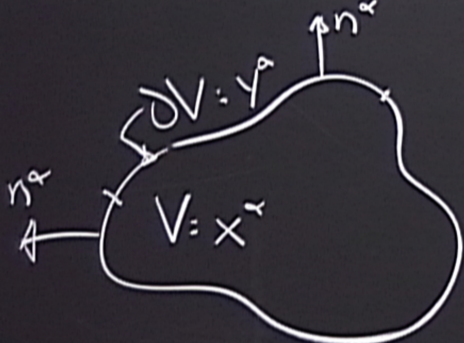
$\ldots \frac{\partial \ldots}{\partial\psi} = 0 \Rightarrow \boxed{\Box\psi = \frac{\partial V}{\partial\psi}}$

GR



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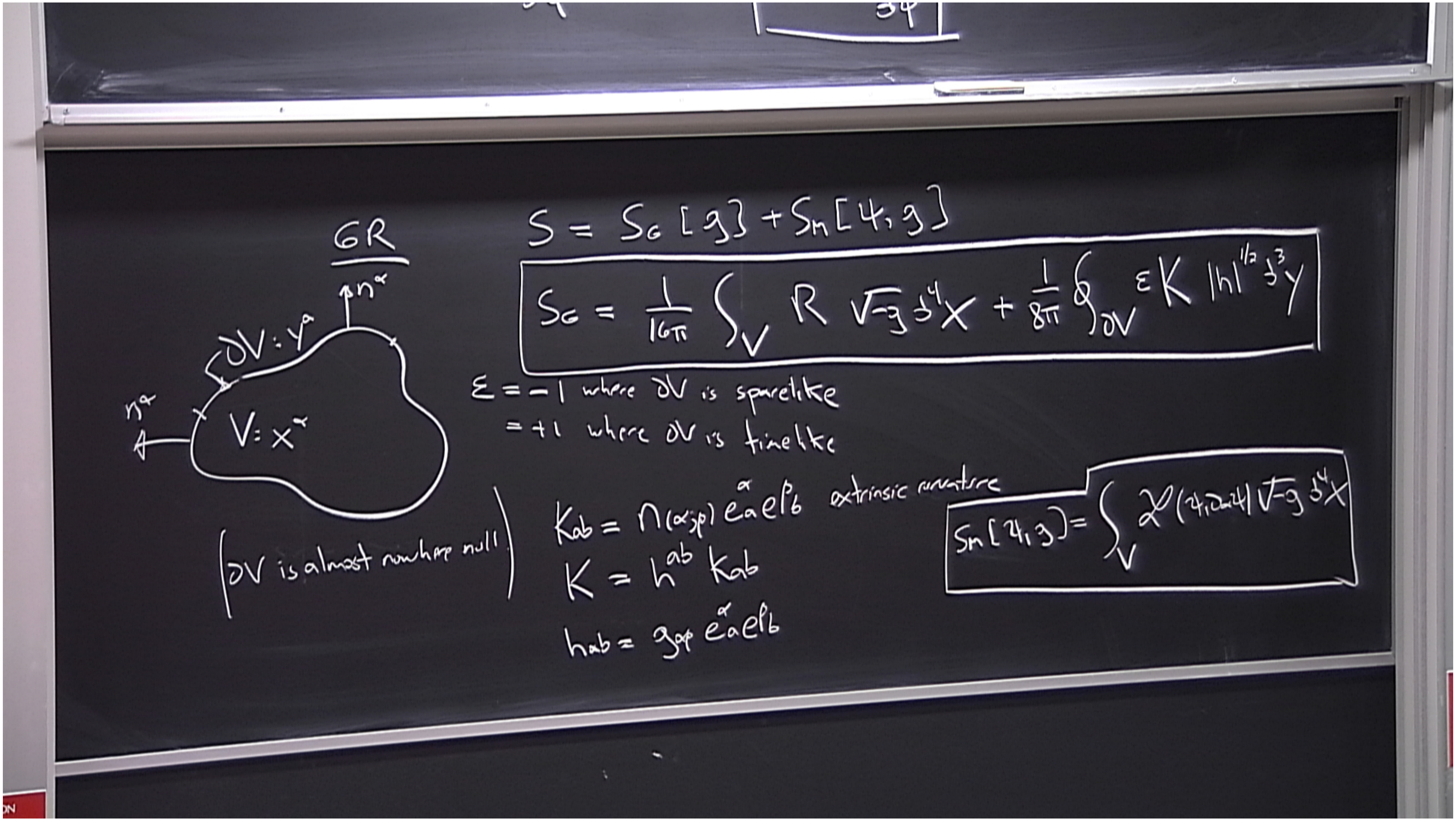
$$S_M[\psi, g] = \int_V \mathcal{L}(\psi, \partial_\alpha\psi)\sqrt{-g}\,d^4x$$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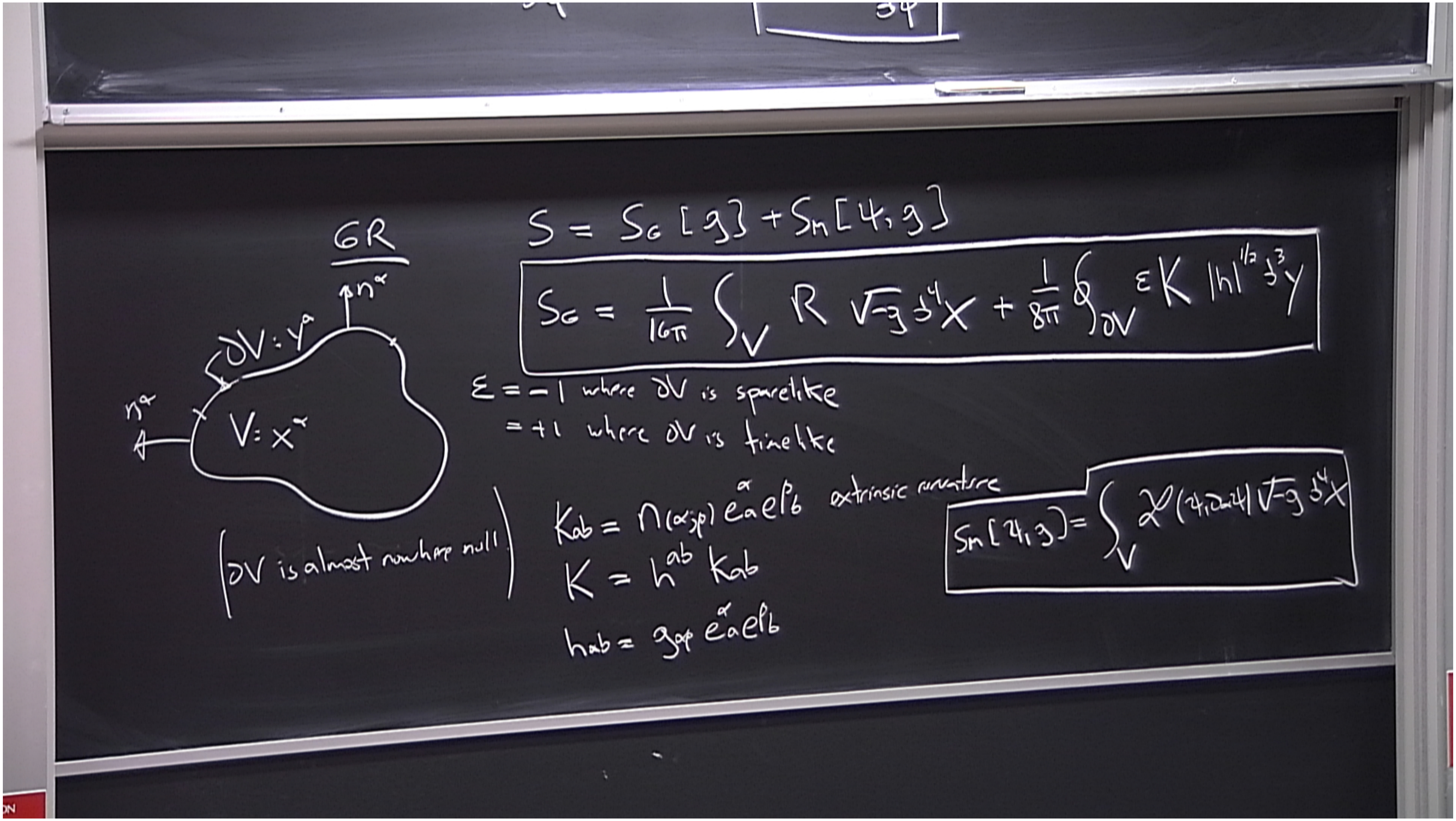
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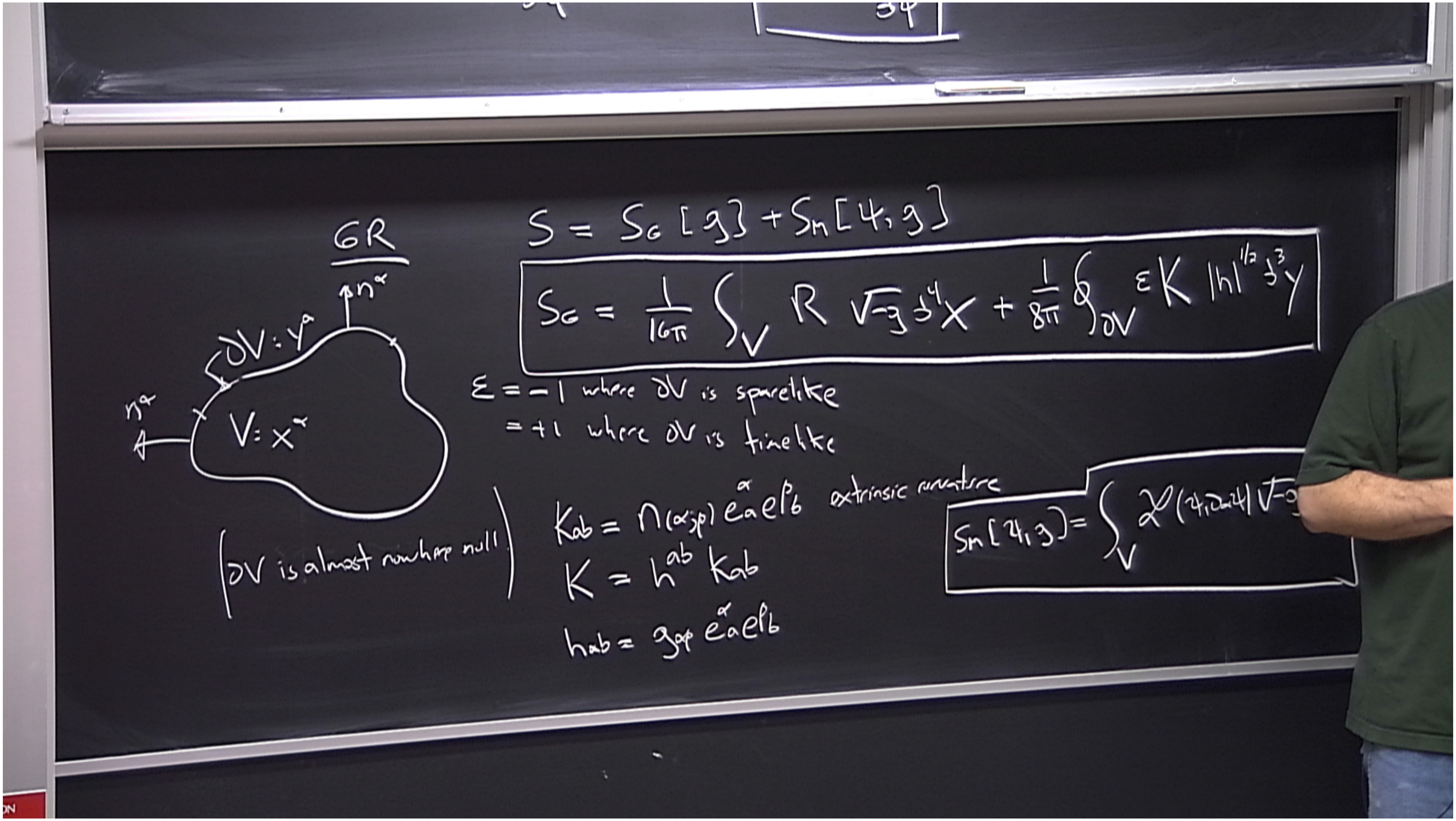
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n^α
V : x^α
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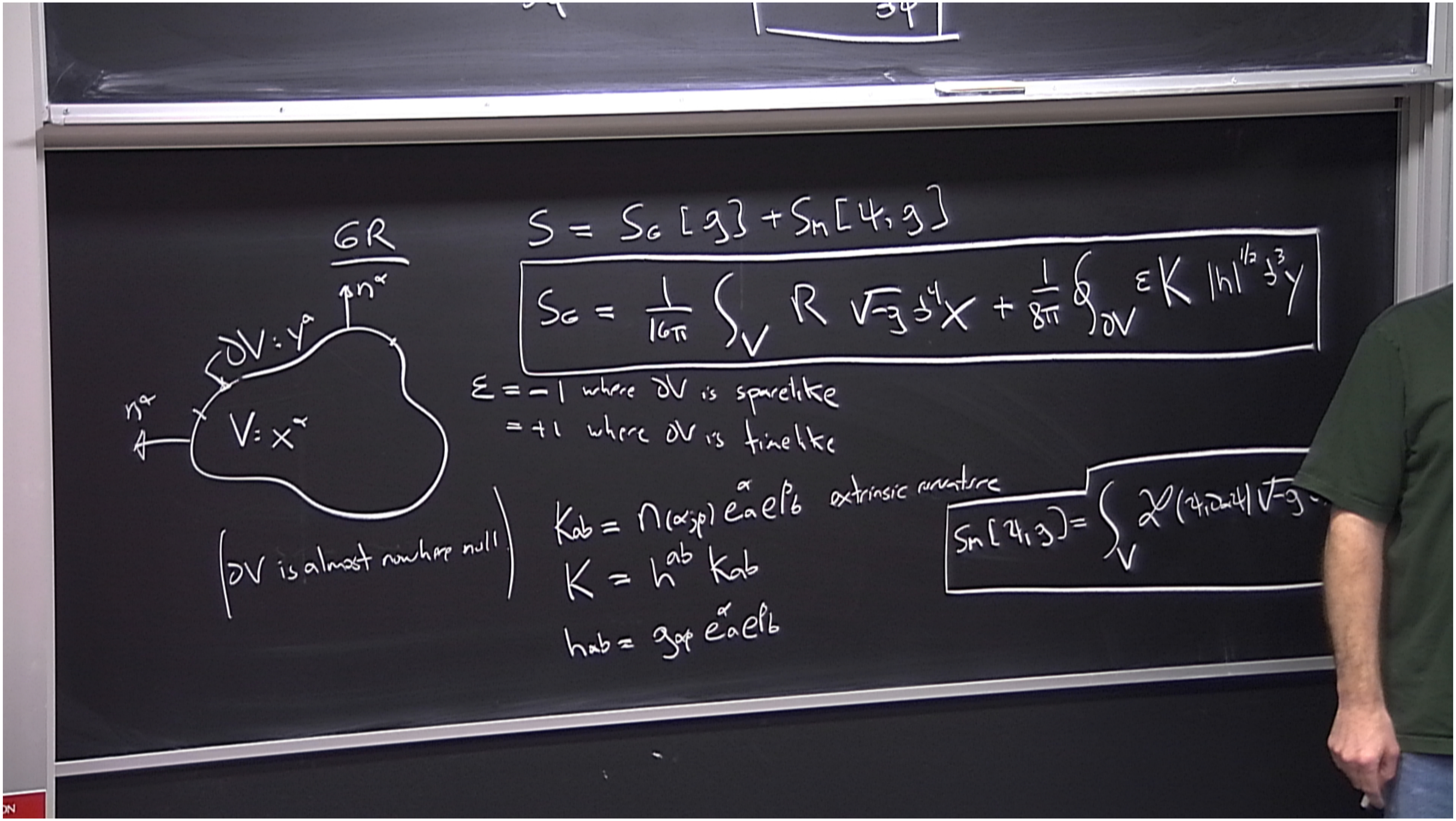
(∂V is almost nowhere null)
S_M[ψ, g] = ∫_V ℒ(ψ, ∂_αψ) √-g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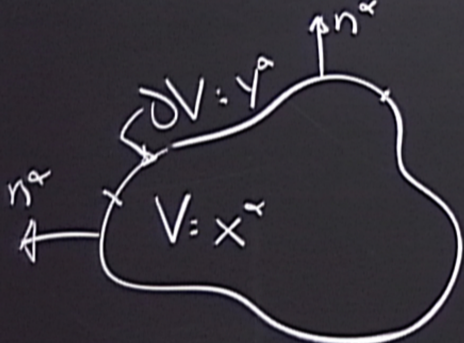
GR



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Variation rules

$$\delta g_{\alpha\beta} = 0 \text{ on } \partial V$$

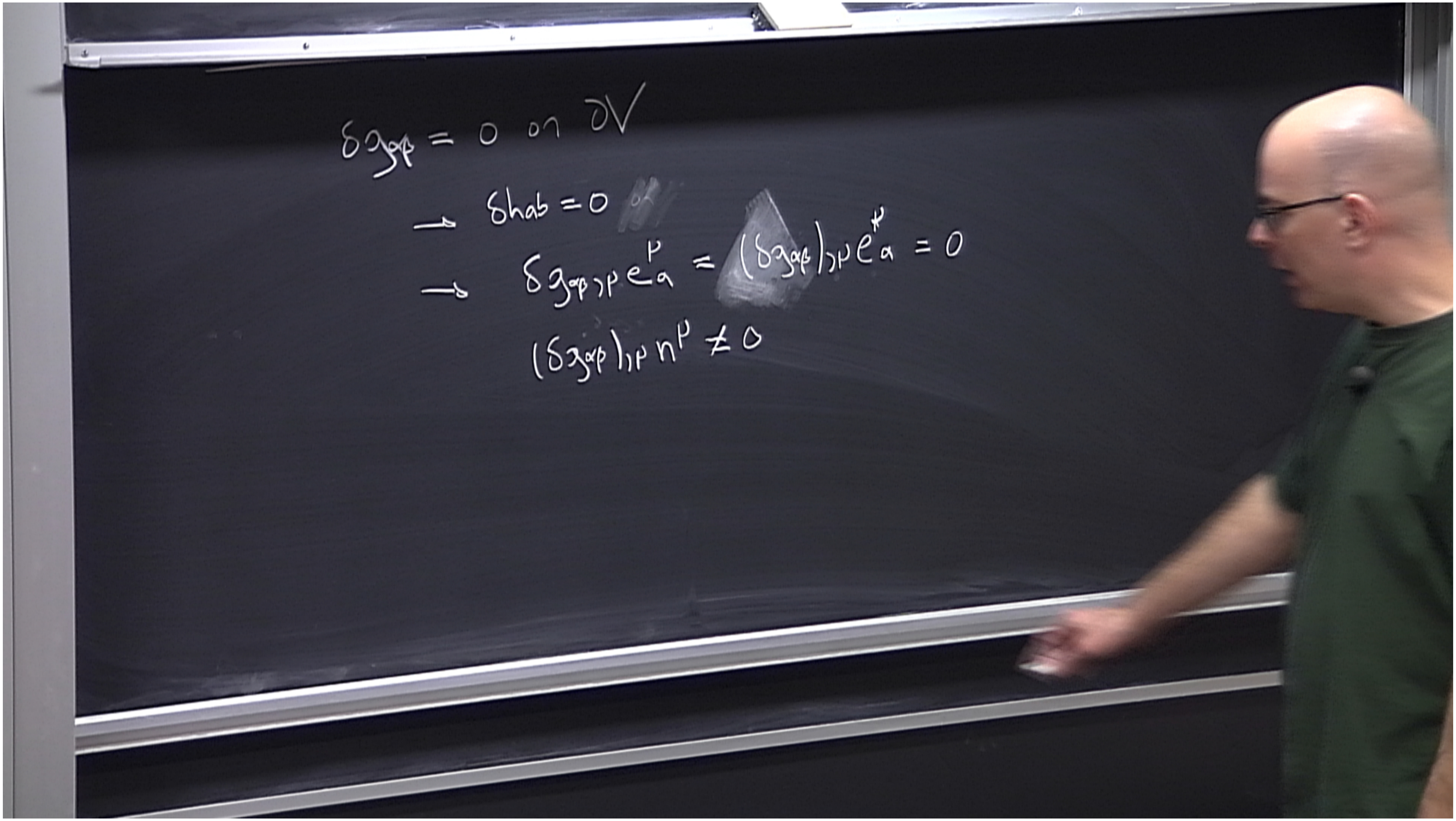
$$\to \delta h_{ab} = 0$$

Variation rules

$$\delta g_{\alpha\beta} = 0 \text{ on } \partial V$$

$$\rightarrow \delta h_{ab} = 0$$

$$\rightarrow \delta g_{\alpha\beta,\mu} e^{\mu}_{a} = (\delta g_{\alpha\beta})_{,\mu} e^{\mu}_{a} = 0$$


$\delta g_{\alpha\beta} = 0$ on ∂V
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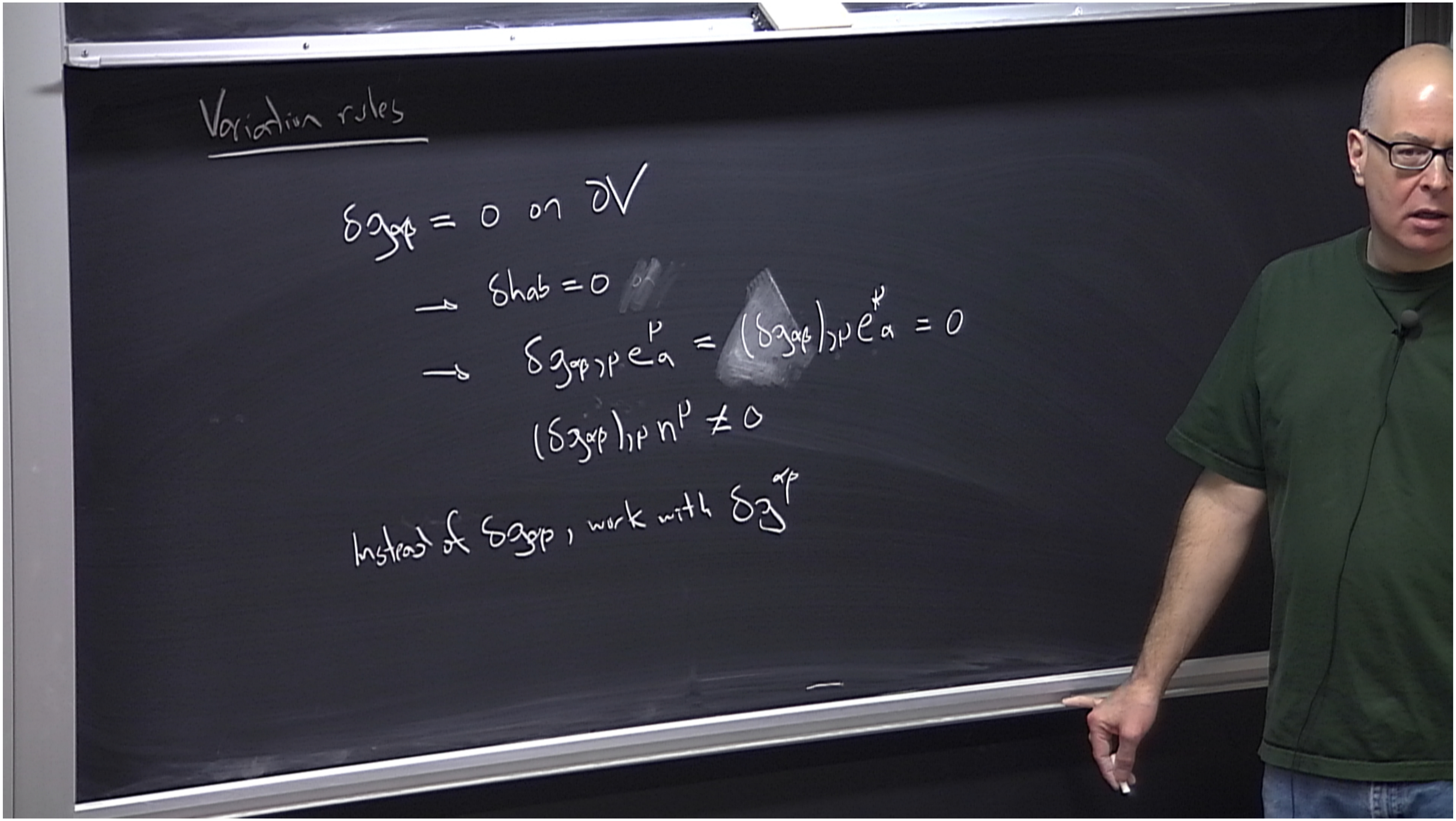
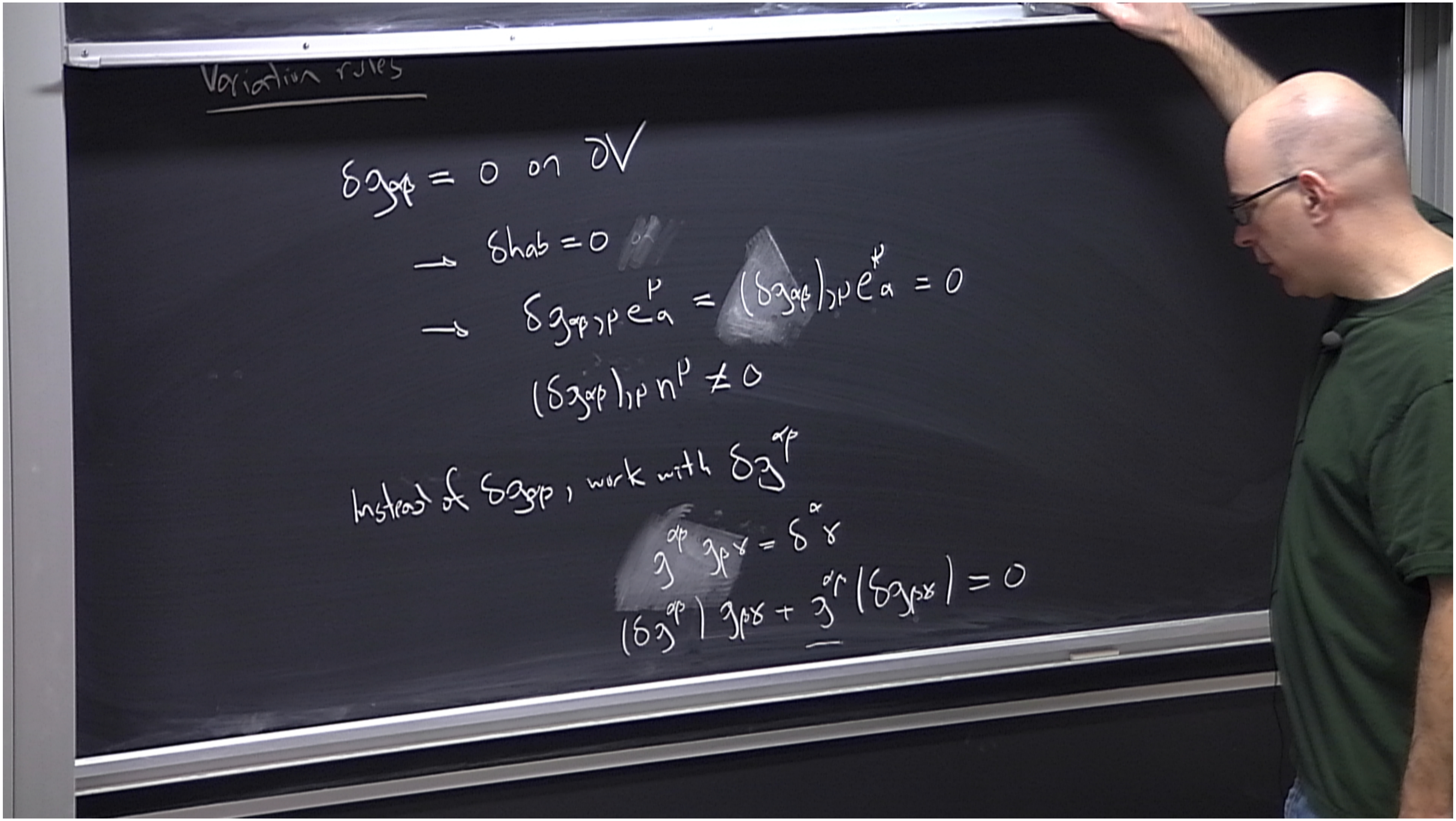
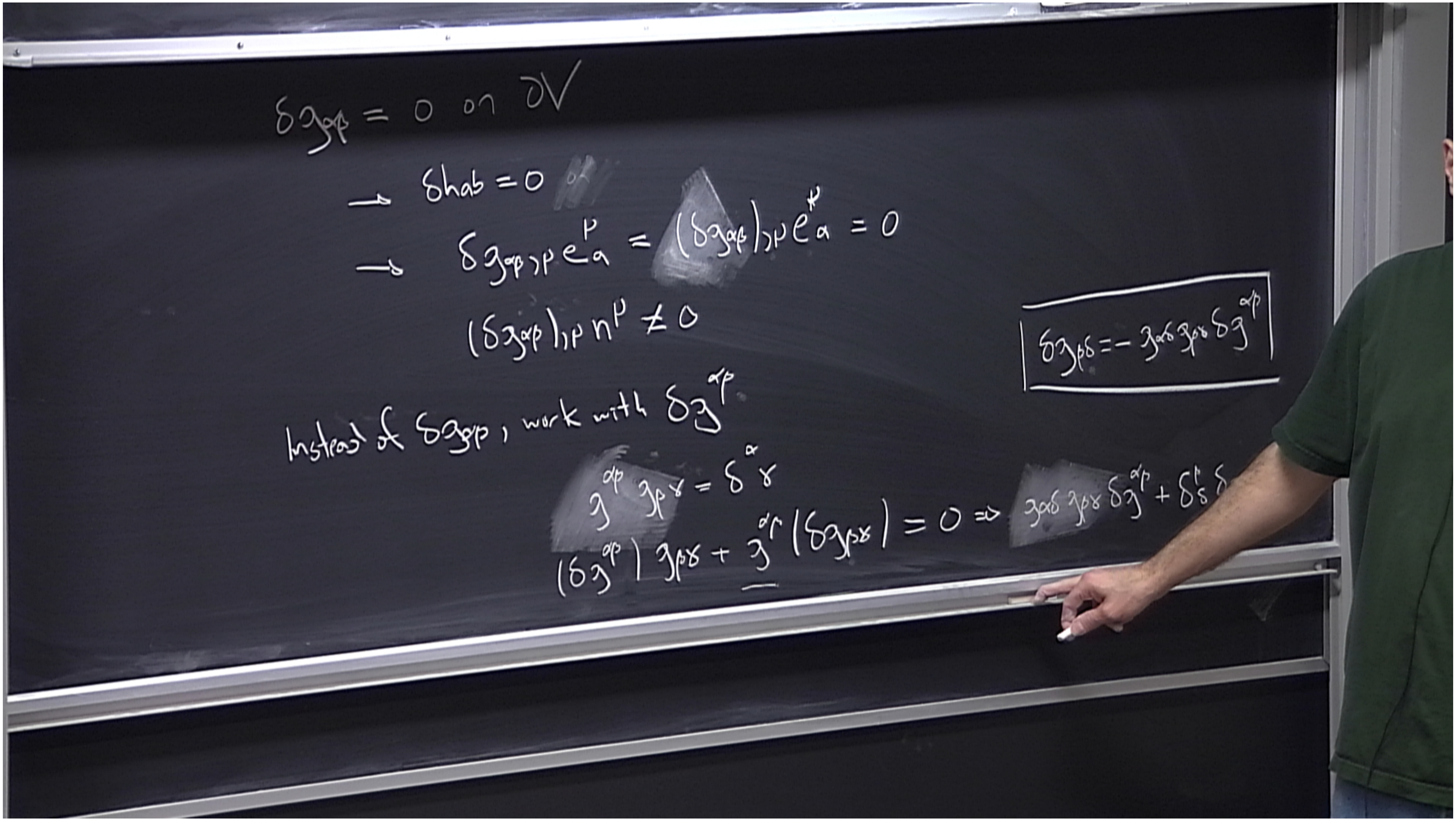
$(\delta g_{\alpha\beta})_{,\mu} n^{\mu} \neq 0$

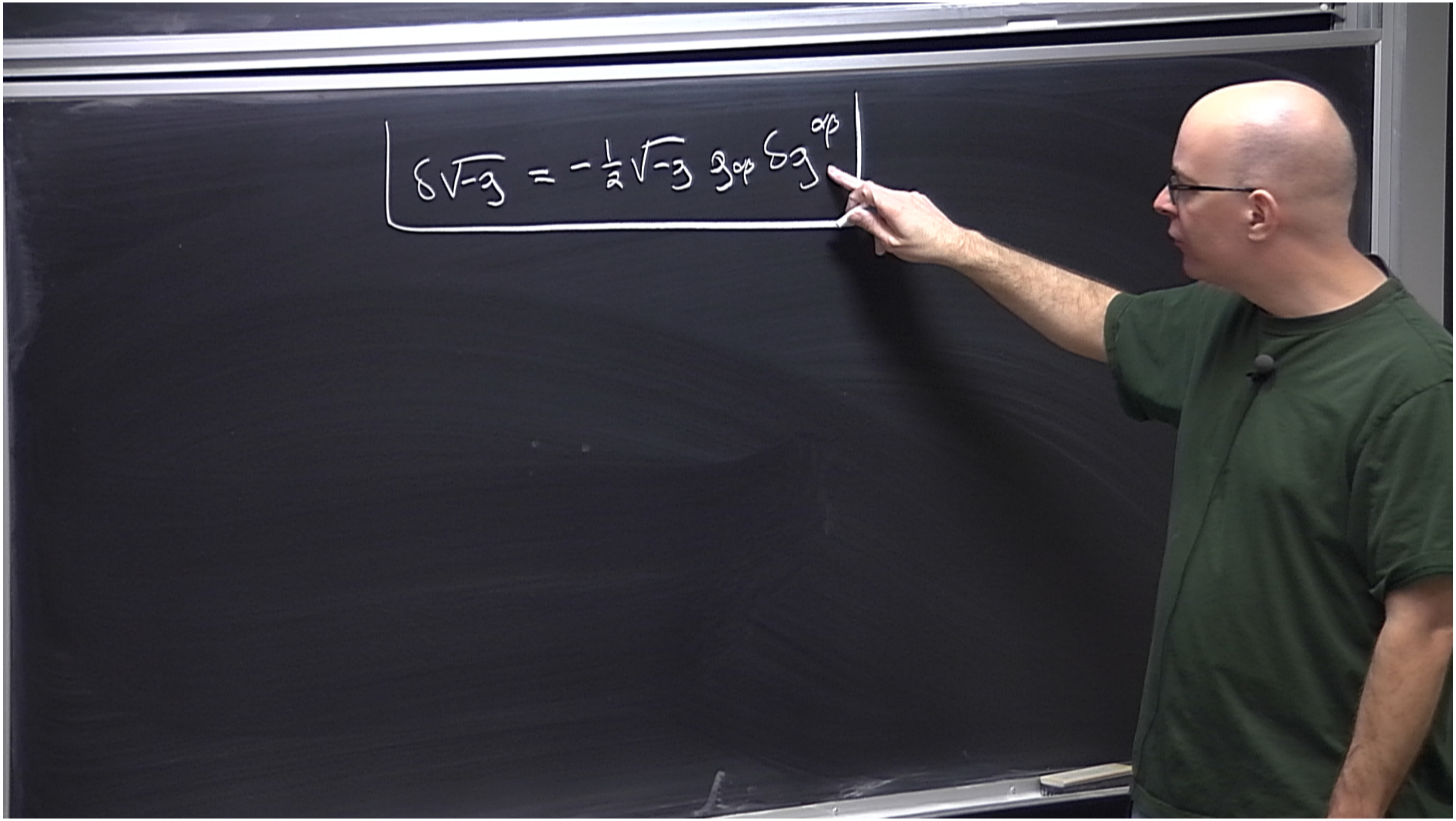

Variation rules
δg_αβ = 0 on ∂V
→ δh_ab = 0
→ δg_αβ,μ e^μ_a = (δg_αβ),μ e^μ_a = 0
(δg_αβ),μ n^μ ≠ 0
Instead of δg_αβ, work with δg^αβ

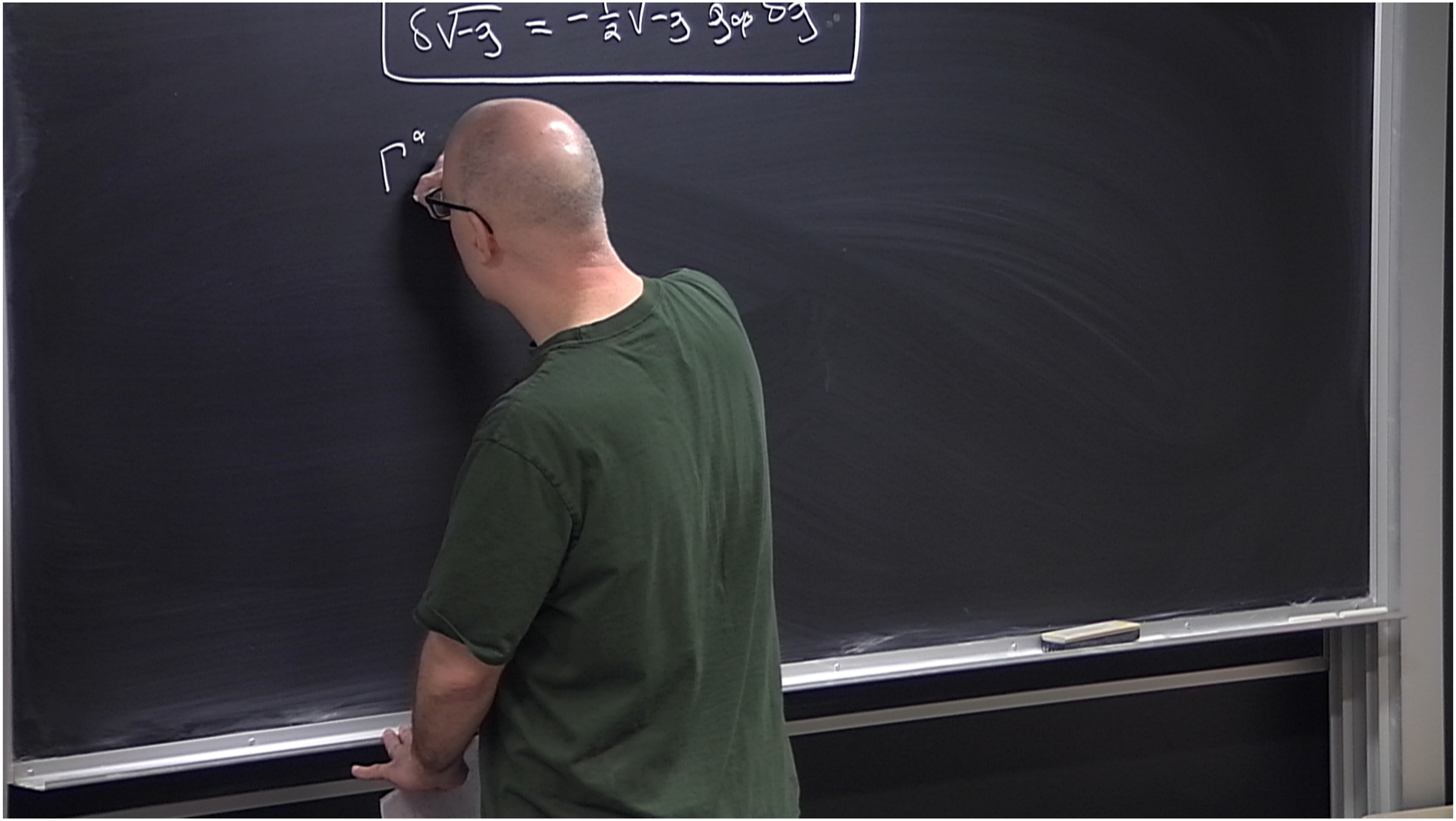

Variation rules
$\delta g_{\alpha\beta} = 0$ on ∂V
$\rightarrow \delta h_{ab} = 0$
$\rightarrow \delta g_{\alpha\beta,\mu} e^{\mu}_{a} = (\delta g_{\alpha\beta})_{,\mu} e^{\mu}_{a} = 0$
$(\delta g_{\alpha\beta})_{,\mu} n^{\mu} \neq 0$
Instead of $\delta g_{\alpha\beta}$, work with $\delta g^{\alpha\beta}$
$g^{\alpha\beta} g_{\beta\gamma} = \delta^{\alpha}_{\gamma}$
$(\delta g^{\alpha\beta}) g_{\beta\gamma} + g^{\alpha\beta}(\delta g_{\beta\gamma}) = 0$

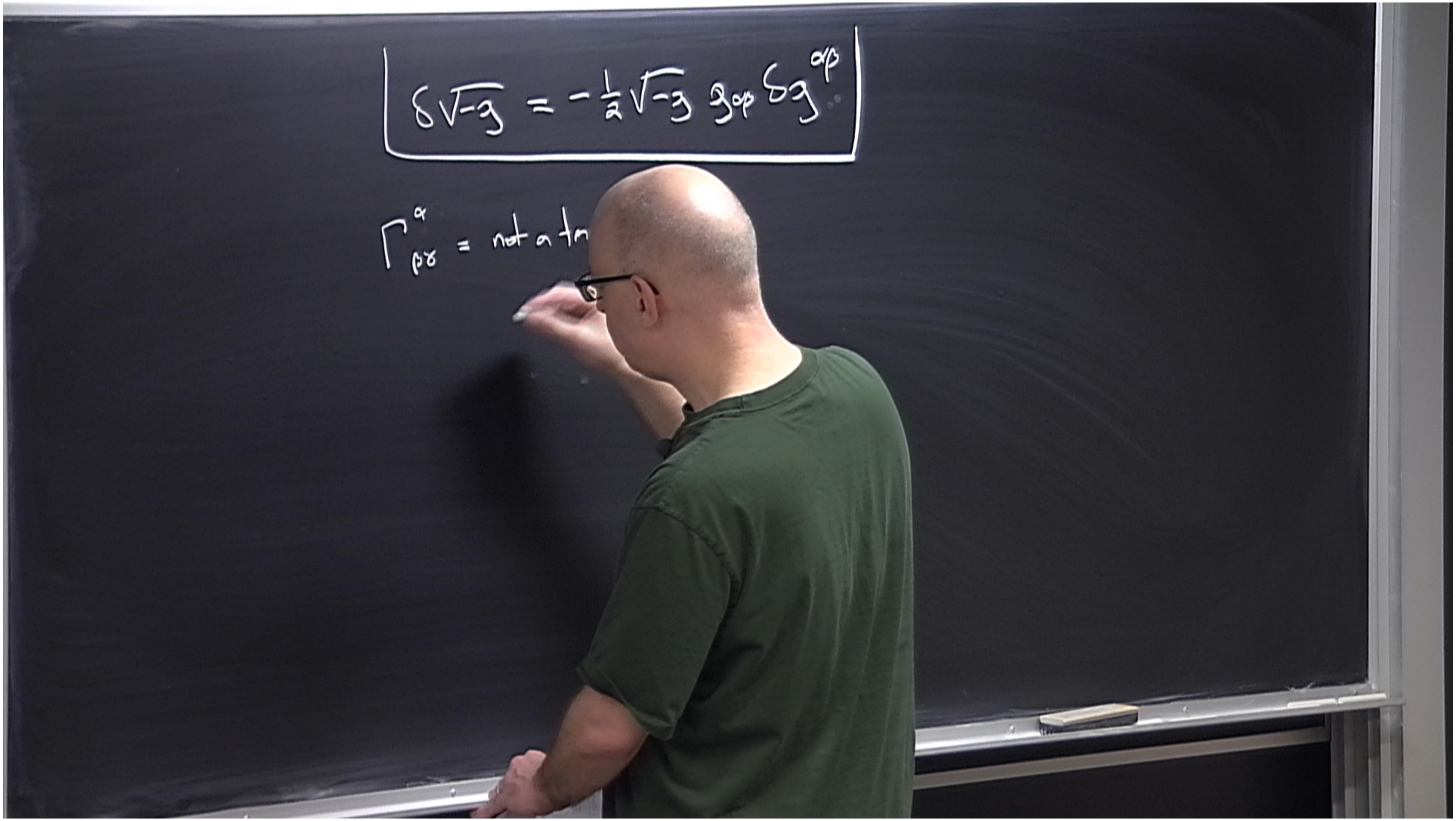


δg_αβ = 0 on ∂V
→ δh_ab = 0
→ δg_αβ,μ e^μ_a = (δg_αβ),μ e^μ_a = 0
(δg_αβ),μ n^μ ≠ 0
δg_βδ = − g_αδ g_βγ δg^αβ
Instead of δg_αβ, work with δg^αβ
g^αβ g_βγ = δ^α_γ
(δg^αβ) g_βγ + g^αβ (δg_βγ) = 0 ⇒ g_αδ g_βγ δg^αβ + δ^β_δ δ


$\delta\sqrt{-g} =$


$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\, g_{\alpha\beta}\, \delta g^{\alpha\beta}$


δ√-g = -½√-g gαβ δg
Γα

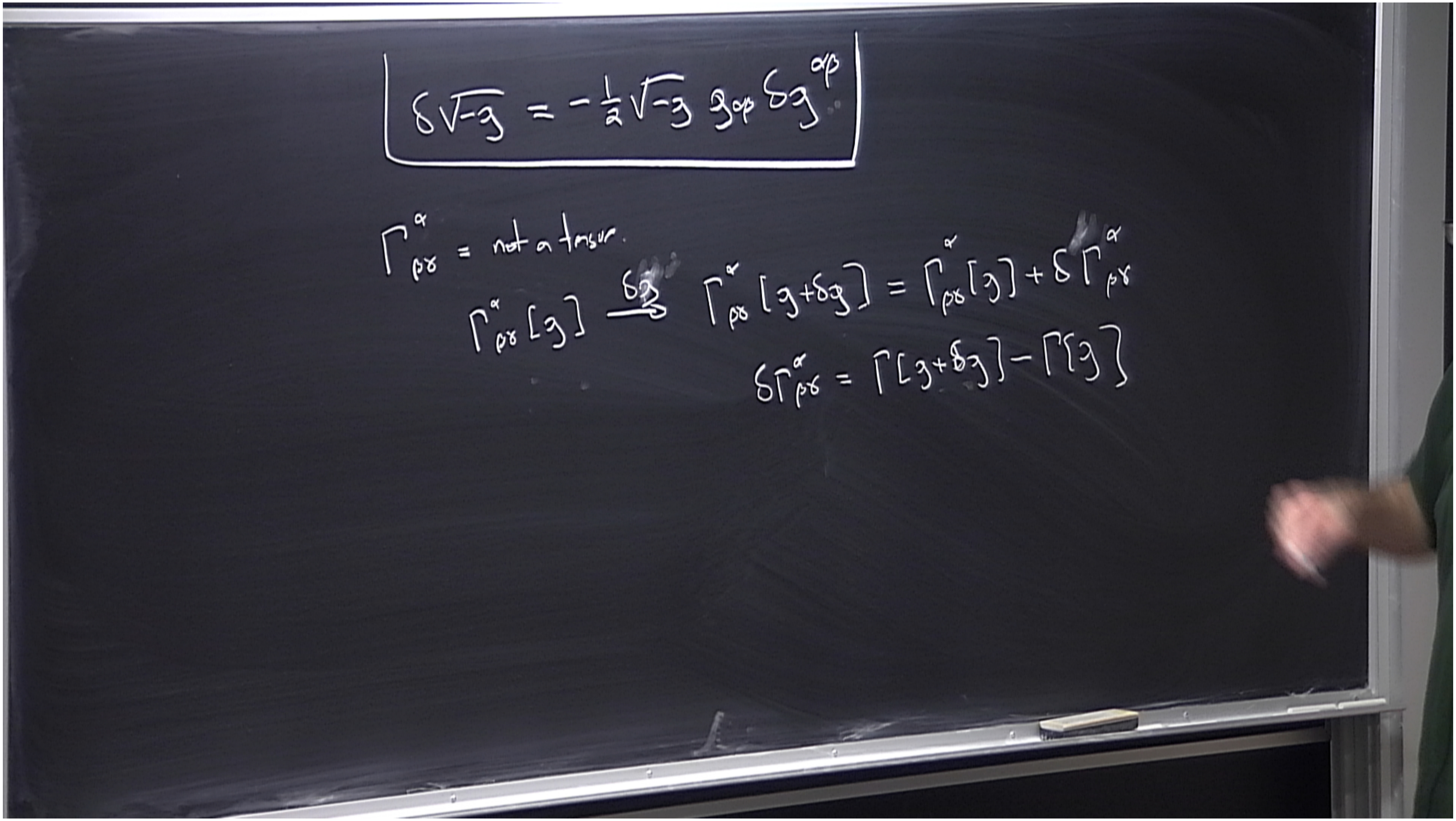


$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\, g_{\alpha\beta}\, \delta g^{\alpha\beta}$
$\Gamma^{\alpha}_{\beta\gamma}$ = not a ten

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\, g_{\alpha\beta}\, \delta g^{\alpha\beta}$$

$\Gamma^{\alpha}_{\beta\gamma}$ = not a tensor.

$$\Gamma^{\alpha}_{\beta\gamma}[g] \xrightarrow{\delta g} \Gamma^{\alpha}_{\beta\gamma}[g+\delta g] = \Gamma^{\alpha}_{\beta\gamma}[g] + \delta\Gamma^{\alpha}_{\beta\gamma}$$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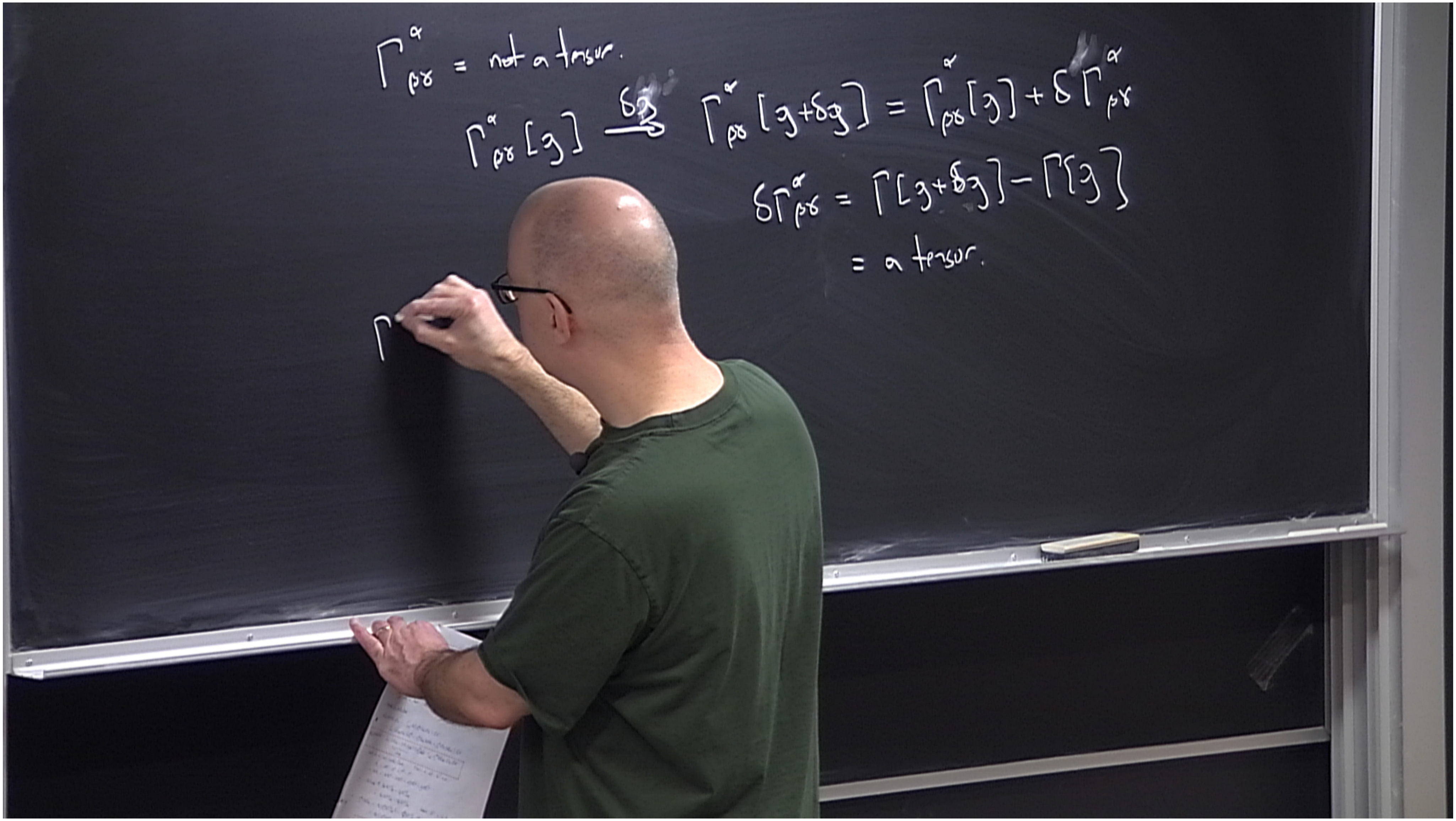
$\Gamma^{\alpha}_{\rho\sigma}$ = not a tensor.
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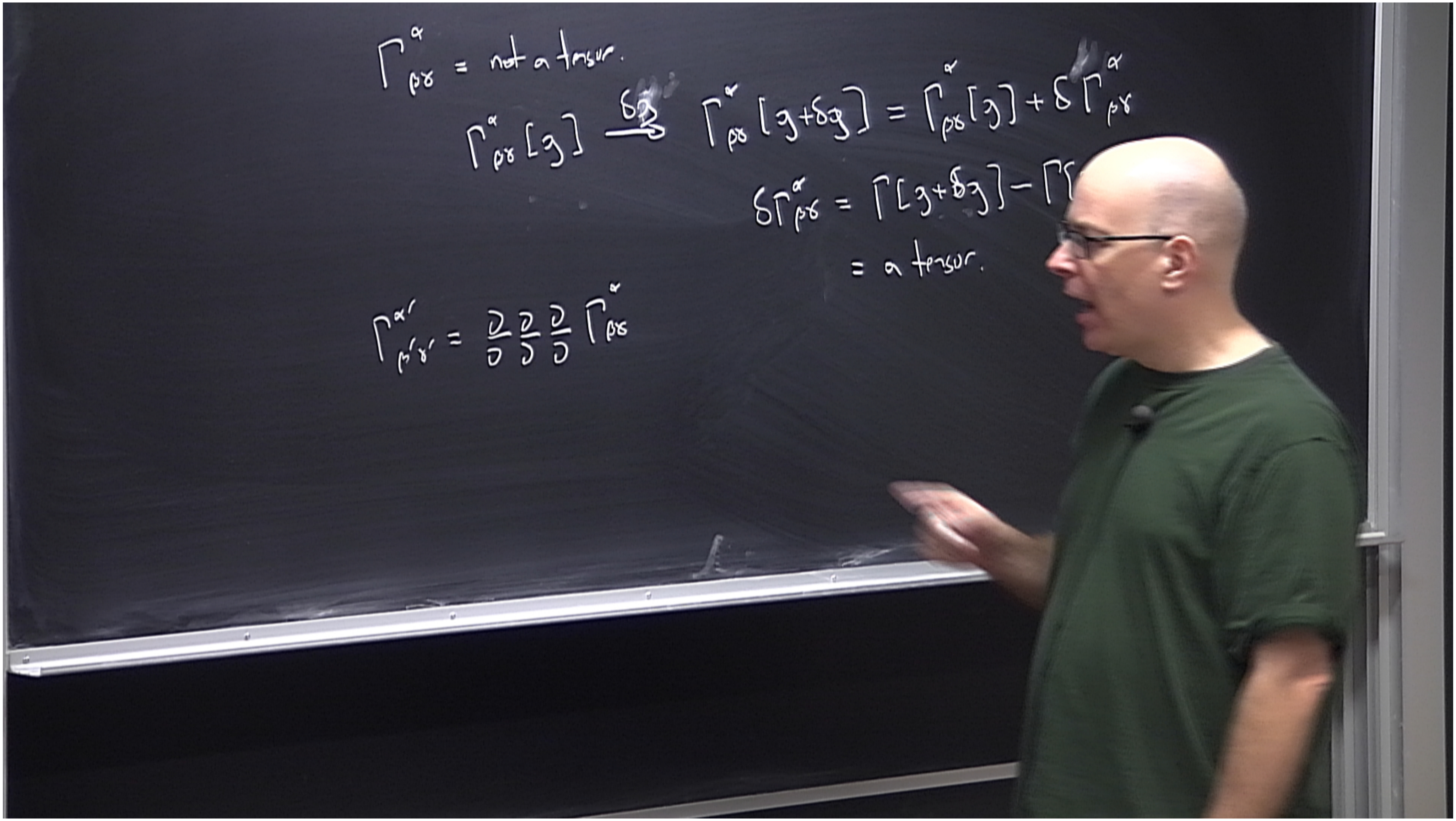
$\delta\Gamma^{\alpha}_{\rho\sigma} = \Gamma[g+\delta g] - \Gamma[g]$
= a tensor.
Γ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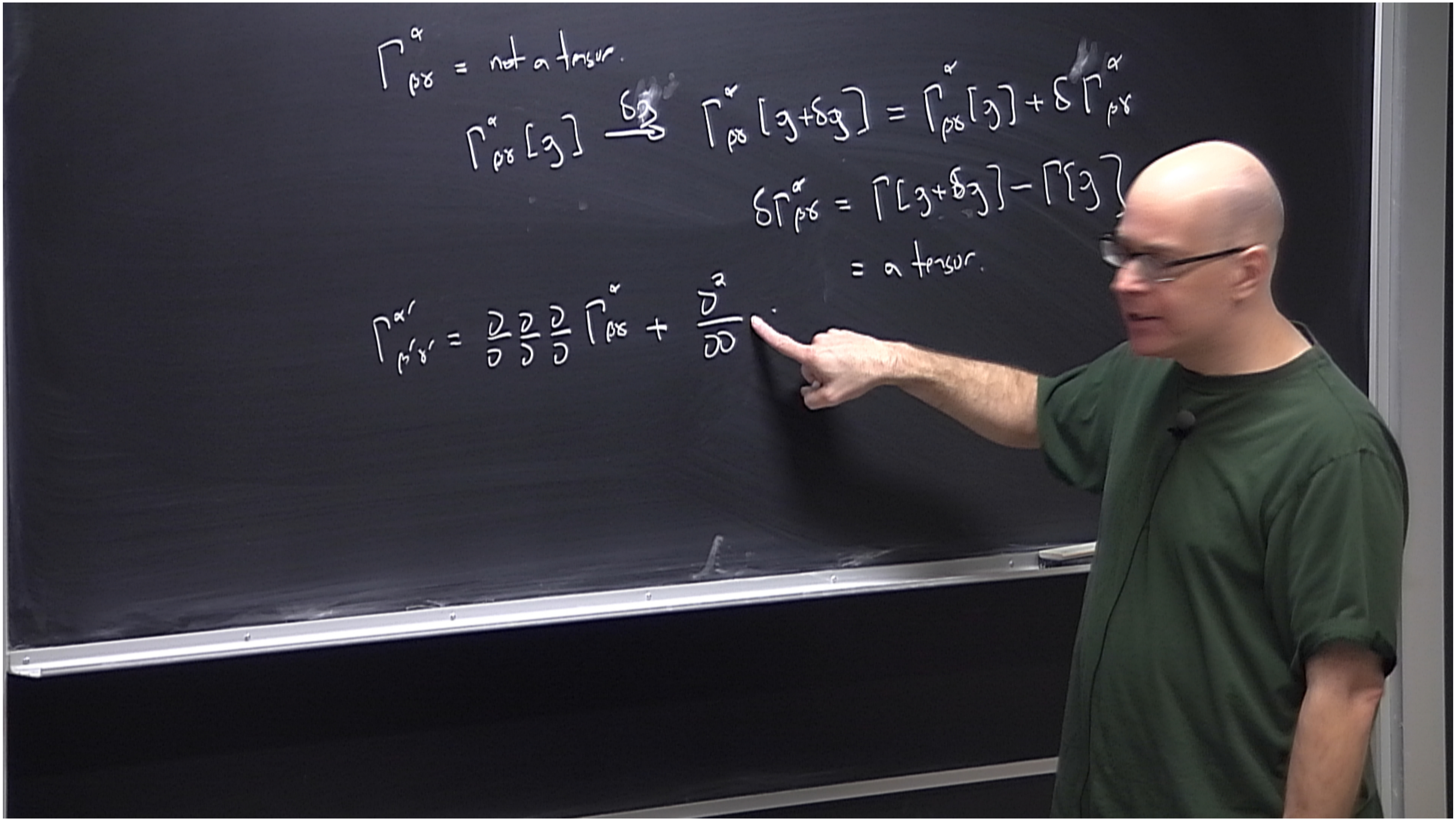
= a tensor.
$\Gamma^{\alpha'}_{\beta'\gamma'} = \frac{\partial}{\partial}\frac{\partial}{\partial}\frac{\partial}{\partial}\Gamma^{\alpha}_{\beta\gamma}$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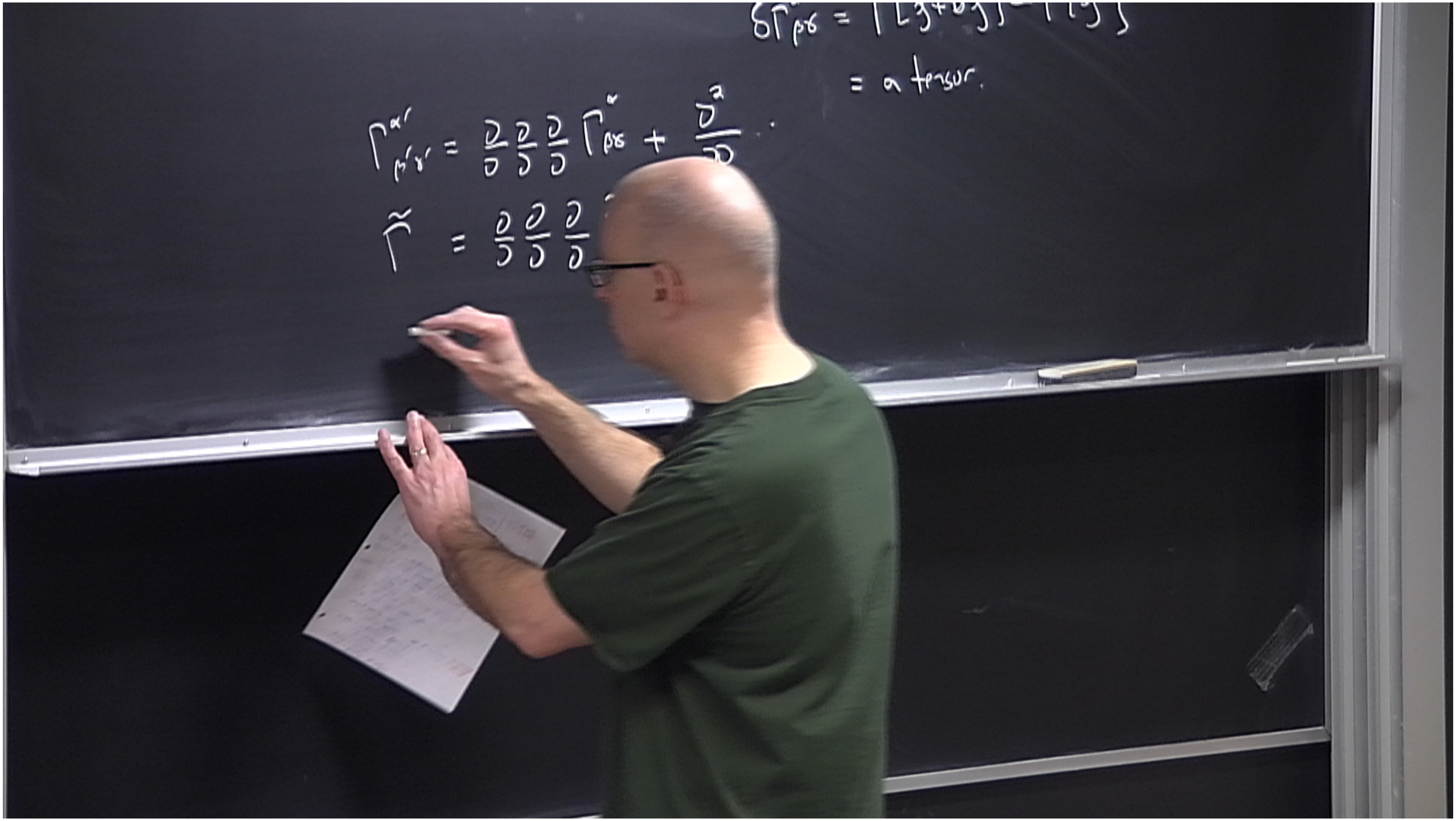
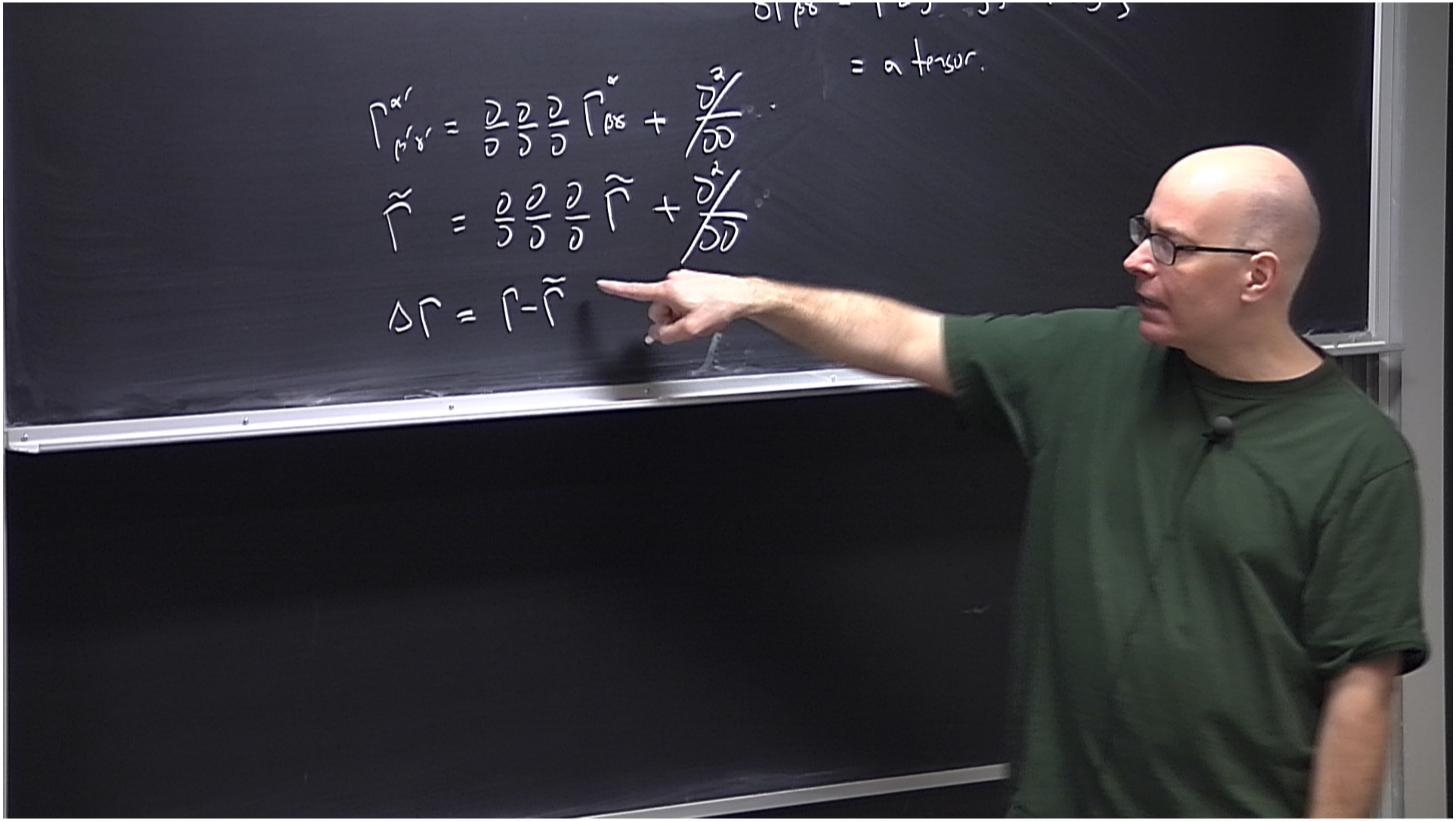
$\delta\Gamma^\alpha_{\beta\gamma} = \Gamma[g+\delta g] - \Gamma[g]$
= a tensor.
$\Gamma^{\alpha'}_{\beta'\gamma'} = \frac{\partial}{\partial}\frac{\partial}{\partial}\frac{\partial}{\partial}\Gamma^\alpha_{\beta\gamma} + \frac{\partial^2}{\partial\partial}$


= a tensor.


= a tensor.
ΔΓ = Γ − Γ̃

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\, g_{\alpha\beta}\, \delta g^{\alpha\beta}$$

$\Gamma^{\alpha}_{\beta\gamma}$ = not a tensor.

$$\Gamma^{\alpha}_{\beta\gamma}[g] \xrightarrow{\delta g} \Gamma^{\alpha}_{\beta\gamma}[g+\delta g] = \Gamma^{\alpha}_{\beta\gamma}[g] + \delta\Gamma^{\alpha}_{\beta\gamma}$$

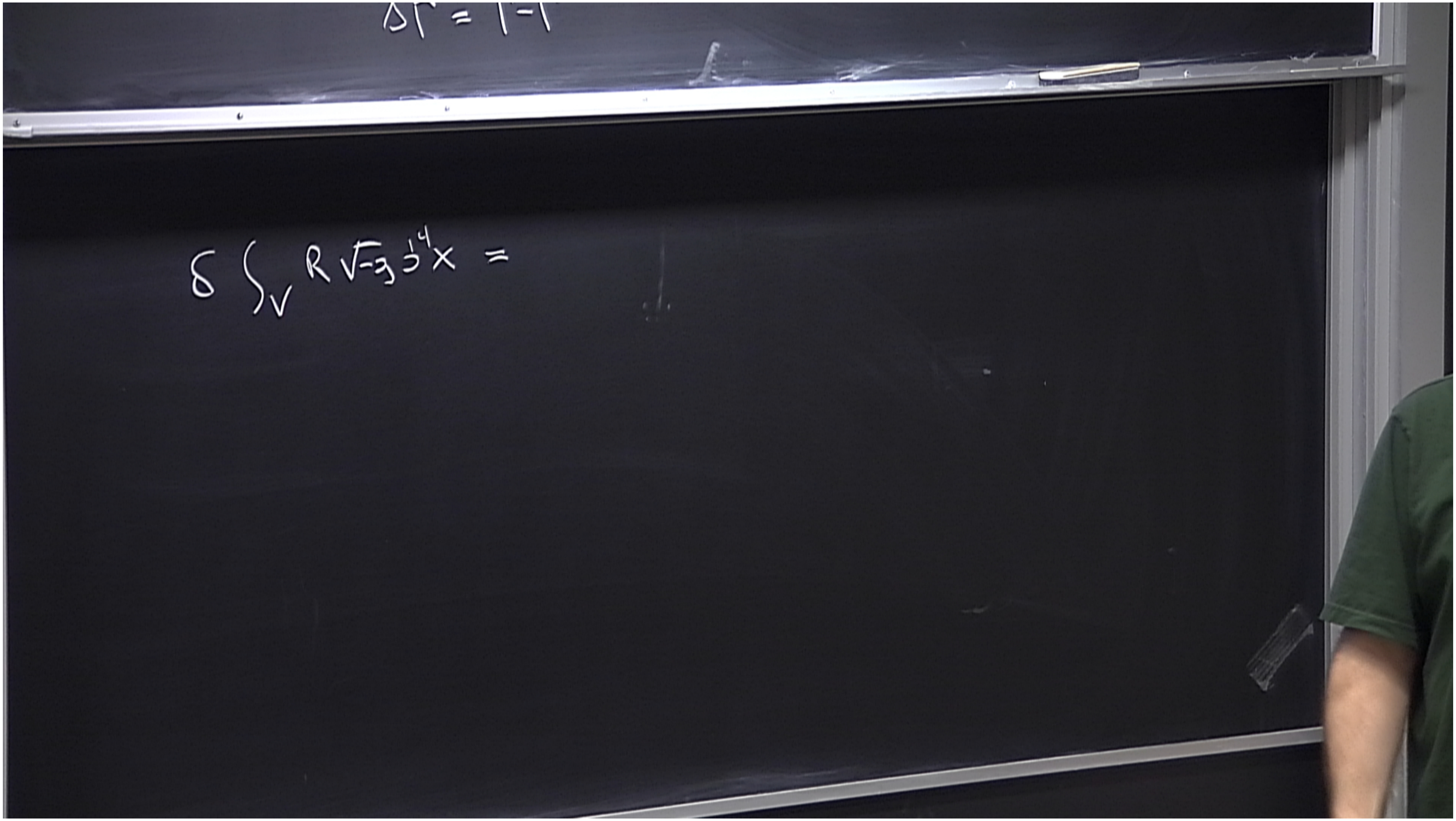
$$\delta\Gamma^{\alpha}_{\beta\gamma} = \Gamma[g+\delta g] - \Gamma[g]$$

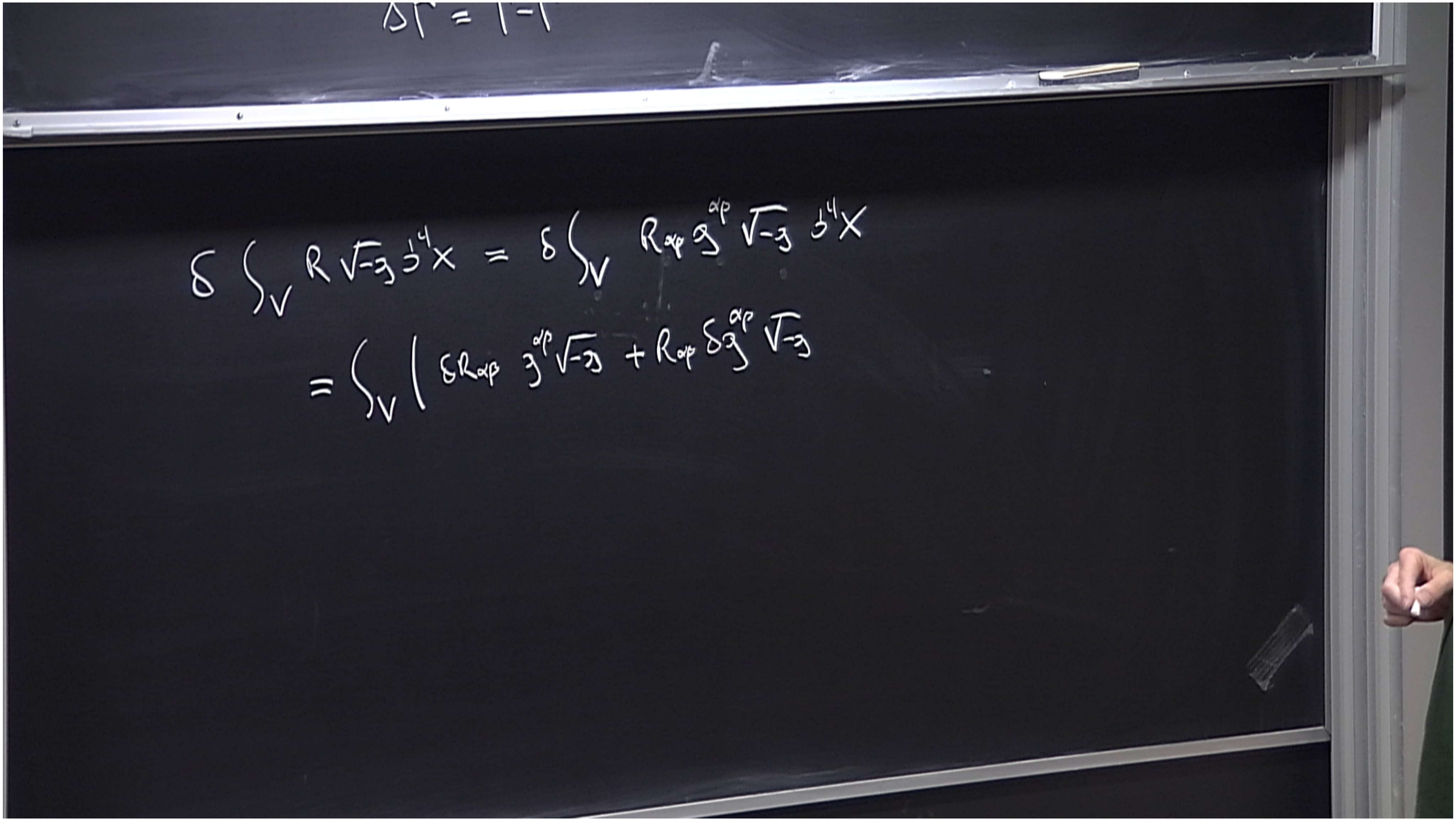
= a tensor

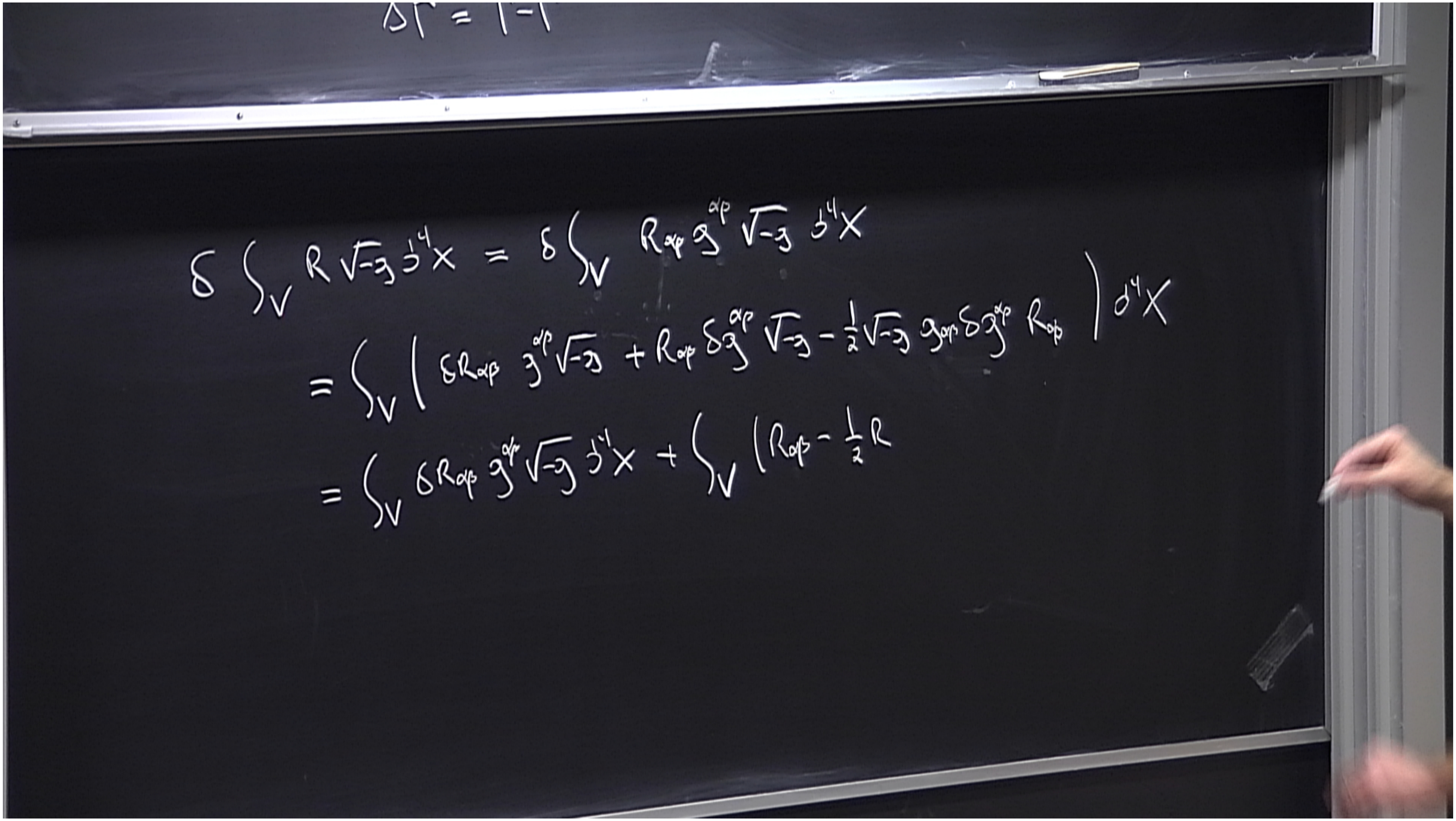
$$\Gamma^{\alpha'}_{\beta'\gamma'} = \frac{\partial}{\partial}\frac{\partial}{\partial}\frac{\partial}{\partial}\Gamma^{\alpha}_{\beta\gamma} + \cancel{\frac{\partial^2}{\partial\partial}}$$

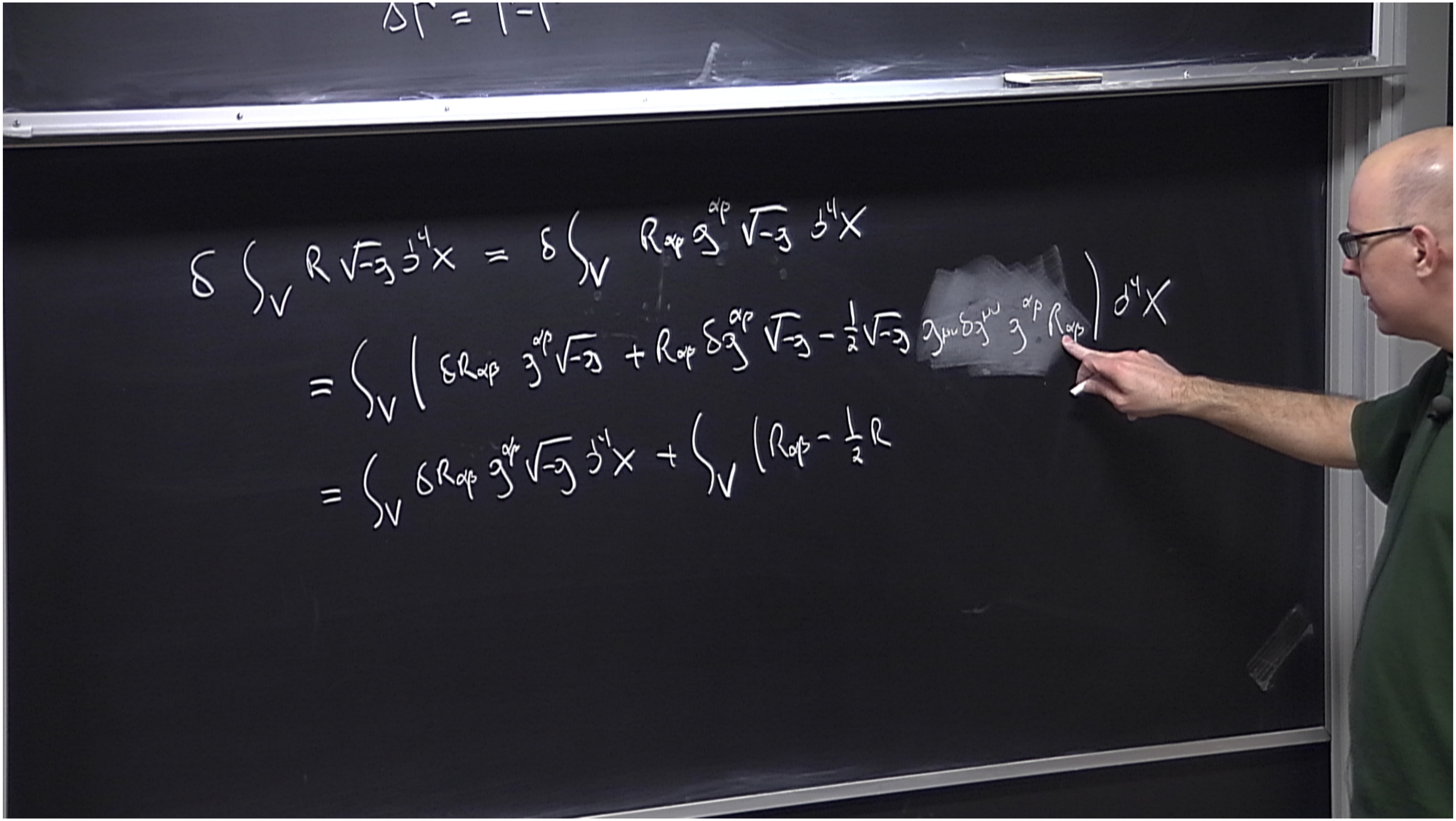
$$\tilde{\Gamma}' = \frac{\partial}{\partial}\frac{\partial}{\partial}\frac{\partial}{\partial}\tilde{\Gamma} + \cancel{\frac{\partial^2}{\partial\partial}}$$

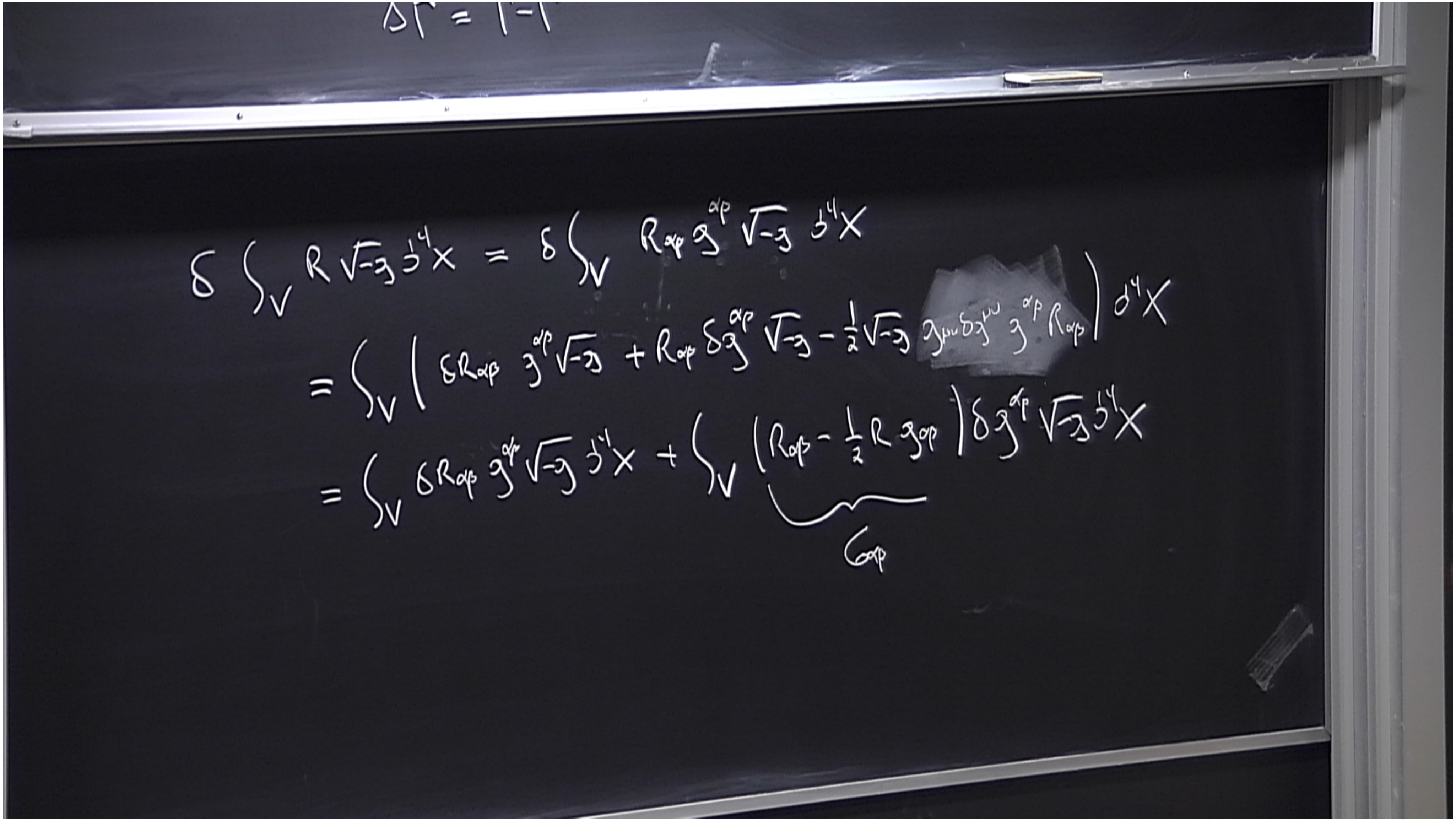
$$\Delta\Gamma = \Gamma - \tilde{\Gamma}$$

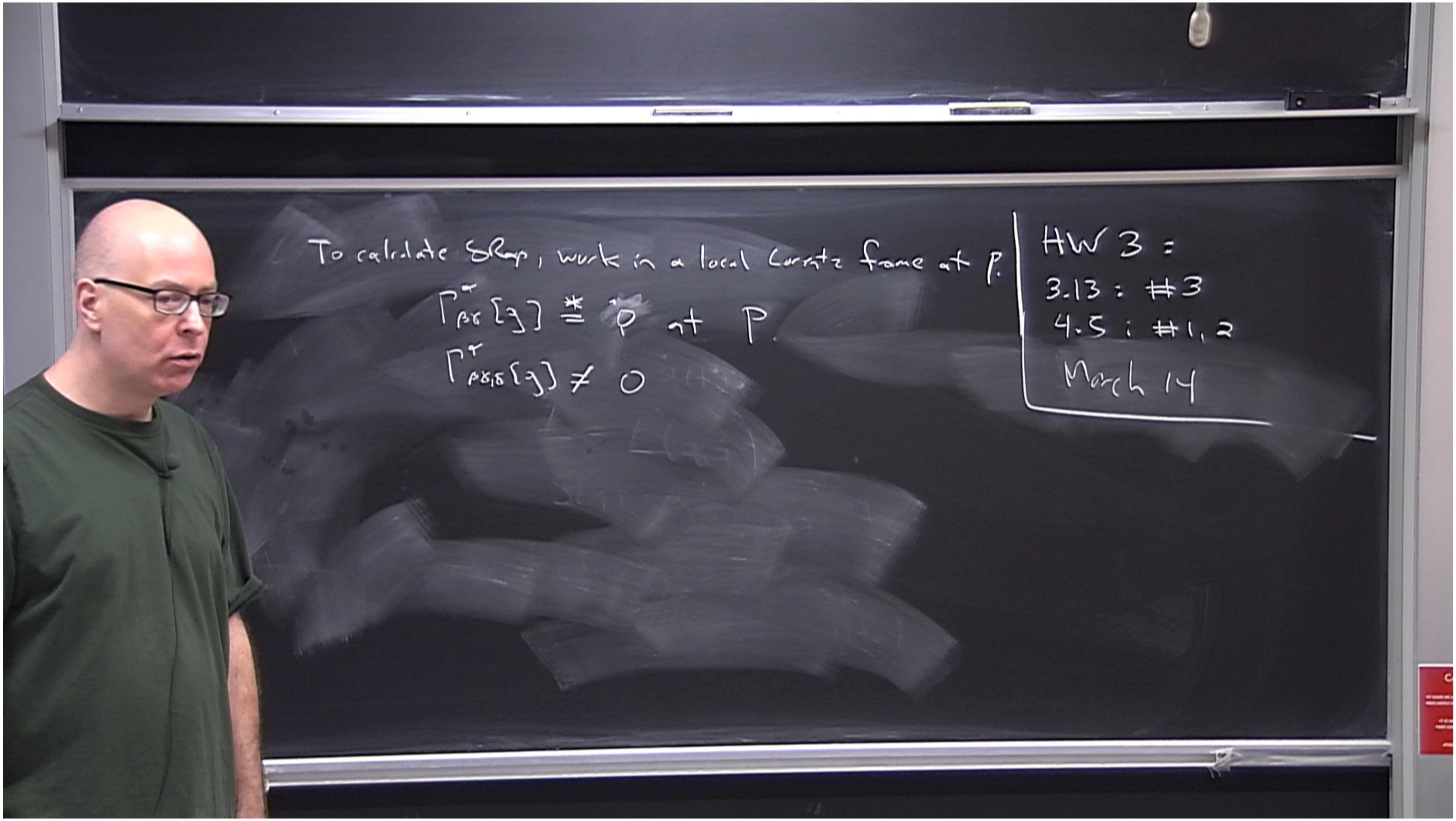

δ ∫_V R √-g d^4x =

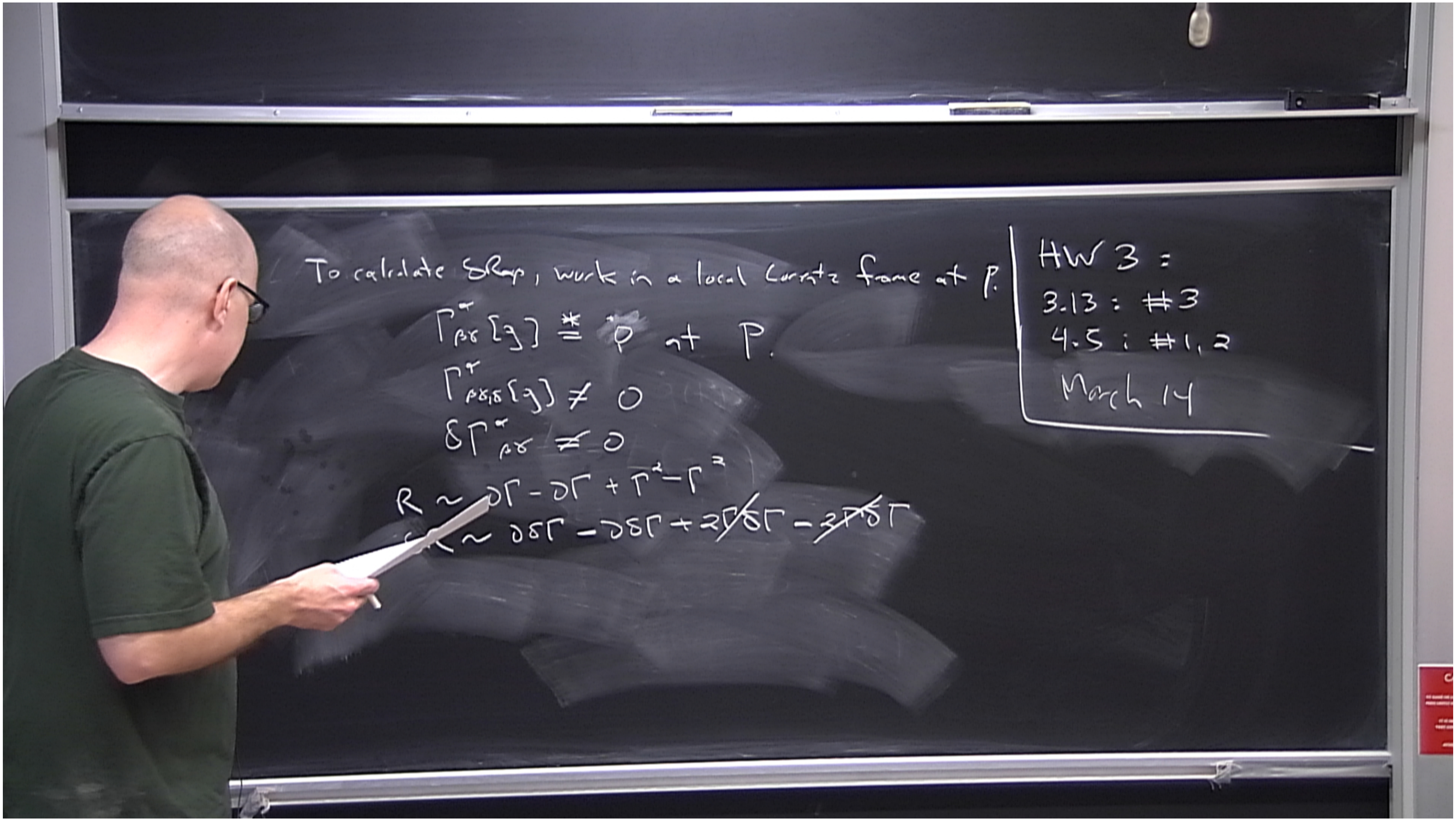

$\delta \int_V R \sqrt{-g}\, d^4x = \delta \int_V R_{\alpha\beta}\, g^{\alpha\beta} \sqrt{-g}\, d^4x$
$= \int_V \big(\delta R_{\alpha\beta}\, g^{\alpha\beta} \sqrt{-g} + R_{\alpha\beta}\, \delta g^{\alpha\beta} \sqrt{-g}$

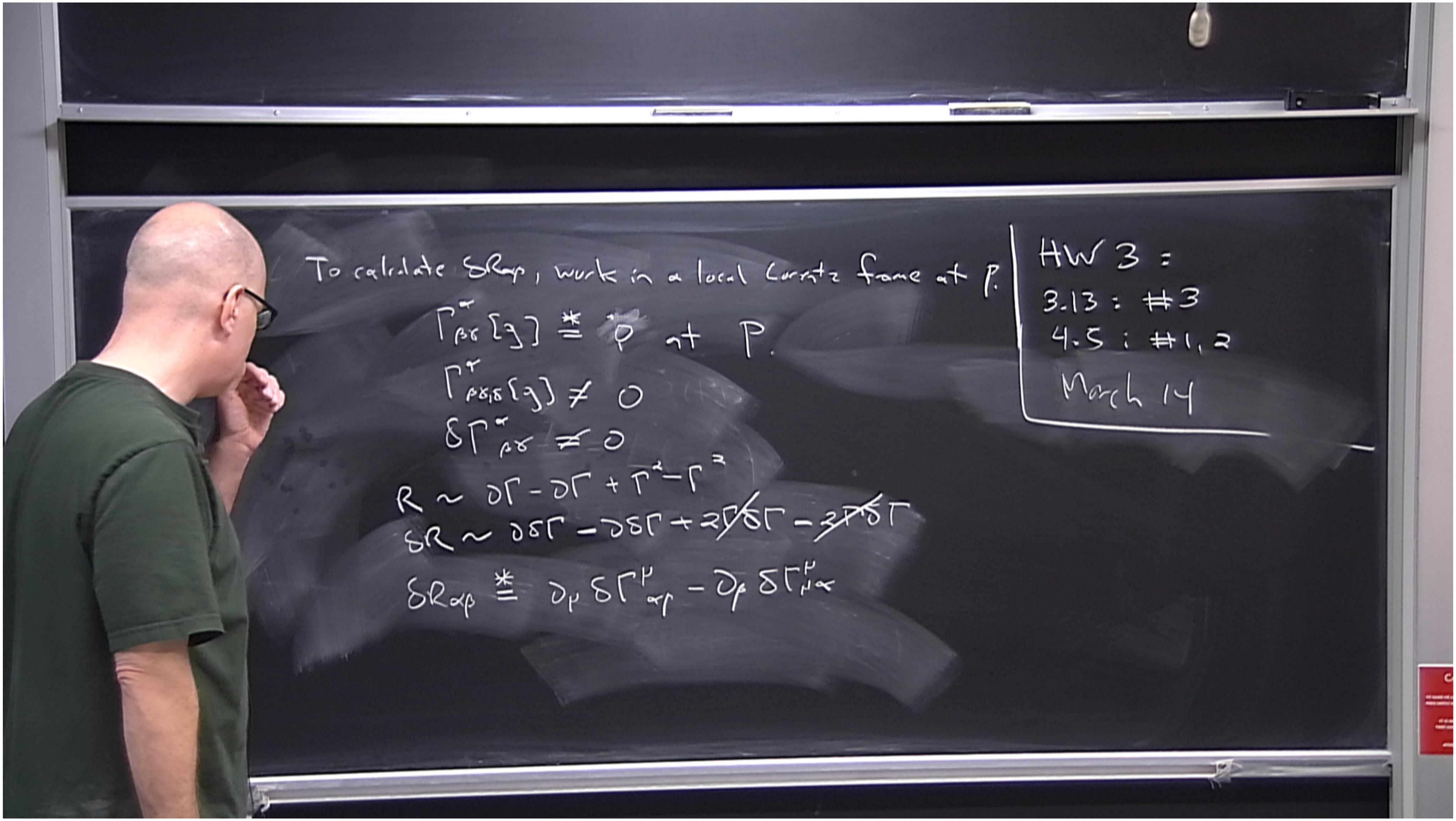

$\delta \int_V R \sqrt{-g}\, d^4x = \delta \int_V R_{\alpha\beta}\, g^{\alpha\beta} \sqrt{-g}\, d^4x$
$= \int_V \left(\delta R_{\alpha\beta}\, g^{\alpha\beta} \sqrt{-g} + R_{\alpha\beta}\, \delta g^{\alpha\beta} \sqrt{-g} - \frac{1}{2} \sqrt{-g}\, g_{\alpha\beta}\, \delta g^{\alpha\beta} R_{\alpha\beta} \right) d^4x$
$= \int_V \delta R_{\alpha\beta}\, g^{\alpha\beta} \sqrt{-g}\, d^4x + \int_V \left(R_{\alpha\beta} - \frac{1}{2} R \right.$

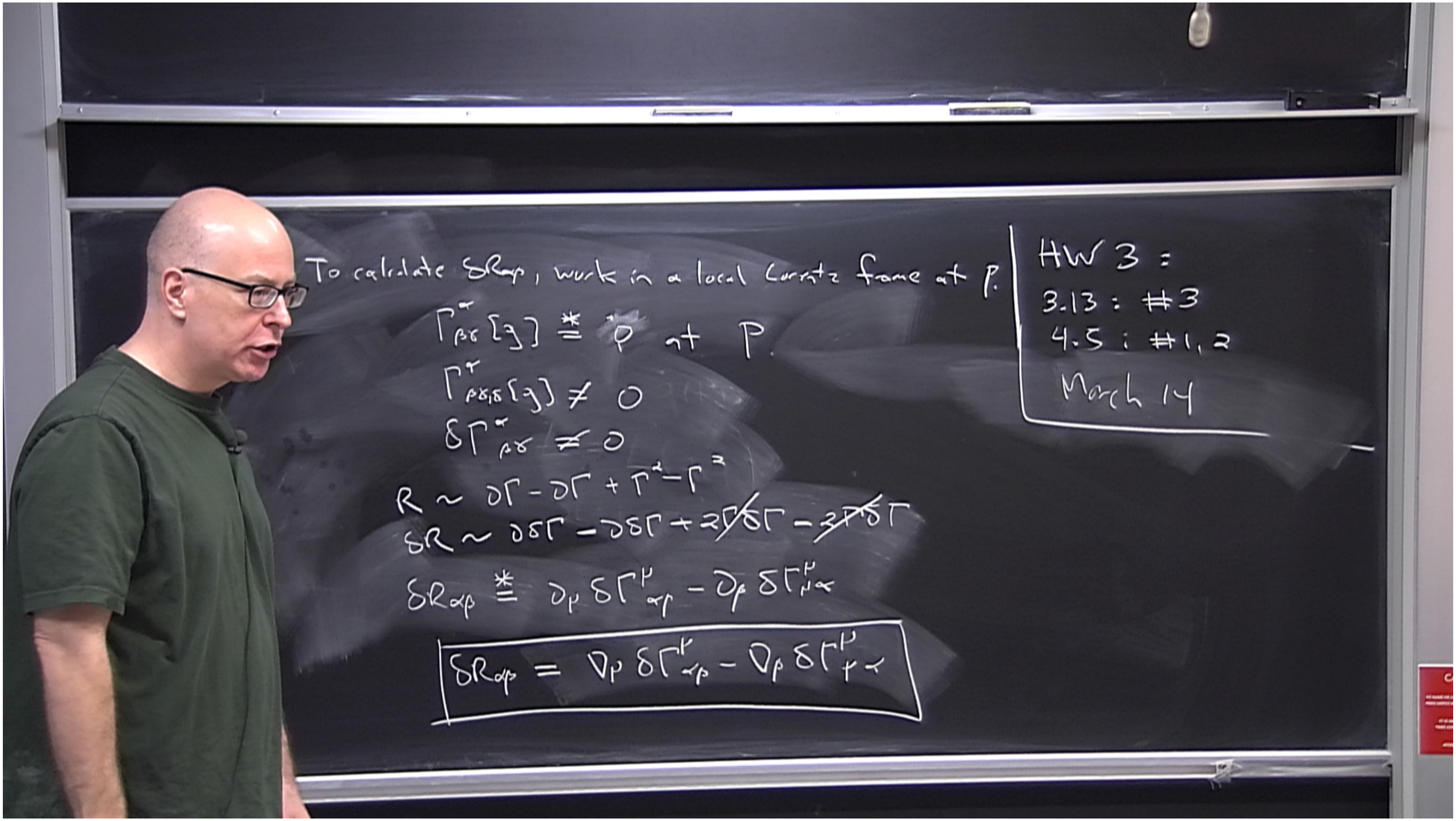

$\delta \int_V R \sqrt{-g}\, d^4x = \delta \int_V R_{\alpha\beta}\, g^{\alpha\beta} \sqrt{-g}\, d^4x$
$= \int_V \left(\delta R_{\alpha\beta}\, g^{\alpha\beta} \sqrt{-g} + R_{\alpha\beta}\, \delta g^{\alpha\beta} \sqrt{-g} - \frac{1}{2} \sqrt{-g}\, g_{\mu\nu}\, \delta g^{\mu\nu} g^{\alpha\beta} R_{\alpha\beta} \right) d^4x$
$= \int_V \delta R_{\alpha\beta}\, g^{\alpha\beta} \sqrt{-g}\, d^4x + \int_V \left(R_{\alpha\beta} - \frac{1}{2} R \right.$

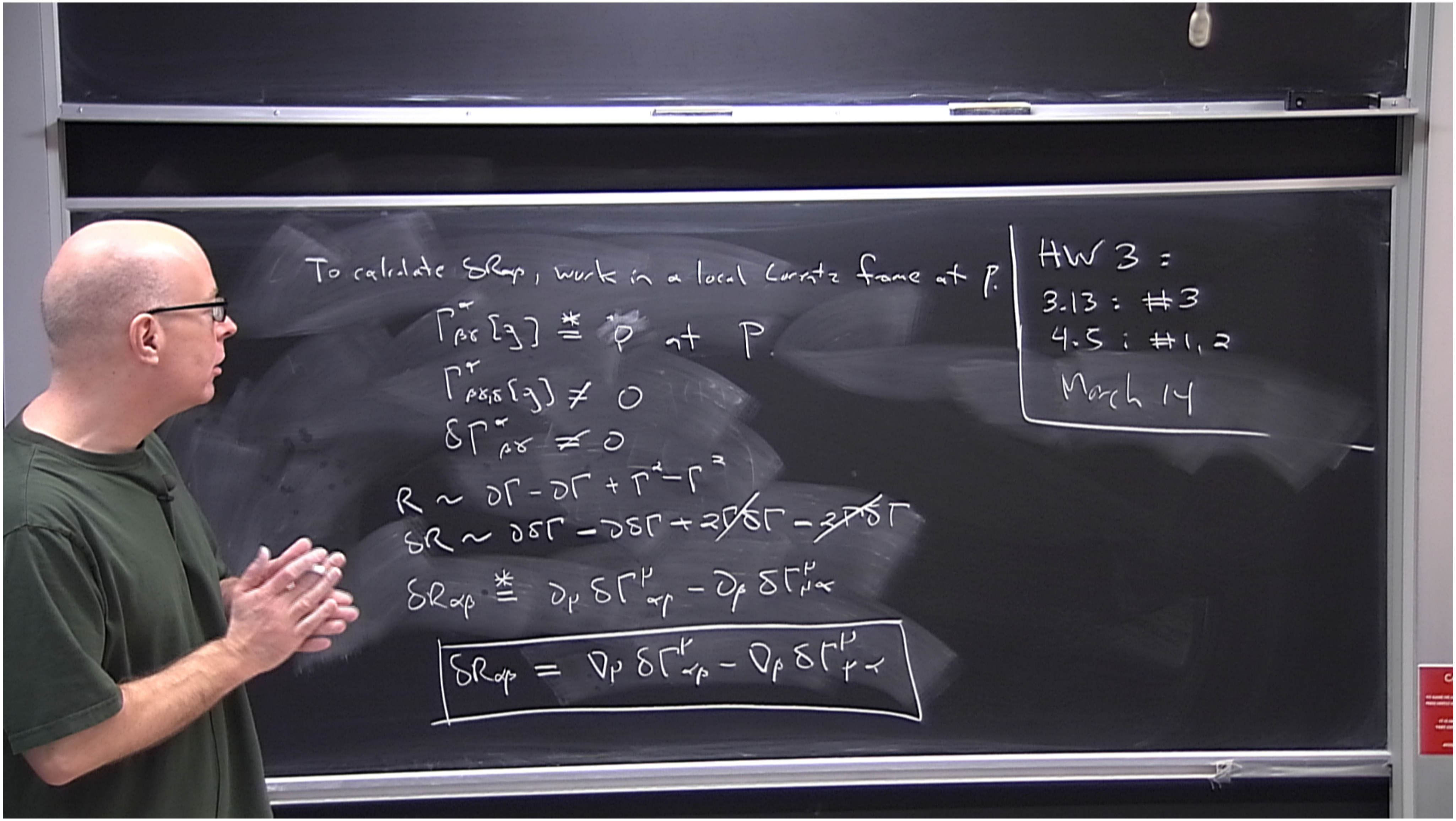

$\delta \int_V R \sqrt{-g}\, d^4x = \delta \int_V R_{\alpha\beta}\, g^{\alpha\beta} \sqrt{-g}\, d^4x$
$= \int_V \left(\delta R_{\alpha\beta}\, g^{\alpha\beta} \sqrt{-g} + R_{\alpha\beta}\, \delta g^{\alpha\beta} \sqrt{-g} - \frac{1}{2}\sqrt{-g}\, g_{\mu\nu}\, \delta g^{\mu\nu}\, g^{\alpha\beta} R_{\alpha\beta} \right) d^4x$
$= \int_V \delta R_{\alpha\beta}\, g^{\alpha\beta} \sqrt{-g}\, d^4x + \int_V \left(R_{\alpha\beta} - \frac{1}{2} R\, g_{\alpha\beta} \right) \delta g^{\alpha\beta} \sqrt{-g}\, d^4x$
$G_{\alpha\beta}$


To calculate δRαβ, work in a local Lorentz frame at p.
Γαβγ[g] ≛ 0 at p
Γαβγ,δ[g] ≠ 0
HW 3 :
3.13 : #3
4.5 : #1, 2
March 14


To calculate δR_αβ, work in a local Lorentz frame at p.
Γ^α_βγ[g] ≐ 0 at p.
Γ^α_βγ,δ[g] ≠ 0
δΓ^α_βγ ≠ 0
R ~ ∂Γ − ∂Γ + Γ² − Γ²
δR ~ ∂δΓ − ∂δΓ + 2ΓδΓ − 2ΓδΓ
HW 3:
3.13: #3
4.5: #1, 2
March 14


To calculate δR_αβ, work in a local Lorentz frame at p.
Γ^α_βγ [g] ≛ 0 at p.
Γ^α_βγ,δ [g] ≠ 0
δΓ^α_βγ ≠ 0
R ~ ∂Γ − ∂Γ + Γ² − Γ²
δR ~ ∂δΓ − ∂δΓ + 2ΓδΓ − 2ΓδΓ
δR_αβ ≛ ∂_μ δΓ^μ_αβ − ∂_β δΓ^μ_μα
HW 3 :
3.13 : #3
4.5 : #1, 2
March 14


To calculate δRαβ, work in a local Lorentz frame at p.
Γ^α_βγ[g] ≐* 0 at p.
Γ^α_βγ,δ[g] ≠ 0
δΓ^α_βγ ≠ 0
R ~ ∂Γ − ∂Γ + Γ² − Γ²
δR ~ ∂δΓ − ∂δΓ + 2ΓδΓ − 2ΓδΓ
δRαβ ≐* ∂μ δΓ^μ_αβ − ∂β δΓ^μ_μα
δRαβ = ∇μ δΓ^μ_αβ − ∇β δΓ^μ_μα
HW 3:
3.13: #3
4.5: #1, 2
March 14


To calculate $\delta R_{\alpha\beta}$, work in a local Lorentz frame at P.
$\Gamma^{\alpha}_{\beta\gamma}[g] \overset{*}{=} 0$ at P.
$\Gamma^{\alpha}_{\beta\gamma,\delta}[g] \neq 0$
$\delta\Gamma^{\alpha}_{\beta\gamma} \neq 0$
$R \sim \partial\Gamma - \partial\Gamma + \Gamma^2 - \Gamma^2$
$\delta R \sim \partial\delta\Gamma - \partial\delta\Gamma + 2\Gamma\delta\Gamma - 2\Gamma\delta\Gamma$
$\delta R_{\alpha\beta} \overset{*}{=} \partial_\mu \delta\Gamma^{\mu}_{\alpha\beta} - \partial_\beta \delta\Gamma^{\mu}_{\mu\alpha}$
$\delta R_{\alpha\beta} = \nabla_\mu \delta\Gamma^{\mu}_{\alpha\beta} - \nabla_\beta \delta\Gamma^{\mu}_{\mu\alpha}$
HW 3:
3.13: #3
4.5: #1, 2
March 14