Title: Topos Quantum Physics - Lecture 17

Date: Feb 17, 2012 01:30 PM

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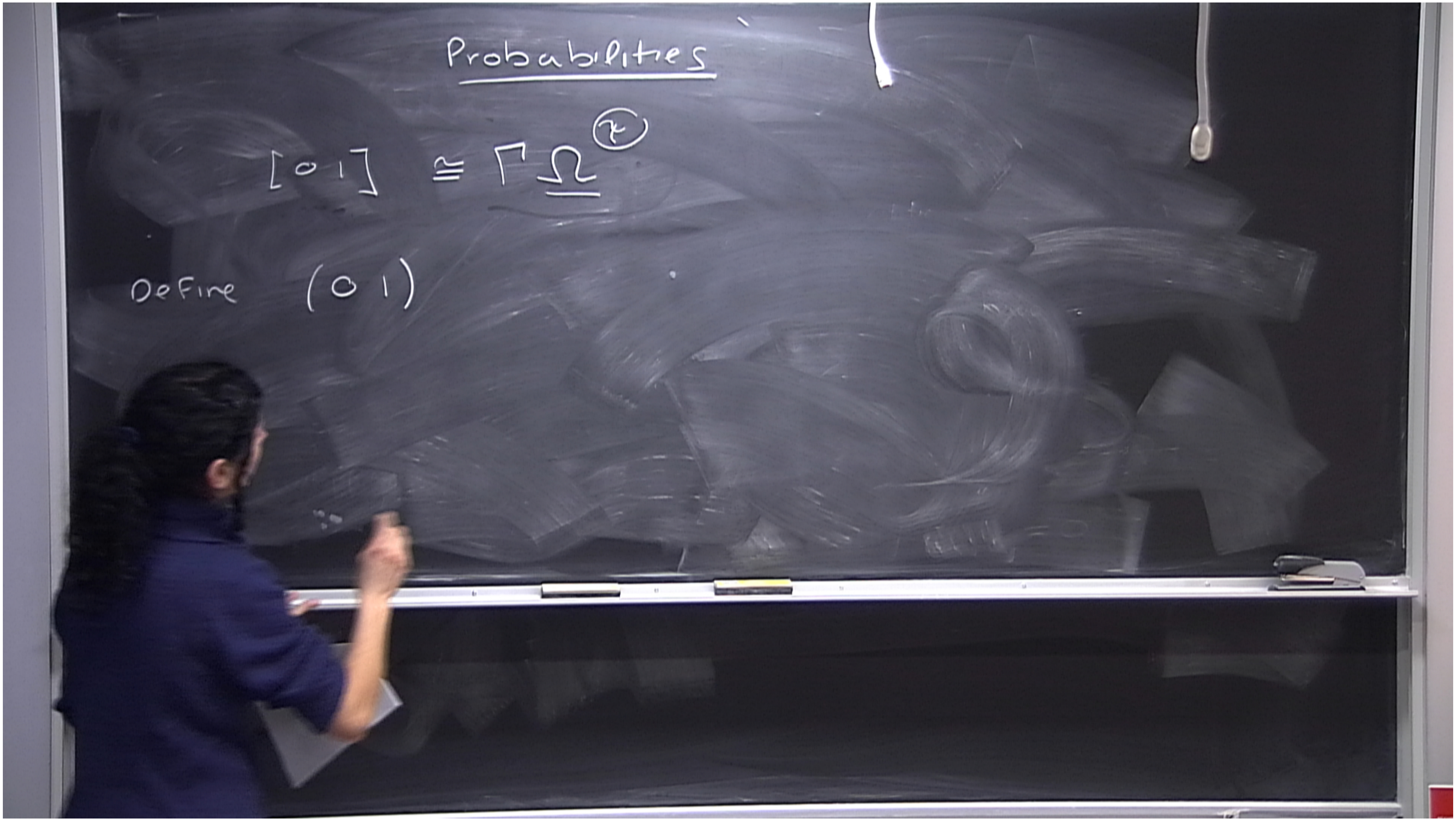
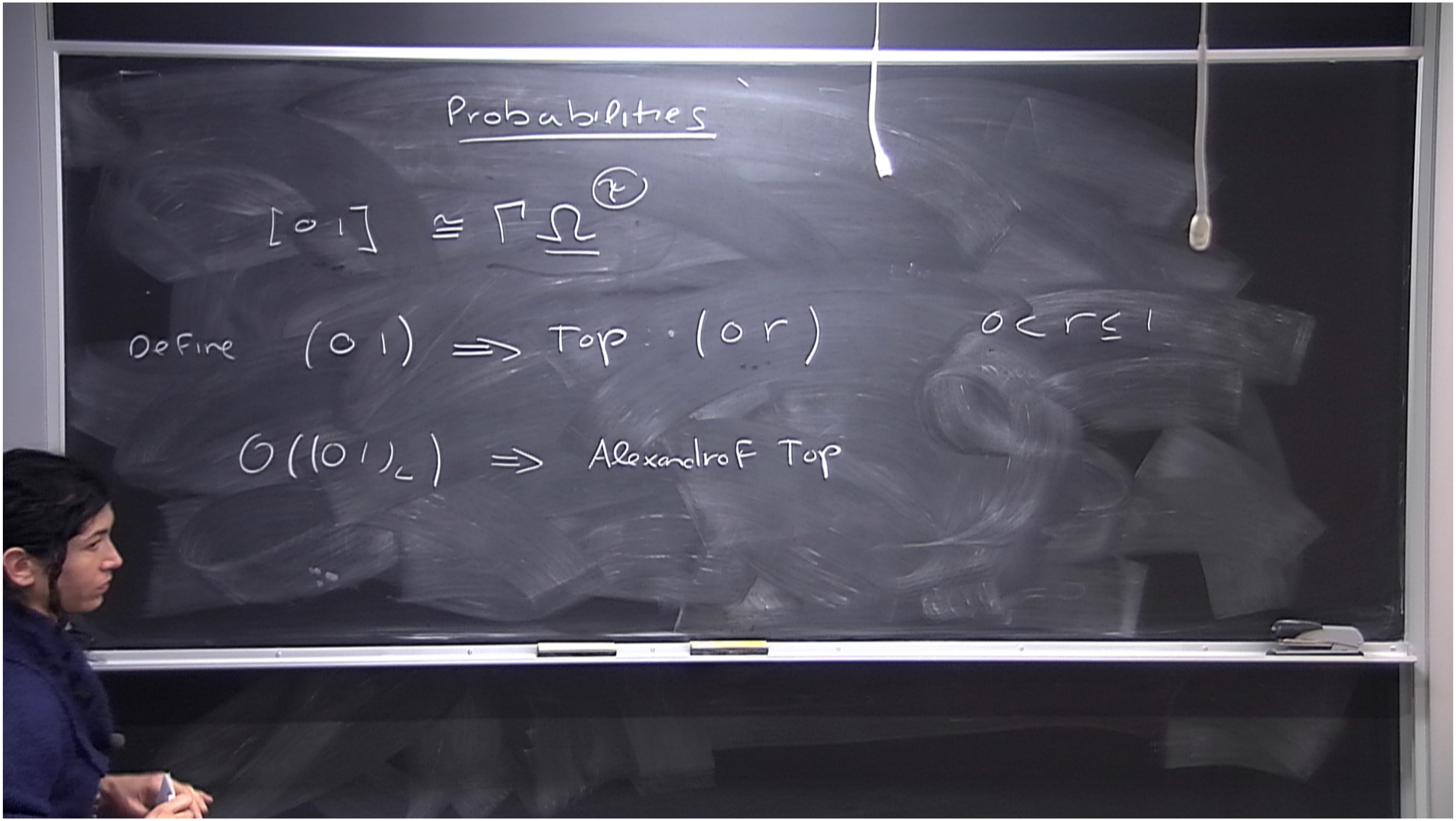
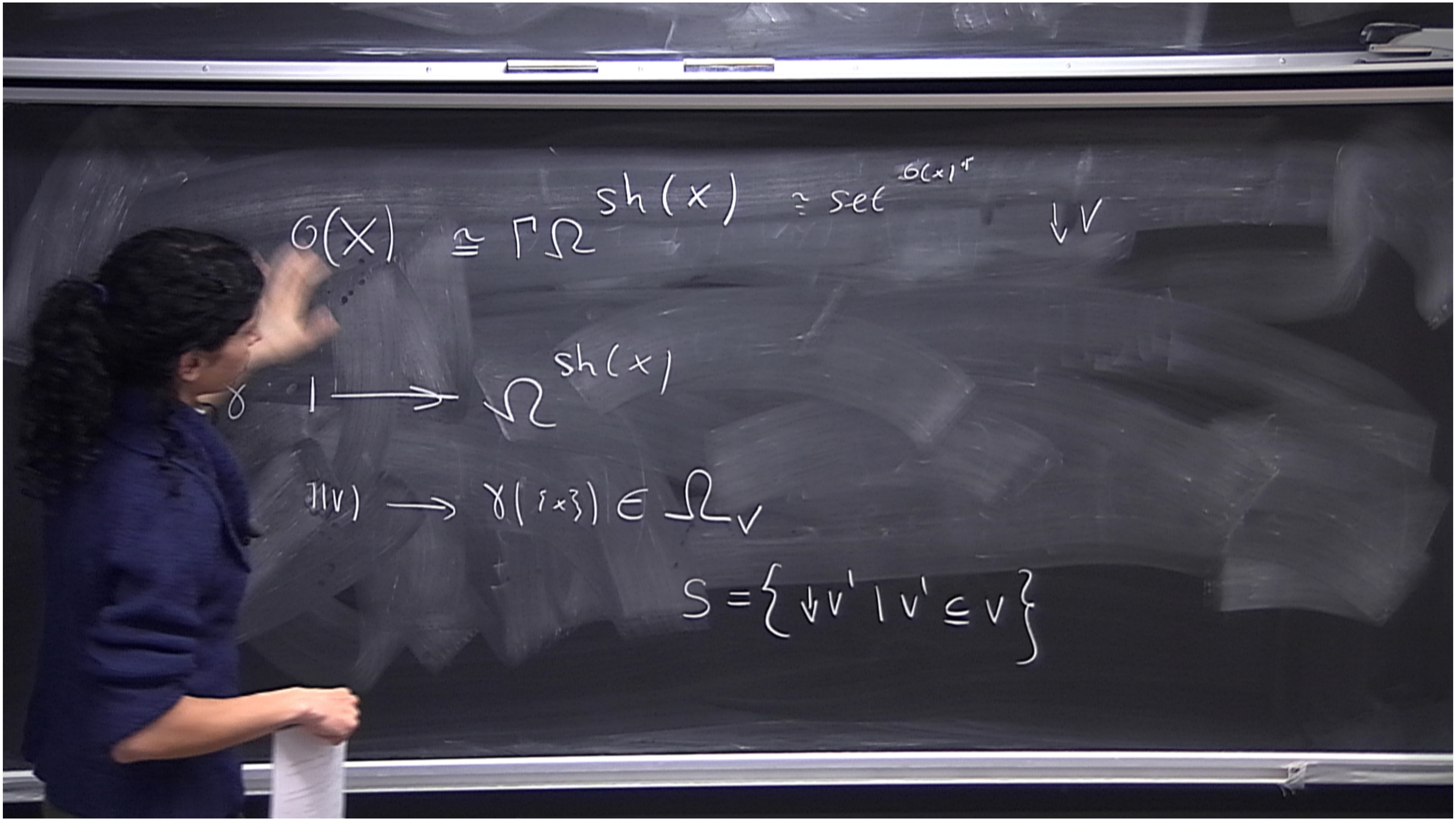
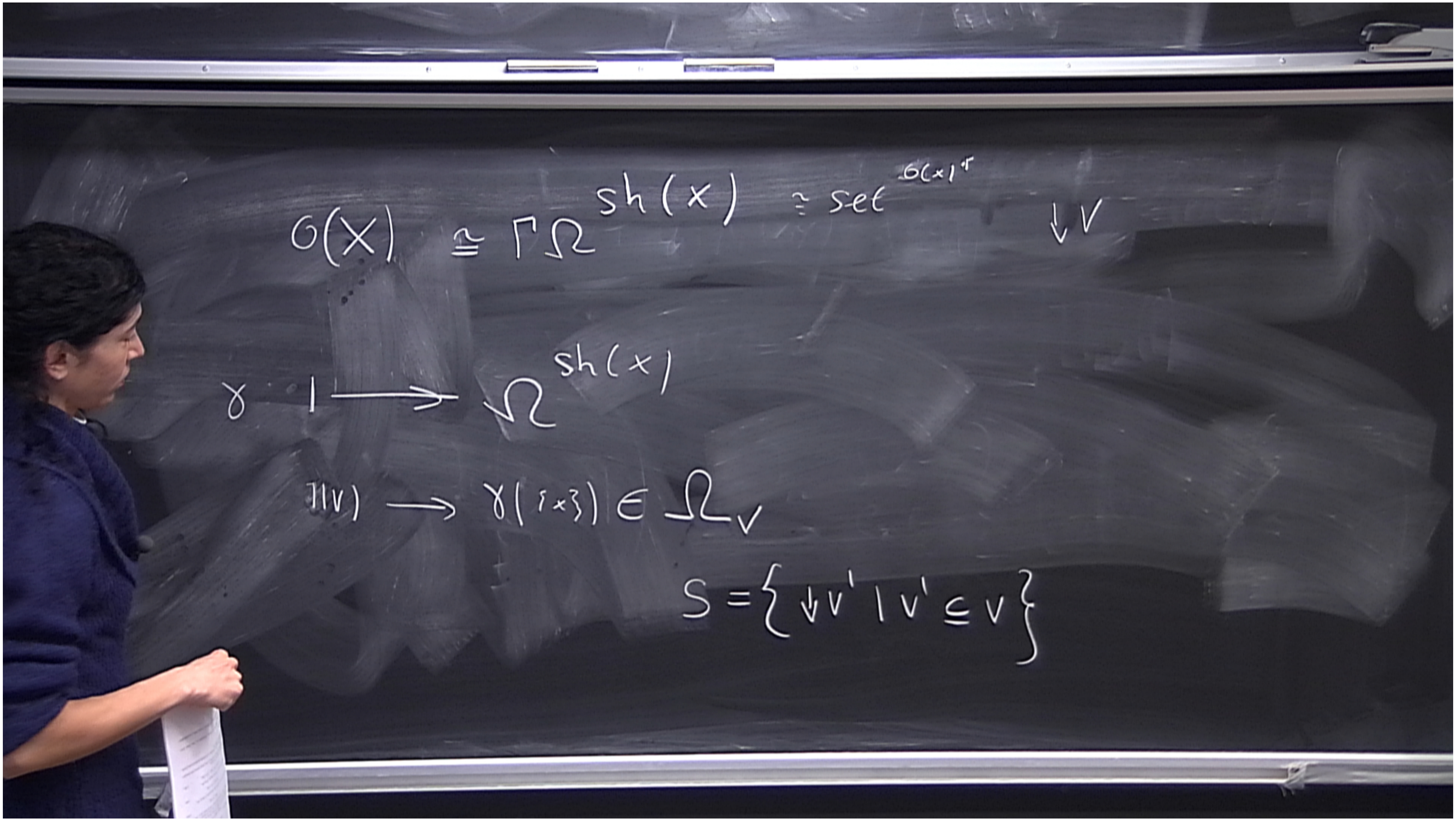
Abstract:

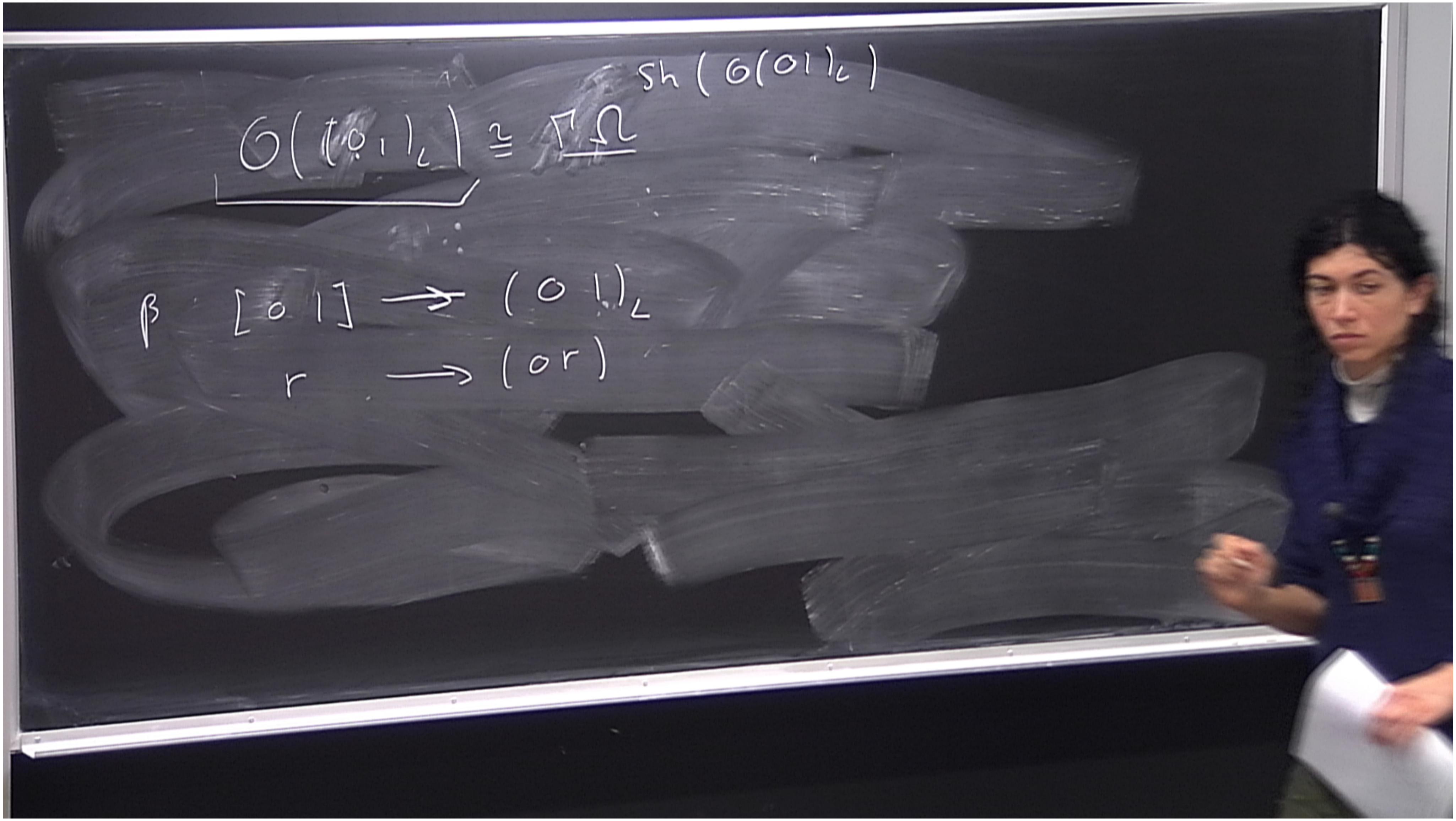

SONY

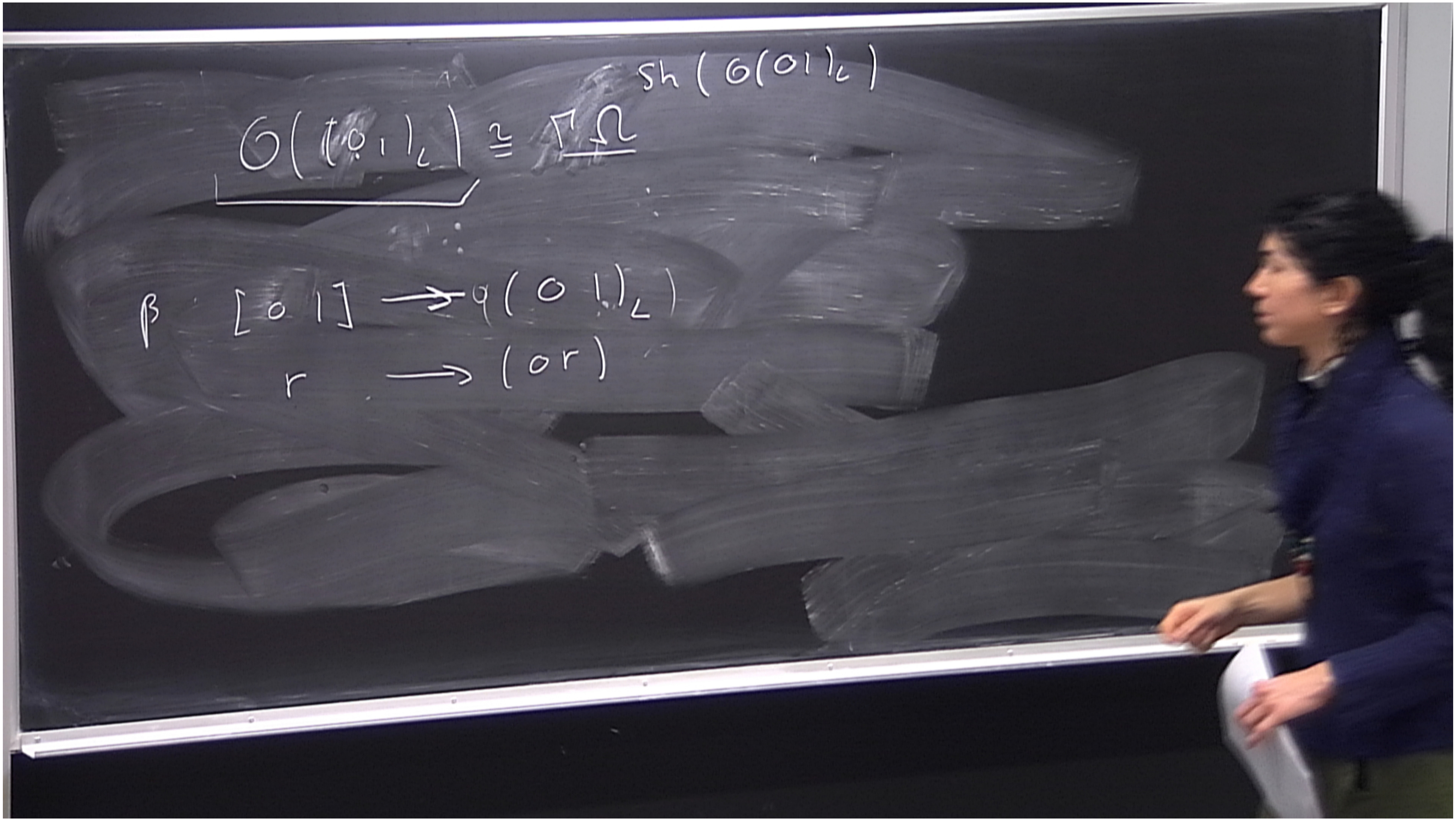

Probabilities
[0 1] ≅ Γ Ω
Define (0 1)


Probabilities
[0 1] ≅ ΓΩ
Define (0 1) ⇒ Top · (0 r)
0<r≤1
O((0 1)_L) ⇒ Alexandrof Top


G(X) ≅ ΓΩ^sh(X) ≅ Set^G(X)^op
↓V
γ : 1 → Ω^sh(X)
γ(V) → γ({x}) ∈ Ω_V
S = {↓V' | V' ⊆ V}


$\mathcal{O}(X) \cong \Gamma\Omega^{sh(x)} \cong set^{\mathcal{O}(x)^{op}}$
$\downarrow V$
$\gamma : 1 \longrightarrow \Omega^{sh(x)}$
$\gamma(V) \longrightarrow \gamma(\{x\}) \in \Omega_V$
$S = \{\downarrow V' \mid V' \subseteq V\}$


Sh (O(0 1)_L)
O((0 1)_L) ≅ Γ Ω
β [0 1] → (0 1)_L
r → (0 r)


Sh (O(0 1)_L)
O((0,1)_L) ≅ Γ Ω
β [0 1] → q(0 1)_L)
r → (or)

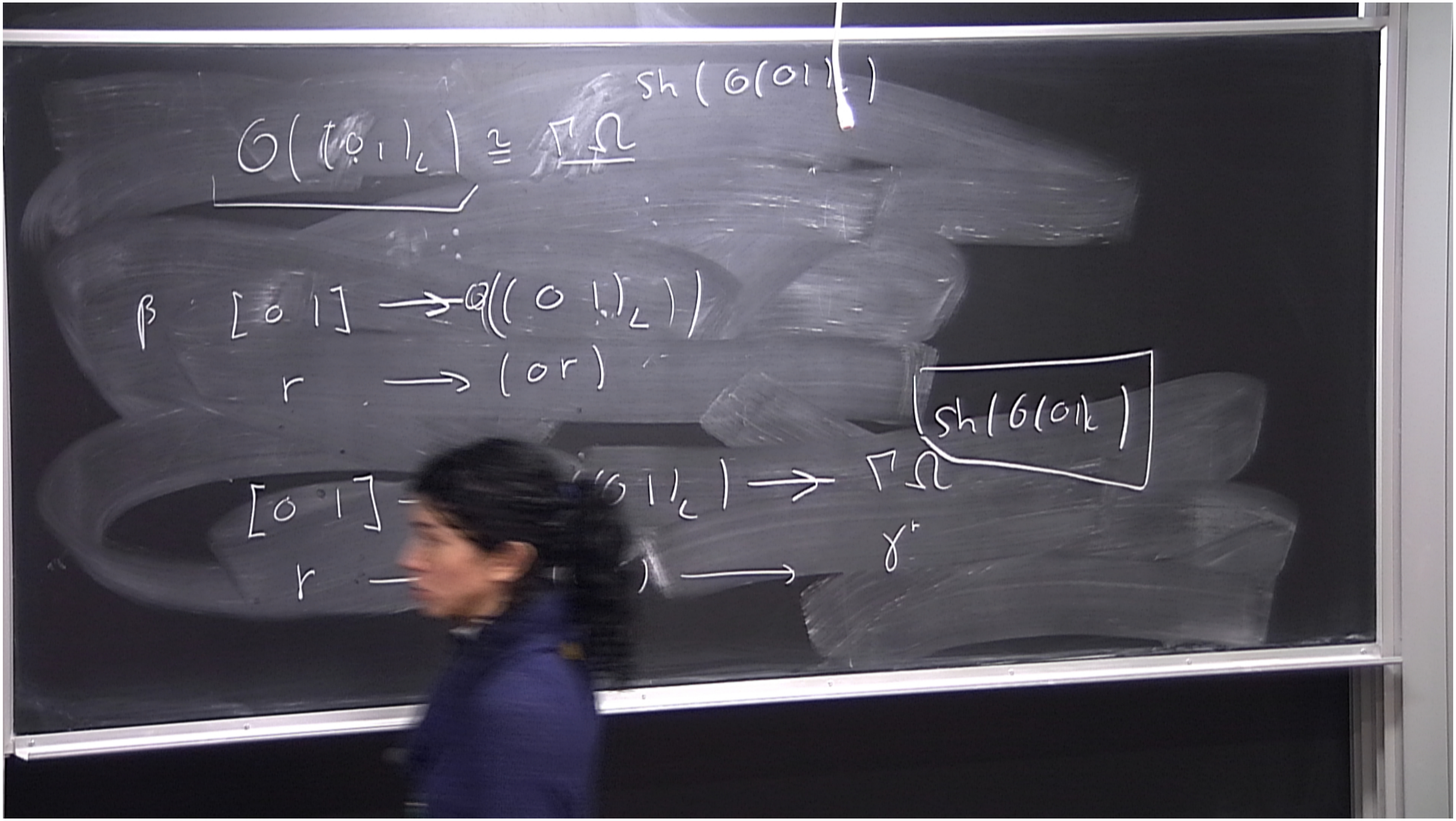
$$G((0,1)_L) \cong \Gamma\Omega^{sh(G(0,1)_L)}$$

$$\beta : [0,1] \longrightarrow G((0,1)_L)$$

$$r \longrightarrow (0,r)$$

$$[0,1] \longrightarrow G((0,1)_L) \longrightarrow \Gamma\Omega^{sh(G(0,1)_L)}$$

$$r \longrightarrow (0,r) \longrightarrow \gamma$$


Sh (G(01)_L)
G([0,1]_L) ≅ ΓΩ
β : [0 1] → G((0 1)_L))
r → (or)
Sh(G(01)_L)
[0 1]
ΓΩ
r
γ+

Classical Theory

$$\nu(A \in \Delta, s) = \begin{cases} 1 & \text{iff } s \in f_A^{-1}(\Delta) \\ 0 & \text{otherwise} \end{cases} \Rightarrow = \begin{cases} 1 & \text{iff } f_A^{-1}(\Delta) \in \Uparrow^s \\ 0 & \text{otherwise} \end{cases}$$

$$\Uparrow^s = \{ S \subseteq X \mid s \in S \}$$

$$\mu : \mathrm{sub}(X) \longrightarrow [0\ 1]$$

$$S \longrightarrow \mu(S)$$

$$\Pi_r^{\mu} = \{ S \subseteq X \mid \mu(S) \geq r \}$$

Classical Theory

$$\nu(A\in\Delta, s) = \begin{cases} 1 & \text{iff } \ s \in f_A^{-1}(\Delta) \\ 0 & \text{otherwise} \end{cases} \Rightarrow \ = \begin{cases} 1 & \text{iff } \ f_A^{-1}(\Delta) \in \mathbb{T}^s \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbb{T}^s = \{ S \subseteq X \mid s \in S \}$$

$$[\![f_A^{-1}(\Delta) \in \mathbb{T}^s]\!]$$

$$X \longrightarrow \underline{X} \in Sh(G(0,1)_c)$$

$$\underline{X}_{(0,r)} = X$$

$$\Delta : \mathrm{sub}(X) \longrightarrow \mathrm{sub}(\underline{X})$$

$$S \longrightarrow \underline{S} \quad \text{st} \quad \forall\ (0,r) \quad \underline{S}_{(0,r)} = S$$

$$\underline{\Pi}^{\mu}_{(0,r)} = \Pi^{\mu}_{r} = \left\{ \underline{S} \subseteq \underline{X} \mid \mu(S_{(0,r)}) \geq r \right\}$$

$$\underline{\Pi}^{\mu}_{(0,r)} = \Pi^{\mu}_{r} = \{ \underline{S} \subseteq \underline{X} \mid \mu(S_{(0,r)}) \geq r \}$$

$$[\![\underline{S} \in \underline{\Pi}^{\mu}]\!]_{(0,r)} = \{ (0,r') \leq (0,r) \mid \underline{S}_{0,r'} \in \Pi^{\mu}_{0,r'} \} =$$

$$= \{ \quad \cdots \quad \mid \mu(\underline{S}) \geq r \}$$

$$
\begin{tikzcd}
\mathrm{Sub}(X) \arrow[r, "\mu"] & {[0,1]} \arrow[d, "\ell"] \\
 & \Gamma\underline{\Omega}^{\mathrm{sh}(\mathcal{O}[0,1]_L)}
\end{tikzcd}
$$

$$\begin{tikzcd}
Sub(X) \arrow[r, "\mu"] \arrow[d, "\Delta"'] & {} \arrow[d, "\ell"] \\
Sub(\underline{X}) \arrow[r, "\mathcal{E}^{\mu}"'] & \Gamma\underline{\Omega}^{sh(\mathcal{O}(\Pi)_c)}
\end{tikzcd}$$

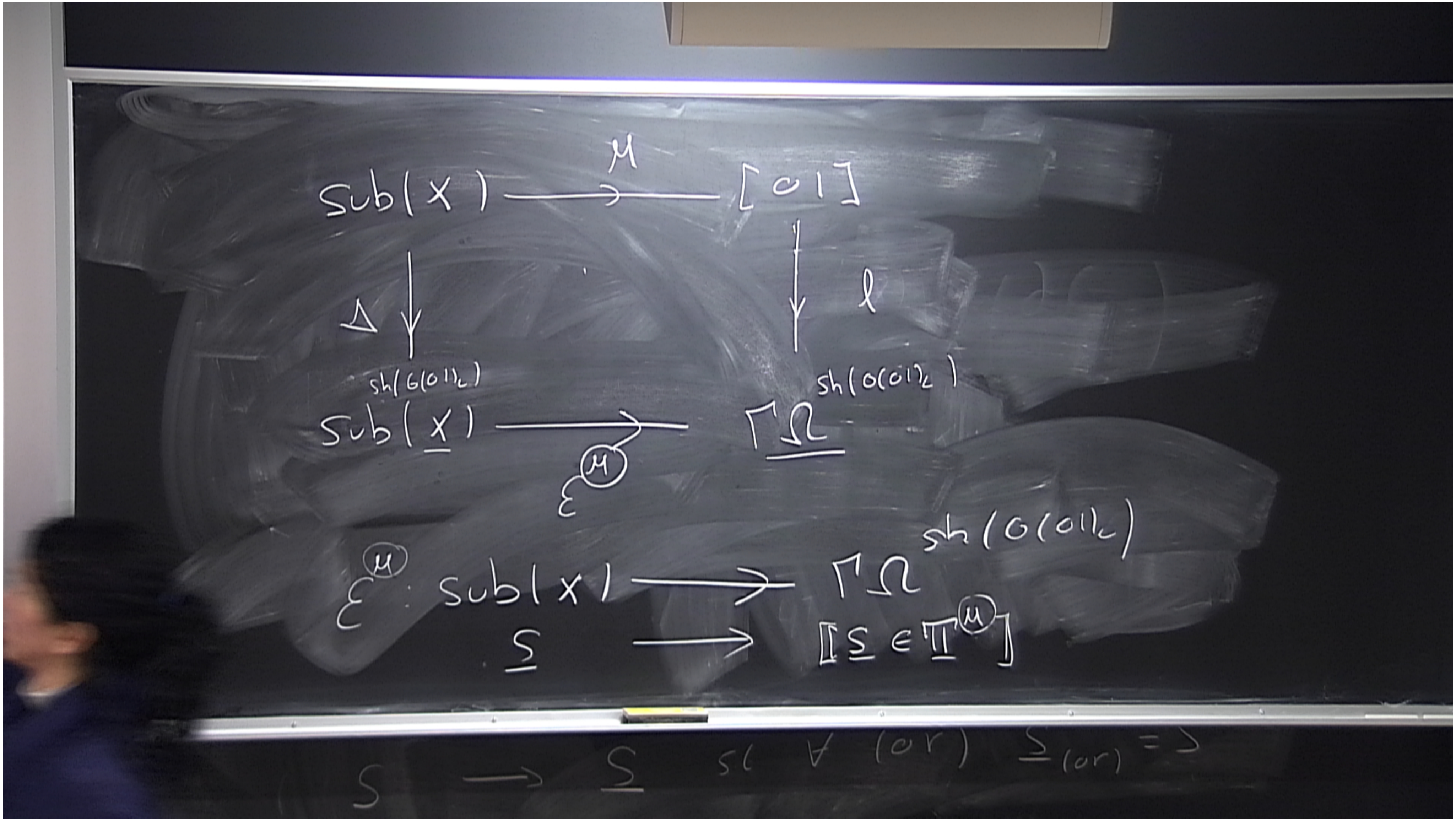
$$\mathcal{E}^{\mu} : Sub(X) \longrightarrow \Gamma\Omega^{sh(\mathcal{O}(\Pi)_c)}$$

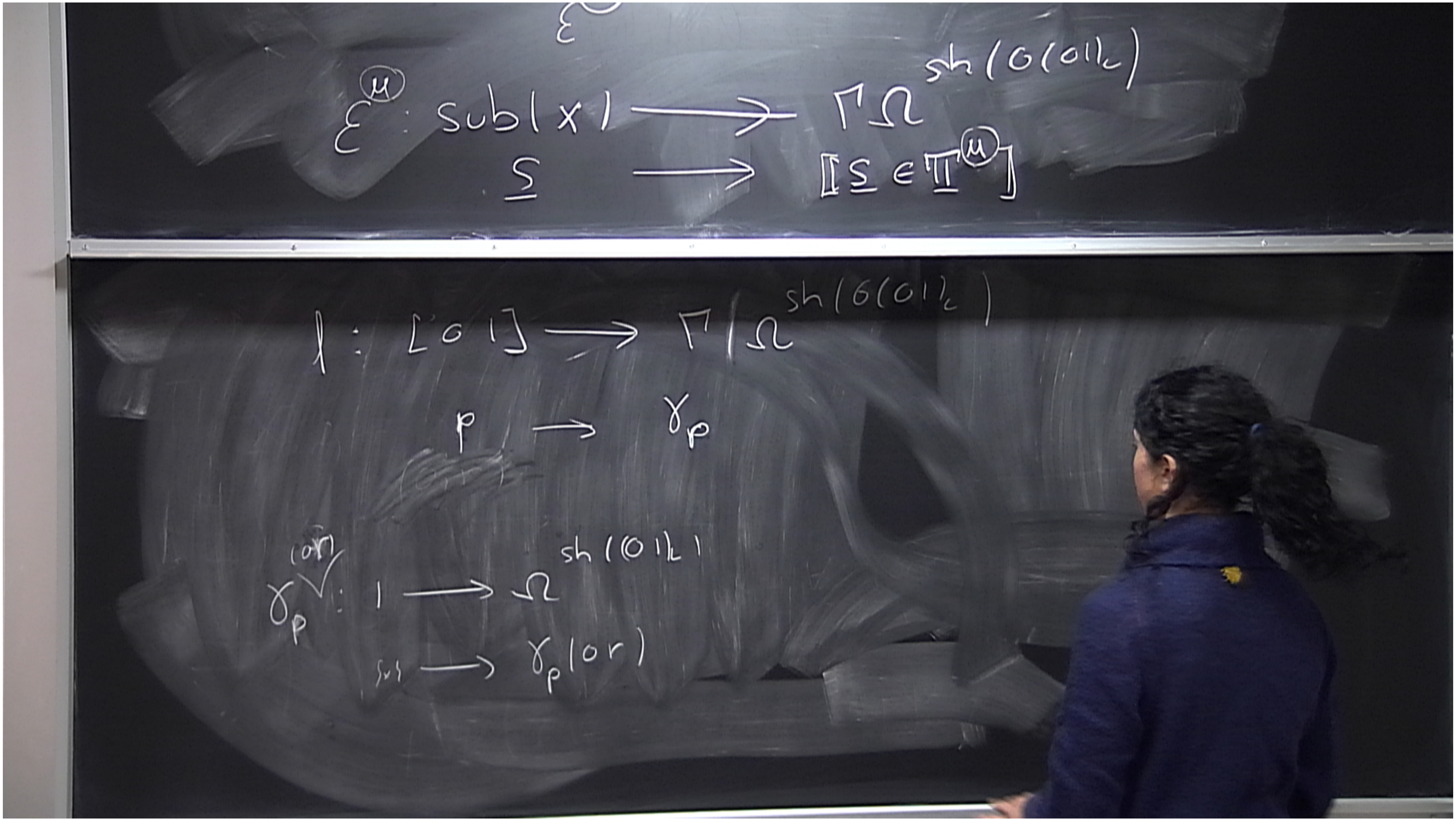
$$\underline{S} \longrightarrow [\![\underline{S} \in \underline{\Pi}^{\mu}]\!]$$

$$\underline{X}_{(o,r)} = X$$

$$\Delta : Sub(X) \longrightarrow Sub(\underline{X})$$

$$S \longrightarrow \underline{S} \quad st \quad \forall (or) \quad \underline{S}_{(or)} = S$$


μ
Sub(X) → [0 1]
Δ
l
sh(O[0 1]_L)
Sub(X) → ΓΩ
sh(O[0 1]_L)
ε^μ
ε^μ : Sub(X) → ΓΩ^sh(O[0 1]_L)
S → [[S ∈ Π^μ]]
S → S st ∀ (or) S_(or) =



$\mathcal{E}^{(\mu)} : \mathrm{sub}(X) \longrightarrow \Gamma\Omega^{sh(O(01)_c)}$
$\underline{S} \longrightarrow [\![\underline{S} \in \underline{\Pi}^{(\mu)}]\!]$
$l : [0\,1] \longrightarrow \Gamma|\Omega^{sh(O(01)_c)}$
$p \longrightarrow \gamma_p$
$\gamma_p^{(or)}\checkmark : 1 \longrightarrow \Omega^{sh(O1)_c}$
$\{*\} \longrightarrow \gamma_p(or)$

$$\mathcal{E}^{(\mu)} : \mathrm{Sub}(X) \longrightarrow \Gamma\Omega^{\mathrm{sh}(\mathcal{O}(0,1)_L)}$$

$$\underline{S} \longrightarrow [\![\underline{S} \in \underline{\Pi}^{(\mu)}]\!]$$

$$l : [0,1] \longrightarrow \Gamma \Omega^{\mathrm{sh}(\mathcal{O}(0,1)_L)}$$

$$p \longrightarrow \gamma_p$$

$$p \geq r$$

$$\gamma_p(0r) = \{ (0,r') \in \mathcal{O}(0,1)_L \mid p \geq r' \}$$

$$\gamma_p^{(0r)} : 1 \longrightarrow \Omega^{\mathrm{sh}(\mathcal{O}(0,1)_L)}$$

$$\{*\} \longrightarrow \gamma_p(0r)$$

$$\mathcal{E}^{(\mu)} : \mathrm{Sub}(X) \longrightarrow \Gamma\Omega^{sh(\mathcal{O}(0,1)_L)}$$

$$\underline{S} \longrightarrow [\![\underline{S} \in \underline{\Pi}^{(\mu)}]\!]$$

$$l : [0,1] \longrightarrow \Gamma\Omega^{sh(\mathcal{O}(0,1)_L)}$$

$$p \longrightarrow \gamma_p$$

$$\gamma_p^{(0,r)} : 1 \longrightarrow \Omega^{sh(\mathcal{O}(0,1)_L)}$$

$$\{*\} \longrightarrow \gamma_p(0,r)$$

$$\gamma_p(0,r) = \begin{cases} \{(0,r') \in \mathcal{O}(0,1)_L \mid r \geq r'\} = \Omega_{(0,r)} & p \geq r \\ \{(0,r') \in \mathcal{O}(0,1)_L \mid r' \leq p\} & 0<p<r \\ \emptyset & p=0 \end{cases}$$

$$(0,r) \longrightarrow$$

$$(l \circ \mu) S = l(\mu(S)) = l_{\mu(S)}$$

$$l_{(\mu(S))}(0,r) = \begin{cases} \{(0,r') \le (0,r) \mid r' \le r\} & \mu(S) \ge r \\ \{(0,r') \le (0,r) \mid r' \le \mu(S)\} & 0 < \mu(S) < r \\ \emptyset & \mu(S) = 0 \end{cases}$$

$$\mathcal{E}^{(\mu)} : \mathrm{Sub}(X) \longrightarrow \Gamma\Omega$$
$$\underline{S} \longrightarrow [\![\underline{S} \in \underline{\Pi}^{(\mu)}]\!]$$

$$\ell(\bigvee_i P_i)_{(0,r)} = \begin{cases} [0 \; r] & \bigvee_i P_i \geq r \\ [0 \; \bigvee_i P_i] & 0 < \bigvee_i P_i < r \\ \phi & \end{cases}$$

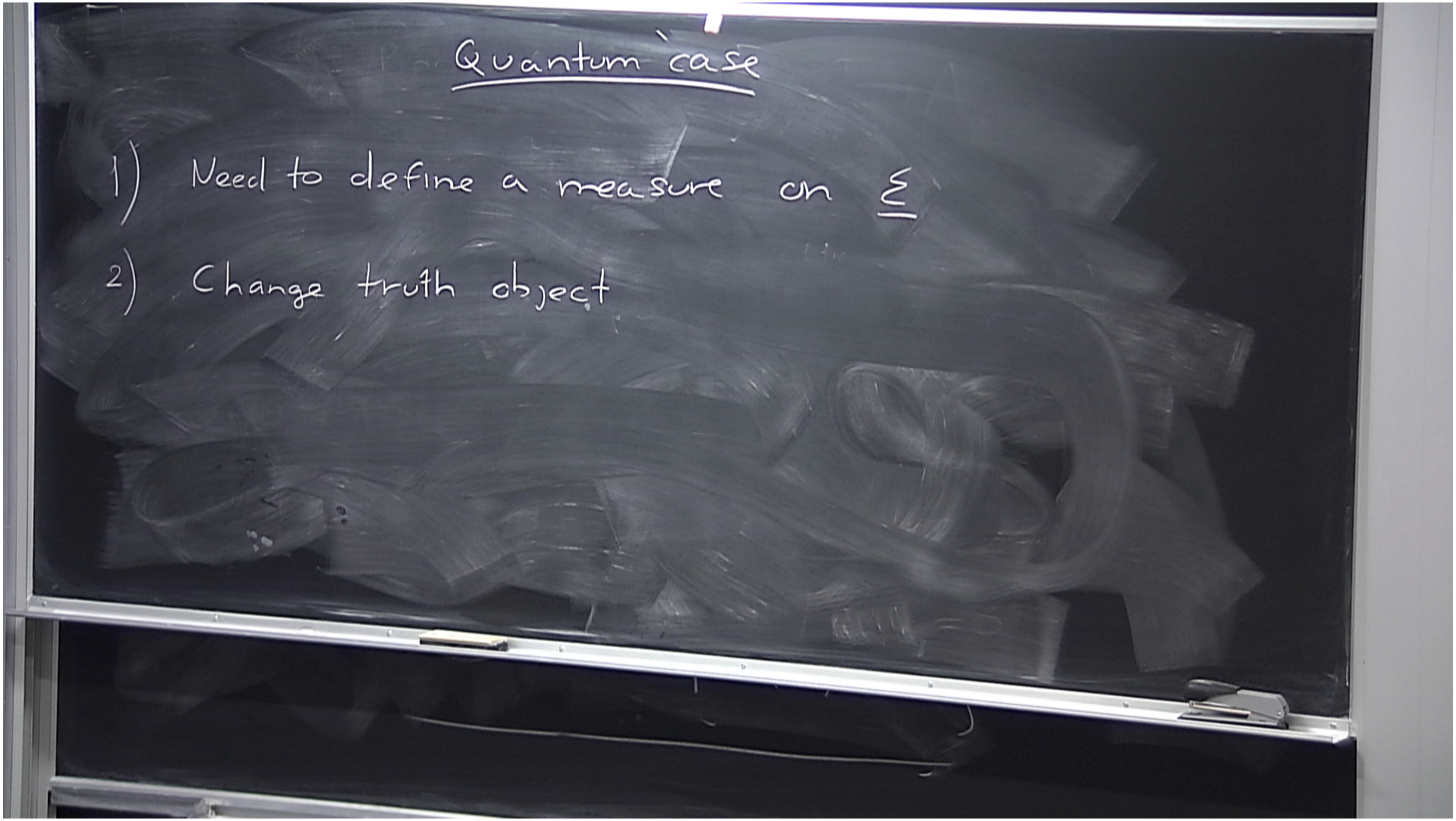
sup P

$$\bigvee_i \ell(P_i)_{(0,r)} = \bigvee_i \begin{cases} [0 \; r] & P_i \geq r \\ [0 \; P_i] & 0 \leq P_i < r \\ \phi & \end{cases}$$

$$(G, r) \longrightarrow g$$

$$\left(\varepsilon^{\mu}_{i} \circ \Delta\right) V_i(\xi_i) =$$

$$(\mathcal{E}^{\mu} \circ \Delta) \bigvee_i (\xi_i) = (l \circ \mu) \bigvee_i \xi_i = \bigvee_i \left((l \circ \mu)(\xi_i) \right)$$

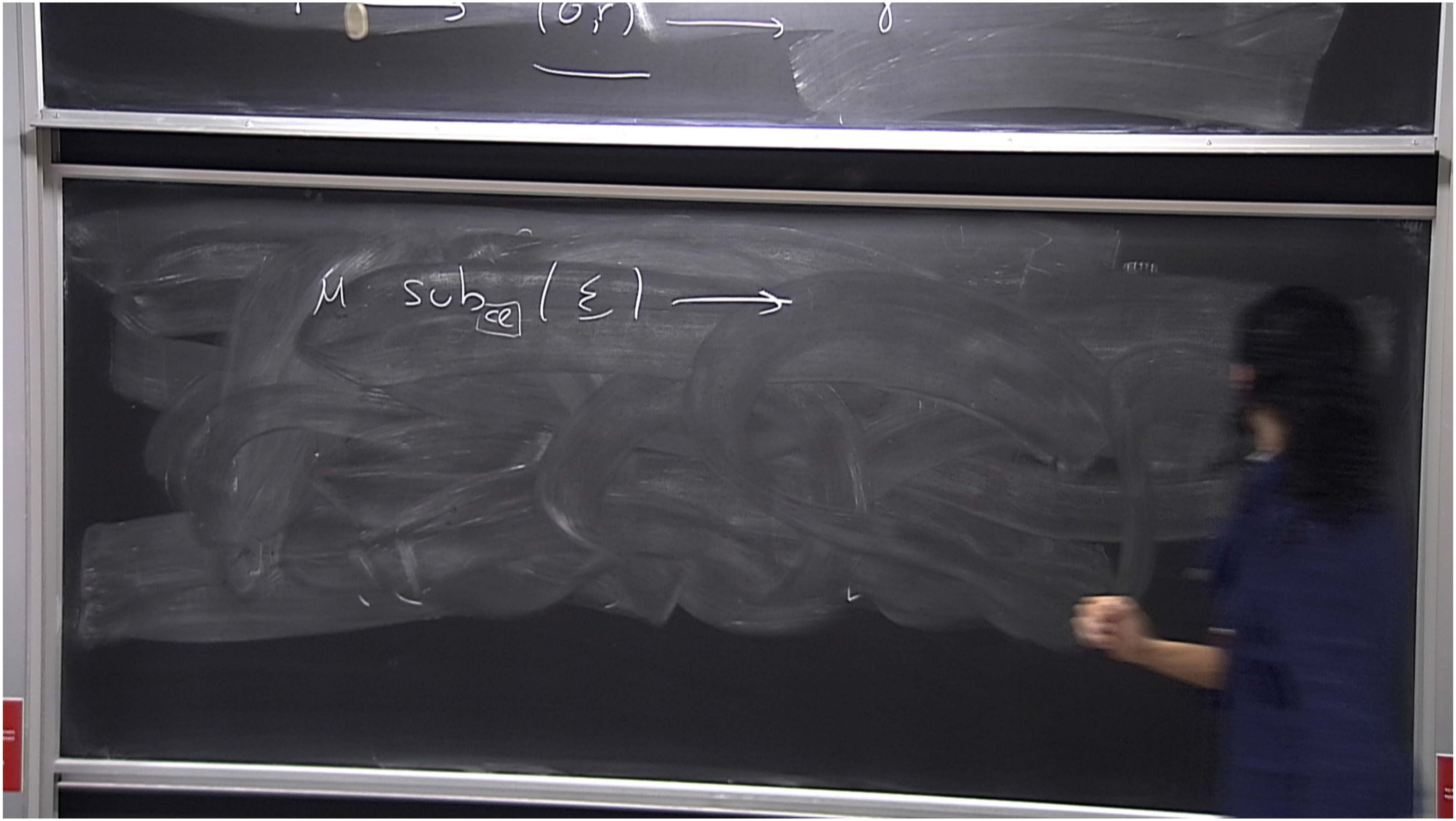

Quantum case
1) Need to define a measure on $\underline{\Sigma}$
2) Change truth object

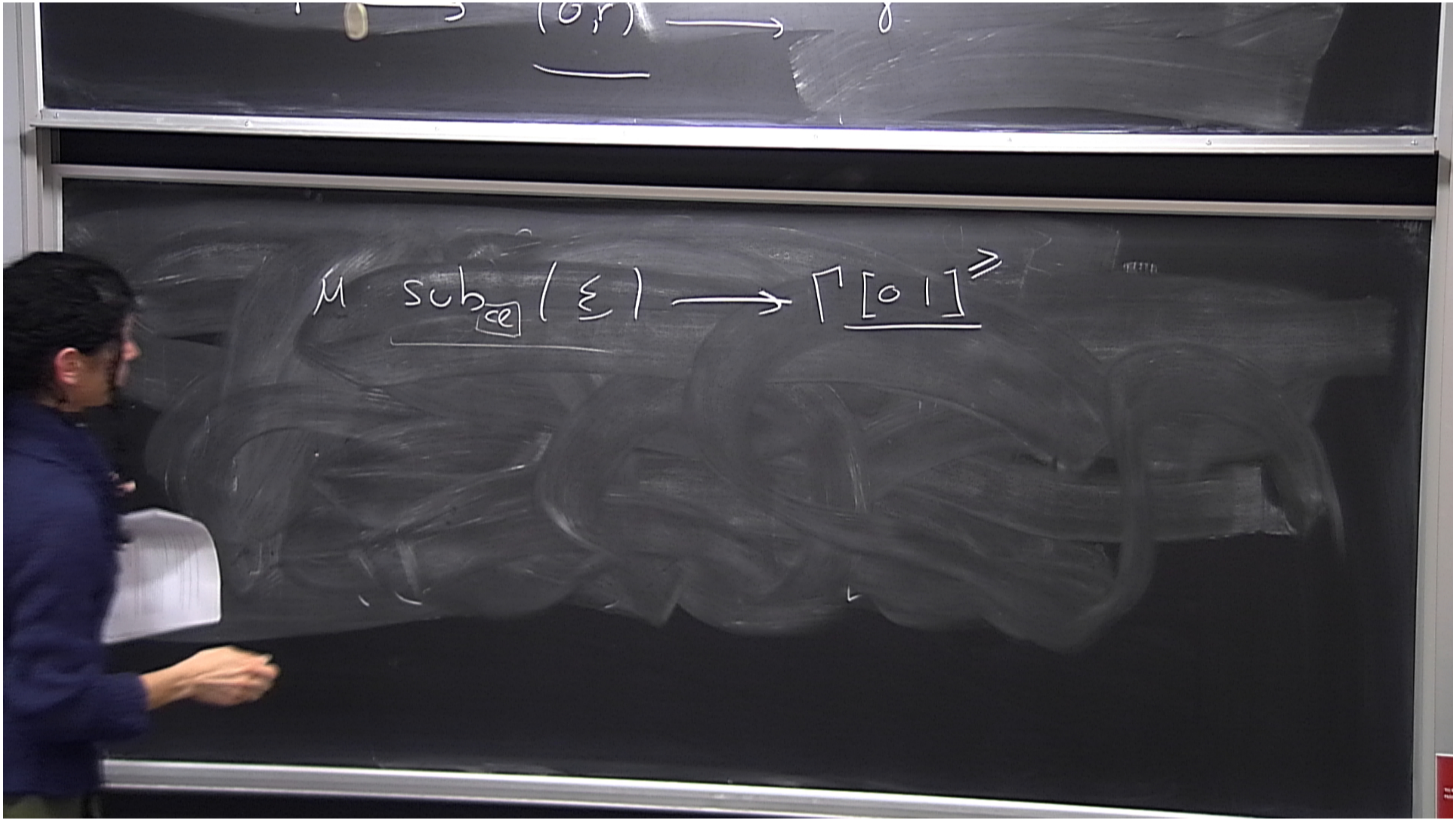
Quantum case

1) Need to define a measure on $\underline{\Sigma}$

2) Change truth object $\Rightarrow$ "diff" Truth values

3) combine 1, 2 so as to get q analogue of commuting diagram


M: sub_ce(ε) →


M sub ce (ε) → Γ[0 1]

Def

$$[0,1]^{\geq} : \mathcal{V}(H) \longrightarrow \mathbf{Sets}$$

$$V \longmapsto \{ f : \downarrow V \to [0,1] \mid f \text{ order reversing} \}$$

$$V' \subseteq V \longrightarrow \quad f \longmapsto f_{|V'}$$

Def

$$\underline{[0,1]^{\geq}} : \mathcal{V}(H) \longrightarrow \mathrm{Sets}$$

$$V \longmapsto \{ f : \downarrow V \longrightarrow [0,1] \mid f \text{ order reversing} \}$$

$$V' \subset V \longrightarrow f \longmapsto f_{|V'} : \downarrow V' \longrightarrow [0,1]$$

$(\sigma, r) \longrightarrow$

$$\mu : \mathrm{Sub}_{\boxed{cl}}(\Sigma) \longrightarrow \Gamma\, \underline{[0\,1]}^{\succeq} \qquad \mathrm{sh}(\mathcal{V}(H)))$$

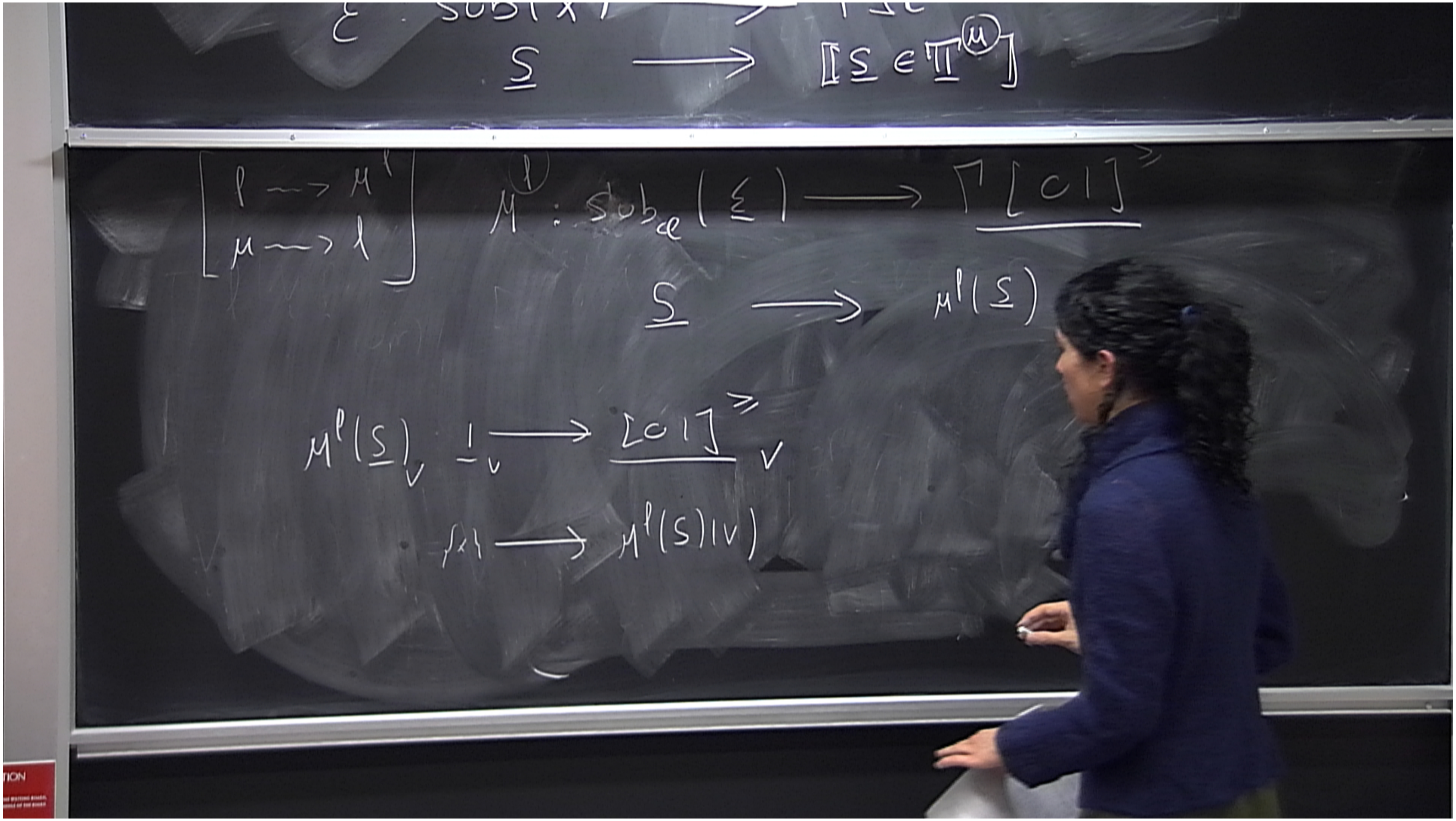
<u>Def</u>

$$\underline{[0\,1]}^{\succeq} \subseteq \underline{\mathbb{R}}^{\leftrightarrow}$$

$$\underline{[0\,1]}^{\succeq} : \mathcal{V}(H) \longrightarrow \mathrm{Sets}$$

$$V \longmapsto \{ f : \downarrow V \to [0\,1] \mid f \text{ order reversing} \}$$

$$V' \subseteq V \longrightarrow \quad f \longmapsto f_{V'} : \downarrow V' \longrightarrow [0\,1]$$


S ⟶ [S ∈ Π(μ)]
l ⇝ μ^l
μ ⇝ l
μ^l : Sub_cl(Σ) ⟶ Γ[0 1]^≥
S ⟶ μ^l(S)
μ^l(S)_v ⟶ [0 1]^≥_v
{x} ⟶ μ^l(S)(v)

$$\varepsilon : \mathrm{Sub}(\underline{\Sigma}) \longrightarrow \Gamma \underline{\Omega}$$

$$\underline{S} \longrightarrow [\![\underline{S} \in \underline{\Pi}^{(\mu)}]\!]$$

$$\begin{bmatrix} \rho \rightsquigarrow \mu^{\rho} \\ \mu \rightsquigarrow \rho \end{bmatrix}$$

$$\mu^{(\rho)} : \mathrm{Sub}_{cl}(\underline{\Sigma}) \longrightarrow \Gamma \underline{[0\,1]^{\geq}}$$

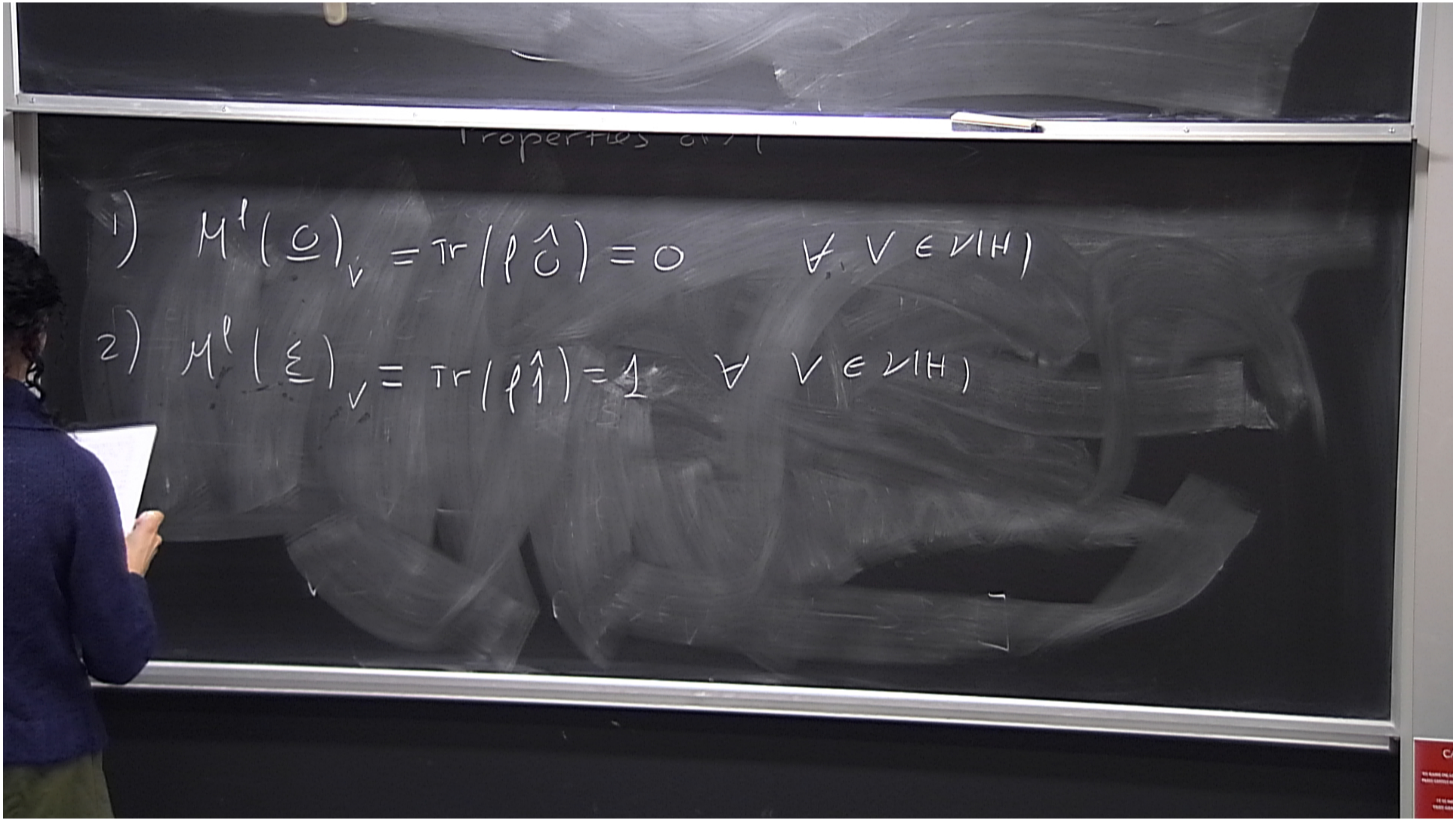
$$\underline{S} \longrightarrow \mu^{\rho}(\underline{S})$$

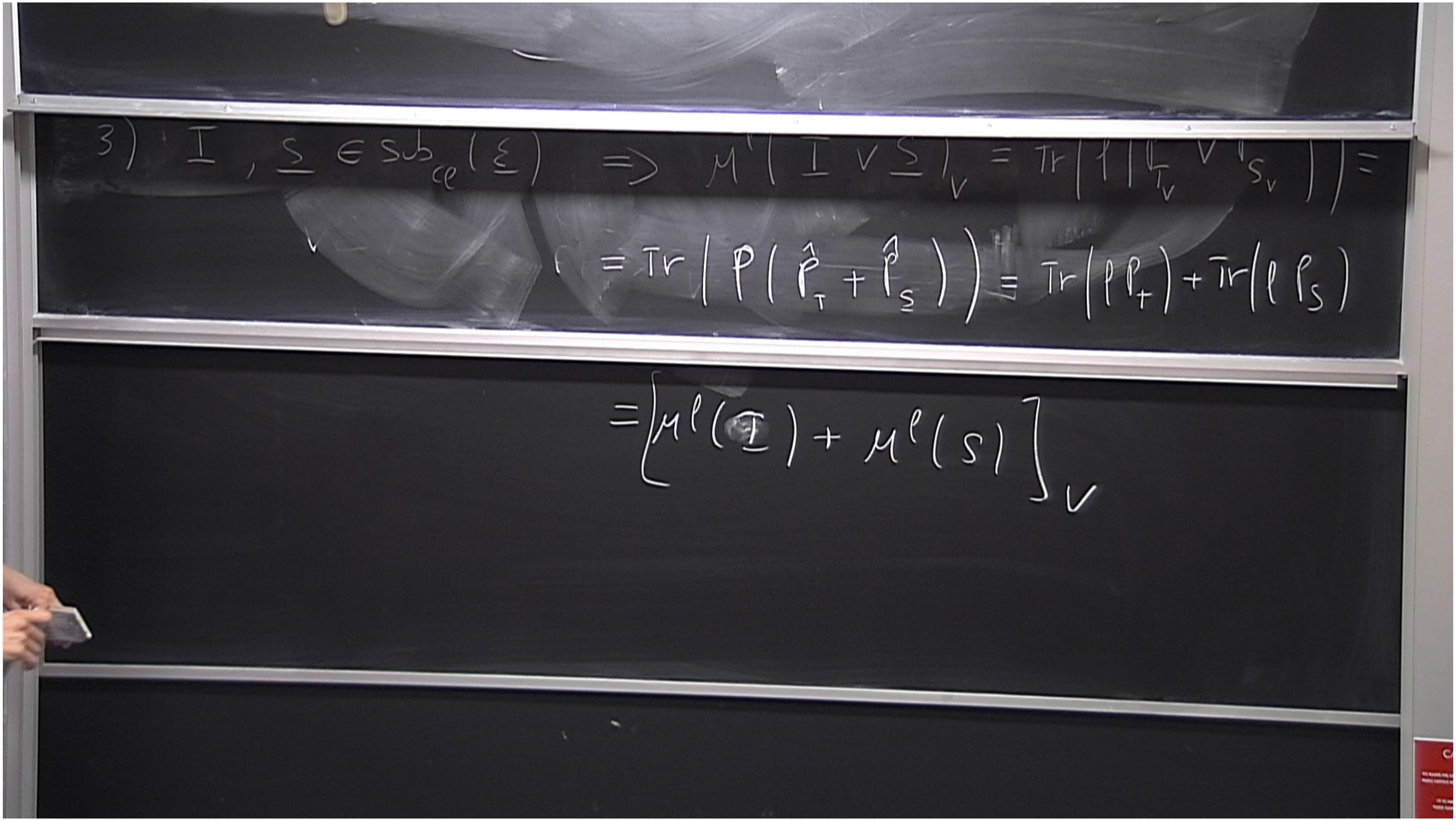
$$\mu^{\rho}(\underline{S})_V : \underline{1}_V \longrightarrow \underline{[0\,1]}^{\geq}_V$$

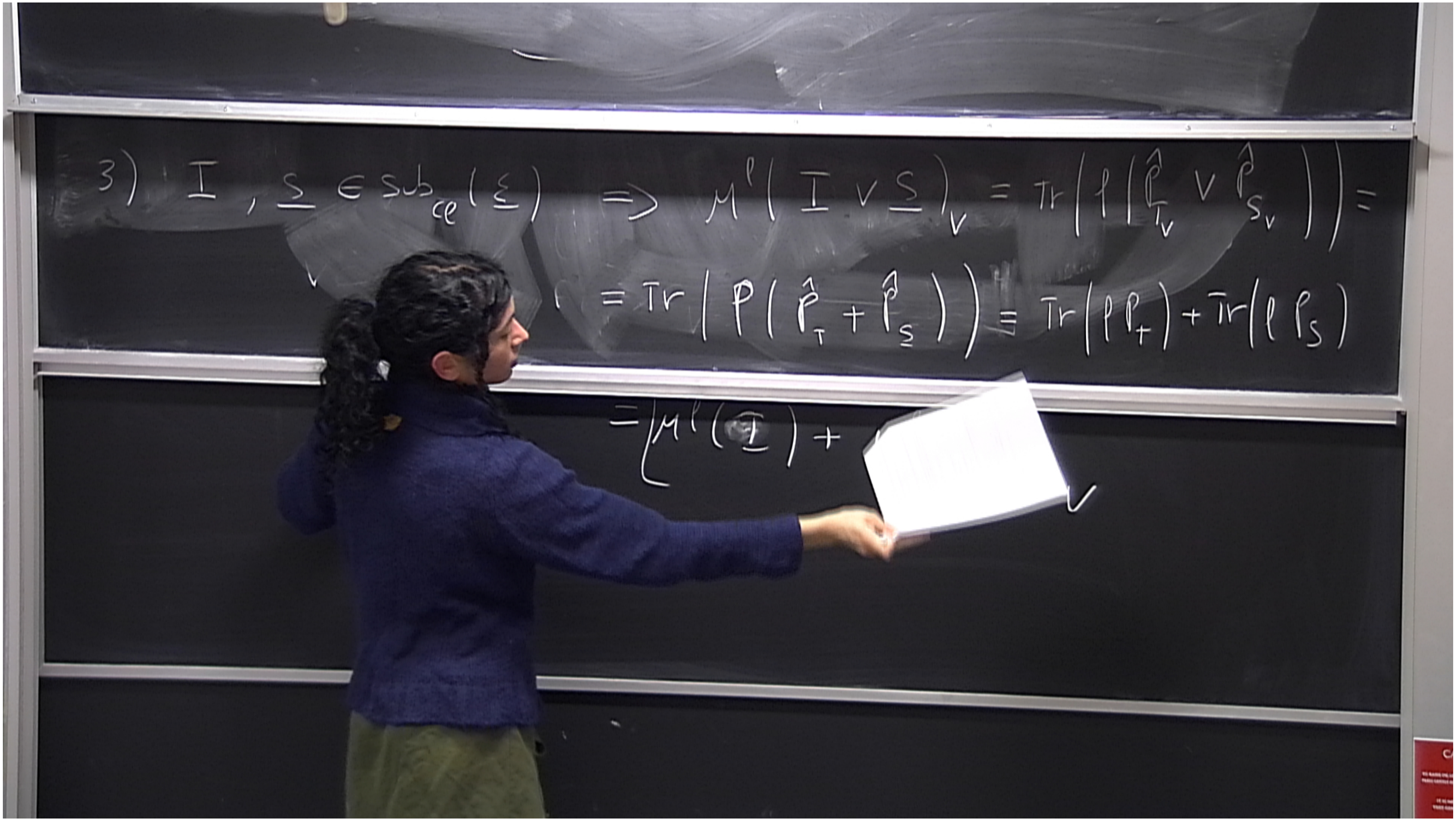
$$\{*\} \longrightarrow \mu^{\rho}(\underline{S})(V) := \mu^{\rho}(\underline{S}_V) = \mathrm{Tr}\left(\rho\, \hat{P}_{\underline{S}_V}\right)$$

$$P(V) \longrightarrow \mathrm{Sub}(\Sigma_V)$$

$$\hat{P} \longmapsto S_P = \{\lambda \in \Sigma_V \mid \lambda(\hat{P}) = 1\}$$


Properties of
1) $\mu^{\rho}(\underline{0})_V = \mathrm{Tr}(\rho \hat{0}) = 0 \quad \forall\, V \in \mathcal{V}(H)$
2) $\mu^{\rho}(\underline{\Sigma})_V = \mathrm{Tr}(\rho \hat{1}) = 1 \quad \forall\, V \in \mathcal{V}(H)$


3) $\underline{I}, \underline{S} \in Sub_{cl}(\underline{\Sigma}) \Rightarrow \mu^{\rho}(\underline{I} \vee \underline{S})_V = Tr(\rho(P_{I_V} \vee P_{S_V})) =$
$= Tr(\rho(\hat{P}_I + \hat{P}_S)) = Tr(\rho P_I) + Tr(\rho P_S)$
$= [\mu^{\rho}(\underline{I}) + \mu^{\rho}(S)]_V$



3) $\underline{I}, \underline{S} \in Sub_{cl}(\underline{\Sigma}) \Rightarrow \mu^{\rho}(\underline{I} \vee \underline{S})_V = Tr(\rho(\hat{P}_{I_V} \vee \hat{P}_{S_V})) =$
$= Tr(\rho(\hat{P}_I + \hat{P}_S)) = Tr(\rho P_I) + Tr(\rho P_S)$
$= \mu^{\rho}(\underline{I}) +$

4) $\mu^{\ell}(\underline{S} \vee \underline{T}) + \mu^{\ell}(S \wedge T) = \mu_{\ell}(\underline{S}) + \mu_{\ell}(\underline{T})$

5)

$$p_S + p_T - p_S p_T \qquad\qquad p\,p$$

$$4)\quad \mu^\rho(\underline{S} \vee \underline{T}) + \mu^\rho(\underline{S} \wedge \underline{T}) = \mu_\rho(\underline{S}) + \mu_\rho(\underline{T})$$

$$5)\quad sub_{ce}(\Sigma) \Longrightarrow \mu^\rho(\underline{S} \vee \neg S) \leq \mu^\rho(\underline{\Sigma})$$

$$\mu^\rho_{S}+\mu^\rho_{T}-\mu^\rho_{S}\mu^\rho_{T} \qquad \rho\ \rho$$

4) $\mu^\rho(\underline{S}\vee\underline{T})+\mu^\rho(\underline{S}\wedge\underline{T})=\mu_\rho(\underline{S})+\mu_\rho(\underline{T})$

5) $\mathrm{sub}_{ce}(\Sigma)\Longrightarrow\mu^\rho(\underline{S}\vee\neg S)\leq\mu^\rho(\underline{\Sigma})$

6) $\mu^\rho(\bigvee_i S_i)|v)=\mathrm{Tr}\left(\rho\left(\bigvee_i P_{S_{iv}}\right)\right)=\mathrm{Tr}\left(\sum_i \hat{P}_{S_{iv}}\right)=\sum\mathrm{Tr}(\quad)$

$$=\sum_i\mu^\rho(S_i)|v)$$

Def

$$M : \mathrm{sub}_{cl}(\mathcal{E}) \to \underline{\Gamma[0,1]^{\geq}}$$

$$\underline{S} \to M(S) \quad \text{s.t.} \ \forall\, V \in \mathcal{V}(H)$$

$$M(S)(V) \in \underline{[0,1]^{\geq}_{V}}$$

1) $\mu(\underline{\Sigma}) = 1$

2) $\forall\ \underline{S}, \underline{T} \in \mathrm{sub}_{cl}(\underline{\Sigma})$

$$\mu(\underline{S} \vee \underline{T}) + \mu(\underline{S} \wedge \underline{T}) = \mu(\underline{S}) + \mu(\underline{T})$$

$$\mu(\underline{S} \vee \underline{T}) + \mu(\underline{S} \wedge \underline{T}) = \mu(\underline{S}) + \mu(\underline{T})$$

$$m : P(H) \longrightarrow [0,1] \qquad m(\hat{P}) = \mu(S_{P})|v|$$

$$P \in V$$

4) $\mu^{\rho}(\underline{S} \vee \underline{T}) + \mu^{\rho}(\underline{S} \wedge \underline{T}) = \mu_{\rho}(\underline{S}) + \mu_{\rho}(\underline{T})$

5) $\mathrm{sub}_{cl}(\Sigma) \Longrightarrow \mu^{\rho}(\underline{S} \vee \neg S) \leq \mu^{\rho}(\underline{\Sigma})$

6) $\mu^{\rho}(\vee_i S_i)|v) = \mathrm{Tr}(\rho(\vee_i P_{S_{iv}})) = \mathrm{Tr}(\sum_i \hat{P}_{S_{iv}}) = \sum \mathrm{Tr}(\quad)$

$$= \sum_i \mu^{\rho}(S_i) \ldots$$

$m : P(H) \longrightarrow [0,1]$ $\quad m(\hat{P}) = \mu(S_P)|V|$ $\quad P \in V$

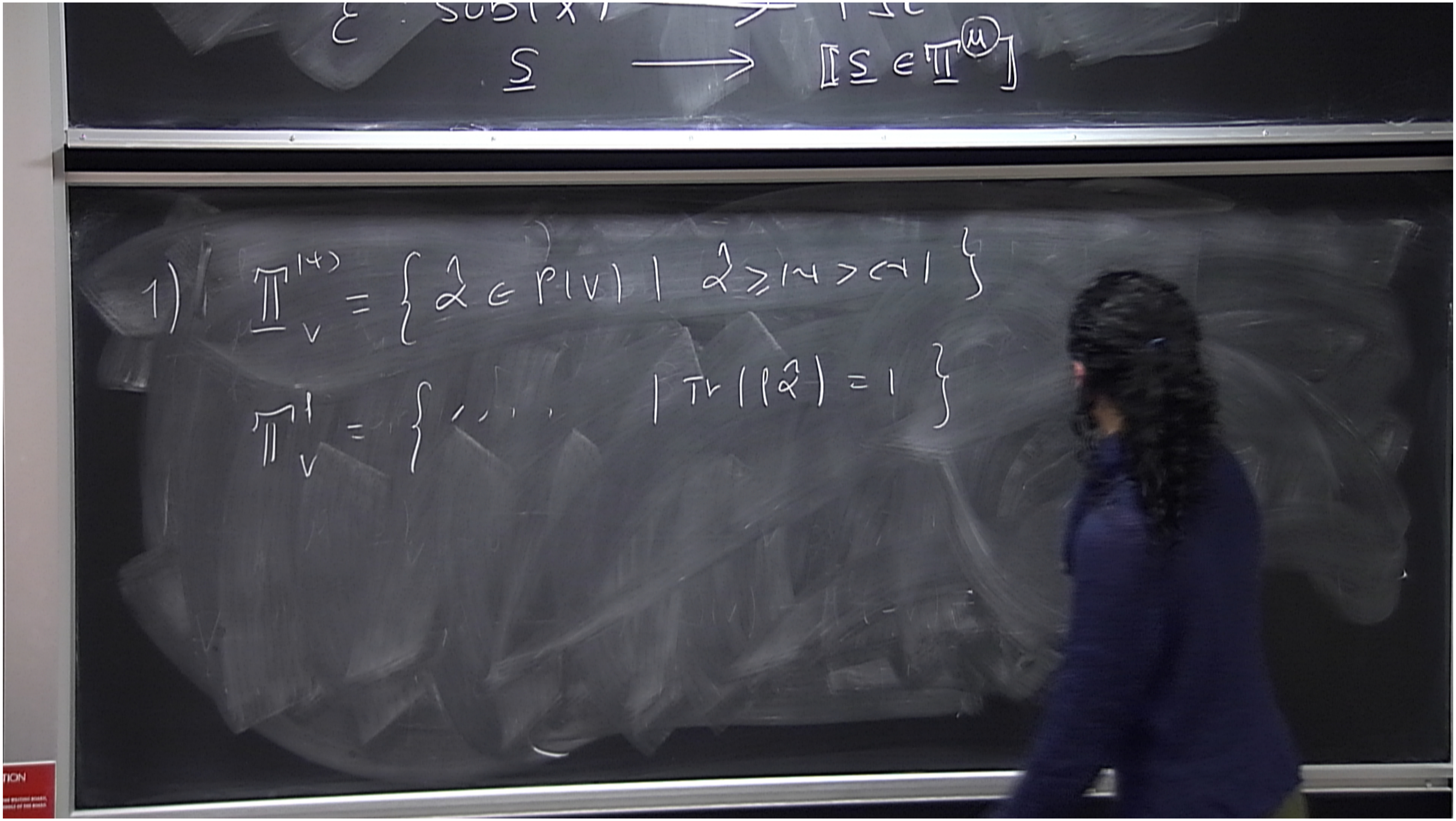
$P_S + P_T - P_S P_T \qquad\qquad P P$

4) $\mu^\rho(\underline{S} \vee \underline{T}) + \mu^\rho(\underline{S} \wedge \underline{T}) = \mu_\rho(\underline{S}) + \mu_\rho(\underline{T})$

5) $\text{sub}_{ce}(\Sigma) \Longrightarrow \mu^\rho(\underline{S} \vee \neg S) \leq \mu^\rho(\underline{\Sigma})$ ✓

6) $\mu^\rho(\bigvee_i S_i)|V| = \mathrm{Tr}\left(\rho\left(\bigvee_i P_{S_{iV}}\right)\right) = \mathrm{Tr}\left(\sum_i \hat{P}_{S_{iV}}\right) = \sum \mathrm{Tr}(\quad)$

$= \sum_i \mu^\rho(S_i)|V|$


S ⟶ [[S ∈ Π^(ψ)]]
1) Π_V^(ψ) = { α ∈ P(V) | α ≥ |ψ⟩⟨ψ| }
Π_V^1 = { , , , | Tr(ρα) = 1 }

$$\varepsilon : \mathrm{SUB}(X) \longrightarrow \ldots$$

$$\underline{S} \longrightarrow [\![\underline{S} \in \underline{\Pi}^{(u)}]\!]$$

1) $\underline{\Pi}^{(+)}_{V} = \{ \underline{\alpha \in P(V)} \mid \alpha \geq |\psi\rangle\langle\psi| \}$ $\subseteq$ $\underline{O}$

$\Pi^{1}_{V} = \{ \underline{\ldots} \mid \mathrm{Tr}(\rho\alpha) = 1 \}$

$\delta(\rho)$

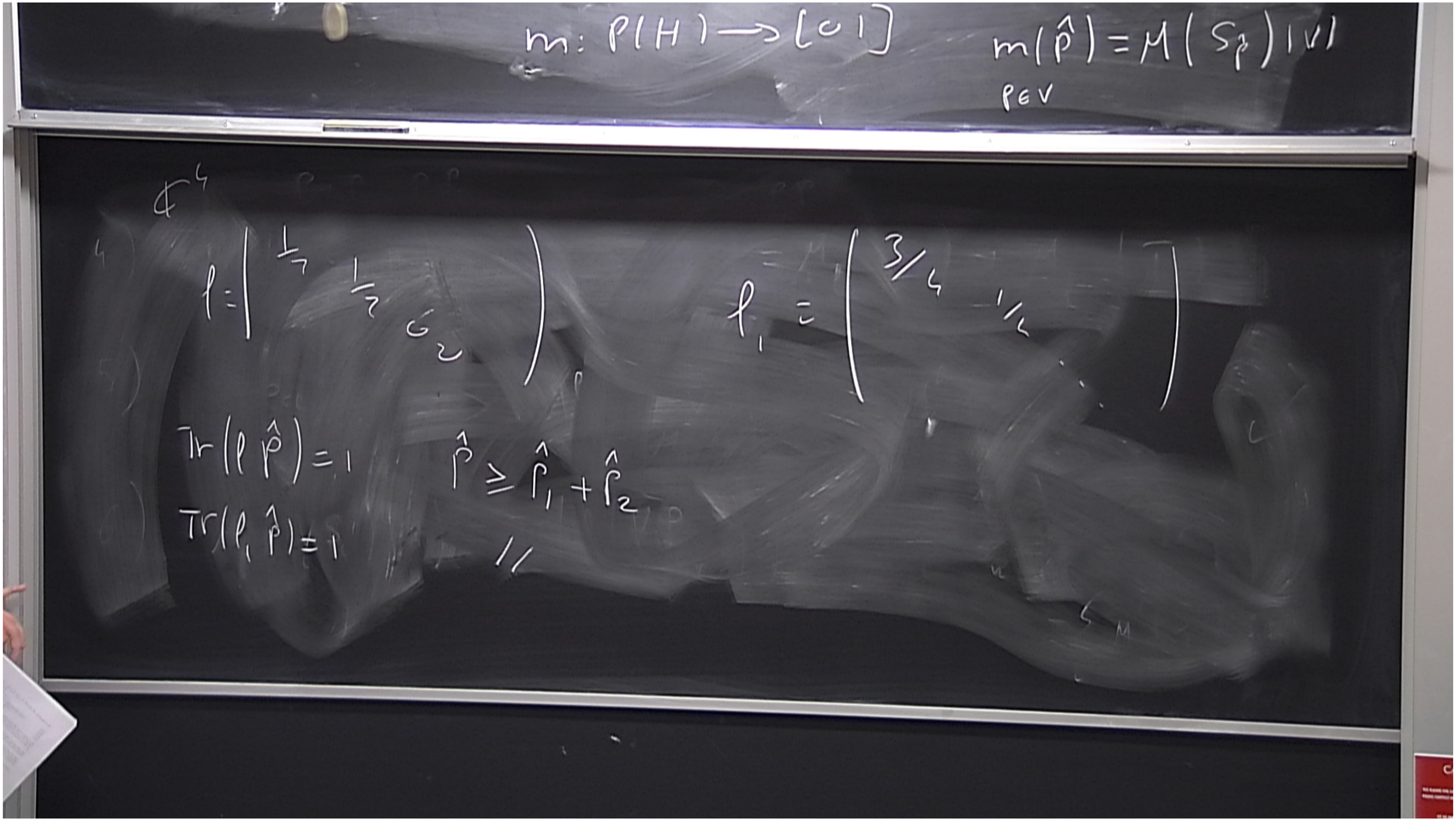
$$\varepsilon : \mathrm{SUB}(\ldots) \longrightarrow \ldots$$

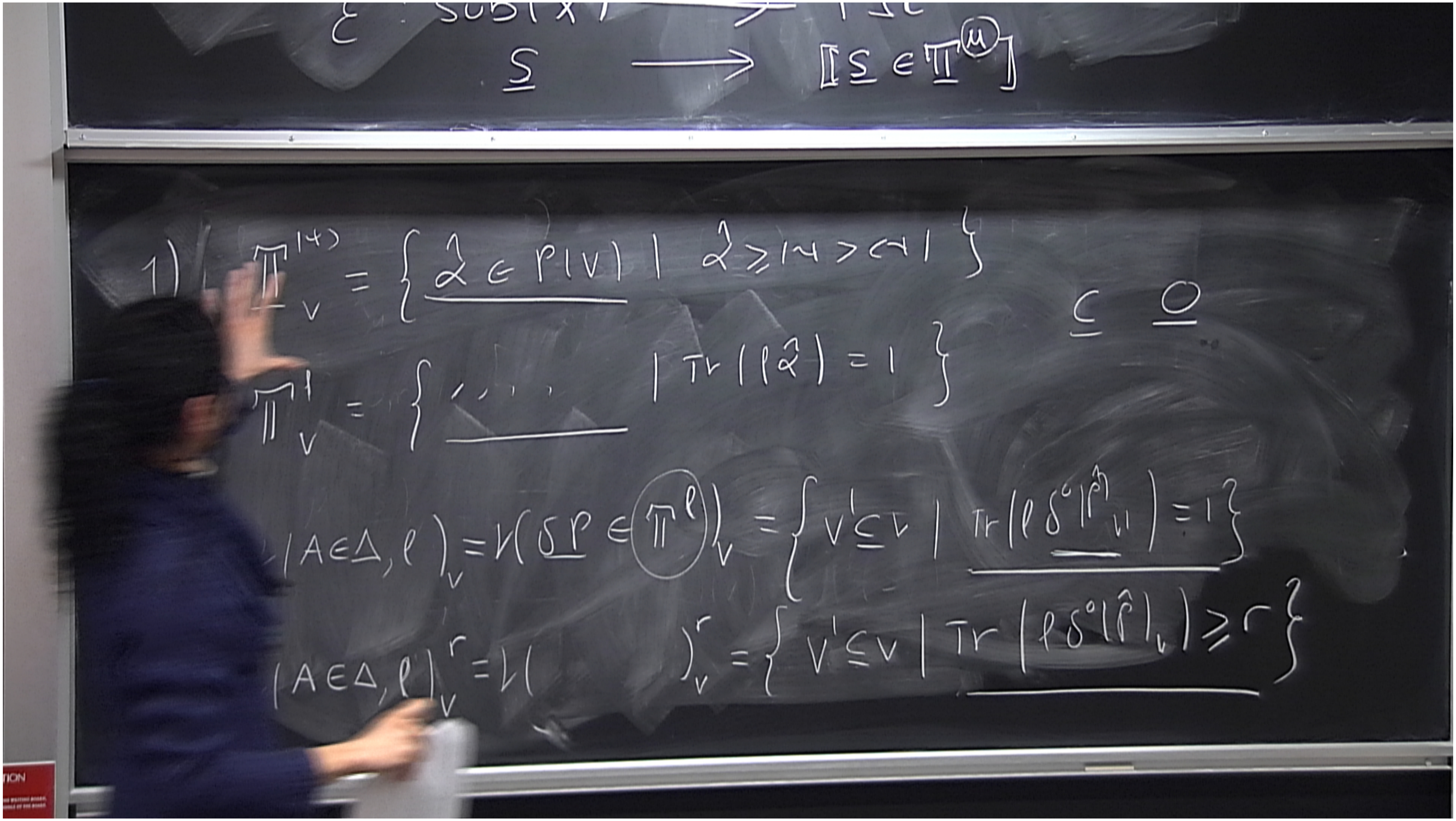
$$\underline{S} \longrightarrow [\![\underline{S} \in \underline{\Pi}^{(u)}]\!]$$

1) $\underline{\Pi}^{(4)}_V = \{ \underline{\hat{\alpha} \in P(V)} \mid \hat{\alpha} \geq |\psi\rangle\langle\psi| \} \subseteq \underline{O}$

$\Pi^{1}_V = \{ \underline{\ldots} \mid Tr(\rho\hat{\alpha}) = 1 \}$

2) $\nu(A \in \Delta, \rho)_V = \nu(\underline{\delta P} \in \Pi^{\rho})_V = \{ V' \subseteq V \mid \underline{Tr(\rho\,\delta^{o}(\hat{P})_{V'}) = 1} \}$


m: P(H) → [0 1]
m(p̂) = μ(S_p)|V|
p∈V
C^4
ρ = 1/2 1/2 0 0
ρ_1 = 3/4 1/4
Tr(ρ p̂) = 1
p̂ ≥ p̂_1 + p̂_2
Tr(ρ_1 p̂) = 1
//


$\underline{S} \longrightarrow [\![\underline{S} \in \underline{\Pi}^{(u)}]\!]$
1) $\underline{\Pi}^{(\psi)}_V = \{ \hat\alpha \in P(V) \mid \hat\alpha \geq |\psi\rangle\langle\psi| \}$ $\subseteq \underline{O}$
$\underline{\Pi}^{\rho}_V = \{ \ldots \mid \mathrm{Tr}(\rho\hat\alpha) = 1 \}$
$\nu(A \in \Delta, \rho)_V = \nu(\underline{\delta P} \in \underline{\Pi}^{\rho})_V = \{ V' \subseteq V \mid \mathrm{Tr}(\rho\,\delta^o(\hat P)_{V'}) = 1 \}$
$\nu(A \in \Delta, \rho)^r_V = \nu(\quad)^r_V = \{ V' \subseteq V \mid \mathrm{Tr}(\rho\,\delta^o(\hat P)_{V'}) \geq r \}$

$$\varepsilon : \mathrm{Sub}(X) \longrightarrow \ldots$$

$$\underline{S} \longrightarrow [\![\underline{S} \in \underline{\Pi}^{(u)}]\!]$$

1) $\underline{\Pi}^{(u)}_V = \{ \hat{\alpha} \in P(V) \mid \hat{\alpha} \geq |u\rangle\langle u| \}$ $\subseteq \underline{O}$

$\underline{\Pi}^{1}_V = \{ \ldots \mid \mathrm{Tr}(\rho\hat{\alpha}) = 1 \}$

2) $\nu(A \in \Delta, \rho)_V = \nu(\underline{\delta P} \in \underline{\Pi}^{\rho})_V = \{ V' \subseteq V \mid \mathrm{Tr}(\rho\, \delta^{o}(\hat{P})_{V'}) = 1 \}$

$\nu(A \in \Delta, \rho)^r_V = \nu(\quad)^r_V = \{ V' \subseteq V \mid \mathrm{Tr}(\rho\, \delta^{o}(\hat{P})_{V'}) \geq r \}$

Def

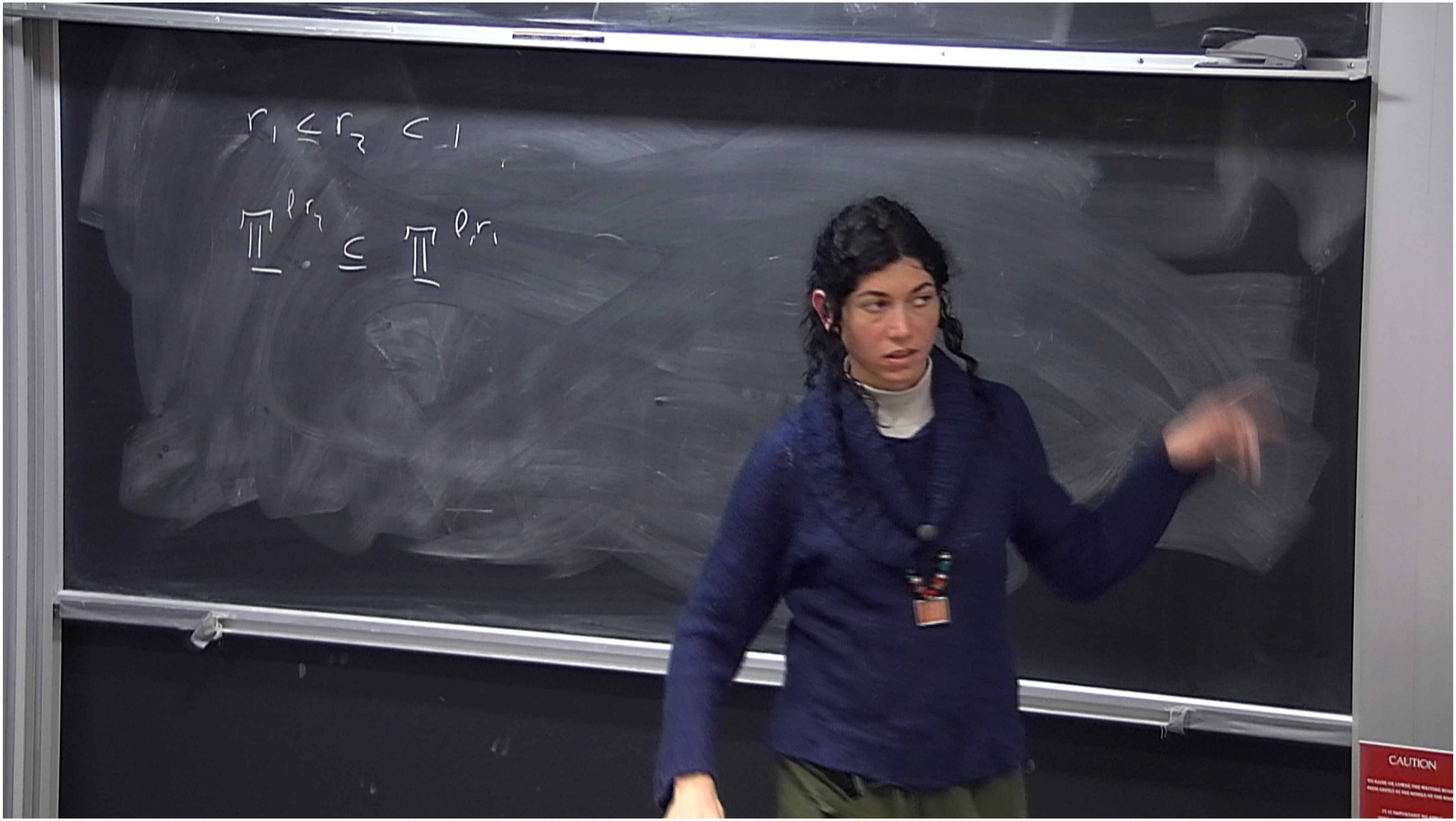
$$\underline{\Pi}^{|\psi\rangle}_{V} = \left\{ \underline{S} \in \mathrm{sub}_{ce}(\mathcal{E}_{\downarrow V}) \mid \forall V' \subseteq V,\ \langle\psi| \hat{P}_{S_{V'}} |\psi\rangle = 1 \right\}$$

$$^{\mathrm{cry}}\underline{\Pi}^{|\psi\rangle} = \Gamma\, \underline{\Pi}^{|\psi\rangle}$$

$$m : P(H) \to [0,1] \qquad m(\hat{p}) = \mu(S_p)\,|V|$$

$p \in V$

$$\Pi^{\ell, r}_{V} = \left\{ \Sigma \in \mathrm{sub}_{ce}(\mathcal{E}_{|V}) \;\middle|\; \forall V' \in V : \mathrm{Tr}(\ell \, \hat{\rho}_{S_{V'}}) \geq r \right\}$$


$r_1 \leq r_2 < 1$
$\underline{\Pi}^{\ell, r_2} \subseteq \underline{\Pi}^{\ell, r_1}$

$r_1 \leq r_2 < 1$

$$\underline{\Pi}^{\rho r_2} \subseteq \underline{\Pi}^{\rho, r_1}$$

<u>Proof</u>

$$\underline{\Pi}^{\rho r_2}_{v} = \{ \underline{S} \in \mathrm{sub}_{cl}(\mathcal{E}_{V}) \mid \forall v' \leq v \ \mathrm{Tr}(\rho \hat{P}_{S_{v'}}) \geq r_2 \}$$

$$= \mathrm{Tr}(\rho(\hat{\rho} + \hat{\rho})) - \mathrm{Tr}(\rho\rho) + \mathrm{Tr}(\rho\rho_c)$$

$$\{(0,r') \leq (0,r) \mid \mu(S_{or'}) \geq r'\} \in \Omega^{Sh(O(0,1)_c)}$$

$$Sh(\nu|H) \times O(0,1)_c))$$

$$= \mathrm{Tr}(P(\hat{\rho}-\hat{\rho})) - \mathrm{Tr}(\rho P) + \mathrm{Tr}(\rho \rho_c)$$

$$\{(0,r') \leq (0,r) \mid \mu(S_{or'}) \geq r'\} \in \Omega^{Sh(\mathcal{O}(01)_c)}$$

$$Sh(V(H) \times \mathcal{O}(0 1)_c))$$

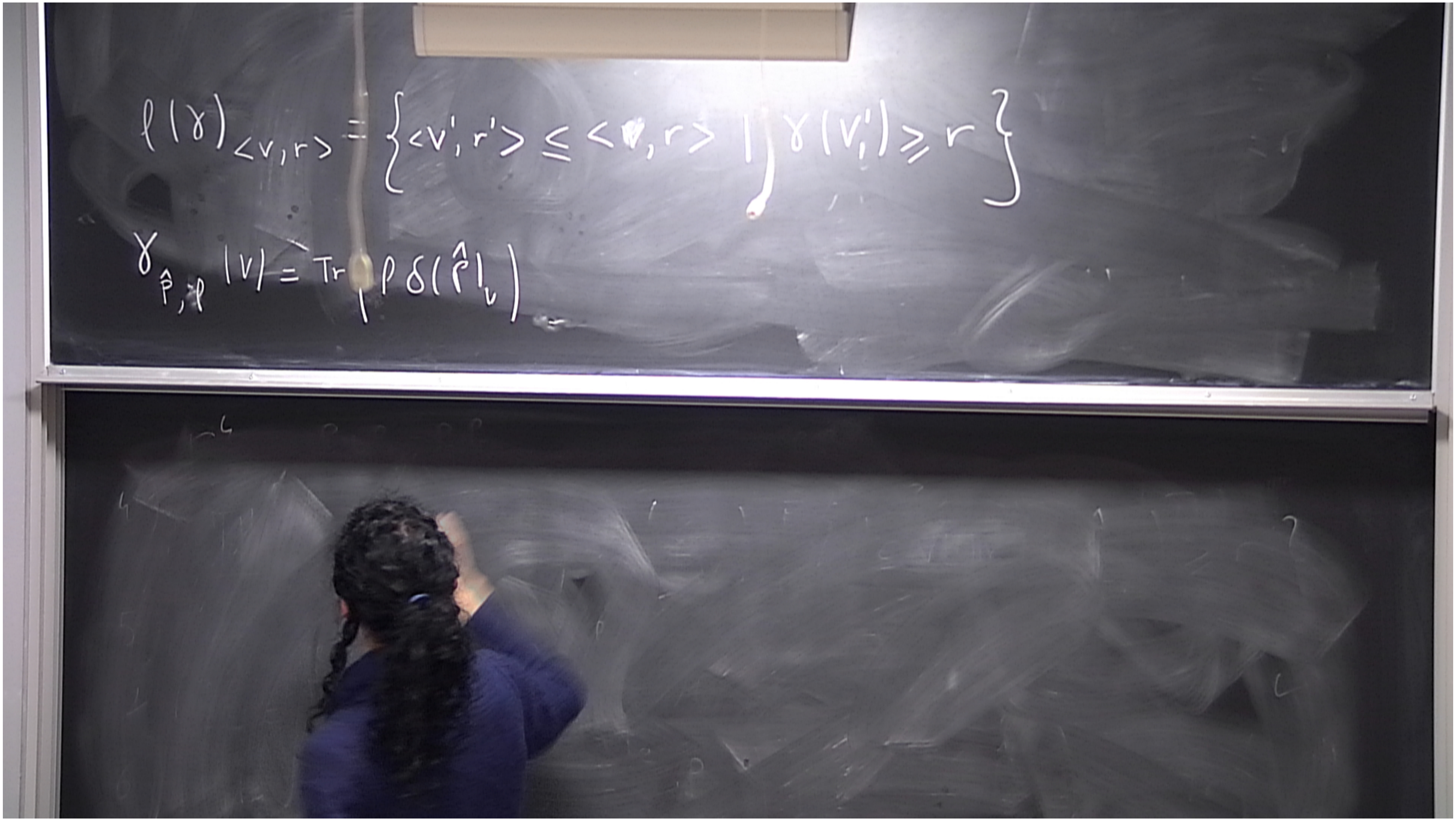
$$(\downarrow V \times (0 r))$$

$$\mathrm{sh}(\mathcal{V}(H) \times \mathcal{O}([0,1]_<))$$

$$l : \frac{\Gamma [0,1]^{\geq}}{\gamma} \longrightarrow \frac{\Gamma \underline{\Omega}}{l(\gamma)} : \underline{1} \longrightarrow \underline{\Omega}$$

$$\mathrm{sh}(\mathcal{V}(H) \times (0,1)_<)$$

$$l(\gamma)_{\langle V, r\rangle} := \{ \langle V', r' \rangle \leq \langle V, r \rangle \mid \gamma(V') \geq r \}$$


$\ell(\gamma)_{\langle v,r\rangle} = \{ \langle v', r'\rangle \leq \langle v, r\rangle \mid \gamma(v') \geq r \}$
$\gamma_{\hat{P},\rho}(V) = \mathrm{Tr}(\rho\,\delta(\hat{P}|_V))$

$$\delta_{\hat{P},\rho}(V) = \mathrm{Tr}(\rho\, \delta(\hat{P})_V)$$

$$\left[(\ell \circ \mu^{\rho})(\Sigma)\right]_{\langle v, r\rangle} = \ell(\gamma_{\mu^{\rho}(\underline{S})})_{\langle v, r\rangle} =$$

$$= \{ \langle v', r' \rangle \leq \langle v, r \rangle \mid \mathrm{Tr}(\rho, \hat{P}_{S_{v'}}) \geq r' \}$$

$$\sigma_{\hat{P},\rho}(V) = \mathrm{Tr}\left(\rho\,\delta(\hat{P}|_V)\right)$$

$$\left[(\ell \circ \mu^{\rho})(\underline{S})\right]_{\langle V,r\rangle} = \ell\left(\gamma_{\mu^{\rho}(\underline{S})}\right)_{\langle V,r\rangle} \equiv \mu^{\rho}(\underline{S}_V) \geq r$$

$$= \left\{ \langle V',r'\rangle \leq \langle V,r\rangle \;\middle|\; \mathrm{Tr}\left(\rho\,\hat{P}_{\underline{S}_{V'}}\right) \geq r' \right\}$$

$$\rho_1 : \mathcal{V}(H) \times G(O1)_c \longrightarrow \mathcal{V}(H)$$

$$\rho_1^* : Sh(\mathcal{V}(H)) \longrightarrow Sh(\mathcal{V}(H) \times G(O1)_c)$$

$$\underline{\Sigma} \longrightarrow \rho_1^*(\underline{\Sigma}) := \underline{\Sigma} \circ \rho_1$$

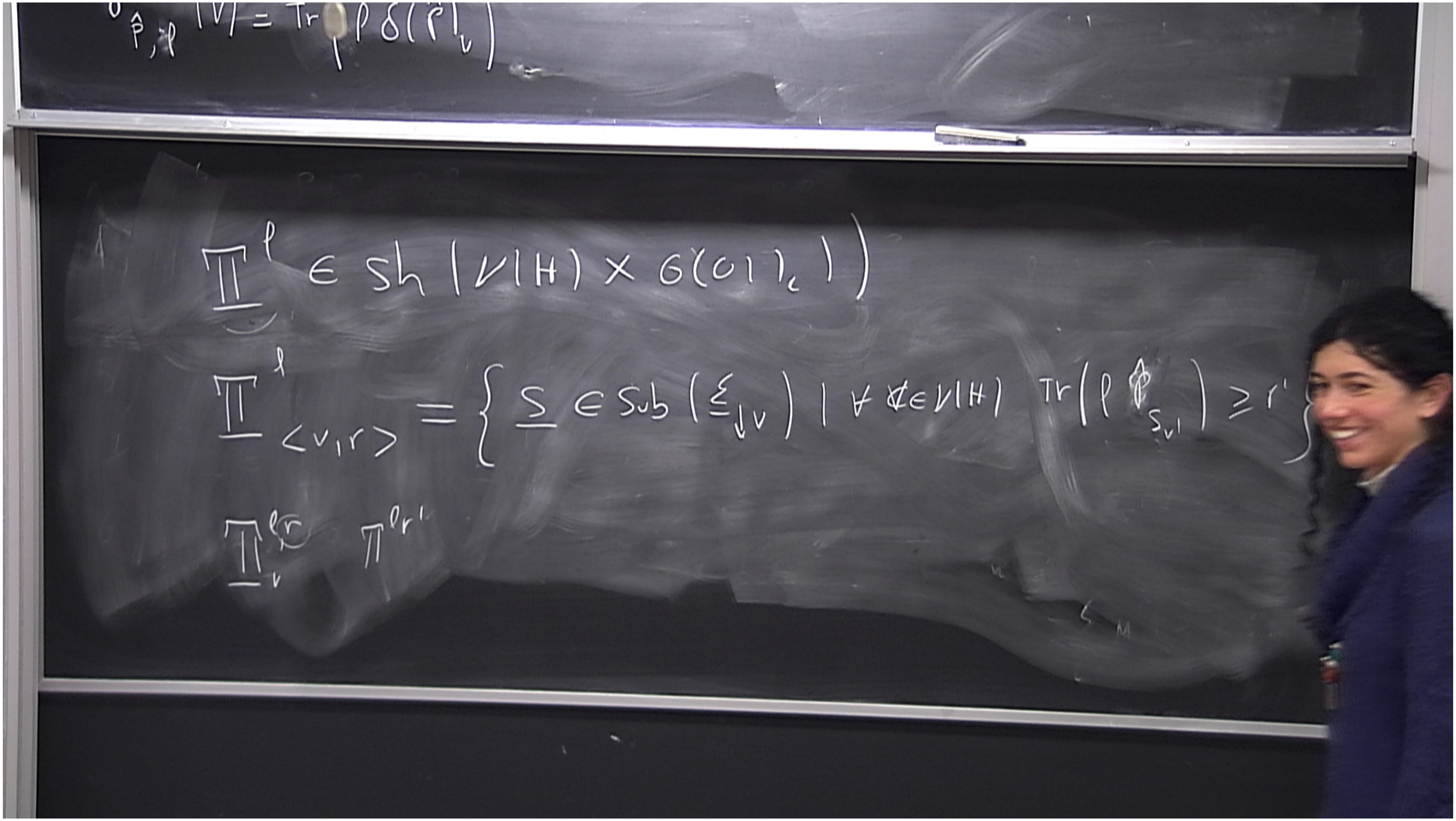
$$(\rho_1^* \underline{\Sigma})_{\langle V, r \rangle} = \underline{\Sigma} \circ \rho_1(\langle V, r \rangle) = \underline{\Sigma}_V$$

$$p_1 : \mathcal{V}(\mathcal{H}) \times \mathcal{O}(0,1)_L \longrightarrow \mathcal{V}(\mathcal{H})$$

$$p_1^* : Sh(\mathcal{V}(\mathcal{H})) \longrightarrow Sh(\mathcal{V}(\mathcal{H}) \times \mathcal{O}(0,1)_L)$$

$$\underline{\Sigma} \longrightarrow p_1^*(\underline{\Sigma}) := \underline{\Sigma} \circ p_1$$

$$(p_1^* \underline{\Sigma})_{\langle V, r \rangle} = \underline{\Sigma} \circ p_1(\langle V, r \rangle) = \underline{\Sigma}_V$$

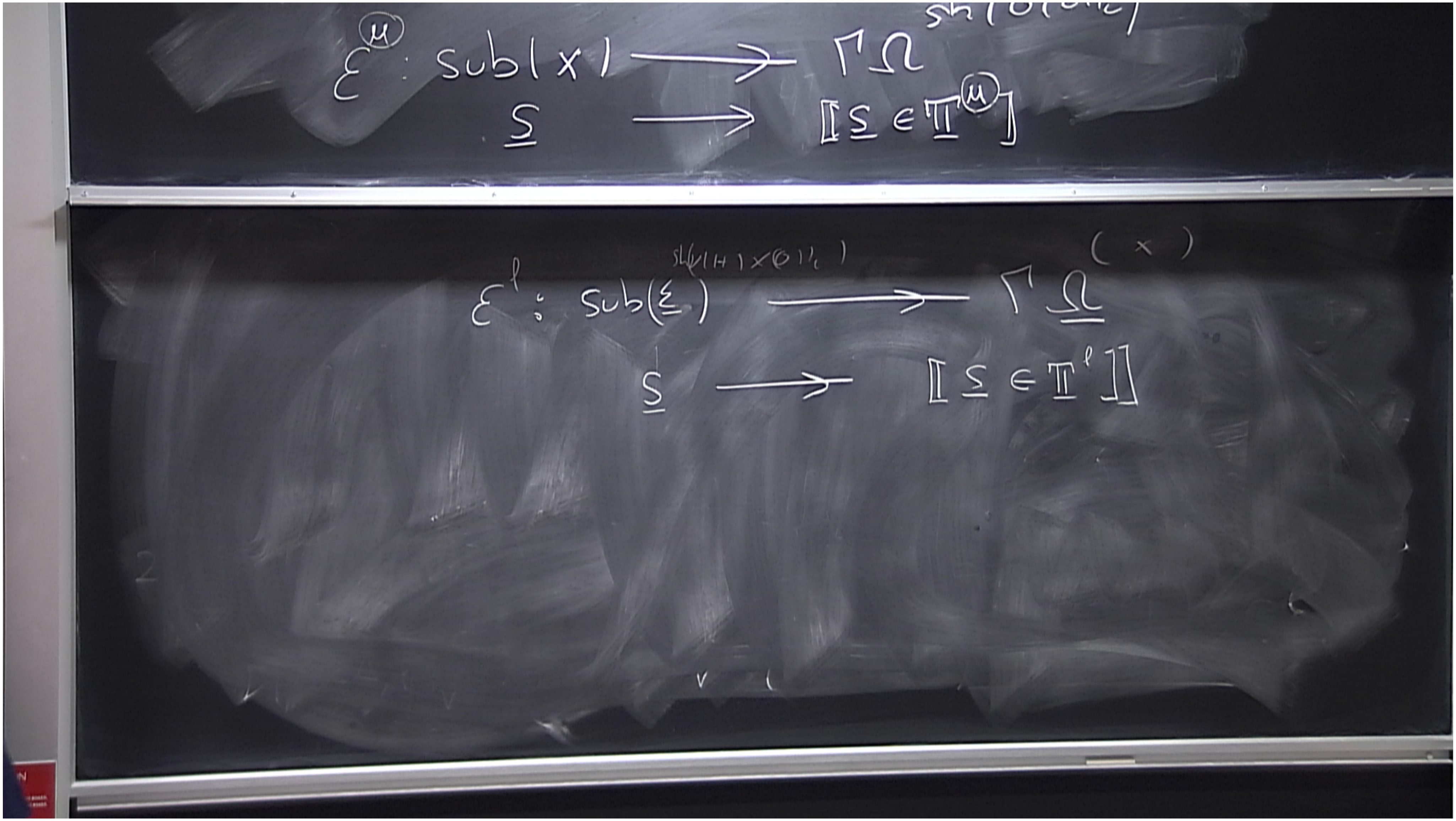


Π^ρ ∈ Sh(V(H) × O((0,1)_c))
Π^ρ_⟨v,r⟩ = { S ∈ Sub(Σ_↓v) | ∀ v′ ∈ V(H) Tr(ρ P̂_{S,v′}) ≥ r′ }

$$Sh(\underline{\Sigma}^{Sh(\mathcal{V}(H)\times \mathcal{O}(\alpha)_c)}) \xrightarrow[\mathcal{E}^l]{} \underline{\Omega}^{Sh(\mathcal{V}(H)\times \mathcal{O}(01)_c)}$$

$$P_1 : \mathcal{V}(H)\times \mathcal{O}(01)_c \longrightarrow \mathcal{V}(H)$$

P^*


$\mathcal{E}^{\mu}$: sub(x) $\longrightarrow$ $\Gamma\Omega$
$\underline{S}$ $\longrightarrow$ $[\![\underline{S} \in \underline{\mathbb{T}}^{\mu}]\!]$
$\mathcal{E}^{\ell}$: sub($\underline{\Sigma}$) $\longrightarrow$ $\Gamma\,\underline{\Omega}$
$\underline{S}$ $\longrightarrow$ $[\![\underline{S} \in \underline{\mathbb{T}}^{\ell}]\!]$

$\hat{P}, \rho \, (V) = \mathrm{Tr}(\rho \, \delta(\hat{P}|_V))$

$$\underline{\Pi}^{\rho} \in \mathrm{Sh}(\mathcal{V}(H) \times \mathcal{O}((0,1)_L)) \qquad \mu^{\rho}(\underline{S}_{V'}) \geq r'$$

$$\underline{\Pi}^{\rho}_{\langle V, r \rangle} = \left\{ \underline{S} \in \mathrm{Sub}(\underline{\Sigma}_{\downarrow V}) \mid \forall V' \in \mathcal{V}(H) \ \mathrm{Tr}(\rho \, \hat{P}_{S_{V'}}) \geq r' \right\}$$

$$\underline{\Pi}^{\rho, r}_{V} \qquad \Pi^{\rho r'}$$

$$
\begin{tikzcd}
Sh(\underline{\Sigma}^{\nu(H)}) \arrow[r, "\mu^{\ell}"] \arrow[d, "p^{*}"'] & \underline{\Gamma[0 1]^{\geq}} \arrow[d, "\ell"] \\
Sh(\underline{\Sigma}^{Sh(\nu(H)\times G(\alpha)_c)}) \arrow[r, "\varepsilon^{\ell}"'] & \underline{\Gamma\Omega}^{sh(\nu(H)\times G(\alpha)_c)}
\end{tikzcd}
$$