Title: Continuous Formulation of the Loop Quantum Gravity Phase Space

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Abstract: We relate the discrete classical phase space of loop gravity to the continuous phase space of general relativity. Our construction shows that the flux variables do not label a unique geometry, but rather a class of gauge-equivalent geometries. We resolve the tension between the loop gravity geometrical interpretation in terms of singular geometry, and the spin foam interpretation in terms of piecewise-flat geometry, showing that both geometries belong to the same equivalence class. We also establish a clear relationship between Regge geometries and the piecewise-flat spin foam geometries. All of this is based on <a href="http://arxiv.org/abs/1110.4833" target="_blank">arXiv:1110.4833</a>.

# Continuous formulation of the Loop Quantum Gravity phase space

*[L. Freidel, M. Geiller, JZ, arXiv:1110.4833]*

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# Some outstanding questions in LQG & spin foams

- What is the precise relationship between spin networks and spatial geometries?
- At the microscopic level, spin foams view space as composed of flat cells, while LQG views space as a polymer geometry. Is one – or somehow both – of these interpretations correct?
- These different views of geometry suggest different forms of volume operators. How do we choose?
- If we can obtain a dynamical theory of quantum gravity, how can we check if it is consistent with GR?

# Loop ‘classical’ gravity

- Although LQG aims to quantize GR directly, there is a non-trivial gap between the two theories.
- LQG uses graphs to discretize the continuous phase space of GR, and in the same stroke, promotes this to a quantum theory.
- The key idea is to disentangle these two steps.


- To address these questions, we need an intermediate theory of loop ‘classical’ gravity to bridge the gap.

# Outline

1. Review the continuous GR phase space in terms of connection and triad variables, and the discrete spin network phase space in terms of holonomy and flux variables.
2. Employ a flatness constraint on the continuous phase space to concretely relate it to the discrete phase space.
3. Explore which types of continuous geometry can be described by the data on a spin network.
4. Summarize the results and discuss how they can be used to further develop LQG.

# Continuous phase space

- A continuous phase space describing GR is $\mathcal{P} = T^*\mathcal{A}$, written in terms of the conjugate pair:

$$A^i_a := \Gamma^i_a + \gamma K^i_a \in \mathfrak{su}(2); \qquad \widetilde{E}^a_i := \tfrac{1}{2}\epsilon^{abc}\epsilon_{ijk} e^j_b e^k_c \in \mathfrak{su}(2);$$
$$\{A^i_a(x), \widetilde{E}^b_j(y)\} = \gamma \delta^i_j \delta^b_a \delta^3(x-y).$$

- We will work with the two-form $E^i := \epsilon_{abc}\widetilde{E}^c_i dx^a \wedge dx^b$, and use the notation $E \equiv E^i\tau^i$, $A \equiv A^i\tau^i$.
- The Poisson algebra is defined by the symplectic potential:

$$\Theta_{\mathcal{P}} = \int_\Sigma \mathrm{Tr}\,(E \wedge \delta A)\,.$$

# Continuous constraints

- The Hamiltonian is defined on a spatial three-geometry $\Sigma$ as a sum of the (smeared) scalar, diffeomorphism and Gauss constraints.
- We will consider in particular the Gauss constraint:

$$\mathcal{G}^i := \mathrm{d}_A E^i = \mathrm{d}E^i + \epsilon^{ijk} A^j E^k.$$

- The finite SU(2) gauge transformations are:

$$g \triangleright A = gAg^{-1} + g\mathrm{d}g^{-1}; \qquad g \triangleright E = gEg^{-1}.$$

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# Symplectic reduction

- How do we impose the Gauss constraint within the continuous phase space?
- We define a constrained space by imposing that the variables $(A, E)$ satisfy the Gauss constraint:

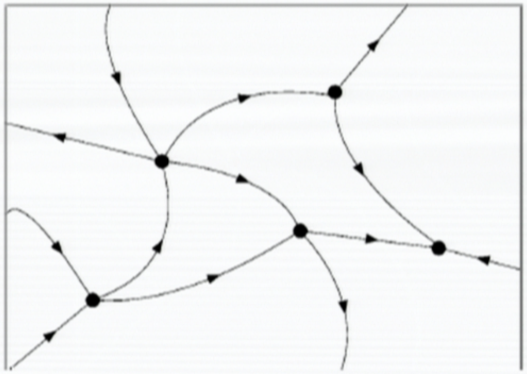
$$\mathcal{C}_{\mathcal{G}} = \{(A, E) \in T^*\mathcal{A} \mid \mathcal{G}(x) = 0 \; \forall x \in \Sigma\} .$$

- We also want fields related by SU(2) transformations to represent the same physical data, so we divide out the action of the Gauss constraint. This yields the reduced space:

$$\mathcal{P}_{\mathcal{G}} := \mathcal{C}_{\mathcal{G}}/\mathcal{G} \equiv \mathcal{P}/\!\!/\mathcal{G}.$$

# Oriented graphs

- The situation in loop gravity is rather different than the continuous setting of GR.
- LQG uses spin network Hilbert spaces $\mathcal{H}_\Gamma$ associated to oriented graphs.
- An oriented graph $\Gamma$ is a collection of oriented edges which meet at vertices.



- The continuous Hilbert space is a direct sum of Hilbert spaces associated to all graphs $\mathcal{H} = \bigoplus_\Gamma \mathcal{H}_\Gamma$.
- A particular $\mathcal{H}_\Gamma$ corresponds to a truncation of the theory.

# Discrete phase space

- $\mathcal{H}_\Gamma$ can be developed from the quantization of a classical phase space $P_\Gamma$ *[Rovelli & Speziale]*.
- To each edge of the graph, we assign a holonomy $h_e \in \mathrm{SU}(2)$ and a flux $X_e \in \mathfrak{su}(2)$, giving a phase space $P_e := T^*\mathrm{SU}(2)$ on each edge.
- Under orientation reversal we have:

$$h_{e^{-1}} = h_e^{-1}, \qquad X_{e^{-1}} = -h_e^{-1} X_e h_e.$$

- The phase space of the entire graph is:

$$P_\Gamma := \times_e T^*\mathrm{SU}(2)_e.$$

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# Poisson algebra of $P_\Gamma$

- The Poisson brackets are:

$$\begin{aligned} \{h_e, h_{e'}\} &= 0; \\ \{X_e^i, X_{e'}^j\} &= \delta_{ee'}\epsilon^{ij}{}_k X_e^k; \\ \{X_e^i, h_{e'}\} &= -\delta_{ee'}\tau^i h_e + \delta_{ee'^{-1}} h_e \tau^i . \end{aligned}$$

- This algebra is defined by the symplectic potential:

*[Alekseev & Malkin]*

$$\Theta_{P_\Gamma} = \sum_e \mathrm{Tr}\left(X_e \delta h_e h_e^{-1}\right).$$

# Discrete constraints

- Since the graph is not (yet) embedded within a manifold, the phase space is manifestly diffeomorphism invariant.
- The Gauss constraint is defined at each vertex as:

$$G_v := \sum_{e|s(e)=v} X_e + \sum_{e|t(e)=v} X_{e^{-1}}.$$

- The finite gauge transformations generated by this constraint are:

$$g_v \triangleright h_e = g_{s(e)} h_e g_{t(e)}^{-1}, \qquad g_v \triangleright X_e = g_{s(e)} X_e g_{s(e)}^{-1}.$$

- The reduced phase space is obtained by taking the double quotient: $P_\Gamma^G := \times_e T^* \mathrm{SU}(2)_e /\!/ G_v^{|V_\Gamma|}$ .

# The story so far

- GR is written in terms of an infinite dimensional continuous phase space $\mathcal{P}$.
- Loop gravity uses a union of finite dimensional discrete phase spaces, and for practical purposes the theory is often truncated to the phase space associated to a single graph $P_\Gamma$.
- Our goal is to find an isomorphism between $P_\Gamma$ and a reduced form of $\mathcal{P}$.
- We will prove the isomorphism by defining a one-to-one map from continuous to discrete data, and showing that this map is invertible.

# Dual graphs

- In order to determine a set of discrete data from the continuous variables, we embed a graph $\Gamma$ within $\Sigma$.
- We then choose a cellular decomposition that is dual to $\Gamma$ such that:
  - There is a single vertex of $\Gamma$ inside each cell $C_v$;
  - Each edge of $\Gamma$ intersects a single face $F_e$ at a point.
- The intersections of dual faces define a dual graph $\Gamma^*$.

# Defining holonomies and fluxes

- The holonomies are given by $h_e(A) := \overrightarrow{\exp} \int_e A$ .
- The traditional definition of flux is $X_e := \int_{F_e} E(x)$. However, this does not transform covariantly, i.e. $g \triangleright X_e \neq g_{s(e)} X_e g_{s(e)}^{-1}$.

  *[Freidel & Speziale, Thiemann, Wieland]*
- We use the definition:

$$X_{(F_e,\pi_e)}(A,E) := \int_{F_e} h_{\pi_e}(x) E(x) h_{\pi_e}^{-1}(x).$$

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# Choice of map

- We have established a one-to-one map:

$$\begin{aligned} \mathcal{J}: \quad \mathcal{P} &\to P_\Gamma \\ (A, E) &\mapsto \left(h_e(A), X_{(F_e,\pi_e)}(A, E)\right) \end{aligned}$$

- The map depends on the following choices:
  - an embedding of $(\Gamma, \Gamma^*)$;
  - a face $F_e$ dual to each edge;
  - a system of paths $\pi_e$ for each flux.
- Different choices lead to different holonomies and fluxes.
- Given only the discrete data, we cannot determine the continuous fields. How then can we invert the map?

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# Partially flat connection

- The key to eliminating the ambiguity in the map $\mathcal{J}$ is a flat connection.
- We impose this within each $C_v$ and $F_e$, but allow curvature on $\Gamma^*$, by using the following smeared constraint:

$$\mathcal{F}_{\Gamma^*}(\phi) := \int_\Sigma \phi_i \wedge F^i(A), \quad \text{where} \quad \phi(x) = 0 \; \forall \; x \in \Gamma^*.$$

- The gauge transformations generated by the flatness constraint are:

$$\delta_\phi^{\mathcal{F}_{\Gamma^*}} A = 0; \qquad \delta_\phi^{\mathcal{F}_{\Gamma^*}} E = \mathrm{d}_A \phi.$$

# Benefits of flatness

- The holonomy along an edge is path independent:

$$h_e = a_{s(e)}(x)^{-1} a_{t(e)}(x), \text{ where } a_v(x) := \overrightarrow{\exp} \int_x^v A.$$

- The flux is the same for any choice of paths $\pi_e$:

$$X_{(F_e, \pi_e)} = X_{F_e} = \int_{F_e} a_v(x)^{-1} E(x) a_v(x).$$

- Moreover, the flux is independent of the choice of face. If $F_e$ and $F'_e$ bound a region $R$:

$$\begin{aligned} 0 &= \int_R a_v(x)^{-1} \mathrm{d}_A E(x) a_v(x) = \int_R \mathrm{d}\left(a_v(x)^{-1} E(x) a_v(x)\right) \\ &= X_{F_e} - X_{F'_e}. \end{aligned}$$

# Flatness and Gauss constraints

- We want to relate the continuous and discrete phase spaces using the flatness and Gauss constraints.
- We enforce SU(2) invariance everywhere except the vertices:

$$\mathcal{G}_\Gamma(\alpha) = \int_\Sigma \alpha^i (\mathrm{d}_A E)_i, \quad \text{where} \quad \alpha(x) = 0 \;\forall\; x \in V_\Gamma .$$

- In fact, with the partially flat connection we can show:

$$d_A E(x) = \sum_{v \in V_\Gamma} G_v \delta^3(x - v).$$

# Reduced continuous phase space

- We will see that the reduced continuous phase space:

$$\mathcal{P}_{\Gamma,\Gamma^*} = \mathcal{P}/\!/(\mathcal{F}_{\Gamma^*} \times \mathcal{G}_\Gamma),$$

  is the continuous analog of $P_\Gamma$.
- We also consider the full Gauss constraint $\mathcal{G} = \mathcal{G}_\Gamma \times G_{V_\Gamma}$.
- We will see that the fully SU(2) invariant phase space:

$$\mathcal{P}^{\mathcal{G}}_{\Gamma,\Gamma^*} = \mathcal{P}/\!/(\mathcal{F}_{\Gamma^*} \times \mathcal{G}),$$

  is the continuous analog of $P^G_\Gamma$.

# Constraint solutions

- Let us look more closely at the continuous variables in the reduced phase space.
- The solution to the flatness and Gauss constraints can be written locally as:

$$A = a_v(x)\mathrm{d}a_v(x)^{-1}; \qquad E = a_v(x)X_v(x)a_v(x)^{-1},$$

where $X_v \in \Omega^2(\Sigma, \mathfrak{su}(2))$ is closed outside of the vertex.

- This solution extends throughout $\Sigma$ by demanding continuity across faces:

$$a_{v_2}(x) = a_{v_1}(x)h_e; \qquad X_{v_2} = h_e^{-1}X_{v_1}(x)h_e.$$

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# Holonomies and fluxes from $\mathcal{P}_{\Gamma,\Gamma^*}$ data

- The holonomies and fluxes are:

$$h_e = a_{s(e)}(x)^{-1} a_{t(e)}(x), \qquad X_e = \int_{F_e} X_v.$$

- $\mathcal{F}_{\Gamma^*} \times \mathcal{G}_\Gamma$ generates the following transformations:

$$\begin{aligned} a_v(x) &\to g_o(x) a_v(x), \\ X_v(x) &\to X_v(x) + \mathrm{d}\left(a_v(x)^{-1}\phi(x) a_v(x)\right). \end{aligned}$$

- These transformations leave the holonomies and fluxes invariant:

$$\begin{aligned} h_e &\to a_{v_1}(x)^{-1} g_o(x)^{-1} g_o(x) a_{v_2}(x) = a_{v_1}(x)^{-1} a_{v_2}(x), \\ X_e &\to \int_{F_e} \left(X_v + \mathrm{d}\left(a_v^{-1}\phi a_v\right)\right) = \int_{F_e} X_v. \end{aligned}$$

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# Mapping $\mathcal{P}_{\Gamma,\Gamma^*}$ to $P_\Gamma$

- We have established a one-to-one map from the constrained continuous data to the discrete data, that is invariant under the action of $\mathcal{F}_{\Gamma^*} \times \mathcal{G}_\Gamma$:

$$\mathcal{I} : \mathcal{P}_{\Gamma,\Gamma^*} \longrightarrow P_\Gamma$$

- A single spin network maps to an equivalence class of continuous geometries:

$$\boxed{(A, E) \sim (g_\circ \triangleright A, g_\circ^{-1}(E + \mathrm{d}_A \phi) g_\circ)}$$

- A similar map exists between the SU(2)-invariant phase spaces: $\mathcal{I}^\mathcal{G} : \mathcal{P}^\mathcal{G}_{\Gamma,\Gamma^*} \longrightarrow P^G_\Gamma$

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- A similar map exists between the SU(2)-invariant phase spaces: $\mathcal{I}^{\mathcal{G}} : \mathcal{P}^{\mathcal{G}}_{\Gamma,\Gamma^*} \longrightarrow P^G_\Gamma$

# Continuous analog of LQG phase space

- Using that $\mathrm{d}X_v = G_v \delta^3(x - v)$ and the compatibility across faces, we can show:

$$\Theta_{\mathcal{P}_{\Gamma,\Gamma^*}} = \int_\Sigma \mathrm{Tr}\,(E \wedge \delta A) = \sum_e \mathrm{Tr}\left(X_e \delta h_e h_e^{-1}\right) = \Theta_{P_\Gamma}.$$

- Since symplectic forms are invertible by definition, this result proves the isomorphism:

$$\boxed{\mathcal{P}_{\Gamma,\Gamma^*} \cong P_\Gamma}$$

- A similar argument proves the fully SU(2) invariant phase spaces are also isomorphic.

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# Gauge choices

- The flatness constraint presents a wealth of gauge choices for the electric field.
- A choice of gauge is a map from the discrete data to the continuous phase space,

$$\begin{array}{rccc} \mathcal{T}: & P_\Gamma & \longrightarrow & \mathcal{C}_{\Gamma,\Gamma^*} \\ & (h_e, X_e) & \longmapsto & (A, E) \end{array}$$

- A good gauge choice must be diffeomorphism covariant, i.e. $\Phi^*\mathcal{T}$ is equivalent to $\mathcal{T}$ defined on $\Phi(\Gamma, \Gamma^*)$
- We have found two gauge choices that satisfy this condition:
  1. spin foam geometry: piecewise flat-metric;
  2. LQG geometry: singular $E$ field.

# Choices of geometry

- Using distributional forms, we can use discrete data to construct a singular electric field as in the LQG interpretation:

$$E_S(x) = \sum_e h_{\pi_e}(x)^{-1} X_e h_{\pi_e}(x) \delta_e(x)$$

where $\delta_e(x) \equiv \int_{e(y)} \delta^3(x-y)\epsilon_{ijk} \mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}y^k$.

- We use an existence proof to show that a spin foam geometry composed of piecewise metric-flat cells is also available.
- In the spin foam gauge, when the faces and edges bounding the cells can be made flat, we obtain a Regge geometry.

# Summary

- We considered the discrete holonomy-flux phase space associated to a graph, whose quantization yields a spin network Hilbert space.
- We reduced the continuous phase space of GR using flatness and Gauss constraints.
- The reduced continuous phase space is isomorphic to the discrete phase space.
- This relates the semiclassical kinematics of LQG to GR in a precise manner: the discrete phase space corresponds to a class of gauge-equivalent continuous geometries.

# Some outstanding questions in LQG & spin foams

- What is the precise relationship between spin networks and spatial geometries?
- At the microscopic level, spin foams view space as composed of flat cells, while LQG views space as a polymer geometry. Is one – or somehow both – of these interpretations correct?
- These different views of geometry suggest different forms of volume operators. How do we choose?
- If we can obtain a dynamical theory of quantum gravity, how can we check if it is consistent with GR?

# Provocative statement

**Is it possible to express the classical dynamics of gravity in terms of holonomies and fluxes (on all possible graphs)?**

If the answer is:

- **Yes** $\Rightarrow$ The quantization of gravity will be reduced to the quantizations of finite-dimensional systems.

- **No** $\Rightarrow$ A quantization in terms of holonomy-flux variables cannot express the quantum dynamics. LQG cannot work.