Title: Fractal Space-times Under the Microscope: a RG View on Monte Carlo Data

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Abstract: The emergence of fractal features in the microscopic structure of space-time is a common theme in many approaches to quantum gravity. In particular the spectral dimension, which measures the return probability of a fictitious diffusion process on space-time, provides a valuable probe which is easily accessible both in the continuum functional renormalization group and discrete Monte Carlo simulations of the gravitational action. In this talk, I will give a detailed exposition of the fractal properties associated with the effective space-times of asymptotically safe Quantum Einstein Gravity (QEG). Comparing these continuum results to three-dimensional Monte Carlo simulations, we demonstrate that the resulting spectral dimensions are in very good agreement. This comparison also provides a natural explanation for the apparent conflicts between the short distance behavior of the spectral dimension reported from Causal Dynamical Triangulations (CDT), Euclidean Dynamical Triangulations (EDT), and Asymptotic Safety.


string


Fractal space-times under the microscope:
a RG view on Monte Carlo data
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M. Reuter and F.S., JHEP 12 (2011) 012, arXiv:1110.5224 [hep-th]
S. Rechenberger and F.S., work in progress
Quantum Gravity Seminar
Perimeter Institute, February 15, 2012

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Outline

introduction

- the gravitational renormalization group: a primer
- quantifying properties of fractals

fractal dimensions in QEG

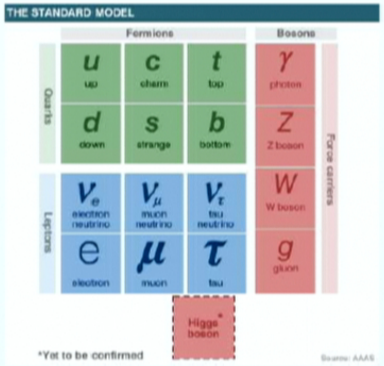
- dimensional flow in the Einstein-Hilbert truncation
- comparison to Monte-Carlo data
- higher-derivative corrections

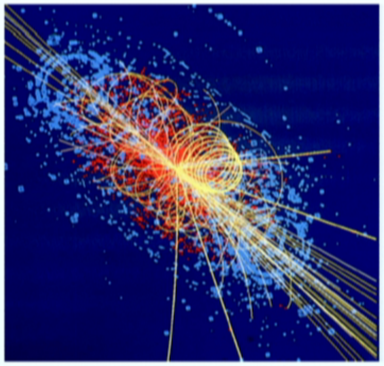
conclusions and outlook

Introduction

standard model of particle physics:

- describes: electromagnetic/strong/weak force + interactions with matter
- theoretical basis: quantum field theory in four-dimensional Minkowski space





works extremely well!

Space-time is four-dimensional, n'est-ce pas?

[B. Müller, A. Schäfer, Phys. Rev. Lett. 56 (1986) 1215]
[M. Haugan, C. Lämmerzahl, Lect. Notes Phys. 562 (2001) 195]
[D. Mattingly, Liv. Rev. Rel. 8 (2005) 5]

- Bounds on dimension from dimensional regularization

$$d_H = 4 - \epsilon$$

  - ϵ: probes fractional space-times which "misses points"

Experimental bounds on $|\epsilon|$:

- anomalous magnetic moment of muon $g-2$:

$$|\epsilon| < 10^{-8}\,, \qquad \ell \approx 10^{-15} m$$

- Lamb shift in hydrogen:

$$|\epsilon| < 10^{-11}\,, \qquad \ell \approx 10^{-11} m$$

- precession of planetry orbits:

$$|\epsilon| < 10^{-9}\,, \qquad \ell \approx 10^{11} m$$

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Spontaneous dimensional reduction of space-time: a survey

dimensional reduction: a common theme in many approaches to quantum gravity

- Renormalization Group Analysis:
  - classical general relativity regime: four-dimensional space-time
  - non-trivial UV fixed point: theory behaves two-dimensional

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- area-spectrum in Loop-Quantum Gravity:

$$A_j \propto \sqrt{\ell_j^2(\ell_j^2 + \ell_P^2)} \propto \begin{cases} \ell_j^2 & \text{for large area} \\ \ell_j \ell_P & \text{for small area} \end{cases}$$

- string theory at high temperatures
- anisotropic scaling models (Horava-Lifshitz Gravity)
- Strong coupling limit Wheeler-de Witt equation . . .

the gravitational renormalization group

a primer

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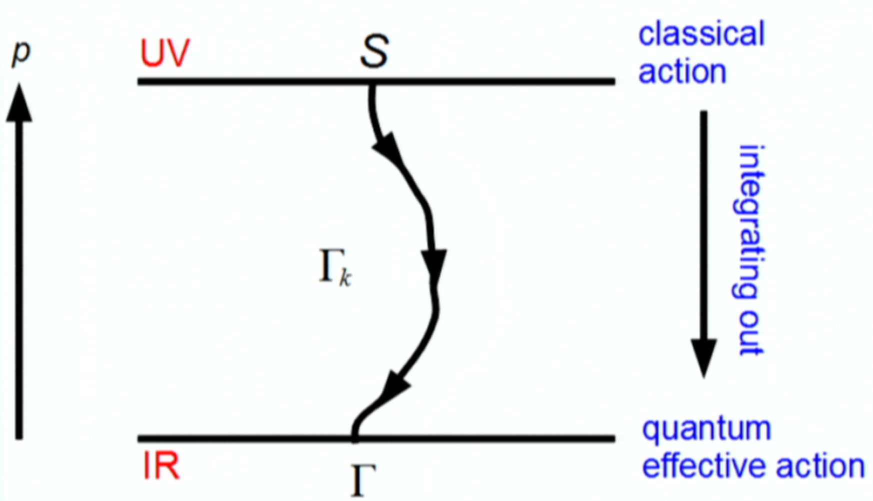
Wilson's modern picture of renormalization

central idea: integrate out quantum fluctuations shell-by-shell in momentum-space



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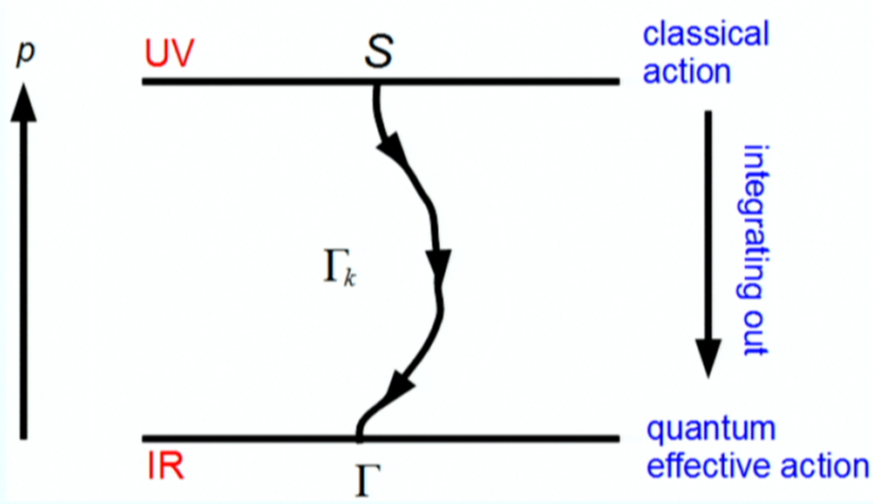


implementation:

- action with scale-dependent couplings $(G_N, \Lambda, \ldots)$: $g_i(k)$
- scale-dependence governed by β-functions: $k\partial_k g_i = \beta_{g_i}(\{g_i\})$

Functional Renormalization Group Equation for gravity

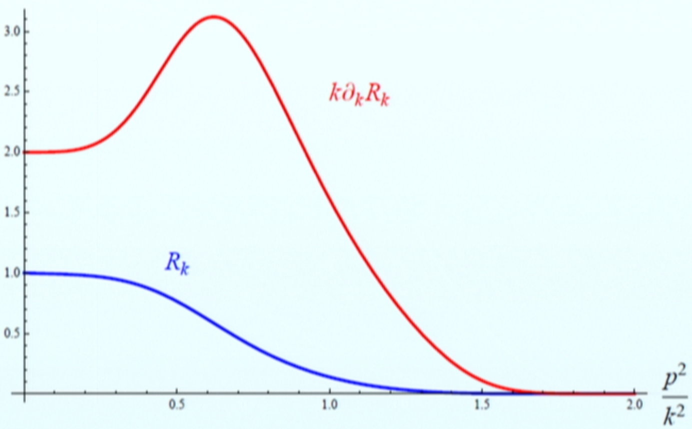
[C. Wetterich, Phys. Lett. **B301** (1993) 90]

[M. Reuter, Phys. Rev. D **57** (1998) 971, hep-th/9605030]

scale-dependence of Γ_k governed by exact RG equation

$$k\partial_k \Gamma_k[\phi, \bar{\phi}] = \frac{1}{2} \mathrm{Tr} \left[\left(\frac{\delta^2 \Gamma_k}{\delta\phi\delta\phi} + \mathcal{R}_k \right)^{-1} k\partial_k \mathcal{R}_k \right]$$

- $\mathcal{R}_k(p^2)$ = IR momentum-cutoff at scale k



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limits of the RG-flow:

- $k = \Lambda$: initial (boundary) condition $\qquad \Gamma_{k=\Lambda} = \Gamma_\Lambda$
- $k = 0$: all quantum fluctuations integrated out $\qquad \Gamma_{k=0} = \Gamma$

$$\Gamma = \Gamma_\Lambda + \lim_{k \to 0} \int_\Lambda^k d\hat{k}\, \partial_{\hat{k}} \Gamma_{\hat{k}} \left[\Gamma^{(2)}_{\hat{k}}, \mathcal{R}_{\hat{k}} \right]$$

in between:

- flow receives contributions from momentum-shell k only

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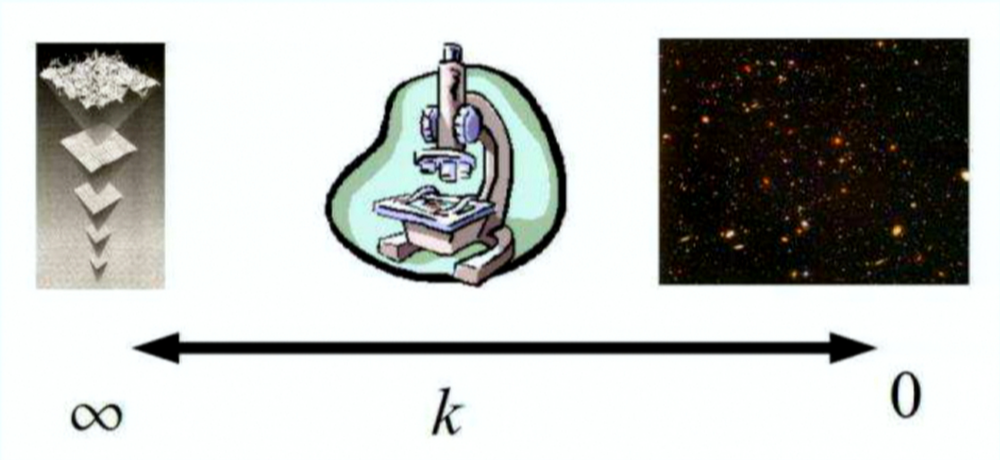
flow equation: microscope for space-time structure with resolution k



Solving the flow equation in the Einstein-Hilbert truncation

Einstein-Hilbert truncation: two running couplings: $G(k), \Lambda(k)$

$$\Gamma_k = \frac{1}{16\pi G(k)} \int d^4x \sqrt{g} \, [-R + 2\Lambda(k)] + S^{\mathrm{gf}} + S^{\mathrm{gh}}$$

- project flow onto G-Λ−plane

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- project flow onto G-$\Lambda-$plane

explicit β-functions for dimensionless couplings $g_k := k^2 G(k)\,,\ \lambda_k := \Lambda(k) k^{-2}$

- Particular choice of $\mathcal{R}_k$ (optimized cutoff)

$$k\partial_k g_k = (\eta_N + 2) g_k\,,$$

$$k\partial_k \lambda_k = -\left(2 - \eta_N\right)\lambda_k - \frac{g_k}{2\pi}\left[5\,\frac{1}{1-2\,\lambda_k} - 4 - \frac{5}{6}\,\frac{1}{1-2\lambda_k}\eta_N\right]$$

- anomalous dimension of Newton's constant:

$$\eta_N = \frac{g B_1}{1 - g B_2}$$

$$B_1 = \frac{1}{3\pi}\left[5\,\frac{1}{1-2\lambda} - 9\,\frac{1}{(1-2\lambda)^2} - 7\right]\,,\ B_2 = -\frac{1}{12\pi}\left[5\,\frac{1}{1-2\lambda} + 6\,\frac{1}{(1-2\lambda)^2}\right]$$

Einstein-Hilbert truncation: Fixed Point structure

β-functions for $g_k := k^2 G(k)\,,\ \lambda_k := \Lambda(k) k^{-2}$

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microscopic theory $\Longleftrightarrow$ fixed points of the β-functions

$$\beta_g(g^*, \lambda^*) = 0\,, \qquad \beta_\lambda(g^*, \lambda^*) = 0$$

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  - at $g^* = 0, \lambda^* = 0 \Longleftrightarrow$ free theory
  - saddle point in the g-λ-plane

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- non-Gaussian Fixed Point ($\eta_N^* = -2$):
  - at $g^* > 0, \lambda^* > 0 \Longleftrightarrow$ "interacting" theory
  - UV attractive in g_k, λ_k

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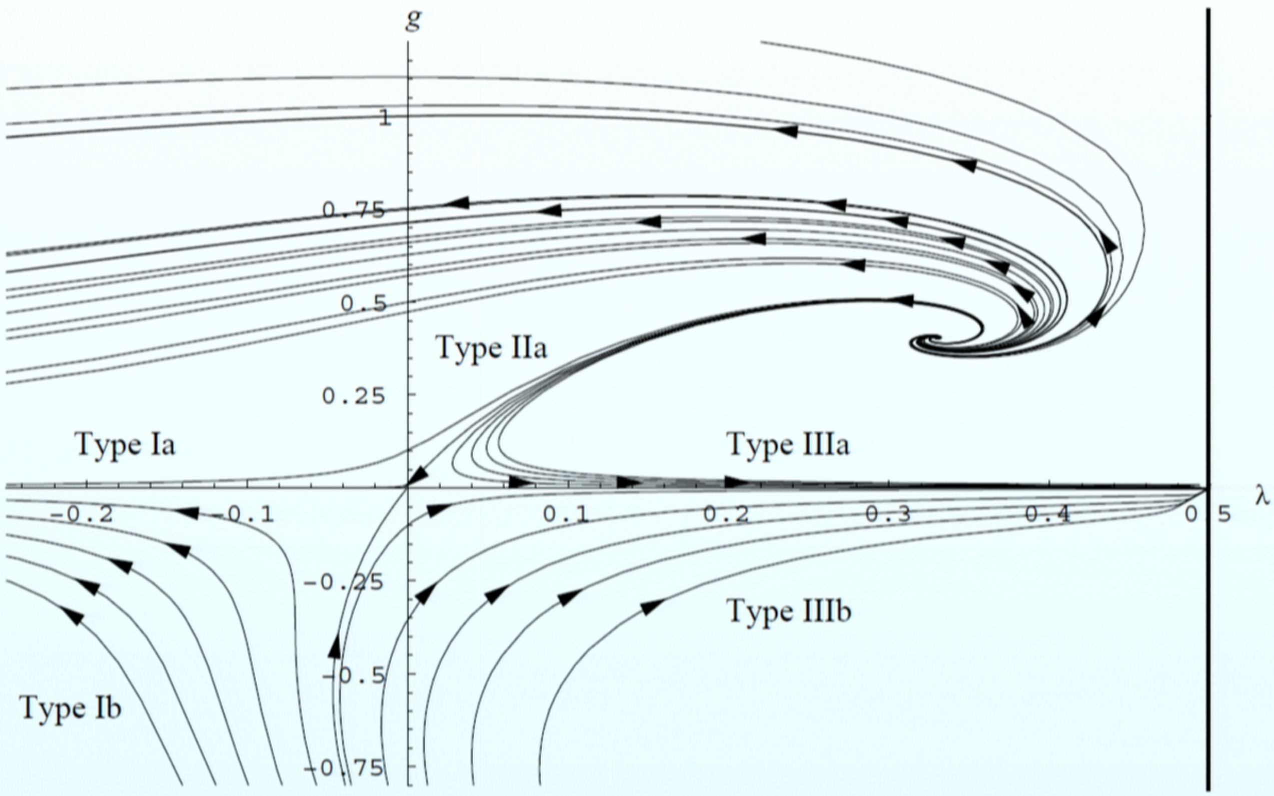
> Asymptotic safety: non-Gaussian Fixed Point is UV completion for gravity

Einstein-Hilbert-truncation: the phase diagram

[M. Reuter, FS, '01]



quantifying properties of fractals

Hausdorff or topological dimension

Determined by number N of balls necessary to cover a point-set:

$$N(R) \propto R^{-D}$$

Hausdorff-dimension d_H:

$$d_H = -\lim_{R\to 0} \frac{\log N(R)}{\log R}$$

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Examples:

- real line: $N(R) \propto R^{-1} \quad \longrightarrow \quad d_H = 1$
- coast-line of England: $d_H \approx 1.2$

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Spectral dimension d_s

- Heat-equation: diffusion of scalar test particle on manifold with metric g

$$\partial_T K_g(x,x';T) = -\Delta_g K_g(x,x';T)$$

- define averaged return probability

$$P_g(T) \equiv \frac{1}{V} \int d^d x \sqrt{g(x)}\, K_g(x,x;T)$$
$$= \frac{1}{V} \mathrm{Tr}\left[\exp(-T\Delta_g)\right]$$
$$= \left(\frac{1}{4\pi T}\right)^{d/2} \sum_{n=0}^{\infty} A_n T^n$$

- generalization: space-time dimension seen by diffusion process

$$d_s = -2 \frac{d \ln P_g(T)}{d \ln T}\bigg|_{T=0}$$

Extension to finite random walks: $\mathcal{D}_s(T)$

Walk dimension d_w

characterizes the fractal properties of the trail left by random walk

- probability density for random walk in flat space

$$K(x, x'; T) = (4\pi T)^{-d/2} \exp\left(-\frac{|x - x'|^2}{4T}\right)$$

- average square displacement characteristic for regular diffusion

$$\langle x^2 \rangle = \int d^d x\, x^2\, K(x, 0; T) \propto T$$

- On fractals: diffusion can be anomalous:

$$\langle x^2 \rangle \propto T^{2/d_w}\Big|_{T=0}$$

definition of walk dimension

Extension to finite random walks: $\mathcal{D}_w(T)$

Alexander-Orbach relation

[S. Alexander, R. Orbach, J. Phys. Lett. (Paris) 43 (1982) L625]

on homogeneous fractals:

$$\frac{d_s}{2} = \frac{d_H}{d_w} \, .$$

- relation between spectral, walk and Hausdorff dimension fractal space-times

from the functional renormalization group

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Classical vs. quantum space-times

[O. Lauscher, M. Reuter, JHEP 10 (2005) 050]

classical space-times from general relativity

$$S^{\rm EH} = \frac{1}{16\pi G_N} \int d^dx \sqrt{g}\,(-R + 2\Lambda)$$

- Einstein equations

$$R_{\mu\nu} = \frac{2}{2-d}\,\Lambda\, g_{\mu\nu}$$

solution: metric $g_{\mu\nu}$ valid at all length scales

effective quantum space-time: replace $S^{\rm EH} \rightarrow$ effective average action $\Gamma_k[g]$

- one-parameter family of equations of motion

$$\frac{\delta \Gamma_k[\langle g_{\mu\nu}\rangle_k]}{\delta g_{\mu\nu}} = 0$$

- solution: metric $\langle g_{\mu\nu}\rangle_k$ seen by physical process with momentum k^2
- proper distance calculated from $\langle g_{\mu\nu}\rangle_k$ depends on k^2

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Diffusion processes on QEG space-times

[O. Lauscher, M. Reuter, JHEP 10 (2005) 050]

basic idea: replace classical return probability by expectation value:

$$P(T) \equiv \langle P_\gamma(T) \rangle \equiv \int \mathcal{D}\gamma \mathcal{D}C \mathcal{D}\bar{C}\, P_\gamma(T)\, e^{-S_{\text{bare}}[\gamma, C, \bar{C}]}$$

average action $\Gamma_k[g]$: description of physics at scale k

- $\langle g \rangle_k$: approximate $\Gamma_k[g]$ by Einstein-Hilbert truncation

$$R_{\mu\nu}(\langle g \rangle_k) = \frac{2}{2-d}\, \Lambda_k \, \langle g_{\mu\nu} \rangle_k$$

- scaling relation between metrics at different scales k:

$$\langle g_{\mu\nu}(x) \rangle_k = [\Lambda_{k_0}/\Lambda_k] \langle g_{\mu\nu}(x) \rangle_{k_0}$$

- spectrum of Laplace-operators at different scales

$$\Delta(k) = \left[\Lambda_k/\Lambda_{k_0}\right] \Delta(k_0)$$

Compute the spectral dimension $\mathcal{D}_s(T)$

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1. spectrum of Laplace-operators at different scales

$$\Delta(k) = \left[\Lambda_k / \Lambda_{k_0}\right] \Delta(k_0)$$

2. solve the k-dependent heat equation

$$\partial_T K(x, x'; T) = -\Delta(k) K(x, x'; T)$$

- assume Λ_{k_0} small $\Longrightarrow$ flat-space approximation

$$K(x, x'; T) = \int \frac{d^d p}{(2\pi)^d} \, e^{ip\cdot(x-x')} \, e^{-p^2 F(p^2) T}\,, \qquad F(p^2) = \Lambda(p)/\Lambda(k_0)$$

3. quantum return probability

$$P(T) = \int \frac{d^d p}{(2\pi)^d} \, e^{-p^2 F(p^2) T}$$

4. spectral dimension for scaling cosmological constant: $\Lambda_k \propto k^\delta$:

$$\boxed{\mathcal{D}_s(T) = \tfrac{2d}{2+\delta}}$$

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Compute the walk dimension $\mathcal{D}_w(T)$

1. flat-space approximation of probability density
   - scaling regime $F(p) = (Lp)^\delta$

$$K(x, x'; T) = \int \frac{d^d p}{(2\pi)^d} \, e^{ip\cdot(x-x')} \, e^{-p^{(2+\delta)} \, L^\delta \, T}$$

2. Rescaling $q_\mu = p_\mu T^{1/(2+\delta)}$, $\xi_\mu = (x_\mu - x'_\mu)/T^{1/(2+\delta)}$:

$$K(x, x'; T) = \frac{1}{T^{d/(2+\delta)}} \int \frac{d^d q}{(2\pi)^d} \, e^{iq\cdot\xi} \, e^{-L^\delta q^{2+\delta}}$$

3. $\langle x^2 \rangle$ scales as $T^{2/(2+\delta)}$
4. walk dimension for scaling cosmological constant: $\Lambda_k \propto k^\delta$:

$$\boxed{\mathcal{D}_w(T) = 2 + \delta}$$

Hausdorff dimension of effective QEG space-times

- Volume of d-ball $\mathcal{B}^d$ computed from $\langle g_{\mu\nu} \rangle_k$

$$V(\mathcal{B}^d) = \int_{\mathcal{B}^d} d^d x \sqrt{g_k} \propto (r_k)^d$$

- compare to definition of d_H:

$$d_H = d$$

Hausdorff dimension of effective QEG space-times

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Conclusion:

- QEG space-times are not sponge-like
- fractal properties: dynamical
- QEG space-times satisfy the Alexander-Orbach relation

$$\frac{\mathcal{D}_s}{2} = \frac{d_H}{\mathcal{D}_w} \, .$$

Spectral and walk dimension on theory space

Scaling law for cosmological constant

$$\Lambda_k \propto k^{\delta}$$

generalization: $\delta(k)$ as scale-dependent quantity

$$\begin{aligned}\delta(k) &\equiv k\partial_k \ln(\Lambda_k) \\ &= 2 + \lambda_k^{-1}\beta_\lambda(g,\lambda)\end{aligned}$$

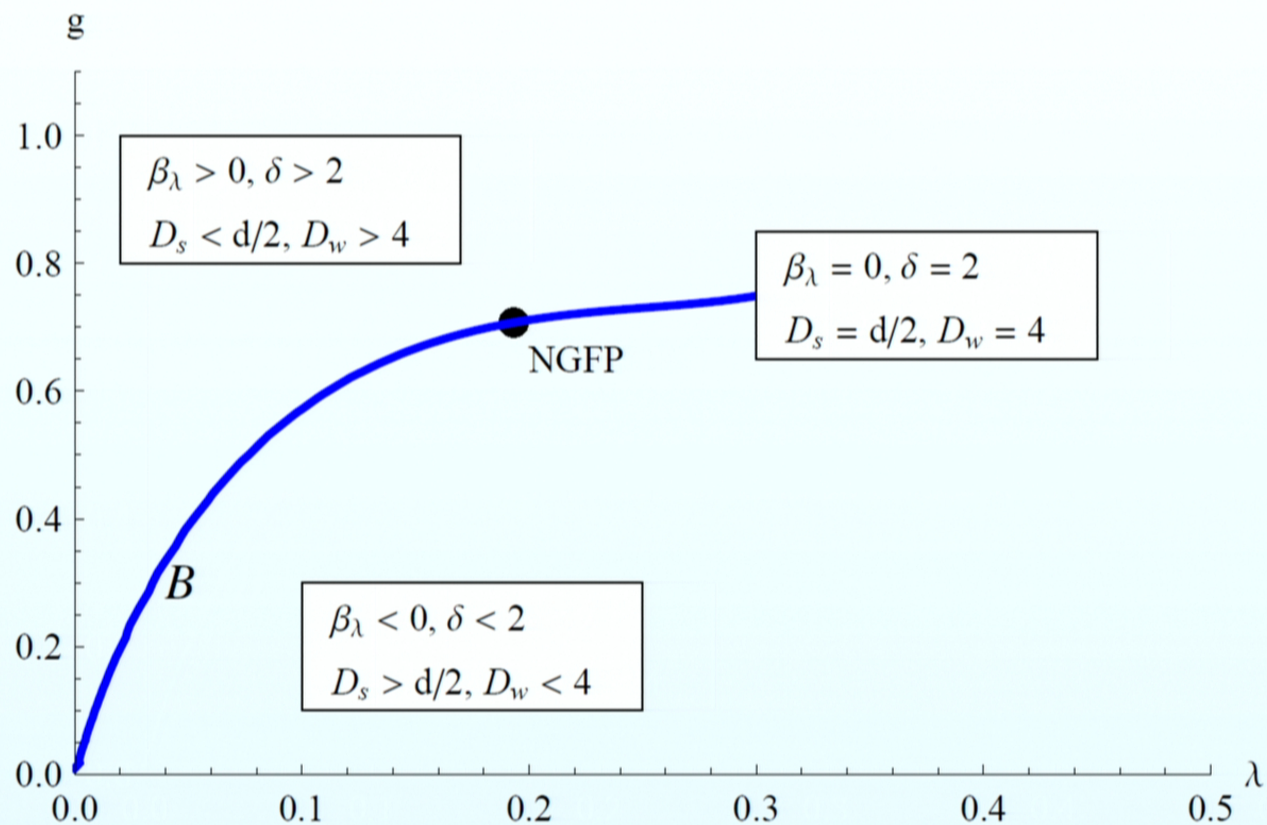
Substitute into fractal dimensions

$$\mathcal{D}_s(g,\lambda) = \frac{2d}{4 + \lambda^{-1}\beta_\lambda(g,\lambda)}$$

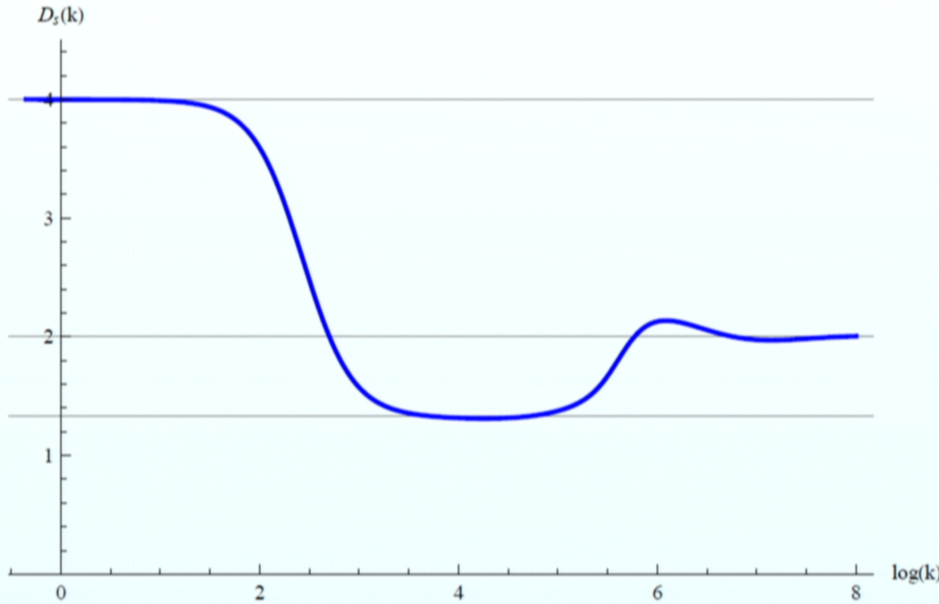
$$\mathcal{D}_w(g,\lambda) = 4 + \lambda^{-1}\beta_\lambda(g,\lambda)$$

$\mathcal{D}_s$ and $\mathcal{D}_w$ are autonomous functions of theory space!

Fractal dimensions on theory space

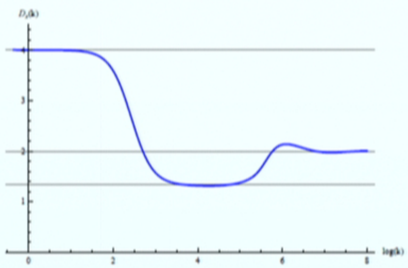


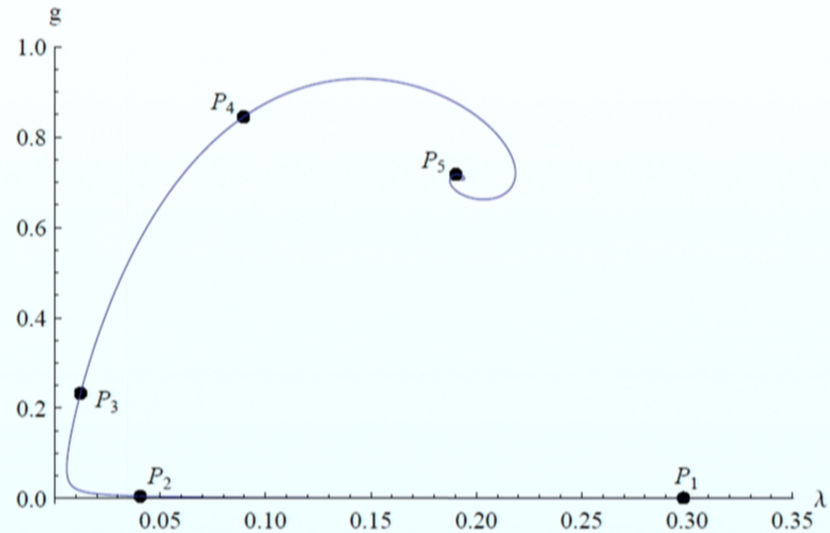
Spectral dimension $\mathcal{D}_s$ along a typical RG-trajectory



- classical regime: $\mathcal{D}_s(T) = 4$
- semi-classical regime: $\mathcal{D}_s(T) = 4/3 \quad \Rightarrow$ branched polymer
- NGFP regime: $\mathcal{D}_s(T) = 2$

Spectral dimension $\mathcal{D}_s$ along a typical RG-trajectory





- classical regime: $\mathcal{D}_s(T) = 4$
- semi-classical regime: $\mathcal{D}_s(T) = 4/3 \quad \Rightarrow$ branched polymer
- NGFP regime: $\mathcal{D}_s(T) = 2$

Spectral Dimension

discrete vs. continuum results

The spectral dimension puzzle

effective QEG space-times [O. Lauscher, M. Reuter '05]

- classical regime $(F(p^2) = 1)$: $\mathcal{D}_s(T) = d$
- NGFP regime $(F(p^2) \propto p^2)$: $\mathcal{D}_s(T) = d/2$

Causal Dynamical Triangulations $(d = 4)$ [J. Ambjorn, J. Jurkiewicz, R. Loll '05]

- classical regime: $\mathcal{D}_s(T) = 4$
- short random walks: $\mathcal{D}_s(T) = 2$

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Causal Dynamical Triangulations ($d = 3$) [D. Benedetti, J. Henson '09]

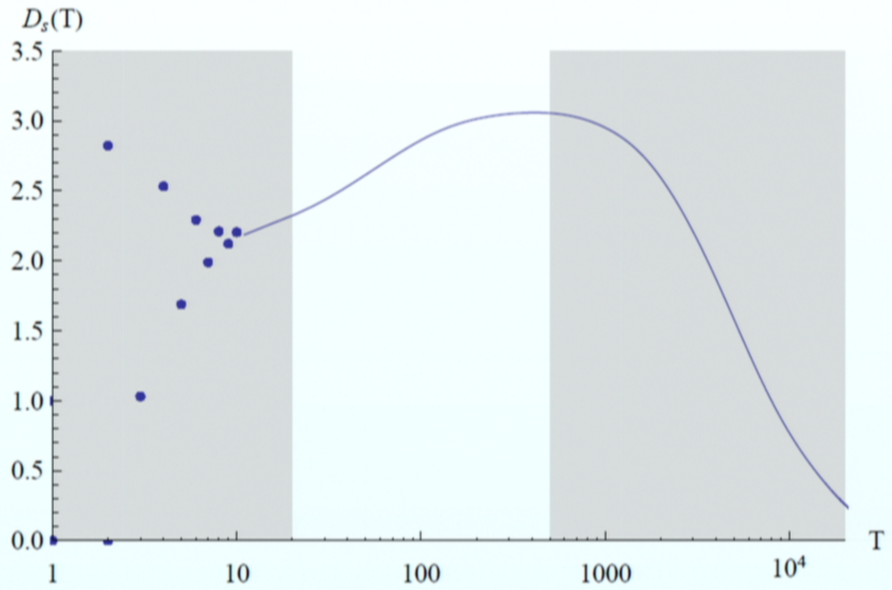
- classical regime: $\mathcal{D}_s(T) = 3$
- short random walks: $\mathcal{D}_s(T) = 2$

Euclidean Dynamical Triangulations ($d = 4$) [J. Laiho, D. Coumbe '11]

- classical regime: $\mathcal{D}_s(T) = 4$
- short random walks: $\mathcal{D}_s(T) = 1.5$

Spectral Dimension measured in 3-dimensional CDT

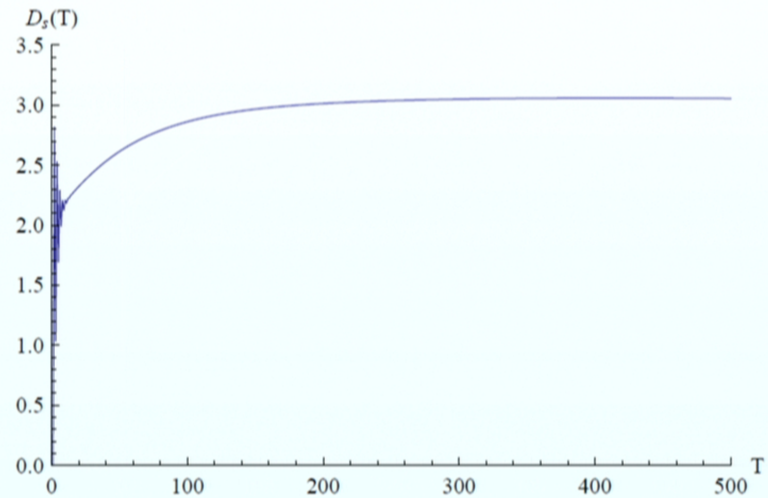
[D. Benedetti, J. Henson, Phys. Rev. D 80 (2009) 124036]



	$T \leq 20$	oscillations (discrete simplex structure)
$20 \leq$	$T \leq 500$	good data
$500 \leq$	T	exponential fall-off (triangulation is compact)

Determining the spectral dimension in CDT

[D. Benedetti, J. Henson, Phys. Rev. D 80 (2009) 124036]



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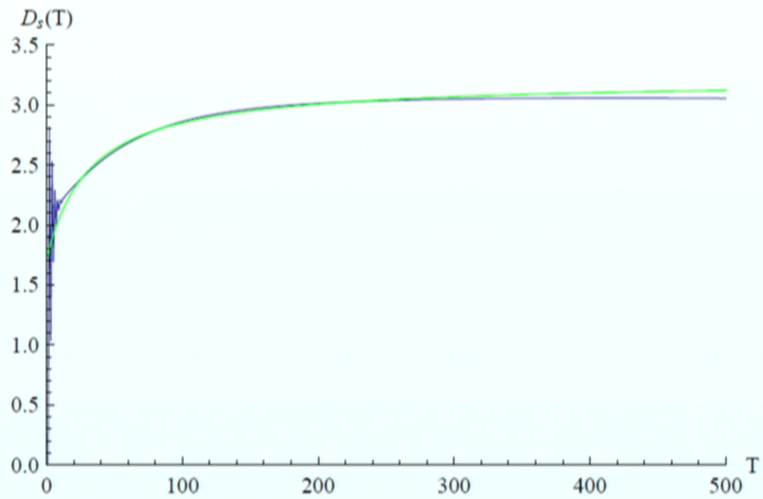

Fit-function I:

$$D_s(T) = a - \frac{b}{c+T} : \qquad a = 3.21\,,\ b = 46.93\,,\ c = 31.43$$

$$D_s(T)|_{T=0} = 1.39\,, \qquad D_s(T)|_{T=\infty} = 3.19$$

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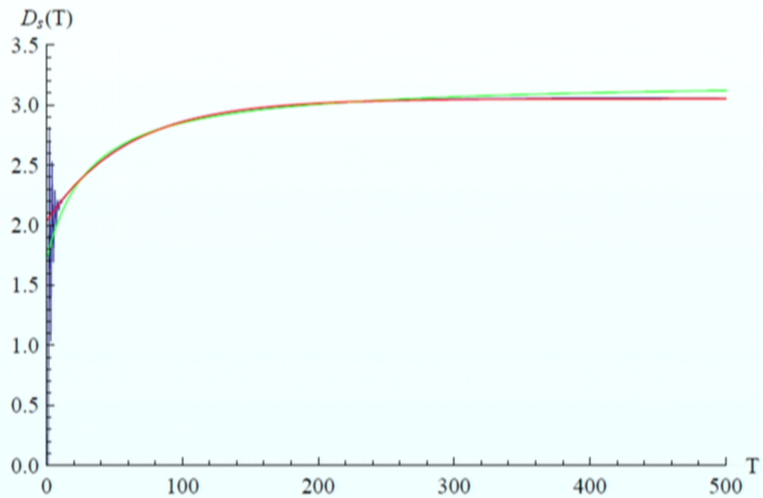
[D. Benedetti, J. Henson, Phys. Rev. D 80 (2009) 124036]



Fit-function II:

$$D_s(T) = a + b\,\mathrm{e}^{-cT}: \qquad a = 3.05\,,\ b = -1.02\,,\ c = 0.02$$

$$D_s(T)|_{T=0} = 2.03\,, \qquad D_s(T)|_{T=\infty} = 3.05$$

– p. 35/

The RG-trajectory underlying the CDT-data

Matching the spectral dimensions of QEG and CDT:

1. integrate β-functions: $g_0, \lambda_0 \mapsto g_k, \lambda_k$
2. substitute RG-trajectory into $\mathcal{D}_s^{\mathrm{QEG}}(T)$:

$$\mathcal{D}_s^{\mathrm{QEG}}(T) \mapsto \mathcal{D}_s^{\mathrm{QEG}}(T; g_0, \lambda_0)$$

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$$(\Delta \mathcal{D}_s)^2 \equiv \sum_{T=20}^{500} \left(\mathcal{D}_s^{\mathrm{QEG}}(T; g_0^{\mathrm{fit}}, \lambda_0^{\mathrm{fit}}) - \mathcal{D}_s^{\mathrm{CDT}}(T) \right)^2$$

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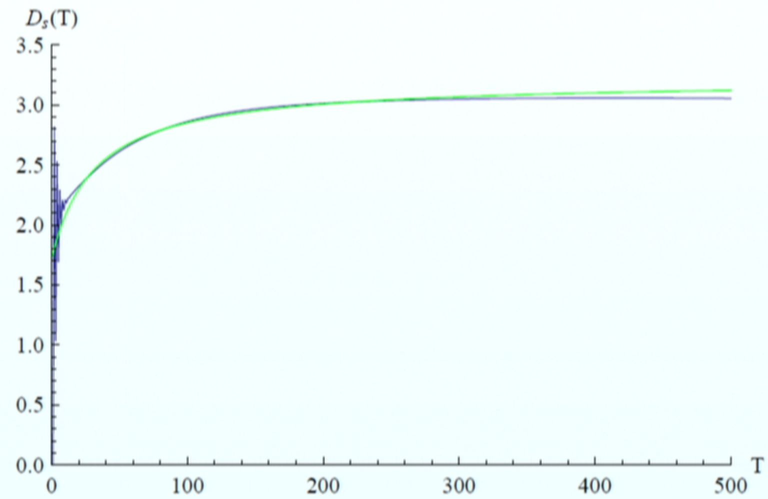

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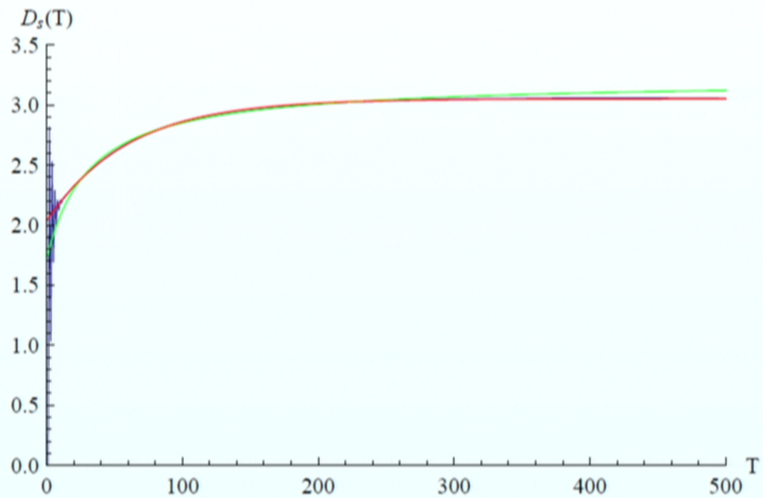

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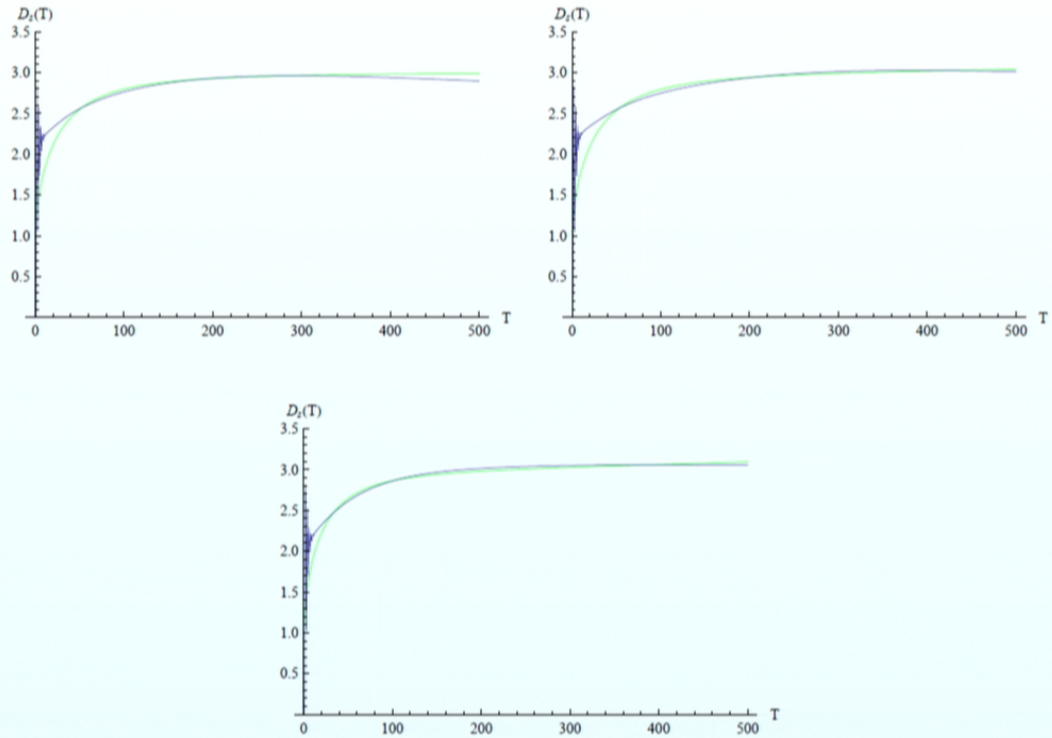
Best-fit values for CDT-data with N simplices:

N	g_0^{fit}	λ_0^{fit}	$(\Delta\mathcal{D}_s)^2$
70k	0.7×10^{-5}	7.5×10^{-5}	0.680
100k	8.8×10^{-5}	39.5×10^{-5}	0.318
200k	13×10^{-5}	61×10^{-5}	0.257

– p. 36/4

Spectral dimension: comparison

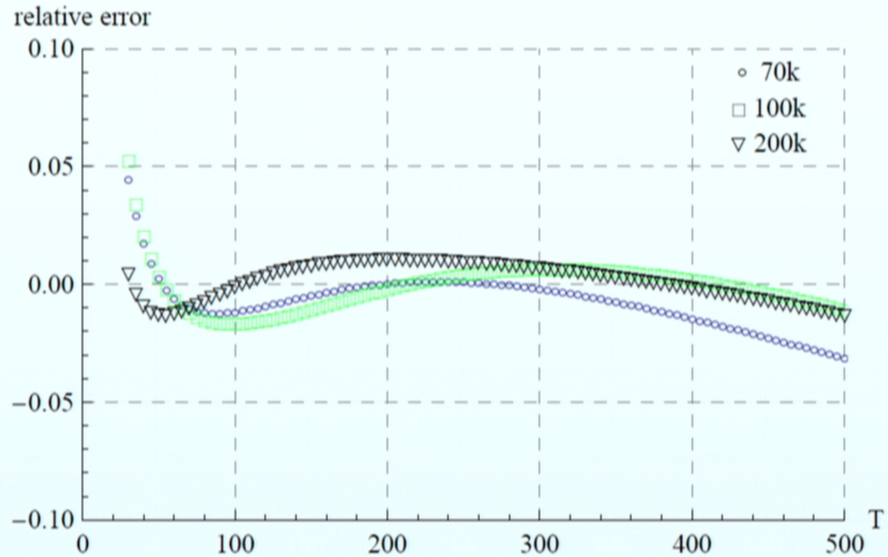


- $\mathcal{D}_s^{\mathrm{QEG}}(T; g_0^{\mathrm{fit}}, \lambda_0^{\mathrm{fit}})$
- $\mathcal{D}_s^{\mathrm{CDT}}(T)$ for $N = 70k, 100k, 200k$-simplices

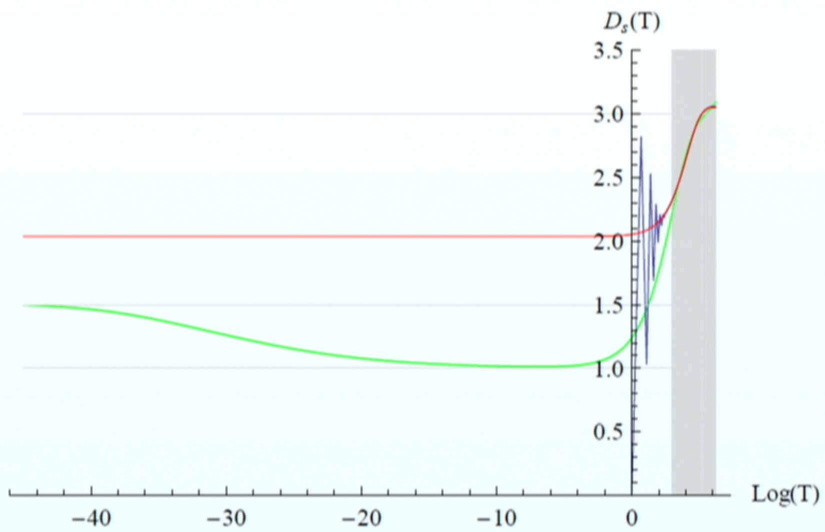
Spectral dimension: fit-quality

relative error captured by residuals:

$$\epsilon \equiv -\frac{\mathcal{D}_s^{\text{QEG}}(T; g_0^{\text{fit}}, \lambda_0^{\text{fit}}) - \mathcal{D}_s^{\text{CDT}}(T)}{\mathcal{D}_s^{\text{QEG}}(T; g_0^{\text{fit}}, \lambda_0^{\text{fit}})}$$



Comparing spectral dimensions in $d = 3$



- CDT and QEG agree with data within 1% accuracy
- no data-points on the semi-classical and NGFP-plateau

resolves puzzle between CDT data and QEG prediction!

dimensional flow

higher-derivative corrections

– p. 40/4

RG-flow of the R^2-truncation

[O. Lauscher, M. Reuter, '02]

Includes "leading" higher-derivative corrections to Einstein-Hilbert truncation

$$\Gamma_k^{\text{grav}}[g] = \int d^4x \sqrt{g} \left[\frac{1}{16\pi G_k} (-R + 2\Lambda_k) + \frac{1}{b_k} R^2 \right]$$

Conclusions . . .

spectral, walk and Hausdorff dimensions

- allow to compare quantum structure of space-time in different QG-models

QEG space-times carry multifractal structure

- classical regime: $\mathcal{D}_s(T) = 4$
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- fixed-point regime: $\mathcal{D}_s(T) = 2$

connection to Causal Dynamical Triangulations

- CDT-data agrees with QEG prediction to one percent accuracy
- no CDT-data probing semi-classical and fixed-point regime

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Conclusions and open questions

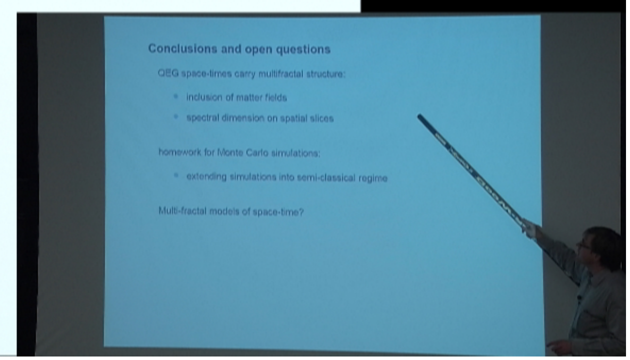
QEG space-times carry multifractal structure:

- inclusion of matter fields
- spectral dimension on spatial slices

homework for Monte Carlo simulations:

- extending simulations into semi-classical regime

Multi-fractal models of space-time?



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Multi-fractal models of space-time?

!!! WORK AHEAD !!!

Diffusion processes on QEG space-times

[O. Lauscher, M. Reuter, JHEP 10 (2005) 050]

basic idea: replace classical return probability by expectation value:

$$P(T) \equiv \langle P_\gamma(T) \rangle \equiv \int \mathcal{D}\gamma \mathcal{D}C \mathcal{D}\bar{C}\, P_\gamma(T)\, e^{-S_{\text{bare}}[\gamma, C, \bar{C}]}$$

average action $\Gamma_k[g]$: description of physics at scale k

- $\langle g \rangle_k$: approximate $\Gamma_k[g]$ by Einstein-Hilbert truncation

$$R_{\mu\nu}(\langle g \rangle_k) = \frac{2}{2-d}\, \Lambda_k \, \langle g_{\mu\nu} \rangle_k$$

- scaling relation between metrics at different scales k:

$$\langle g_{\mu\nu}(x) \rangle_k = [\Lambda_{k_0}/\Lambda_k] \langle g_{\mu\nu}(x) \rangle_{k_0}$$

- spectrum of Laplace-operators at different scales

$$\Delta(k) = \left[\Lambda_k/\Lambda_{k_0}\right] \Delta(k_0)$$

Compute the spectral dimension $\mathcal{D}_s(T)$

[O. Lauscher, M. Reuter, JHEP 10 (2005) 050]

1. spectrum of Laplace-operators at different scales

$$\Delta(k) = \left[\Lambda_k / \Lambda_{k_0}\right] \Delta(k_0)$$

2. solve the k-dependent heat equation

$$\partial_T K(x, x'; T) = -\Delta(k) K(x, x'; T)$$

- assume Λ_{k_0} small $\Longrightarrow$ flat-space approximation

$$K(x, x'; T) = \int \frac{d^d p}{(2\pi)^d} \, e^{ip\cdot(x-x')} \, e^{-p^2 F(p^2) T}\,, \qquad F(p^2) = \Lambda(p)/\Lambda(k_0)$$

3. quantum return probability

$$P(T) = \int \frac{d^d p}{(2\pi)^d} \, e^{-p^2 F(p^2) T}$$

4. spectral dimension for scaling cosmological constant: $\Lambda_k \propto k^\delta$:

$$\boxed{\mathcal{D}_s(T) = \tfrac{2d}{2+\delta}}$$

– p. 22/4