Title: Gravitational Physics (Review) - Lecture 8

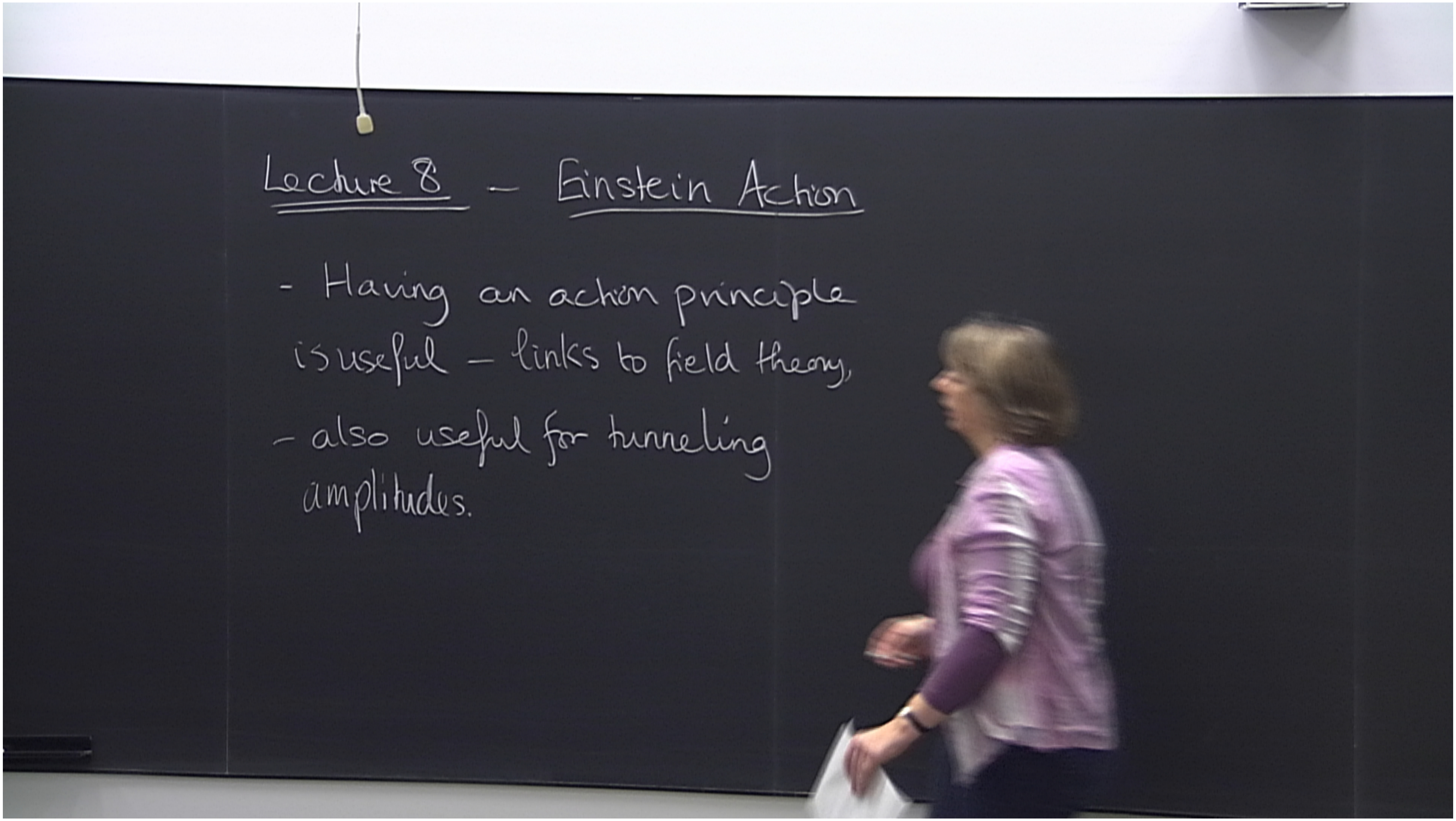
Date: Jan 31, 2012 09:00 AM

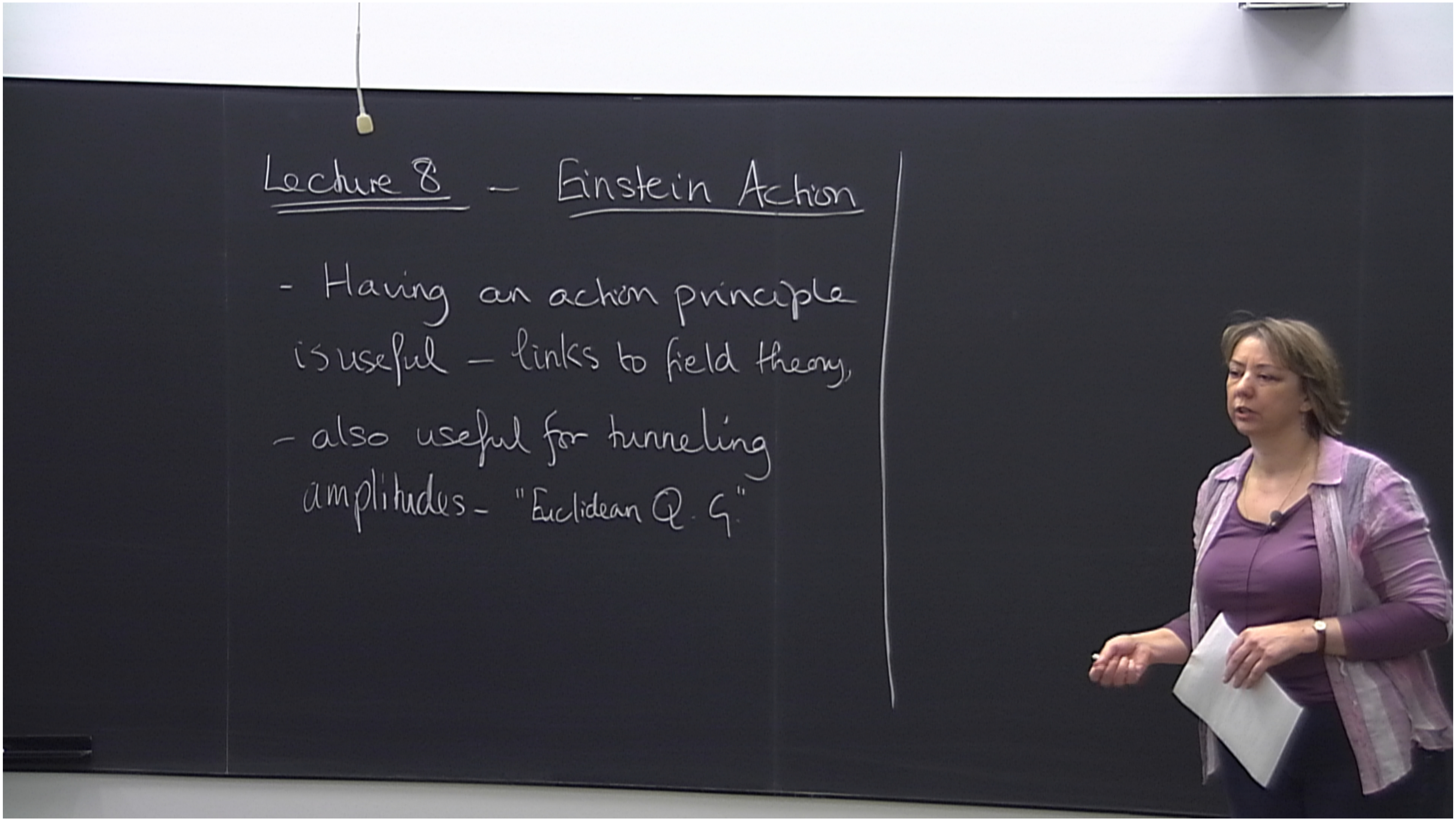
URL: http://pirsa.org/12020010

Abstract:

Lecture 8 – Einstein Action

- Having an action principle is useful – links to field theory,


Lecture 8 — Einstein Action
- Having an action principle is useful — links to field theory,
- also useful for tunneling amplitudes.


Lecture 8 – Einstein Action
- Having an action principle is useful – links to field theory,
- also useful for tunneling amplitudes – "Euclidean Q. G."

[Le]cture 8 – Einstein Action

Having an action principle [i]s useful – links to field theory, [a]lso useful for tunneling [a]mplitudes – "Euclidean Q. G"

Recall, under a g.c.t

$$d^4x \to \det\left(\frac{\partial x}{\partial \tilde{x}}\right) d^4\tilde{x}$$

cture 8 – Einstein Action

Having an action principle
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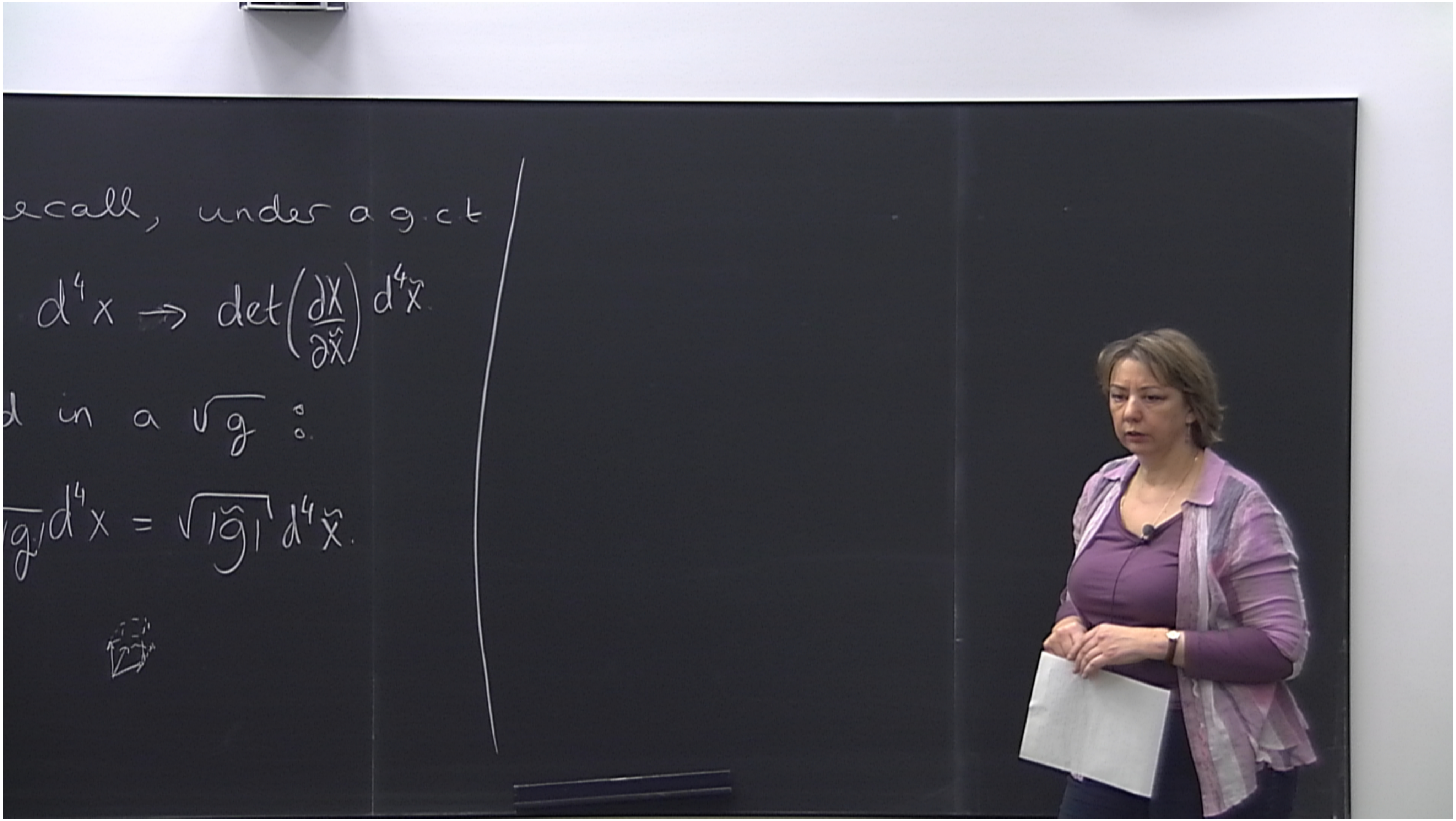
Recall, under a g.c.t

$$d^4x \to \det\left(\frac{\partial X}{\partial \tilde{X}}\right) d^4\tilde{X}$$

Add in a $\sqrt{g}$:

$$\sqrt{|g|}\, d^4x = \sqrt{|\tilde{g}|}\, d^4\tilde{x}.$$

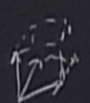

ecall, under a g.c.t
d⁴x → det(∂X/∂X̃) d⁴X̃
d in a √g :
√g d⁴x = √|g̃| d⁴x̃.

ecall, under a g.c.t

$$d^4x \to \det\left(\frac{\partial x}{\partial \tilde{x}}\right) d^4\tilde{x}$$

d in a $\sqrt{g}$:

$$\sqrt{|g|}\, d^4x = \sqrt{|\tilde{g}|}\, d^4\tilde{x}.$$



In gravity, we vary w.r.t g_{ab}.

To get $\delta\sqrt{|-g|}$ use a useful identity.

$$\det M = \exp \operatorname{tr} \log M$$

$$\Rightarrow \quad \delta \det M = (\det M)\operatorname{tr}(M^{-1}\delta M)$$

ecall, under a g.c.t

$$d^4x \Rightarrow \det\left(\frac{\partial X}{\partial \tilde{X}}\right) d^4\tilde{x}$$

d in a $\sqrt{g}$:

$$\sqrt{|g|}\, d^4x = \sqrt{|\tilde{g}|}\, d^4\tilde{x}.$$

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$$\delta \det g = \det g \; g^{ab}\, \delta g_{ab}.$$

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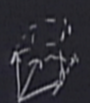


In gravity, we vary w.r.t $\underline{g_{ab}}$.

To get $\delta\sqrt{|-g|}$ use a useful identity.

$$\det M = \exp \operatorname{tr} \log M$$

$$\Rightarrow \quad \delta \det M = (\det M)\operatorname{tr}(M^{-1}\delta M)$$

$$\delta \det g = \det g \; g^{ab}\delta g_{ab}.$$

Thus $\quad \partial_a \sqrt{-g} = \frac{1}{2} \frac{1}{\sqrt{-g}} \cdot -g \cdot g^{cd} g_{cd,a}$

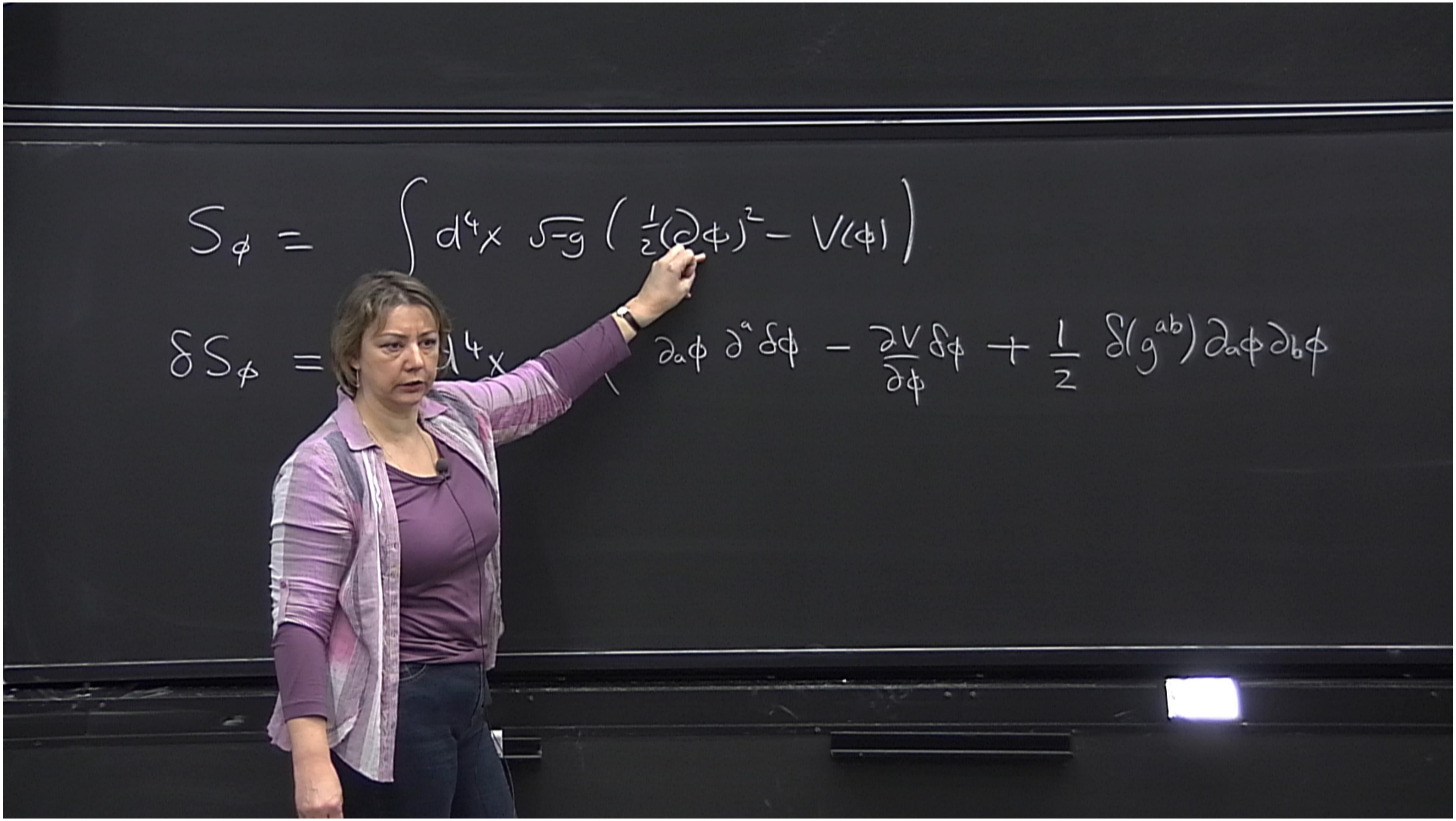
$$= \sqrt{-g}\, \Gamma^{d}_{a\,d}$$

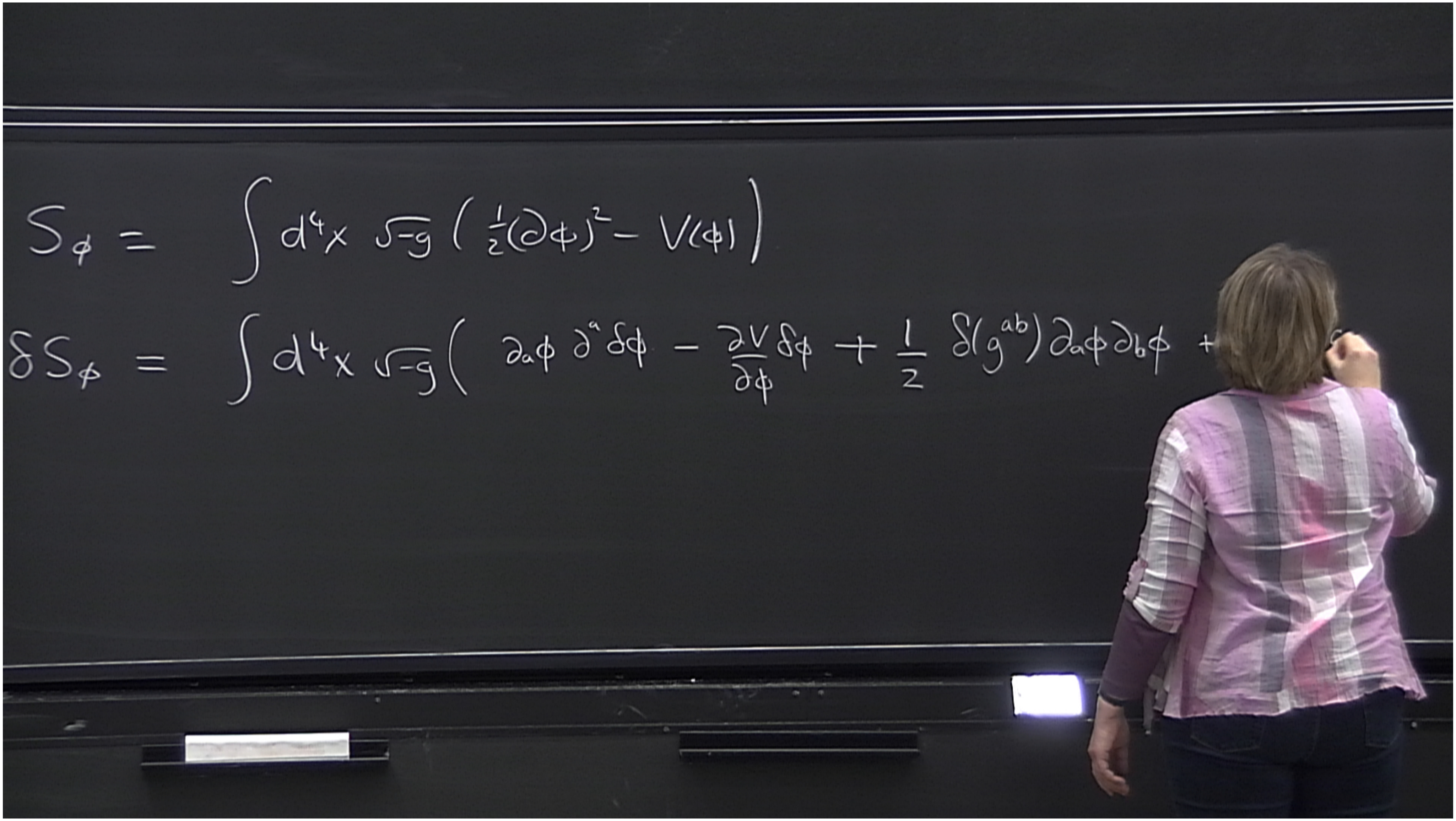
Thus $\partial_a \sqrt{-g} = \frac{1}{2} \frac{1}{\sqrt{-g}} \cdot -g \cdot g^{cd} g_{cd,a}$

$$= \sqrt{-g}\, \Gamma^{d}_{ad}$$

Allows simple expressions in eqns of motion.

e.g. $\Box \varphi = \frac{1}{\sqrt{g}} \partial_a \left(g^{ab} \sqrt{g}\, \partial_b \phi \right)$


$S_\phi = \int d^4x \sqrt{-g} \left(\frac{1}{2}(\partial\phi)^2 - V(\phi) \right)$
$\delta S_\phi = \int d^4x \; \partial_a\phi \, \partial^a \delta\phi - \frac{\partial V}{\partial \phi}\delta\phi + \frac{1}{2} \delta(g^{ab}) \partial_a\phi \, \partial_b\phi$


$S_\phi = \int d^4x \sqrt{-g} \left(\frac{1}{2}(\partial\phi)^2 - V(\phi) \right)$
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$$\phi = \int d^4x \sqrt{-g}\left(\frac{1}{2}(\partial\phi)^2 - V(\phi) \right)$$

$$\phi = \int d^4x \sqrt{-g}\left(\partial_a\phi\, \partial^a \delta\phi - \frac{\partial V}{\partial \phi}\delta\phi + \frac{1}{2}\, \delta(g^{ab})\, \partial_a\phi\, \partial_b\phi + \frac{1}{2} g^{cd}\delta g_{cd}\left(\frac{1}{2}(\partial\phi)^2 - V \right) \right)$$

$$\phi = \int d^4x \sqrt{-g} \left(\tfrac{1}{2}(\partial\phi)^2 - V(\phi) \right)$$

$$\phi = \int d^4x \sqrt{-g} \left(\partial_a\phi\, \partial^a \delta\phi - \frac{\partial V}{\partial \phi}\delta\phi + \frac{1}{2}\, \delta(g^{ab})\, \partial_a\phi\, \partial_b\phi + \frac{1}{2} g^{cd}\delta g_{cd} \left(\tfrac{1}{2}(\partial\phi)^2 - V \right) \right)$$

$$= \int d^4x \sqrt{-g} \left[-\left(\frac{1}{\sqrt{-g}} \partial_a \sqrt{-g}\, \partial^a\phi + \frac{\partial V}{\partial\phi} \right) \delta\phi + \underset{\uparrow \atop \delta(g^{-1})}{\delta g^{ab}} \left(\tfrac{1}{2}\phi_{,a}\phi_{,b} - \tfrac{1}{2}\left(\tfrac{1}{2}(\partial\phi)^2 - V \right) \right) \right] +$$

$$\frac{1}{2}\,\delta(g^{ab})\,\partial_a\phi\,\partial_b\phi \;+\; \frac{1}{2}\,g^{cd}\,\delta g_{cd}\left(\tfrac{1}{2}(\partial\phi)^2 - V\right)\Big)\Big]$$

$$+\;\underbrace{\delta g^{ab}}_{\delta(g^{-1})}\left(\tfrac{1}{2}\phi_{,a}\phi_{,b} - \tfrac{1}{2}\left(\tfrac{1}{2}(\partial\phi)^2 - V\right)\right)\Big] + \text{boundary } \delta\phi \text{ terms.}$$

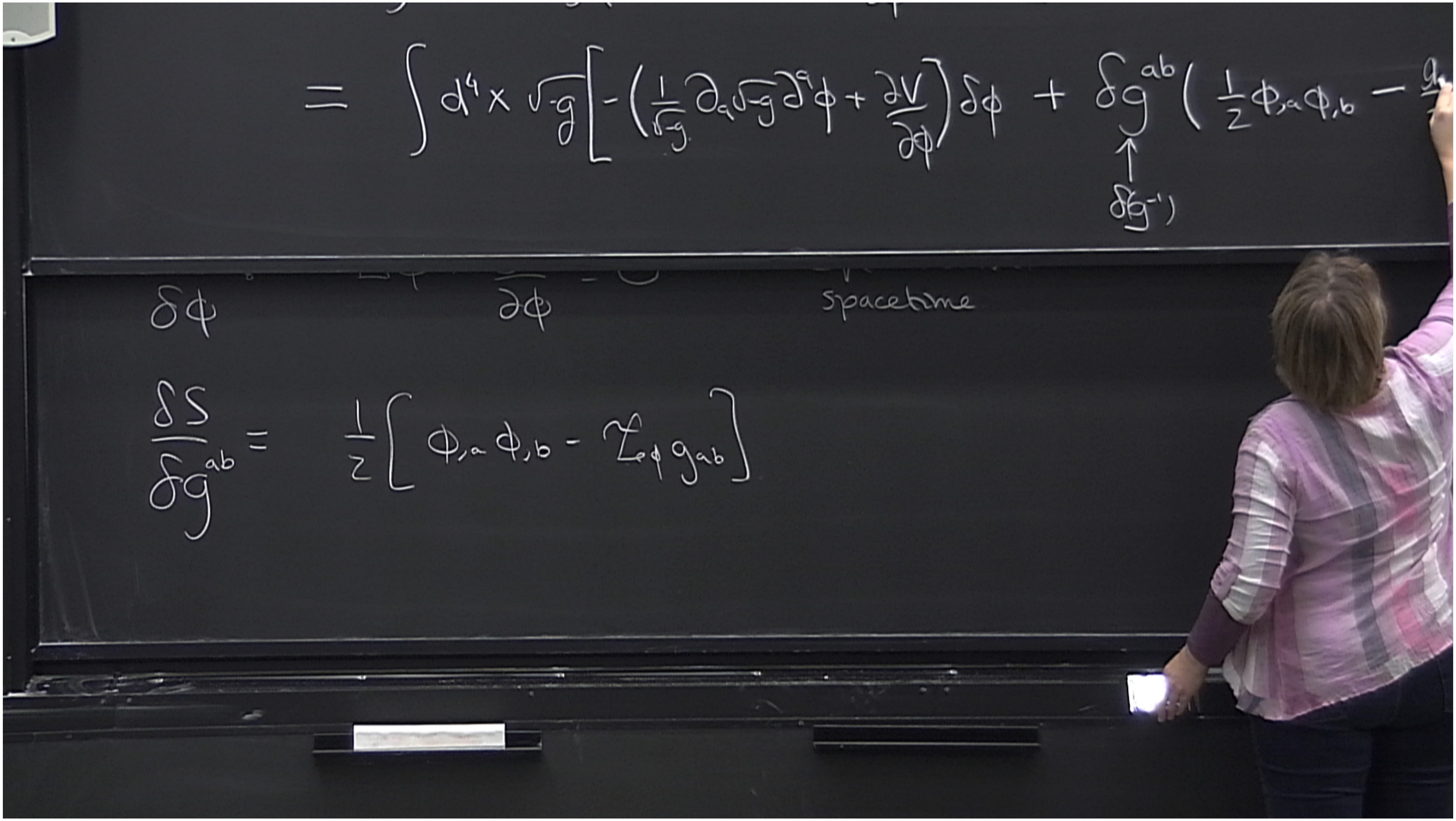
Here use

$$\delta(g^{-1}) = -g^{-1}\,\delta g\, g^{-1}$$

$$\frac{1}{2}\,\delta(g^{ab})\,\partial_a\phi\,\partial_b\phi + \frac{1}{2}\, g^{cd}\,\delta g_{cd}\left(\tfrac{1}{2}(\partial\phi)^2 - V\right)\Big)\Big]$$

Keep $\delta\phi|_{\partial M} = 0$

$$+\ \underset{\uparrow \atop \delta(g^{-1})}{\delta g^{ab}}\left(\tfrac{1}{2}\phi_{,a}\phi_{,b} - \tfrac{1}{2}\left(\tfrac{1}{2}(\partial\phi)^2 - V\right)\right)\Big] + \text{boundary } \delta\phi \text{ terms.}$$


$= \int d^4x \sqrt{-g} \left[-\left(\frac{1}{\sqrt{-g}} \partial_a \sqrt{-g}\, \partial^a \phi + \frac{\partial V}{\partial \phi} \right) \delta\phi + \delta g^{ab} \left(\frac{1}{2} \phi_{,a} \phi_{,b} - \right.\right.$
$\delta(g^{-1})$
$\delta\phi$
$\partial\phi$
spacetime
$\frac{\delta S}{\delta g^{ab}} = \frac{1}{2} \left[\phi_{,a} \phi_{,b} - \mathcal{L}_\phi g_{ab} \right]$

$$= \int d^4x \sqrt{-g} \left[-\left(\frac{1}{\sqrt{-g}} \partial_a \sqrt{-g} \, \partial \phi + \frac{\partial V}{\partial \phi}\right) \delta\phi + \underset{\uparrow \ \delta(g^{-1})}{\delta g} \left(\tfrac{1}{2} \phi_{,a} \phi_{,b} \right. \right. \quad 2$$

$$\frac{\delta S}{\delta \phi} : \quad \Box \phi + \frac{\partial V}{\partial \phi} = 0$$

Scalar wave eqn on curved spacetime

$$\frac{\delta S}{\delta g^{ab}} = \frac{1}{2} \left[\phi_{,a} \phi_{,b} - \mathcal{L}_\phi \, g_{ab} \right]$$

$$= \int d^4x \sqrt{-g}\left[-\left(\frac{1}{\sqrt{-g}}\partial_a\sqrt{-g}\,\partial\phi + \frac{\partial V}{\partial\phi}\right)\delta\phi + \underset{\delta(g^{-1})}{\delta g}\left(\frac{1}{2}\phi_{,a}\phi_{,b} \right.\right.$$

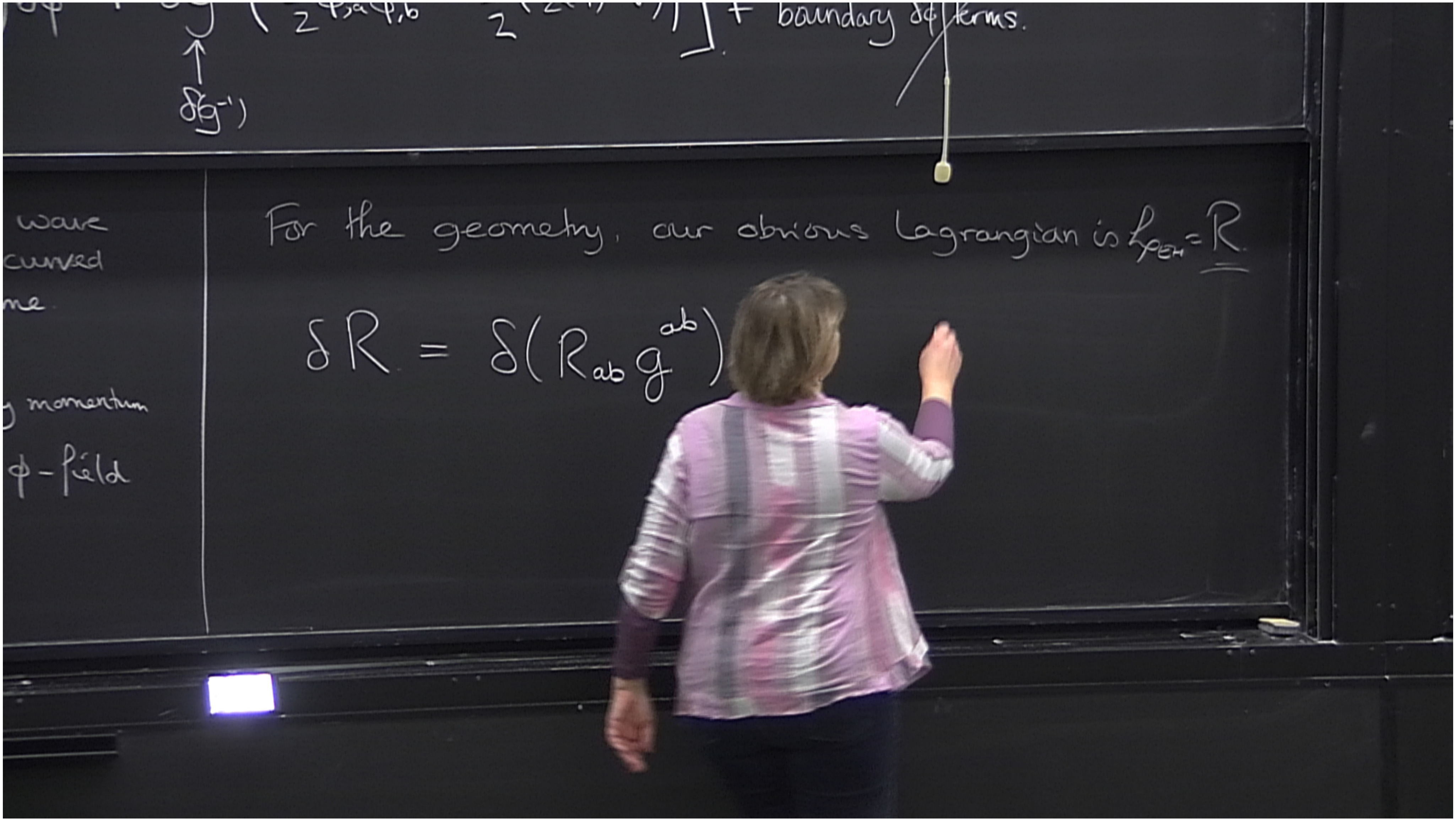
$$\frac{\delta S}{\delta\phi} : \quad \Box\phi + \frac{\partial V}{\partial\phi} = 0$$

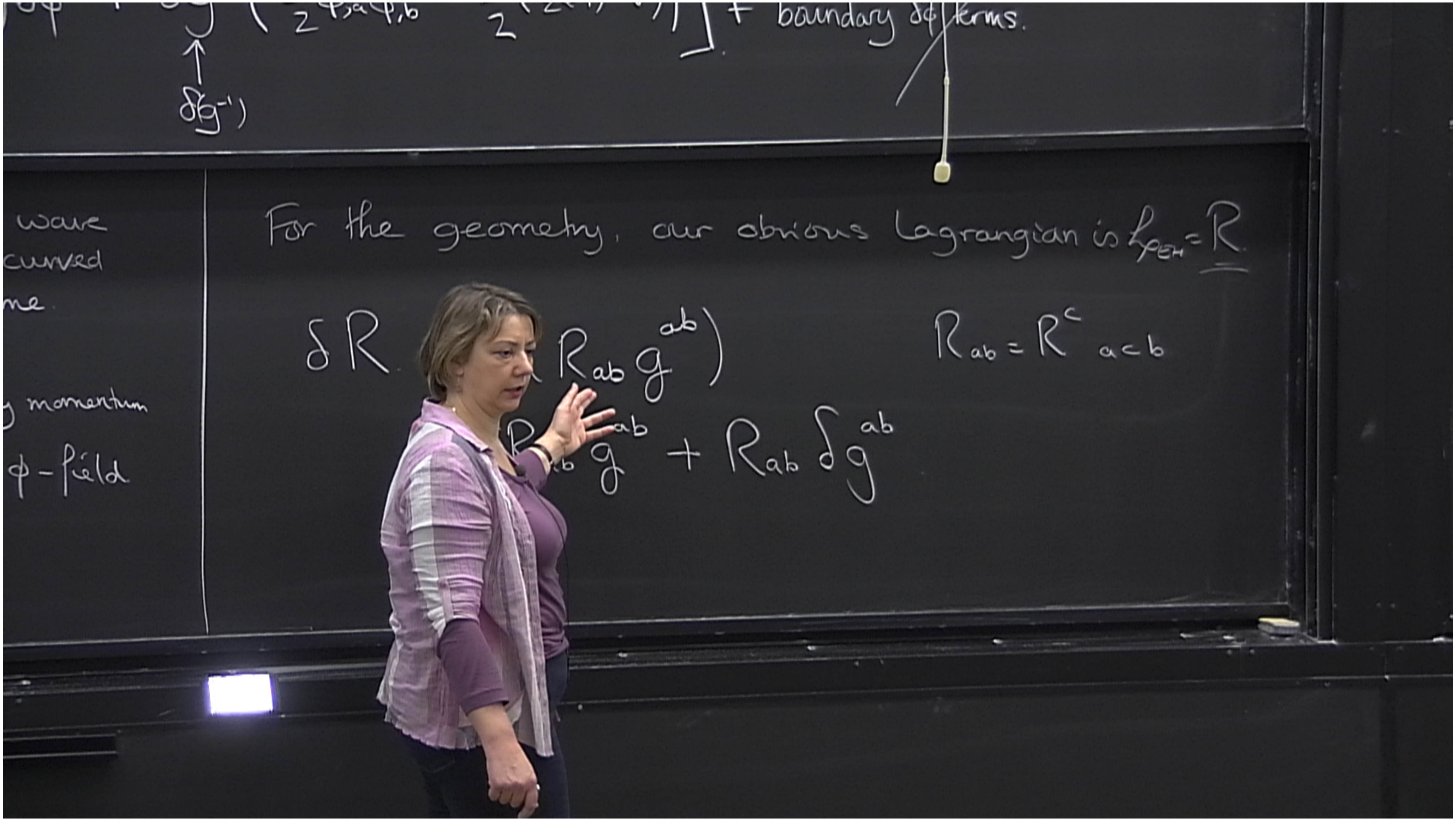
Scalar wave eqn on curved spacetime

$$\frac{\delta S}{\delta g^{ab}} = \frac{1}{2}\left[\phi_{,a}\phi_{,b} - \mathcal{L}_\phi g_{ab}\right]$$

$$= \frac{1}{2} T_{ab}$$

Energy momentum of ϕ-field


+ boundary δφ terms.
δ(g⁻¹)
wave
curved
me.
momentum
φ - field
For the geometry, our obvious Lagrangian is $\mathcal{L}_{EH} = R$.
$\delta R = \delta(R_{ab} g^{ab})$



boundary δφ terms.
δ(g⁻¹)
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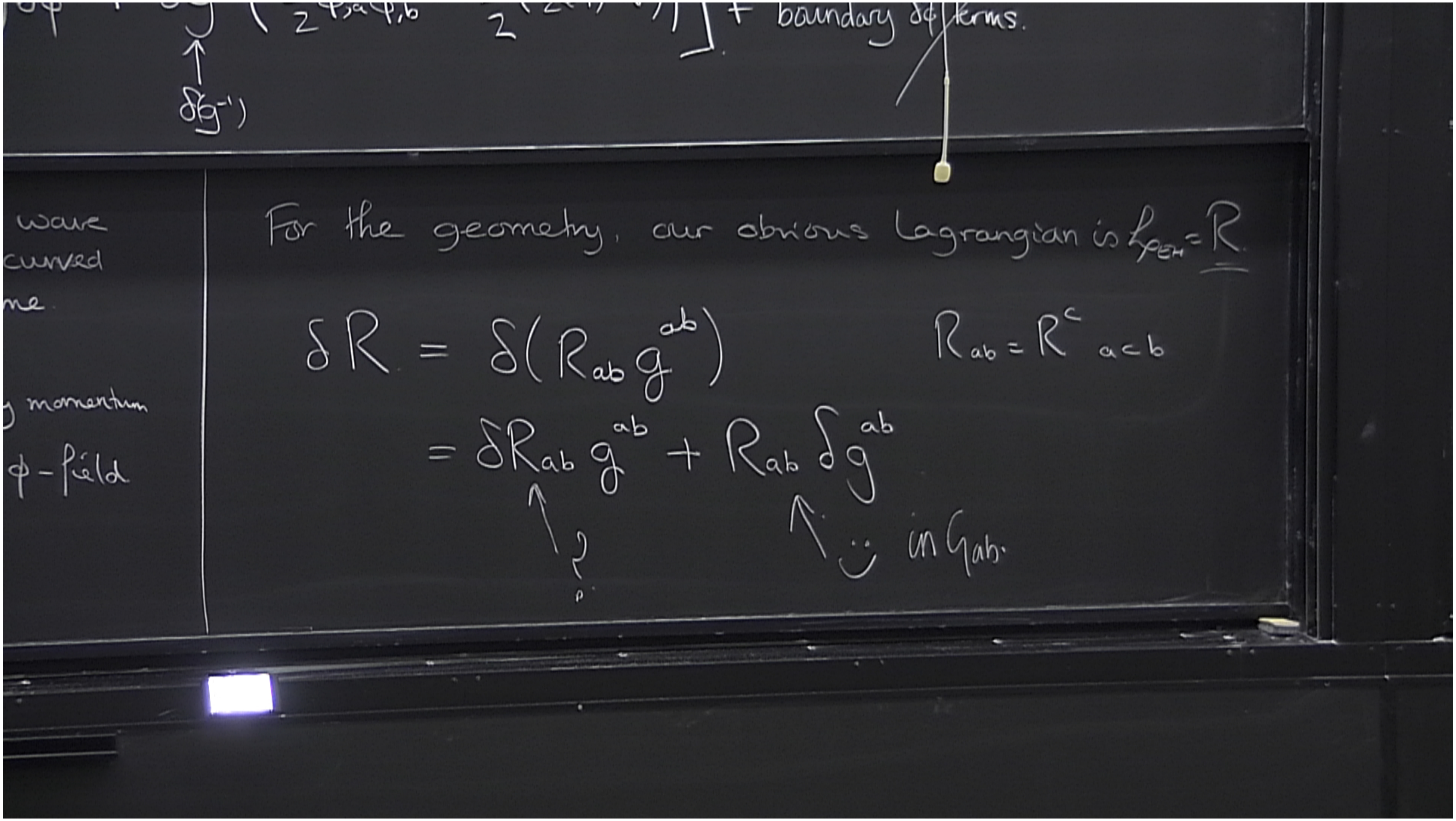
For the geometry, our obvious Lagrangian is L_EH = R.
δR
R_ab g^ab)
R_ab = R^c_acb
+ R_ab δg^ab


+ boundary $\delta\phi$ terms.
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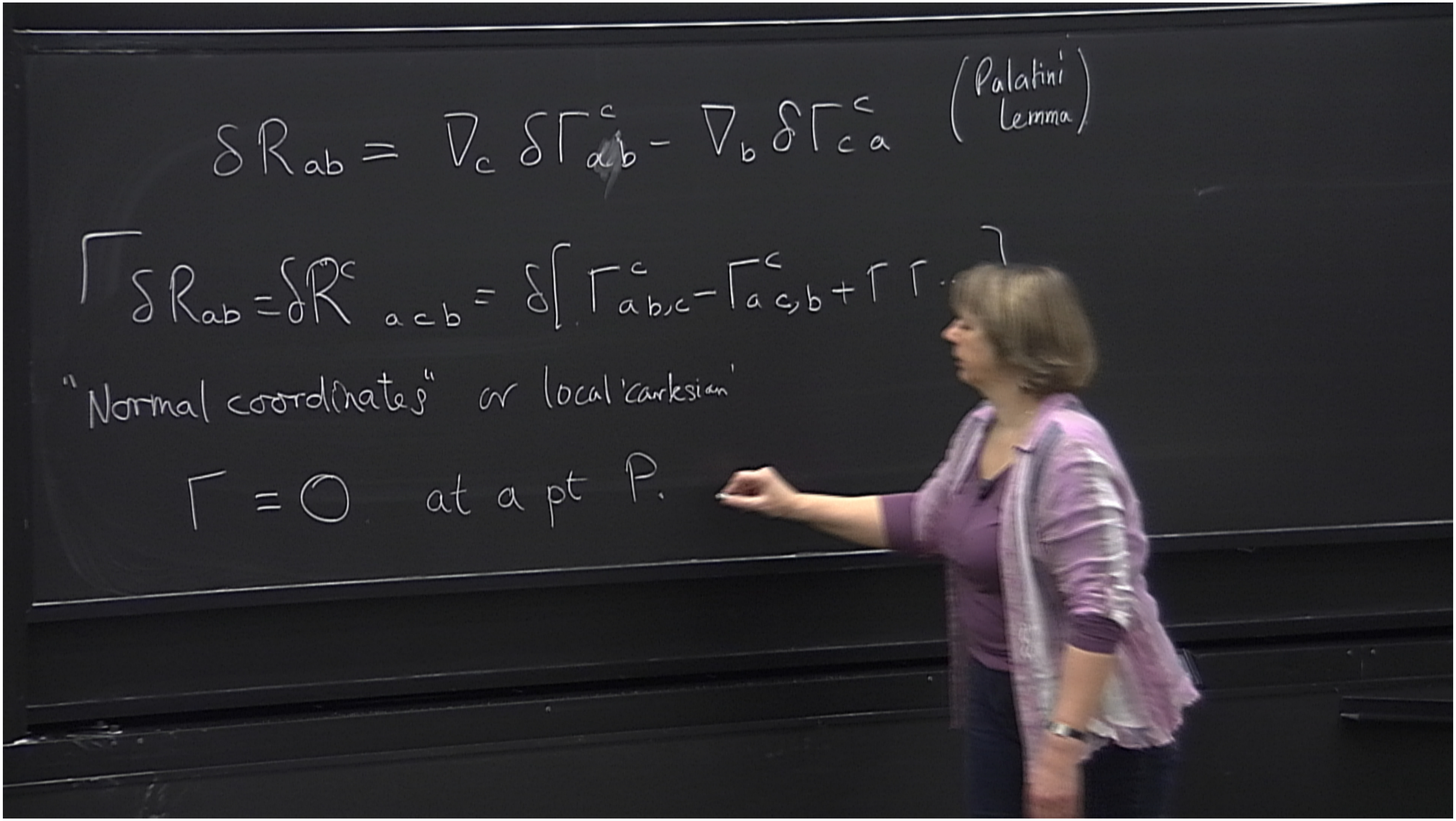
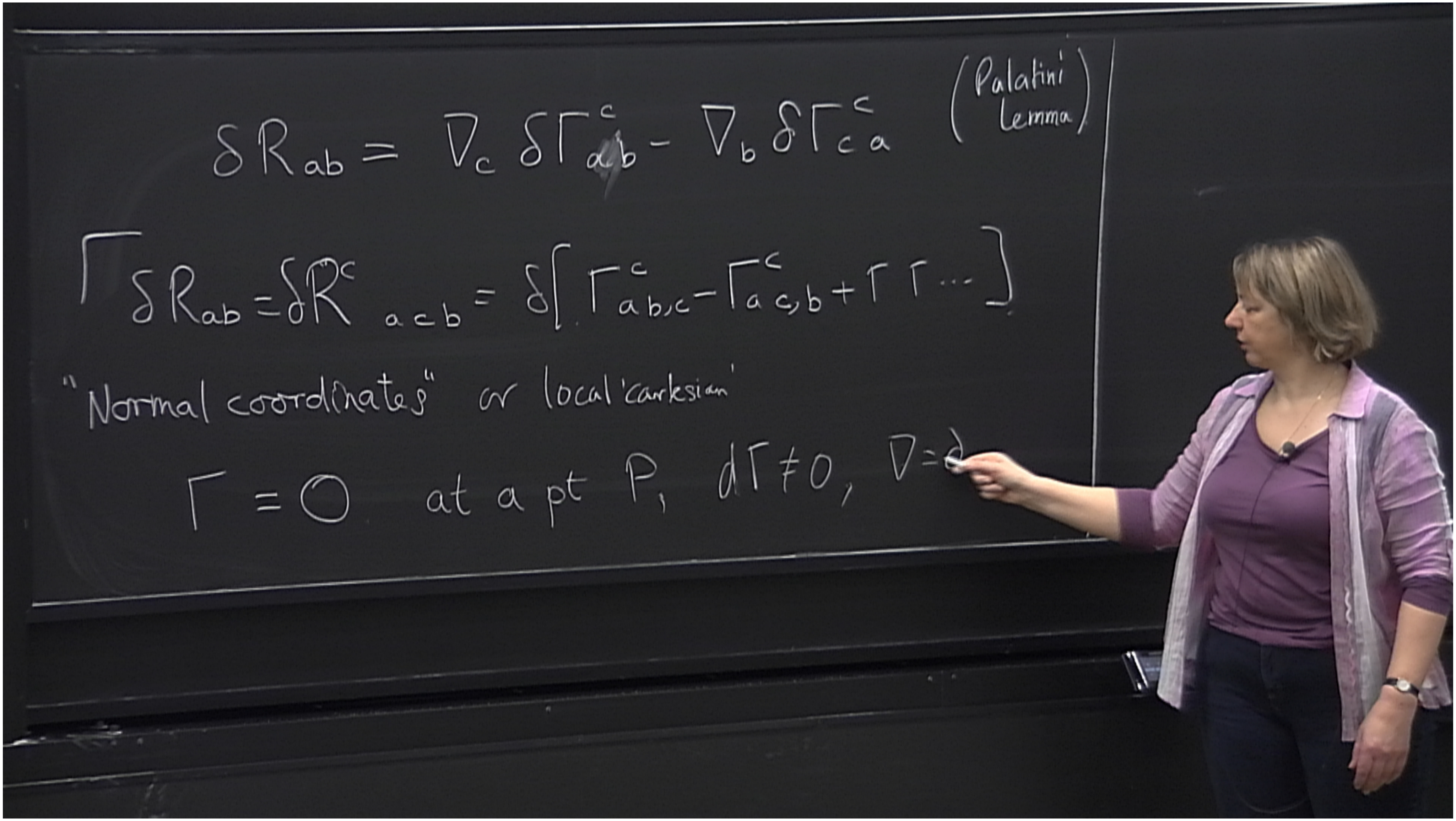
For the geometry, our obvious Lagrangian is $\mathcal{L}_{EH} = R$.
$\delta R = \delta(R_{ab} g^{ab})$
$R_{ab} = R^c{}_{acb}$
$= \delta R_{ab} g^{ab} + R_{ab} \delta g^{ab}$
?
in G_{ab}.

$$\delta R_{ab} = \nabla_c \delta \Gamma^c_{ab} - \nabla_b \delta \Gamma^c_{ca} \quad \text{(Palatini lemma)}$$

$$\delta R_{ab} = \delta R^c{}_{acb} = \delta\left[\Gamma^c_{ab,c} - \Gamma^c_{ac,b} + \Gamma\Gamma\ldots\right]$$

"Normal coordinates"


$\delta R_{ab} = \nabla_c \delta\Gamma^c_{ab} - \nabla_b \delta\Gamma^c_{ca}$ (Palatini lemma)
Γ $\delta R_{ab} = \delta R^c{}_{acb} = \delta[\Gamma^c_{ab,c} - \Gamma^c_{ac,b} + \Gamma\Gamma \ldots]$
"Normal coordinates" or local 'cartesian'
$\Gamma = 0$ at a pt P.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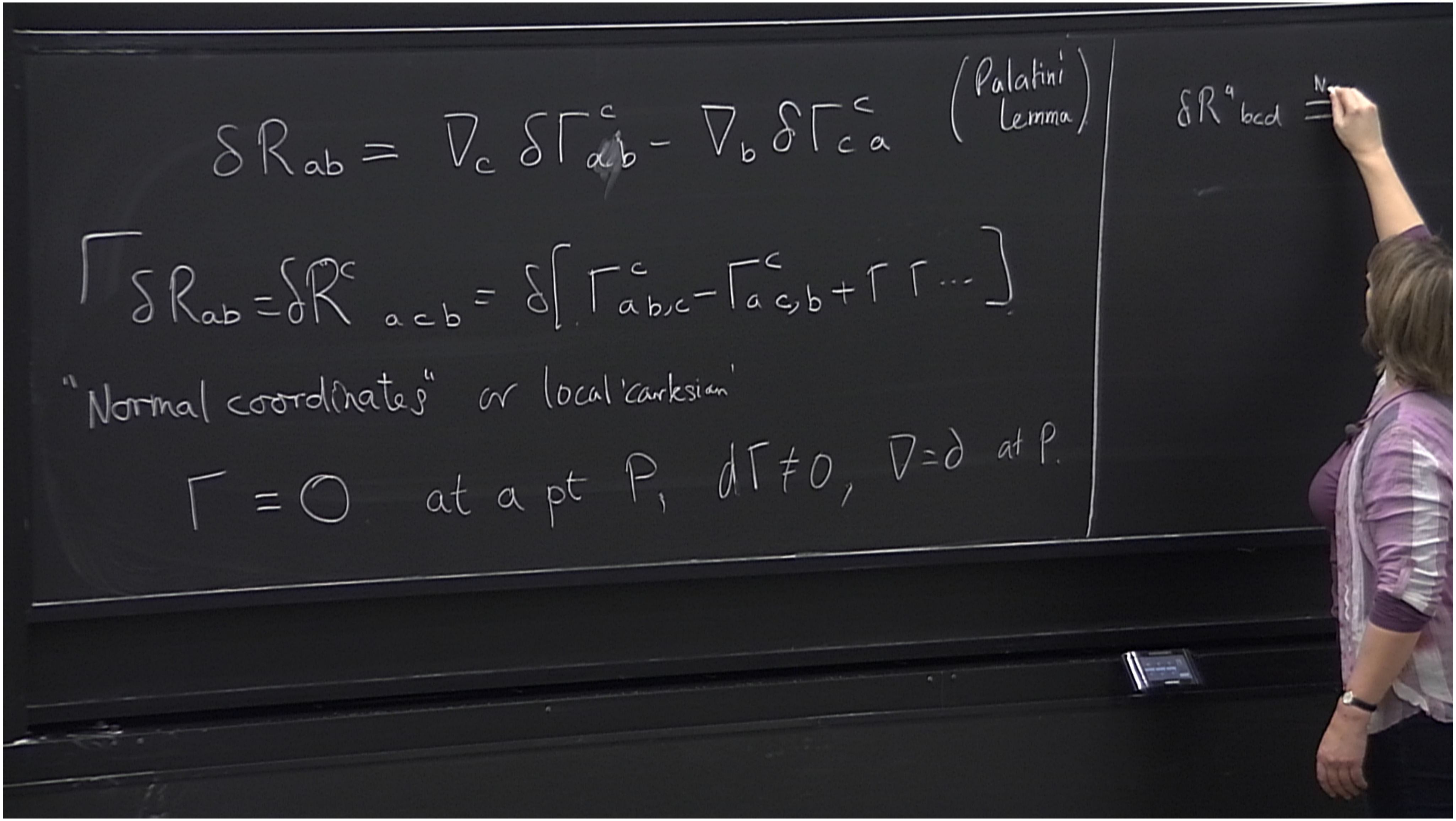
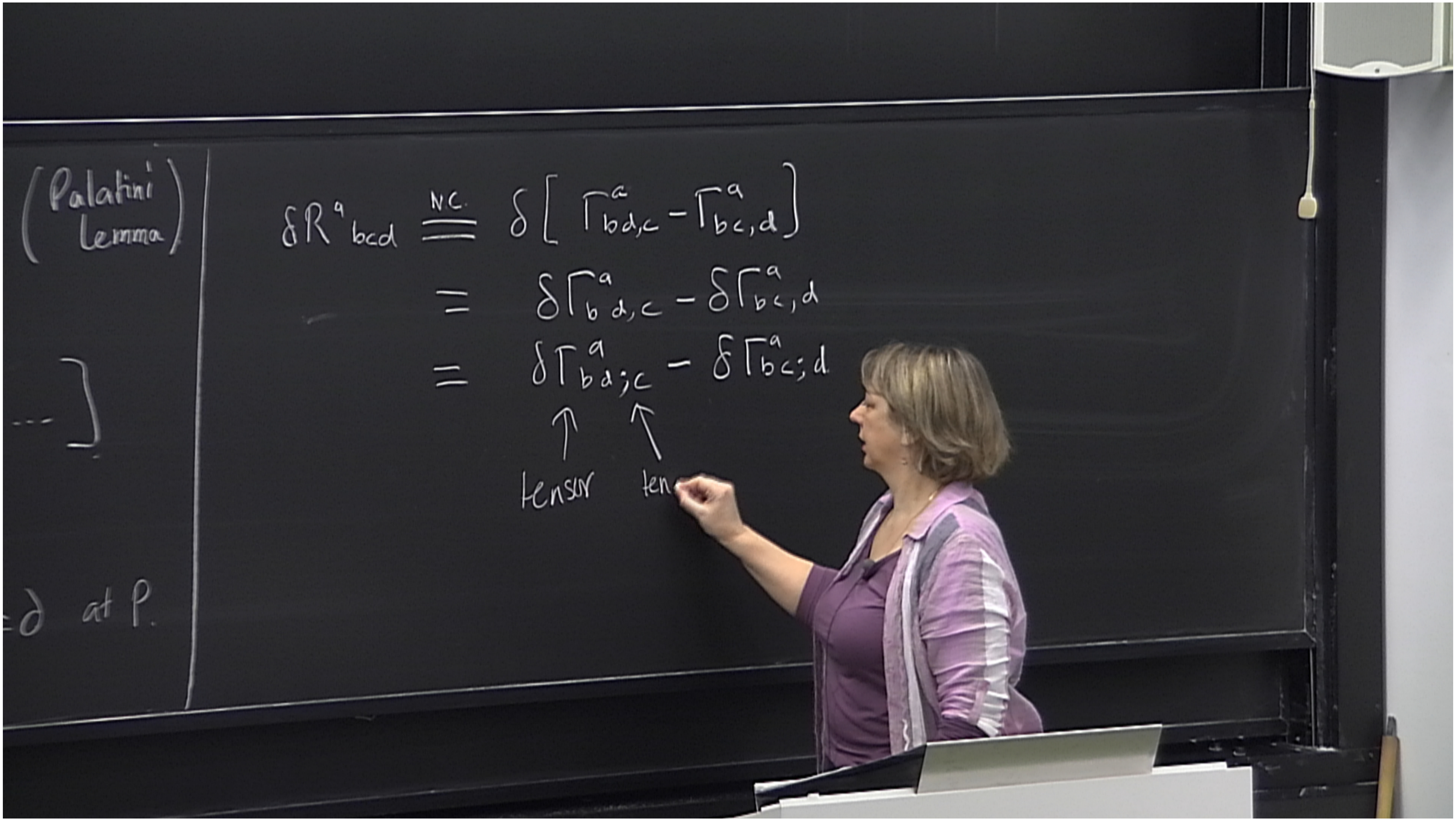
"Normal coordinates" or local 'cartesian'
$\Gamma = 0$ at a pt P, $d\Gamma \neq 0$, $\nabla = \partial$


$\delta R_{ab} = \nabla_c \delta \Gamma^c_{ab} - \nabla_b \delta \Gamma^c_{ca}$
(Palatini Lemma)
$\delta R_{ab} = \delta R^c{}_{acb} = \delta[\Gamma^c_{ab,c} - \Gamma^c_{ac,b} + \Gamma\Gamma \dots]$
"Normal coordinates" or local 'cartesian'
$\Gamma = 0$ at a pt P, $\partial\Gamma \neq 0$, $\nabla = \partial$ at P.
$\delta R^a{}_{bcd}$


(Palatini Lemma)
$\delta R^a{}_{bcd} \overset{N.C.}{=} \delta[\Gamma^a_{bd,c} - \Gamma^a_{bc,d}]$
$= \delta\Gamma^a_{bd,c} - \delta\Gamma^a_{bc,d}$
$= \delta\Gamma^a_{bd;c} - \delta\Gamma^a_{bc;d}$
tensor
ten
...]
=∂ at P.

(Palatini Lemma)

$$\delta R^a{}_{bcd} \overset{\text{N.C.}}{=\!=} \delta\left[\Gamma^a_{bd,c} - \Gamma^a_{bc,d}\right] \quad (\text{NC})$$

$$= \delta\Gamma^a_{bd,c} - \delta\Gamma^a_{bc,d} \quad (\text{NC})$$

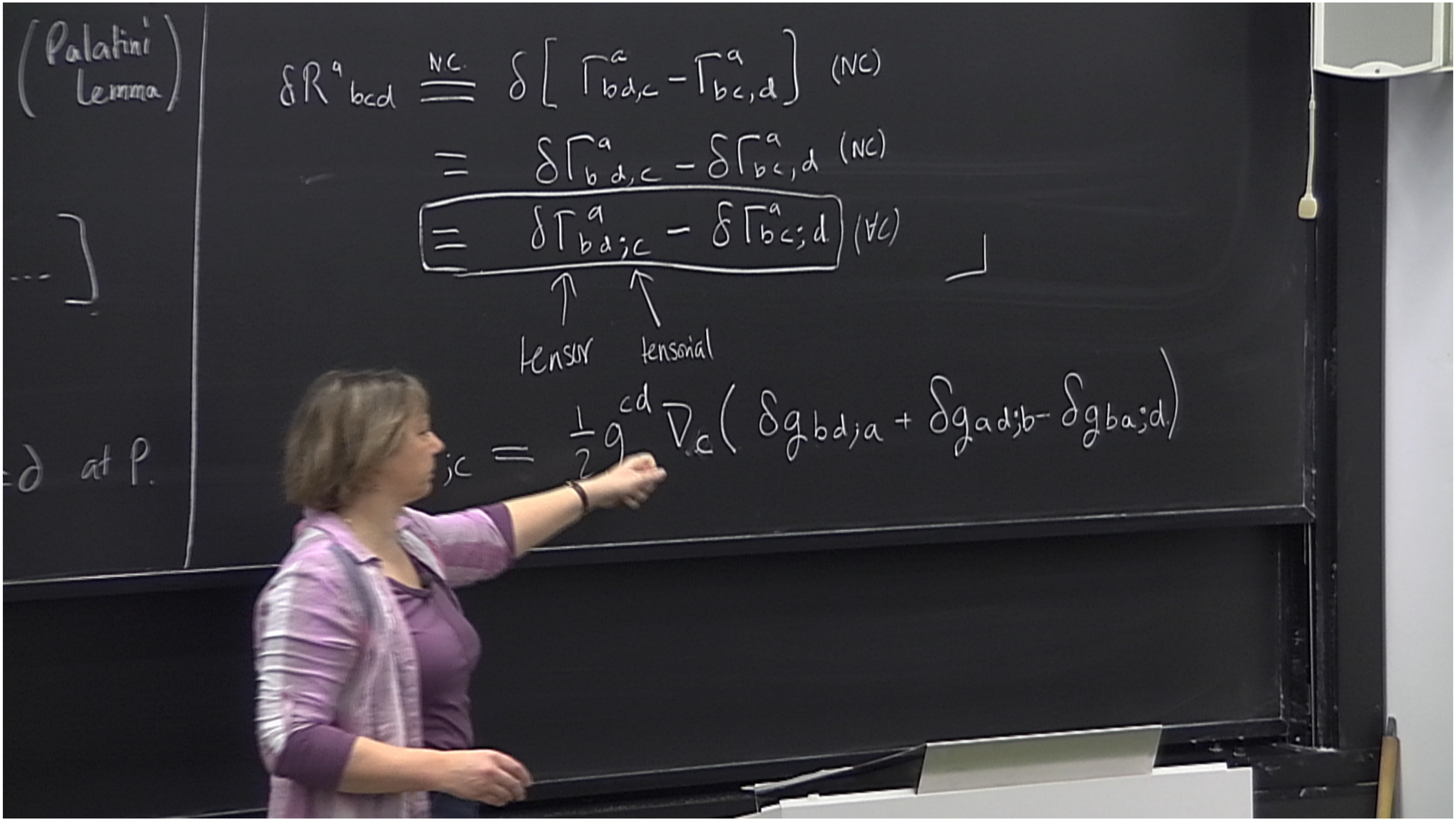
$$\boxed{= \delta\Gamma^a_{bd;c} - \delta\Gamma^a_{bc;d}} \quad (\forall C)$$

tensor tensorial

$\dots]$

$= \partial$ at P.

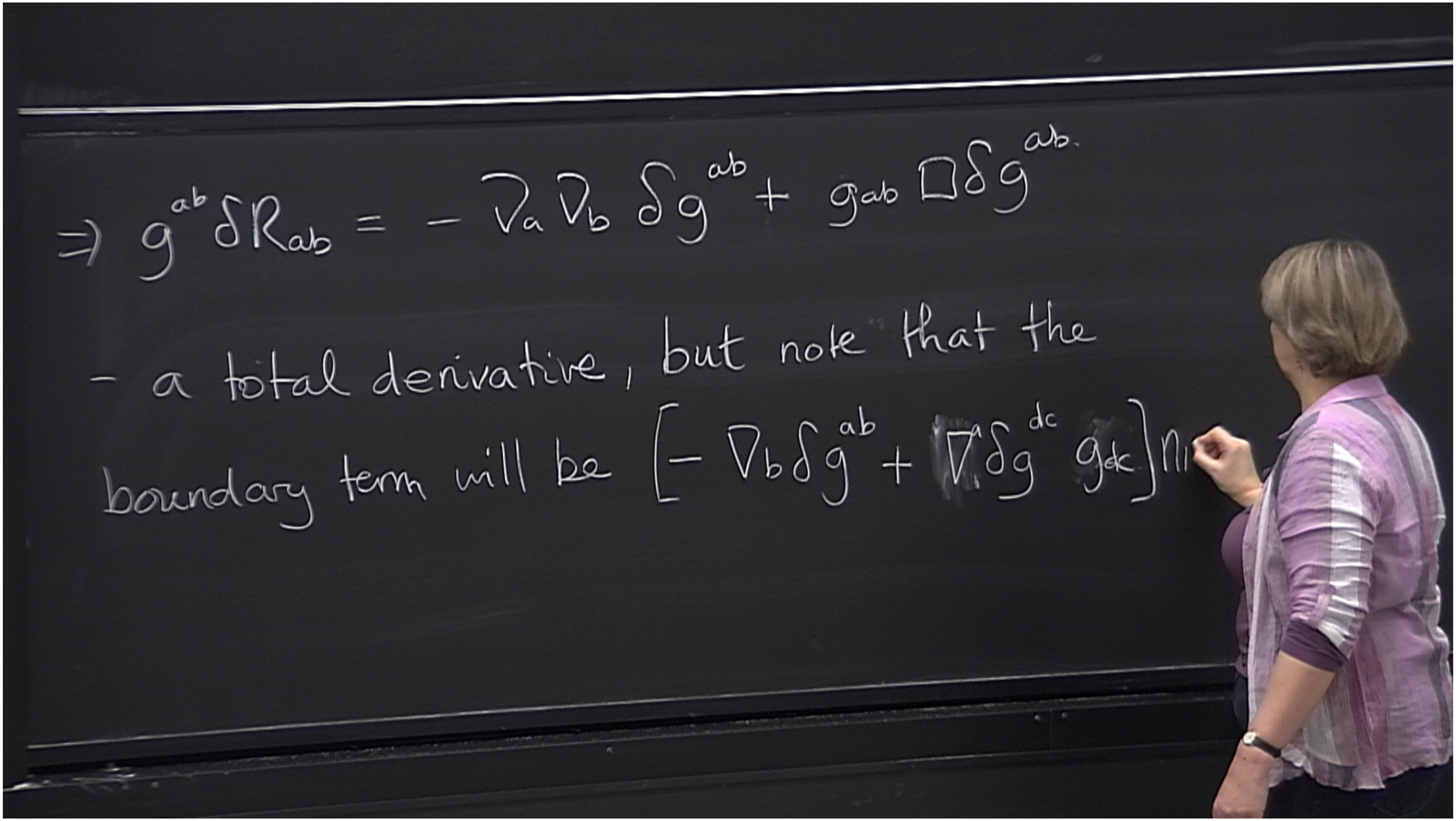
C


(Palatini lemma)
$\delta R^a{}_{bcd} \overset{N.C.}{=\!=} \delta\left[\Gamma^a_{bd,c} - \Gamma^a_{bc,d}\right]$ (NC)
$= \delta\Gamma^a_{bd,c} - \delta\Gamma^a_{bc,d}$ (NC)
$\boxed{= \delta\Gamma^a_{bd;c} - \delta\Gamma^a_{bc;d}}$ ($\forall$C)
tensor
tensorial
$;c = \frac{1}{2} g^{cd} \nabla_c \left(\delta g_{bd;a} + \delta g_{ad;b} - \delta g_{ba;d} \right)$
at P.

$$\Rightarrow g^{ab}\delta R_{ab} = -\nabla_a \nabla_b \delta g^{ab} + g_{ab} \Box \delta g^{ab}.$$

$$\Rightarrow g^{ab}\delta R_{ab} = -\nabla_a \nabla_b \delta g^{ab} + g_{ab} \Box \delta g^{ab}.$$

- a total derivative



$\Rightarrow g^{ab}\,\delta R_{ab} = -\nabla_a \nabla_b\, \delta g^{ab} + g_{ab}\,\Box\, \delta g^{ab}.$
− a total derivative, but note that the
boundary term will be $\left[-\nabla_b\, \delta g^{ab} + \nabla^a \delta g^{dc}\, g_{dc}\right] n$

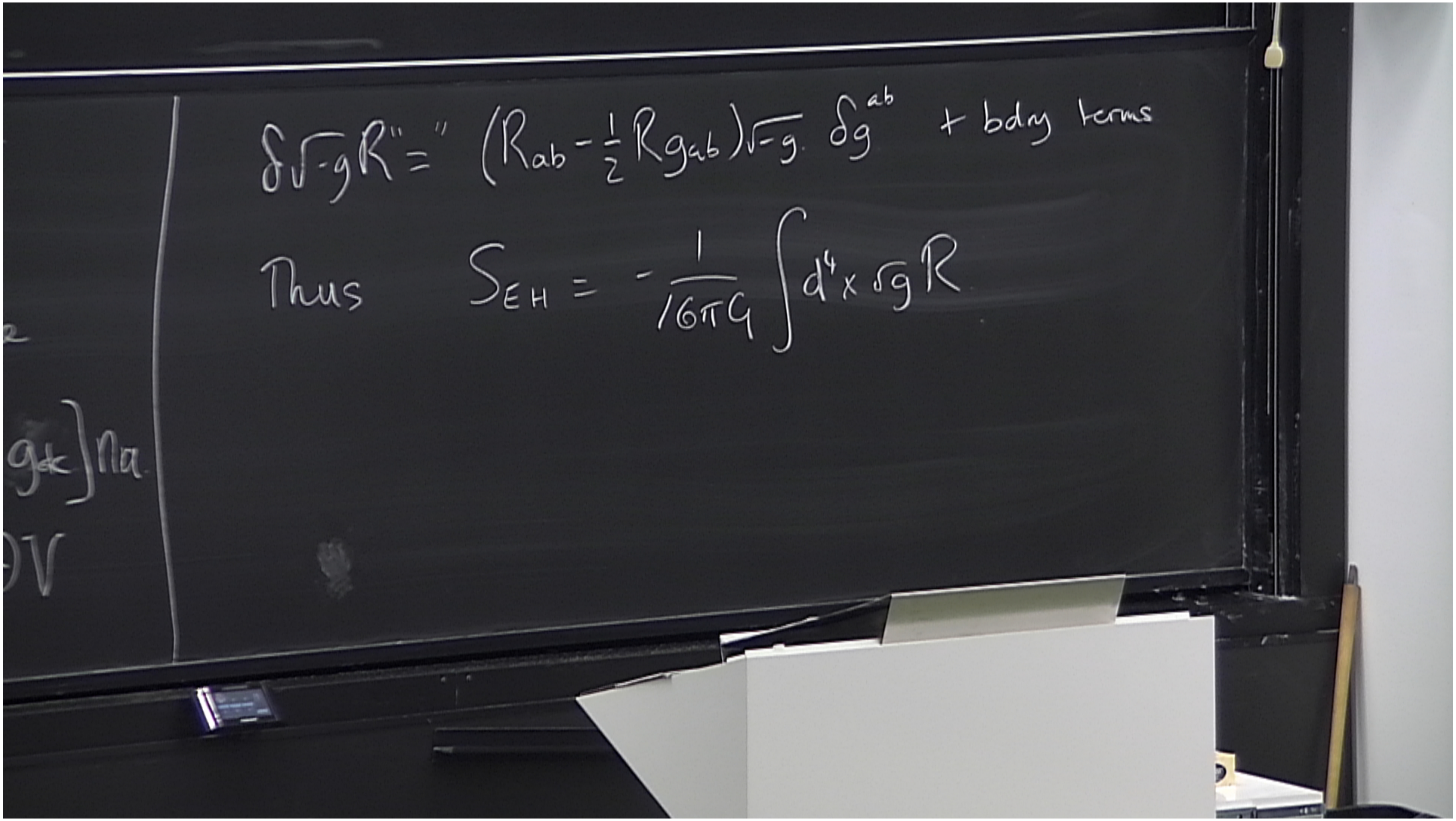
$$\Rightarrow g^{ab}\,\delta R_{ab} = -\nabla_a \nabla_b\, \delta g^{ab} + g_{ab}\,\Box\,\delta g^{ab}.$$

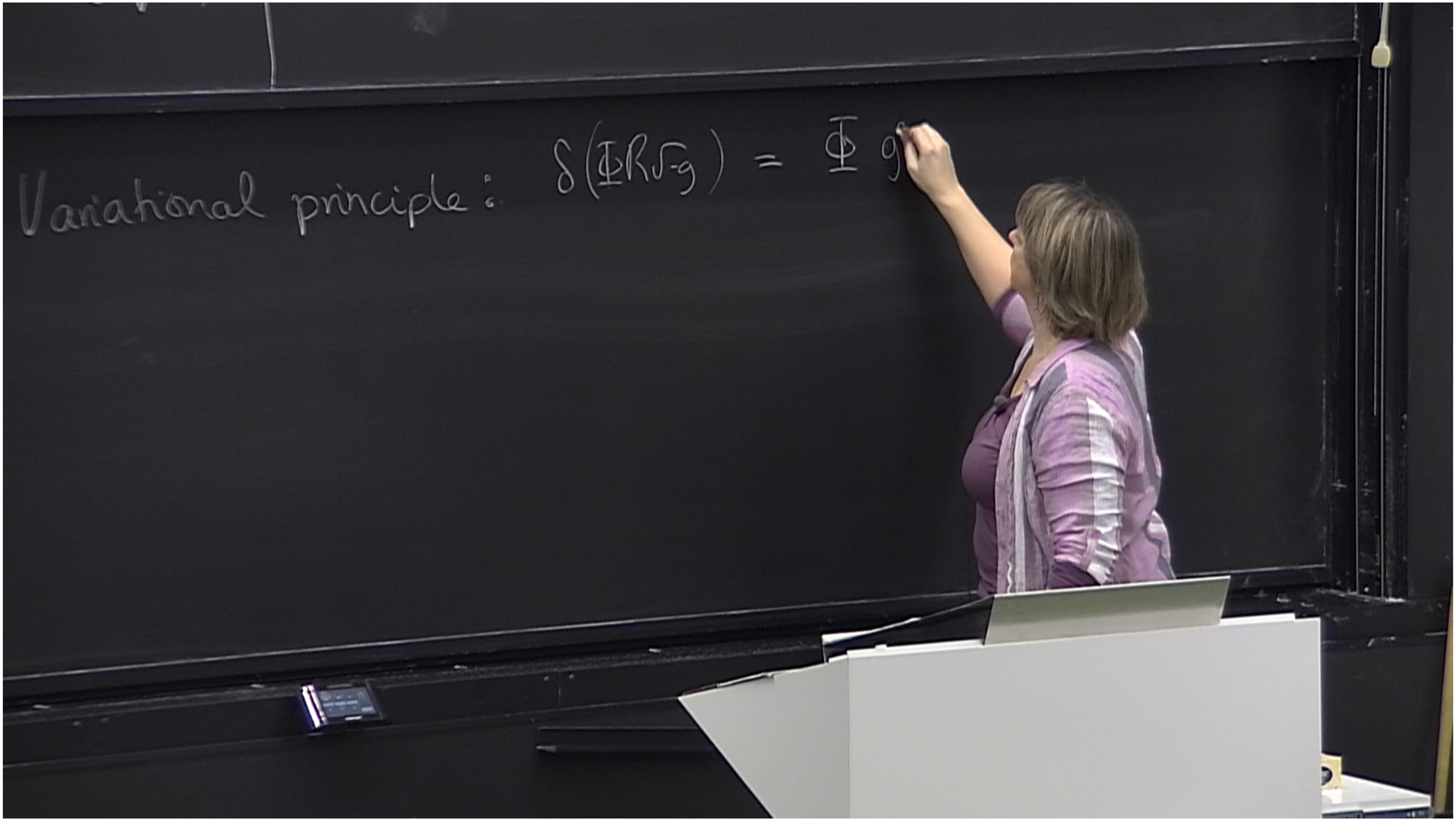
– a total derivative, but note that the boundary term will be $\left[-\nabla_b\,\delta g^{ab} + \nabla^a\,\delta g^{dc}\,g_{dc}\right] n_a$

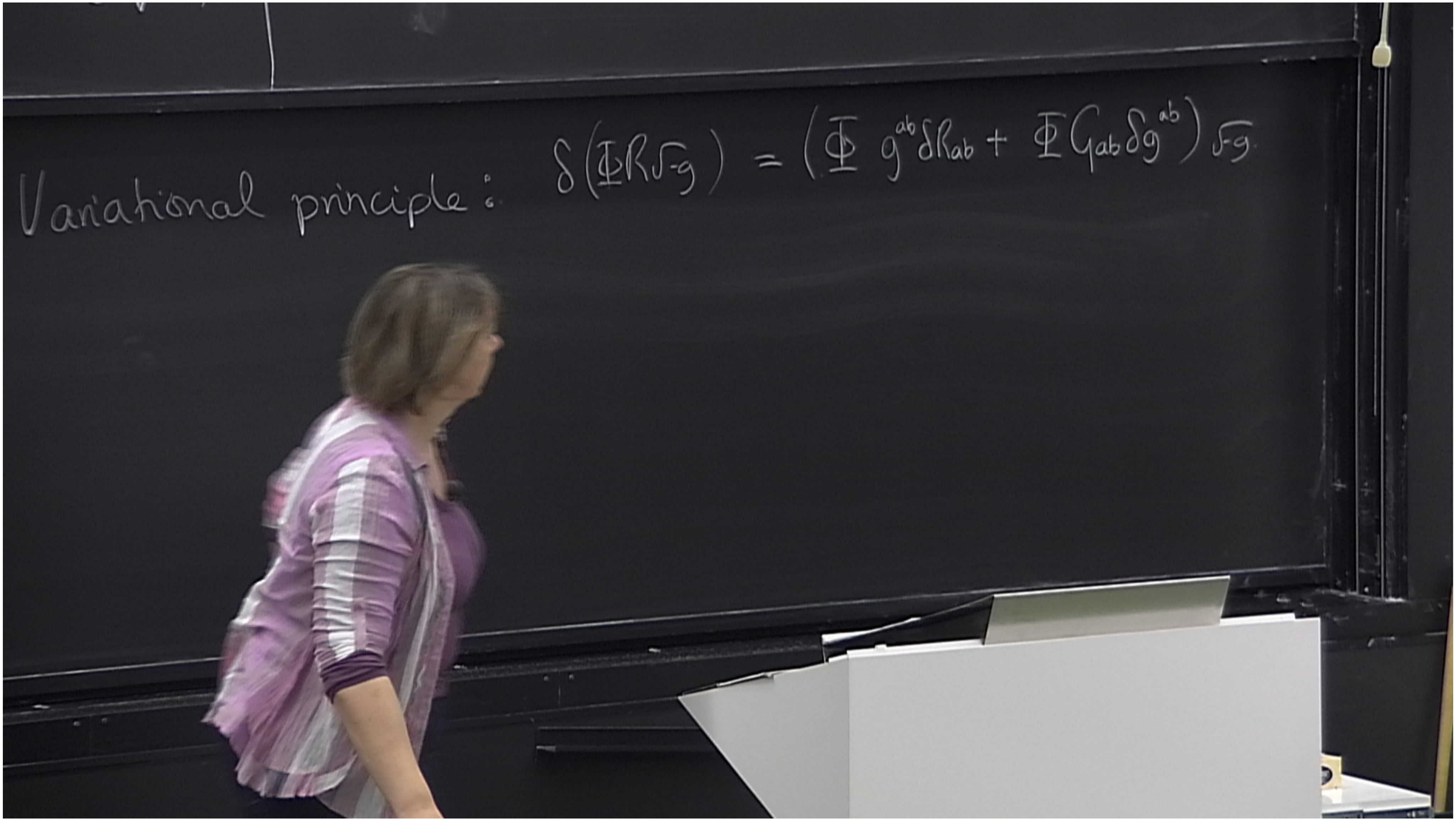
$$\Rightarrow g^{ab}\,\delta R_{ab} = -\nabla_a \nabla_b\, \delta g^{ab} + g_{ab}\,\Box\,\delta g^{ab}.$$

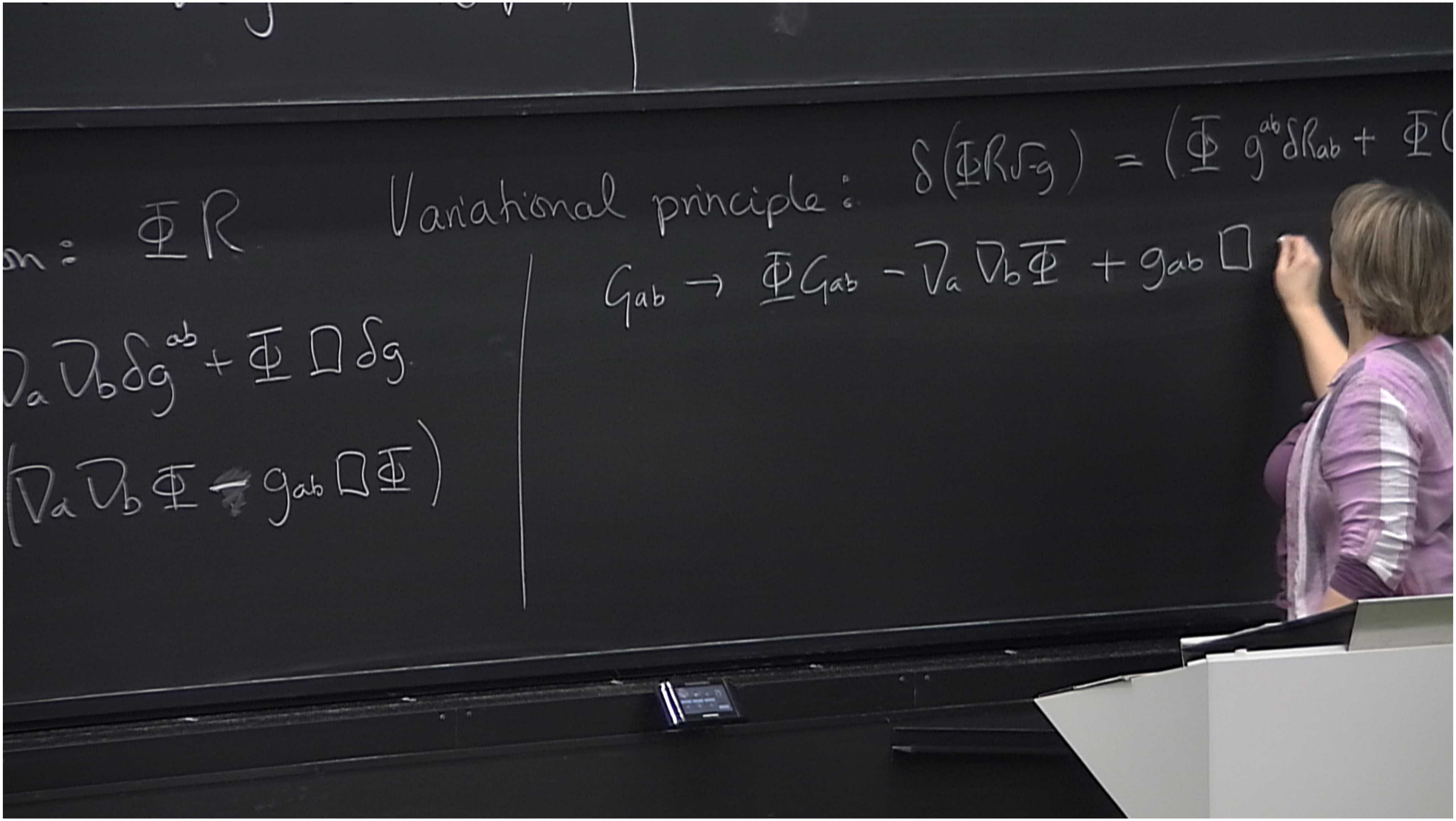
– a total derivative, but note that the boundary term will be $\left[-\nabla_b\,\delta g^{ab} + \nabla^a\,\delta g^{dc}\, g_{dc}\right] n_a$.

– ie have to set δg and $\nabla \delta g = 0$ on ∂V


$\delta \sqrt{-g} R$ "=" $(R_{ab} - \frac{1}{2} R g_{ab}) \sqrt{-g}\, \delta g^{ab}$ + bdry terms
Thus $S_{EH} = -\frac{1}{16\pi G} \int d^4x \sqrt{g} R$


Variational principle: $\delta(\Phi R\sqrt{-g}) = \Phi g$


Variational principle: $\delta(\Phi R\sqrt{-g}) = (\Phi\, g^{ab}\delta R_{ab} + \Phi\, G_{ab}\delta g^{ab})\sqrt{-g}$


Variational principle: $\delta(\Phi R\sqrt{-g}) = (\Phi g^{ab}\delta R_{ab} + \Phi($
on: ΦR
$G_{ab} \to \Phi G_{ab} - \nabla_a \nabla_b \Phi + g_{ab} \Box$
$\nabla_a \nabla_b \delta g^{ab} + \Phi \Box \delta g$
$(\nabla_a \nabla_b \Phi - g_{ab} \Box \Phi)$

- Scalar-tensor gravity

$$\mathcal{L}_0 = -\Phi R + \frac{\omega(\partial\Phi)^2}{\Phi}$$

(Jordan) - Brans-Dicke gravity

$$\Phi G_{ab} = \nabla_a \nabla_b \Phi - g_{ab} \Box \Phi + \frac{\omega}{\Phi}\left(\Phi_{,a}\Phi_{,b} - \frac{1}{2}(\partial\Phi)^2 g_{ab}\right)$$

$$\frac{\delta}{\delta\Phi} : -R - 2\omega \frac{\Box\Phi}{\Phi} + \omega\frac{(\nabla\Phi)^2}{\Phi} = 0$$

$$\Rightarrow \frac{\Box\Phi}{\Phi} = \frac{8\pi T}{(3+2\omega)}$$

- Scalar-tensor gravity

$$\mathcal{L}_0 = -\Phi R + \omega\frac{(\partial\Phi)^2}{\Phi}$$

(Jordan)-Brans-Dicke gravity

$$\Phi G_{ab} = \nabla_a \nabla_b \Phi - g_{ab}\Box\Phi + \frac{\omega}{\Phi}\left(\Phi_{,a}\Phi_{,b} - \frac{1}{2}(\partial\Phi)^2 g_{ab}\right)$$

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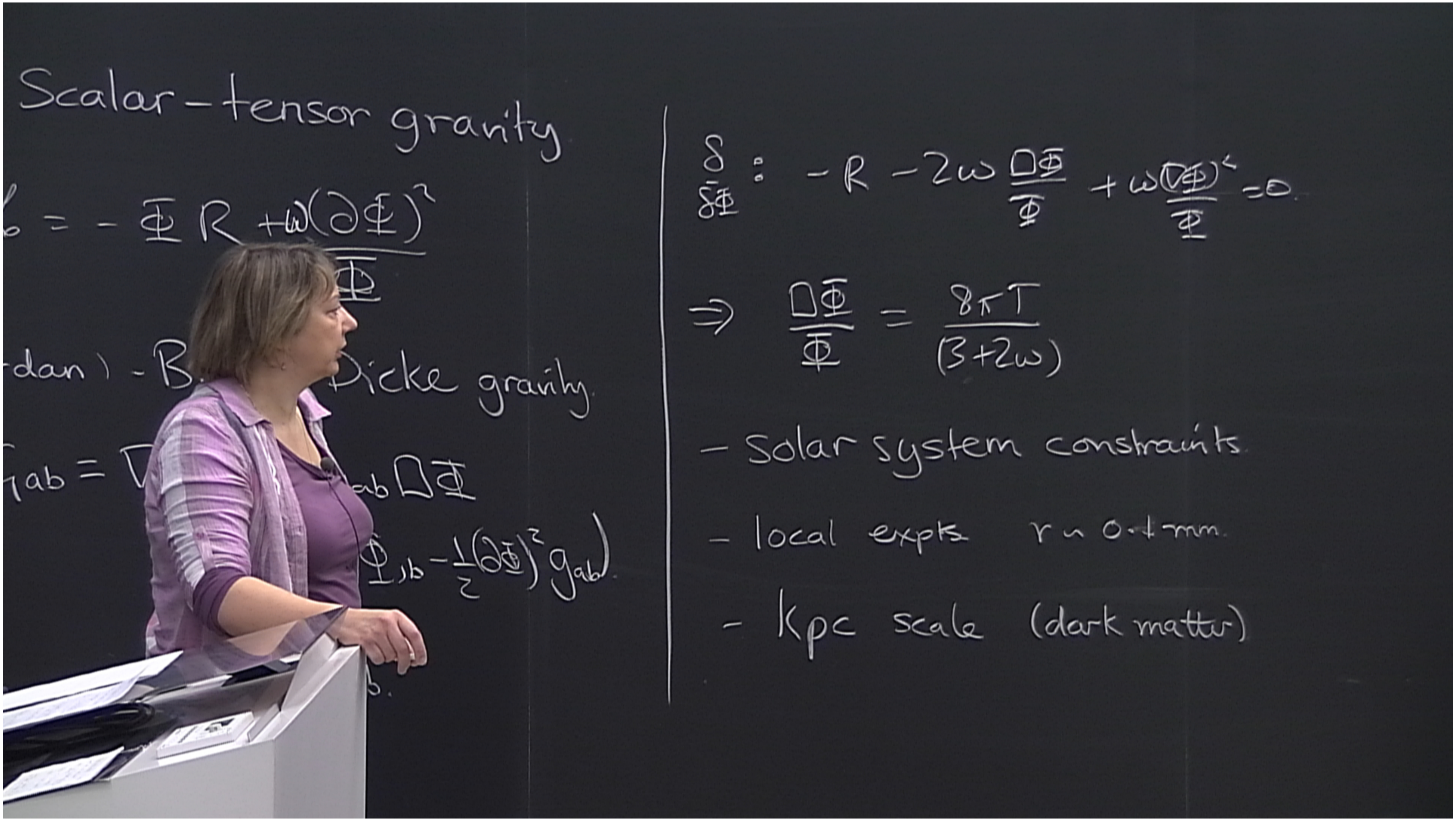
Scalar-tensor gravity
Dicke gravity
$\frac{\delta}{\delta\Phi}$: $-R - 2\omega \frac{\Box\Phi}{\Phi} + \omega\frac{(\nabla\Phi)^2}{\Phi^2} = 0$
$\Rightarrow \frac{\Box\Phi}{\Phi} = \frac{8\pi T}{(3+2\omega)}$
– Solar system constraints
– local expts r ~ 0.1 mm
– kpc scale (dark matter)

Scalar-tensor gravity

$$\mathcal{L} = -\Phi R + \omega \frac{(\partial\Phi)^2}{\Phi}$$

(Jordan) - Brans-Dicke gravity

$$G_{ab} = \nabla_a \nabla_b \Phi - g_{ab} \Box \Phi + \frac{\omega}{\Phi}\left(\Phi_{,a}\Phi_{,b} - \frac{1}{2}(\partial\Phi)^2 g_{ab}\right)$$

$$\frac{\delta}{\delta\Phi} : \quad -R - 2\omega \frac{\Box\Phi}{\Phi} + \omega \frac{(\nabla\Phi)^2}{\Phi^2} = 0$$

$$\Rightarrow \quad \frac{\Box\Phi}{\Phi} = \frac{8\pi T}{(3+2\omega)}$$

- Solar system constraints
- local expts $r \sim 0.1$ mm
- kpc scale (dark matter)