Title: The Gravity Dual of the Ising Model

Date: Jan 31, 2012 02:00 PM

URL: http://pirsa.org/12010164

Abstract: We evaluate the partition function of three dimensional general relativity with a negative cosmological constant, including all known perturbative and non-perturbative contributions to the sum over geometries.

The Problem:

The two coupling constants

- Newton constant G
- Cosmological constant $\Lambda \sim -1/\ell^2 < 0$

can be combined into a dimensionless coupling $c = \frac{3\ell}{2G}$.

We will study the theory with asymptotically AdS_3 boundary conditions.

If pure quantum gravity exists, it should be dual to two dimensional CFT with central charge c.

Can we identify this CFT?

The Problem:

The two coupling constants

- Newton constant G
- Cosmological constant $\Lambda \sim -1/\ell^2 < 0$

can be combined into a dimensionless coupling $c = \frac{3\ell}{2G}$.

We will study the theory with asymptotically AdS_3 boundary conditions.

If pure quantum gravity exists, it should be dual to two dimensional CFT with central charge c.

Can we identify this CFT?

The Computation:

The Euclidean sum over geometries

$$Z = \int Dg \; e^{-S[g]}$$

can be computed exactly for general relativity if we can

- find all classical saddles (instantons)
- compute all perturbative corrections around each saddle

This computes the *exact* spectrum of quantum gravity in asymptotically AdS_3 spacetimes.

It should equal the Hilbert space trace

$$Z = \mathrm{Tr}_{\mathcal{H}} \; e^{-\beta H}$$

The Result:

For a unitary quantum theory this should take the form

$$Z = \sum_E N(E) e^{-\beta E}$$

where the $N(E)$ are positive integers.

This is not the case for general values of $c = \frac{3\ell}{2G}$. Pure quantum gravity does not exist.

But this works for certain values of c! The result is the partition function of a known, unitary CFT.

The spectrum of quantum general relativity with $G = 3\ell$ that of the critical Ising model.

The Result:

For a unitary quantum theory this should take the form

$$Z = \sum_E N(E) e^{-\beta E}$$

where the $N(E)$ are positive integers.

This is not the case for general values of $c = \frac{3\ell}{2G}$. Pure quantum gravity does not exist.

But this works for certain values of c! The result is the partition function of a known, unitary CFT.

The spectrum of quantum general relativity with $G = 3\ell$ equals that of the critical Ising model.

The Dualities:

Pure theories of gravity conjectured to be dual to exactly solvable CFTs:

Bulk Theory	$\frac{3\ell}{2G} = c$	CFT
General Relativity	$\frac{1}{2}$	Ising Model
General Relativity	$\frac{7}{10}$	Tricritical Ising Model
$SL(3)$ gravity	$\frac{4}{5}$	Potts Model
E_6 gravity	$\frac{6}{7}$	Tricritical Potts Model
$SL(N)$ Gravity	$2\frac{N-1}{N+2}$	Parafermions
Chiral Gravity	24	Monster CFT

These are strongly coupled theories with $c \sim \mathcal{O}(1)$. We cannot use perturbative methods.

The Dualities:

Pure theories of gravity conjectured to be dual to exactly solvable CFTs:

Bulk Theory	$\frac{3\ell}{2G} = c$	CFT
General Relativity	$\frac{1}{2}$	Ising Model
General Relativity	$\frac{7}{10}$	Tricritical Ising Model
$SL(3)$ gravity	$\frac{4}{5}$	Potts Model
E_6 gravity	$\frac{6}{7}$	Tricritical Potts Model
$SL(N)$ Gravity	$2\frac{N-1}{N+2}$	Parafermions
Chiral Gravity	24	Monster CFT

These are strongly coupled theories with $c \sim \mathcal{O}(1)$. We cannot use perturbative methods.

The Dualities:

Pure theories of gravity conjectured to be dual to exactly solvable CFTs:

Bulk Theory	$\frac{3\ell}{2G} = c$	CFT
General Relativity	$\frac{1}{2}$	Ising Model
General Relativity	$\frac{7}{10}$	Tricritical Ising Model
$SL(3)$ gravity	$\frac{4}{5}$	Potts Model
E_6 gravity	$\frac{6}{7}$	Tricritical Potts Model
$SL(N)$ Gravity	$2\frac{N-1}{N+2}$	Parafermions
Chiral Gravity	24	Monster CFT

These are strongly coupled theories with $c \sim \mathcal{O}(1)$. We cannot use perturbative methods.

The Plan for Today:

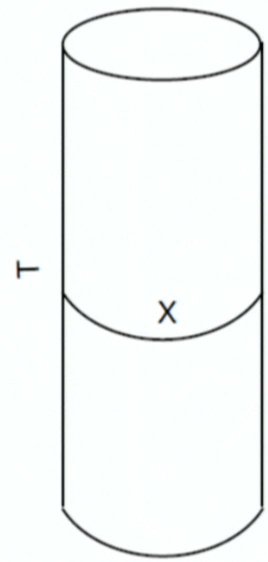
- Setup
- Gravitational Exclusion Principle
- Sum over Geometries
- Diatribe

The Partition Function:

The spectrum of quantum gravity in AdS_3 is encoded in the partition function

$$Z(\tau) = \mathrm{Tr}_{\mathcal{H}}\ q^{L_0}\bar{q}^{\bar{L}_0}, \qquad q = e^{2\pi i\tau}$$

where $L_0 = H + J$, $\bar{L}_0 = H - J$.

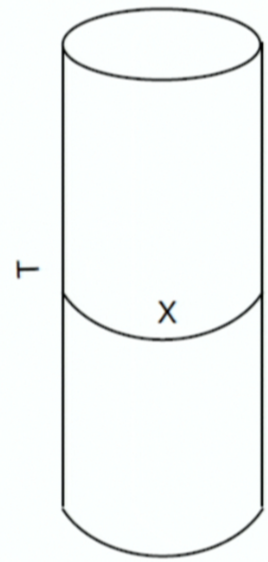


The Partition Function:

The spectrum of quantum gravity in AdS_3 is encoded in the partition function

$$Z(\tau) = \mathrm{Tr}_{\mathcal{H}}\, q^{L_0}\bar{q}^{\bar{L}_0}, \qquad q = e^{2\pi i\tau}$$

where $L_0 = H + J$, $\bar{L}_0 = H - J$.



The Path Integral:

This partition function is computed by the path integral

$$Z(\tau) = \int Dg \; e^{-S[g]}$$

over Euclidean manifolds with T^2 boundary with conformal structure τ. This includes a sum over topologies

$$Z(\tau) = \sum_{\mathcal{M}} Z_{\mathcal{M}}(\tau)$$

In the semiclassical ($c \to \infty$) limit only $\mathcal{M}$ which admit a classical solution contribute

$$Z = \sum_{g_0} e^{-c\, S^{(0)} + S^{(1)} + c^{-1} S^{(2)} + \ldots}$$

We will assume this is true even when $c \sim \mathcal{O}(1)$.

The Path Integral:

This partition function is computed by the path integral

$$Z(\tau) = \int Dg \; e^{-S[g]}$$

over Euclidean manifolds with T^2 boundary with conformal structure τ. This includes a sum over topologies

$$Z(\tau) = \sum_{\mathcal{M}} Z_{\mathcal{M}}(\tau)$$

In the semiclassical ($c \to \infty$) limit only $\mathcal{M}$ which admit a classical solution contribute

$$Z = \sum_{g_0} e^{-c\, S^{(0)} + S^{(1)} + c^{-1} S^{(2)} + \ldots}$$

We will assume this is true even when $c \sim \mathcal{O}(1)$.

The Path Integral:

This partition function is computed by the path integral

$$Z(\tau) = \int Dg \; e^{-S[g]}$$

over Euclidean manifolds with T^2 boundary with conformal structure τ. This includes a sum over topologies

$$Z(\tau) = \sum_{\mathcal{M}} Z_{\mathcal{M}}(\tau)$$

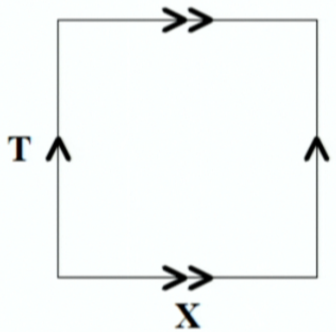
In the semiclassical ($c \to \infty$) limit only $\mathcal{M}$ which admit a classical solution contribute

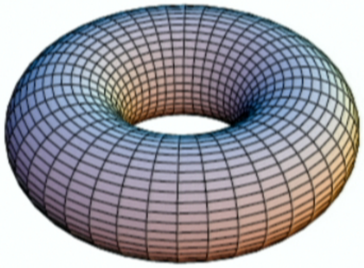
$$Z = \sum_{g_0} e^{-c\, S^{(0)} + S^{(1)} + c^{-1} S^{(2)} + \dots}$$

We will assume this is true even when $c \sim \mathcal{O}(1)$.

The Classical Solutions:

The smooth classical solutions are solid tori M_γ, labelled by elements $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$





All M_γ are related to thermal AdS by modular transformations

$$Z_\gamma(\tau) = Z_{vac}(\gamma\tau), \qquad \gamma\tau = \frac{a\tau + b}{c\tau + d}$$

where Z_{vac} is the contribution from thermal AdS.

Maldacena & Strominger
Dijkgraaf, Moore, Maldacena & Verlinde

The Black Hole Farey Tail:

The partition function is a sum over the modular group

$$Z(\tau) = \sum_{\gamma \in SL(2,\mathbb{Z})} Z_{vac}(\gamma\tau)$$

where Z_{vac} is the contribution from geometries continuously connected to AdS, i.e. the trace

$$Z_{vac}(\tau) = Tr_{\mathcal{H}_{\rm vac}}\ q^{L_0}\bar{q}^{\bar{L}_0}$$

over the Hilbert space $\mathcal{H}_{\rm vac}$ of fluctuations around AdS.

There are no local degrees of freedom, but there are "global" excitations coming from non-trivial diffeomorphisms applied to empty AdS.

The Hilbert space $\mathcal{H}_{\rm vac}$ is obtained by quantizing the phase space *diff* $S^1 \times$ *diff* S^1.

A. M. & Witten

Boundary Gravitons:

The diffeomorphisms obey the Virasoro algebra with central charge $c = \frac{3\ell}{2G}$. The states in $\mathcal{H}_{\rm vac}$ are descendants

$$L_{-n_1} \dots L_{-n_i}|0\rangle$$

These are "boundary gravitons."

When $c < 1$ they appear to outnumber black hole microstates:

$$S_{BH} = 2\pi\sqrt{\frac{c}{6}L_0} < S_{gravitons} = 2\pi\sqrt{\frac{1}{6}L_0}$$

This the AdS_3 version of the usual holographic puzzle

- Black Hole entropy grows like area, but the entropy of a gas of gravitons grows like volume.

Brown & Henneaux

The Gravitational Exclusion Principle:

Let's compute the norms of boundary graviton states. When $c < 1$:

- There are negative norm states unless $c = 1 - \frac{6}{m(m+1)}$
- Most states have zero norm

So the gauge symmetry of AdS_3 gravity is *enhanced* when $c < 1$.

Removing the null states, we find the contribution from the vacuum topology

$$Z_{vac} = |\chi_{1,1}(\tau)|^2$$

is a character of an irreducible representation of Virasoro.

This rescues holography:

$$S_{gravitons} = 2\pi\sqrt{\frac{c}{6}L_0}$$

Castro, Hartman & A. M.

The Sum over Geometries:

We must compute

$$Z = \sum_{\gamma \in SL(2,\mathbb{Z})} |\chi_{1,1}(\gamma\tau)|^2$$

Under a modular transformation γ, $\chi_{1,1}$ transforms into other characters χ_h.

These χ_h are traces of other Virasoro representations built on non-trivial primary states.

They form a finite dimensional representation of $SL(2, \mathbb{Z})$

A subgroup $\Gamma_c \subset SL(2, \mathbb{Z})$ acts trivially. This is a manife

the enhanced gauge symmetry.

The Miracle:

The subgroup Γ_c has finite index. The sum over $\gamma \in \Gamma_c \backslash SL(2,\mathbb{Z})$ is now finite!

The result is explicitly computable

$$Z(\tau) = \sum_{\gamma} |\chi_{1,1}(\gamma\tau)|^2 \;= \sum_{h,\bar{h}} N_{h,\bar{h}}\, \chi_h(\tau)\chi_{\bar{h}}(\bar{\tau})$$

where the χ_h are Virasoro characters.

The gravity path integral gives a particular spectrum $N_{h,\bar{h}}$ of conformal representations.

Are the $N_{h,\bar{h}}$ positive integers?

Is the result the partition function of a unitary CFT?

The Result:

When $c = \frac{1}{2}$ ($c = \frac{7}{10}$) the result equals the partition function of the (tri-)critical Ising model.

All other cases have non-integer coefficients.

Conjecture:

- Three dimensional general relativity can be quantized only for these values of the coupling constants.
- When it exists it is dual to the (tri-)critical Ising model.

These are highly quantum theories of gravity, with

- No semi-classical limit
- Finite number of primary states (i.e. black holes)

The Results II:

For the certain theories of gravity the exact quantum spectrum is computable and matches a known CFT:

Bulk Theory	$\frac{3\ell}{2G} = c$	CFT
General Relativity	$\frac{1}{2}$	Ising Model
General Relativity	$\frac{7}{10}$	Tricritical Ising Model
$SL(3)$ gravity	$\frac{4}{5}$	Potts Model
E_6 gravity	$\frac{6}{7}$	Tricritical Potts Model
$SL(N)$ Gravity	$2\frac{N-1}{N+2}$	Parafermions

This list is probably not exhaustive.

The Results II:

For the certain theories of gravity the exact quantum spectrum is computable and matches a known CFT:

Bulk Theory	$\frac{3\ell}{2G} = c$	CFT
General Relativity	$\frac{1}{2}$	Ising Model
General Relativity	$\frac{7}{10}$	Tricritical Ising Model
$SL(3)$ gravity	$\frac{4}{5}$	Potts Model
E_6 gravity	$\frac{6}{7}$	Tricritical Potts Model
$SL(N)$ Gravity	$2\frac{N-1}{N+2}$	Parafermions

This list is probably not exhaustive.

Conclusions:

Quantum gravity as a sum over geometries makes sense.

The sum can be computed precisely if we

- know all the classical solutions
- know all the "perturbative" corrections

This allows us to determine the exact spectrum of quantum gravity, if it exists.

It matches the spectrum of a known, unitary, exactly solvable CFT.

But the duality is not proven.

We must check correlation functions/higher genus partition functions.

Speculations:

Pure quantum gravity exists only for sporadic, $\mathcal{O}(1)$ values of the coupling constant $c = \frac{3\ell}{2G}$.

The duals are certain special minimal models:

- They are "extremal CFTs" in the sense of Witten.
- Primary states are in the Cardy (black hole) regime: $L_0 > \frac{c}{24}$.

This may help us understand the

- Path integral of quantum gravity
- Emergence of space-time in AdS/CFT

Can we find a family of dualities with a semi-classical limit?

What matter content is necessary for consistent quantum gravity?