Title: Causal Structure and Categorical Quantum Mechanics

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Abstract: Categorical quantum mechanics (CQM) uses symmetric monoidal categories to formalize quantum theory, in order to extract the key structures that yield protocols such as teleportation in an abstract way. This formalism admits a purely graphical calculus, but the causal structure of these diagrams, and the formalism in general, is unclear. We begin by considering the signaling abilities of probabilistic devices with inputs and outputs and we show how a non-signaling device can become a perfect signaling device under time-reversal. This conflicts with the causal structure of relativity, and suggests that an `asymmetry' is needed when formalizing causality in CQM. We then show how a fixed causal structure within CQM corresponds to topological connectedness in the graphical language, and that correlations, either classical or quantum, force terminality of the tensor unit. We also show that well-definedness of a global state forces the monoidal product to be only partially defined, which in turn results in a covariance theorem. These structural results lead to a mathematical entity which we call a `causal category'.


Causal structure and categorical quantum mechanics
Raymond Lal
Oxford University Computing Laboratory
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R. Lal (OUCL)
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1 / 55

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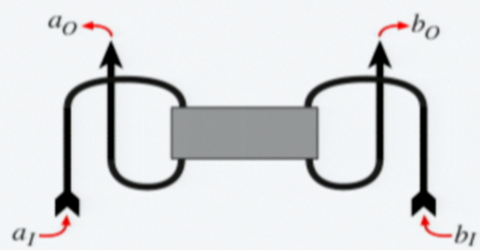
The broad theme

Categorical quantum mechanics uses a certain type of category and its graphical calculus to describe the structure of information flow in quantum theory.

But protocols need a background causal framework:



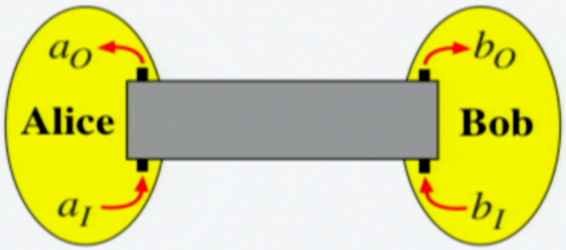
- How should we add causal structure?
- What does this mean for the compatibility between quantum theory and special relativity?

Outline

1. Signalling under time-reversal
2. Categorical quantum mechanics
3. Causal categories
   - Connectedness
   - Terminality of the tensor unit
   - Partiality of the monoidal product
   - Defining a new structure

Bipartite correlations

Consider bipartite correlations, with binary inputs and outputs:



We represent probabilities of the outputs of this device, for each input pair, as a stochastic matrix, e.g.:

$$g = \begin{pmatrix} \frac{1}{4} & 0 & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & \frac{1}{2} \\ 0 & \frac{1}{4} & 0 & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{pmatrix}$$

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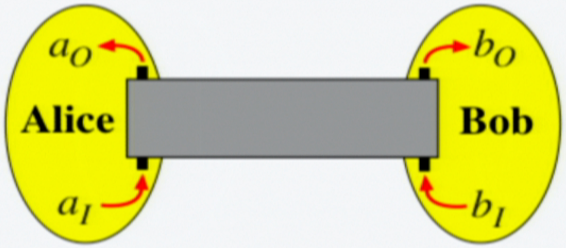

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No-signalling conditions

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Definition

A correlation matrix enables *signalling from Alice to Bob* iff

$$\exists(b_I, b_O) : g_{0,b_I}^{0,b_O} + g_{0,b_I}^{1,b_O} \neq g_{1,b_I}^{0,b_O} + g_{1,b_I}^{1,b_O} .$$

In words, Alice signals to Bob if, from the statistics of his output b_O and input b_I, he can deduce Alice's input a_I.

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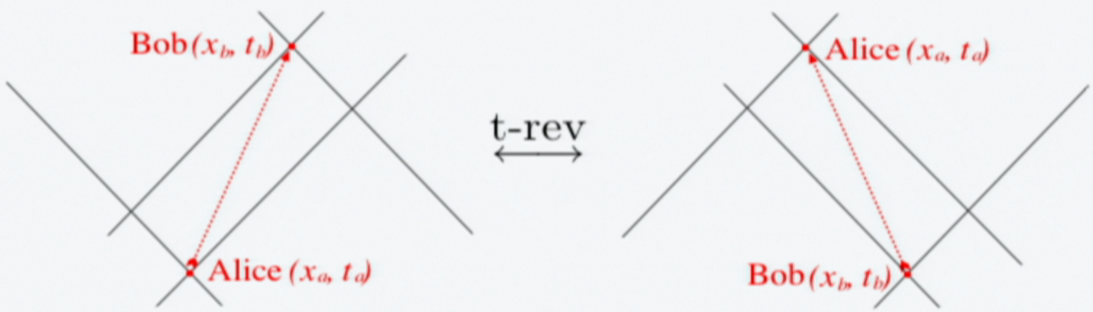
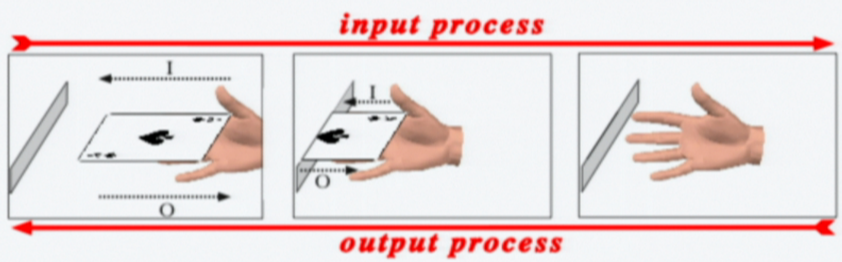
Relativistic causal structure is time-symmetric: reversing the time-orientation on the manifold does not change whether two points are spacelike separated.



Defining time-reversal

Is this symmetry is respected by our bipartite correlation devices?

Note that we can make the implementation of these devices symmetric between inputs and outputs:



Bayesian inversion

A natural definition of time-reversal for probabilistic devices is *Bayesian inversion:*

$$P(I|O) = \frac{P(O|I)P(I)}{P(O)}.$$

But this requires knowing the prior probability distribution $P(I)$ on the inputs of the device:

- Boundary conditions?
- Prior is free choice of experimenter.

Signalling under time-reversal

Theorem

There exist classical correlation matrices for which the time-reverse, for any total prior, is signaling. More specifically, each such time reverse of

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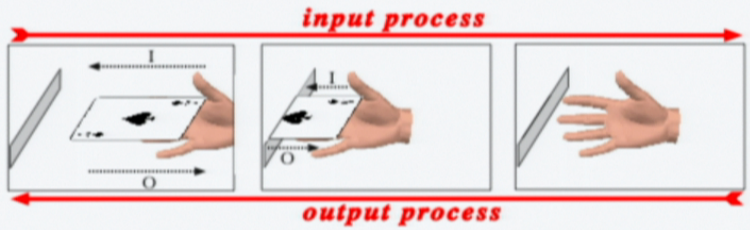
enables perfect signalling from Alice to Bob, which is achieved when Bob fixes his input to always be 0.

Hence this device becomes a *perfect channel* under time-reversal.

An arrow of time

Using the 'videotape' metaphor of time-reversal, we can record the statistics of the inputs and outputs.



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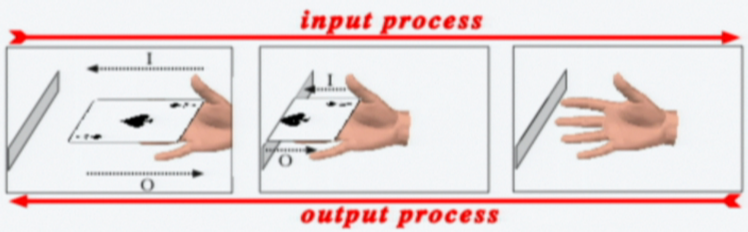

Playing the tape backwards gives the statistics from Bayesian inversion.

But just by the empirical records of the correlations of these devices, we detect the backward direction in time:

> *This is the direction which enables probabilistic signalling between spacelike separated regions.*

Relationship to thermodynamic arrow of time

Necessary conditions for producing signalling under time-reversal: process must be

- input-dependent, and
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Is there a connection to thermodynamics and the 2nd law?

These processes can either increase or decrease Shannon entropy, depending on $P(I) \rightsquigarrow$ these are not 'irreversible' computations, even when considered in isolation.

Entropic arrow

- To realize the process, expand the system to include the environment of the device, i.e. a heat bath.

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$$\Delta S_{th} \geq -\Delta H k \ln 2$$

- This is how consistency with 2nd law is met in an *entropy-increasing universe*.

Independence from 2nd law

1. Does 2nd law imply backwards-signaling?
   - No: counterexample is any input-independent process.
2. Does backwards-signaling imply 2nd law?
   - No: by Maroney's work*
     - define an entropy-decreasing universe, i.e. low-entropy *future* boundary condition
     - any computation in an entropy-increasing universe can be performed in an entropy-decreasing universe
     - difference is that in the latter $\Delta S_{th} \leq -\Delta H k \ln 2$.

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- *Bell*: no local hidden variable theory $\rightsquigarrow$ intrinsic probabilities for consistency with relativity
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Issues raised

What now?

- The question of how correlations are constrained by relativity is less compelling if they do not share the same symmetries anyway.
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What is a category?

A category **C** has:

- a set of *objects* $A, B, \ldots$
- a set of *morphisms* $f, g, \ldots$ between these objects, e.g. $f : A \to B$. The set of morphisms between A and B is denoted $\mathbf{C}(A, B)$.

Crucially, there is a composition law for morphisms

$$\begin{aligned} \circ : \mathbf{C}(A,B) \times \mathbf{C}(B,C) &\longrightarrow \mathbf{C}(A,C) \\ (f,g) &\longmapsto g \circ f \end{aligned}$$

Examples:

- the category of sets (the objects) and functions (the morphisms) **Set**, composition $\circ$ is given by function composition
- the category of groups and group homomorphisms **Grp**

Monoidal categories

Categorical quantum mechanics uses *symmetric monoidal categories* $(\mathbf{C}, \otimes, I)$, which have *two* notions of composition:

- the usual composition of morphisms in hom-sets $\mathbf{C}(A, B)$

$$\circ : \mathbf{C}(A, B) \times \mathbf{C}(B, C) \to \mathbf{C}(A, C)$$

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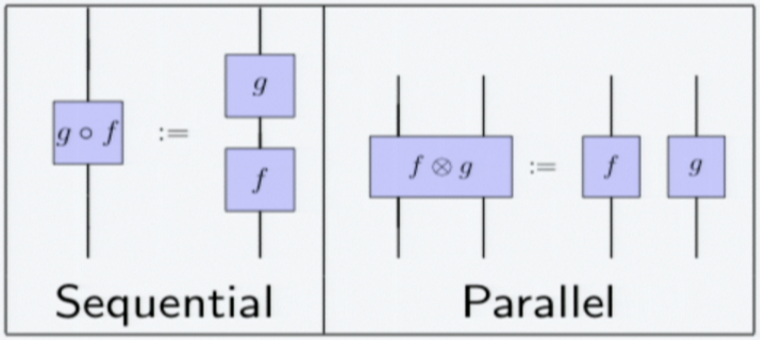
Example: the category of real vector spaces (the objects), and linear maps between them (the morphisms):

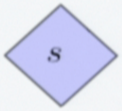
- $\circ$ is given by usual composition of linear maps
- $\otimes$ is given by tensor product of vector spaces
- the monoidal unit I is $\mathbb{R}$ (the *scalars*).

Graphical language

Monoidal categories carry a 2D graphical calculus:



The monoidal unit I is represented by empty space, so scalars $s : I \to I$ are represented as having no input or output:



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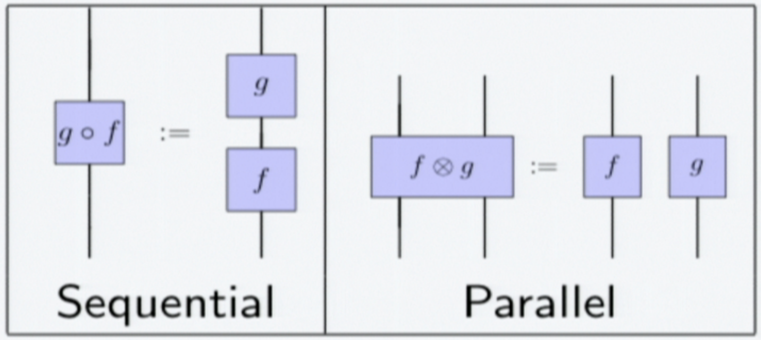

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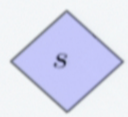
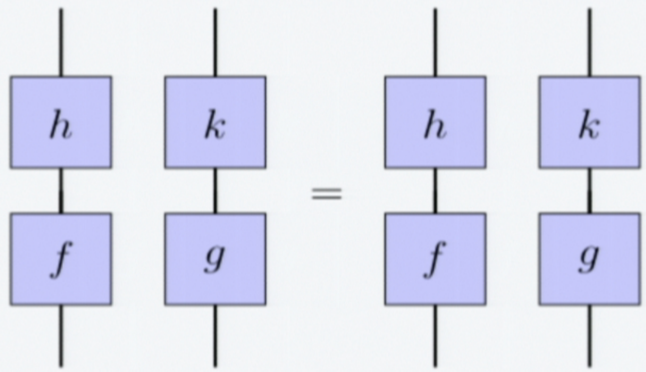


Graphical calculus

The two types of composition interact via

$$(h \otimes k) \circ (f \otimes g) = (h \circ f) \otimes (k \circ g),$$

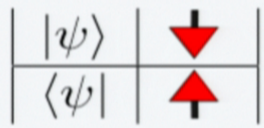
which has a tautologous graphical counterpart:



Dagger

A *dagger SMC* is an SMC with a non-trivial identity-on-objects involutive endofunctor:

This extends Dirac notation.



The category **fHilb** (objects = f.d. Hilbert spaces; morphisms = linear maps) is the canonical example of a dagger SMC:

- dagger functor is given by adjoint
- monoidal product is tensor product of Hilbert spaces.

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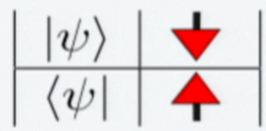

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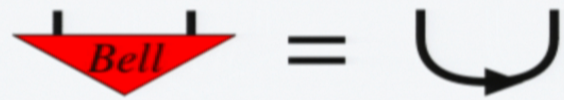
Dynamics: compactness and teleportation

A *dagger compact category* is a dagger SMC which for each object A there exists a morphism $\eta : I \to A \otimes A^*$, depicted as



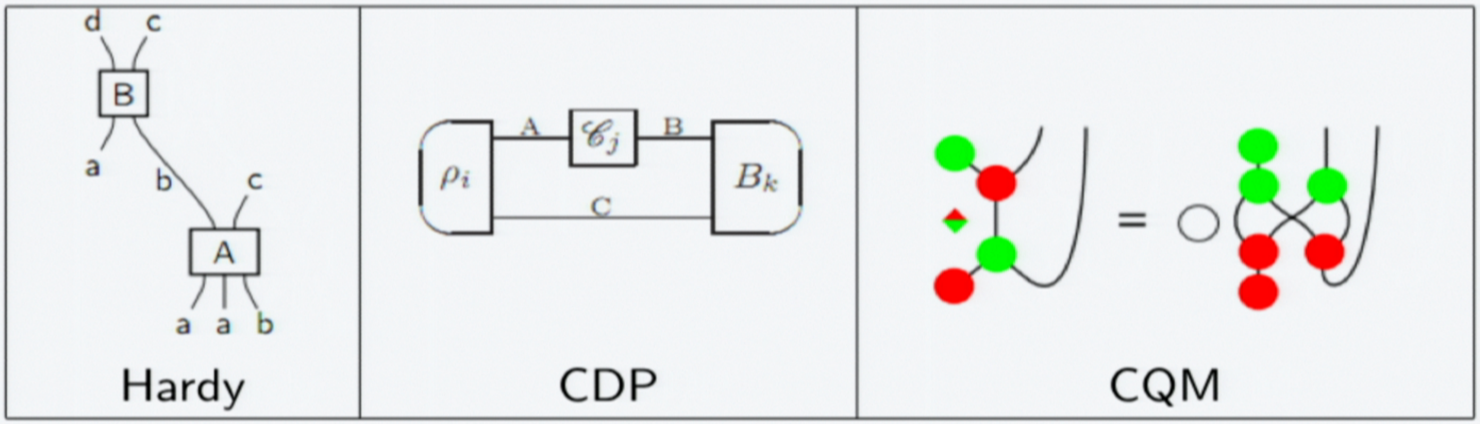
satisfying

In **fHilb** the morphism η is the unnormalised Bell state $|00\rangle + |11\rangle$.

Again, topology of diagrams subsumes symbolic calculations.

Quantum picturalism

Formal graphical methods are now used by various researchers in QF.



This new view of quantum theory forces the question: what do the diagrams mean causally?

Causal structure in relativity

Malament showed that the causal structure of a relativistic spacetime determines its metric up to a conformal factor:

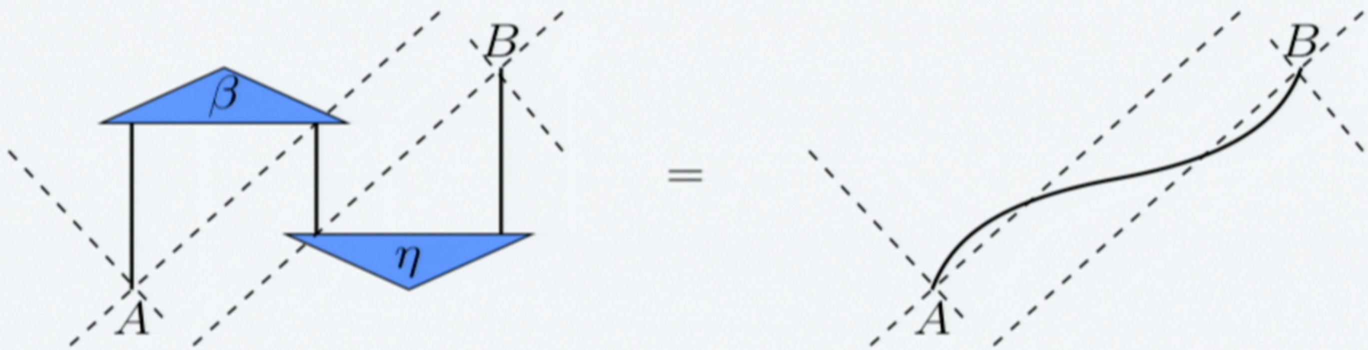
- For spacetime manifolds (M, g) and (M', g') if there exists a causal isomorphism $f : M \to M'$, then f is a conformal diffeomorphism.

This justifies taking causal structure to be the essence of relativity; cf. Zeeman definition of automorphism group, and Kronheimer-Penrose axioms.

A problem

Goal: We want to identify in CQM the kind of structure that will encode causally-respecting processes.

Consider again post-selected teleportation in CQM, now with causal structure added.



Deriving the right structure

- *Dagger compact categories describe CQM, e.g. teleportation.*
- *What type of categories describe CQM + causal structure?*

We want to derive the necessary structure from physical considerations:

1. using the graphical calculus to define information flow;
2. the role of measurement;
3. how to define states at some point in evolution.

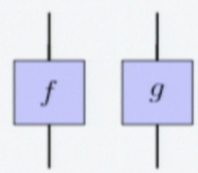
Will refer to the diagram corresponding to a morphism $\mathcal{F} : A \rightarrow C$ as a *scenario*.

Graphical no-signalling

- Can we understand this topologically? We want:

information flow	no information flow
	

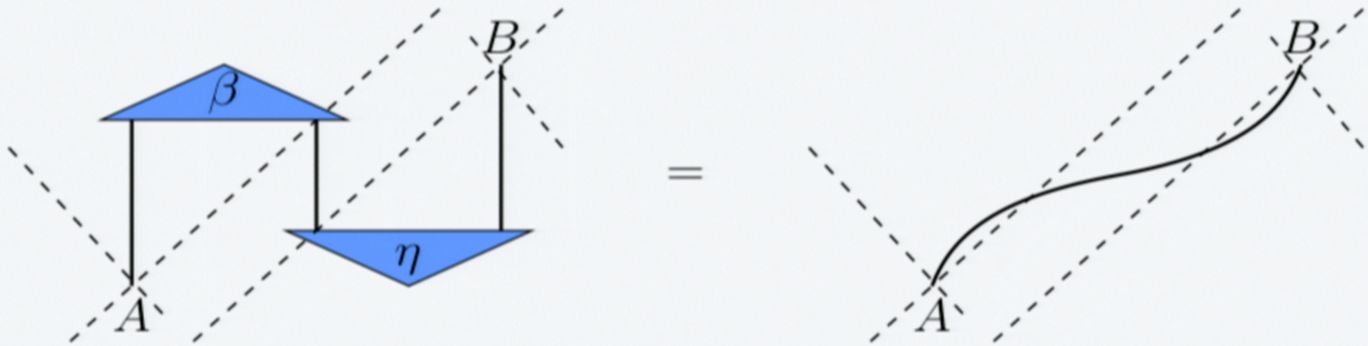
- That is, we want lack of information flow to correspond to graphical *disconnectedness.*
- The monoidal product is also depicted by graphical disconnectedness:



Encoding background causal structure

We use shall use information flow to encode a given background causal structure in a category.

Suppose our diagrams are occurring in an ambient poset of points of a spacetime.



Then given a dagger SMC for CQM, **C**, we will throw away the non-causal morphisms, to obtain a subcategory $\mathbf{CC} \hookrightarrow \mathbf{C}$.

Causal categories defined

A *causal category* **CC** encodes the structural features described above:

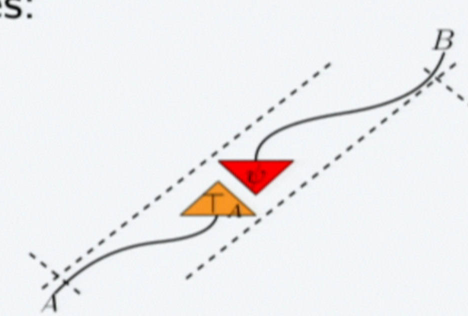
- it is a partial monoidal category, where

$$A \otimes B \text{ exists} \iff \mathbf{CC}(A, B) = \{\psi \circ \top_A \mid \psi : I \to B\}$$

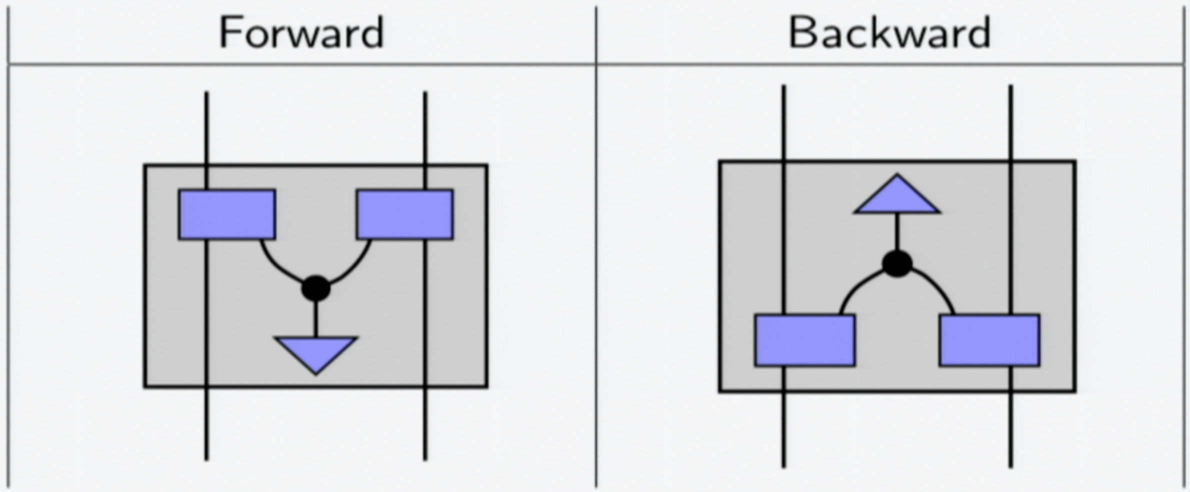
- it has a terminal unit object.

Hence the hom-sets can now encode light cones:

$$\mathbf{CC}(A, B) = \{\psi \circ \top_A \mid \psi : I \to B\} \quad \rightsquigarrow$$



Time-asymmetry of causal structure



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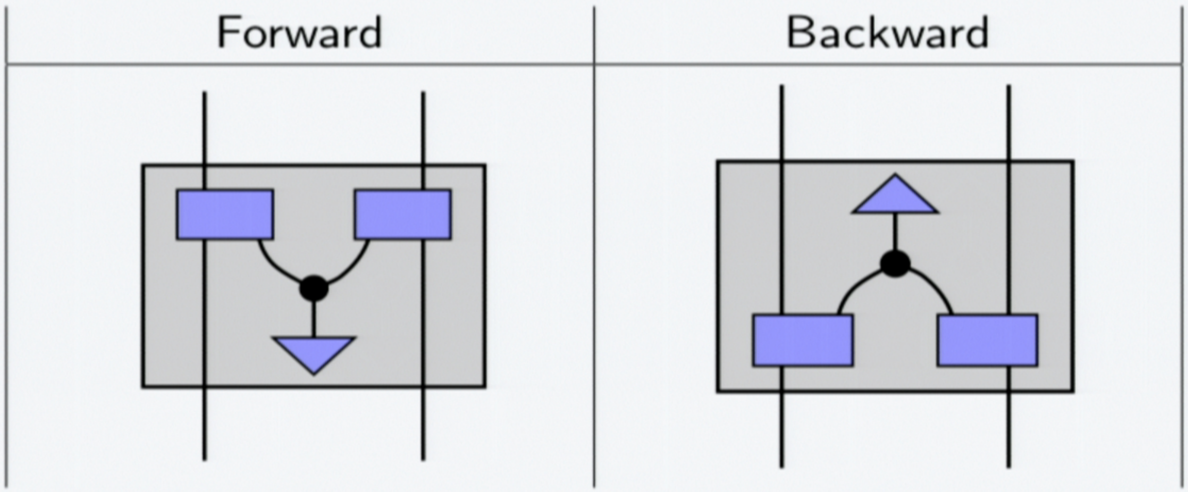

Outlook

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- The structures here have close links to the CDP reconstruction of quantum theory, especially compactness and terminality.

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Papers

- B. Coecke and R. Lal (to appear), *Categorical quantum mechanics meets the Pavia principles.* Electronic Notes in Computer Science.
- B. Coecke and R. Lal (2011), *Time-asymmetry of probabilities versus relativistic causal structure: an arrow of time.* arXiv:1108.1988.
- B. Coecke and R. Lal (to appear), *Causal categories: relativistically interacting processes.* Foundations of Physics. arXiv:1107.6019

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