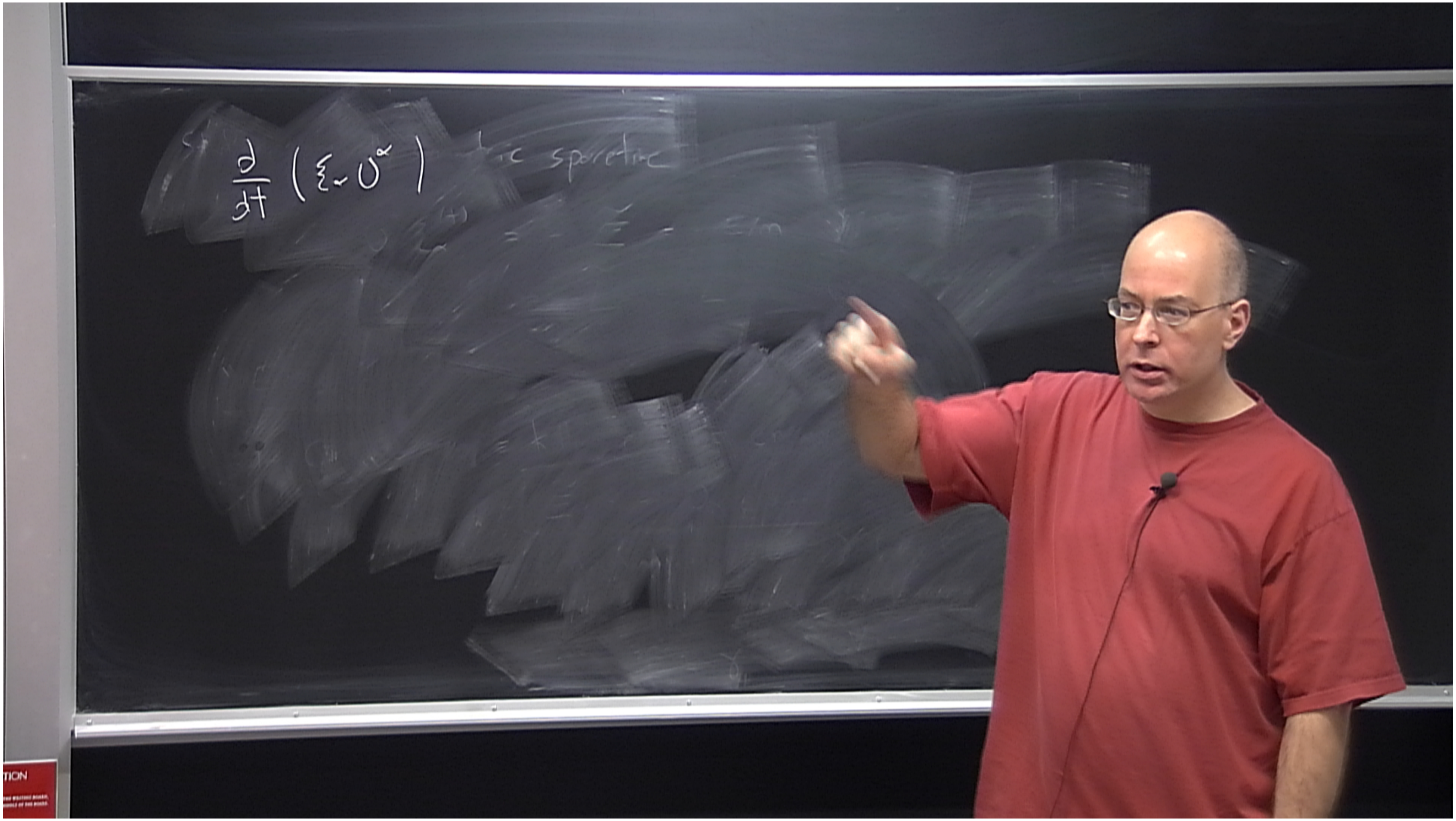
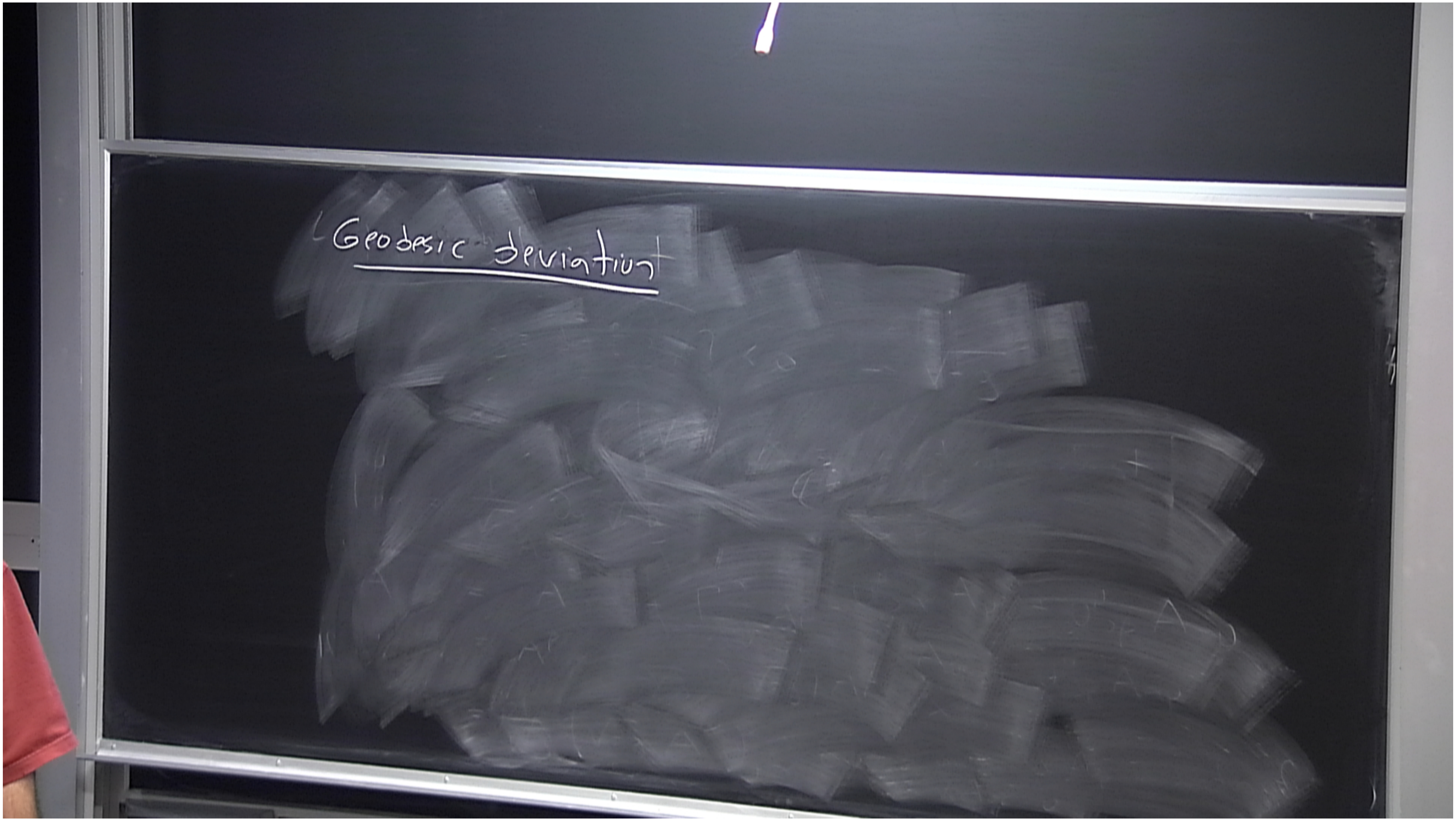
Title: Advanced General Relativity - Lecture 4

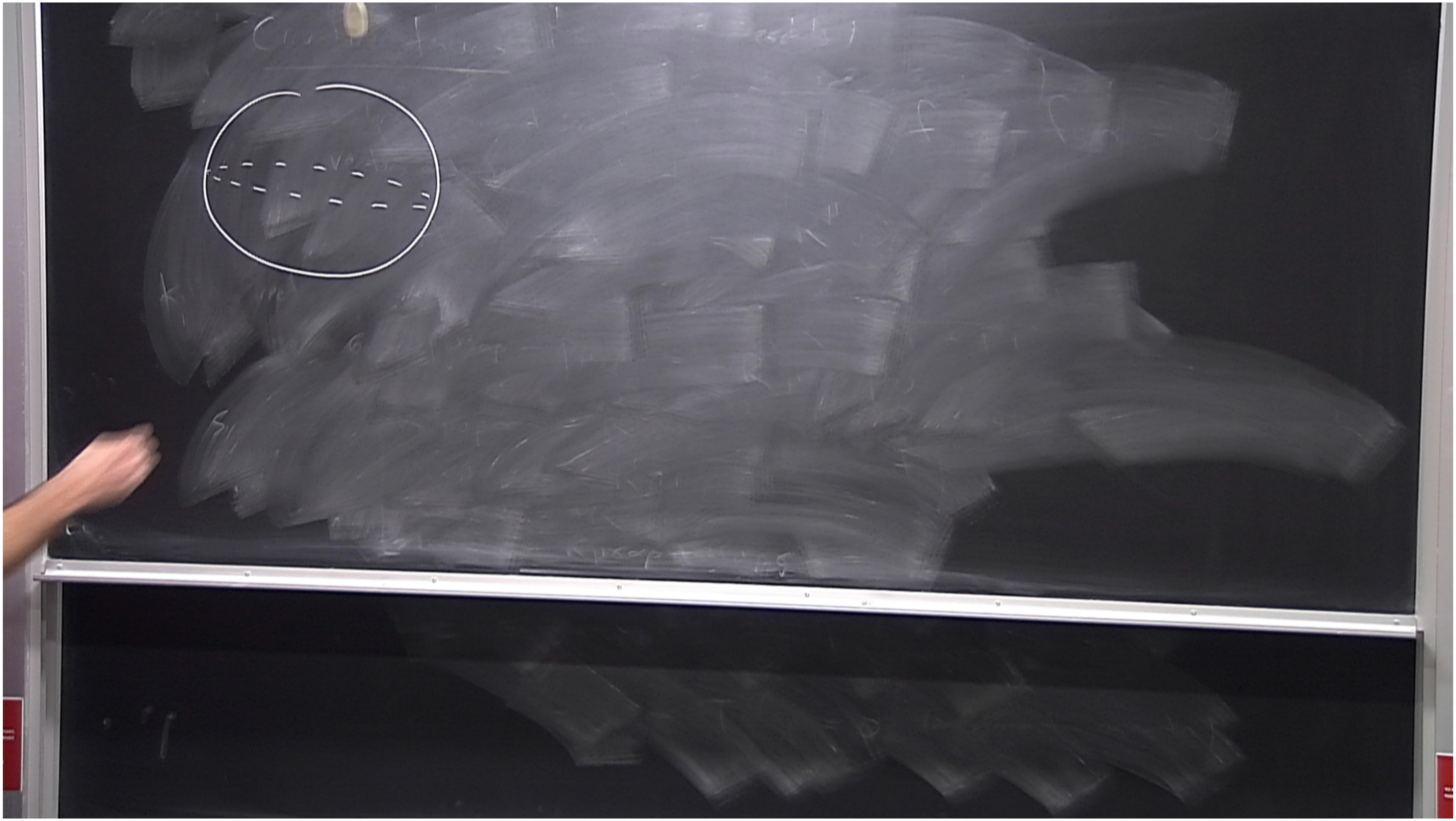
Date: Jan 18, 2012 03:30 PM

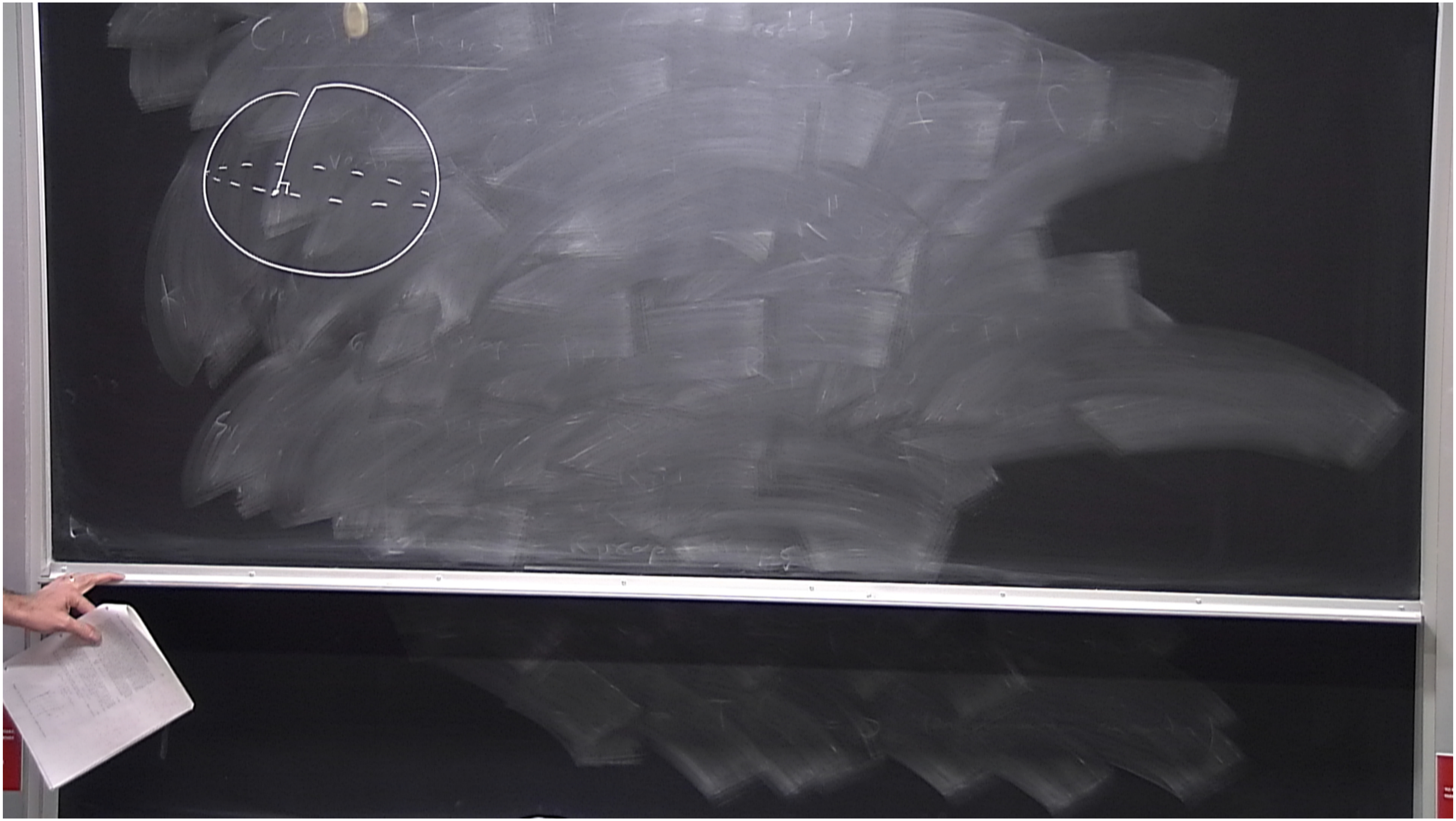
URL: http://pirsa.org/12010155

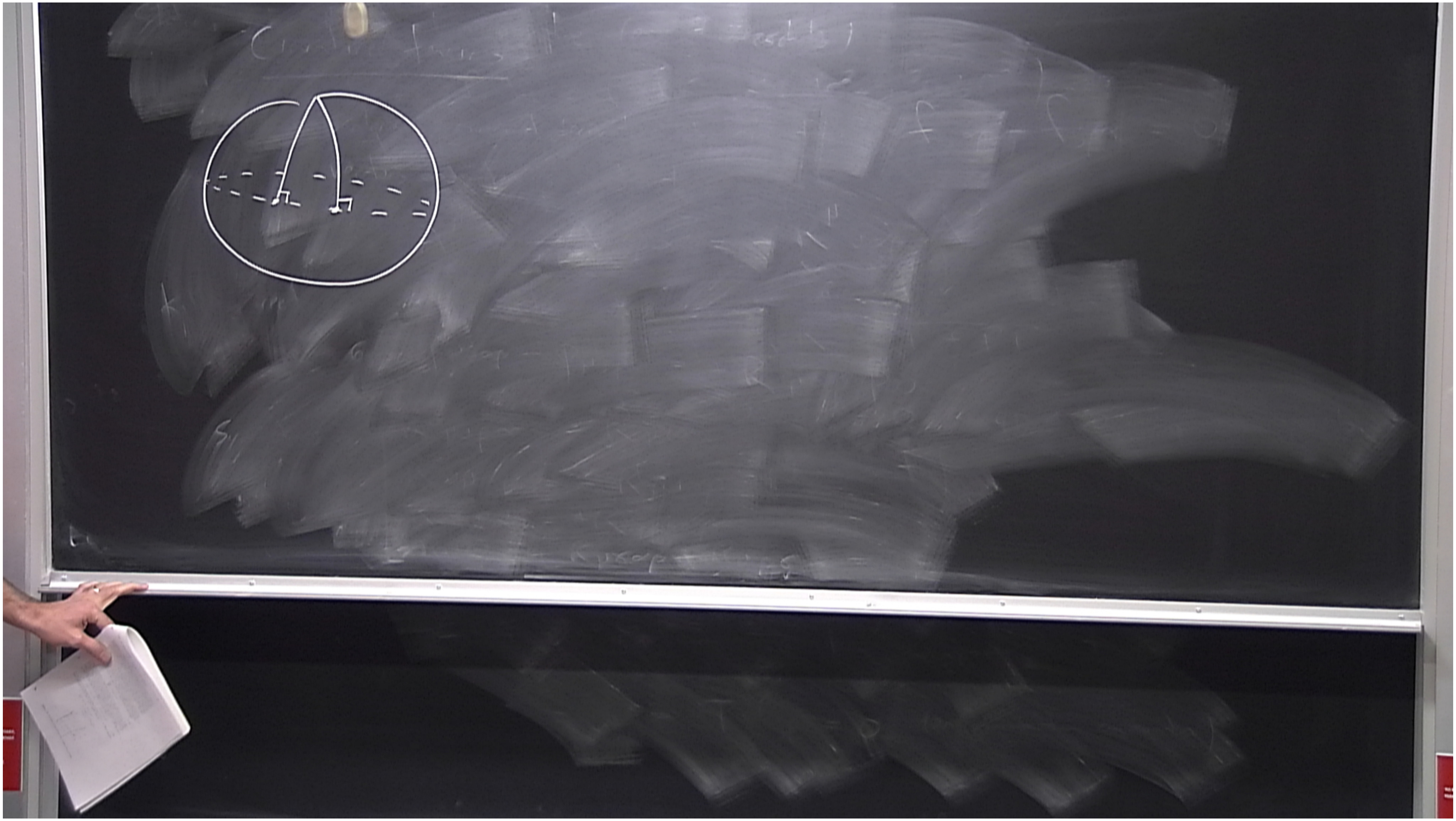
Abstract:

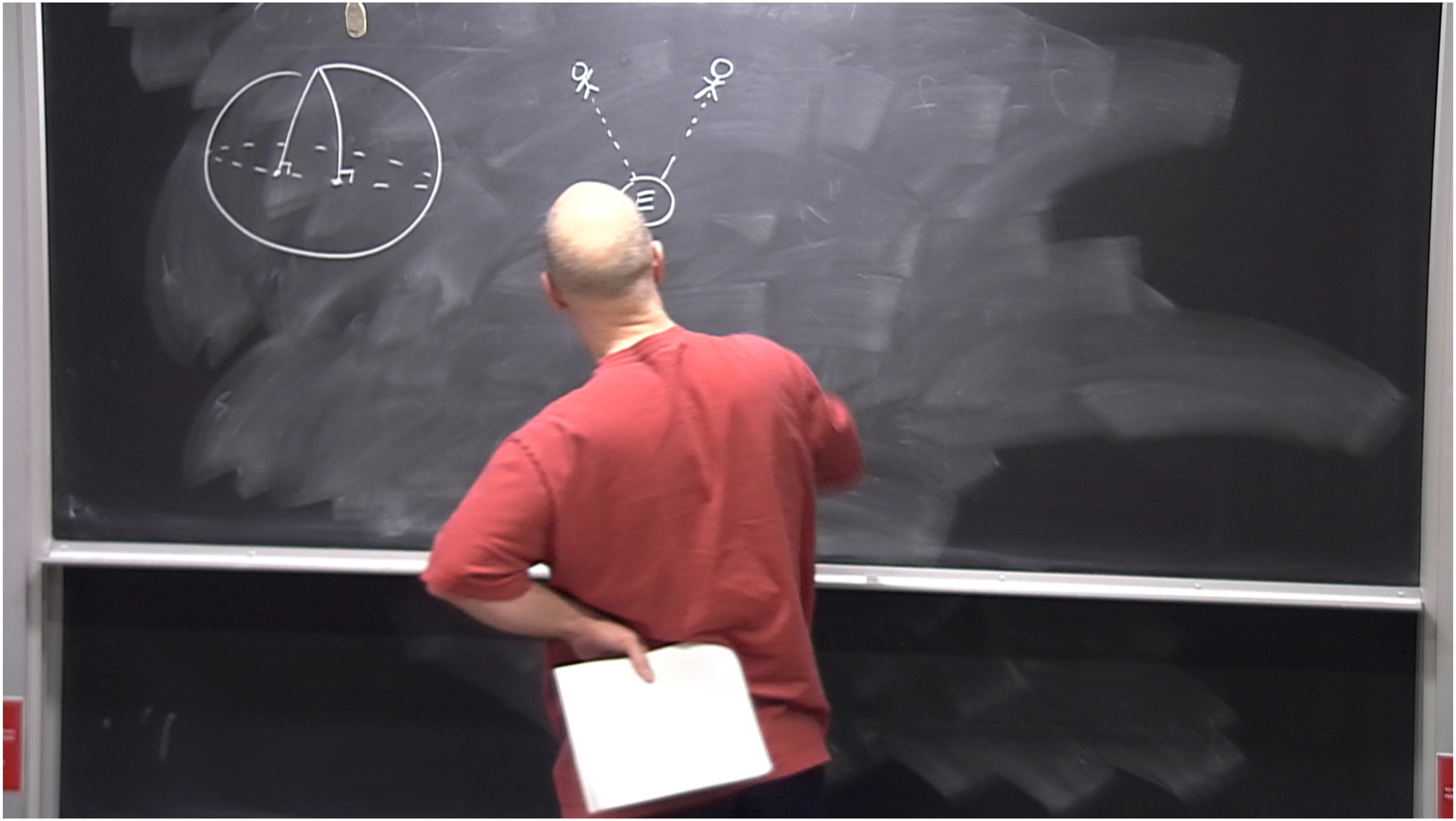

$\frac{d}{dt}(\xi_\alpha U^\alpha)$
spacetime

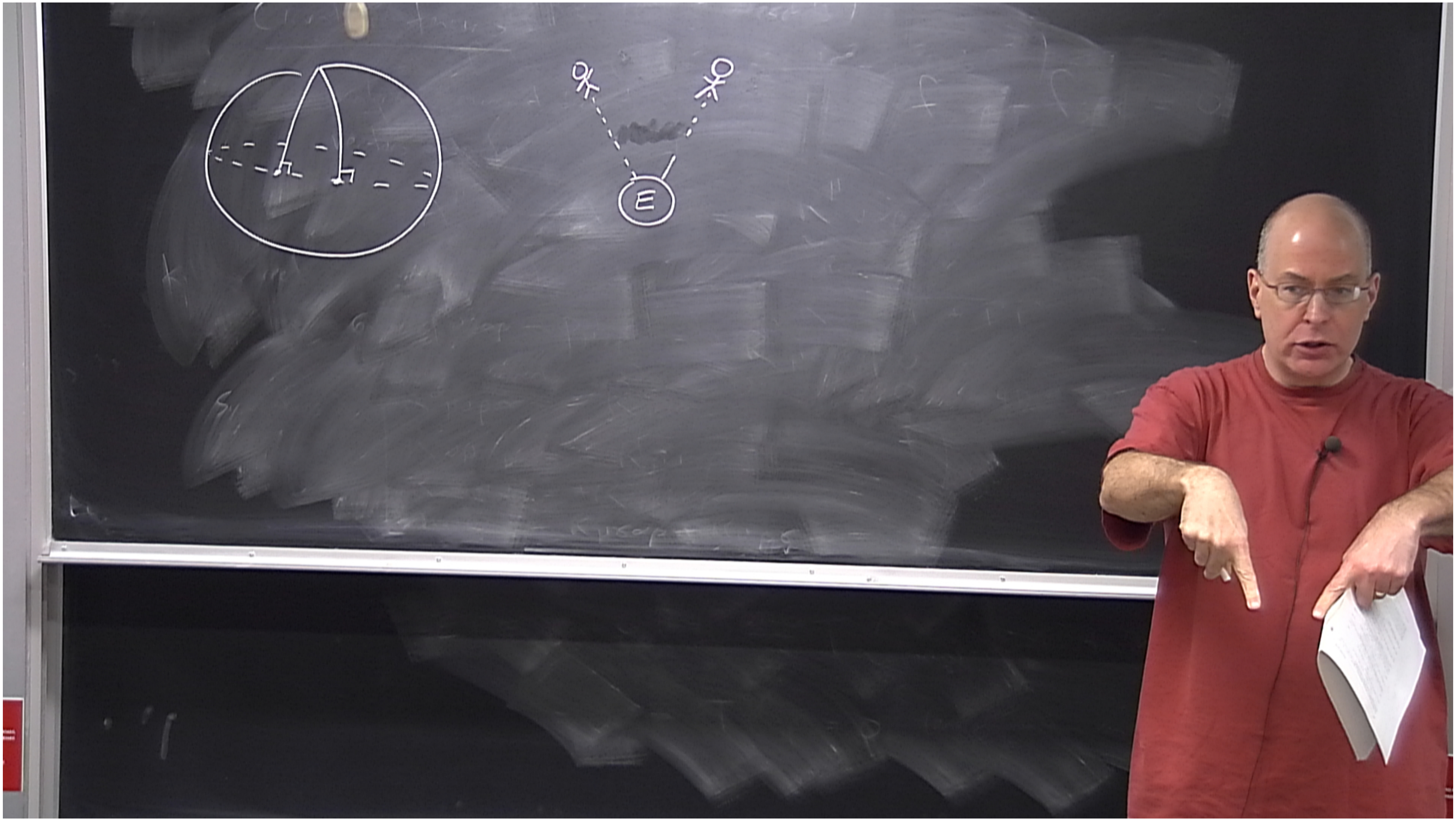

Geodesic deviation

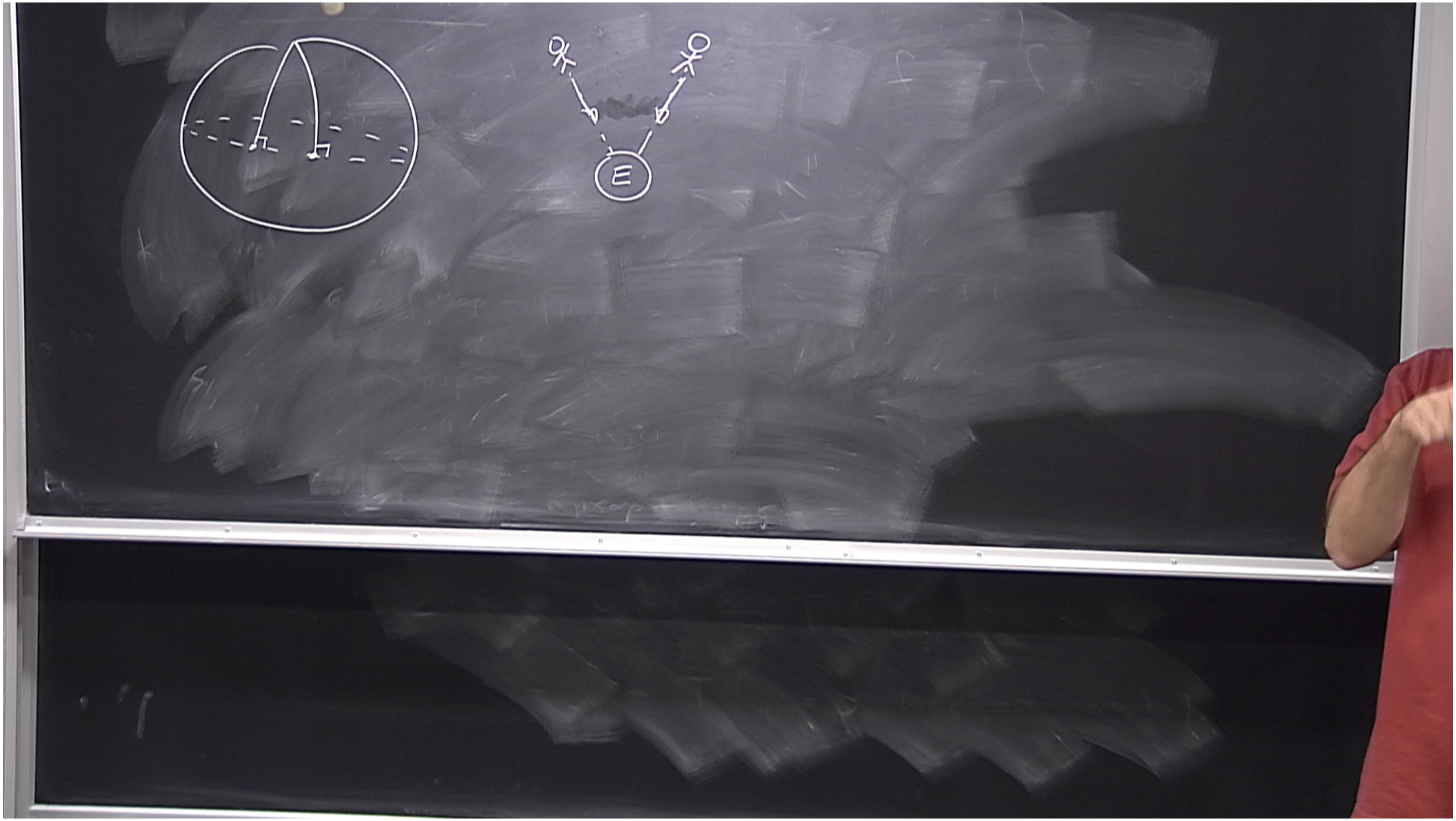


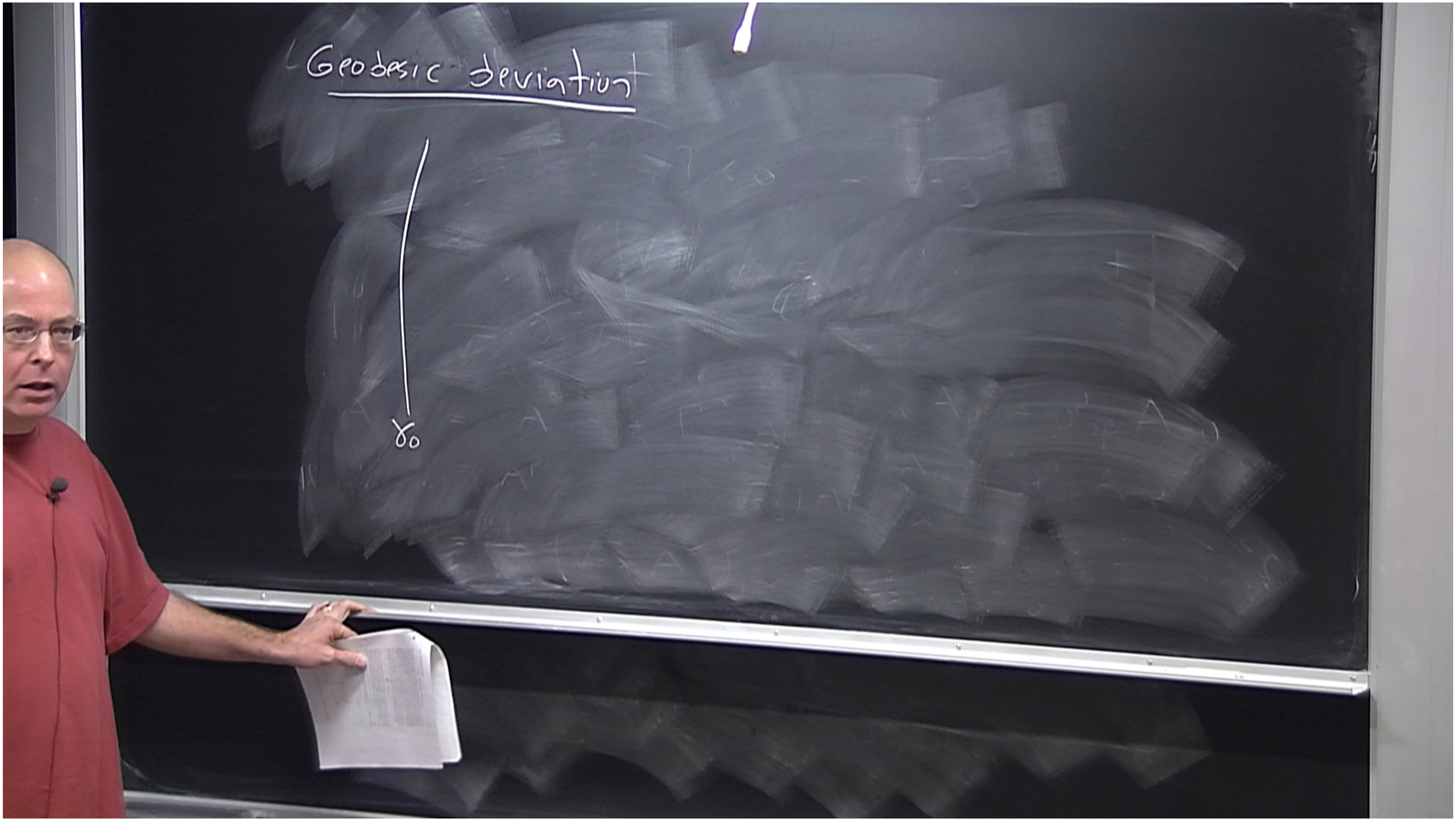


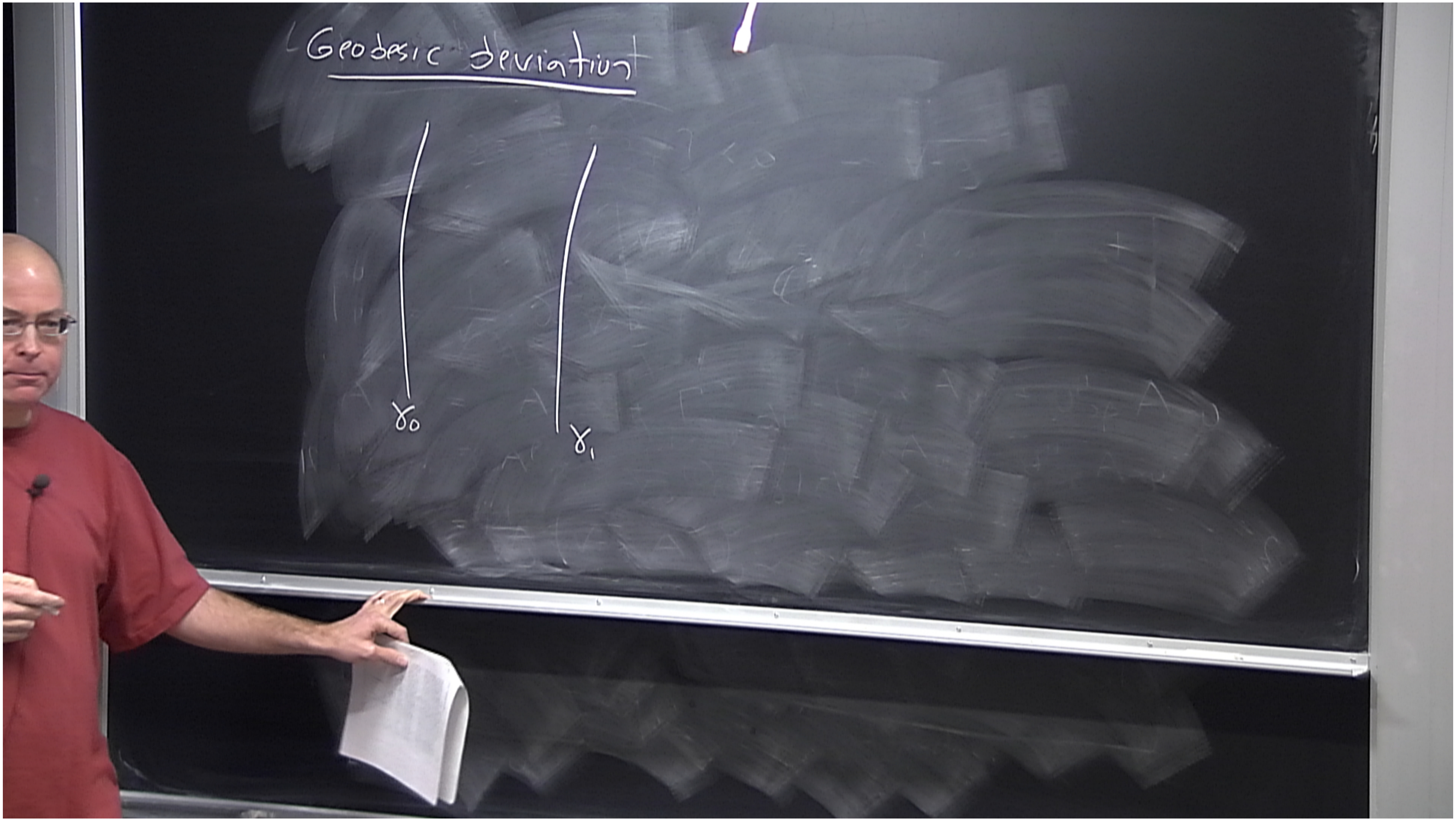




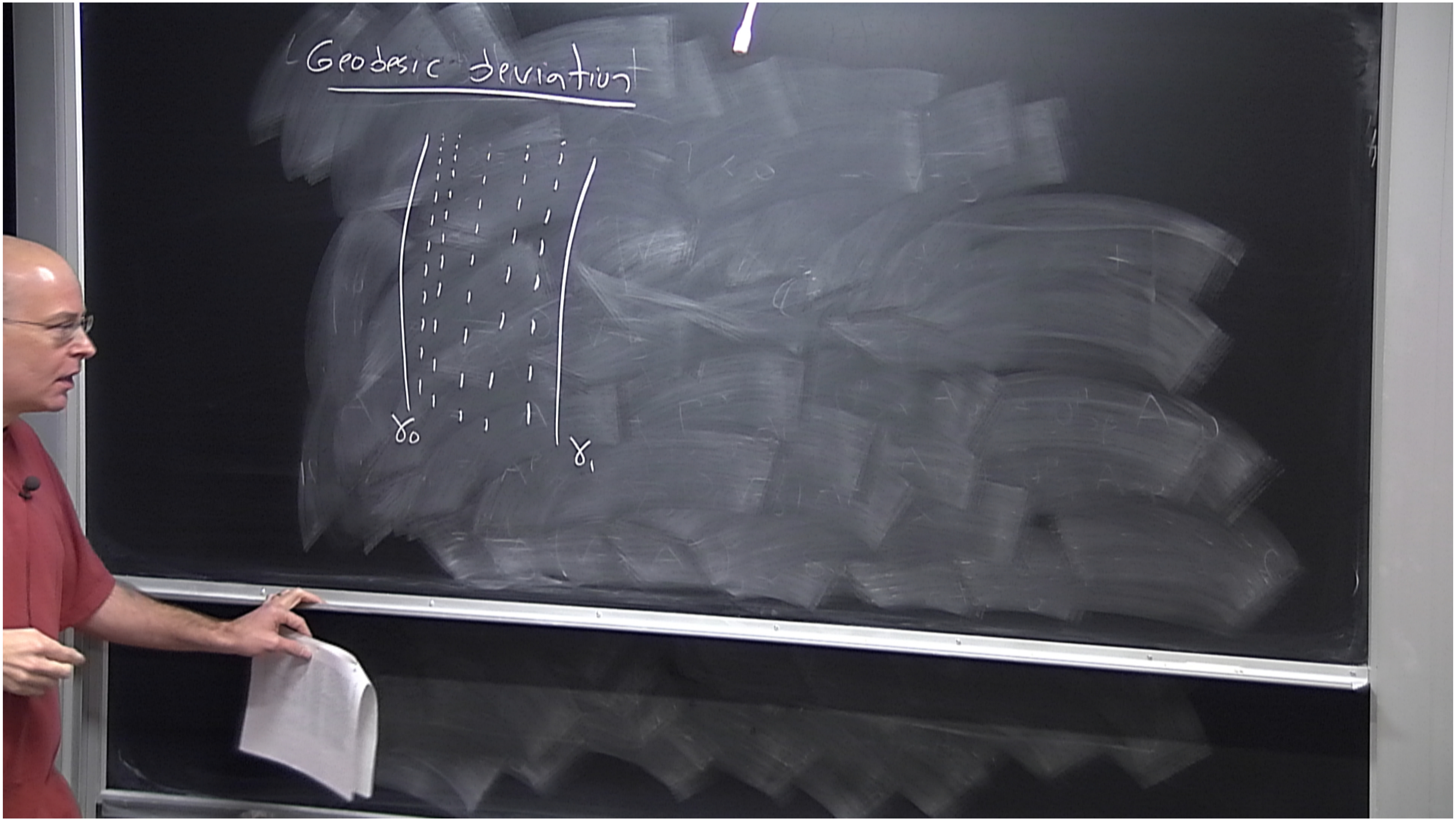

E

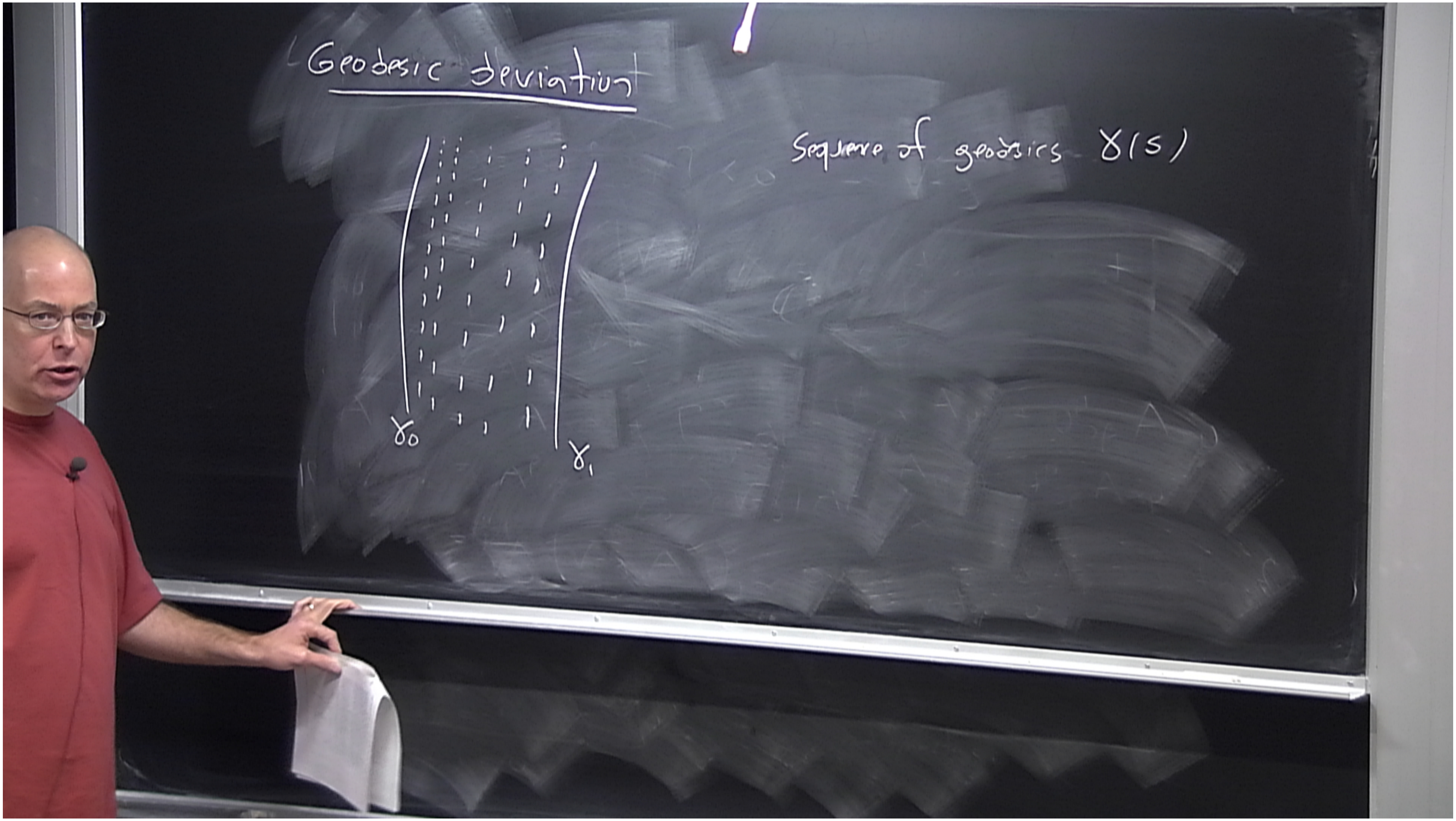


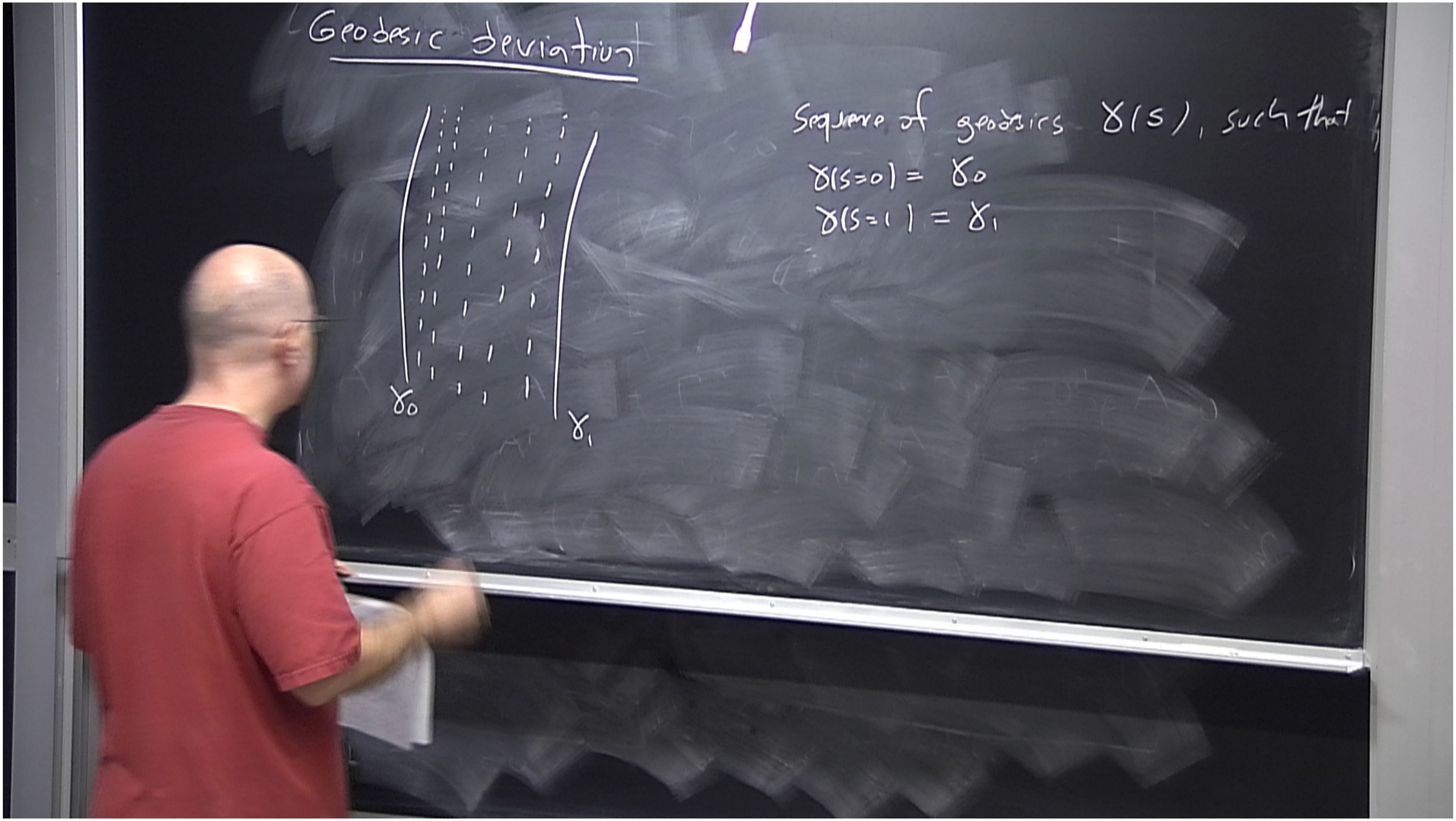

Geodesic deviation
γ0

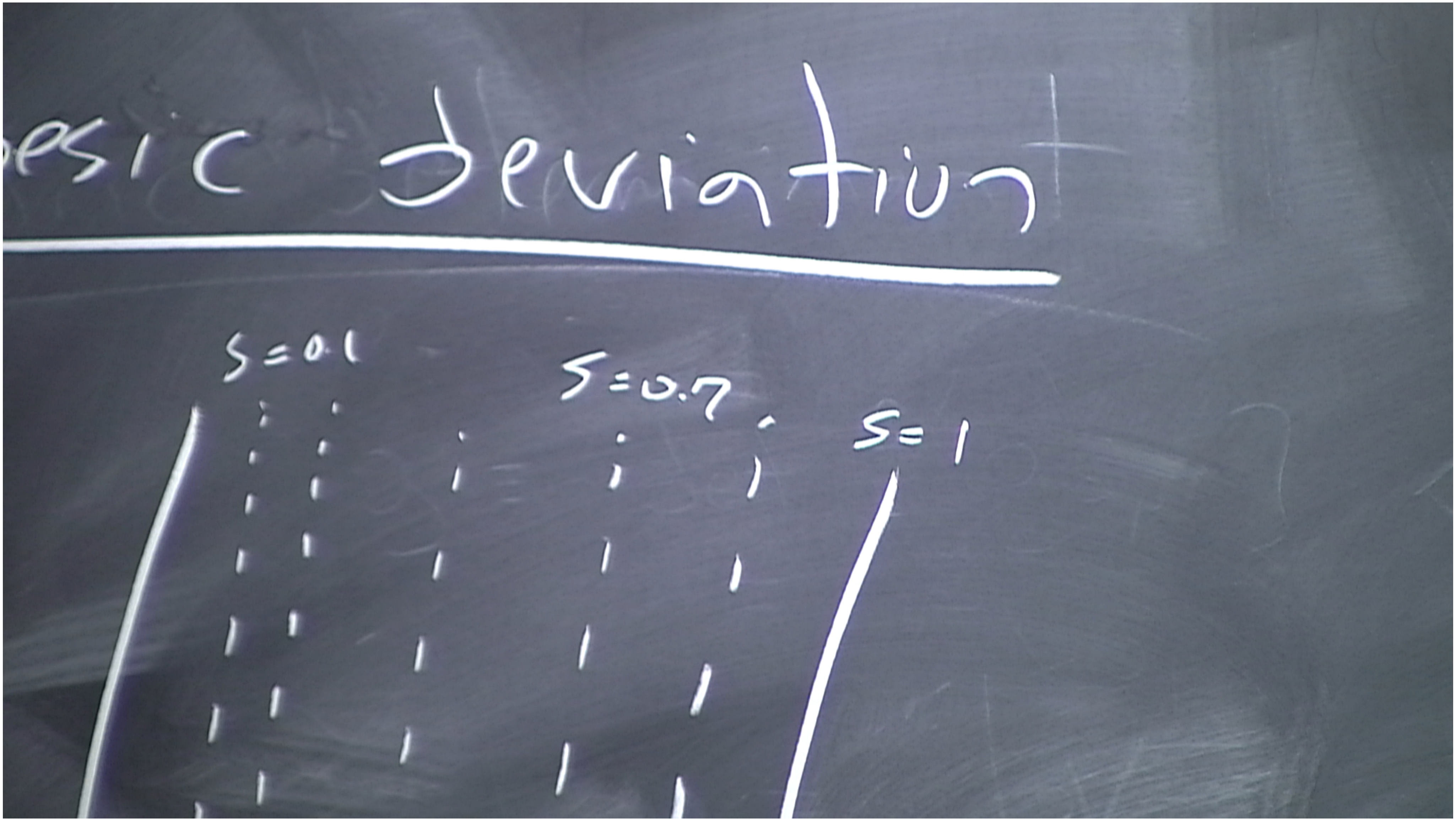


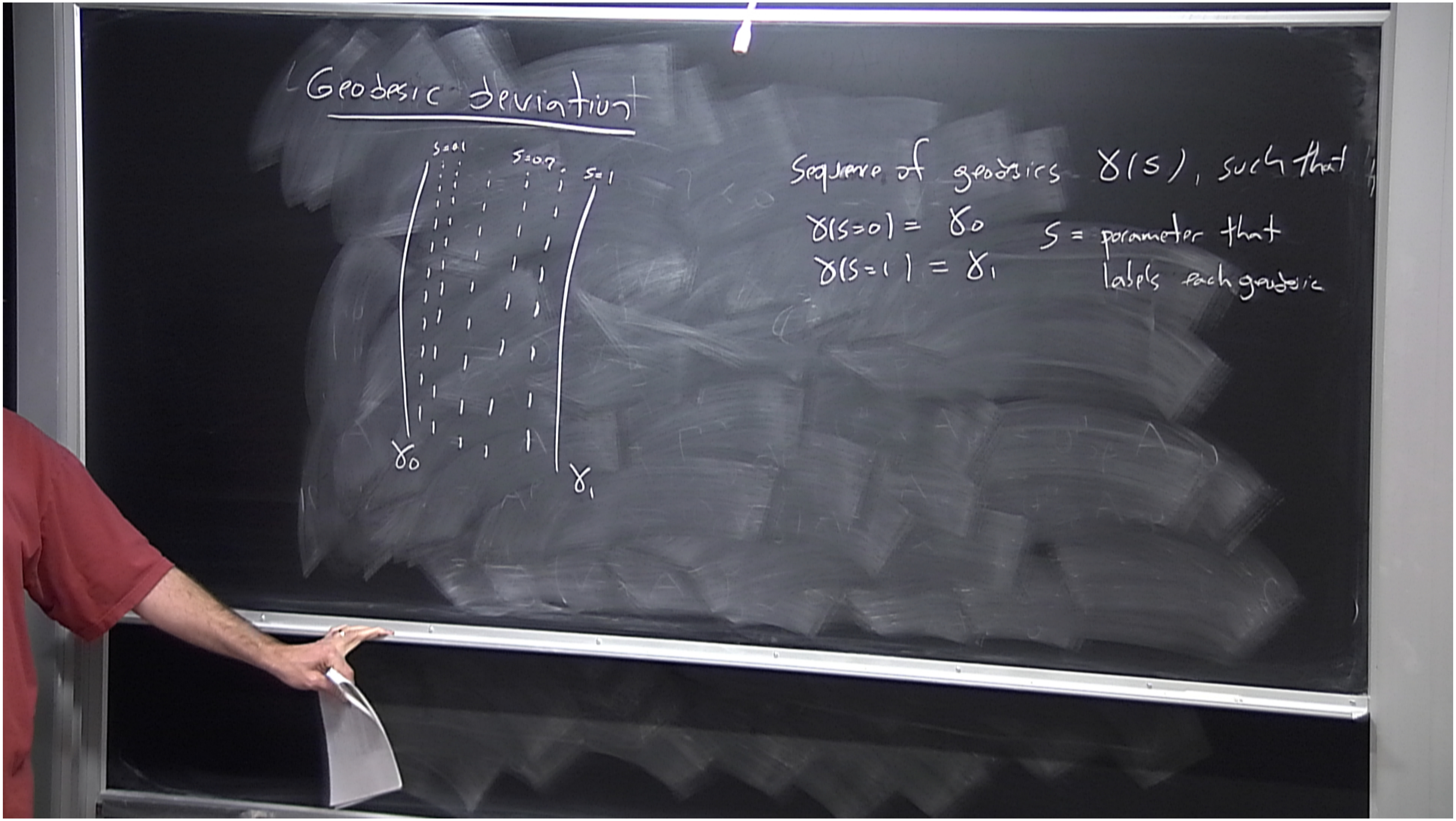
Geodesic deviation
γ_0
γ_1

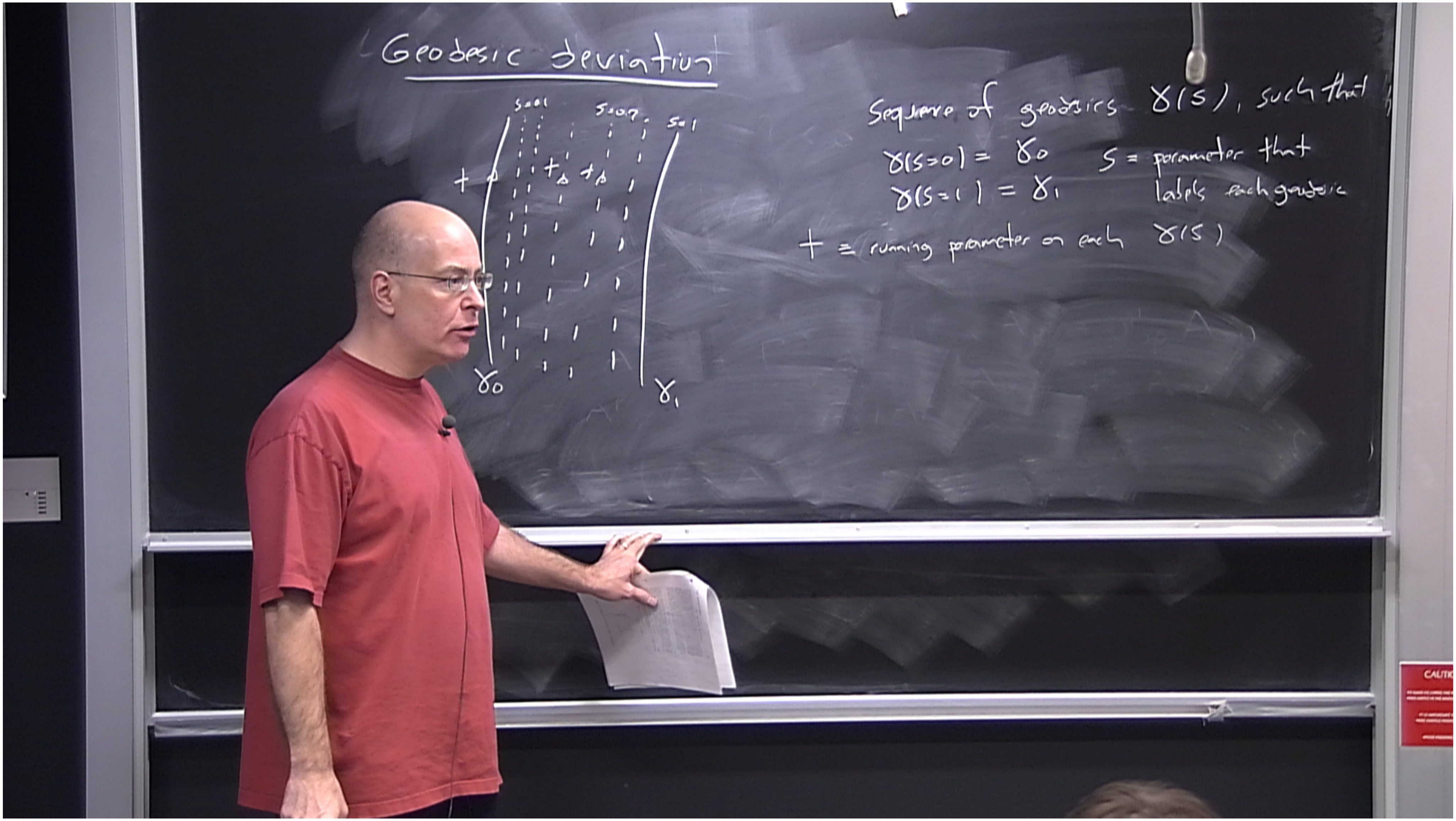

Geodesic deviation
γ0
γ1

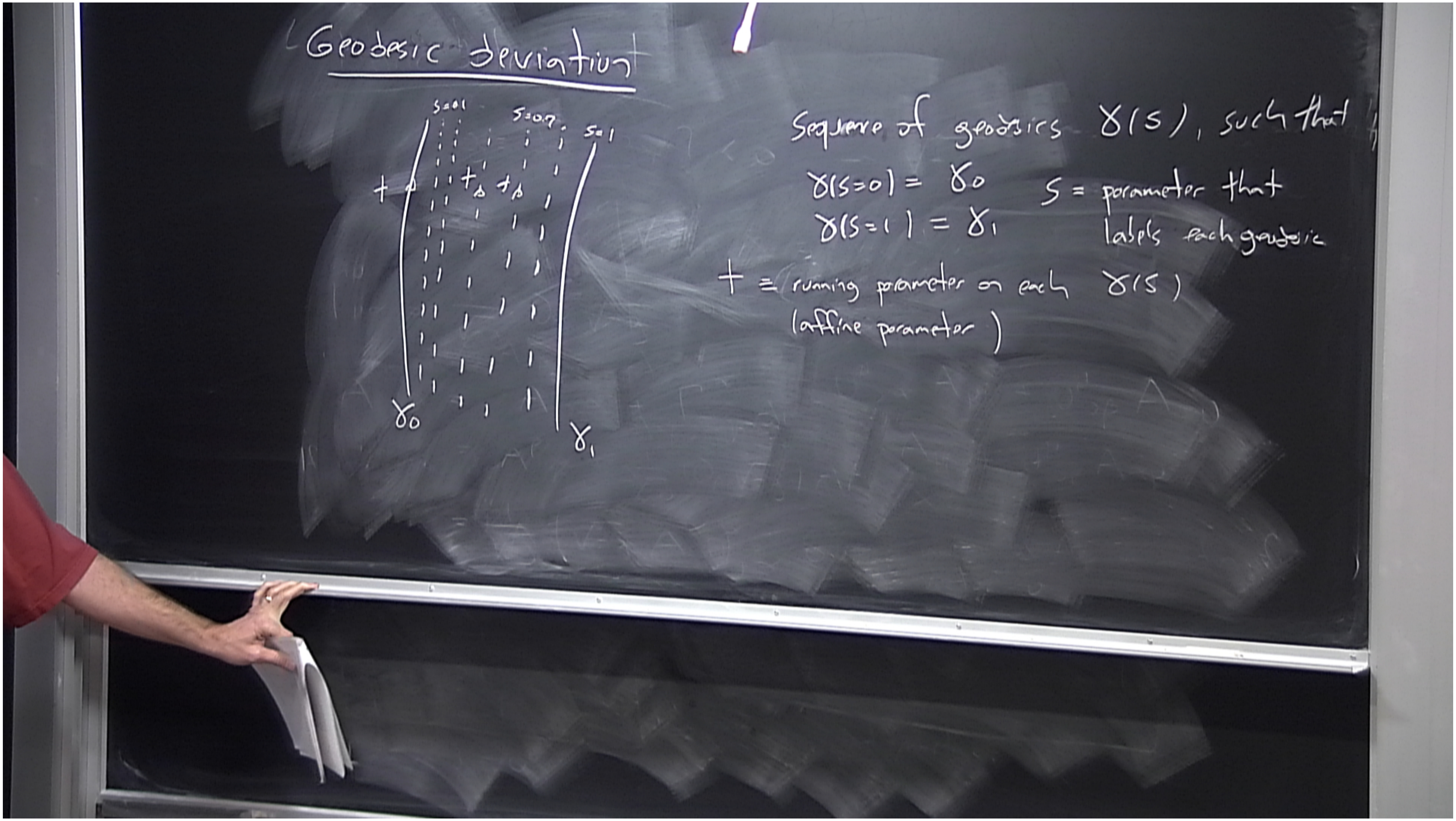

Geodesic deviation
Sequence of geodesics $\gamma(s)$
γ_0
γ_1

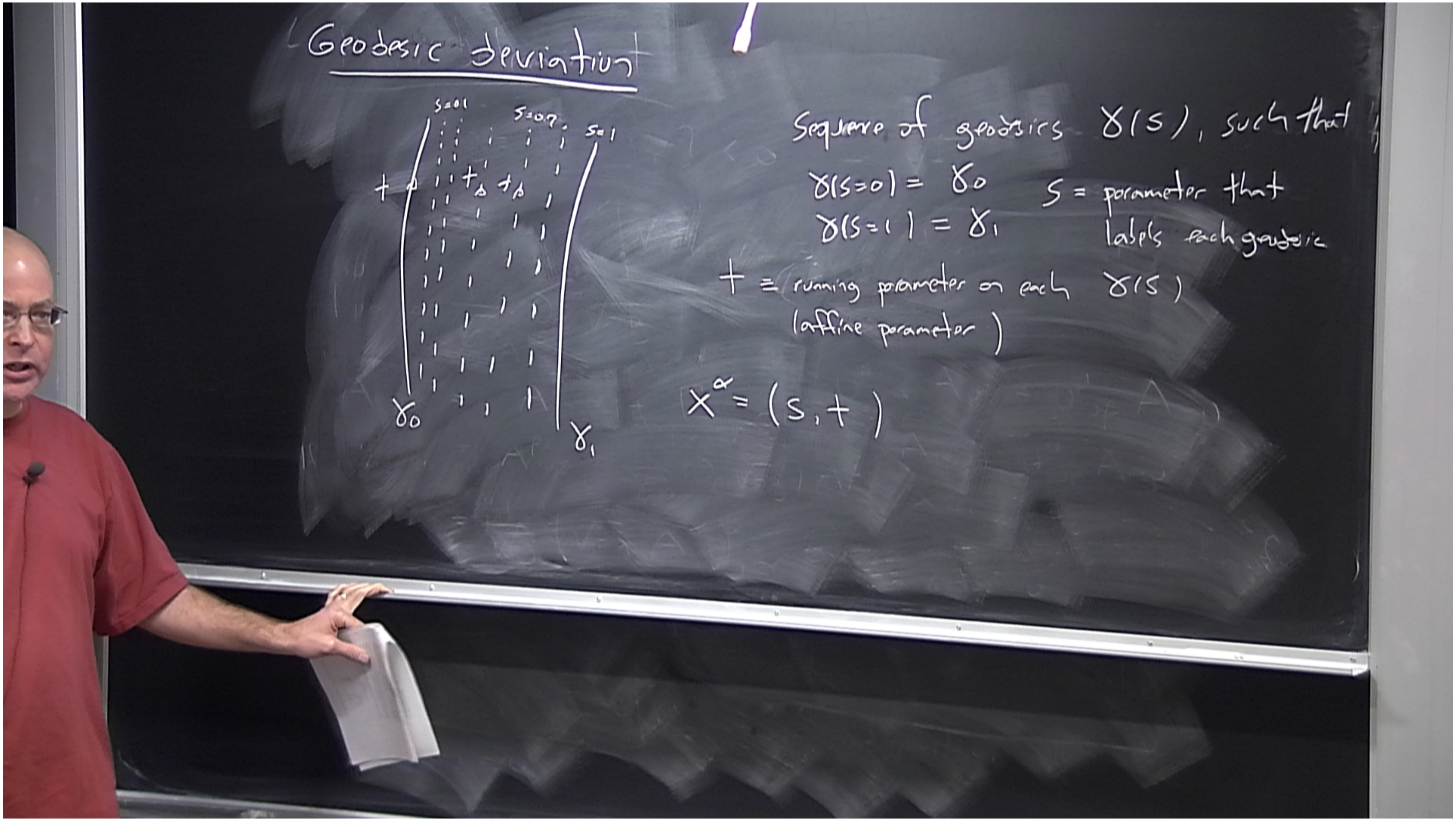

Geodesic deviation
Sequence of geodesics γ(s), such that
γ(s=0) = γ0
γ(s=1) = γ1
γ0
γ1

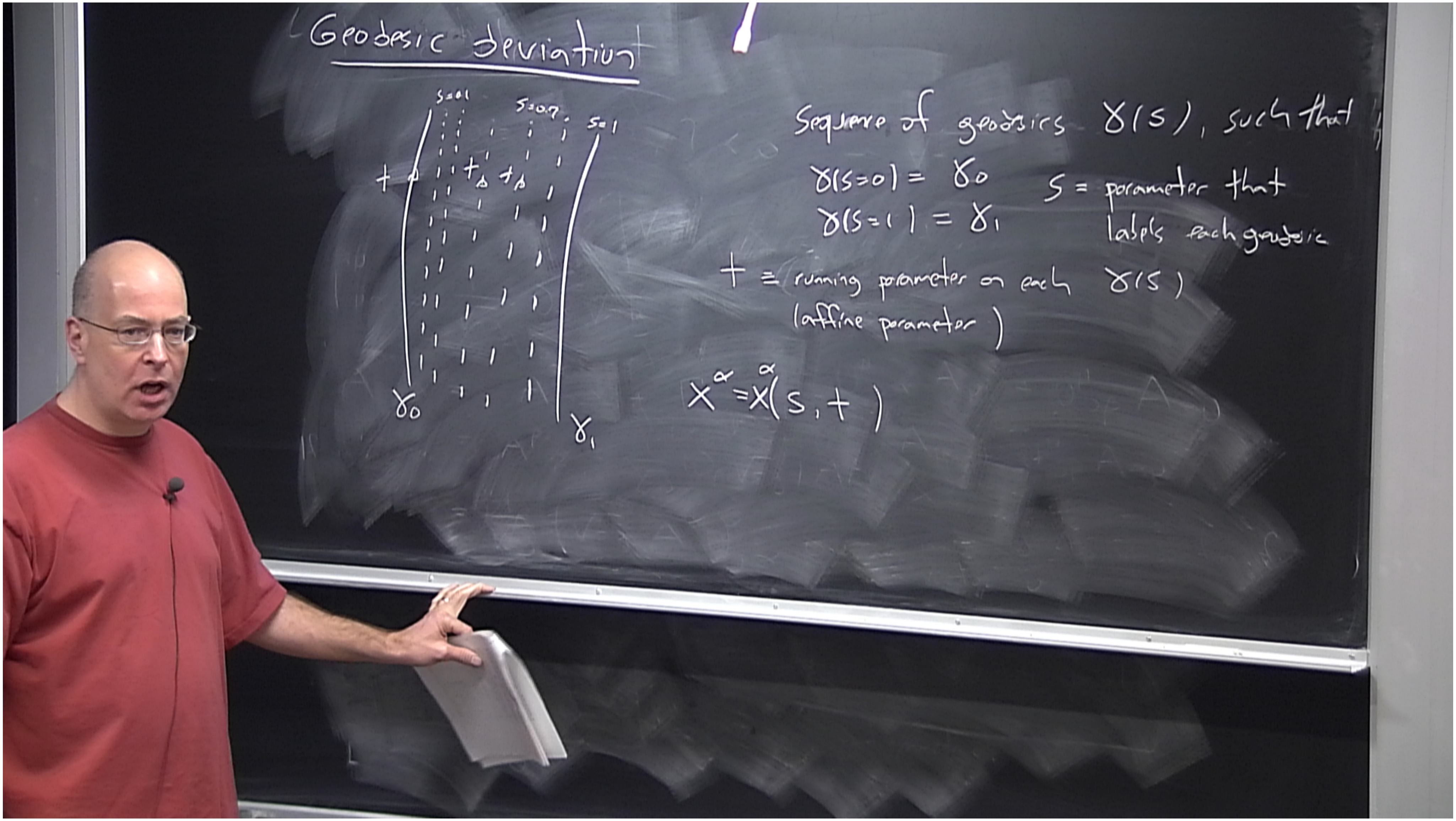

esic deviation
s=0.1
s=0.7
s=1

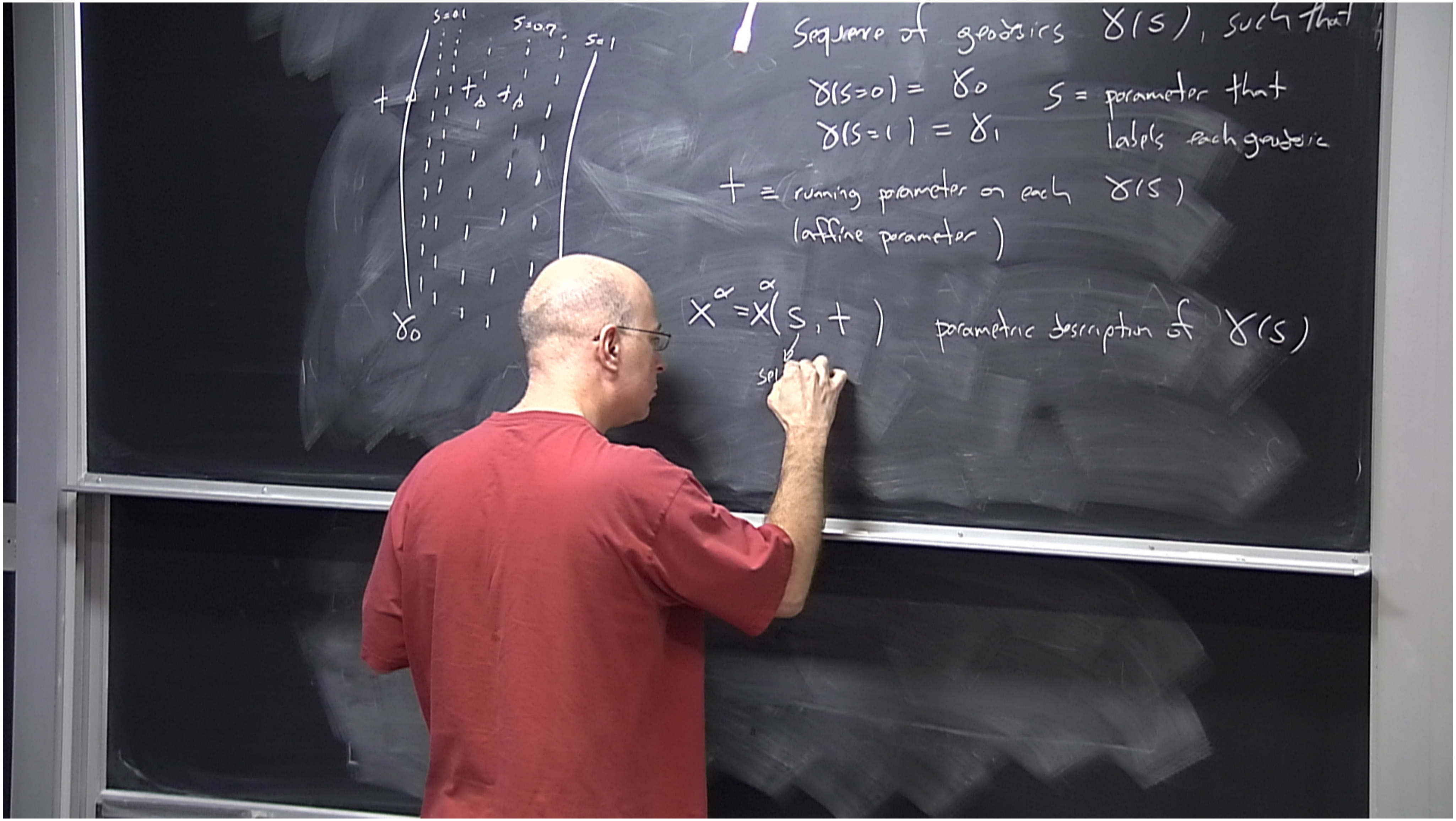

Geodesic deviation
s=1
γ0
γ1
Sequence of geodesics γ(s), such that
γ(s=0) = γ0
γ(s=1) = γ1
s = parameter that
labels each geodesic

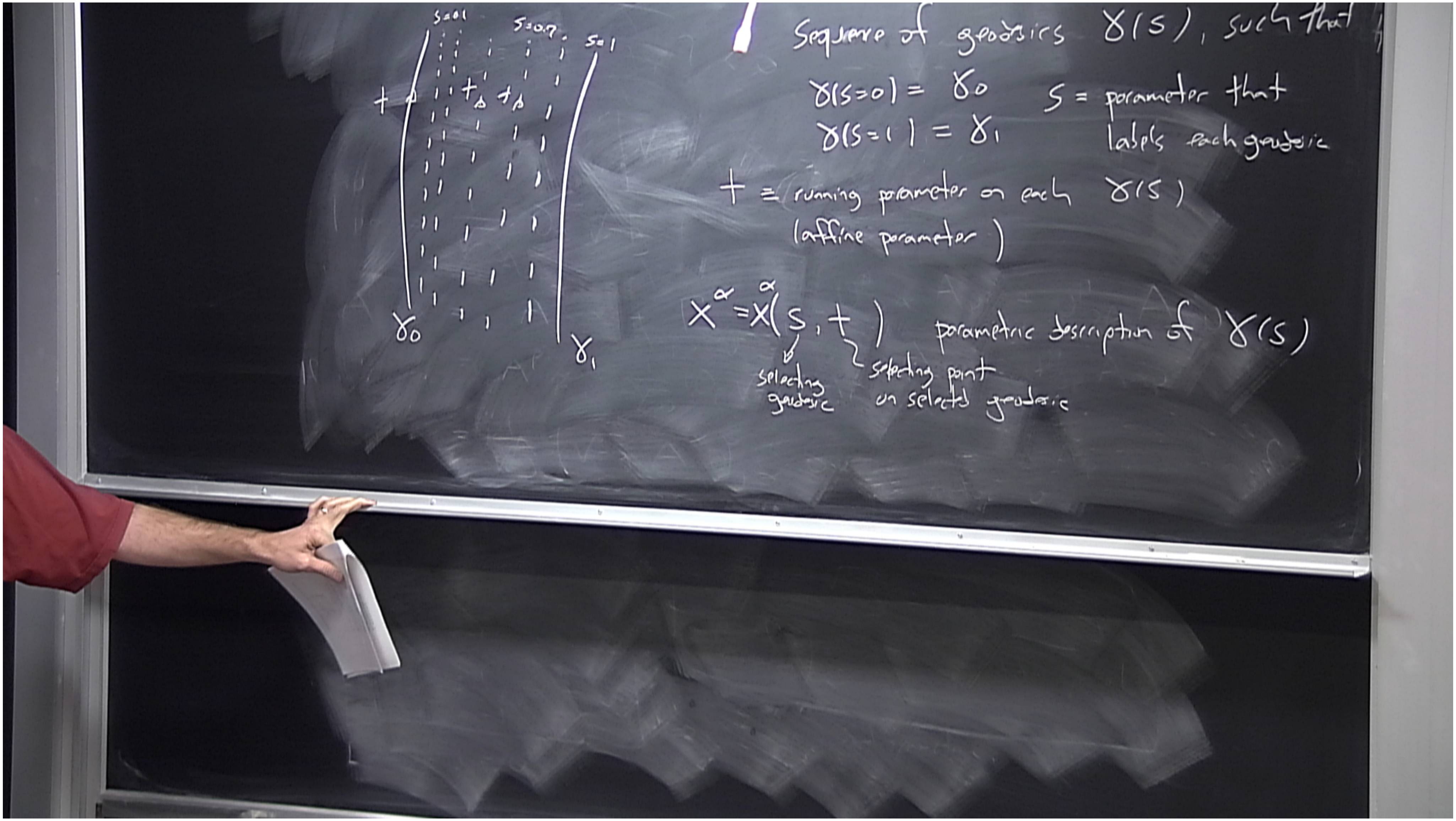

Geodesic deviation
s=0.1
s=0.2
s=1
γ0
γ1
Sequence of geodesics γ(s), such that
γ(s=0) = γ0
γ(s=1) = γ1
s = parameter that
labels each geodesic
t = running parameter on each γ(s)

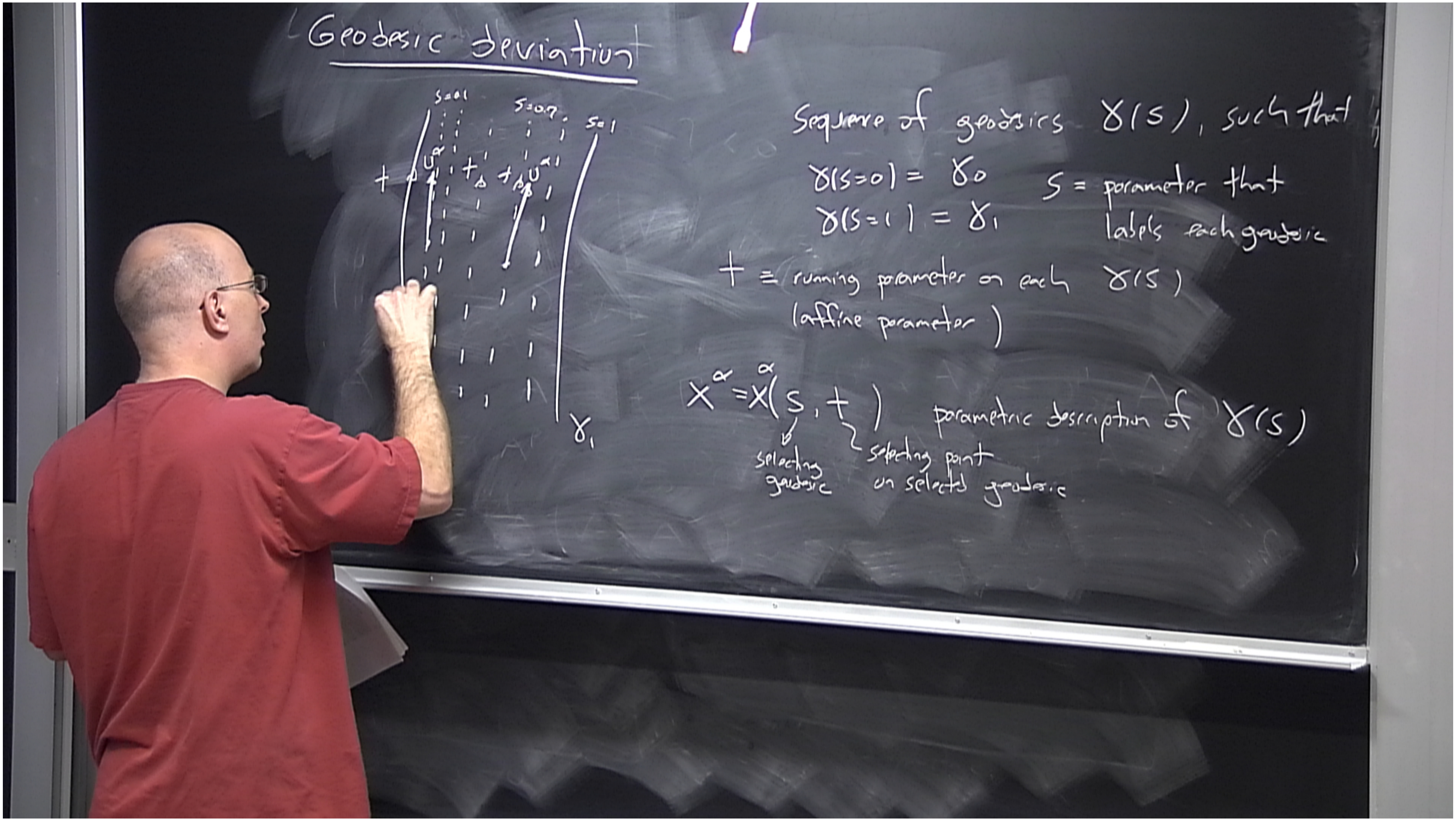

Geodesic deviation
Sequence of geodesics γ(s), such that
γ(s=0) = γ₀
γ(s=1) = γ₁
s = parameter that labels each geodesic
t = running parameter on each γ(s)
(affine parameter)
s=0.1
s=0.7
s=1
γ₀
γ₁

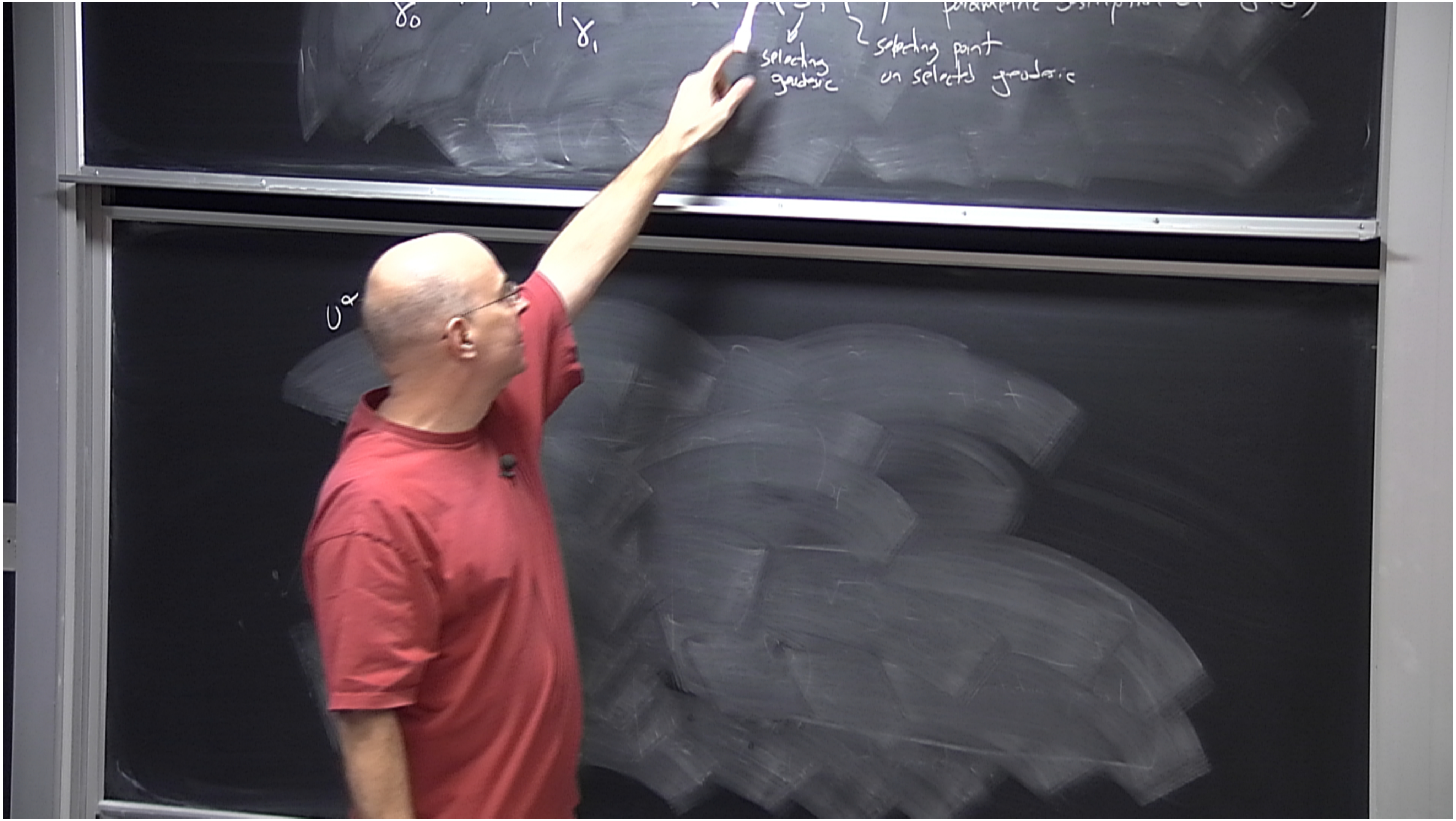

Geodesic deviation
s=0.1
s=0.7
s=1
t
γ0
γ1
Sequence of geodesics γ(s), such that
γ(s=0) = γ0
γ(s=1) = γ1
s = parameter that labels each geodesic
t = running parameter on each γ(s)
(affine parameter)
x^α = (s, t)


Geodesic deviation
s=1
γ0
γ1
Sequence of geodesics γ(s), such that
γ(s=0) = γ0
γ(s=1) = γ1
s = parameter that labels each geodesic
t = running parameter on each γ(s)
(affine parameter)
x^α = x^α(s, t)

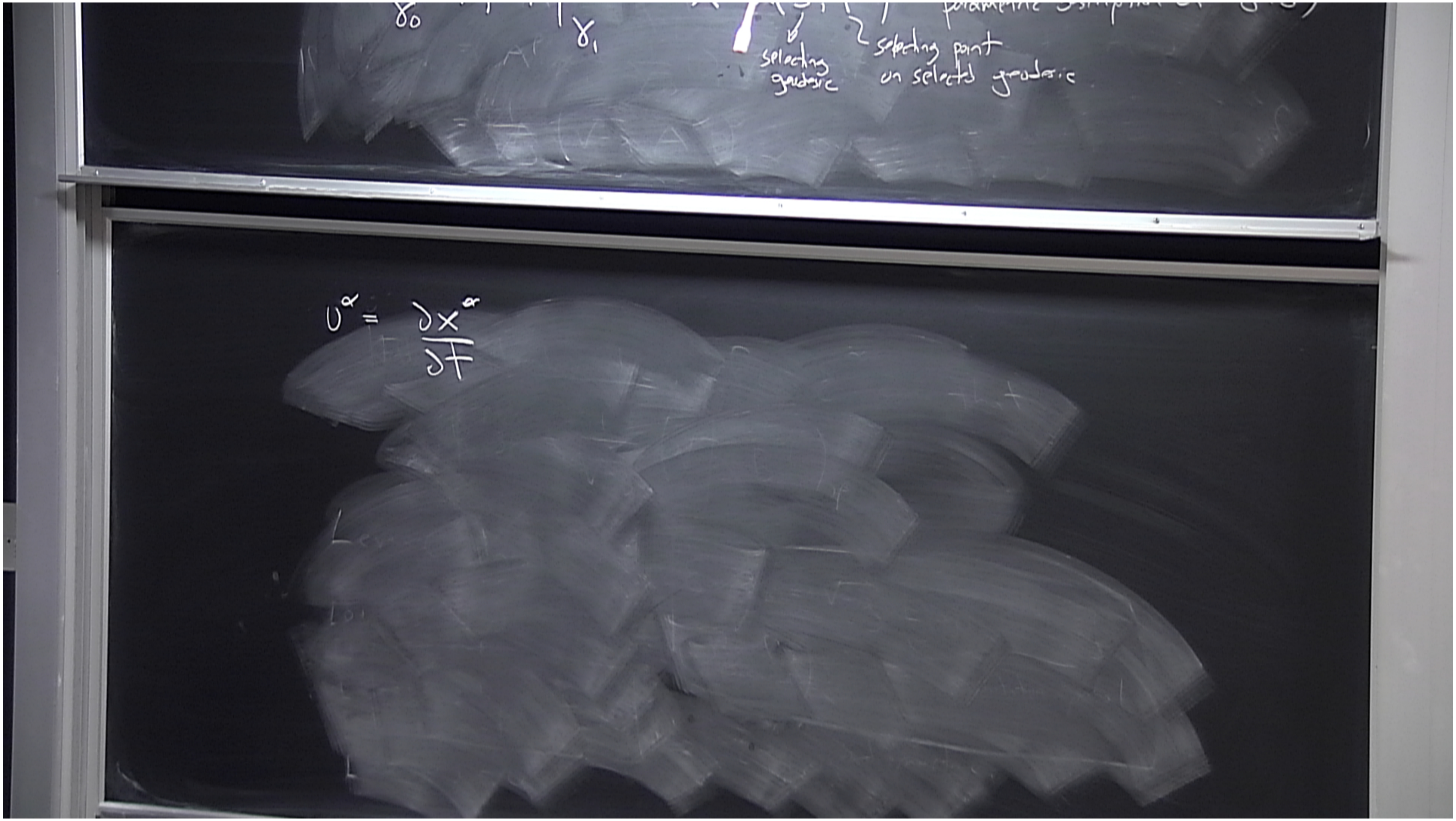

Sequence of geodesics γ(s), such that
γ(s=0) = γ₀
γ(s=1) = γ₁
s = parameter that labels each geodesic
t = running parameter on each γ(s) (affine parameter)
x^α = x^α(s, t) parametric description of γ(s)
s=0.1
s=0.7
s=1
γ₀

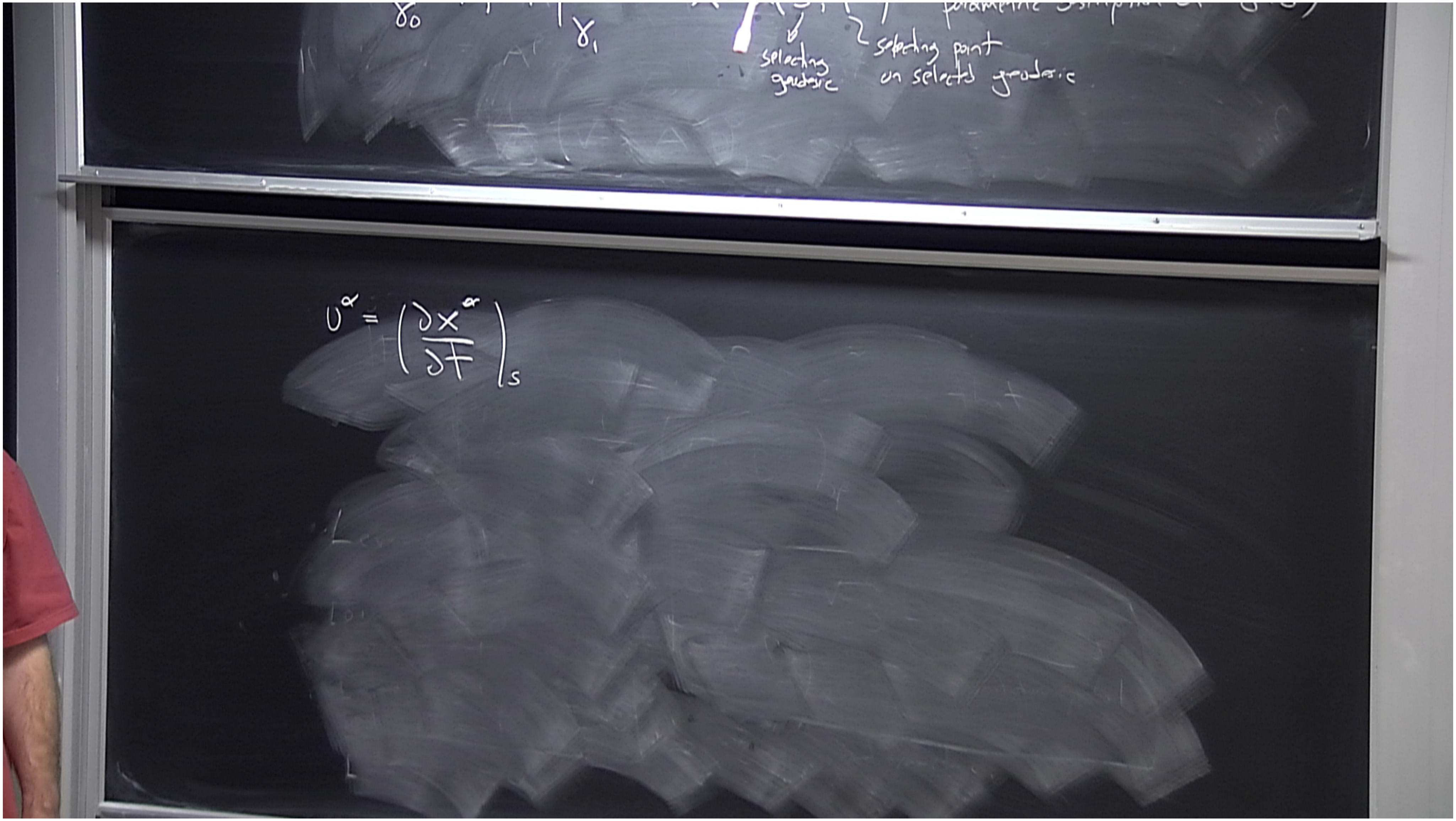

Sequence of geodesics γ(s), such that
γ(s=0) = γ0
γ(s=1) = γ1
s = parameter that labels each geodesic
t = running parameter on each γ(s)
(affine parameter)
x^α = x^α(s, t)
parametric description of γ(s)
selecting geodesic
selecting point on selected geodesic
s=1
γ0
γ1

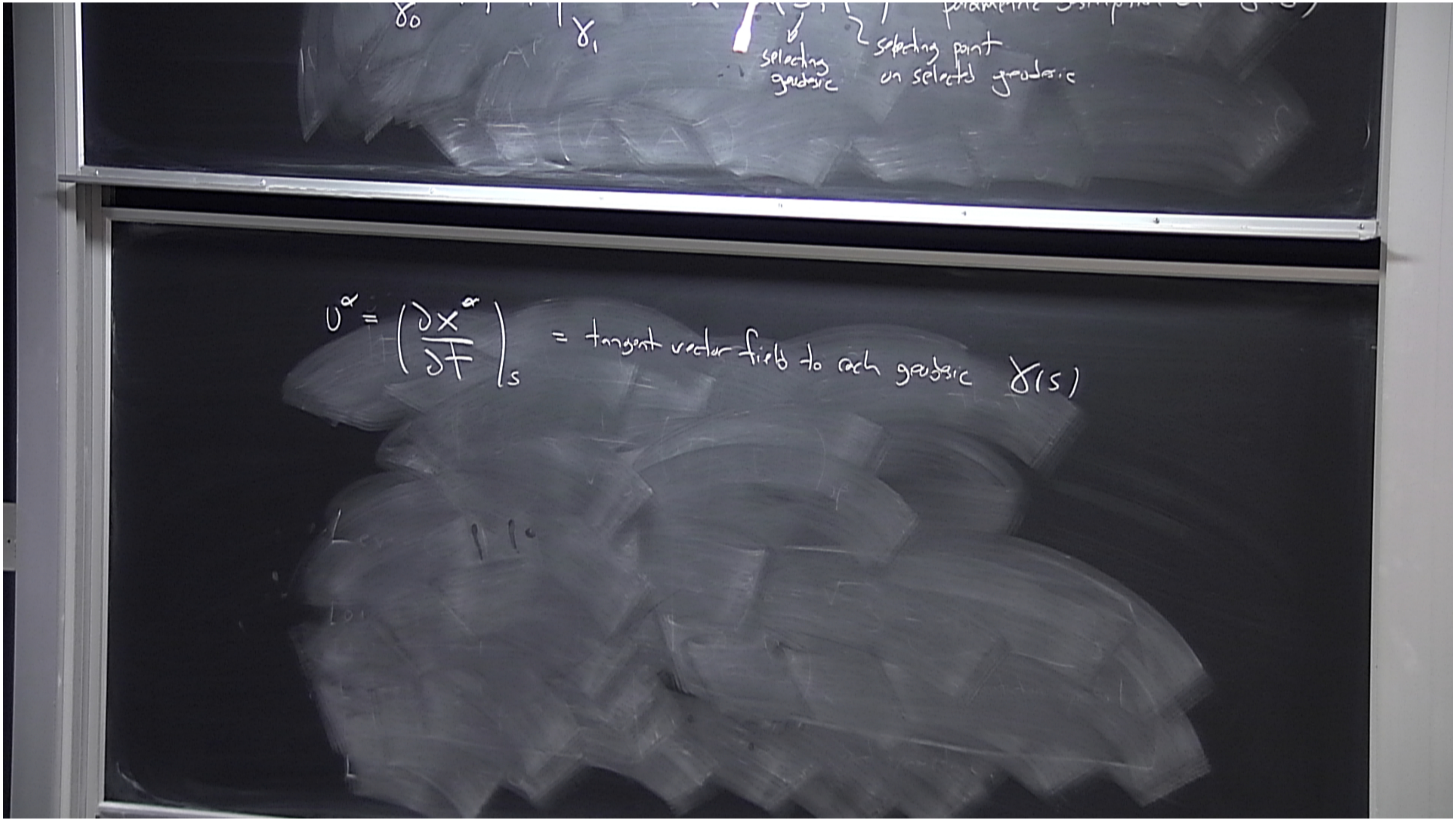

Geodesic deviation
Sequence of geodesics γ(s), such that
γ(s=0) = γ0
γ(s=1) = γ1
s = parameter that labels each geodesic
t = running parameter on each γ(s)
(affine parameter)
x^α = x^α(s, t) parametric description of γ(s)
selecting geodesic
selecting point on selected geodesic
γ1



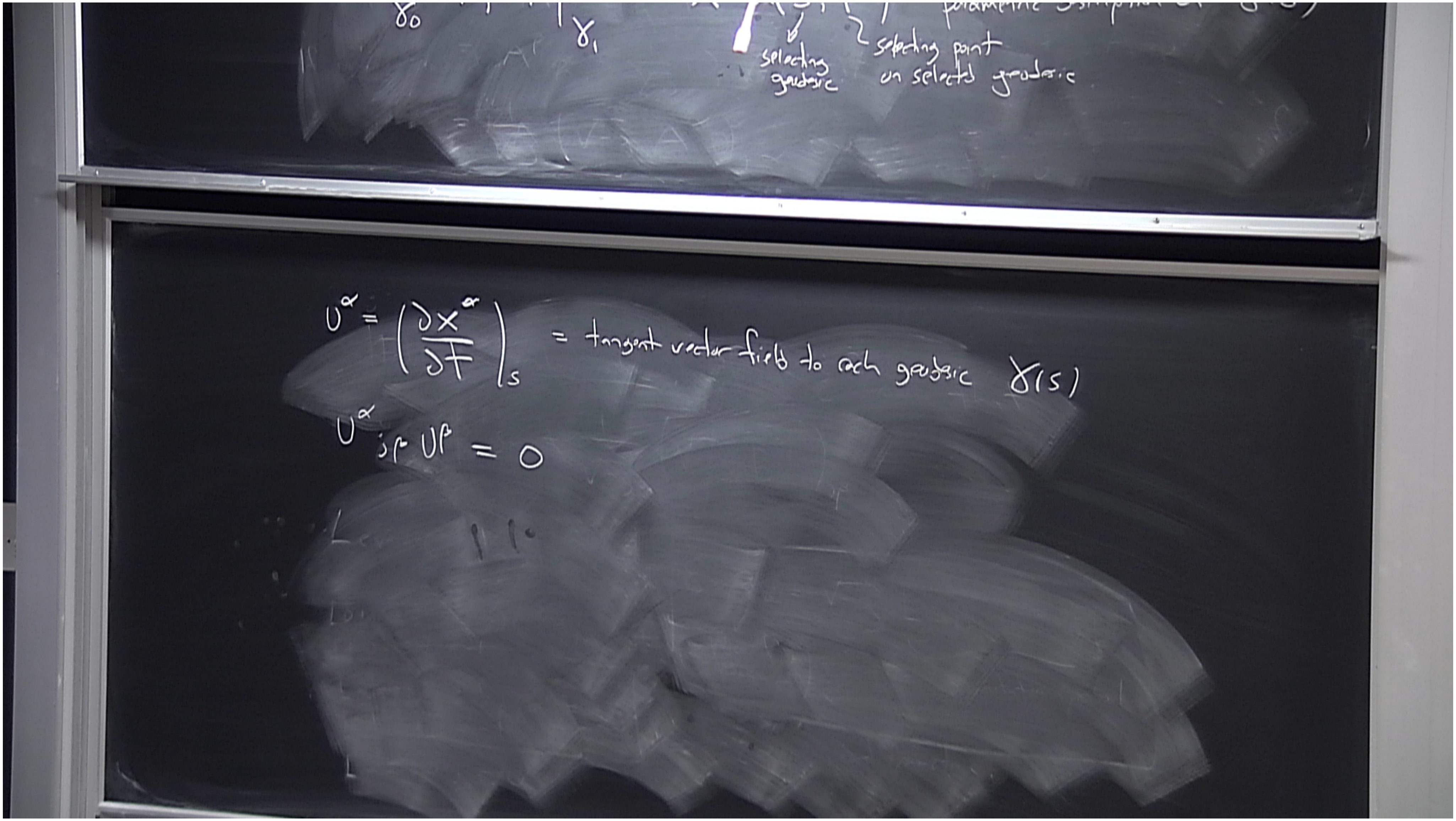
selecting geodesic
selecting point on selected geodesic


selecting geodesic
selecting point on selected geodesic
$U^\alpha = \frac{\partial x^\alpha}{\partial \tau}$

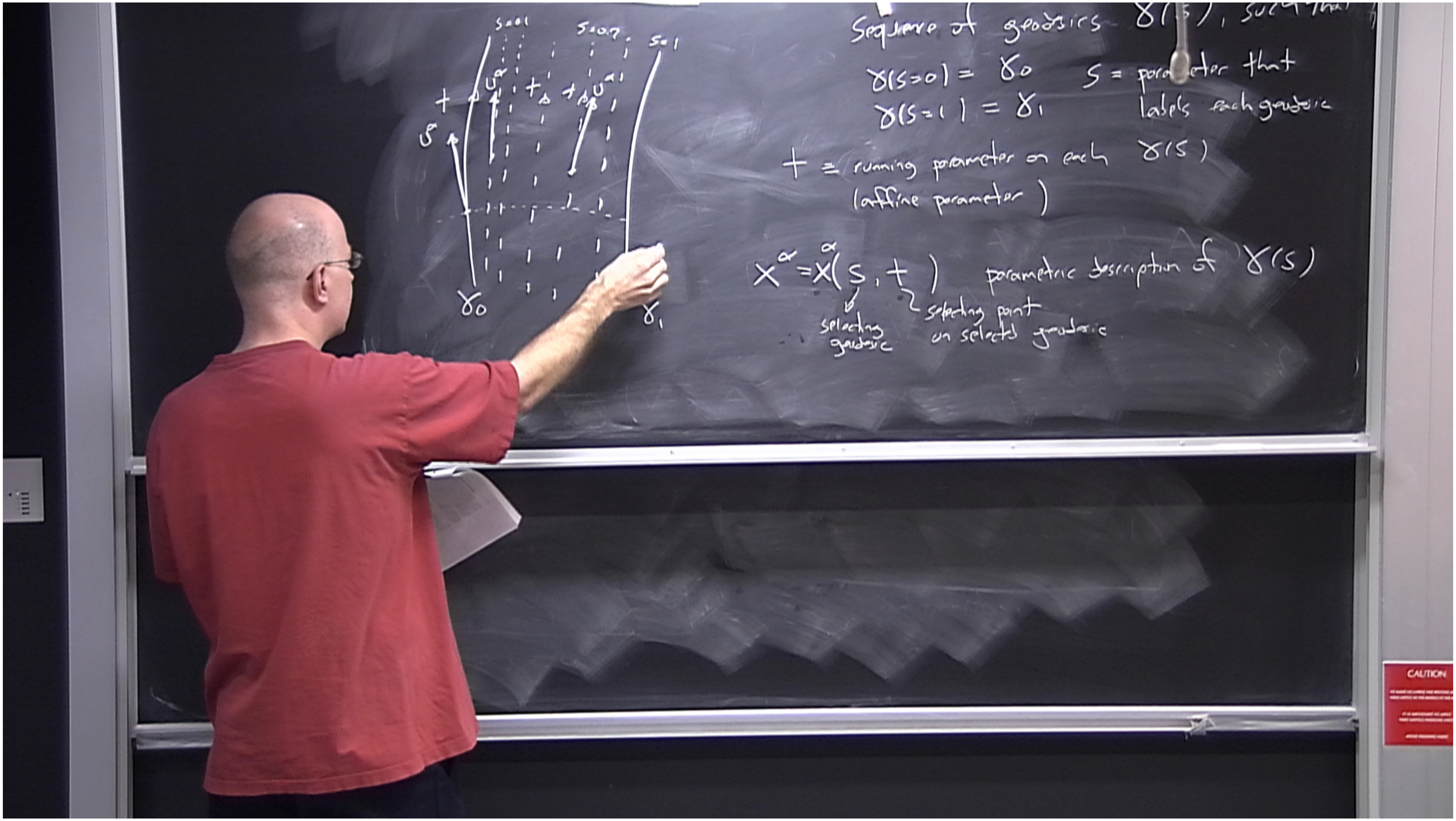

γ0
γ1
selecting geodesic
selecting point on selected geodesic
U^α = (∂x^α/∂τ)_s



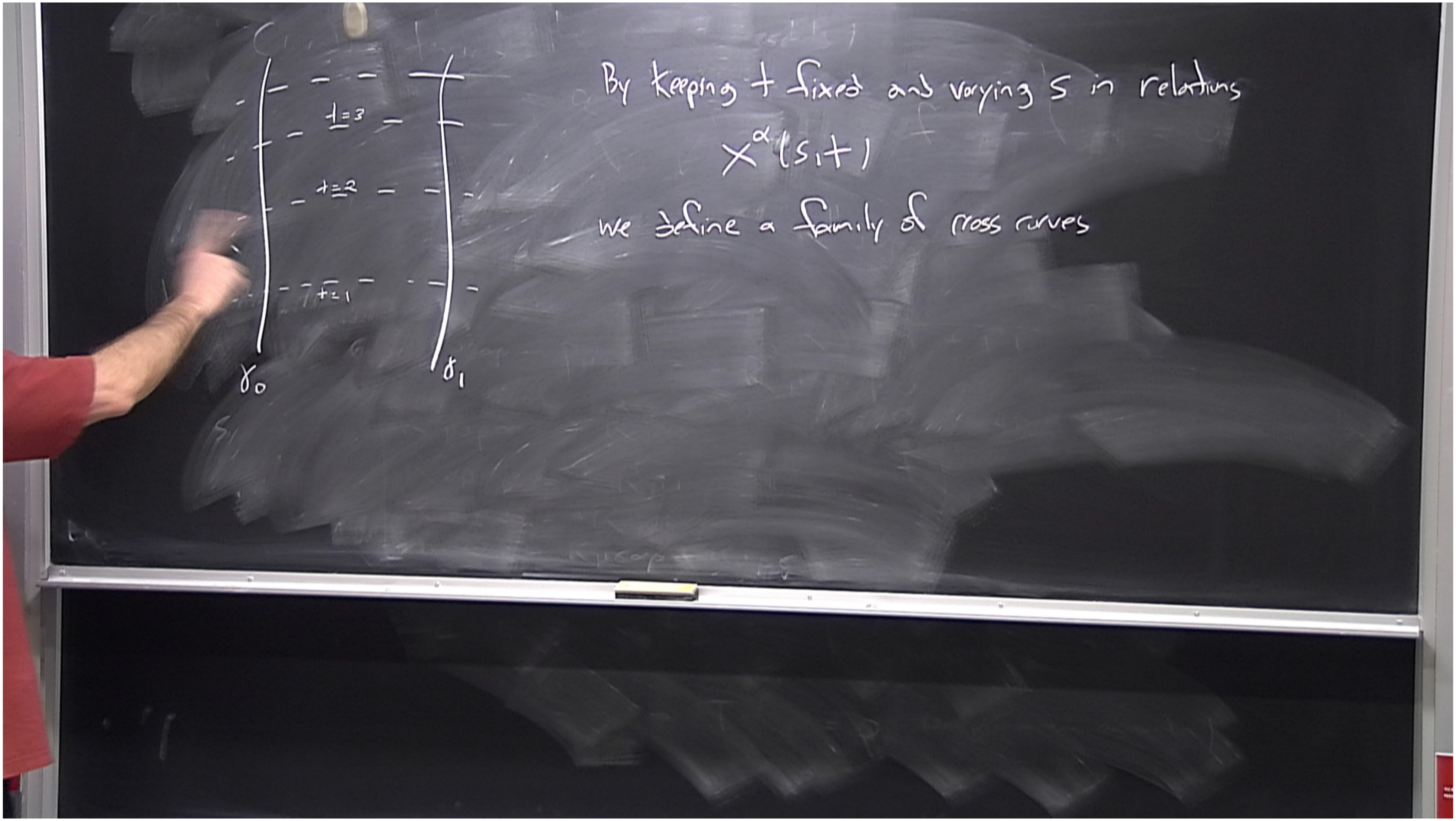
selecting geodesic
selecting point on selected geodesic
$u^\alpha = \left(\frac{\partial x^\alpha}{\partial \tau} \right)_s$ = tangent vector field to each geodesic $\gamma(s)$

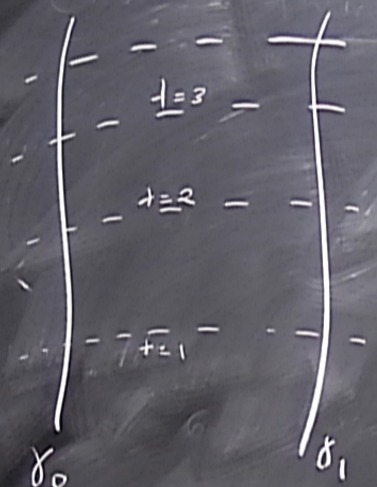

γ0
γ1
selecting geodesic
selecting point on selected geodesic
U^α = (∂x^α/∂τ)_s = tangent vector field to each geodesic γ(s)
U^α_;β U^β = 0


γ0
γ1



s=0.1
s=0.7
s=1
γ_0
γ_1
Sequence of geodesics $\gamma(s)$, such that
$\gamma(s=0) = \gamma_0$
$\gamma(s=1) = \gamma_1$
s = parameter that labels each geodesic
t = running parameter on each $\gamma(s)$
(affine parameter)
$x^\alpha = x^\alpha(s, t)$ parametric description of $\gamma(s)$
selecting geodesic
selecting point on selected geodesic

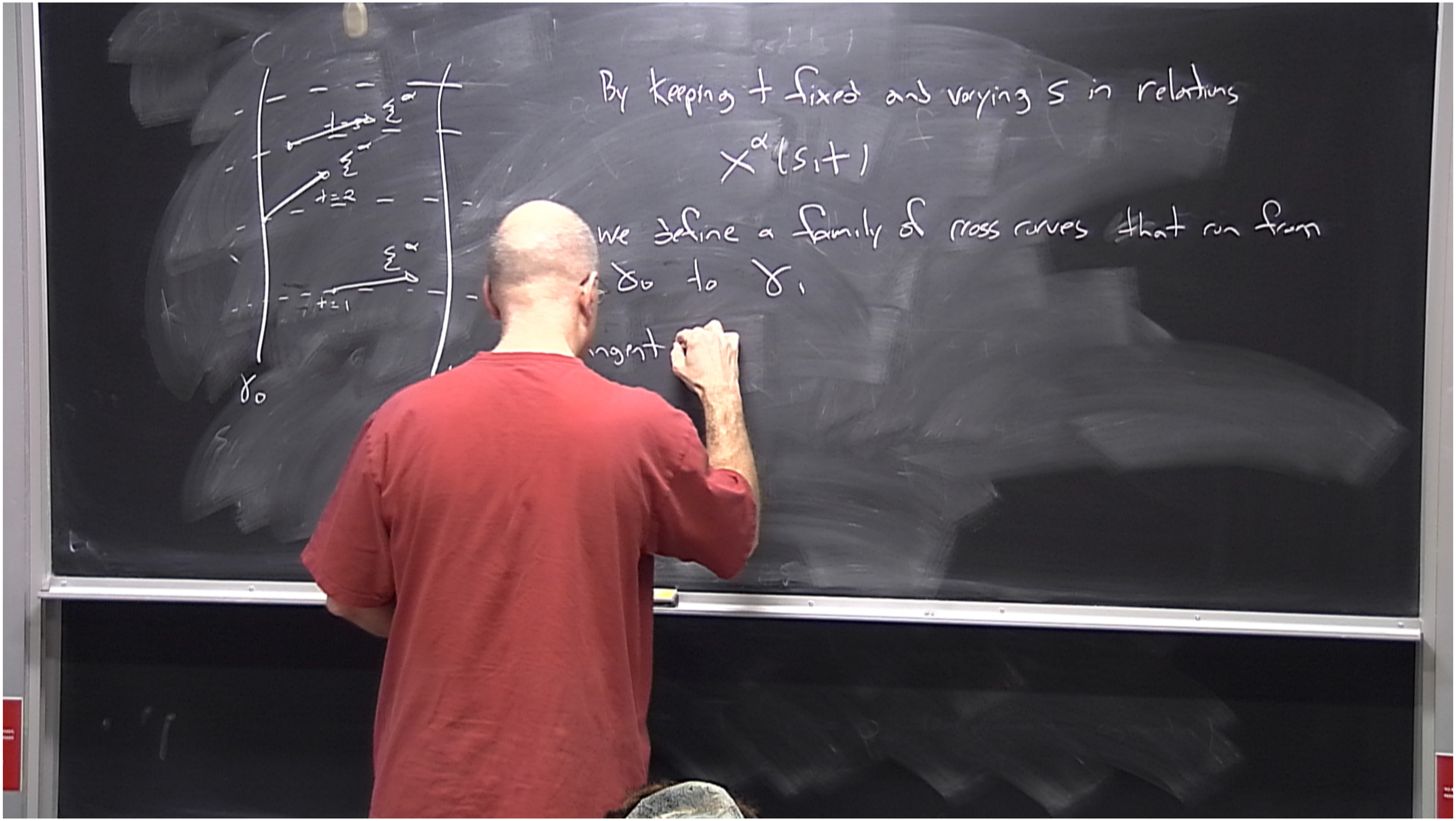

t=3
t=2
t=1
γ_0
γ_1
By keeping t fixed and varying s in relations
$x^\alpha(s,t)$
we define a family of cross curves



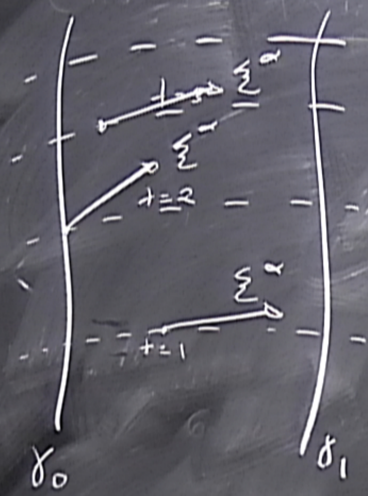
By keeping t fixed and varying s in relations

$$x^\alpha(s,t)$$

we define a family of cross curves that run from γ_0 to γ_1.



By keeping t fixed and varying s in relations
$x^\alpha(s,t)$
we define a family of cross curves that run from
γ_0 to γ_1
t=2
t=1
ξ^α
γ_0



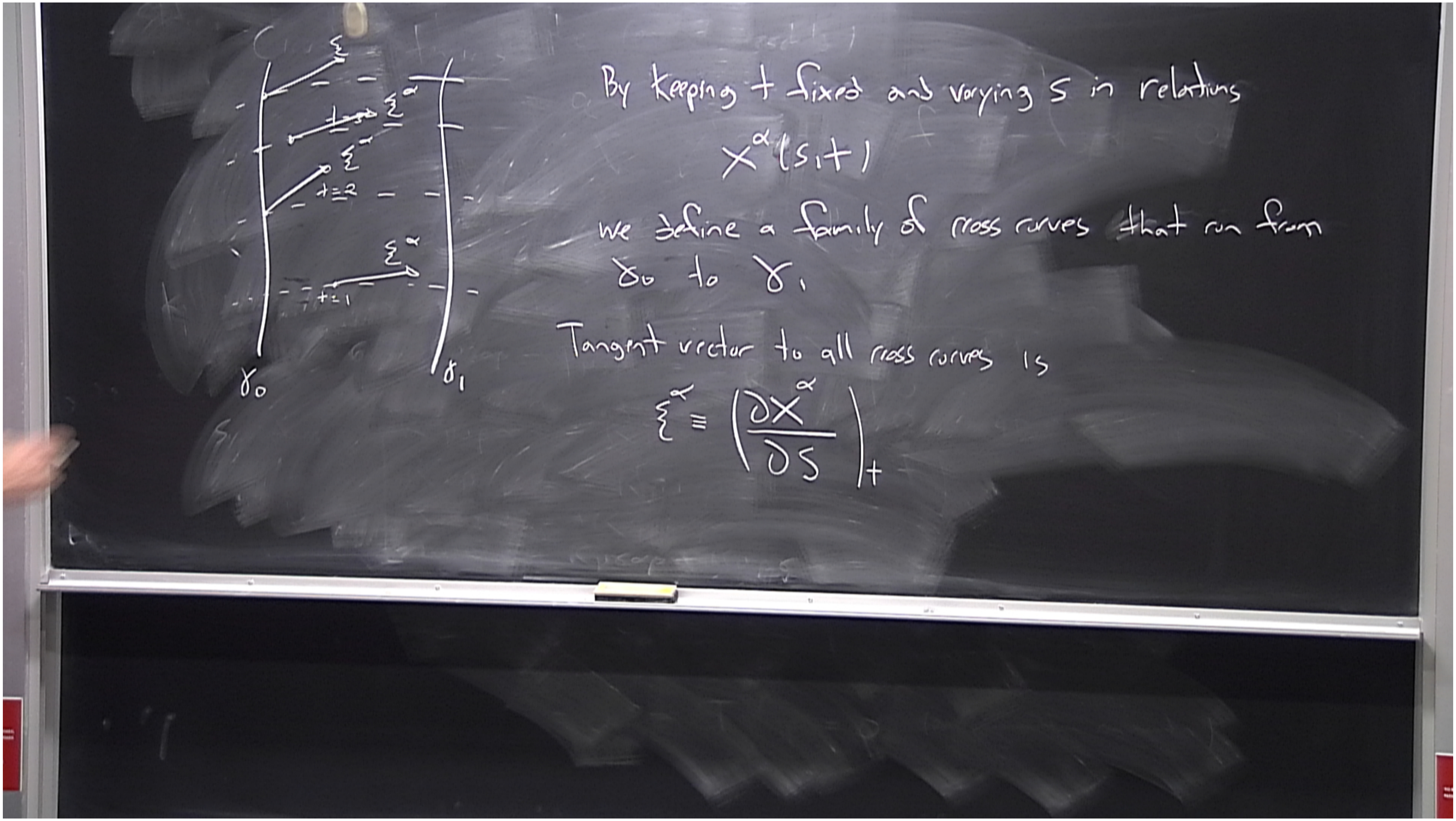
By keeping t fixed and varying s in relations

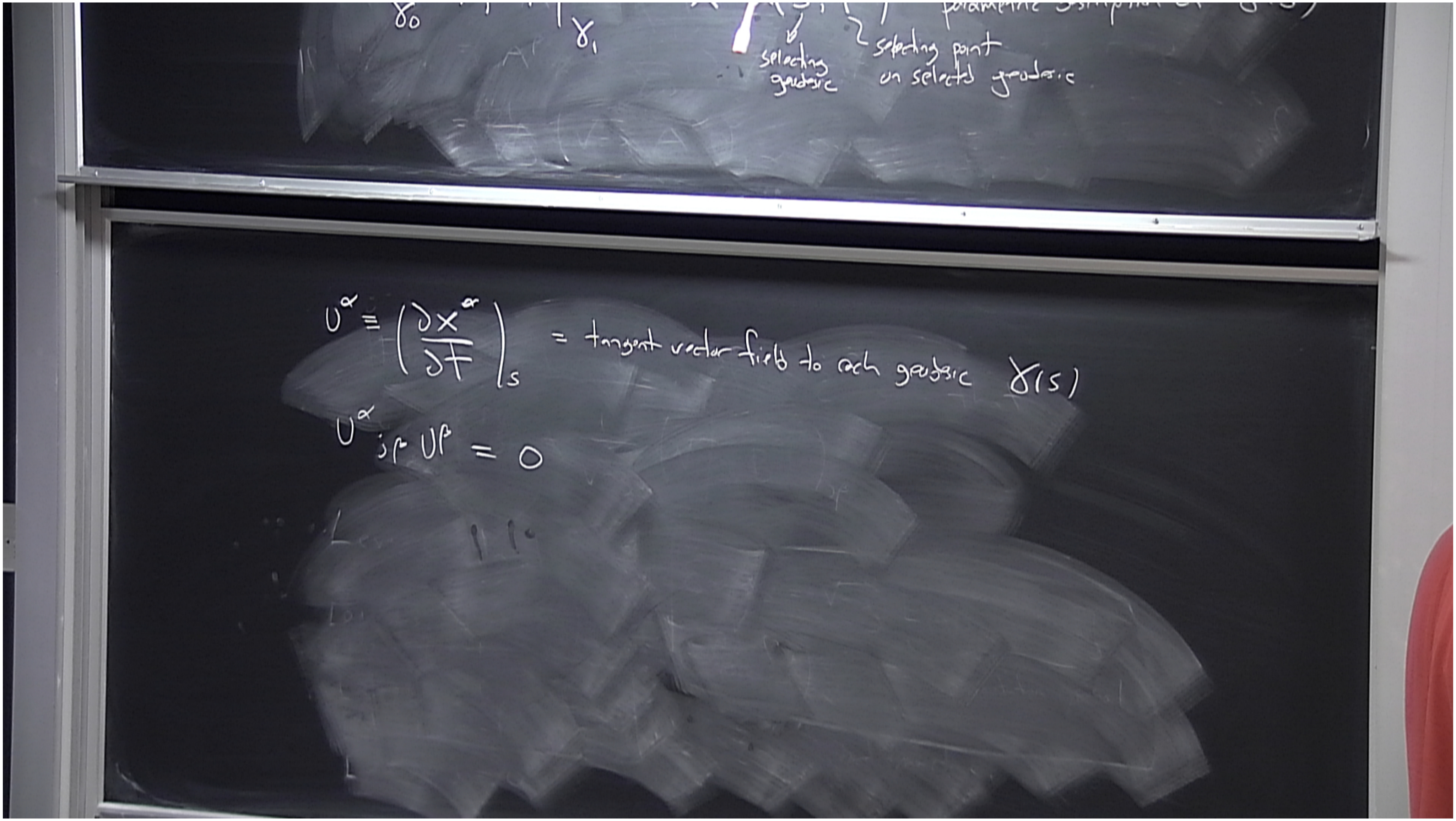
$$x^\alpha(s,t)$$

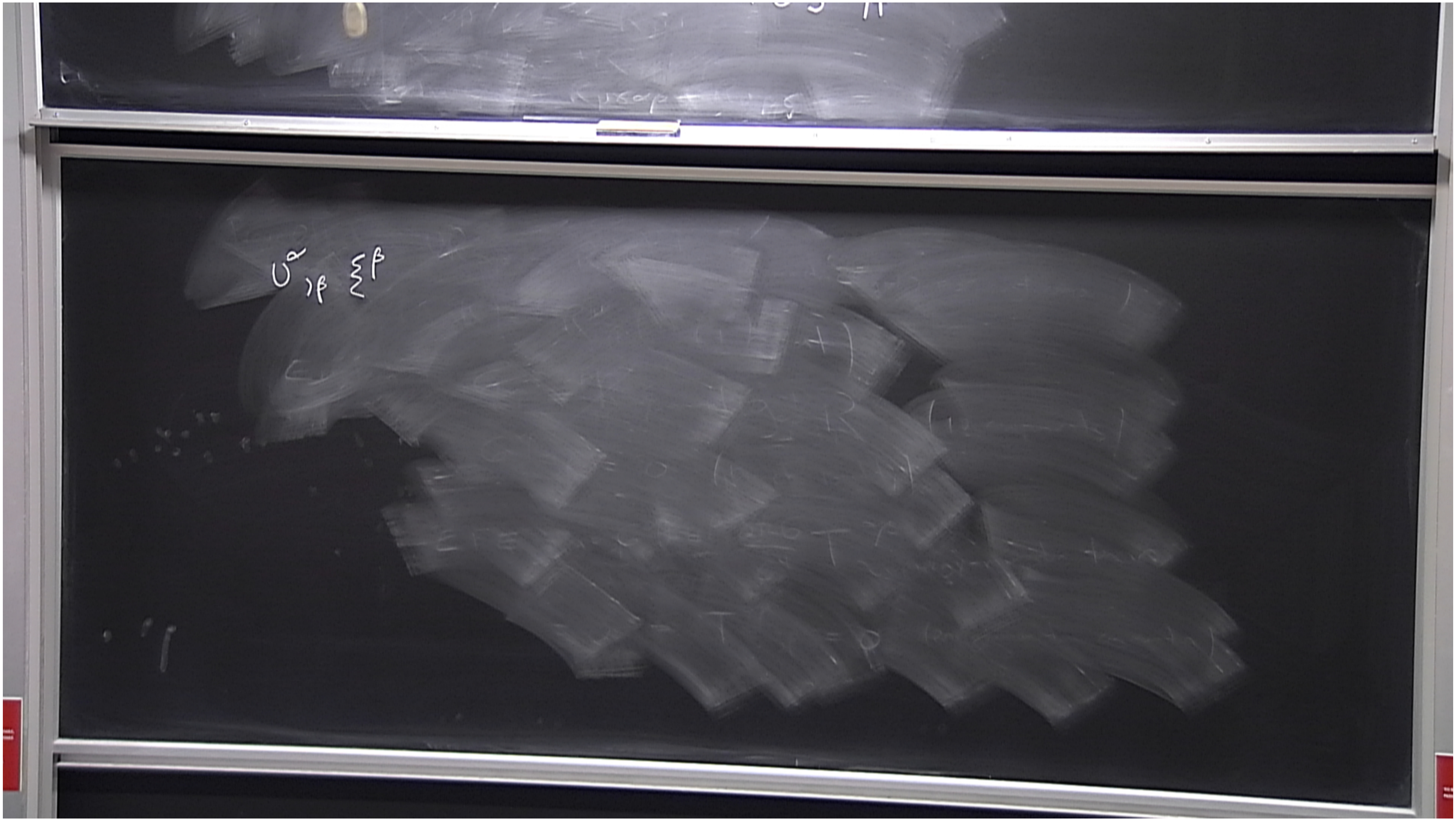
we define a family of cross curves that run from γ_0 to γ_1

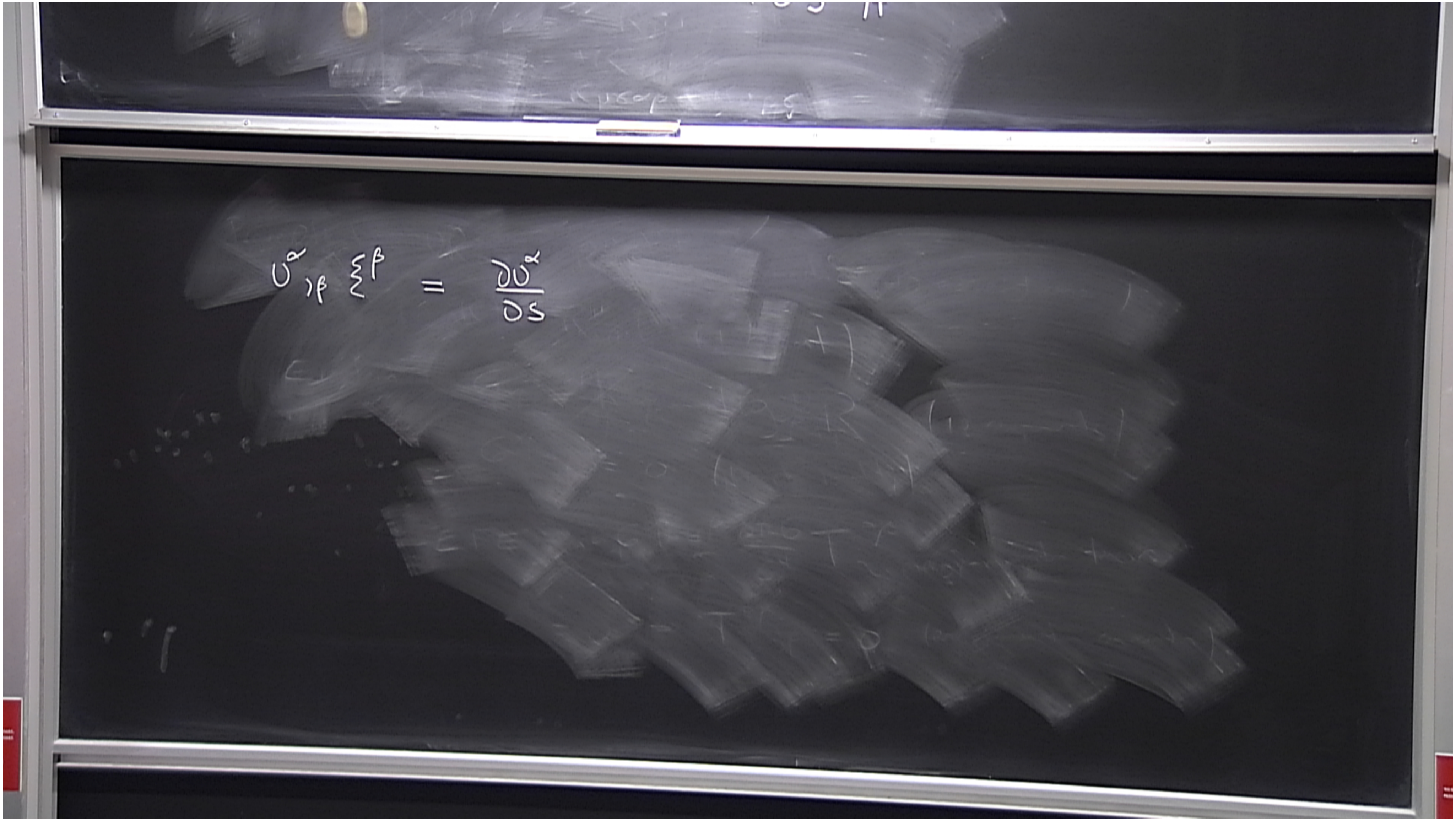
Tangent vector to all cross curves is

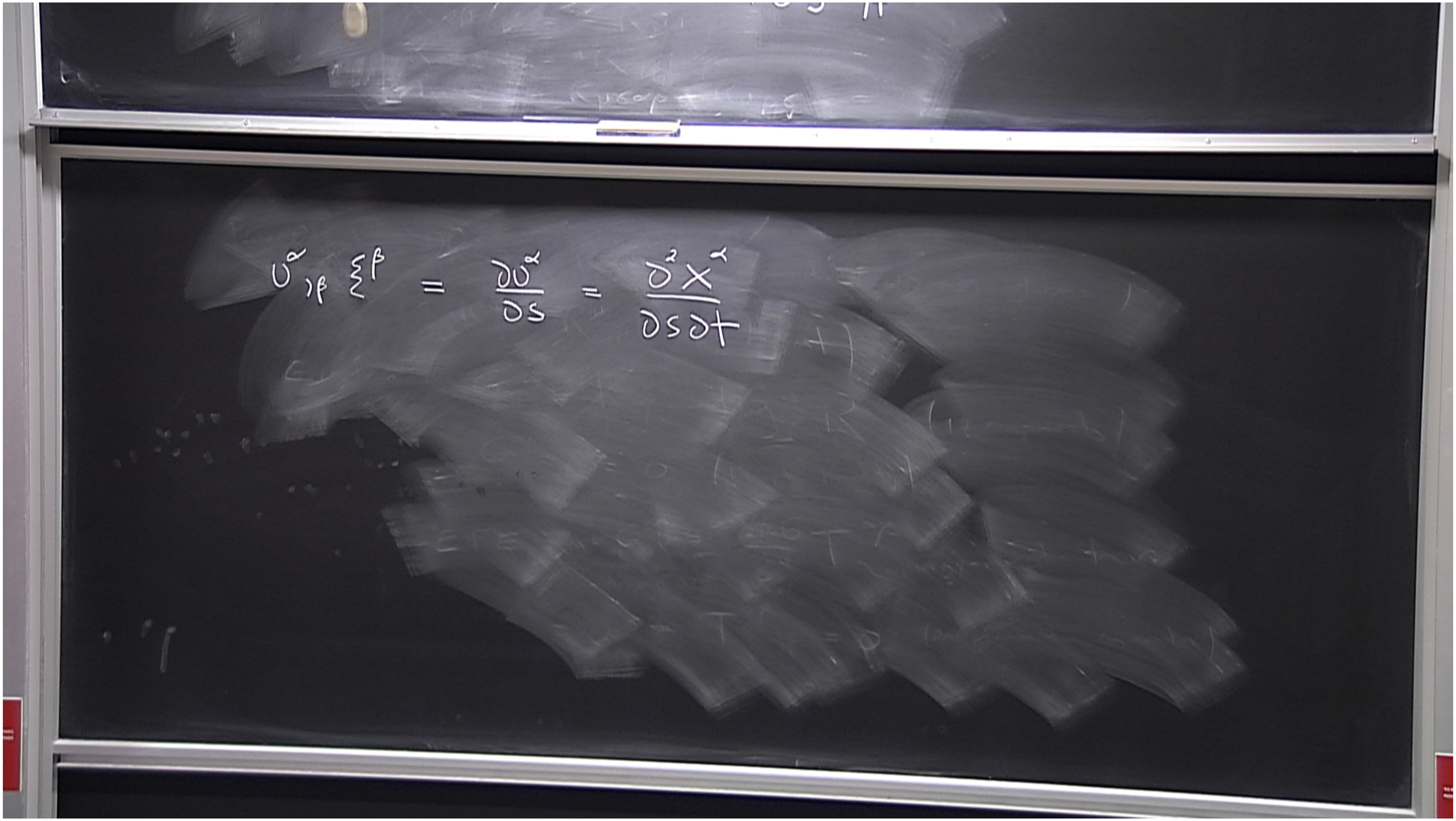
$$\xi^\alpha \equiv \left(\frac{\partial x^\alpha}{\partial s}\right)_t$$

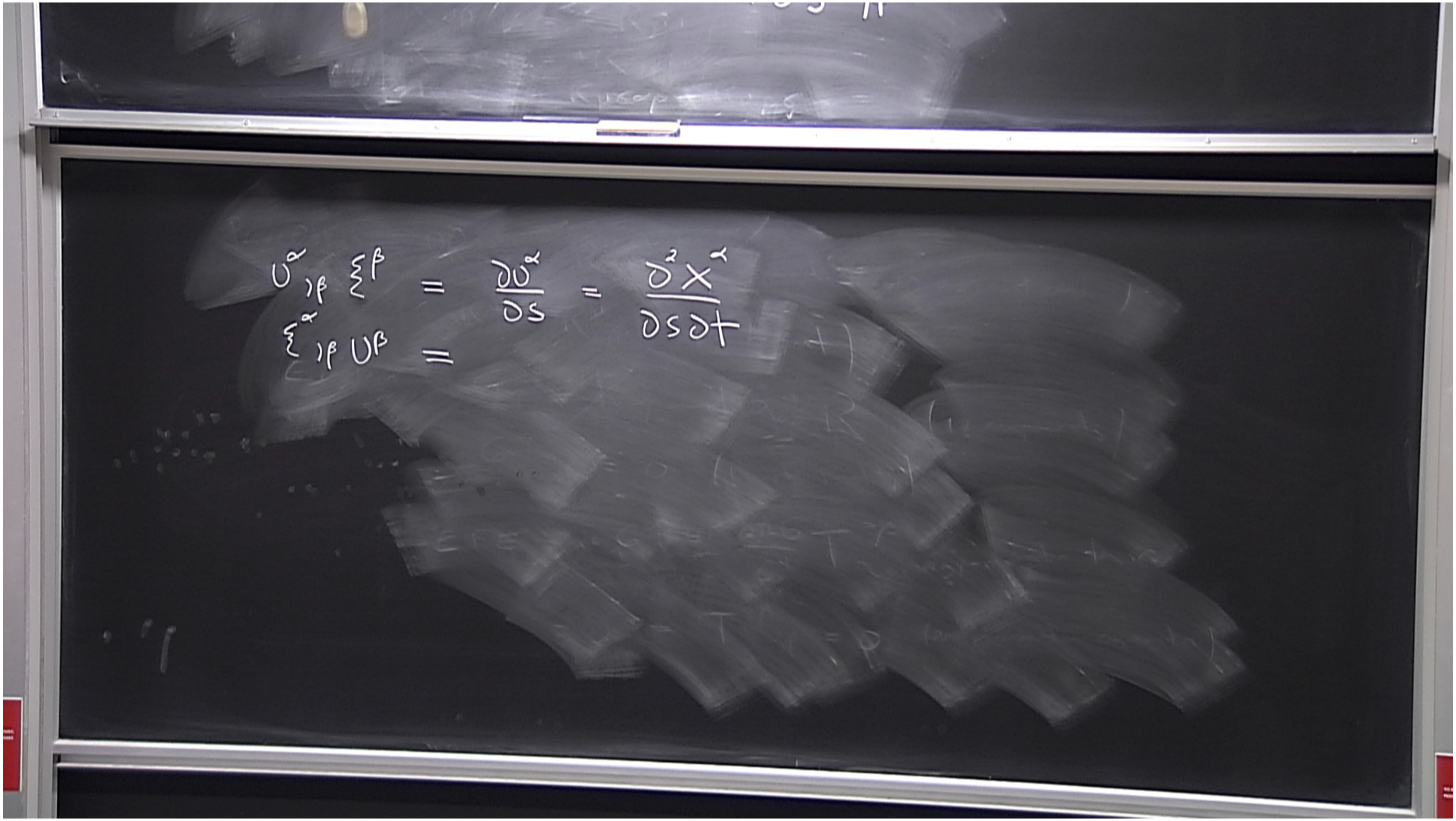

By keeping t fixed and varying s in relations
$x^\alpha(s,t)$
we define a family of cross curves that run from
γ_0 to γ_1
Tangent vector to all cross curves is
$\xi^\alpha \equiv \left(\frac{\partial x^\alpha}{\partial s}\right)_t$
ξ^α
t=2
t=1
γ_0
γ_1


γ0
γ1
selecting geodesic
selecting point on selected geodesic
U^α ≡ (∂x^α/∂τ)_s = tangent vector field to each geodesic γ(s)
U^α ;β U^β = 0


$U^\alpha{}_{,\beta}\,\xi^\beta$

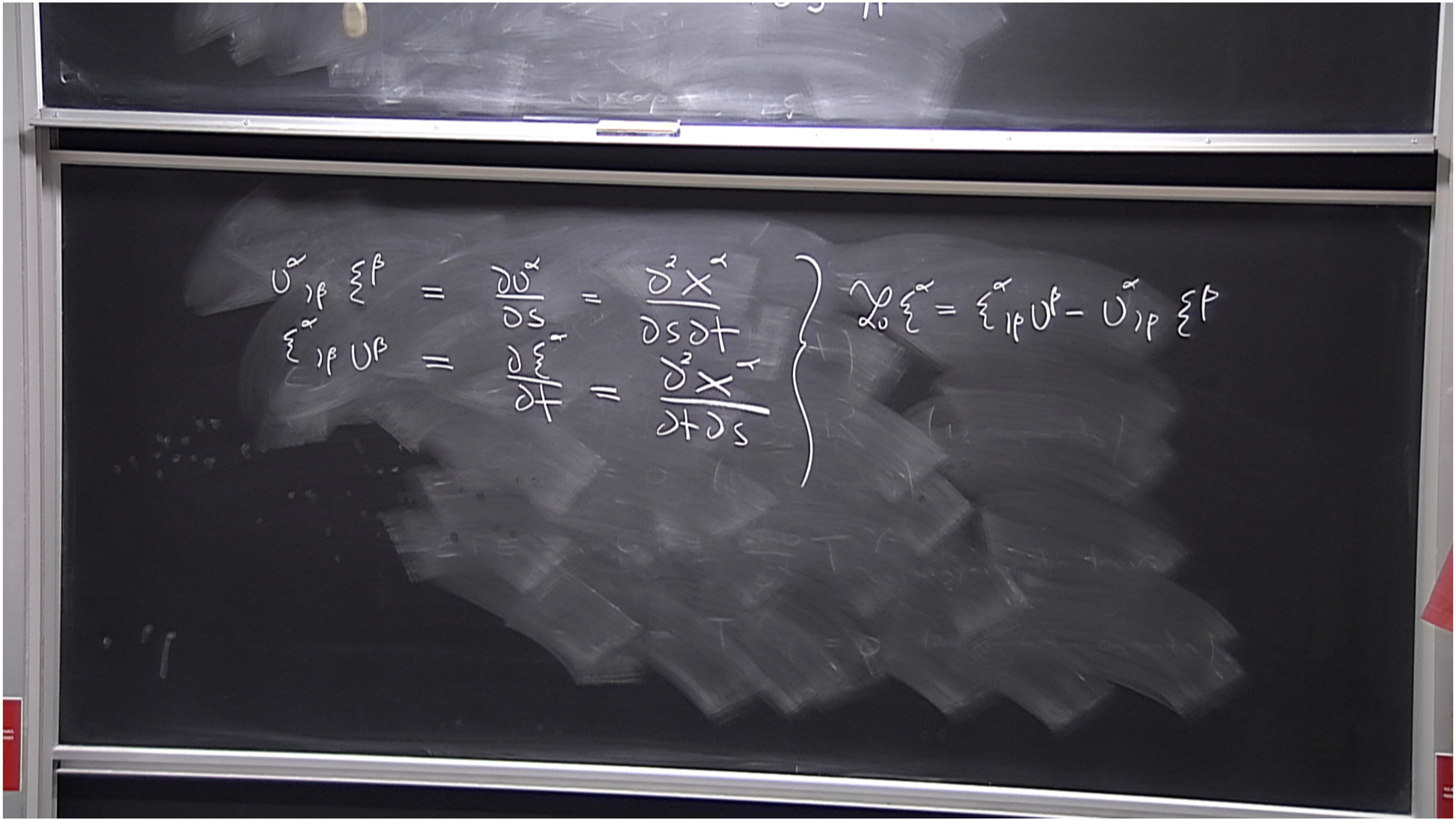

$U^{\alpha}{}_{,\beta}\,\xi^{\beta} = \frac{\partial U^{\alpha}}{\partial s}$


$u^{\alpha}{}_{,\beta}\,\xi^{\beta} = \frac{\partial u^{\alpha}}{\partial s} = \frac{\partial^2 x^{\alpha}}{\partial s\,\partial t}$


$U^{\alpha}{}_{;\beta}\,\xi^{\beta} = \frac{\partial U^{\alpha}}{\partial s} = \frac{\partial^2 x^{\alpha}}{\partial s\,\partial t}$
$\xi^{\alpha}{}_{;\beta}\,U^{\beta} =$

$$U^\alpha{}_{;\beta}\,\xi^\beta = \frac{\partial U^\alpha}{\partial s} = \frac{\partial^2 x^\alpha}{\partial s\,\partial t}$$

$$\xi^\alpha{}_{;\beta}\,U^\beta = \frac{\partial \xi^\alpha}{\partial t} = \frac{\partial^2 x^\alpha}{\partial t\,\partial s}$$



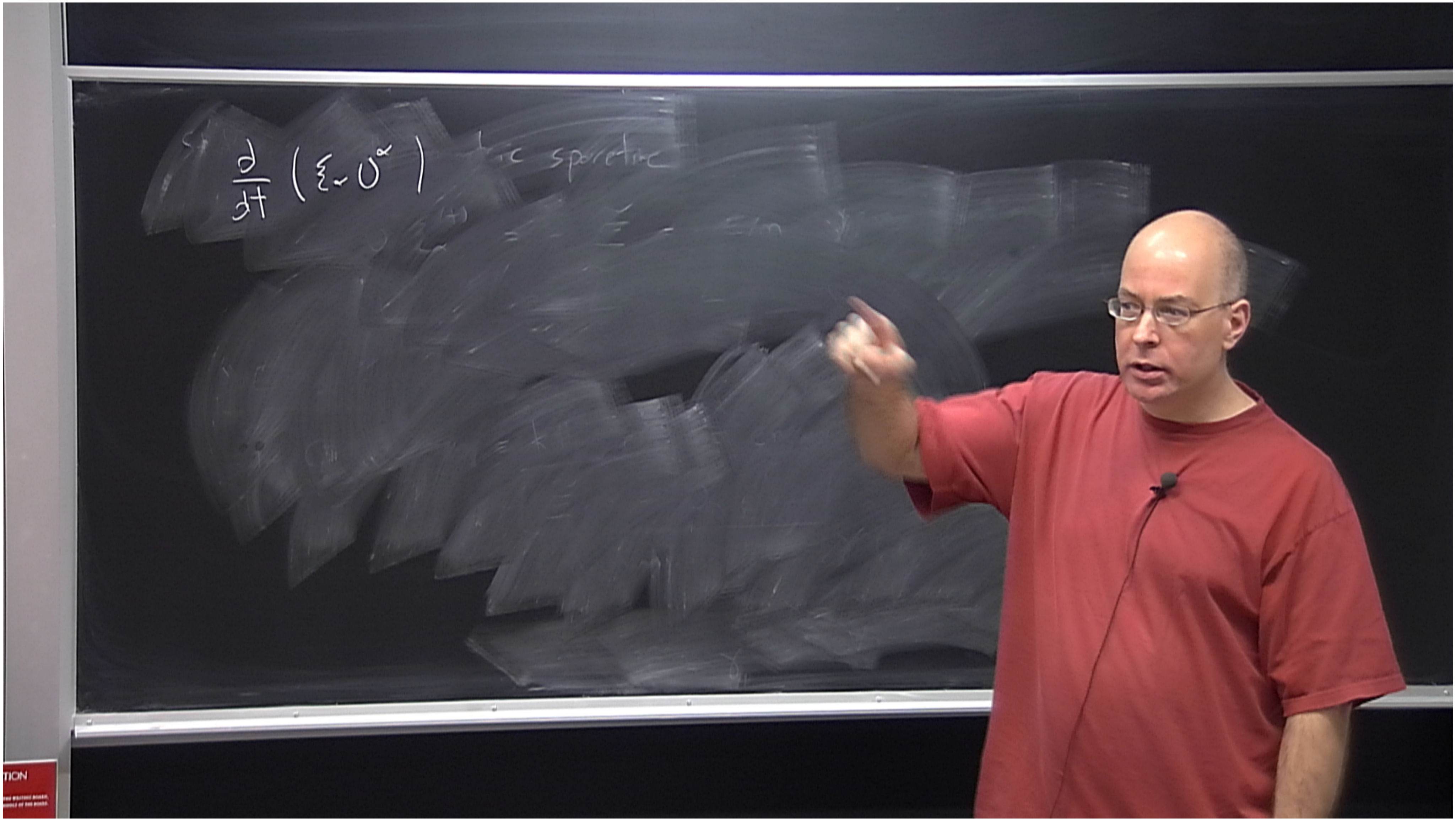
$u^\alpha{}_{;\beta}\,\xi^\beta = \frac{\partial u^\alpha}{\partial s} = \frac{\partial^2 x^\alpha}{\partial s\,\partial t}$
$\xi^\alpha{}_{;\beta}\,u^\beta = \frac{\partial \xi^\alpha}{\partial t} = \frac{\partial^2 x^\alpha}{\partial t\,\partial s}$
$\mathcal{L}_u \xi^\alpha = \xi^\alpha{}_{;\beta}\,u^\beta - u^\alpha{}_{;\beta}\,\xi^\beta$

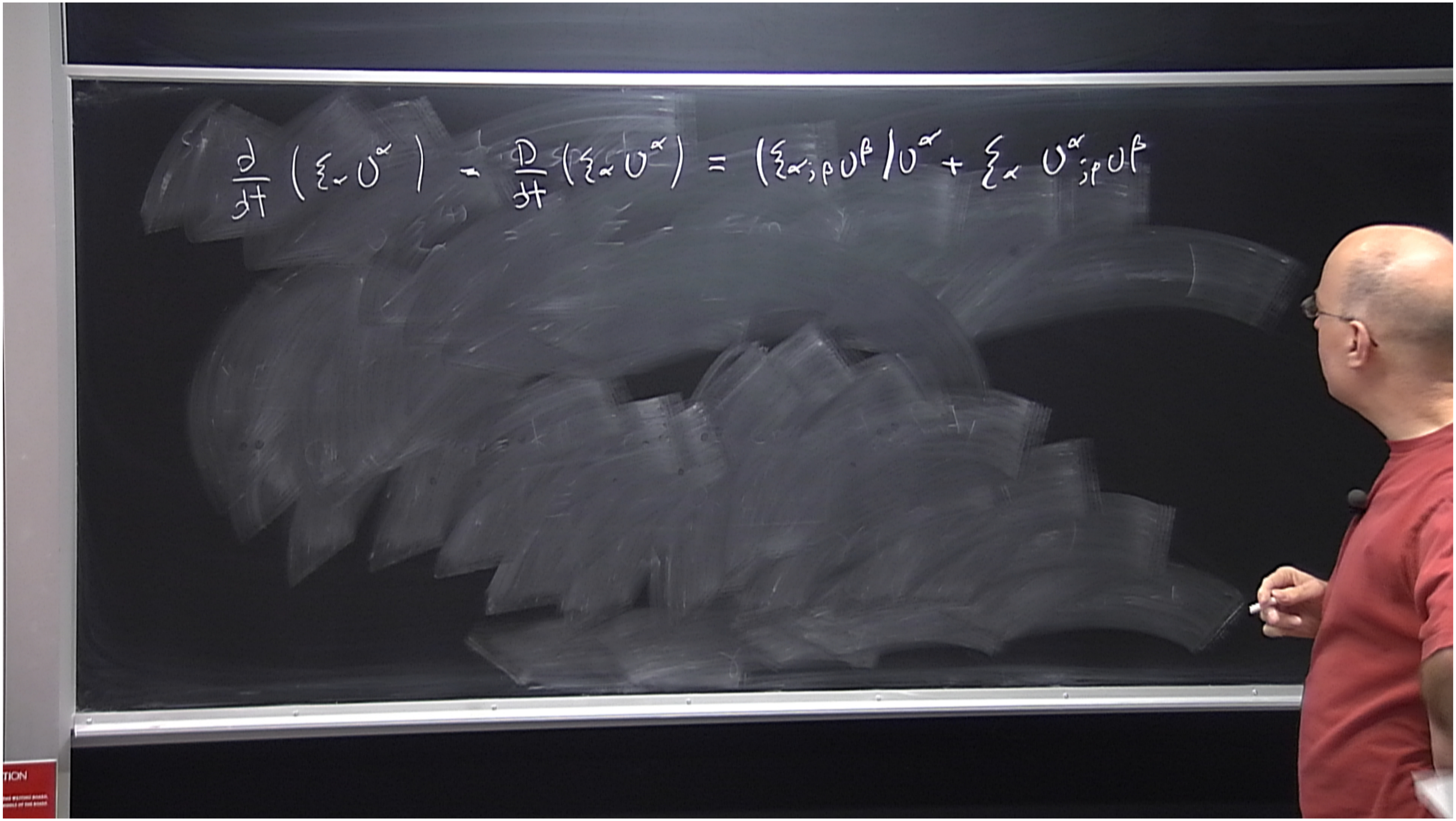
$$U^{\alpha}{}_{;\beta}\, \xi^{\beta} = \frac{\partial U^{\alpha}}{\partial s} = \frac{\partial^2 x^{\alpha}}{\partial s\, \partial t}$$

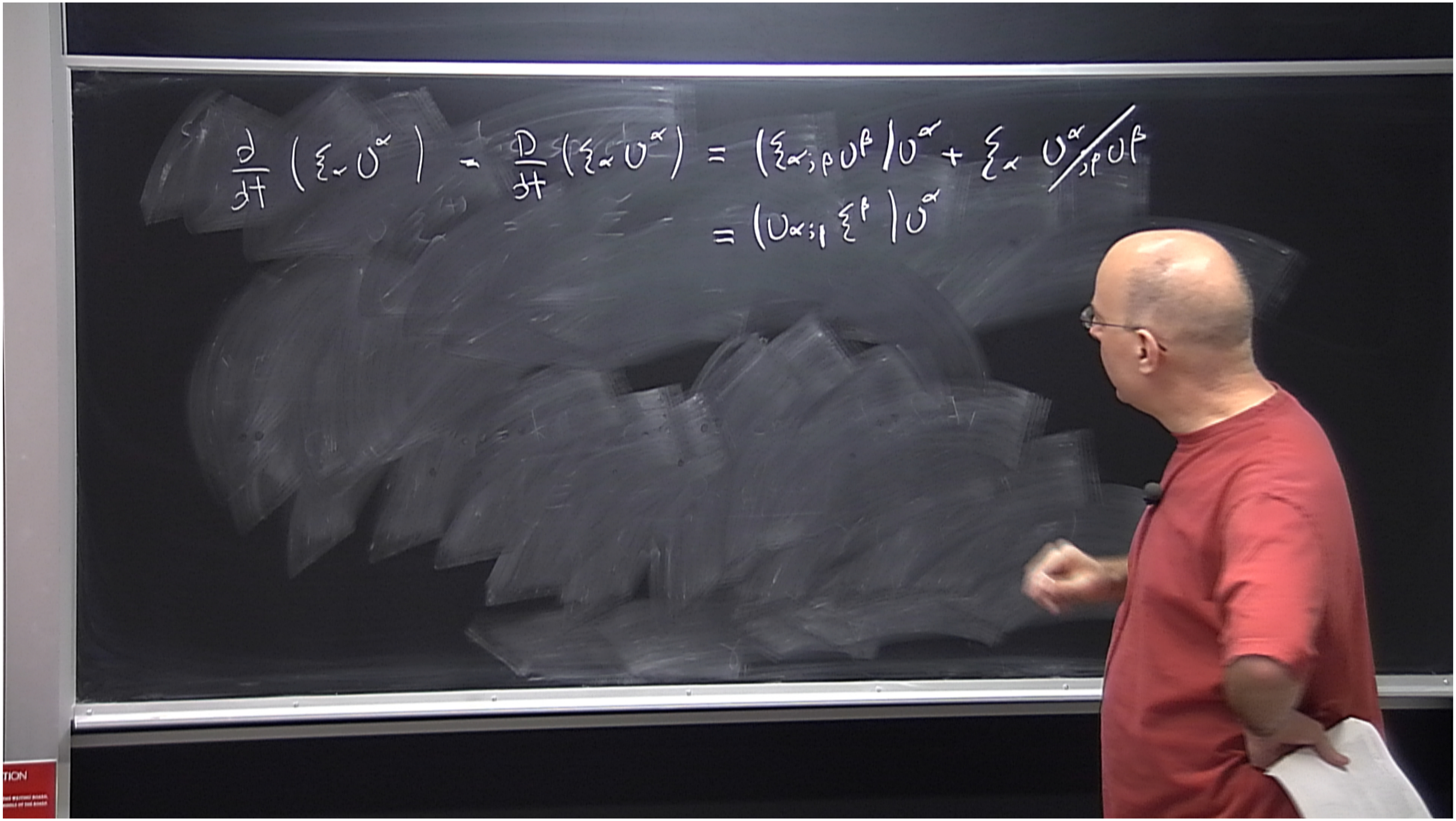
$$\xi^{\alpha}{}_{;\beta}\, U^{\beta} = \frac{\partial \xi^{\alpha}}{\partial t} = \frac{\partial^2 x^{\alpha}}{\partial t\, \partial s}$$

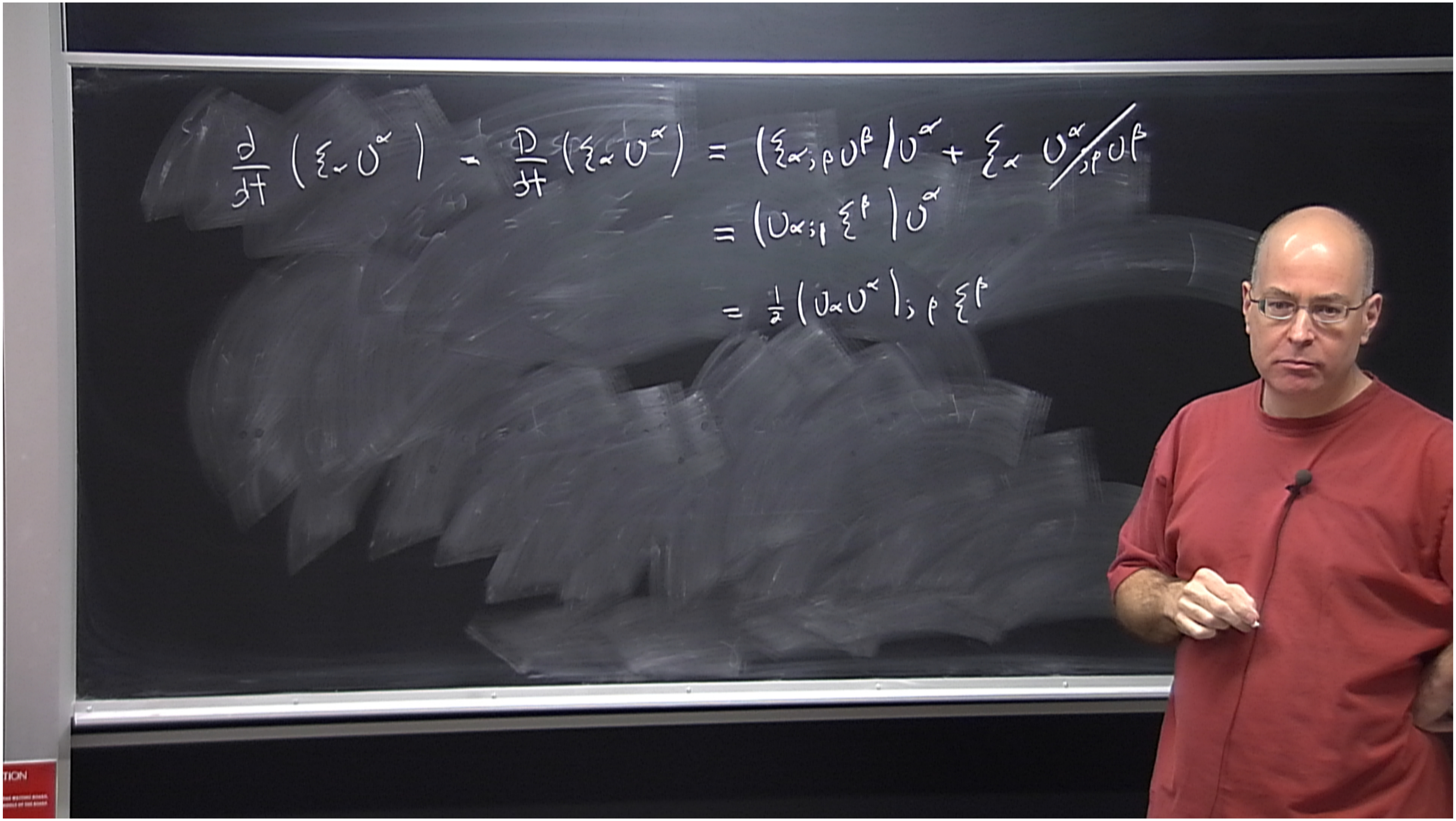
$$\mathcal{L}_U \xi^{\alpha} = \xi^{\alpha}{}_{;\beta} U^{\beta} - U^{\alpha}{}_{;\beta}\, \xi^{\beta}$$

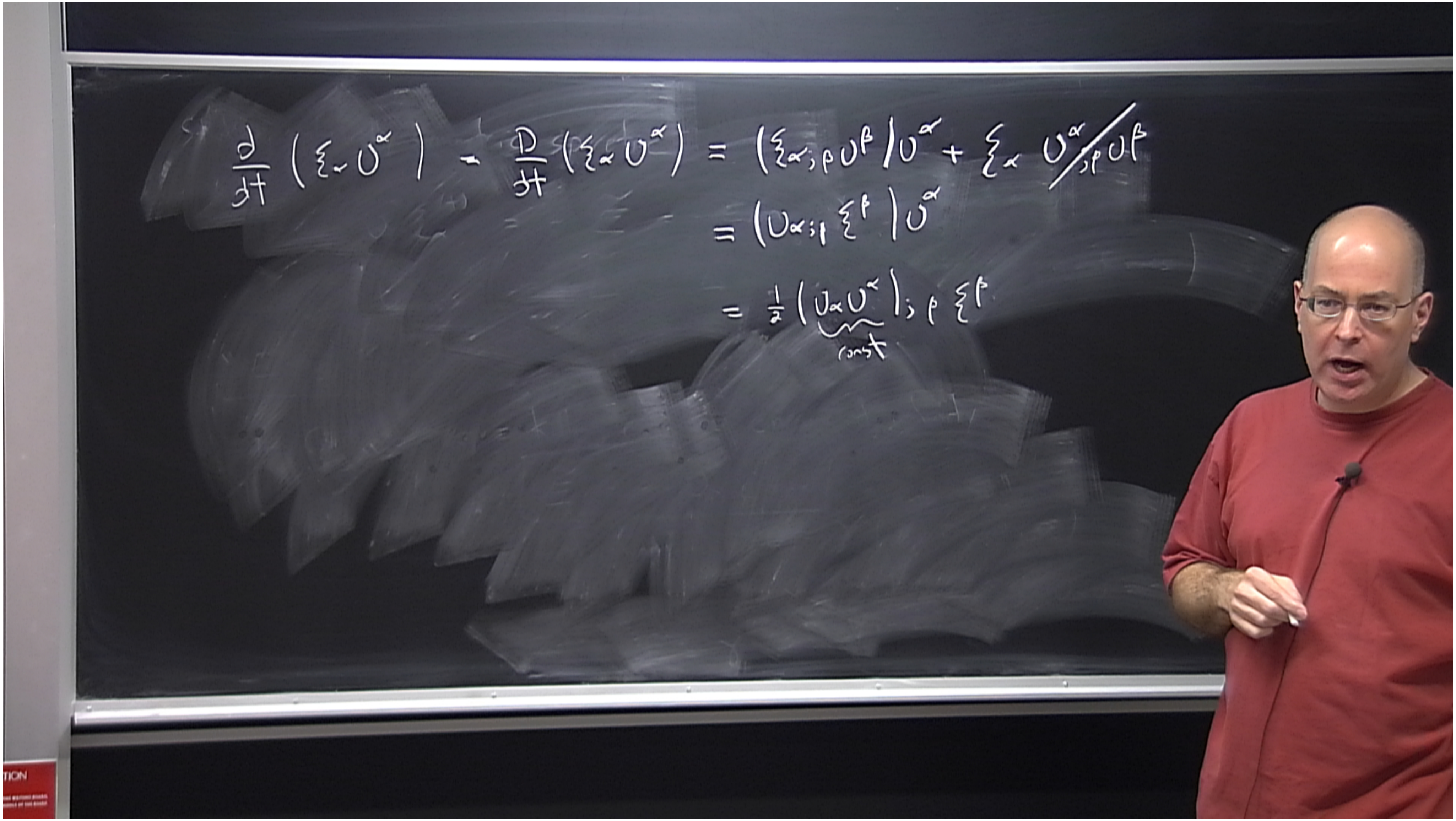
$$\boxed{\begin{aligned} \mathcal{L}_U \xi^{\alpha} &= -\mathcal{L}_{\xi} U^{\alpha} = 0 \\ \xi^{\alpha}{}_{;\beta} U^{\beta} &= U^{\alpha}{}_{;\beta}\, \xi^{\beta} \end{aligned}}$$

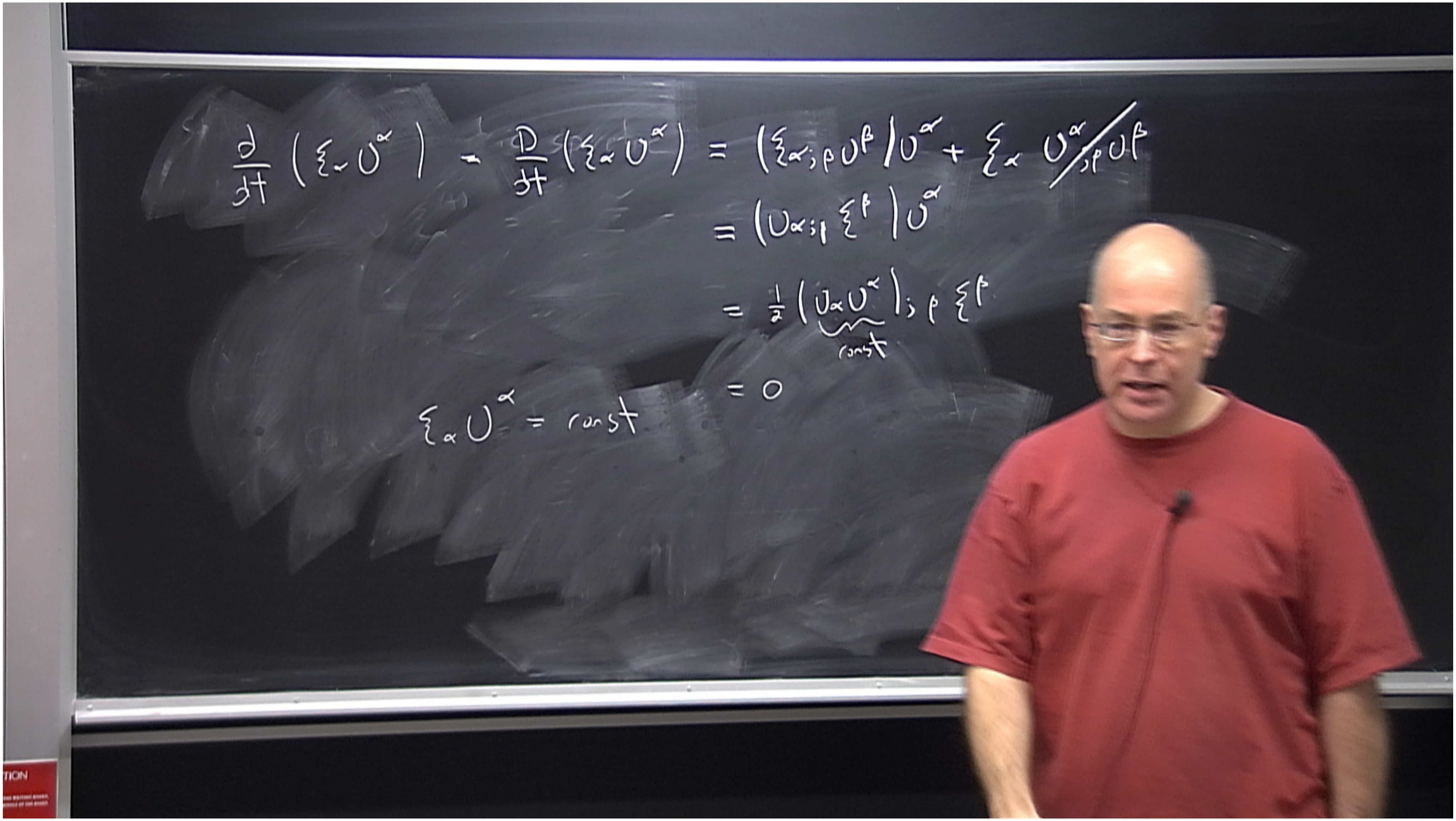

$\frac{d}{dt}(\xi_\alpha U^\alpha)$
spacetime

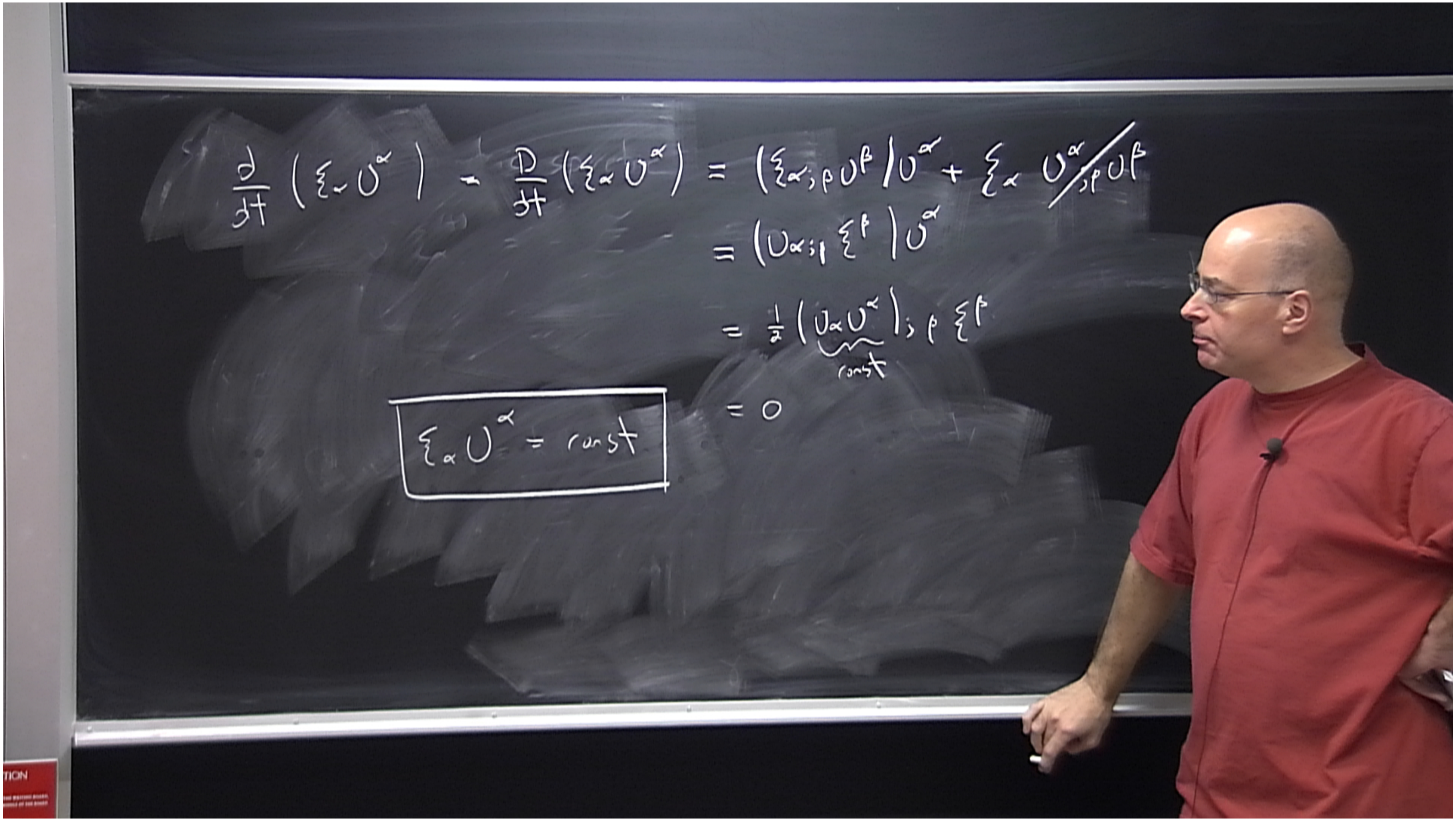

$\frac{d}{dt}(\xi_\alpha U^\alpha) = \frac{D}{dt}(\xi_\alpha U^\alpha) = (\xi_{\alpha;\beta} U^\beta) U^\alpha + \xi_\alpha U^\alpha{}_{;\beta} U^\beta$


$\frac{d}{dt}(\xi_\alpha U^\alpha) = \frac{D}{d\tau}(\xi_\alpha U^\alpha) = (\xi_{\alpha;\beta} U^\beta) U^\alpha + \xi_\alpha \cancel{U^\alpha{}_{;\beta} U^\beta}$
$= (U_{\alpha;\beta}\, \xi^\beta) U^\alpha$

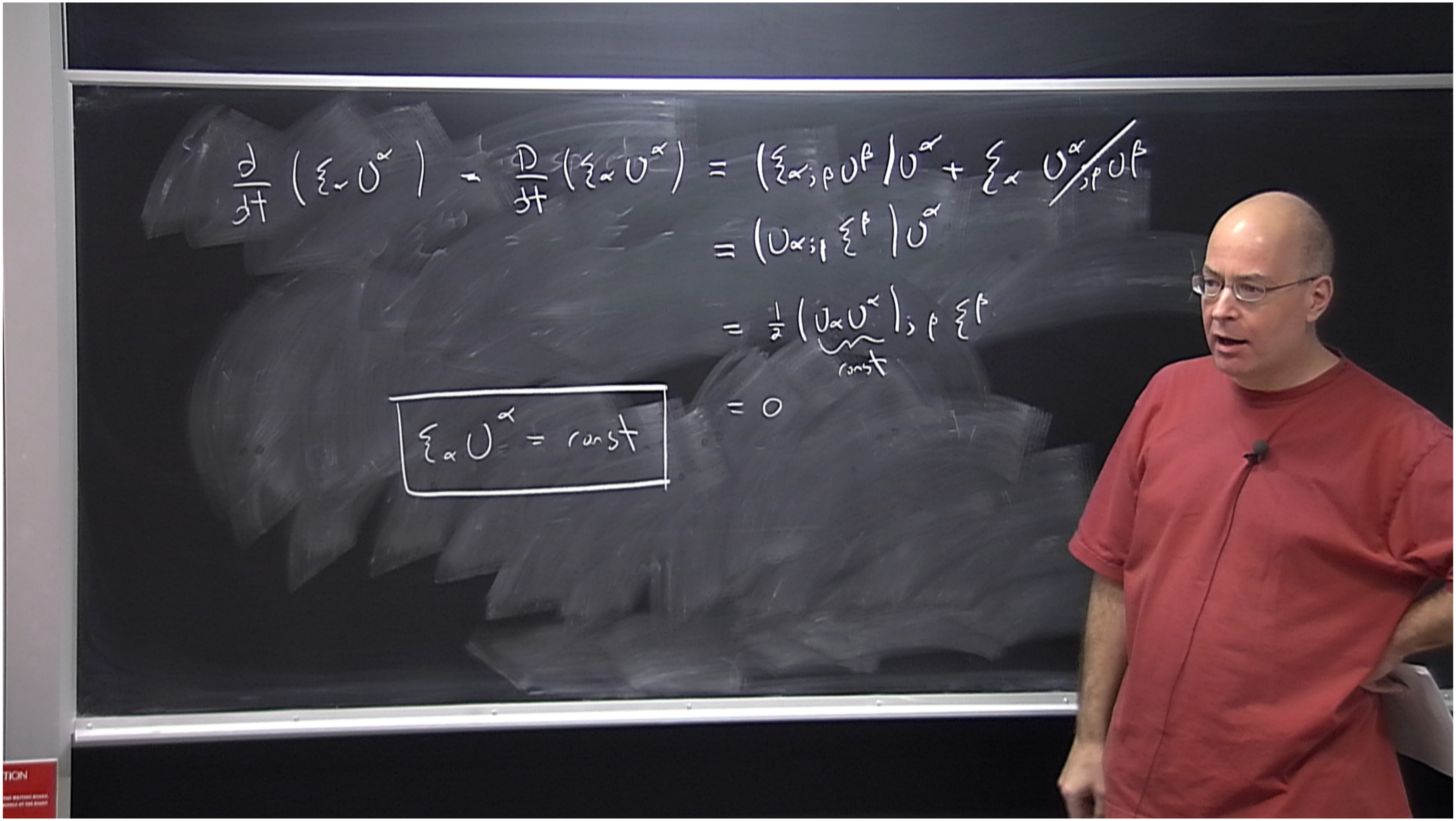

$\frac{d}{dt}\left(\xi_\alpha U^\alpha\right) = \frac{D}{dt}\left(\xi_\alpha U^\alpha\right) = \left(\xi_{\alpha;\beta} U^\beta\right) U^\alpha + \xi_\alpha U^\alpha{}_{;\beta} U^\beta$
$= \left(U_{\alpha;\beta}\, \xi^\beta\right) U^\alpha$
$= \frac{1}{2}\left(U_\alpha U^\alpha\right)_{;\beta}\, \xi^\beta$

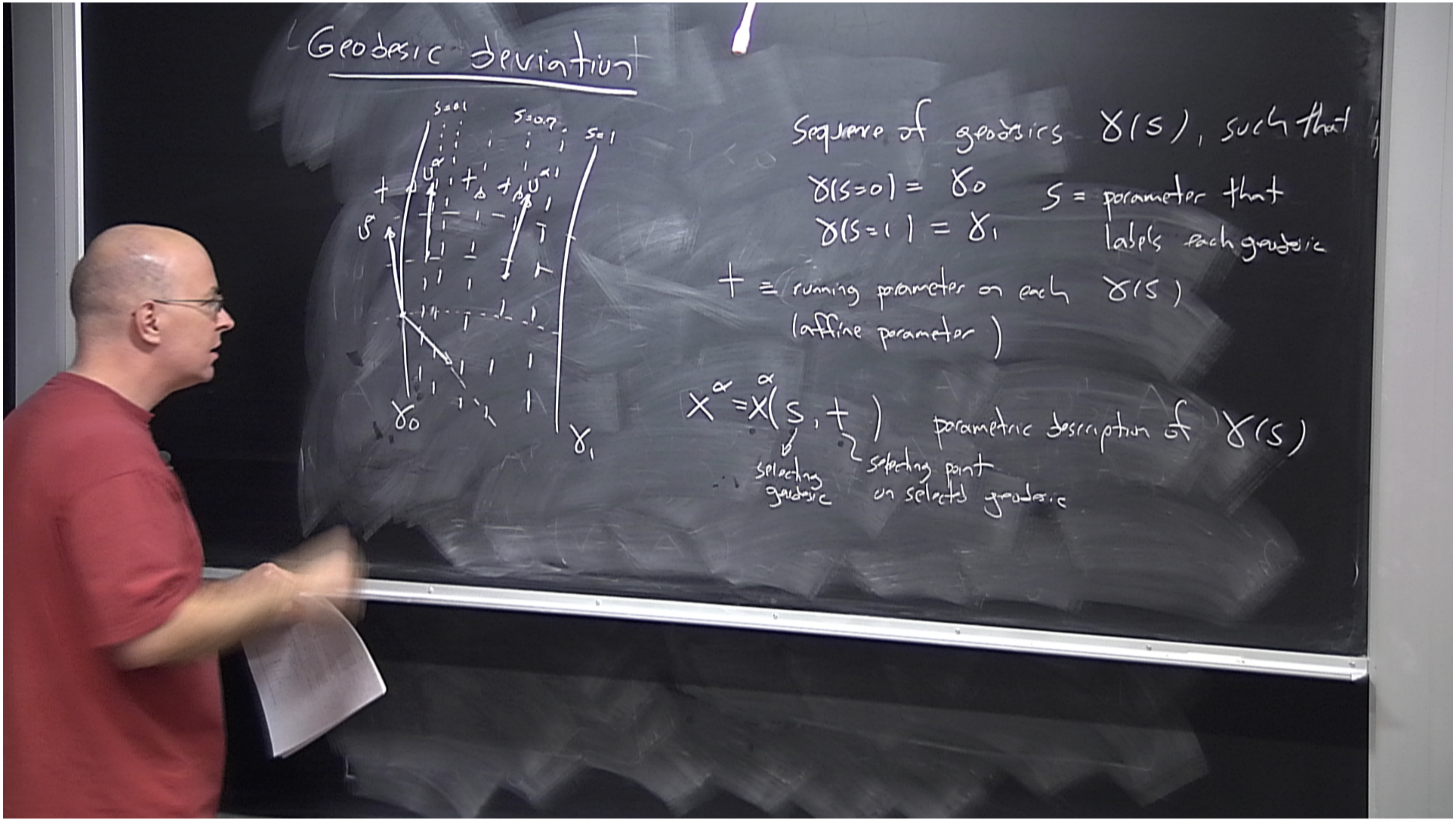

$\frac{d}{dt}(\xi_\alpha U^\alpha) = \frac{D}{dt}(\xi_\alpha U^\alpha) = (\xi_{\alpha;\beta} U^\beta) U^\alpha + \xi_\alpha U^\alpha{}_{;\beta} U^\beta$
$= (U_{\alpha;\beta}\xi^\beta) U^\alpha$
$= \frac{1}{2}(U_\alpha U^\alpha)_{;\beta}\xi^\beta$
const


d/dt (ξ_α U^α) = D/dτ (ξ_α U^α) = (ξ_α;β U^β) U^α + ξ_α U^α;β U^β
= (U_α;β ξ^β) U^α
= 1/2 (U_α U^α);β ξ^β
const
= 0
ξ_α U^α = const

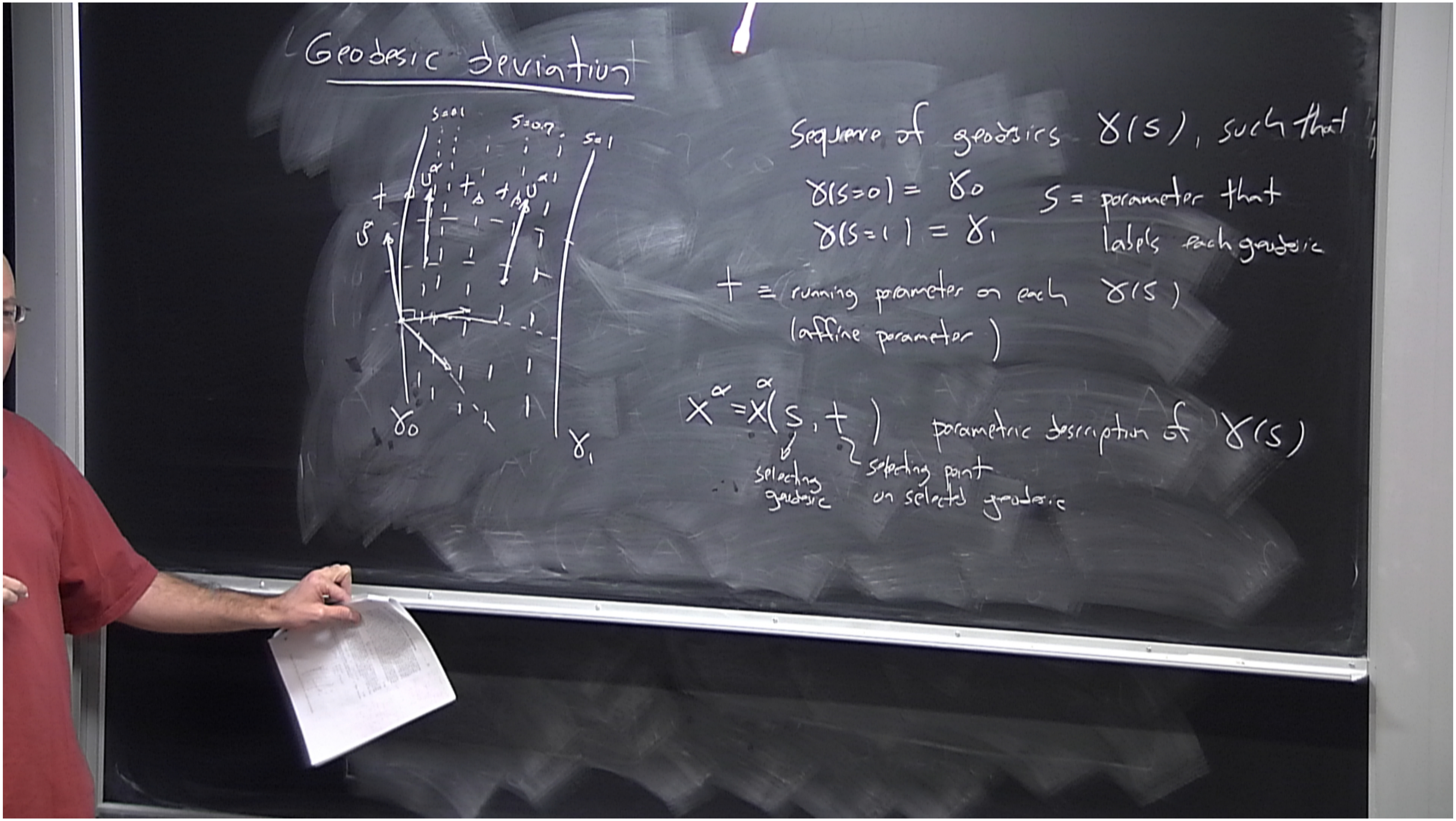


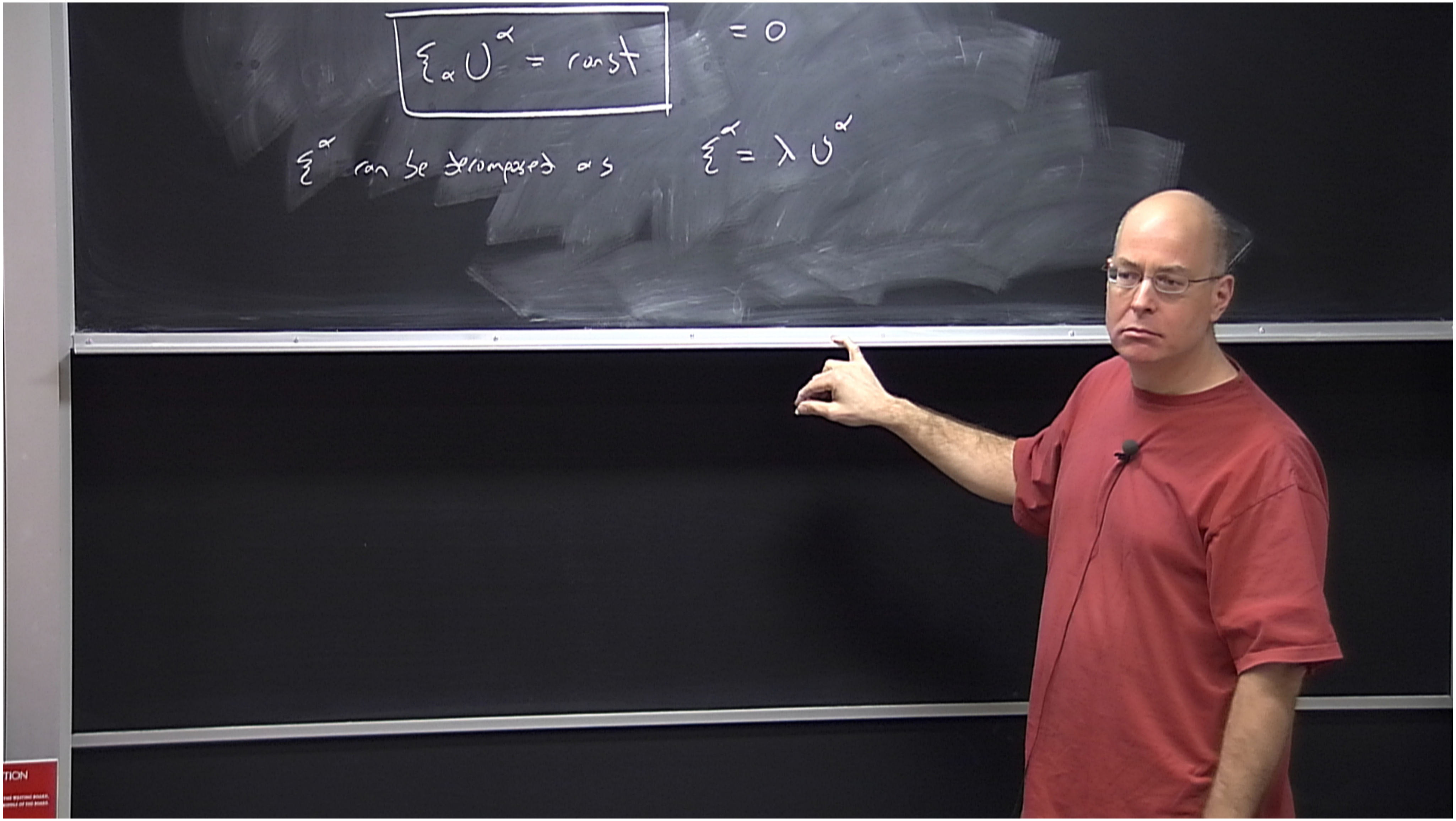
$\frac{d}{d\tau}(\xi_\alpha U^\alpha) = \frac{D}{d\tau}(\xi_\alpha U^\alpha) = (\xi_{\alpha;\beta} U^\beta) U^\alpha + \xi_\alpha U^\alpha{}_{;\beta} U^\beta$
$= (U_{\alpha;\beta}\, \xi^\beta) U^\alpha$
$= \frac{1}{2} (U_\alpha U^\alpha)_{;\beta}\, \xi^\beta$
const
$= 0$
$\xi_\alpha U^\alpha = \text{const}$

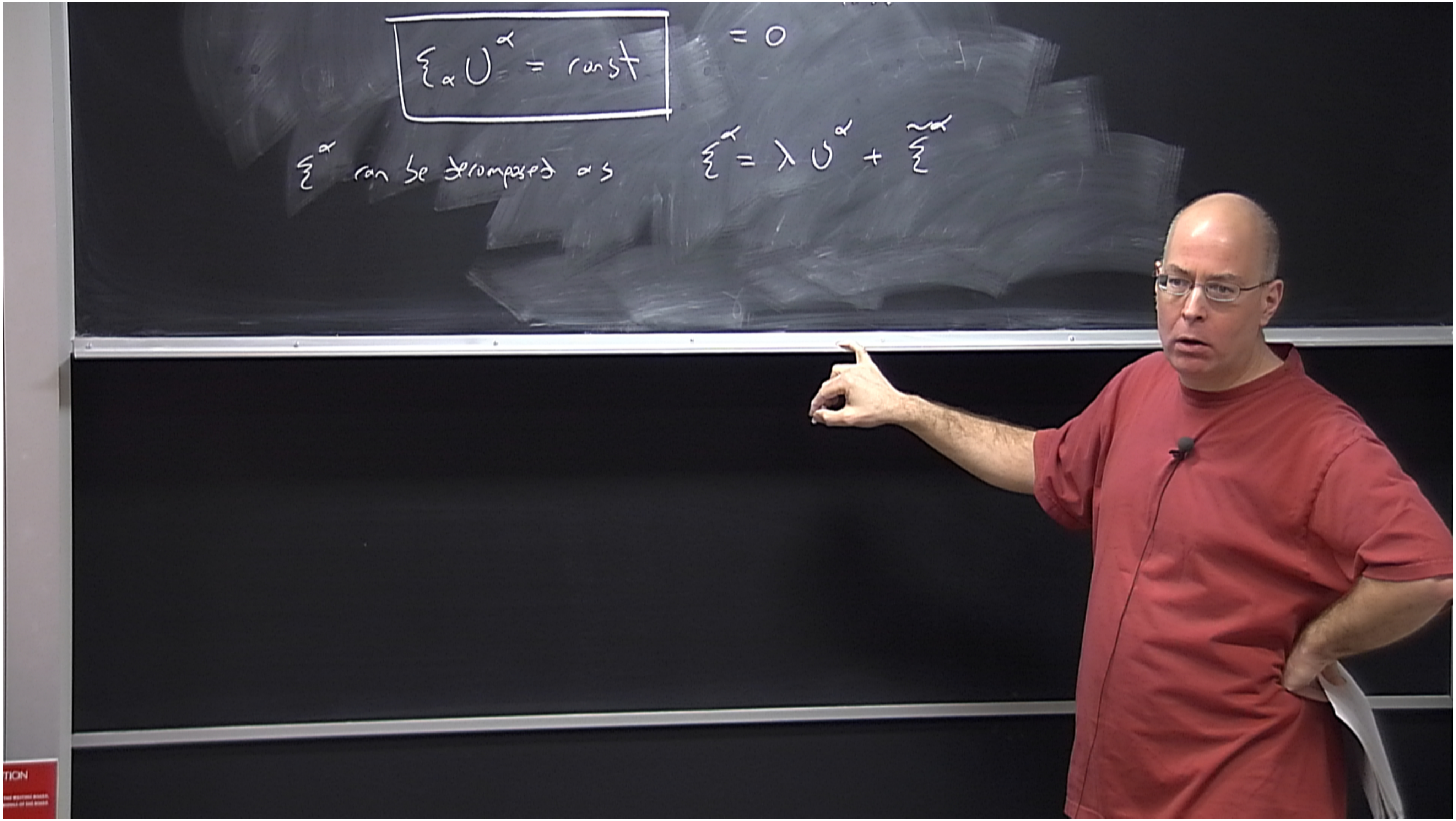

$\frac{d}{dt}(\xi_\alpha U^\alpha) = \frac{D}{dt}(\xi_\alpha U^\alpha) = (\xi_{\alpha;\beta} U^\beta) U^\alpha + \xi_\alpha U^\alpha{}_{;\beta} U^\beta$
$= (U_{\alpha;\beta}\, \xi^\beta) U^\alpha$
$= \frac{1}{2} (U_\alpha U^\alpha)_{;\beta}\, \xi^\beta$
const
$= 0$
$\xi_\alpha U^\alpha = \text{const}$



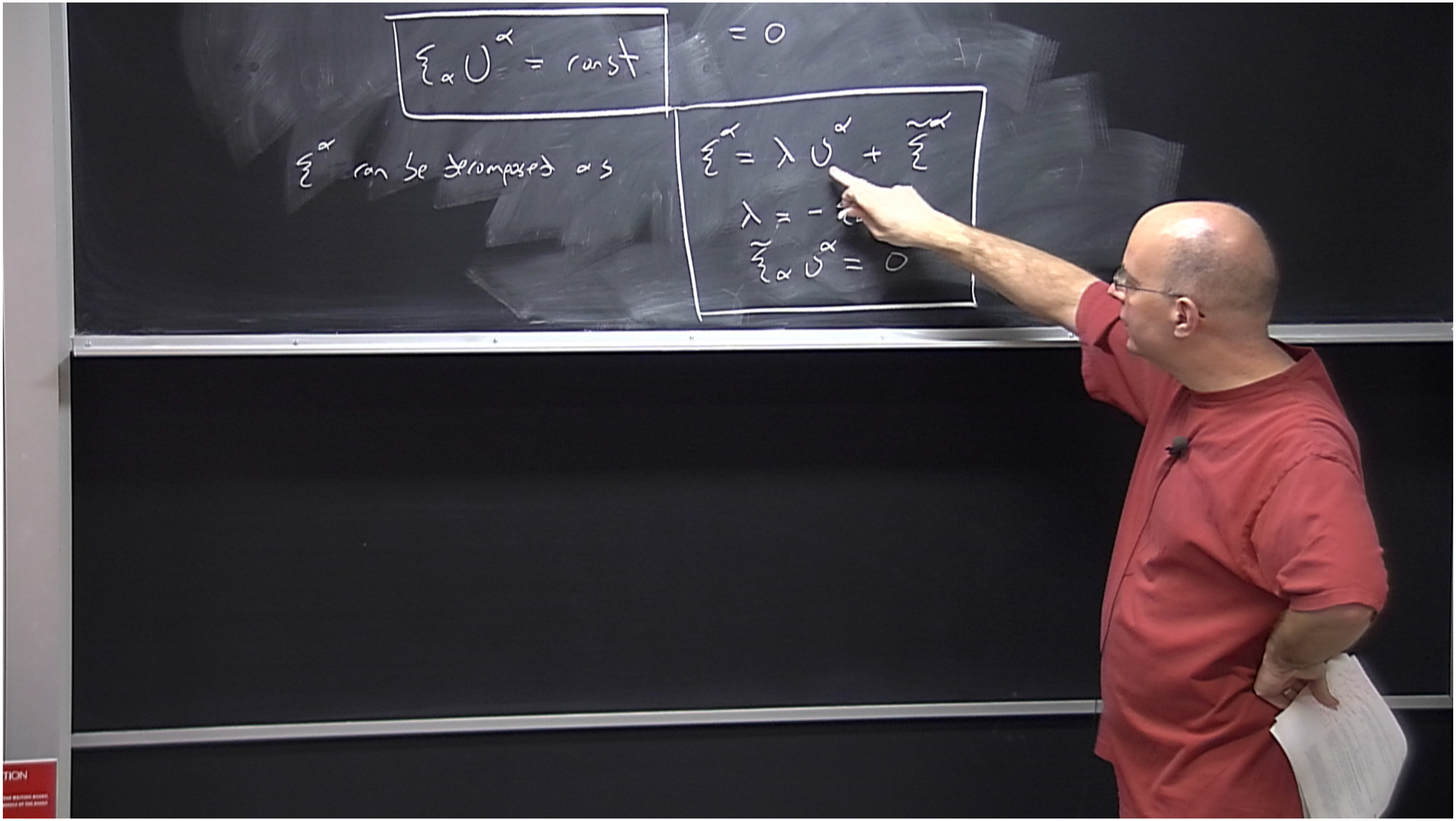
Geodesic deviation
Sequence of geodesics γ(s), such that
γ(s=0) = γ0
γ(s=1) = γ1
s = parameter that labels each geodesic
t = running parameter on each γ(s)
(affine parameter)
x^α = x^α(s, t) parametric description of γ(s)
selecting geodesic
selecting point on selected geodesic
s=1
γ0
γ1

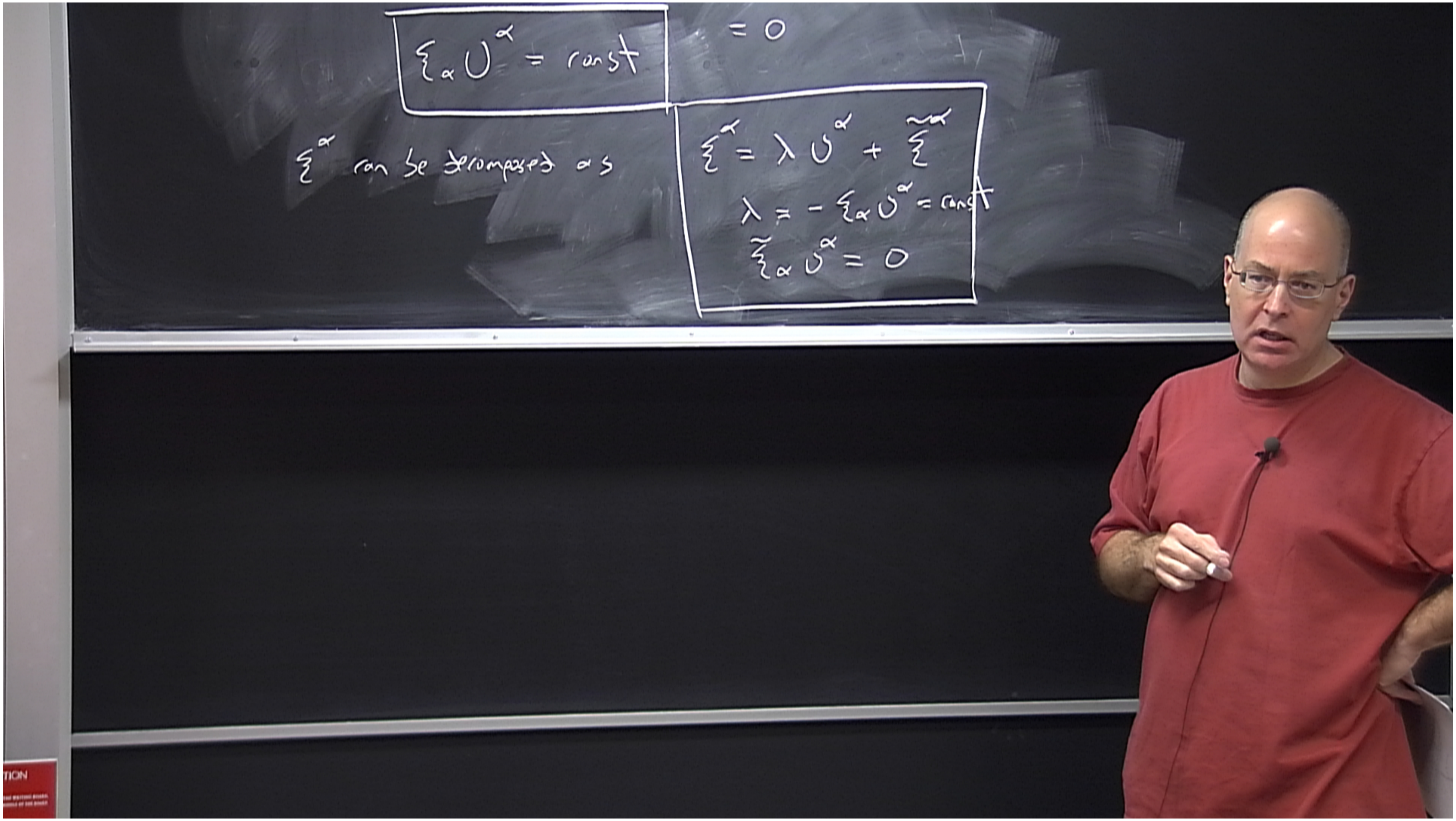

Geodesic deviation
s=0.1
s=0.7
s=1
γ_0
γ_1
Sequence of geodesics $\gamma(s)$, such that
$\gamma(s=0) = \gamma_0$
$\gamma(s=1) = \gamma_1$
s = parameter that labels each geodesic
t = running parameter on each $\gamma(s)$
(affine parameter)
$x^\alpha = x^\alpha(s, t)$ parametric description of $\gamma(s)$
selecting geodesic
selecting point on selected geodesic

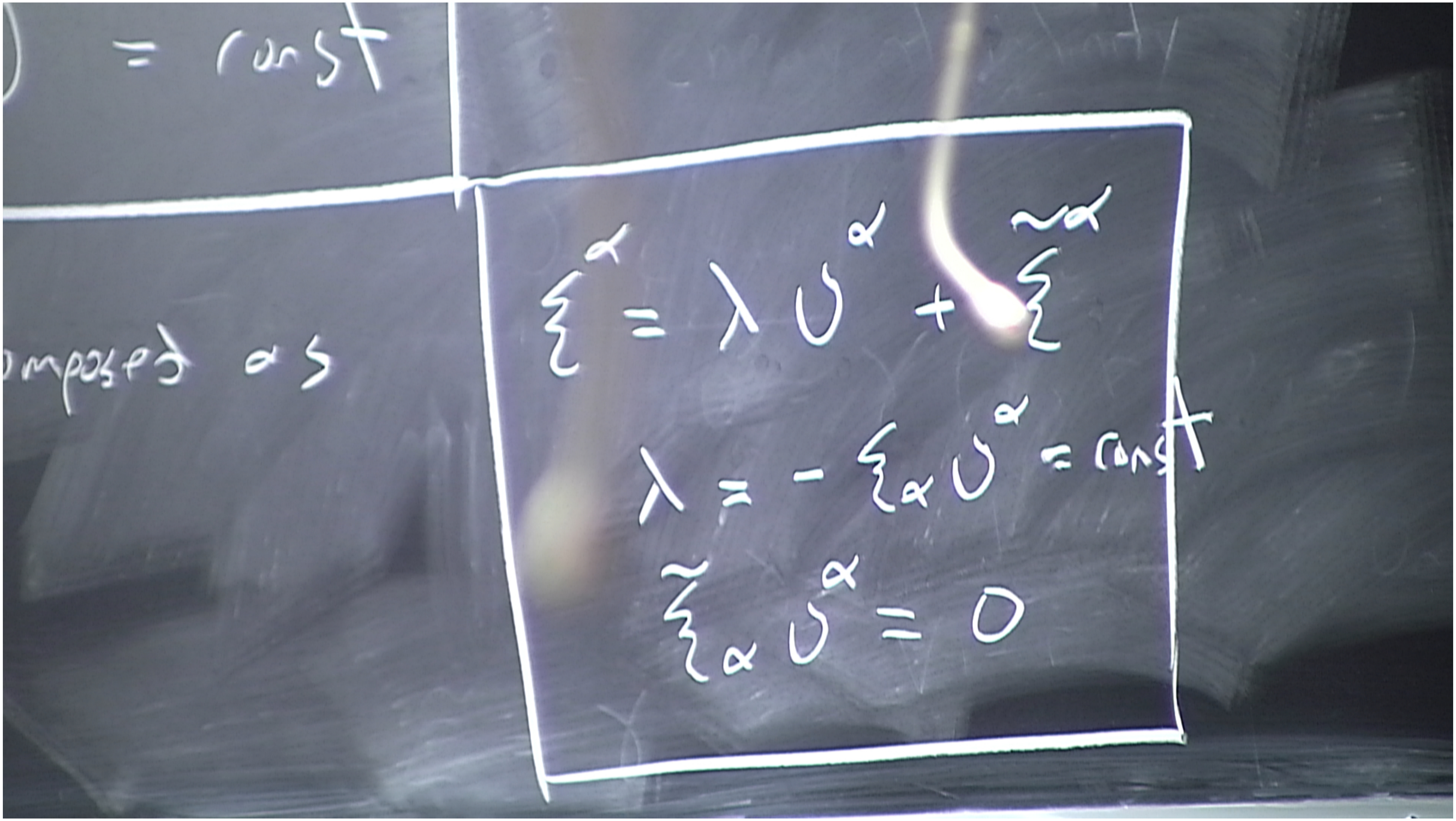

$\xi_\alpha U^\alpha = \text{const}$
$= 0$
ξ^α can be decomposed as $\xi^\alpha = \lambda U^\alpha$
TION

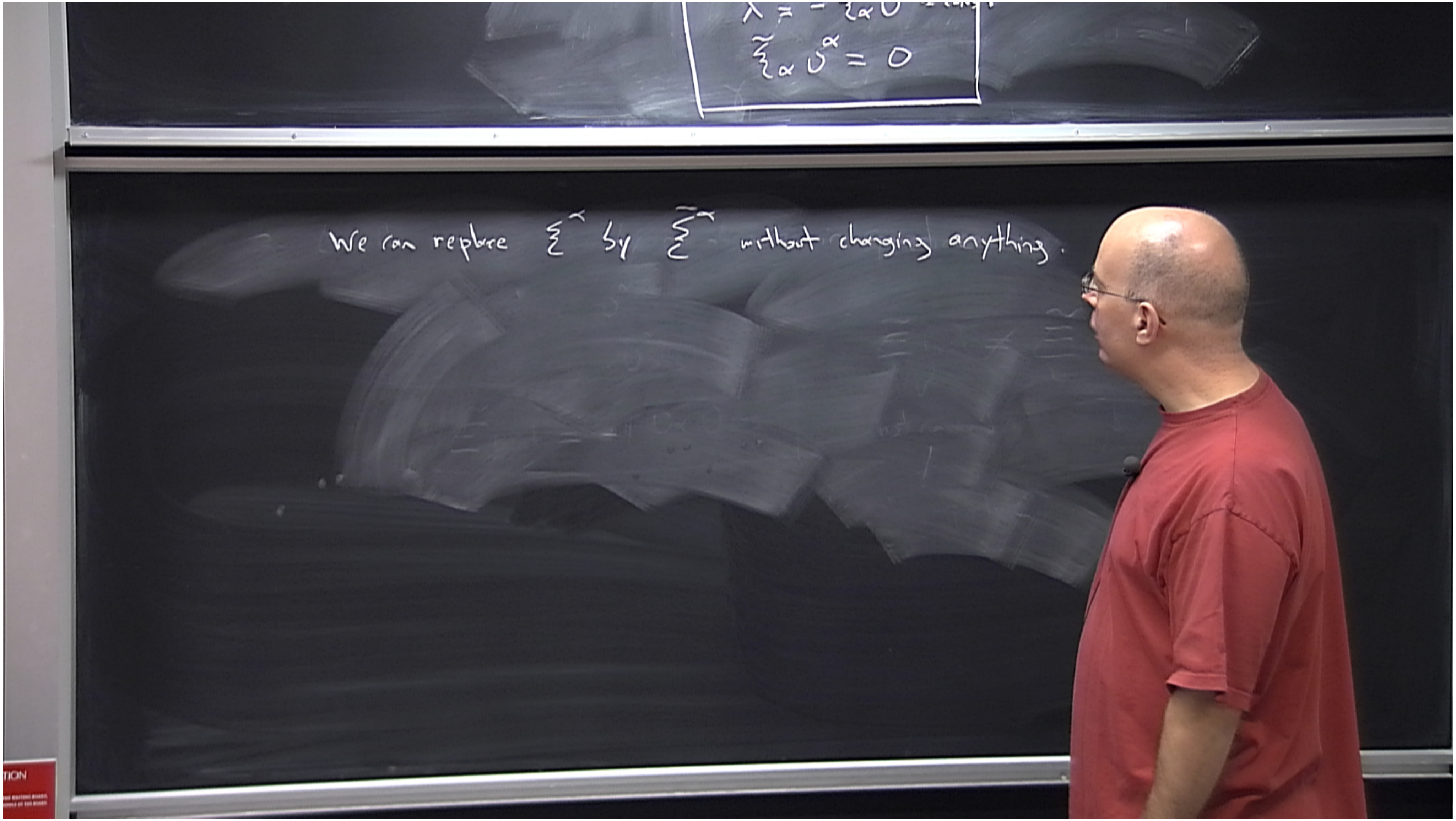


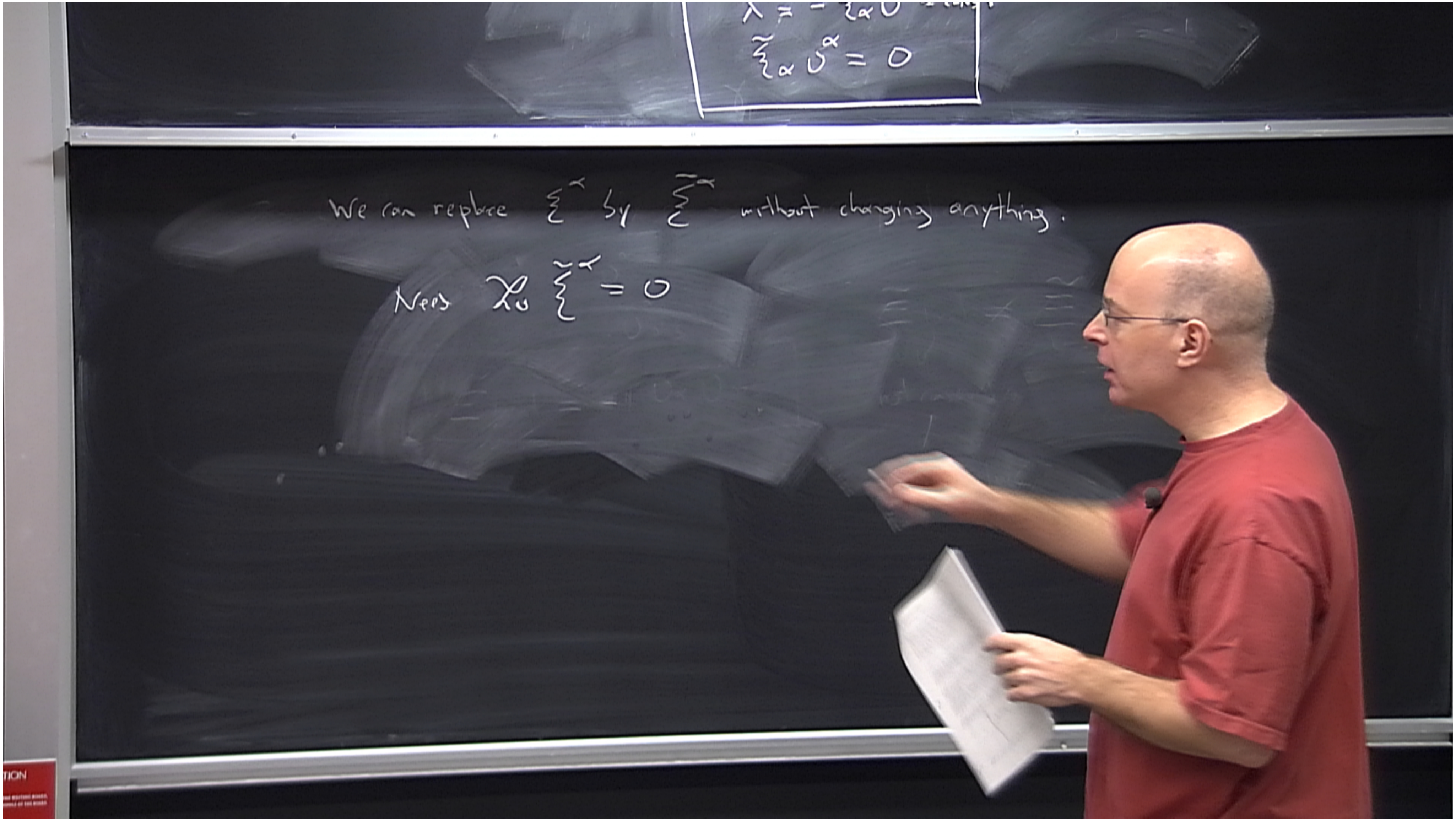
$\xi_\alpha U^\alpha = \text{const}$
$= 0$
ξ^α can be decomposed as $\xi^\alpha = \lambda U^\alpha + \tilde{\xi}^\alpha$


$\xi_\alpha U^\alpha = \text{const}$
$= 0$
ξ^α can be decomposed as
$\xi^\alpha = \lambda U^\alpha + \tilde{\xi}^\alpha$
$\lambda = -$
$\tilde{\xi}_\alpha U^\alpha = 0$

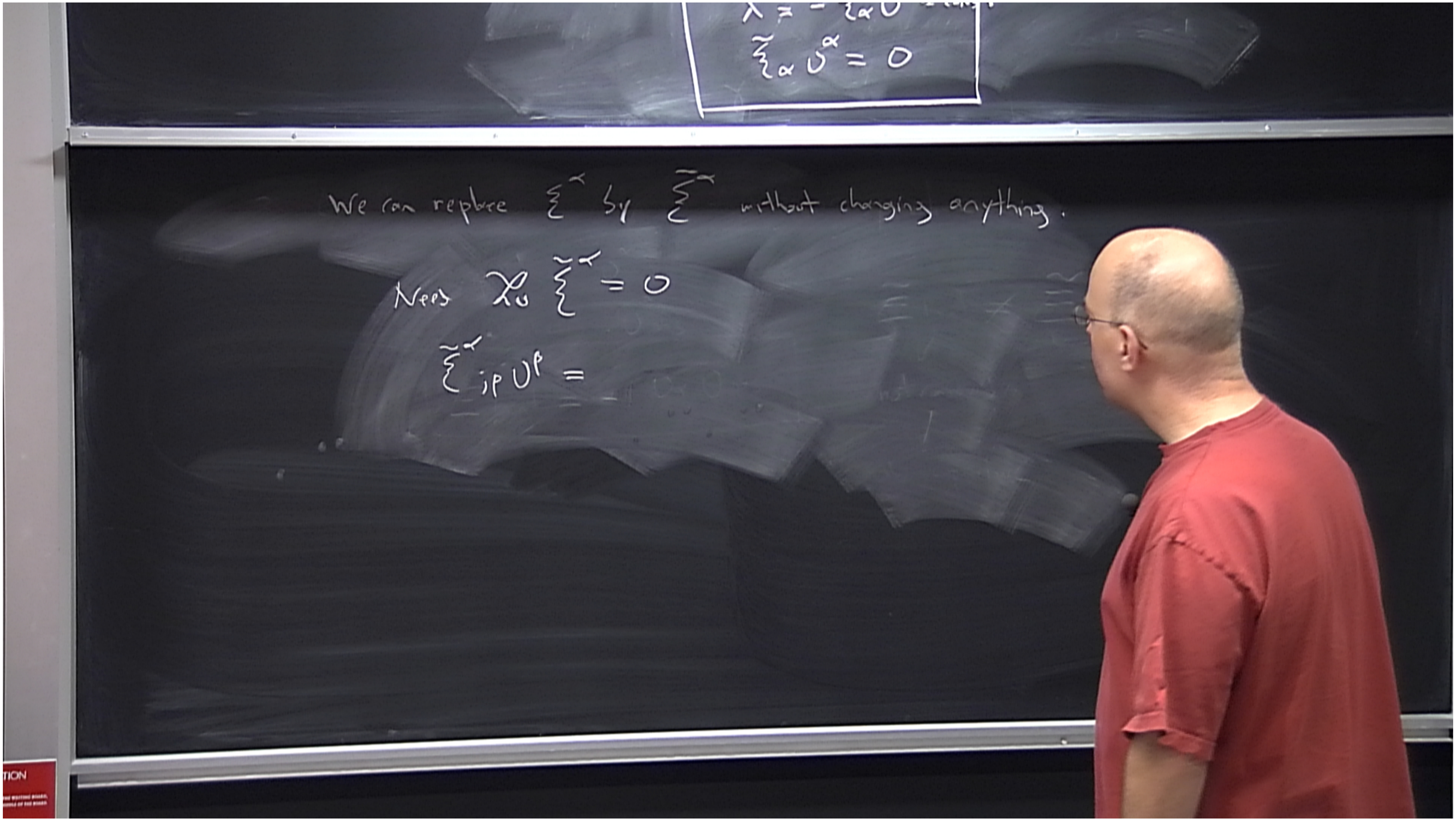

$\xi_\alpha U^\alpha = \text{const}$
$= 0$
ξ^α can be decomposed as
$\xi^\alpha = \lambda U^\alpha + \tilde{\xi}^\alpha$
$\lambda = - \xi_\alpha U^\alpha = \text{const}$
$\tilde{\xi}_\alpha U^\alpha = 0$

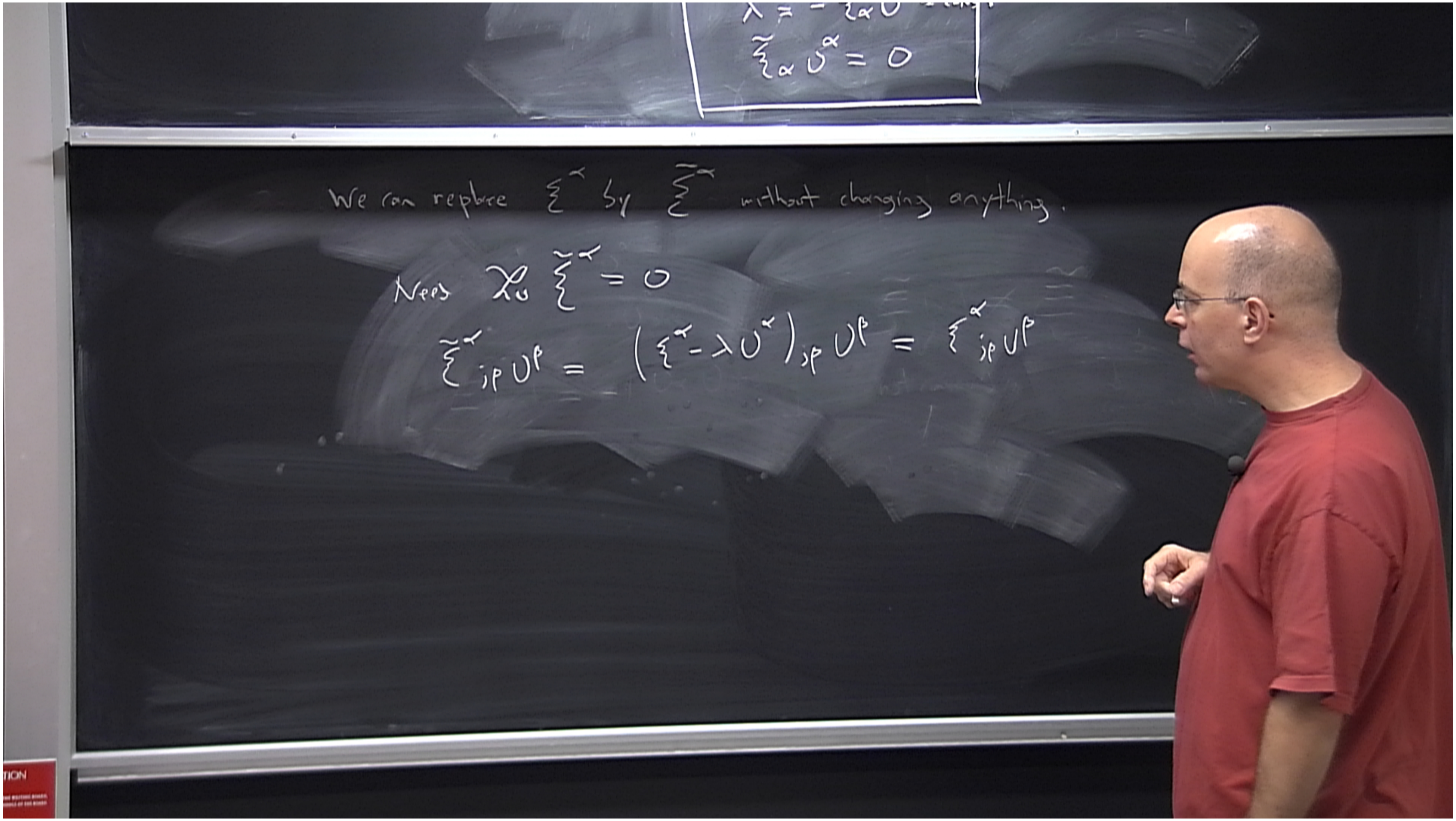

= const
omposed as
$\xi^\alpha = \lambda u^\alpha + \tilde{\xi}^\alpha$
$\lambda = -\xi_\alpha u^\alpha = \text{const}$
$\tilde{\xi}_\alpha u^\alpha = 0$

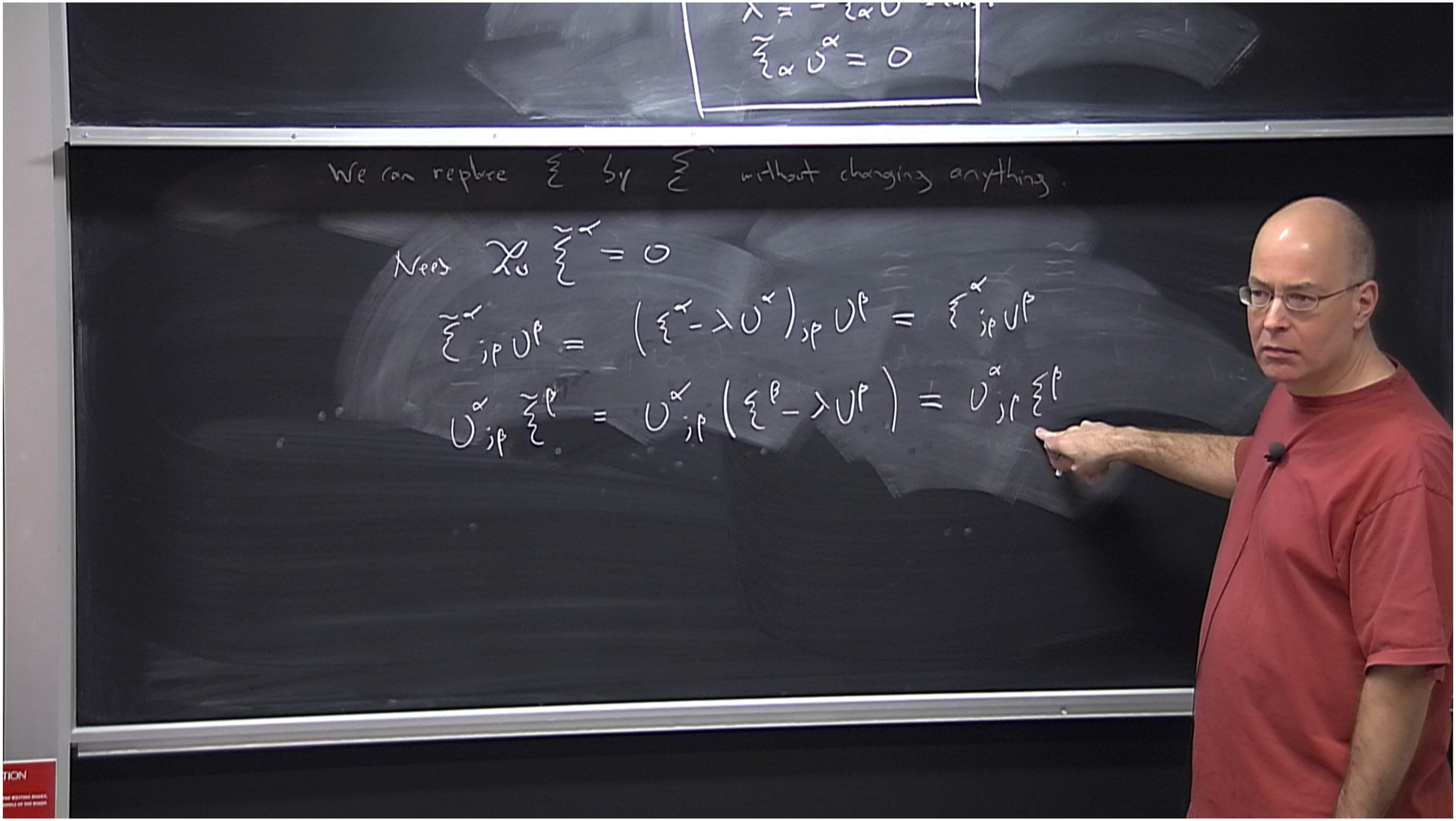

$\tilde{\xi}_\alpha U^\alpha = 0$
We can replace ξ^α by $\tilde{\xi}^\alpha$ without changing anything.

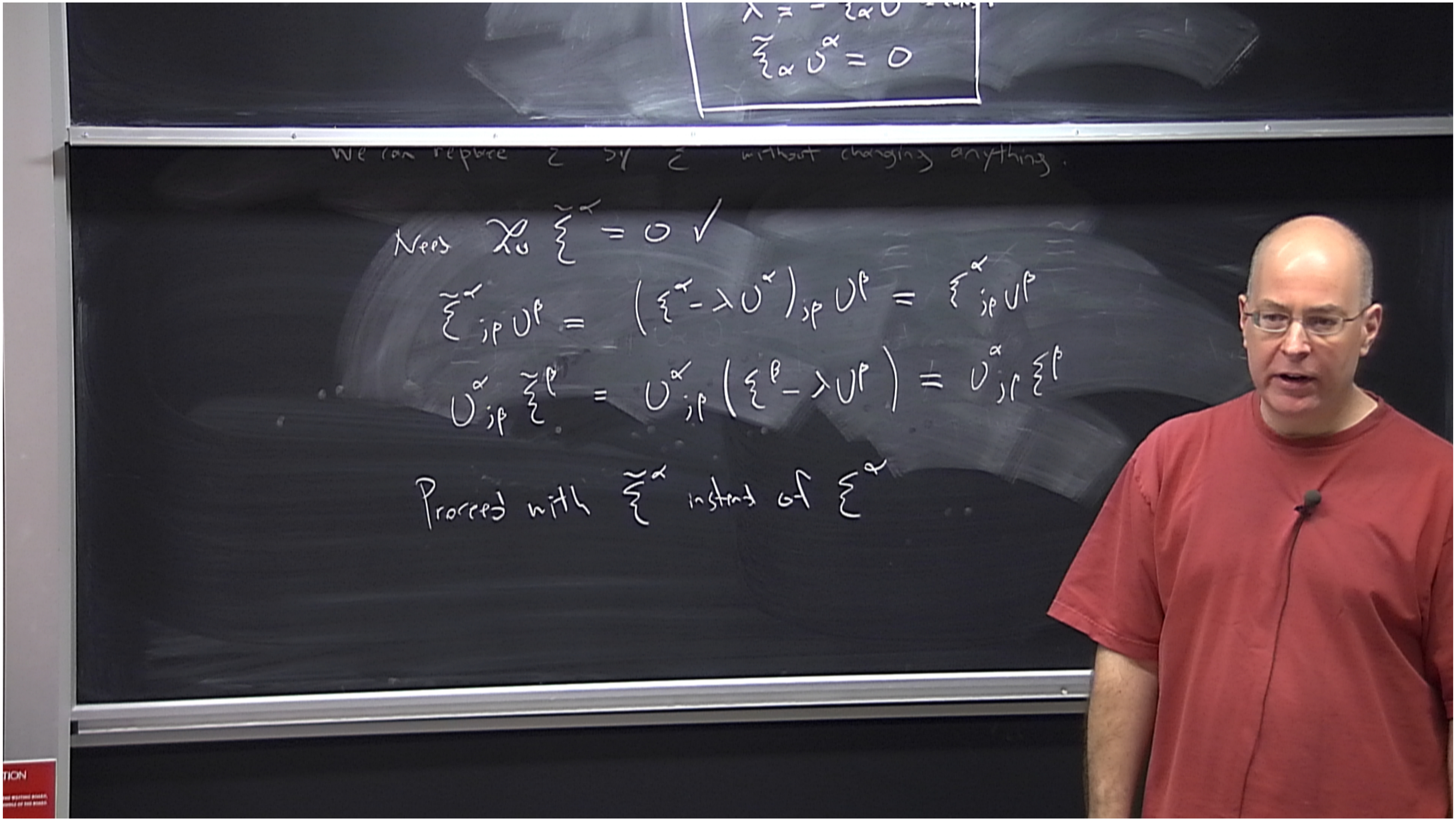


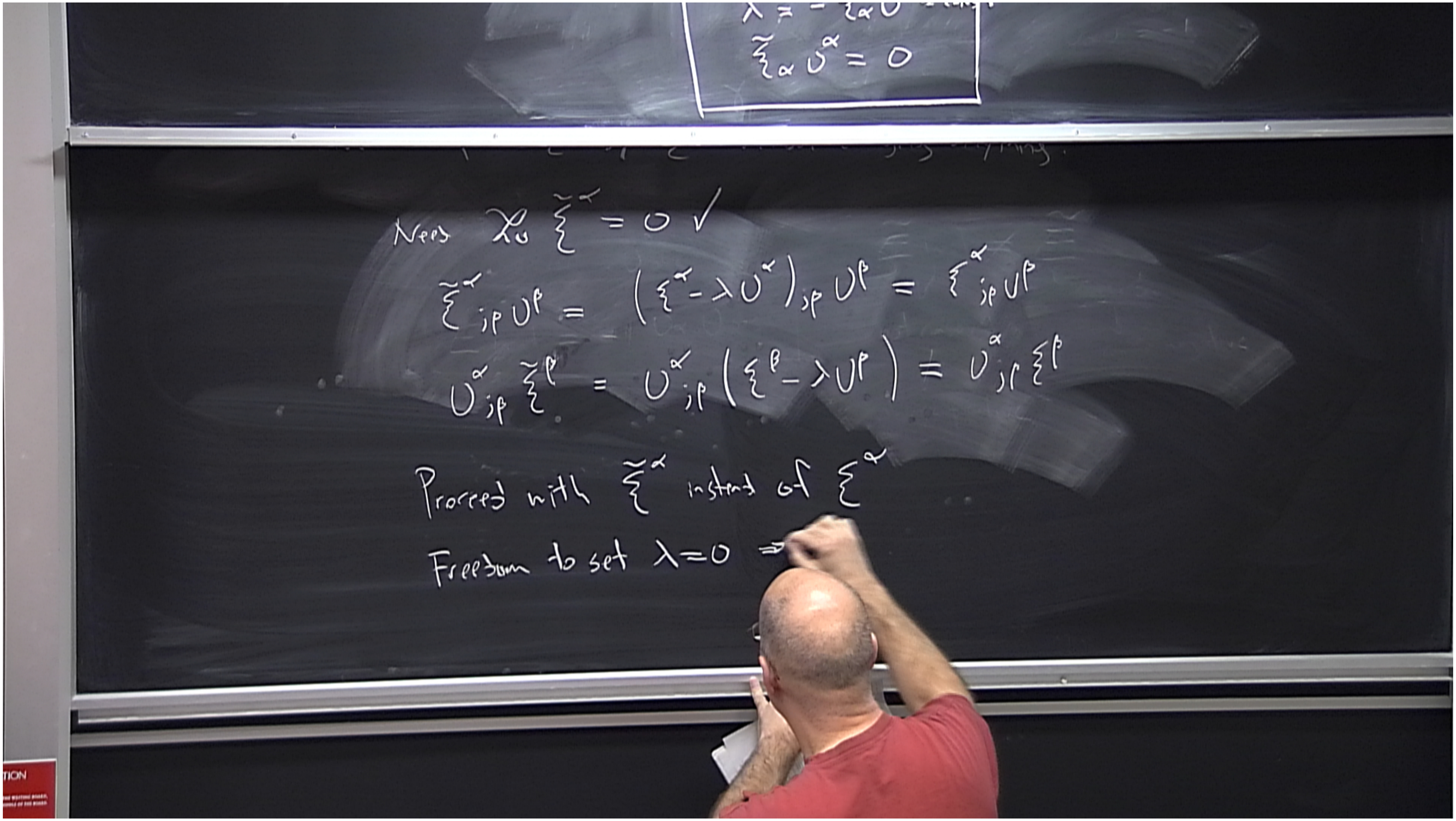
$\tilde{\xi}_\alpha U^\alpha = 0$
We can replace ξ^α by $\tilde{\xi}^\alpha$ without changing anything.

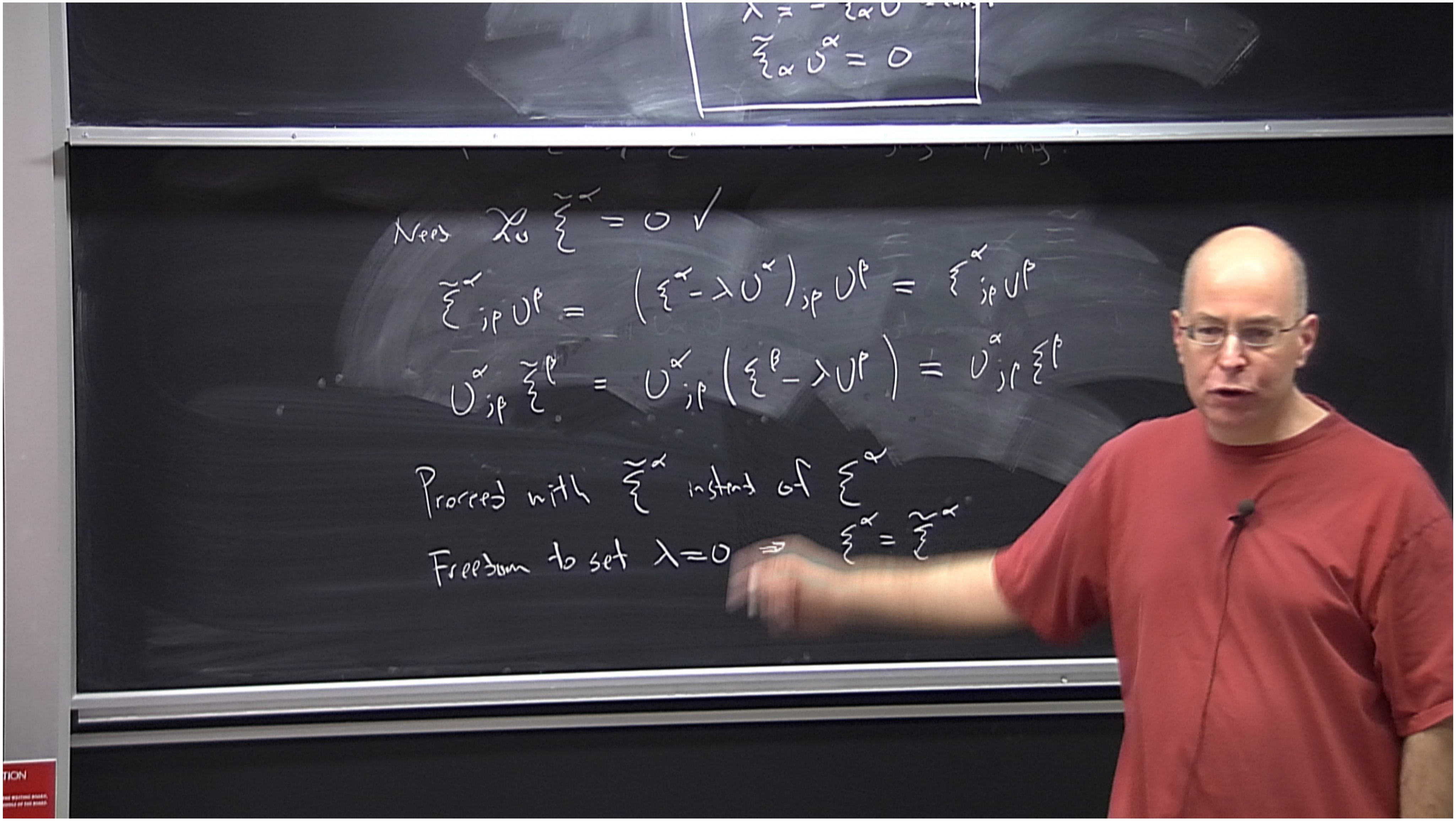

We can replace ξ^α by ξ̃^α without changing anything.
Need 𝓛_u ξ̃^α = 0
ξ̃^α_;β U^β =
ξ̃_α U^α = 0

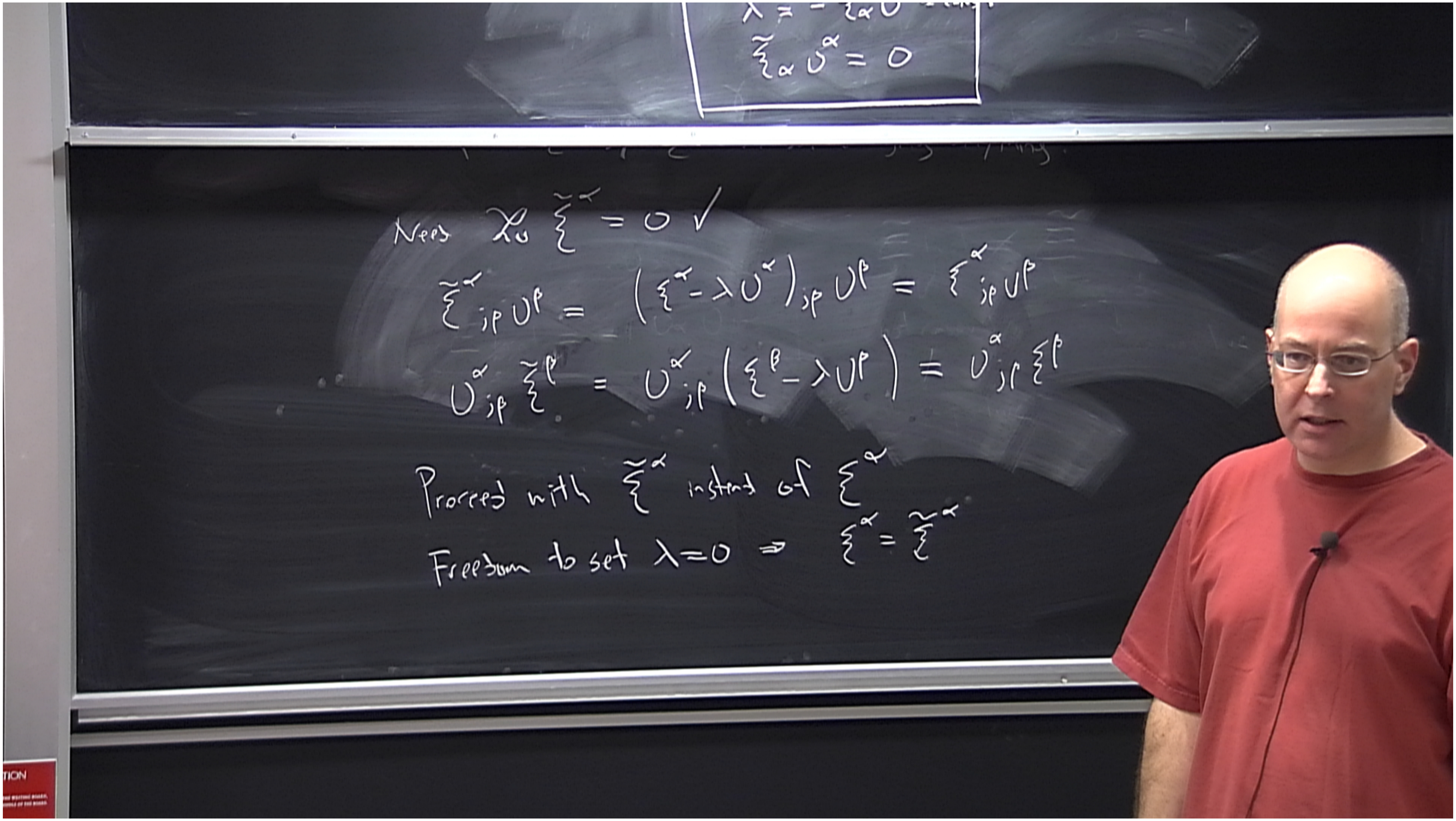

ξ̃_α U^α = 0
We can replace ξ^α by ξ̃^α without changing anything.
Need £_U ξ̃^α = 0
ξ̃^α_;β U^β = (ξ^α − λU^α)_;β U^β = ξ^α_;β U^β


$\tilde{\xi}_\alpha U^\alpha = 0$
We can replace ξ by ξ without changing anything.
Need $\mathcal{L}_u \tilde{\xi}^\alpha = 0$
$\tilde{\xi}^\alpha{}_{;\beta} U^\beta = (\xi^\alpha - \lambda U^\alpha)_{;\beta} U^\beta = \xi^\alpha{}_{;\beta} U^\beta$
$U^\alpha{}_{;\beta} \tilde{\xi}^\beta = U^\alpha{}_{;\beta} (\xi^\beta - \lambda U^\beta) = U^\alpha{}_{;\beta} \xi^\beta$

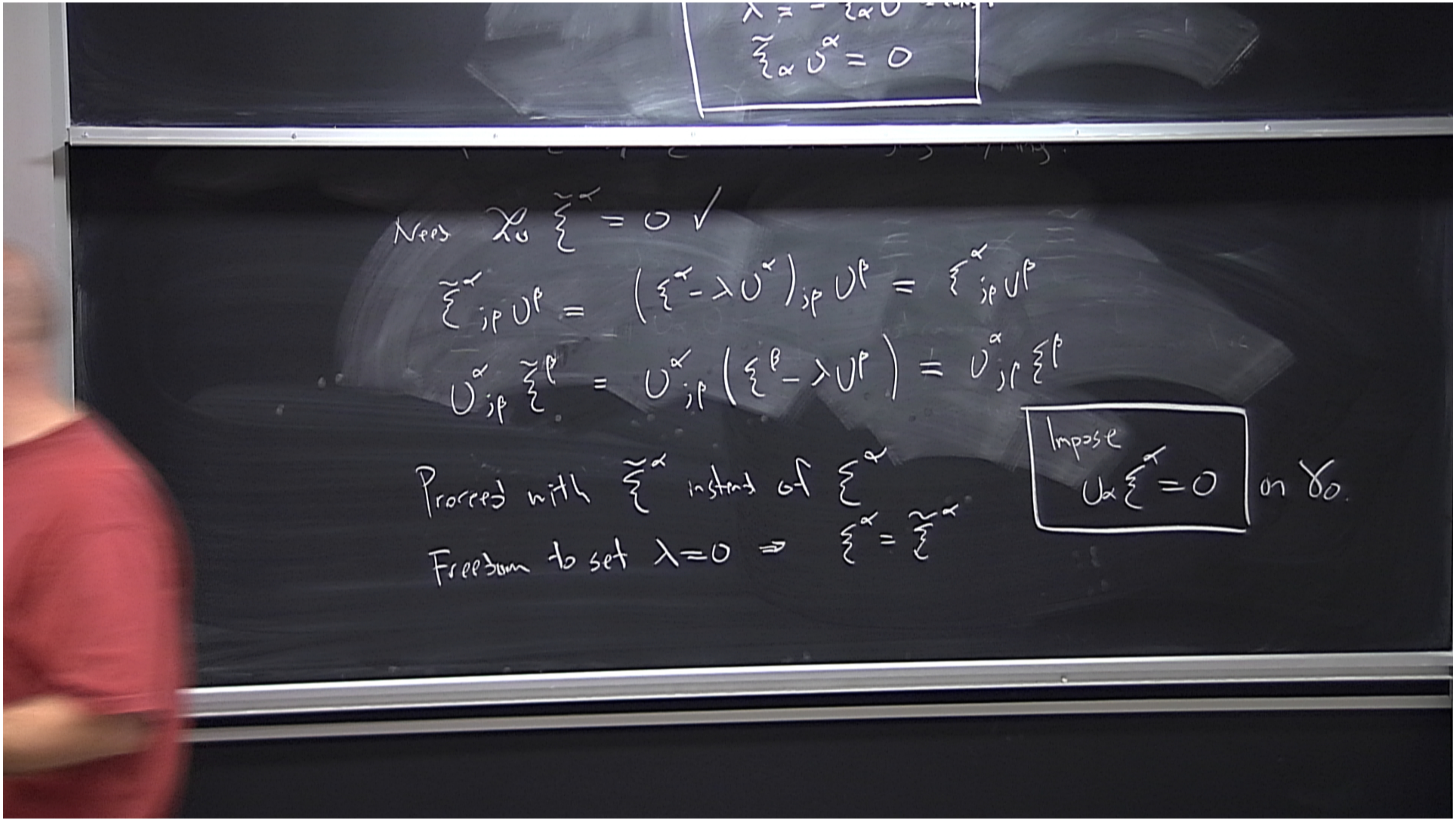

$\tilde{\xi}_\alpha U^\alpha = 0$
We can replace ξ by $\tilde{\xi}$ without changing anything.
Need $\mathcal{L}_U \tilde{\xi}^\alpha = 0$ ✓
$\tilde{\xi}^\alpha{}_{;\beta} U^\beta = (\xi^\alpha - \lambda U^\alpha)_{;\beta} U^\beta = \xi^\alpha{}_{;\beta} U^\beta$
$U^\alpha{}_{;\beta} \tilde{\xi}^\beta = U^\alpha{}_{;\beta} (\xi^\beta - \lambda U^\beta) = U^\alpha{}_{;\beta} \xi^\beta$
Proceed with $\tilde{\xi}^\alpha$ instead of ξ^α

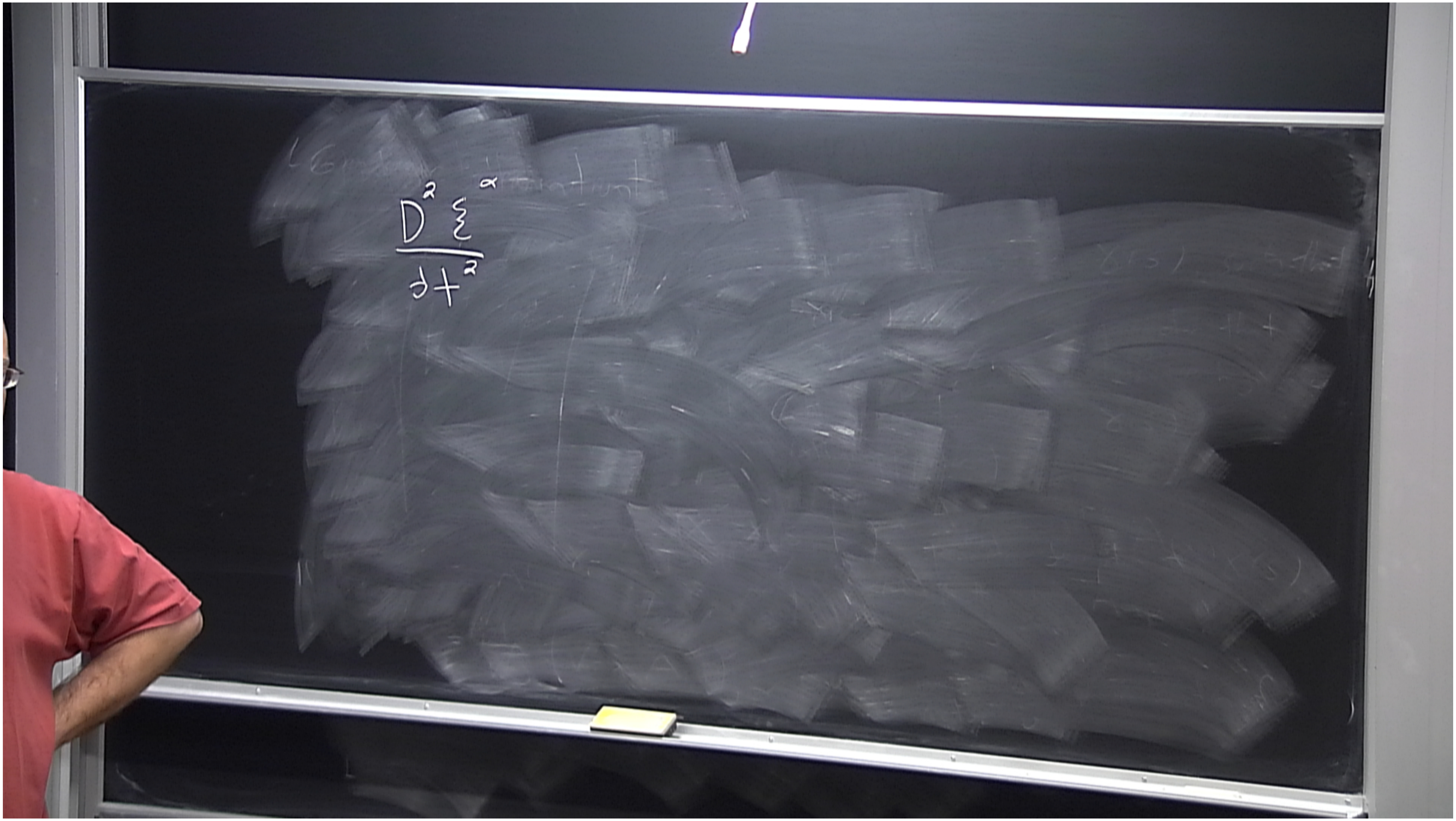

$\tilde{\xi}_\alpha U^\alpha = 0$
Needs $\mathcal{L}_U \tilde{\xi}^\alpha = 0$ ✓
$\tilde{\xi}^\alpha{}_{;\beta} U^\beta = (\xi^\alpha - \lambda U^\alpha)_{;\beta} U^\beta = \xi^\alpha{}_{;\beta} U^\beta$
$U^\alpha{}_{;\beta} \tilde{\xi}^\beta = U^\alpha{}_{;\beta} (\xi^\beta - \lambda U^\beta) = U^\alpha{}_{;\beta} \xi^\beta$
Proceed with $\tilde{\xi}^\alpha$ instead of ξ^α
Freedom to set $\lambda = 0$ ⇒

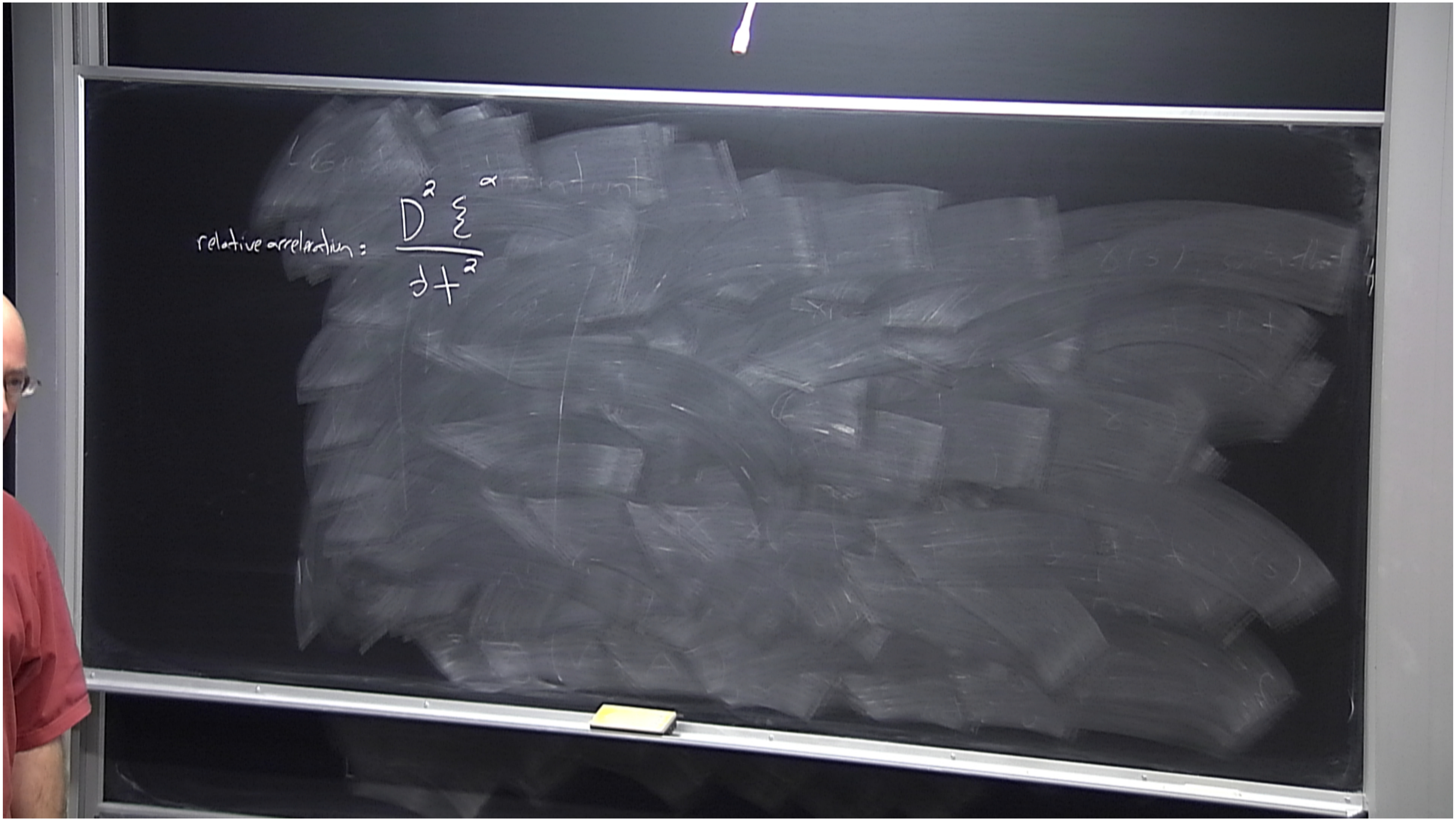

$\tilde{\xi}_\alpha U^\alpha = 0$
Need $\mathcal{L}_U \tilde{\xi}^\alpha = 0$ ✓
$\tilde{\xi}^\alpha{}_{;\beta} U^\beta = (\xi^\alpha - \lambda U^\alpha)_{;\beta} U^\beta = \xi^\alpha{}_{;\beta} U^\beta$
$U^\alpha{}_{;\beta} \tilde{\xi}^\beta = U^\alpha{}_{;\beta} (\xi^\beta - \lambda U^\beta) = U^\alpha{}_{;\beta} \xi^\beta$
Proceed with $\tilde{\xi}^\alpha$ instead of ξ^α
Freedom to set $\lambda = 0 \Rightarrow \xi^\alpha = \tilde{\xi}^\alpha$

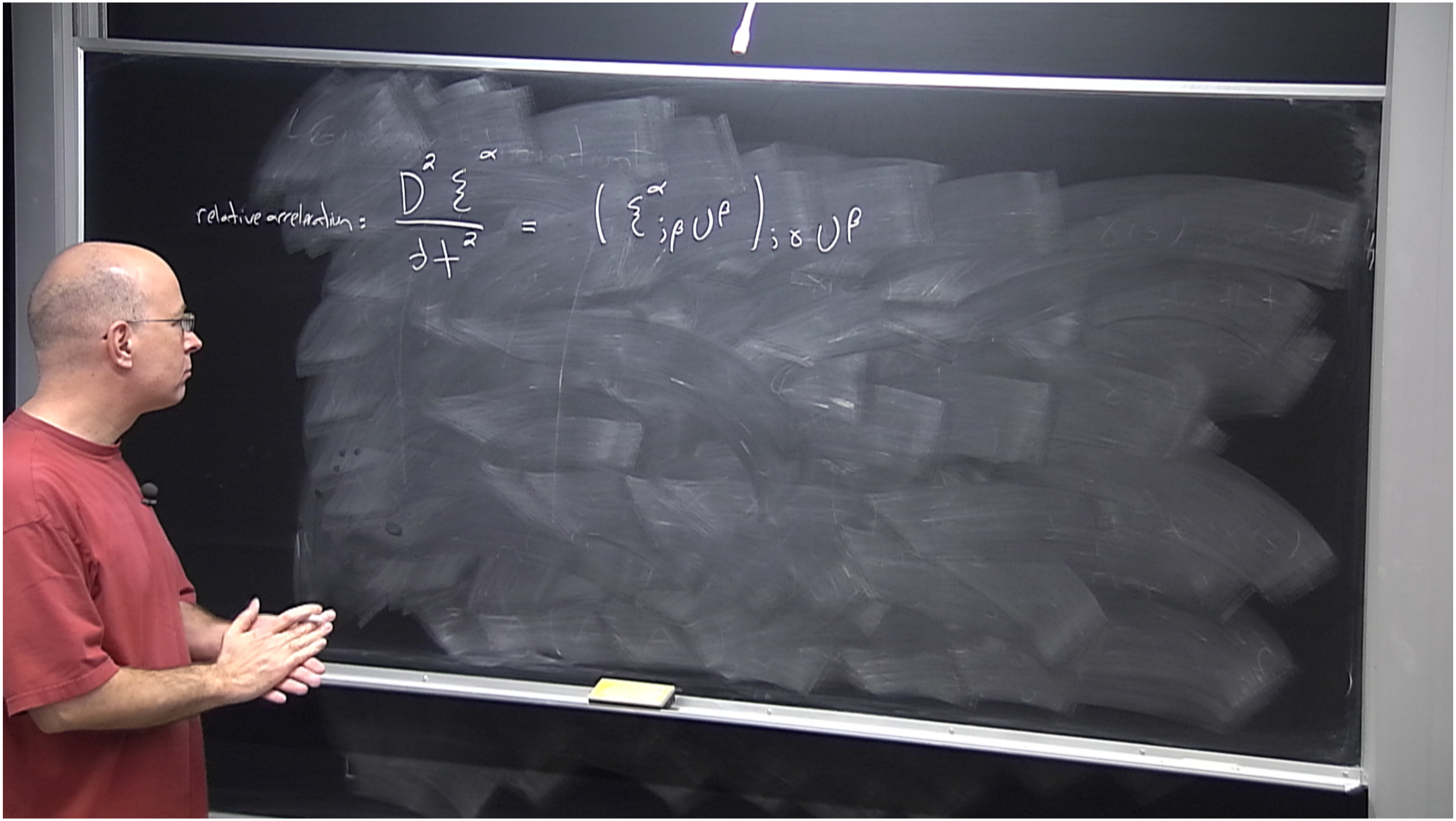


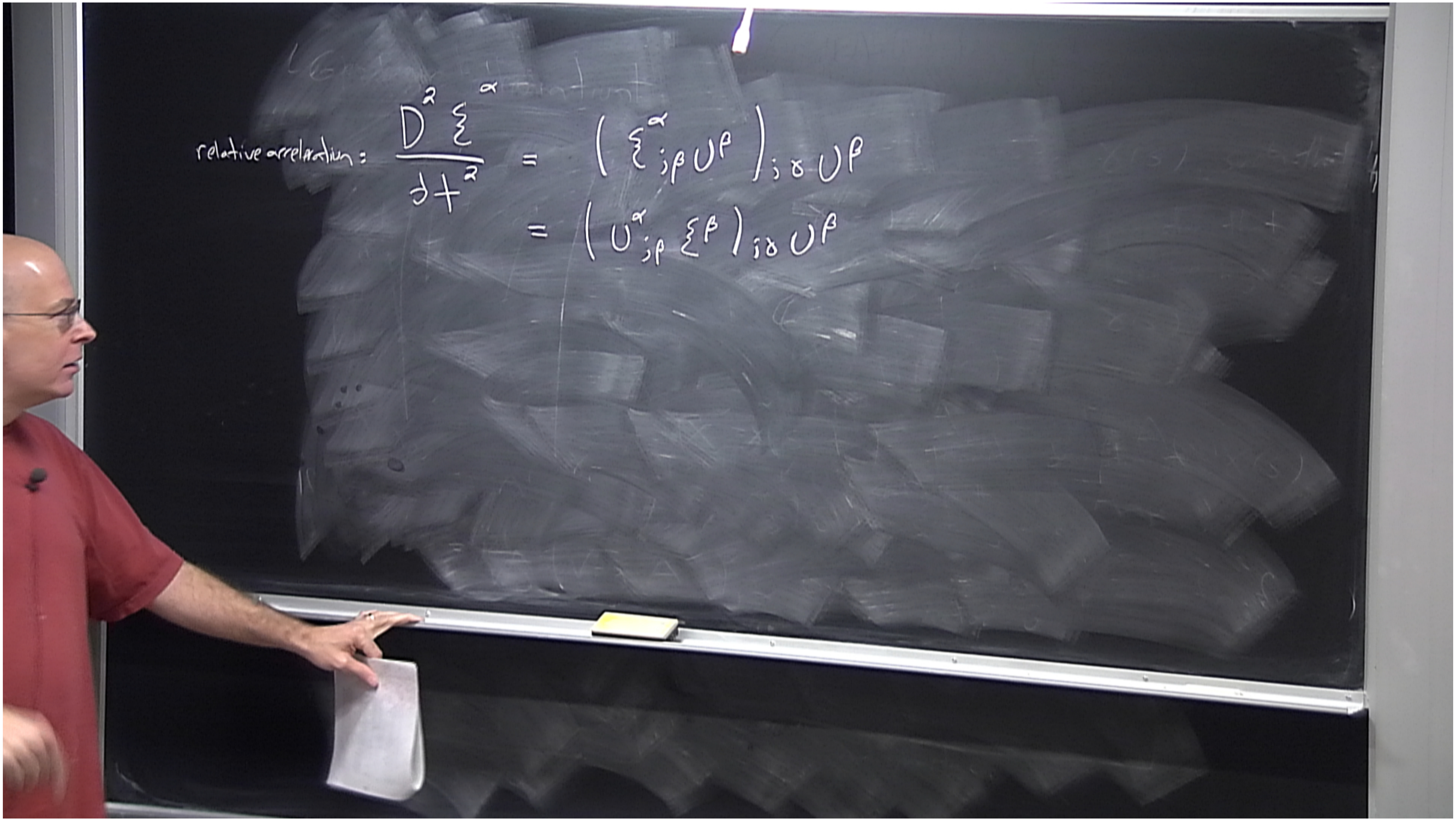
$\tilde{\xi}_\alpha U^\alpha = 0$
Need $\mathcal{L}_U \tilde{\xi}^\alpha = 0$ ✓
$\tilde{\xi}^\alpha{}_{;\beta} U^\beta = (\xi^\alpha - \lambda U^\alpha)_{;\beta} U^\beta = \xi^\alpha{}_{;\beta} U^\beta$
$U^\alpha{}_{;\beta} \tilde{\xi}^\beta = U^\alpha{}_{;\beta} (\xi^\beta - \lambda U^\beta) = U^\alpha{}_{;\beta} \xi^\beta$
Proceed with $\tilde{\xi}^\alpha$ instead of ξ^α
Freedom to set $\lambda = 0 \Rightarrow \xi^\alpha = \tilde{\xi}^\alpha$

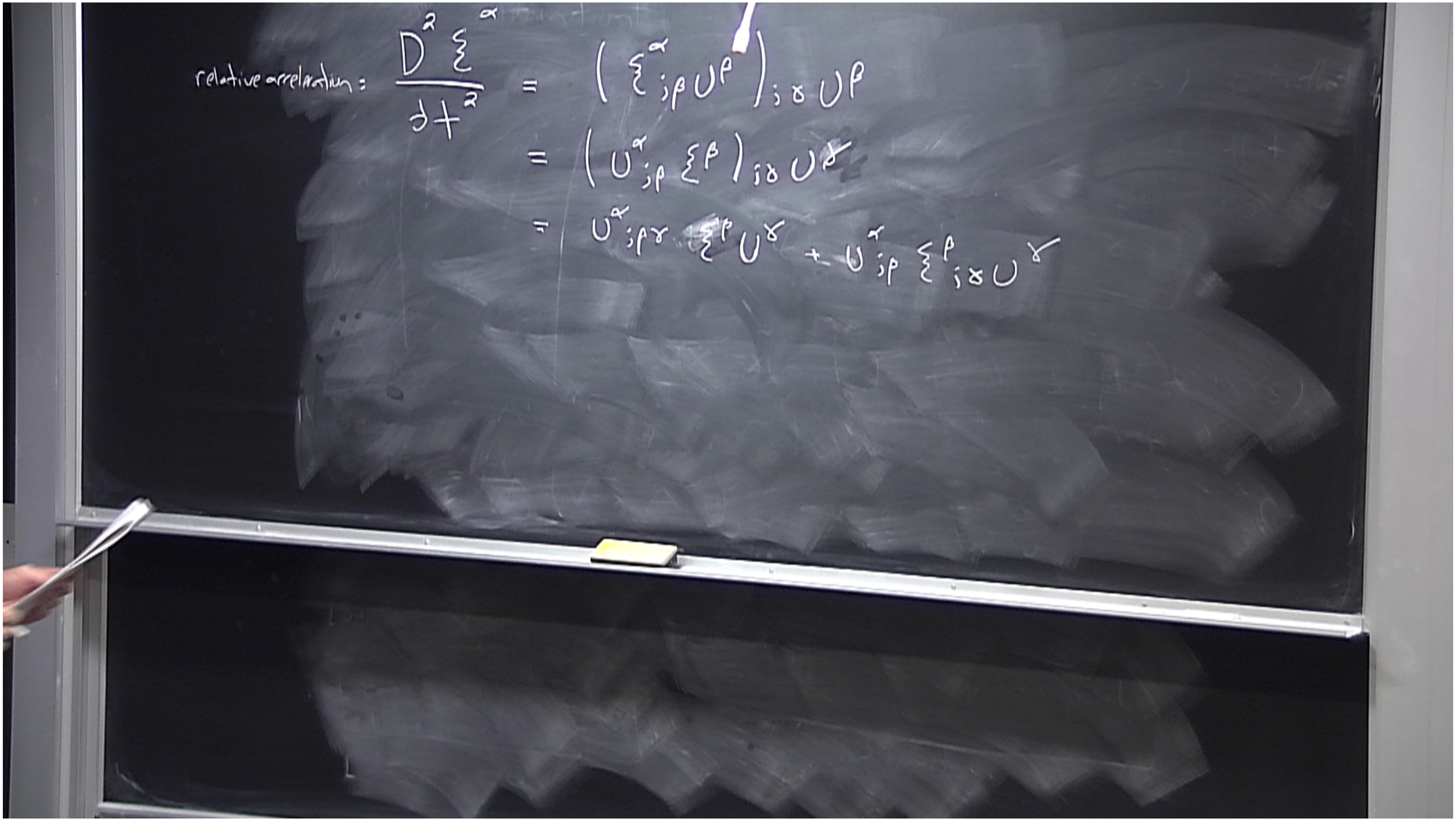

$\tilde{\xi}_\alpha u^\alpha = 0$
Needs $\mathcal{L}_u \tilde{\xi}^\alpha = 0$ ✓
$\tilde{\xi}^\alpha{}_{;\beta} u^\beta = (\xi^\alpha - \lambda u^\alpha)_{;\beta} u^\beta = \xi^\alpha{}_{;\beta} u^\beta$
$u^\alpha{}_{;\beta} \tilde{\xi}^\beta = u^\alpha{}_{;\beta} (\xi^\beta - \lambda u^\beta) = u^\alpha{}_{;\beta} \xi^\beta$
Proceed with $\tilde{\xi}^\alpha$ instead of ξ^α
Freedom to set $\lambda = 0 \Rightarrow \xi^\alpha = \tilde{\xi}^\alpha$
Impose $u_\alpha \xi^\alpha = 0$ on γ_0.

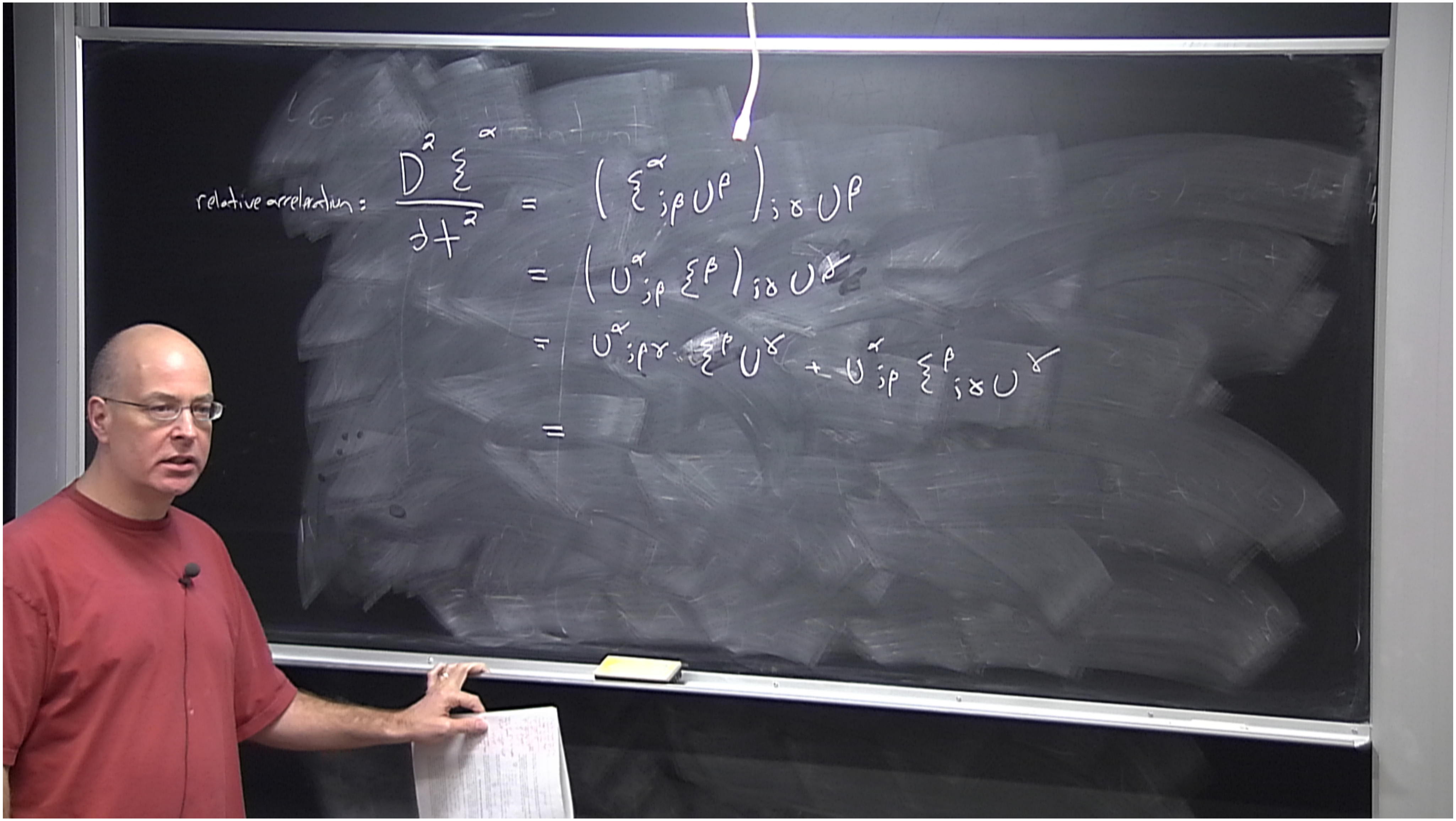

$\frac{D^2 \xi^\alpha}{d\tau^2}$

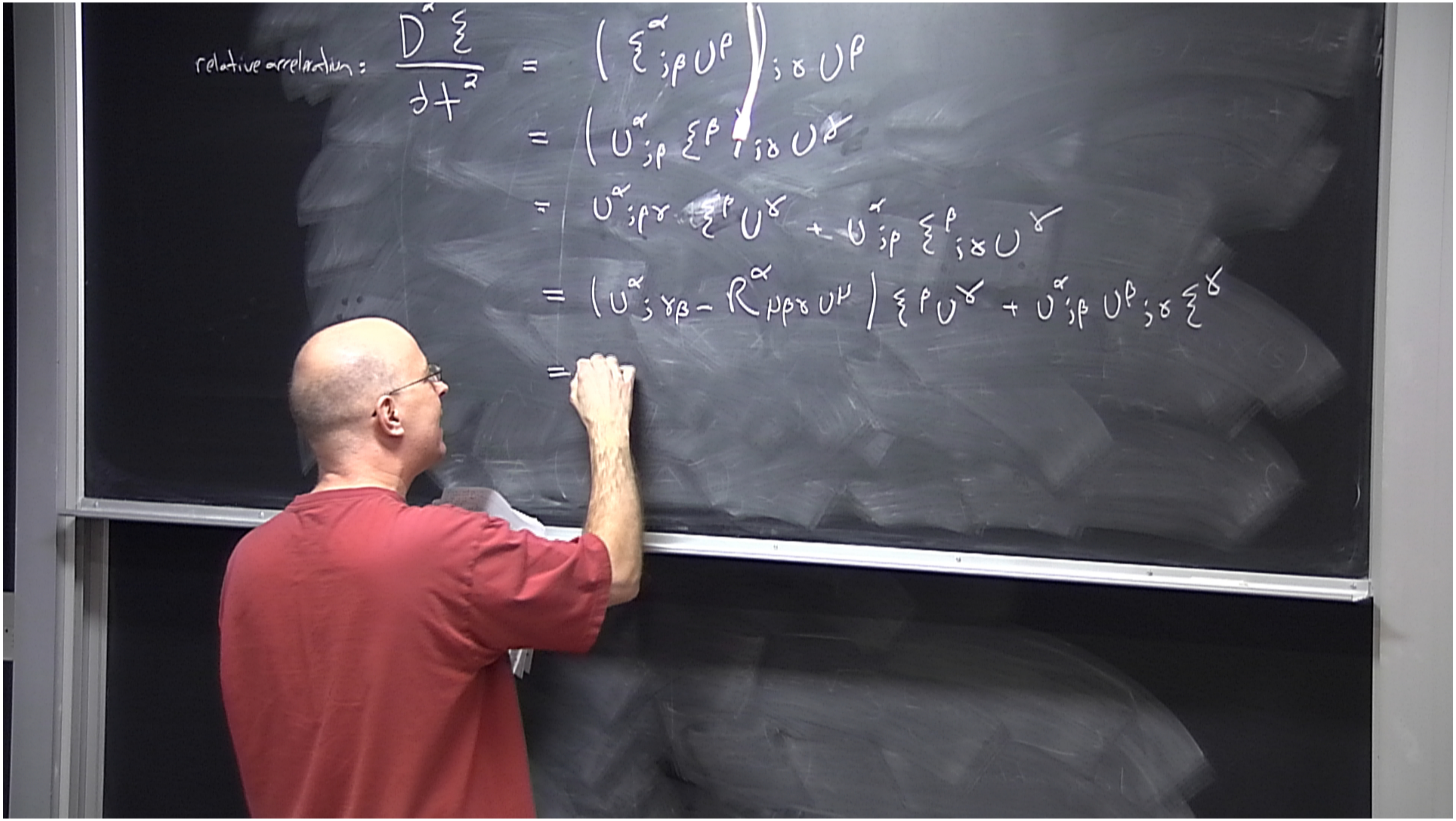

relative acceleration: $\frac{D^2 \xi^\alpha}{d\tau^2}$


relative acceleration:
$\frac{D^2 \xi^\alpha}{d\tau^2} = \left(\xi^\alpha{}_{;\beta} U^\beta \right)_{;\gamma} U^\gamma$

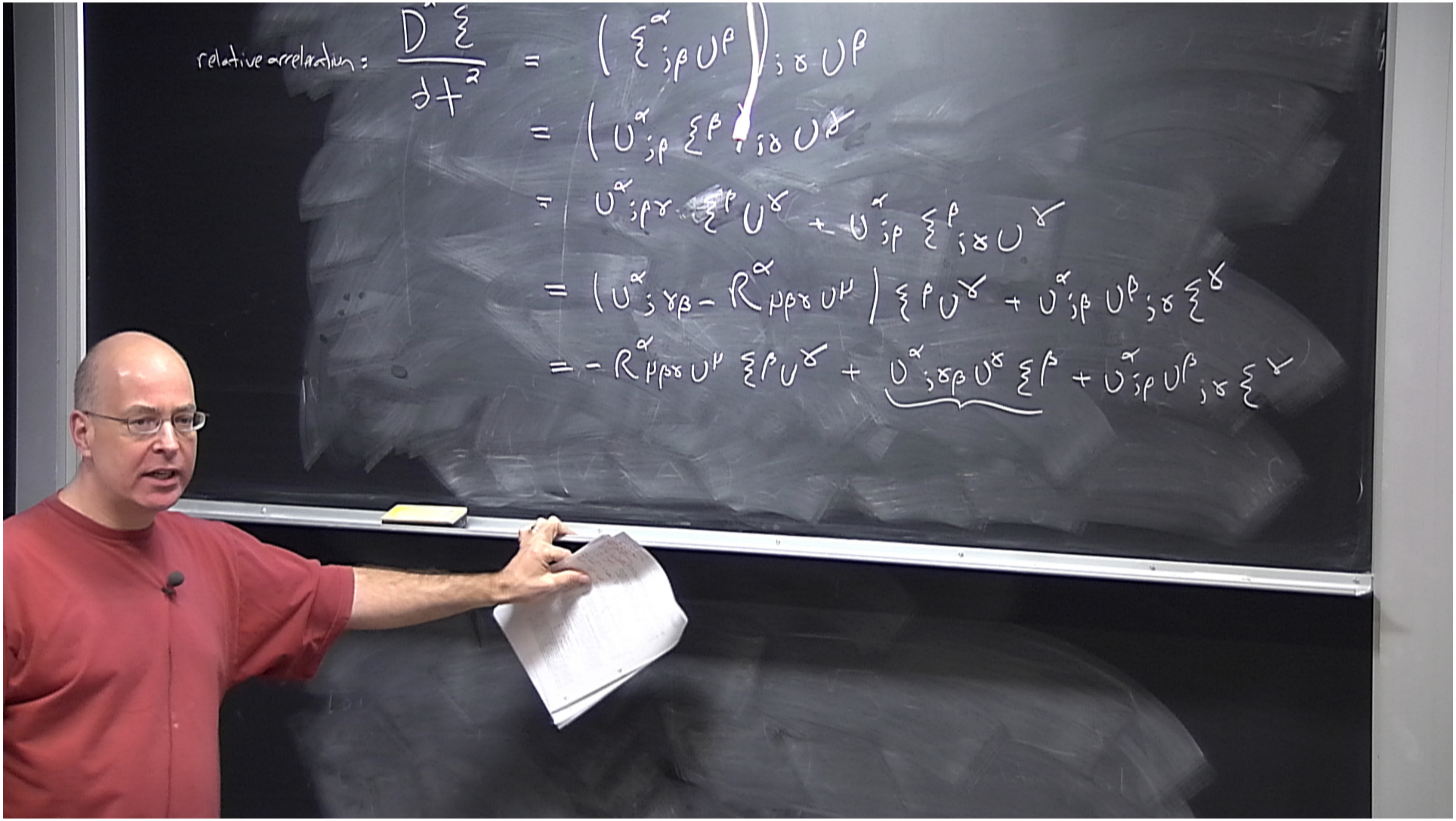

relative acceleration: $\frac{D^2 \xi^\alpha}{dt^2} = \left(\xi^\alpha{}_{;\beta} U^\beta \right)_{;\gamma} U^\gamma$
$= \left(U^\alpha{}_{;\beta} \xi^\beta \right)_{;\gamma} U^\gamma$


relative acceleration: $\frac{D^2 \xi^\alpha}{d\tau^2} = (\xi^\alpha{}_{;\beta} U^\beta)_{;\gamma} U^\beta$
$= (U^\alpha{}_{;\beta} \xi^\beta)_{;\gamma} U^\gamma$
$= U^\alpha{}_{;\beta\gamma} \xi^\beta U^\gamma + U^\alpha{}_{;\beta} \xi^\beta{}_{;\gamma} U^\gamma$

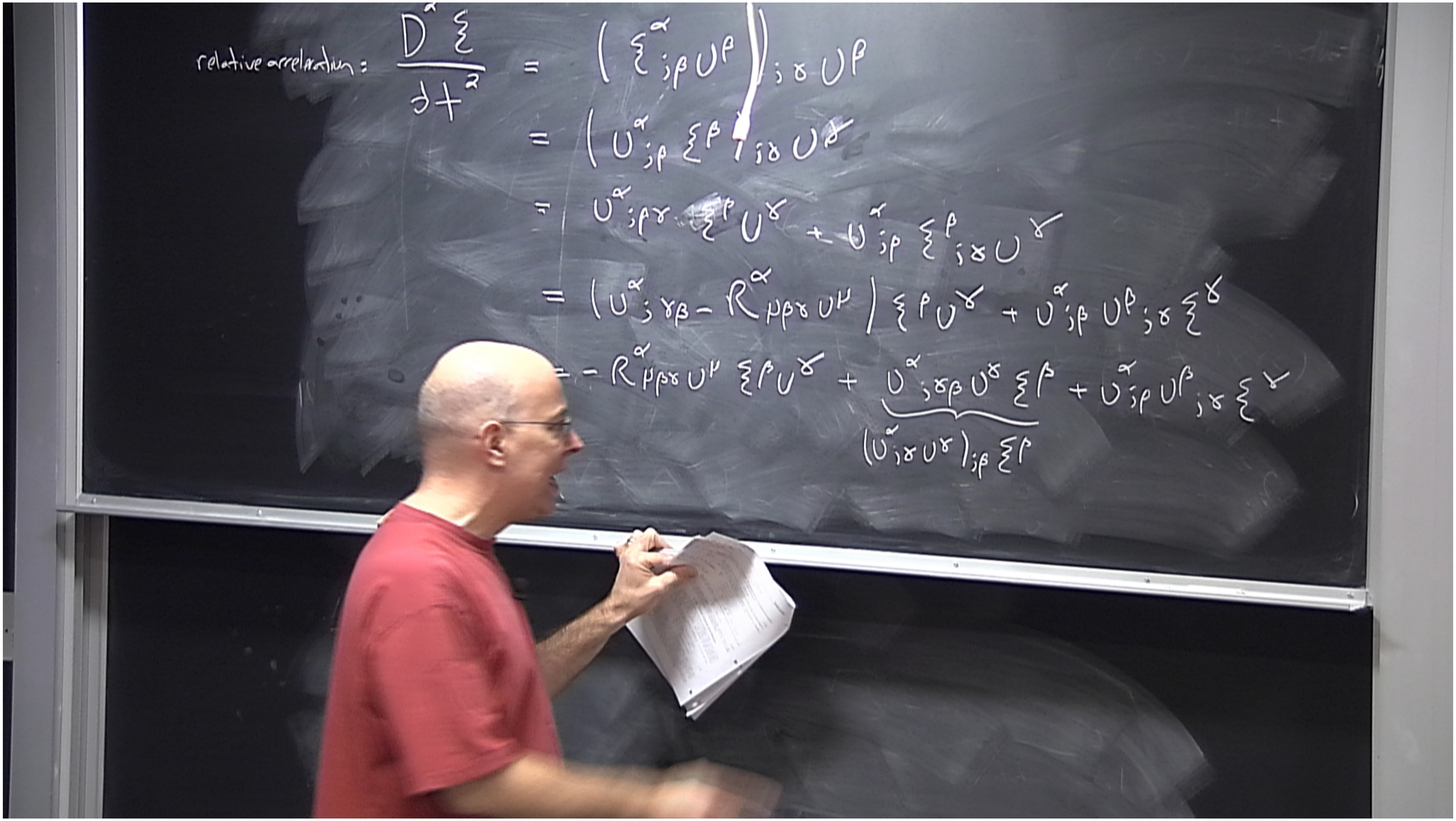

relative acceleration: $\frac{D^2\xi^\alpha}{d\tau^2} = (\xi^\alpha{}_{;\beta}U^\beta)_{;\gamma}U^\beta$
$= (U^\alpha{}_{;\beta}\xi^\beta)_{;\gamma}U^\gamma$
$= U^\alpha{}_{;\beta\gamma}\xi^\beta U^\gamma + U^\alpha{}_{;\beta}\xi^\beta{}_{;\gamma}U^\gamma$
$=$

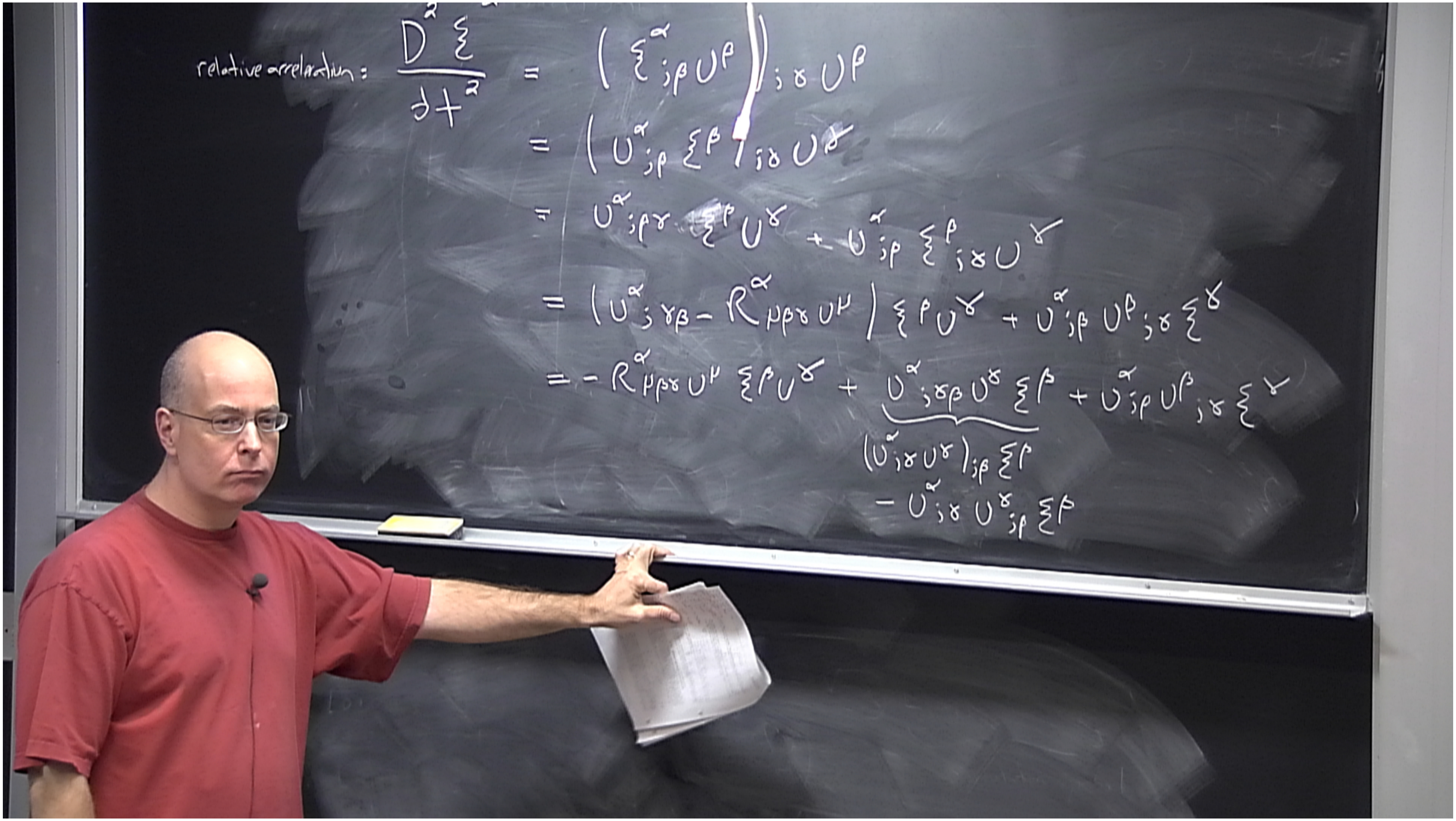


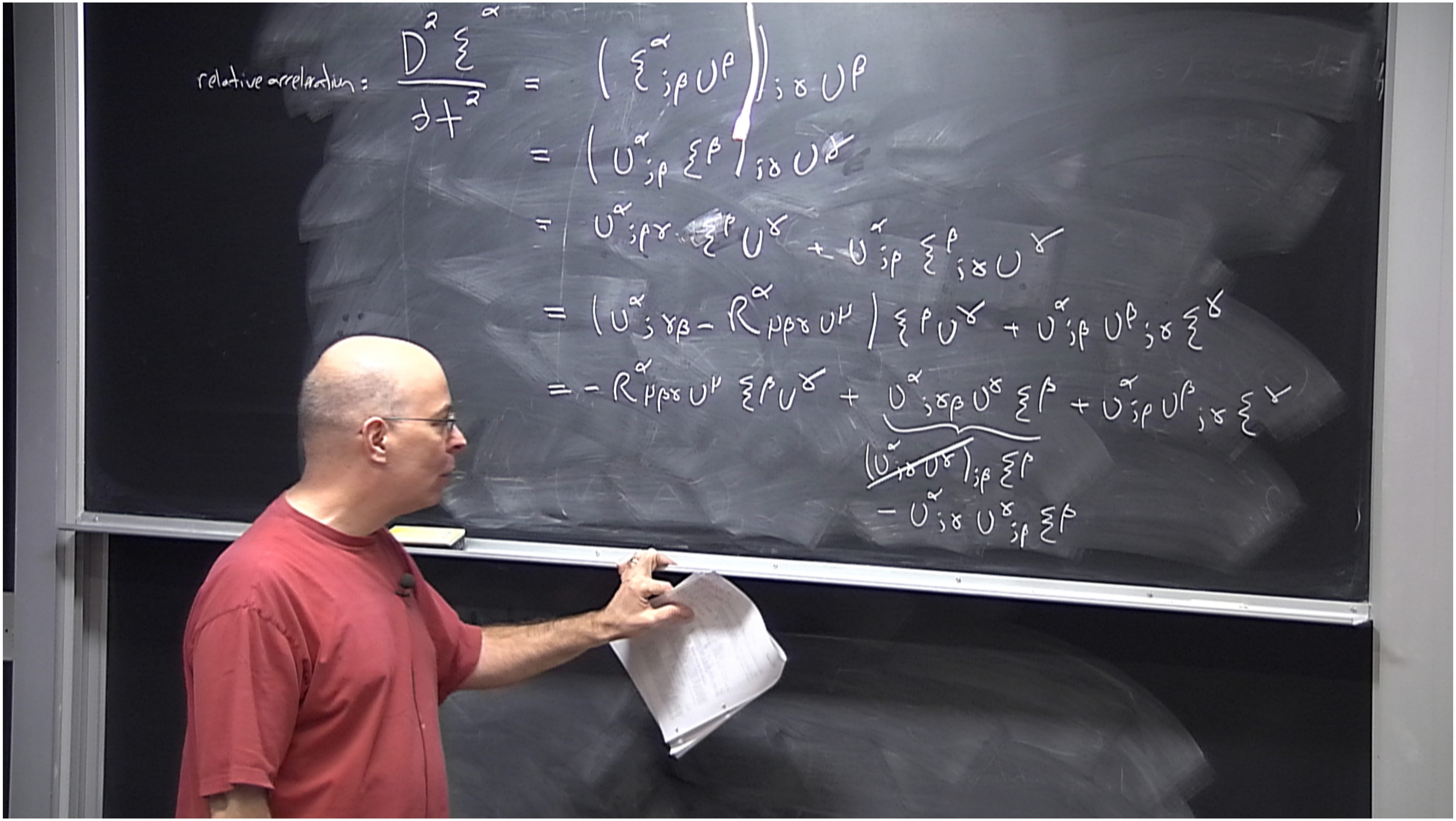
relative acceleration: $\frac{D^2\xi^\alpha}{d\tau^2} = (\xi^\alpha{}_{;\beta}U^\beta)_{;\gamma}U^\gamma$
$= (U^\alpha{}_{;\beta}\xi^\beta)_{;\gamma}U^\gamma$
$= U^\alpha{}_{;\beta\gamma}\xi^\beta U^\gamma + U^\alpha{}_{;\beta}\xi^\beta{}_{;\gamma}U^\gamma$
$= (U^\alpha{}_{;\gamma\beta} - R^\alpha{}_{\mu\beta\gamma}U^\mu)\xi^\beta U^\gamma + U^\alpha{}_{;\beta}U^\beta{}_{;\gamma}\xi^\gamma$
$=$

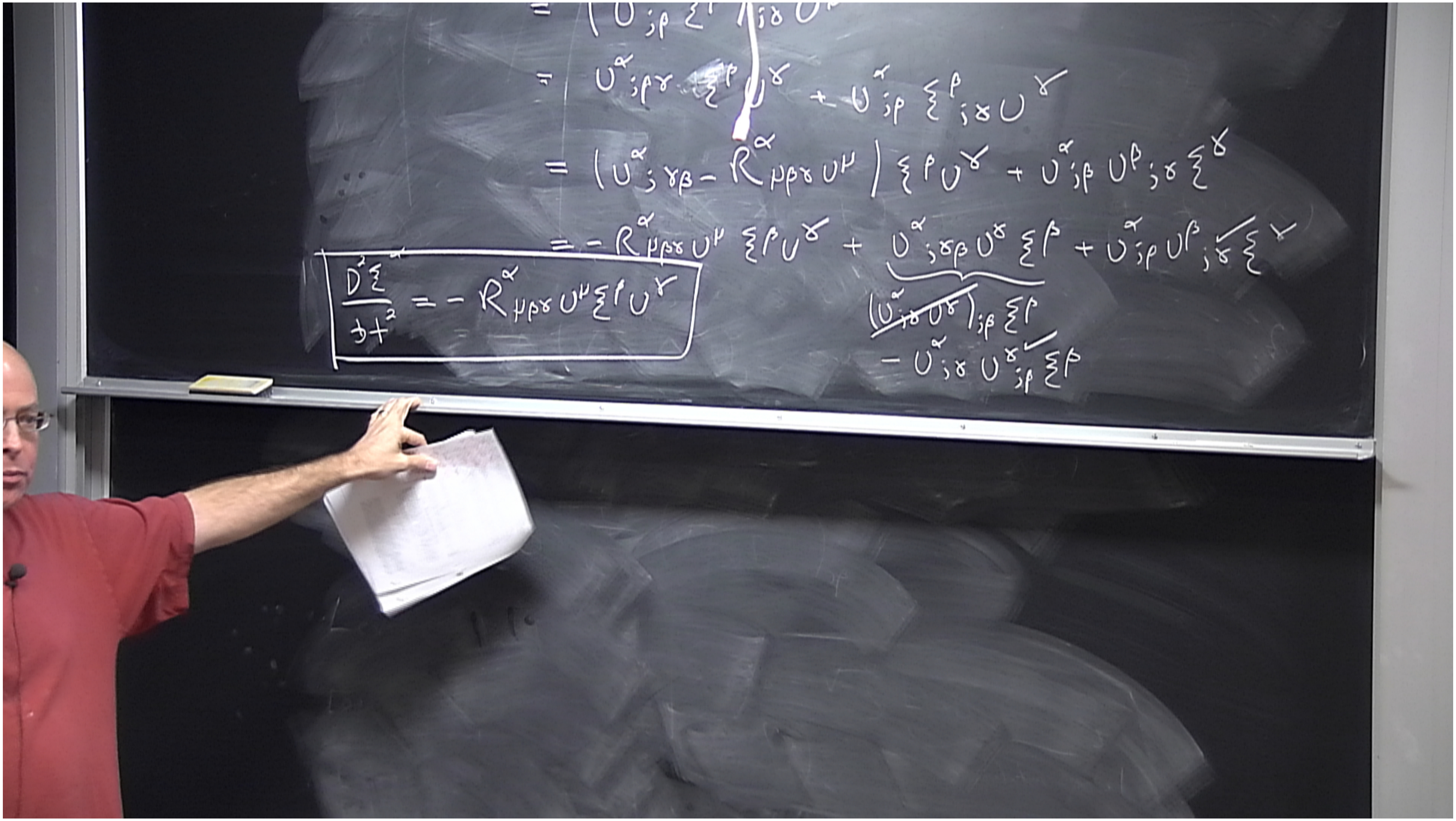

relative acceleration: D²ξ/dτ² = (ξ^α_;β U^β)_;γ U^γ
= (U^α_;β ξ^β)_;γ U^γ
= U^α_;βγ ξ^β U^γ + U^α_;β ξ^β_;γ U^γ
= (U^α_;γβ − R^α_μβγ U^μ) ξ^β U^γ + U^α_;β U^β_;γ ξ^γ
= − R^α_μβγ U^μ ξ^β U^γ + U^α_;γβ U^γ ξ^β + U^α_;β U^β_;γ ξ^γ

relative acceleration: $\frac{D \xi}{d t^2}$

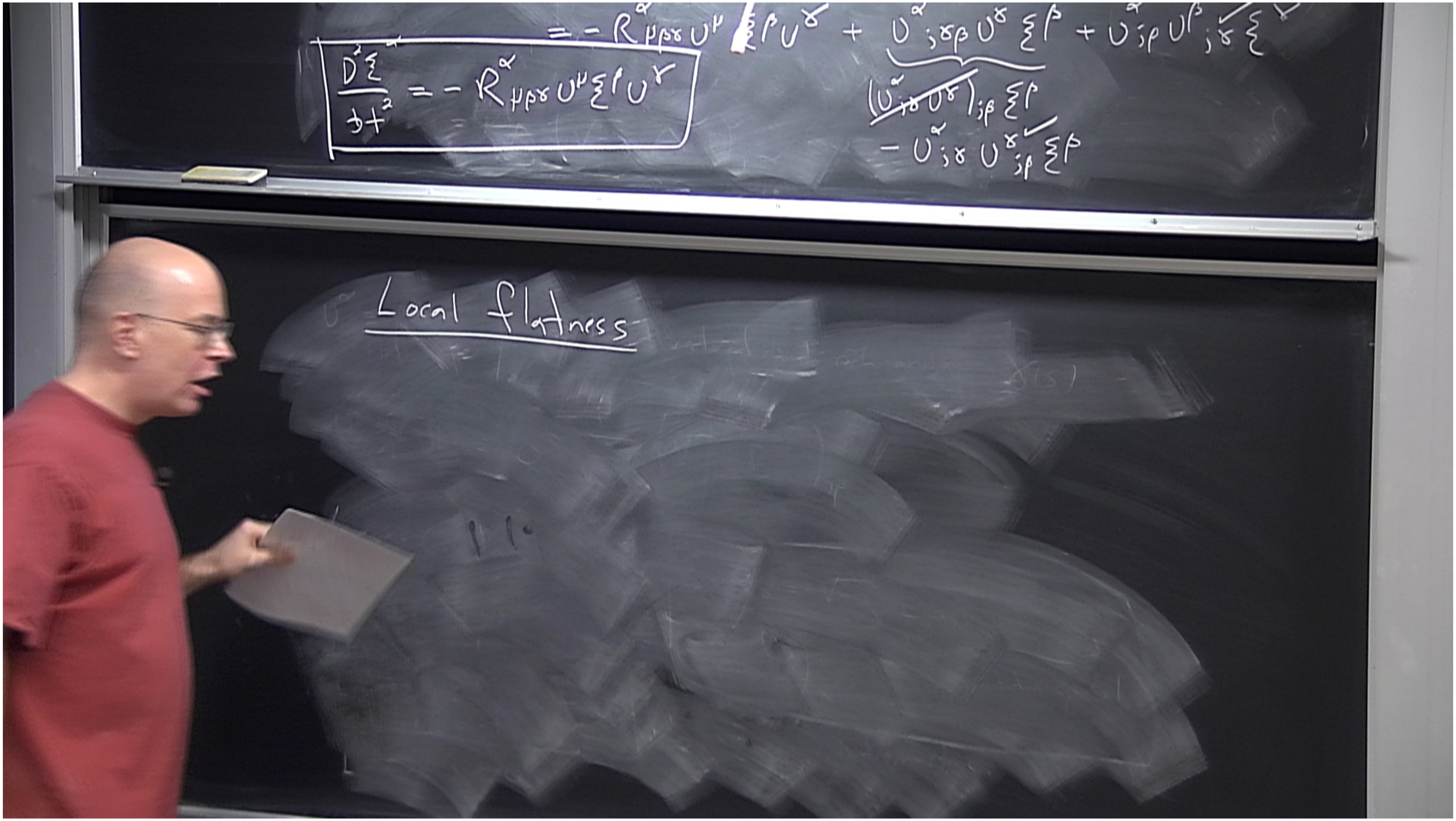

relative acceleration: $\frac{D^2\xi^\alpha}{d\tau^2} = (\xi^\alpha{}_{;\beta}U^\beta)_{;\gamma}U^\gamma$
$= (U^\alpha{}_{;\beta}\xi^\beta)_{;\gamma}U^\gamma$
$= U^\alpha{}_{;\beta\gamma}\xi^\beta U^\gamma + U^\alpha{}_{;\beta}\xi^\beta{}_{;\gamma}U^\gamma$
$= (U^\alpha{}_{;\gamma\beta} - R^\alpha{}_{\mu\beta\gamma}U^\mu)\xi^\beta U^\gamma + U^\alpha{}_{;\beta}U^\beta{}_{;\gamma}\xi^\gamma$
$= -R^\alpha{}_{\mu\beta\gamma}U^\mu\xi^\beta U^\gamma + U^\alpha{}_{;\gamma\beta}U^\gamma\xi^\beta + U^\alpha{}_{;\beta}U^\beta{}_{;\gamma}\xi^\gamma$
$(U^\alpha{}_{;\gamma}U^\gamma)_{;\beta}\xi^\beta$

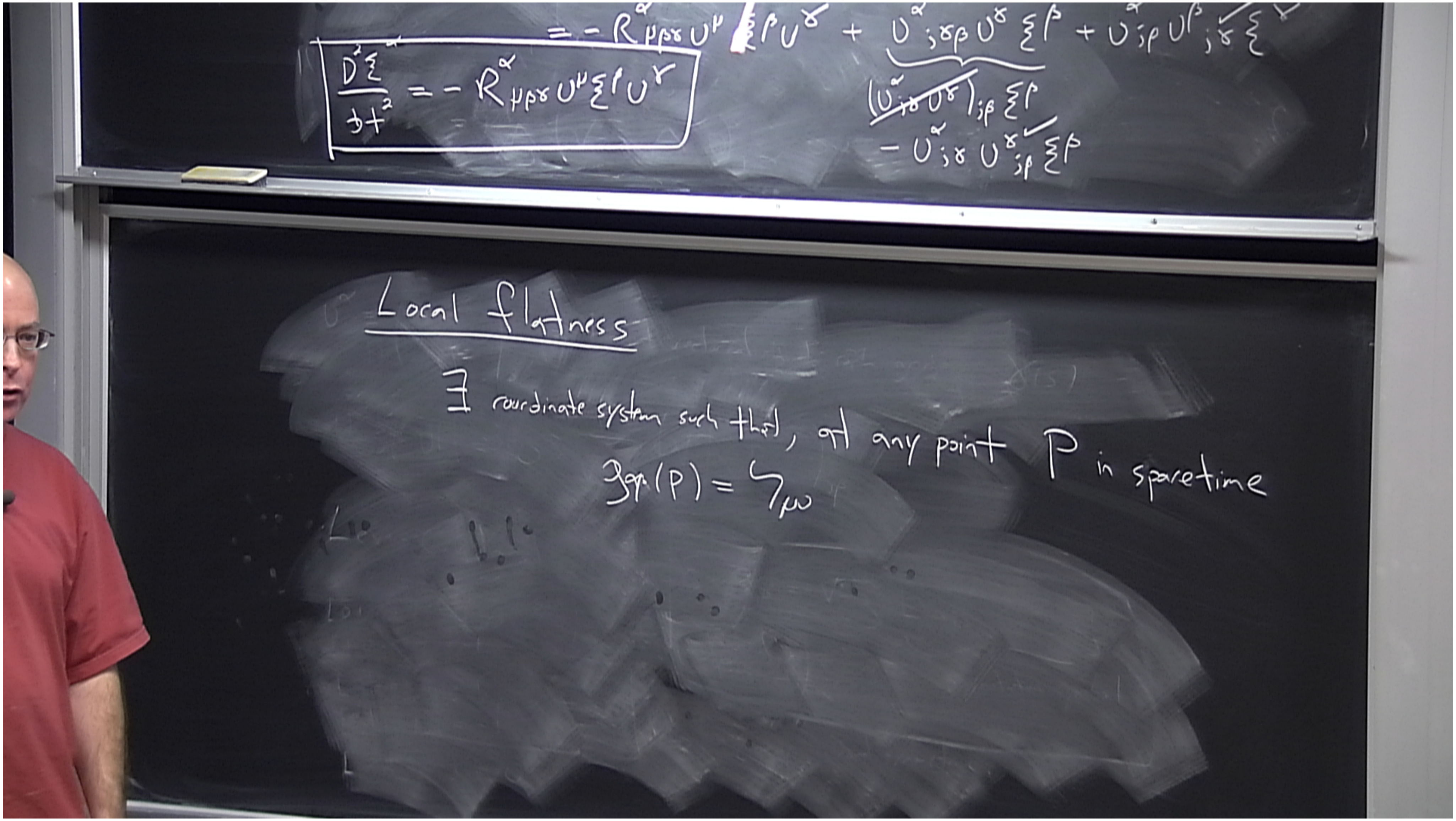

relative acceleration: $\frac{D^2\xi^\alpha}{dt^2} = (\xi^\alpha{}_{;\beta}U^\beta)_{;\gamma}U^\gamma$
$= (U^\alpha{}_{;\beta}\xi^\beta)_{;\gamma}U^\gamma$
$= U^\alpha{}_{;\beta\gamma}\xi^\beta U^\gamma + U^\alpha{}_{;\beta}\xi^\beta{}_{;\gamma}U^\gamma$
$= (U^\alpha{}_{;\gamma\beta} - R^\alpha{}_{\mu\beta\gamma}U^\mu)\xi^\beta U^\gamma + U^\alpha{}_{;\beta}U^\beta{}_{;\gamma}\xi^\gamma$
$= -R^\alpha{}_{\mu\beta\gamma}U^\mu\xi^\beta U^\gamma + U^\alpha{}_{;\gamma\beta}U^\gamma\xi^\beta + U^\alpha{}_{;\beta}U^\beta{}_{;\gamma}\xi^\gamma$
$(U^\alpha{}_{;\gamma}U^\gamma)_{;\beta}\xi^\beta$
$- U^\alpha{}_{;\gamma}U^\gamma{}_{;\beta}\xi^\beta$


relative acceleration: $\frac{D^2 \xi^\alpha}{d\tau^2} = (\xi^\alpha{}_{;\beta} U^\beta)_{;\gamma} U^\gamma$
$= (U^\alpha{}_{;\beta} \xi^\beta)_{;\gamma} U^\gamma$
$= U^\alpha{}_{;\beta\gamma} \xi^\beta U^\gamma + U^\alpha{}_{;\beta} \xi^\beta{}_{;\gamma} U^\gamma$
$= (U^\alpha{}_{;\gamma\beta} - R^\alpha{}_{\mu\beta\gamma} U^\mu) \xi^\beta U^\gamma + U^\alpha{}_{;\beta} U^\beta{}_{;\gamma} \xi^\gamma$
$= - R^\alpha{}_{\mu\beta\gamma} U^\mu \xi^\beta U^\gamma + U^\alpha{}_{;\gamma\beta} U^\gamma \xi^\beta + U^\alpha{}_{;\beta} U^\beta{}_{;\gamma} \xi^\gamma$
$(U^\alpha{}_{;\gamma} U^\gamma)_{;\beta} \xi^\beta$
$- U^\alpha{}_{;\gamma} U^\gamma{}_{;\beta} \xi^\beta$

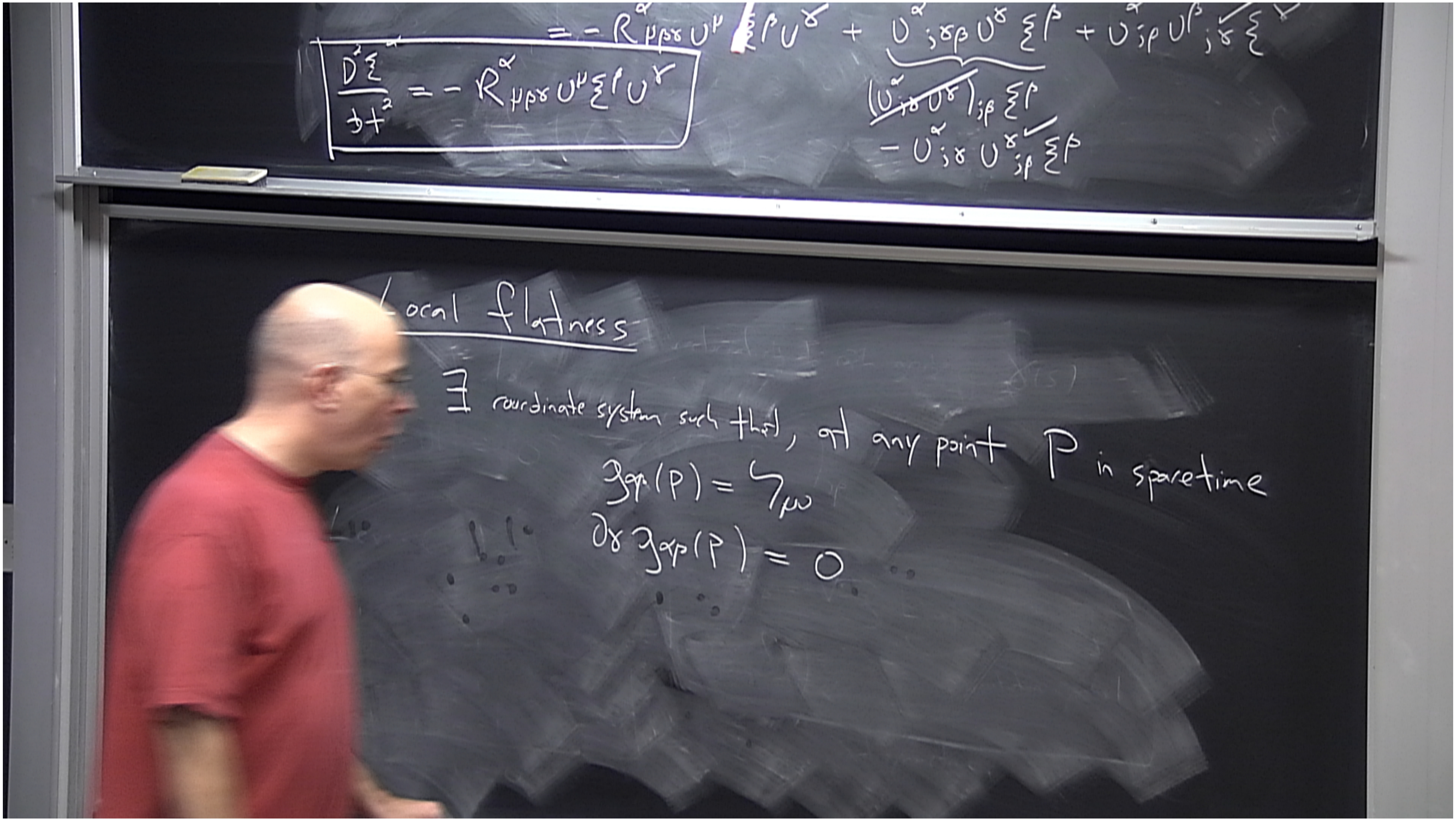


$= U^\alpha{}_{;\beta\gamma}\, \xi^\beta U^\gamma + U^\alpha{}_{;\beta}\, \xi^\beta{}_{;\gamma} U^\gamma$
$= (U^\alpha{}_{;\gamma\beta} - R^\alpha{}_{\mu\beta\gamma} U^\mu)\, \xi^\beta U^\gamma + U^\alpha{}_{;\beta}\, U^\beta{}_{;\gamma}\, \xi^\gamma$
$= - R^\alpha{}_{\mu\beta\gamma} U^\mu \xi^\beta U^\gamma + U^\alpha{}_{;\gamma\beta} U^\gamma \xi^\beta + U^\alpha{}_{;\beta} U^\beta{}_{;\gamma} \xi^\gamma$
$(U^\alpha{}_{;\gamma} U^\gamma)_{;\beta}\, \xi^\beta$
$- U^\alpha{}_{;\gamma}\, U^\gamma{}_{;\beta}\, \xi^\beta$
$\frac{D^2\xi^\alpha}{d\tau^2} = - R^\alpha{}_{\mu\beta\gamma} U^\mu \xi^\beta U^\gamma$


Local flatness



$= - R^\alpha{}_{\mu\rho\gamma} U^\mu \xi^\rho U^\gamma + U^\alpha{}_{;\gamma\beta} U^\gamma \xi^\beta + U^\alpha{}_{;\beta} U^\beta{}_{;\gamma} \xi^\gamma$
$(U^\alpha{}_{;\gamma} U^\gamma)_{;\beta} \xi^\beta$
$- U^\alpha{}_{;\gamma} U^\gamma{}_{;\beta} \xi^\beta$
$\frac{D^2 \xi^\alpha}{d\tau^2} = - R^\alpha{}_{\mu\rho\gamma} U^\mu \xi^\rho U^\gamma$
Local flatness
∃ coordinate system such that, at any point P in spacetime
$g_{\mu\nu}(P) = \eta_{\mu\nu}$


$= -R^{\alpha}{}_{\mu\beta\gamma} U^{\mu} \xi^{\beta} U^{\gamma} + U^{\alpha}{}_{;\gamma\beta} U^{\gamma} \xi^{\beta} + U^{\alpha}{}_{;\beta} U^{\beta}{}_{;\gamma} \xi^{\gamma}$
$\frac{D^2 \xi^{\alpha}}{d\tau^2} = -R^{\alpha}{}_{\mu\beta\gamma} U^{\mu} \xi^{\beta} U^{\gamma}$
$(U^{\alpha}{}_{;\gamma} U^{\gamma})_{;\beta} \xi^{\beta}$
$- U^{\alpha}{}_{;\gamma} U^{\gamma}{}_{;\beta} \xi^{\beta}$
Local flatness
∃ coordinate system such that, at any point P in spacetime
$g_{\alpha\beta}(P) = \eta_{\mu\nu}$
$\partial_{\gamma} g_{\alpha\beta}(P) = 0$

$$= -R^\alpha{}_{\mu\beta\gamma} U^\mu \xi^\beta U^\gamma + U^\alpha{}_{;\gamma\beta} U^\gamma \xi^\beta + U^\alpha{}_{;\beta} U^\beta{}_{;\gamma} \xi^\gamma$$

$$\boxed{\frac{D^2 \xi^\alpha}{d\tau^2} = -R^\alpha{}_{\mu\beta\gamma} U^\mu \xi^\beta U^\gamma}$$

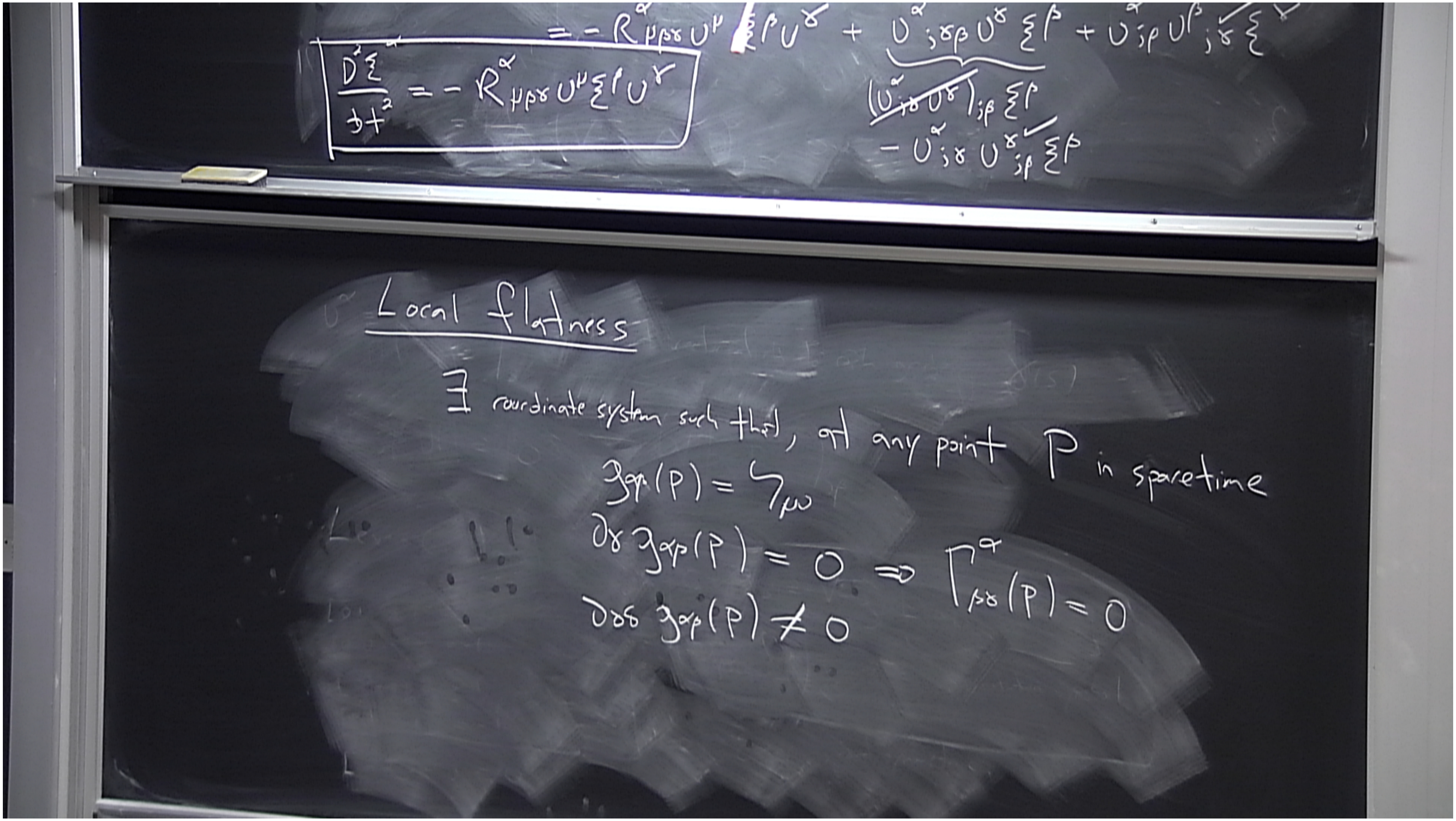
$$U^\alpha{}_{;\gamma\beta} U^\gamma \xi^\beta = (U^\alpha{}_{;\gamma} U^\gamma)_{;\beta} \xi^\beta - U^\alpha{}_{;\gamma} U^\gamma{}_{;\beta} \xi^\beta$$

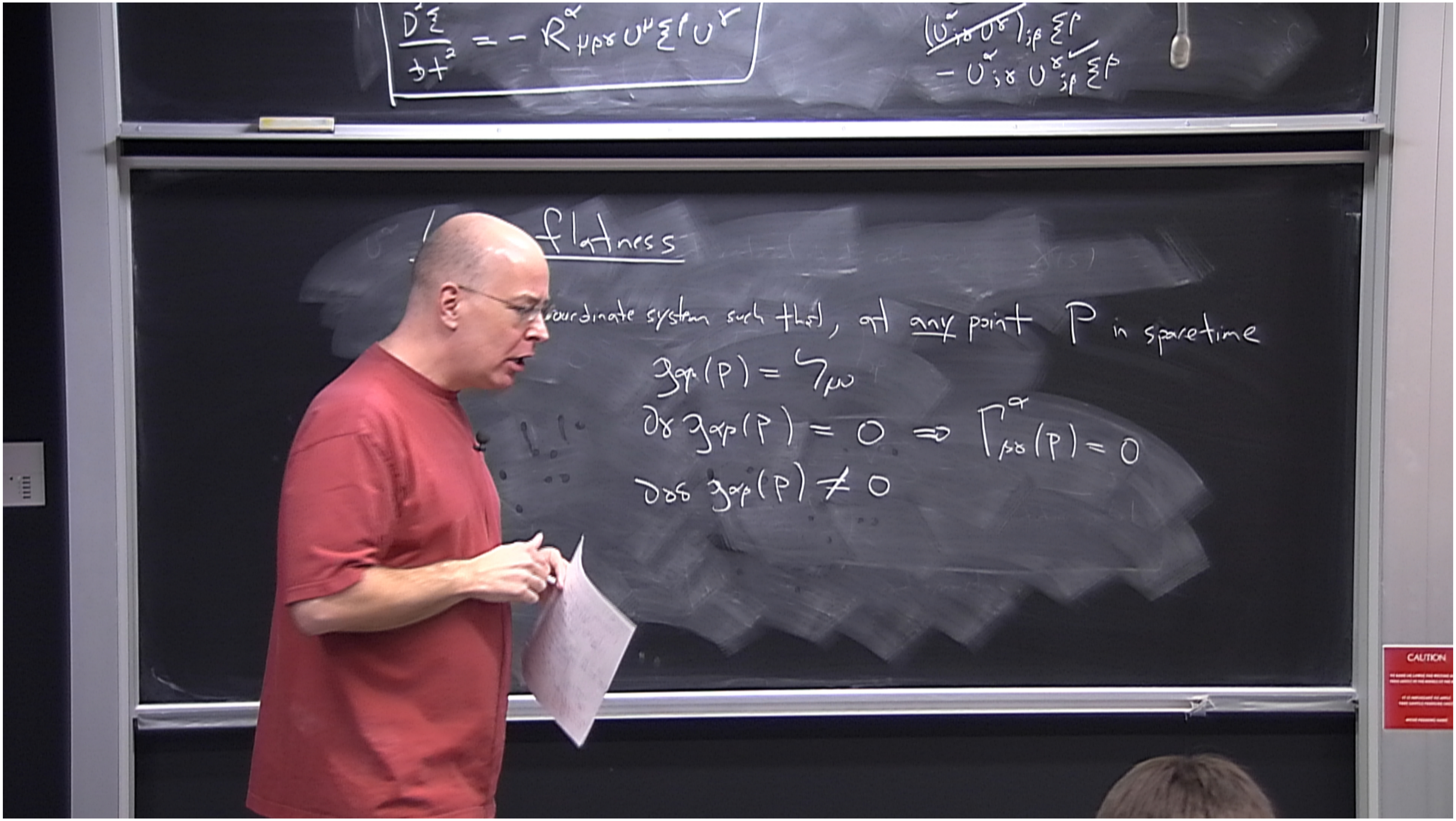
Local flatness

$\exists$ coordinate system such that, at any point P in spacetime

$$g_{\alpha\beta}(P) = \eta_{\mu\nu}$$

$$\partial_\gamma g_{\alpha\beta}(P) = 0 \Rightarrow \Gamma^\alpha_{\beta\gamma}(P) = 0$$


$= -R^{\alpha}{}_{\mu\beta\gamma} U^{\mu} \xi^{\beta} U^{\gamma} + U^{\alpha}{}_{;\gamma\beta} U^{\gamma} \xi^{\beta} + U^{\alpha}{}_{;\beta} U^{\beta}{}_{;\gamma} \xi^{\gamma}$
$(U^{\alpha}{}_{;\gamma} U^{\gamma})_{;\beta} \xi^{\beta}$
$- U^{\alpha}{}_{;\gamma} U^{\gamma}{}_{;\beta} \xi^{\beta}$
$\frac{D^2 \xi^{\alpha}}{d\tau^2} = -R^{\alpha}{}_{\mu\beta\gamma} U^{\mu} \xi^{\beta} U^{\gamma}$
Local flatness
∃ coordinate system such that, at any point P in spacetime
$g_{\alpha\beta}(P) = \eta_{\mu\nu}$
$\partial_\gamma g_{\alpha\beta}(P) = 0 \Rightarrow \Gamma^{\alpha}_{\beta\gamma}(P) = 0$
$\partial_{\gamma\delta} g_{\alpha\beta}(P) \neq 0$


$\frac{D^2 \xi^\alpha}{d\tau^2} = - R^\alpha{}_{\mu\rho\gamma} U^\mu \xi^\rho U^\gamma$
$- U^\alpha{}_{;\gamma} U^\gamma{}_{;\rho} \xi^\rho$
flatness
oordinate system such that, at any point P in spacetime
$g_{\mu\nu}(P) = \eta_{\mu\nu}$
$\partial_\gamma g_{\alpha\beta}(P) = 0 \Rightarrow \Gamma^\alpha_{\beta\gamma}(P) = 0$
$\partial_{\gamma\delta} g_{\alpha\beta}(P) \neq 0$
CAUTION

$$= -R^\alpha{}_{\mu\rho\gamma} U^\mu \xi^\rho U^\gamma + U^\alpha{}_{;\gamma\beta} U^\gamma \xi^\beta + U^\alpha{}_{;\beta} U^\beta{}_{;\gamma} \xi^\gamma$$

$$U^\alpha{}_{;\gamma\beta} U^\gamma \xi^\beta = (U^\alpha{}_{;\gamma} U^\gamma)_{;\beta} \xi^\beta - U^\alpha{}_{;\gamma} U^\gamma{}_{;\beta} \xi^\beta$$

$$\boxed{\frac{D^2 \xi^\alpha}{d\tau^2} = -R^\alpha{}_{\mu\rho\gamma} U^\mu \xi^\rho U^\gamma}$$

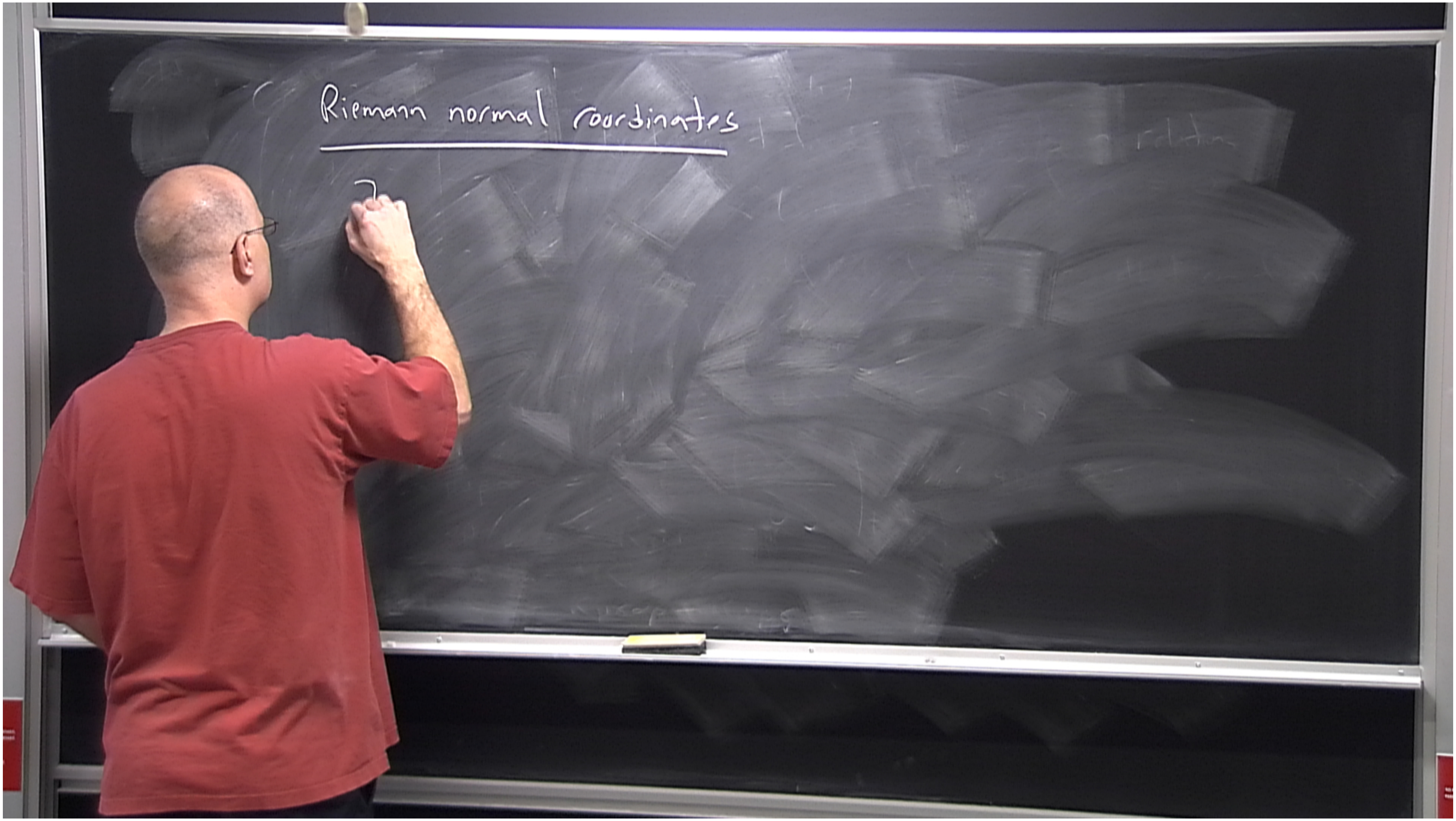
Local flatness

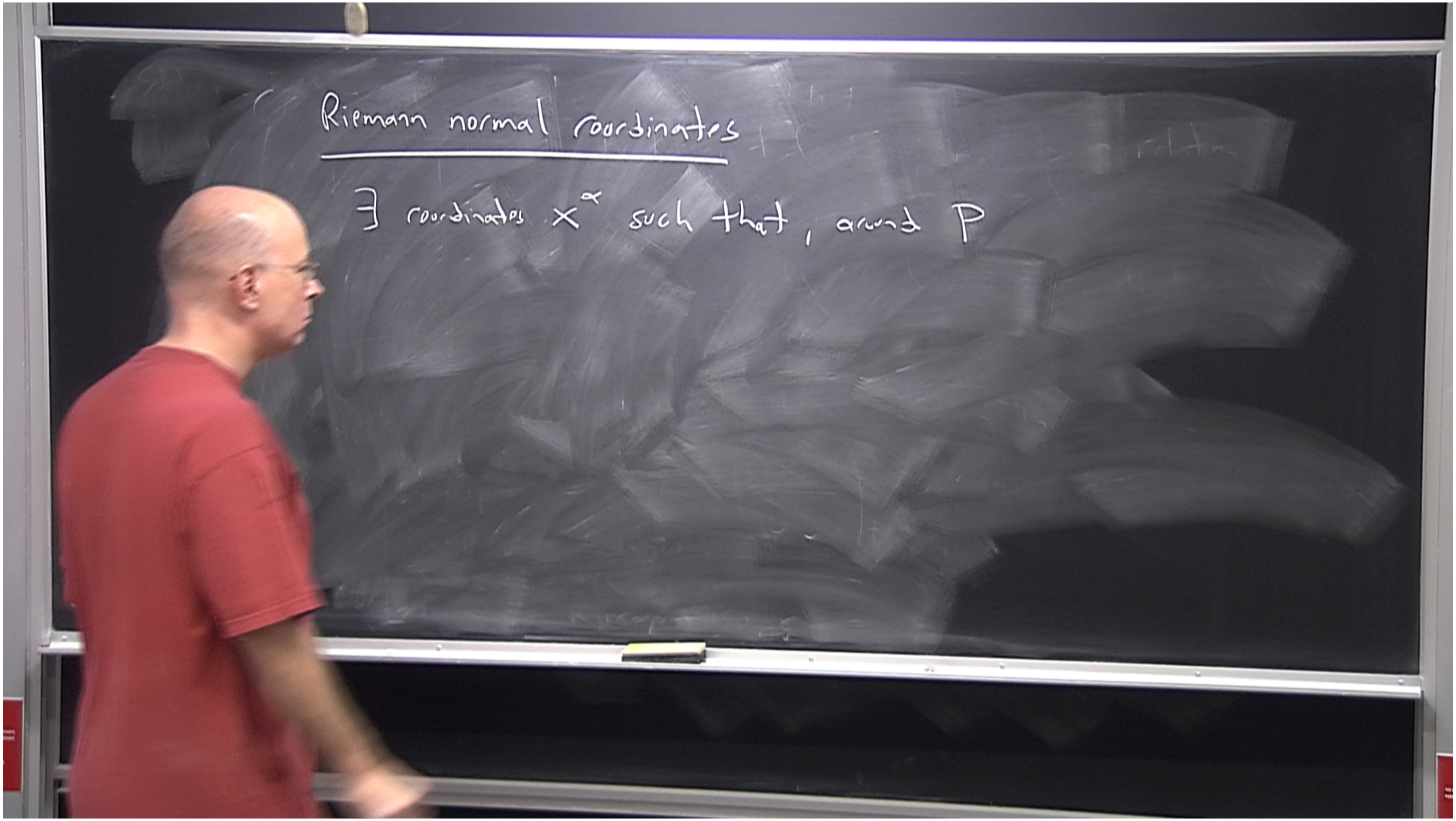
∃ coordinate system such that, at any point P in spacetime

$$g_{\alpha\beta}(P) = \eta_{\mu\nu}$$

$$\partial_\gamma g_{\alpha\beta}(P) = 0 \Rightarrow \Gamma^\alpha_{\beta\gamma}(P) = 0$$

$$\partial_{\delta\gamma} g_{\alpha\beta}(P) \neq 0$$


Riemann normal coordinates


Riemann normal coordinates
∃ coordinates x^α such that, around P

Riemann normal coordinates

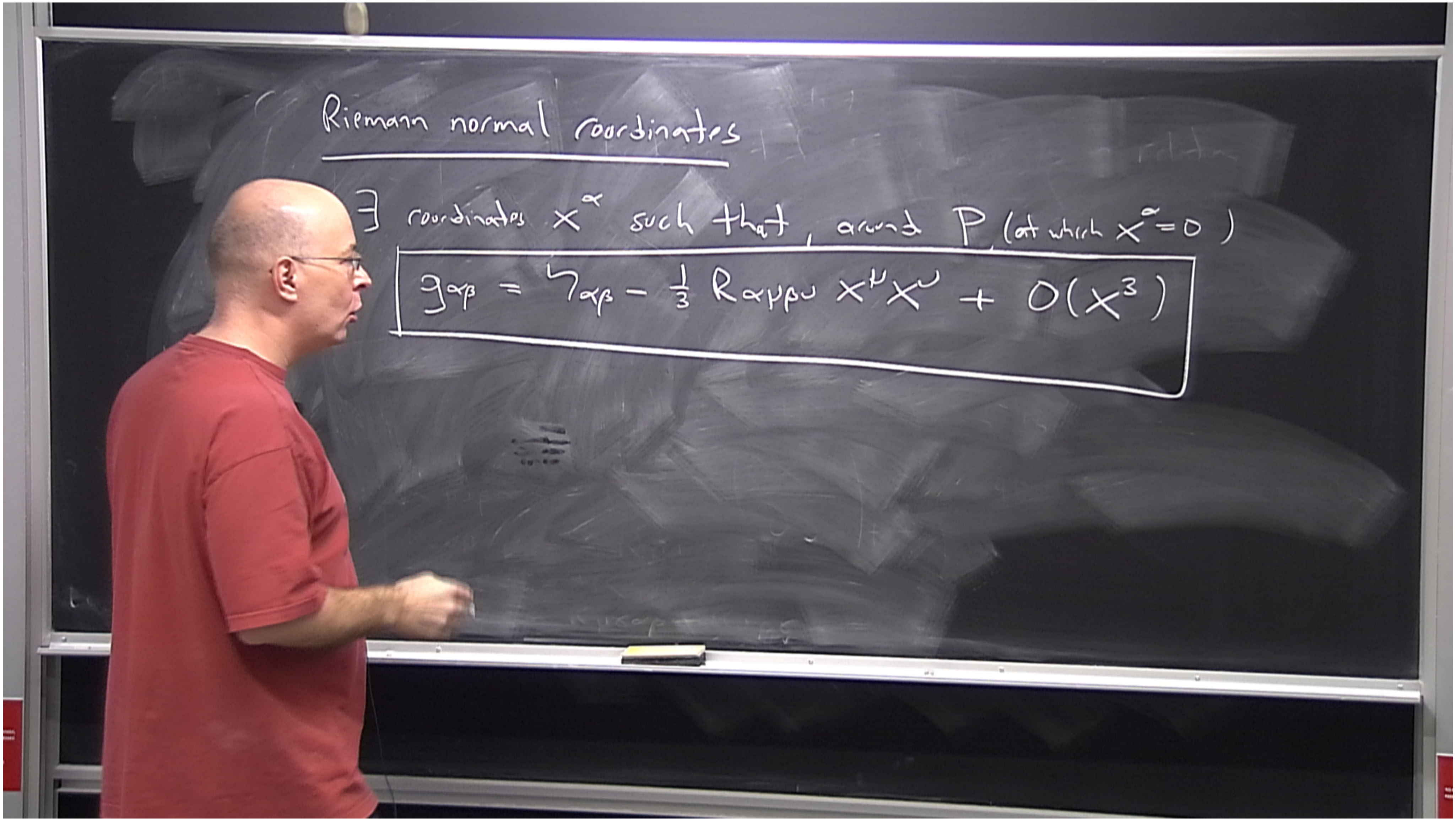
$\exists$ coordinates x^α such that, around P,

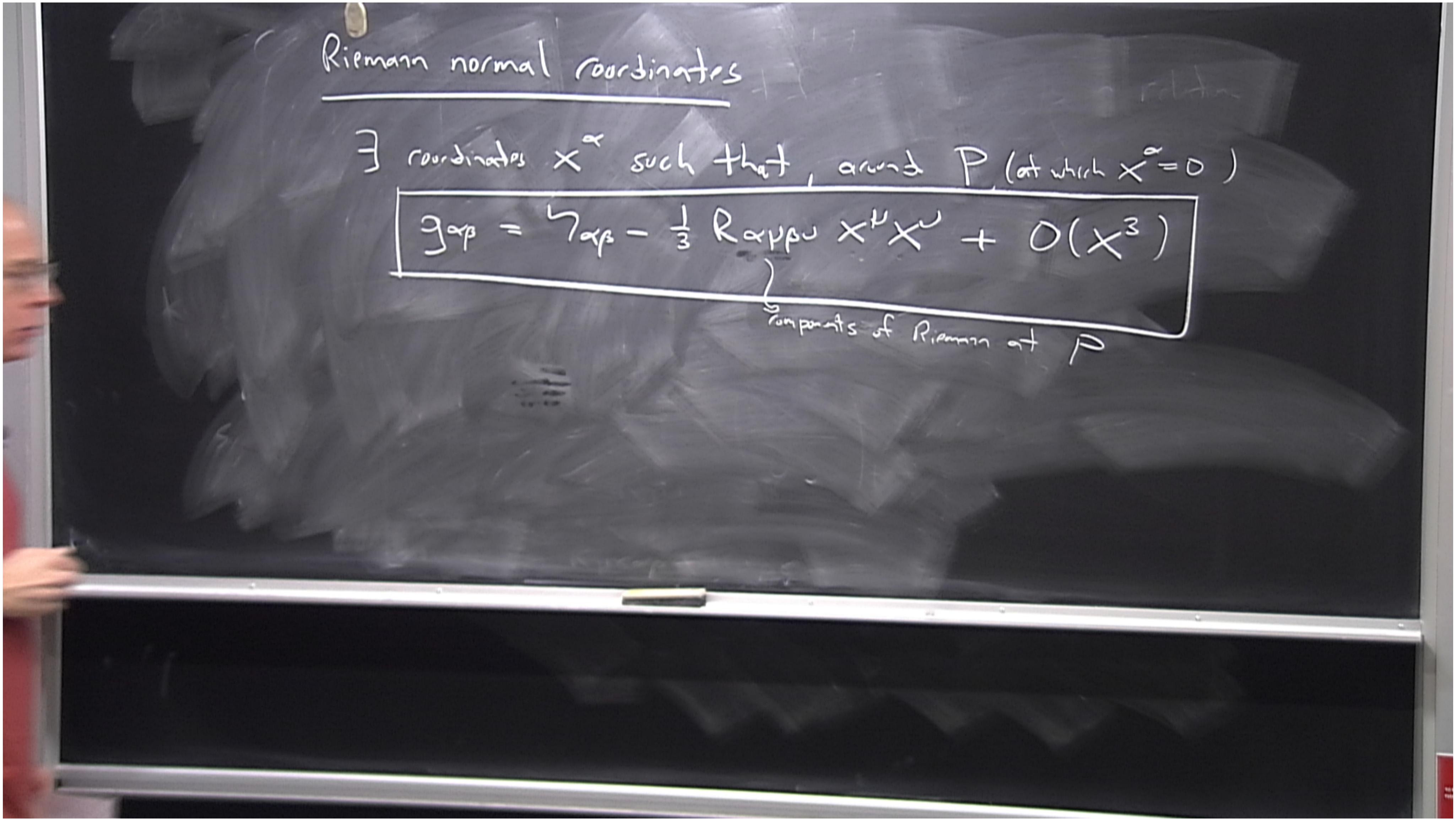
$$g_{\alpha\beta} = \eta_{\alpha\beta} - \frac{1}{3} R_{\alpha\mu\beta\nu}\, x^\mu x^\nu + O(x^3)$$

Riemann normal coordinates

$\exists$ coordinates x^α such that, around P

$$g_{\alpha\beta} = \eta_{\alpha\beta} - \frac{1}{3} R_{\alpha\mu\beta\nu} x^\mu x^\nu + O(x^3)$$


Riemann normal coordinates
∃ coordinates x^α such that, around P (at which $x^\alpha = 0$)
$g_{\alpha\beta} = \eta_{\alpha\beta} - \frac{1}{3} R_{\alpha\mu\beta\nu} x^\mu x^\nu + O(x^3)$

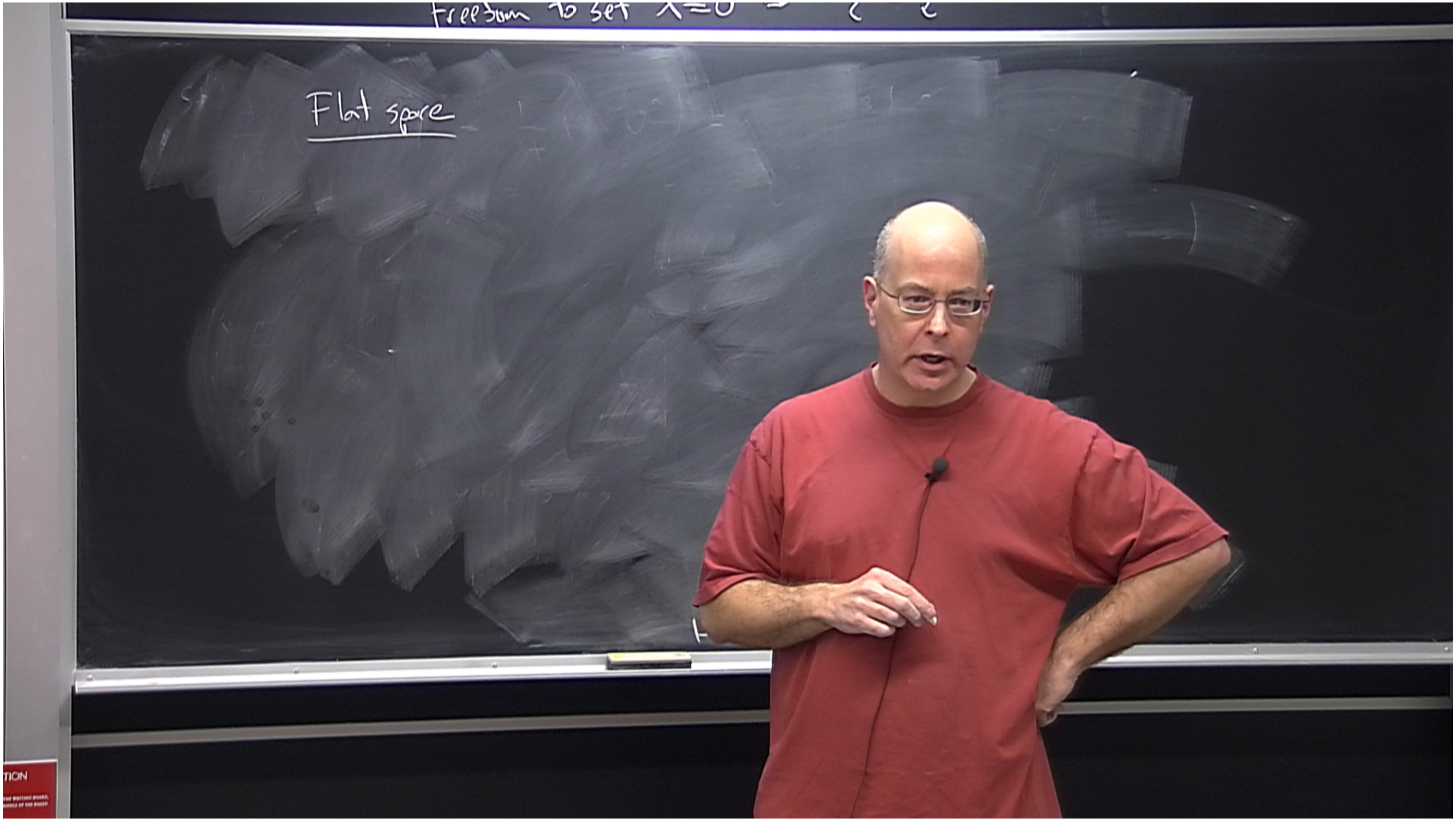

Riemann normal coordinates
∃ coordinates x^α such that, around P (at which $x^\alpha = 0$)
$g_{\alpha\beta} = \eta_{\alpha\beta} - \frac{1}{3} R_{\alpha\mu\beta\nu} x^\mu x^\nu + O(x^3)$
components of Riemann at P

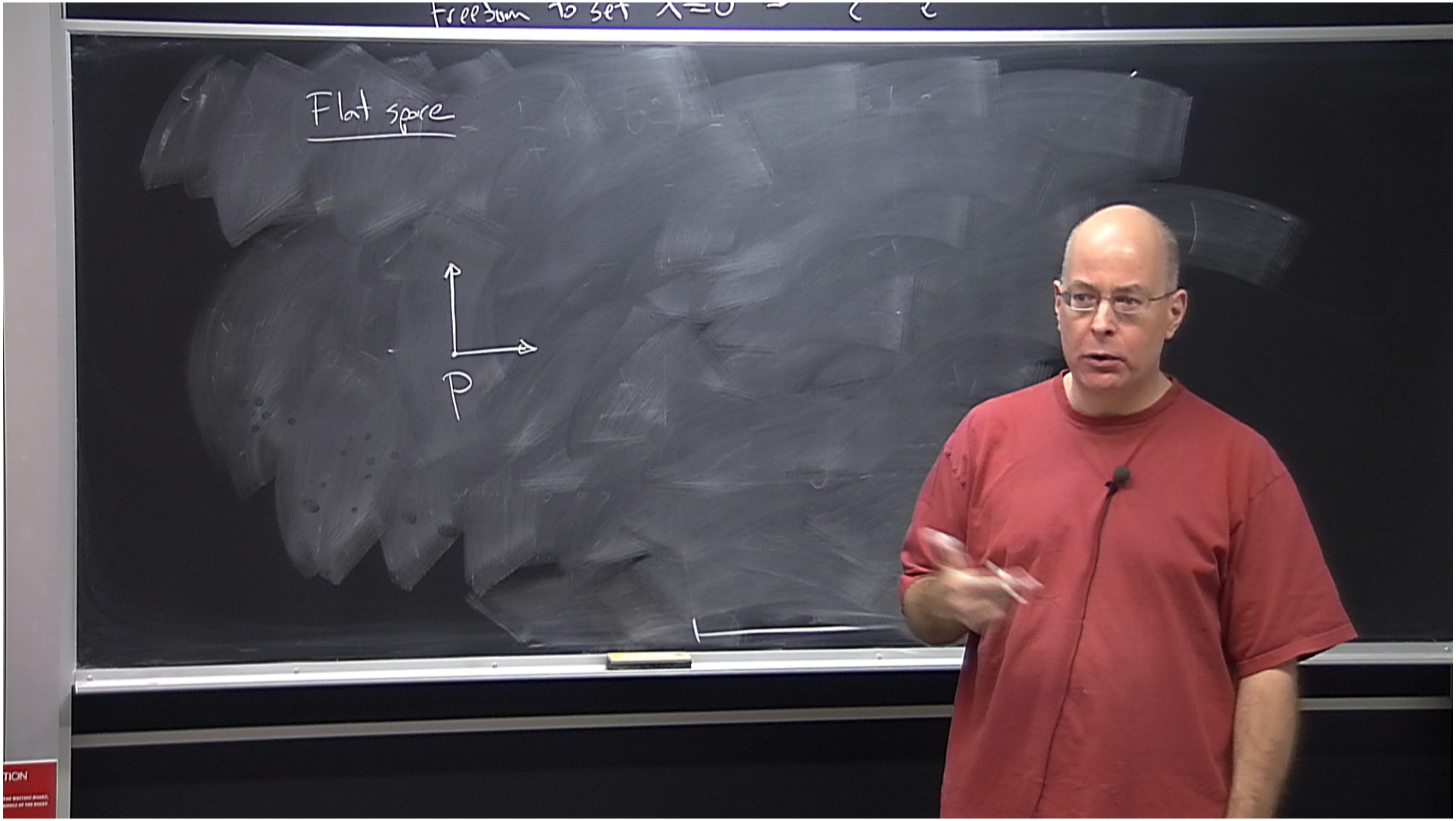
Riemann normal coordinates

$\exists$ coordinates x^α such that, around P (at which $x^\alpha = 0$)

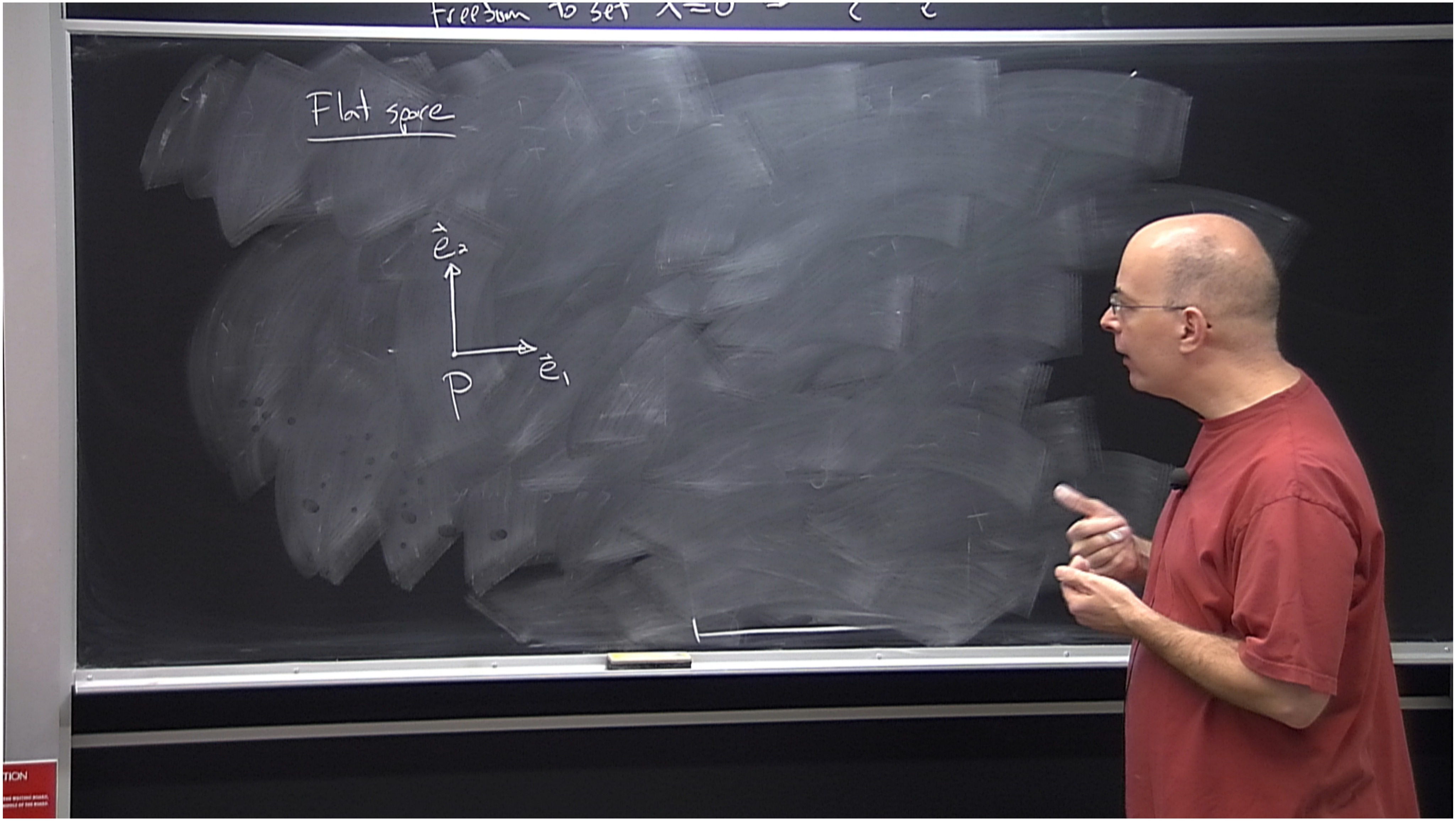
$$\boxed{g_{\alpha\beta} = \eta_{\alpha\beta} - \frac{1}{3} R_{\alpha\mu\beta\nu} x^\mu x^\nu + O(x^3)}$$

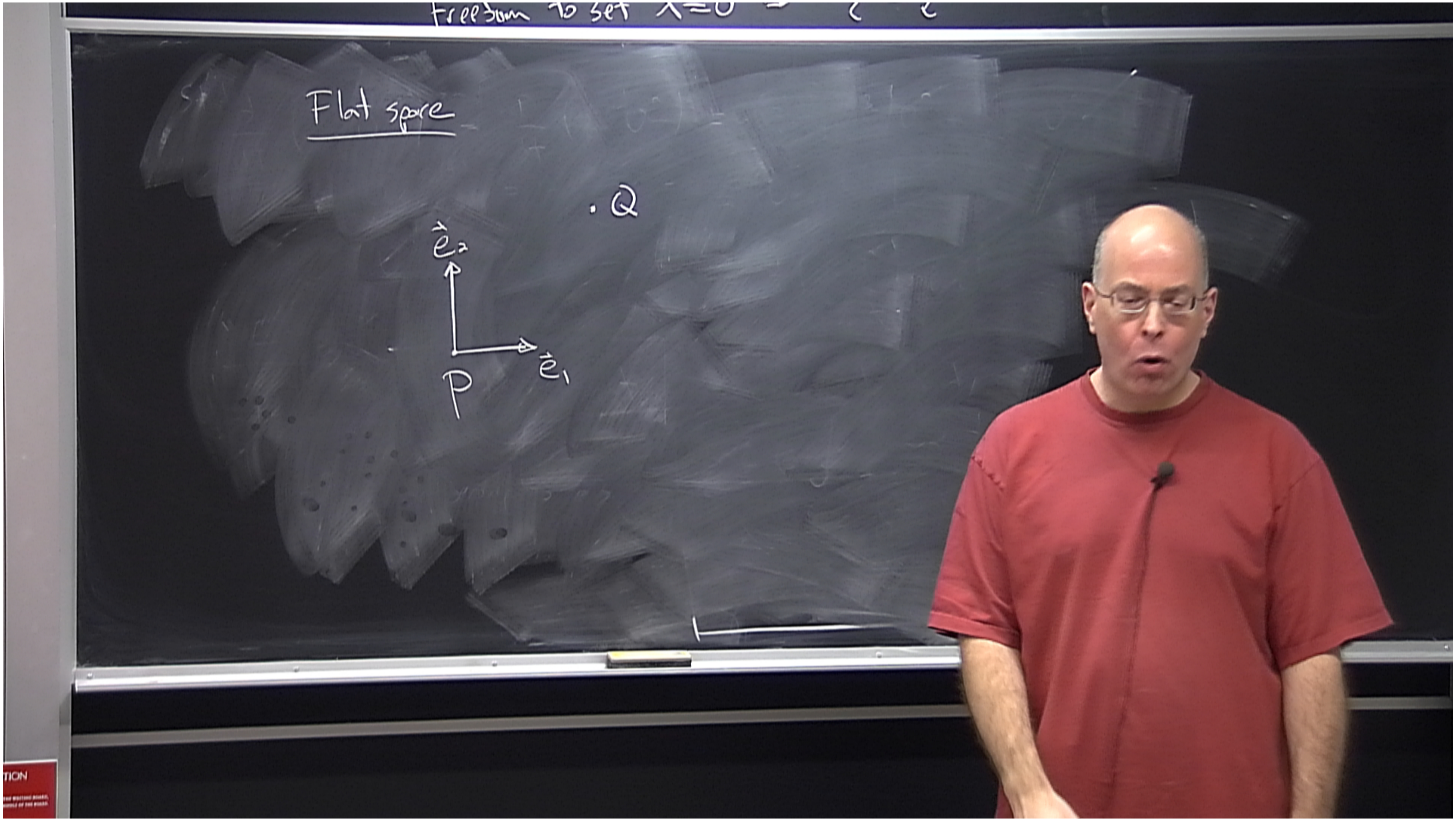
($R_{\alpha\mu\beta\nu}$: components of Riemann at P)


Flat space

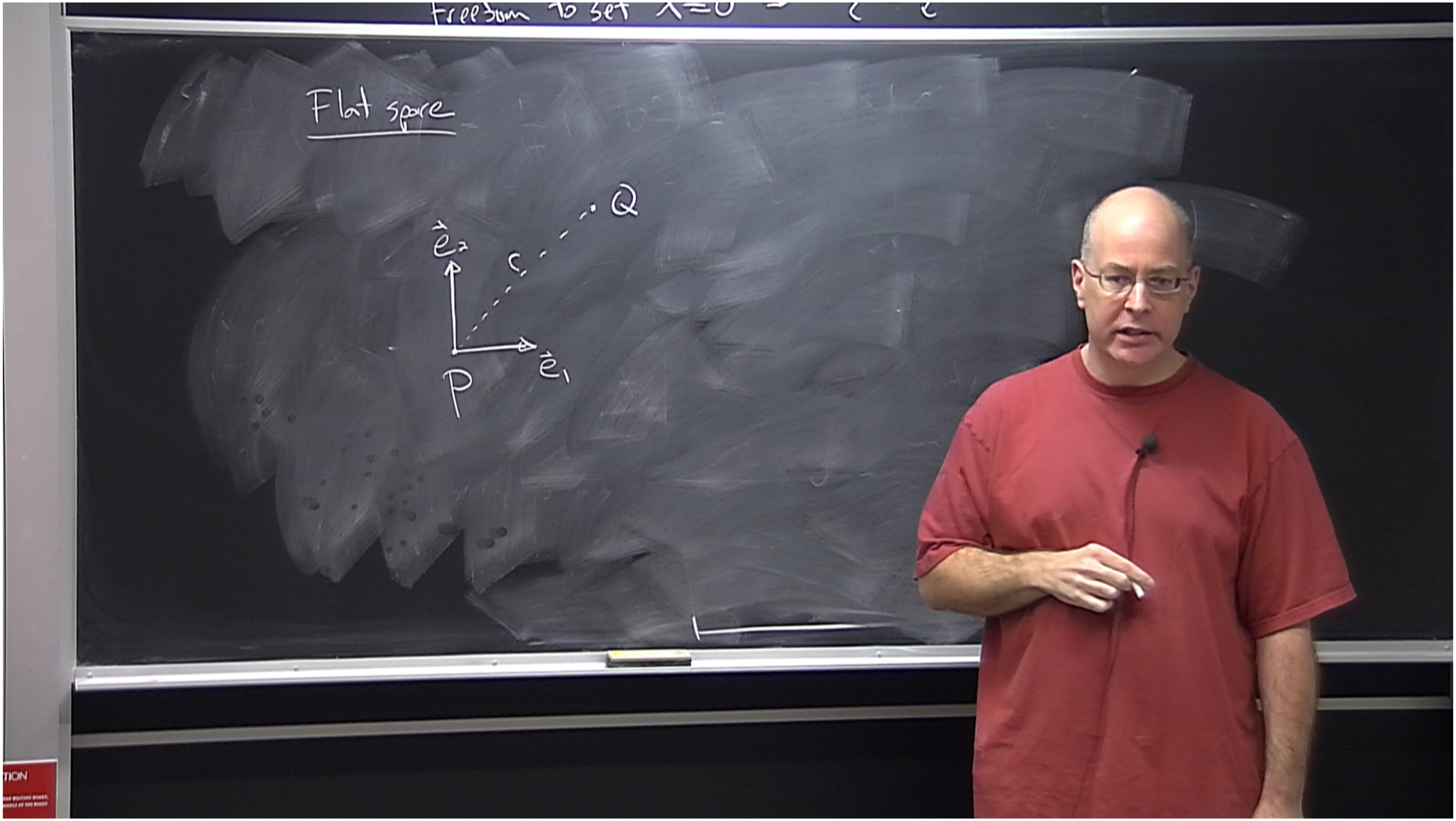


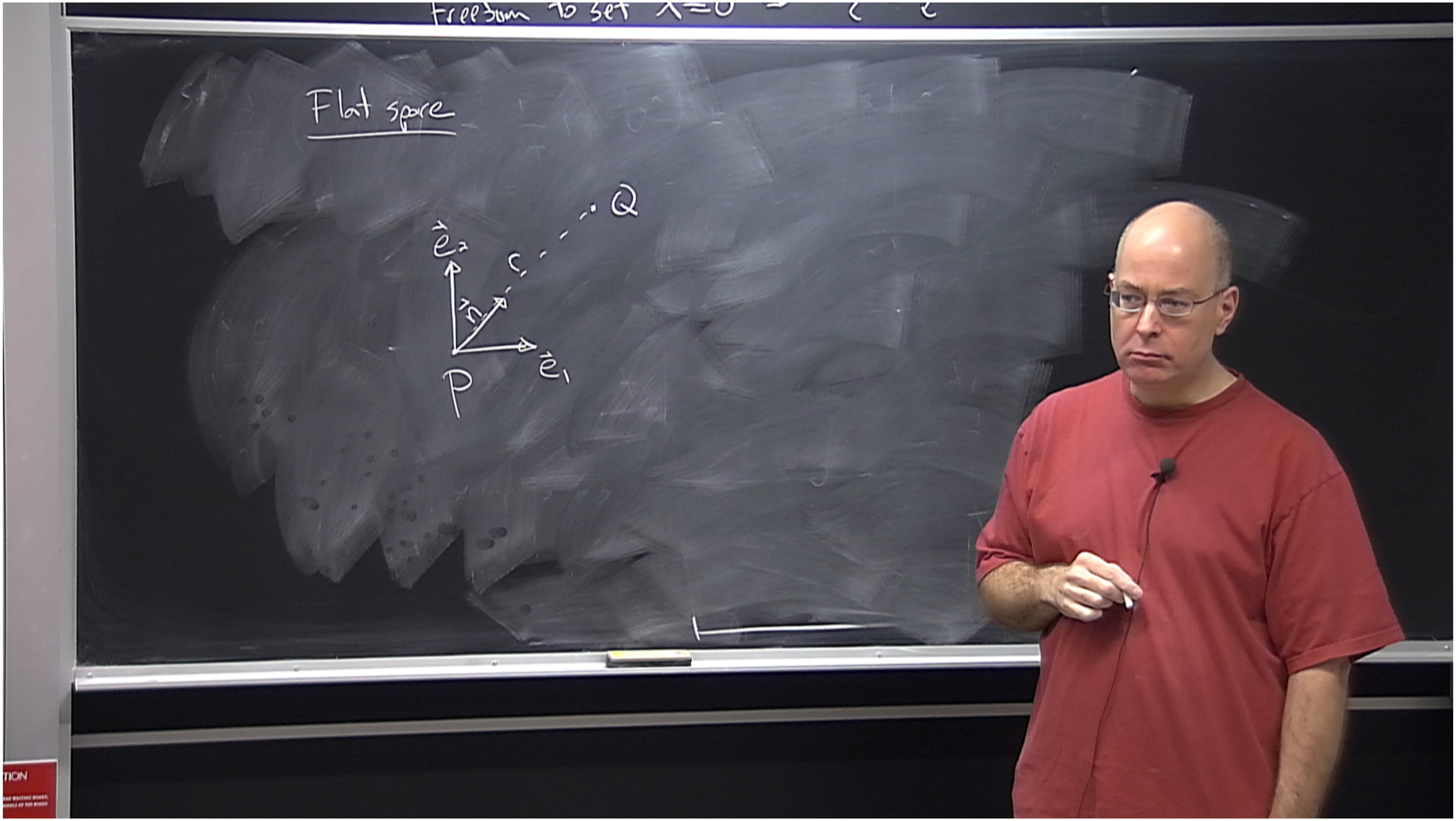
Flat space
P

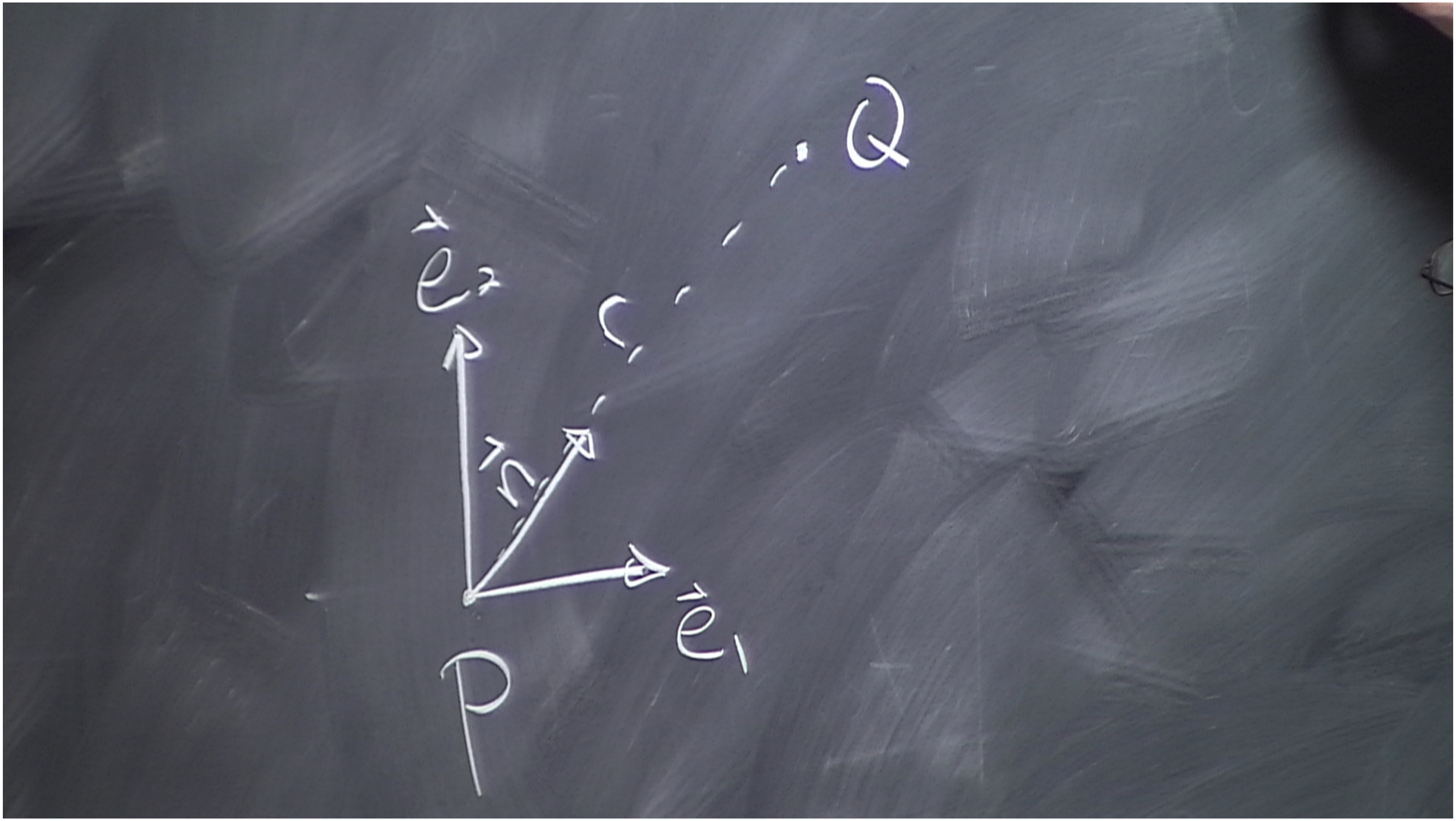

Flat space
$\vec{e}_2$
$\vec{e}_1$
P

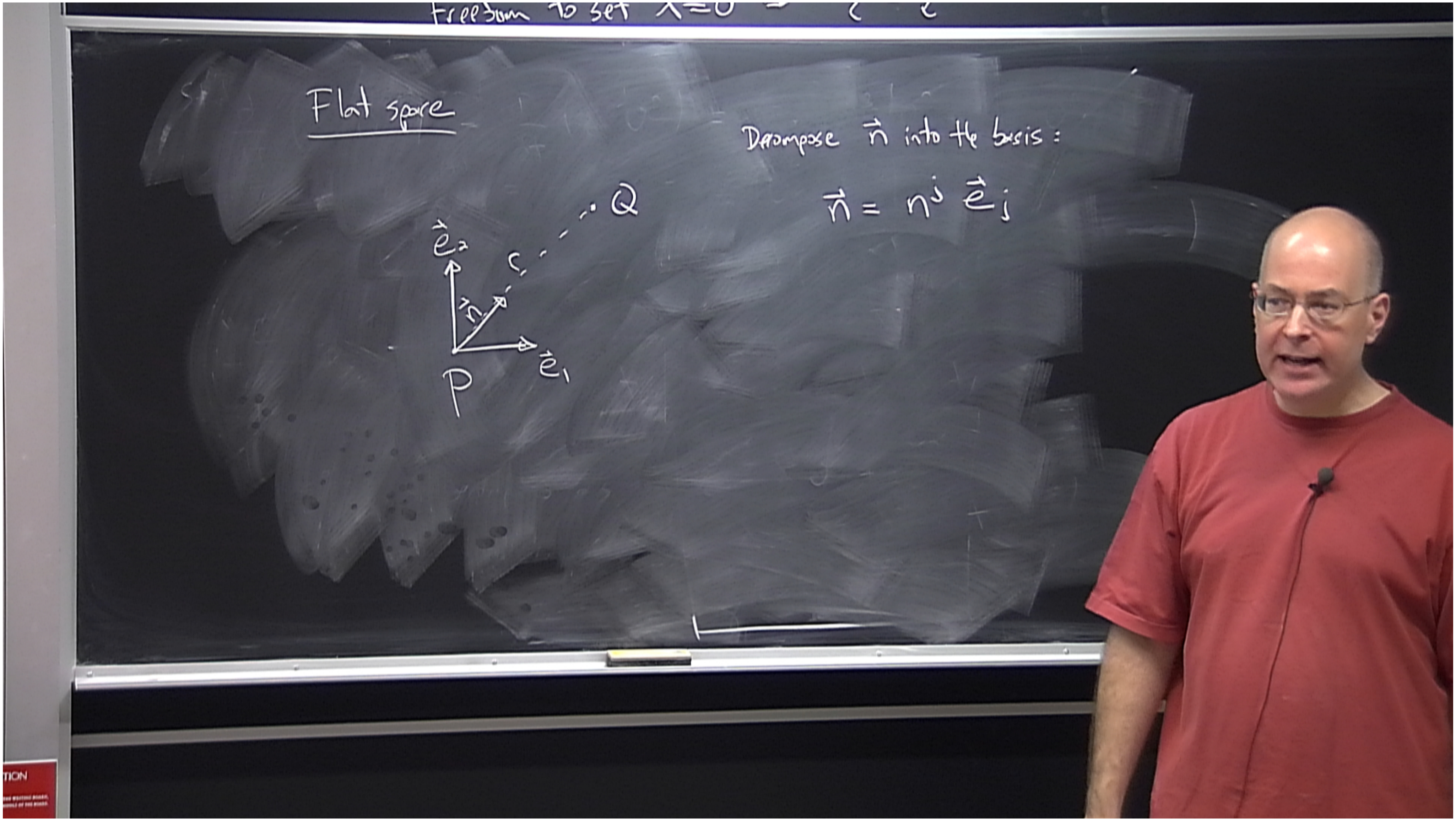

Flat space
Q
e2
e1
P

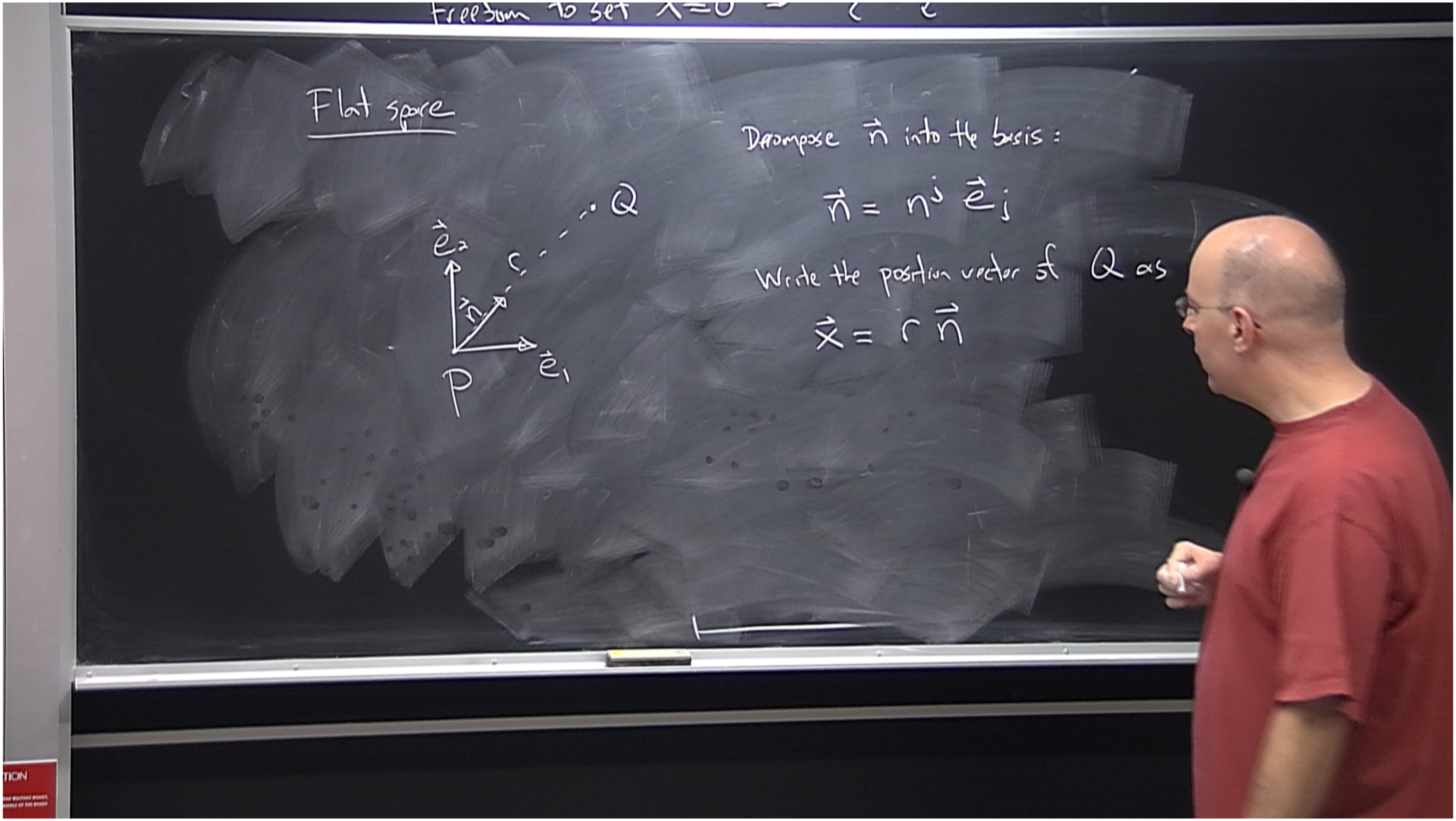

freedom to set
Flat space
Q
$\vec{e}_2$
$\vec{e}_1$
P

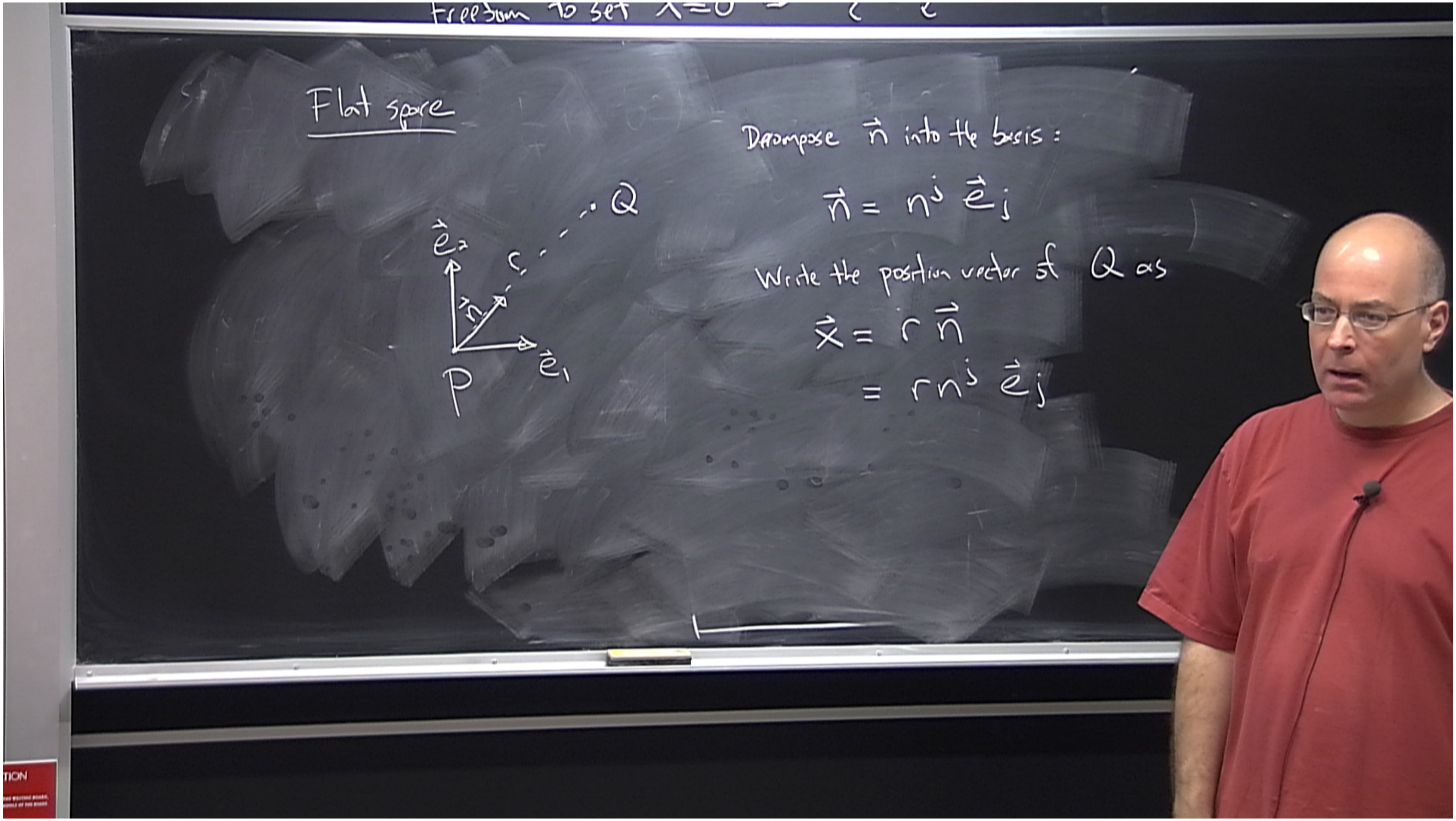

Flat space
Q
$\vec{e}_2$
c
$\vec{e}_1$
P

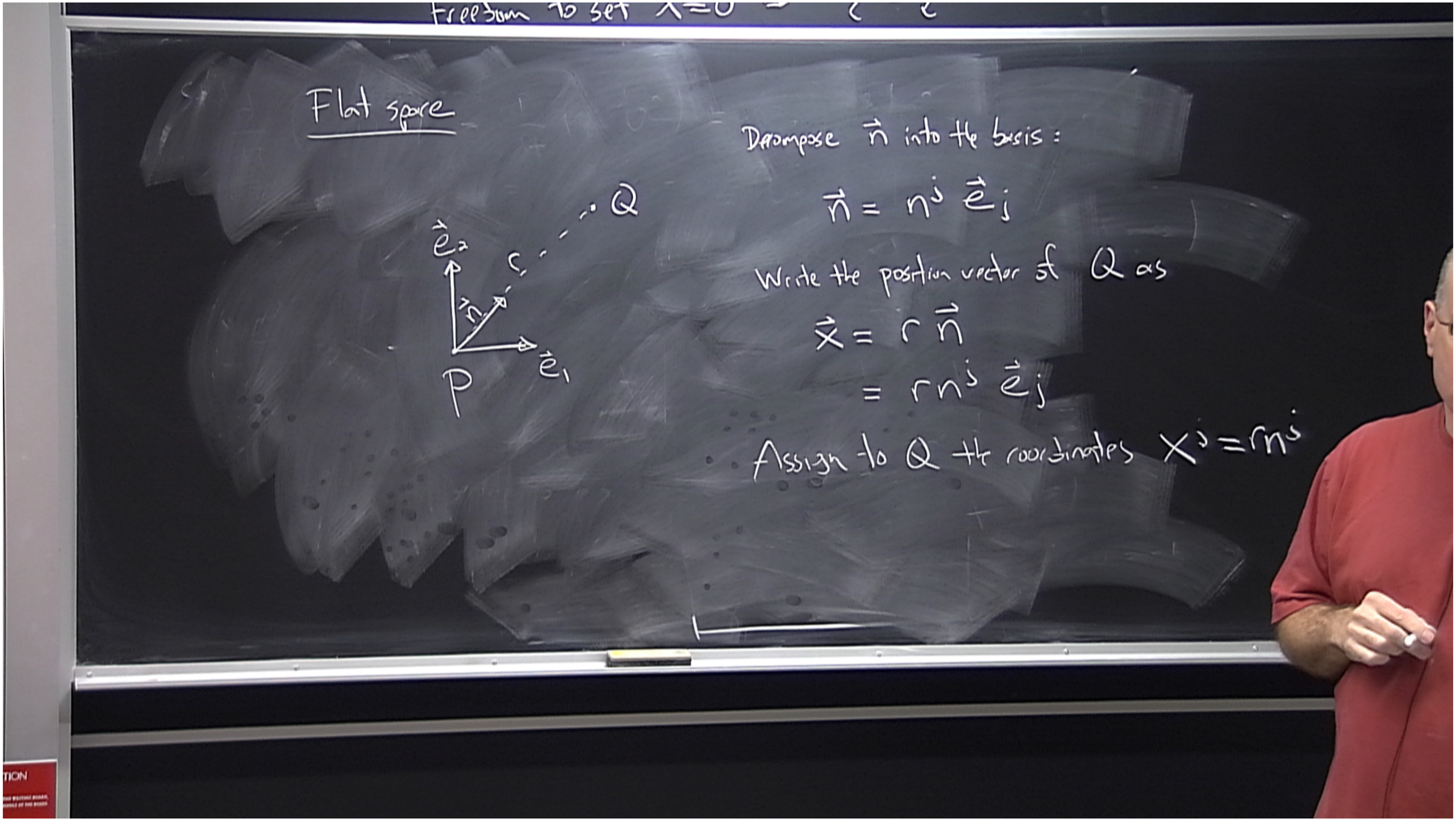

Flat space
Q
$\vec{e}_2$
$\vec{n}$
$\vec{e}_1$
P

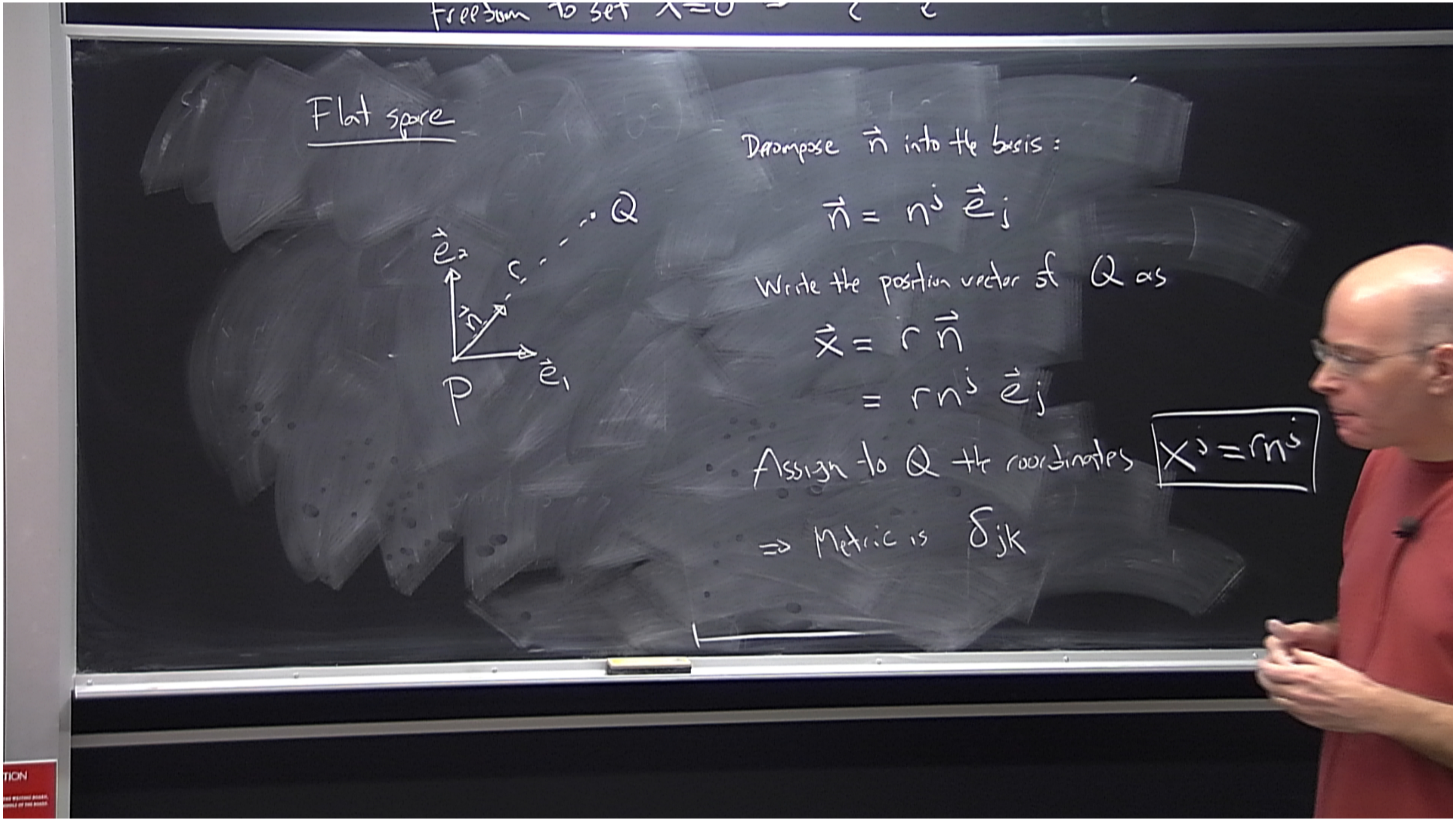

Q
$\vec{e}_2$
$\vec{r}$
$\vec{e}_1$
P

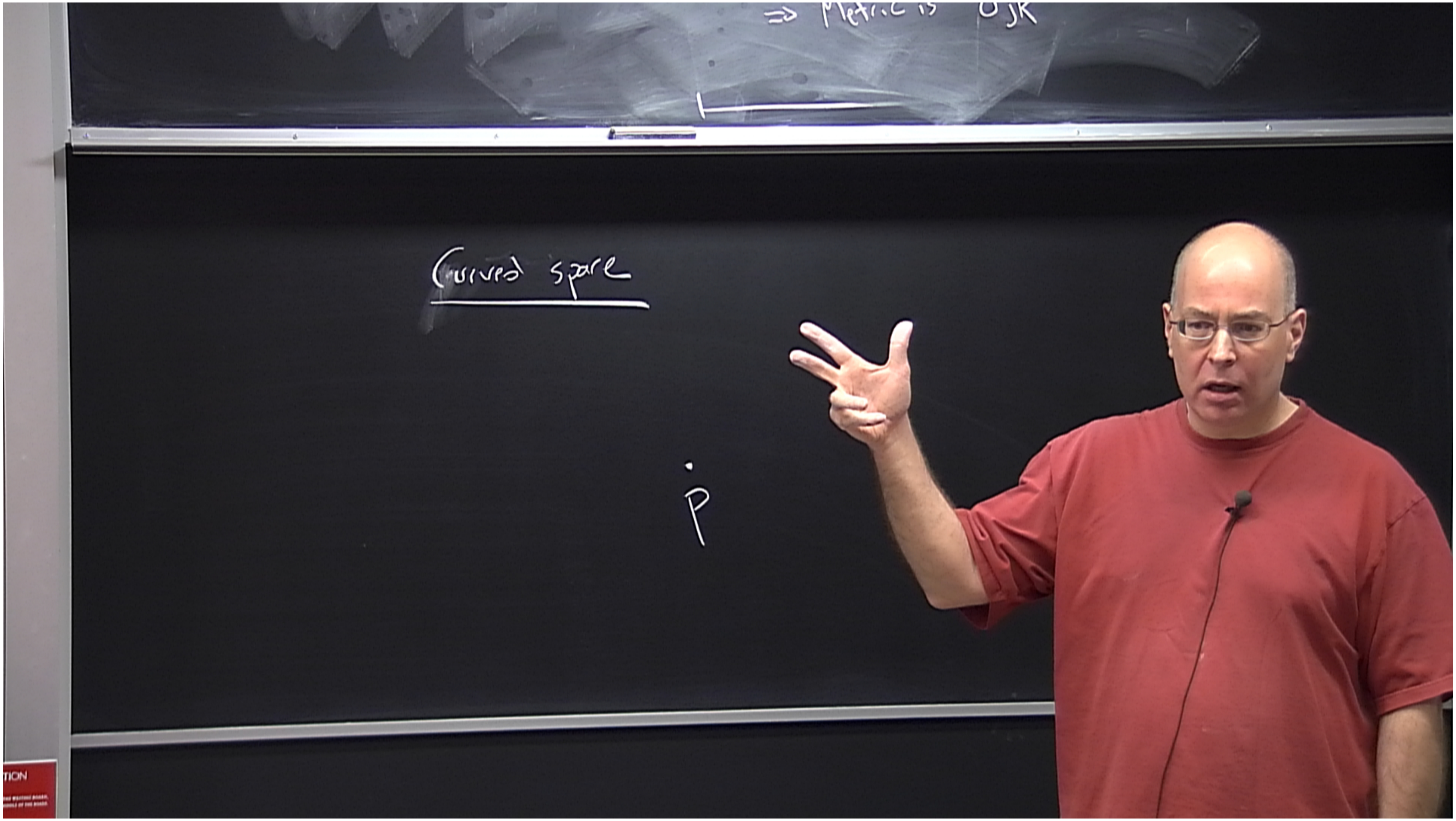

Flat space
Decompose $\vec{n}$ into the basis:
$\vec{n} = n^j \vec{e}_j$
$\vec{e}_2$
$\vec{n}$
$\vec{e}_1$
P
Q

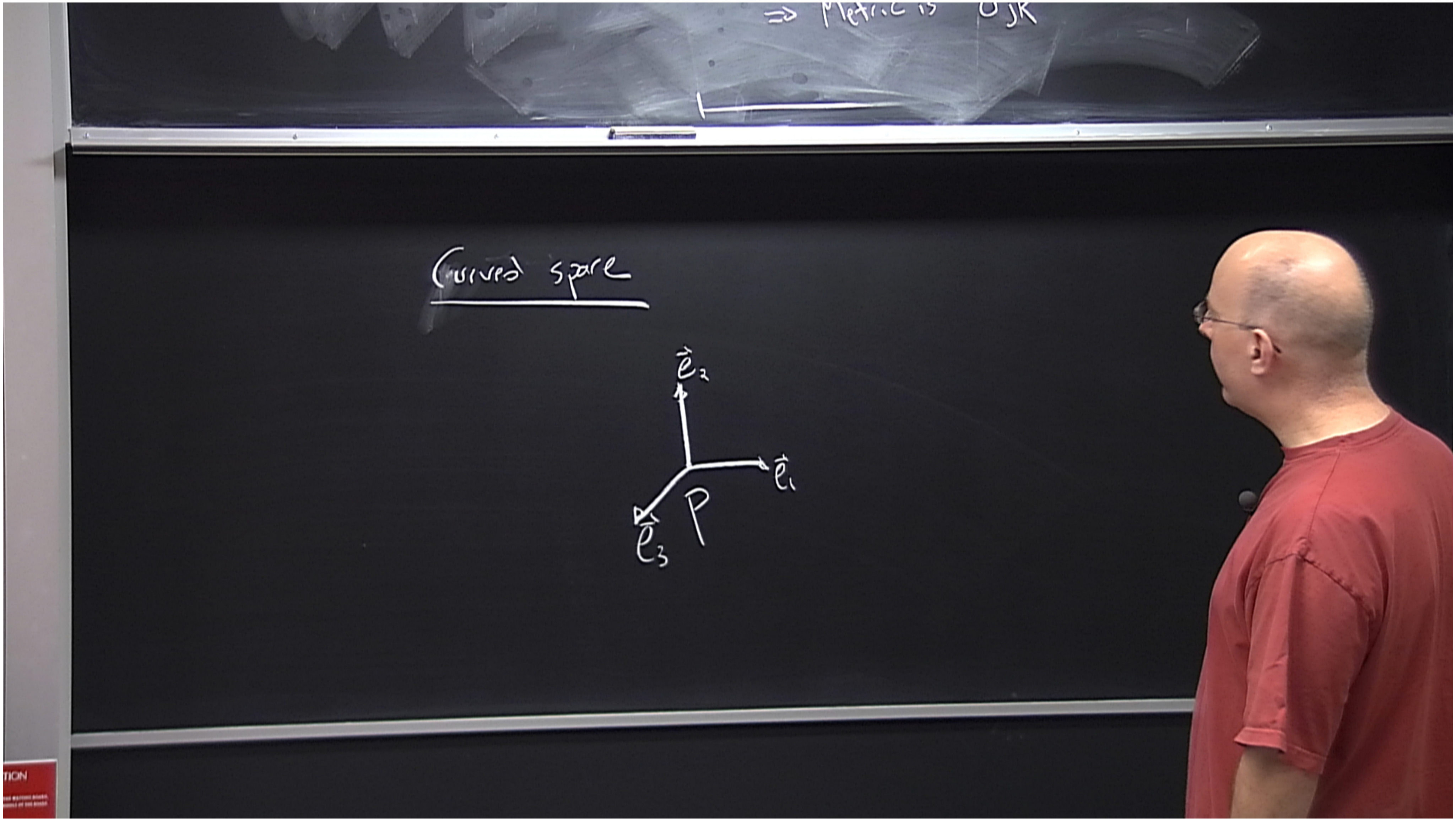

Flat space
Decompose $\vec{n}$ into the basis:
$\vec{n} = n^j \vec{e}_j$
Write the position vector of Q as
$\vec{x} = r \vec{n}$
Q
$\vec{e}_2$
r
$\vec{n}$
$\vec{e}_1$
P

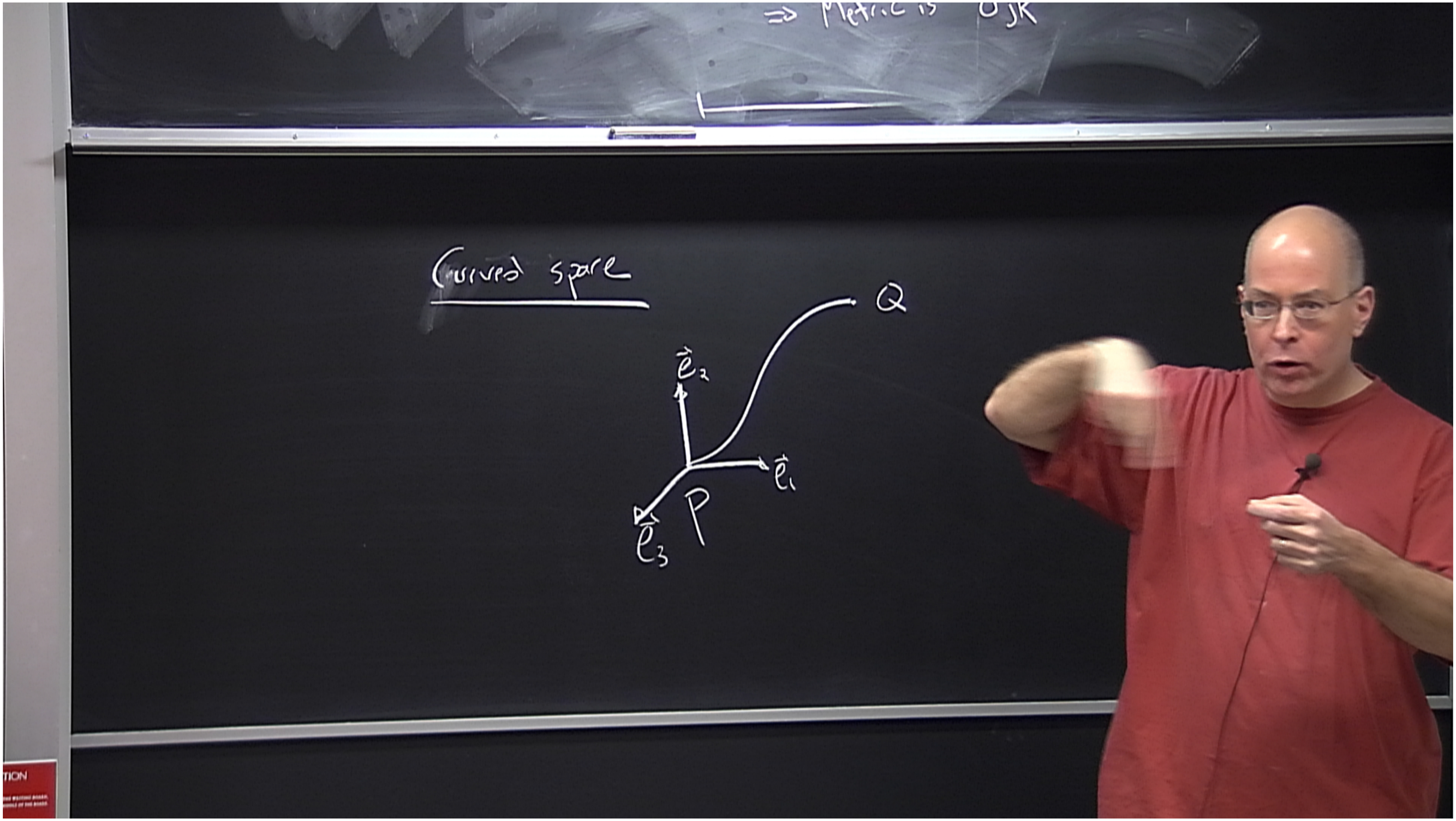

Flat space
Q
r
n
e2
e1
P
Decompose n into the basis:
n = n^j e_j
Write the position vector of Q as
x = r n
= r n^j e_j

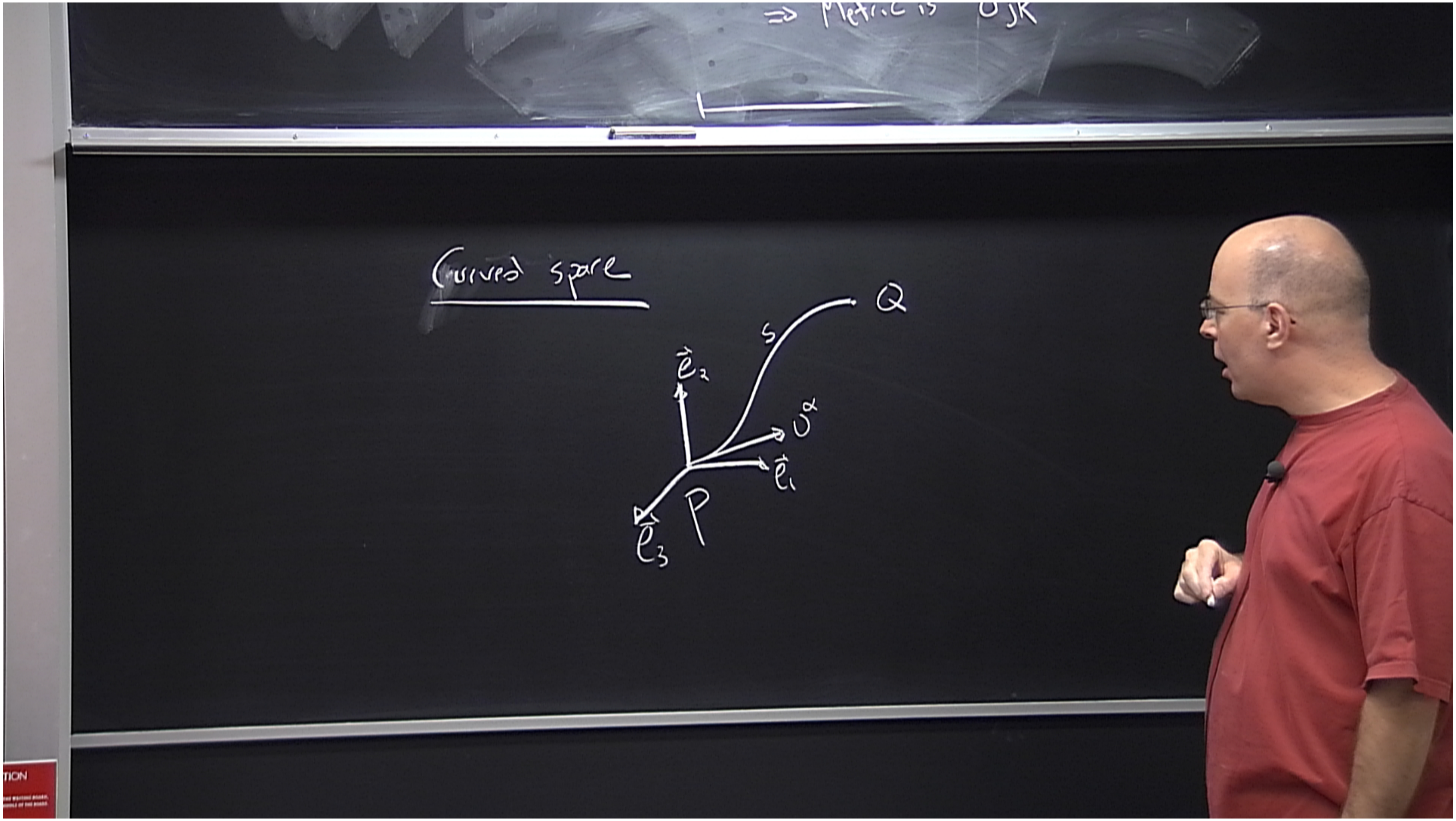

Flat space
Decompose $\vec{n}$ into the basis:
$\vec{n} = n^j \vec{e}_j$
Write the position vector of Q as
$\vec{x} = r\vec{n}$
$= r n^j \vec{e}_j$
Assign to Q the coordinates $x^j = r n^j$
Q
r
$\vec{n}$
$\vec{e}_2$
$\vec{e}_1$
P


Flat space
Q
r
P
Decompose n into the basis :
n = n^j e_j
Write the position vector of Q as
x = r n
= r n^j e_j
Assign to Q the coordinates X^j = rn^j
⇒ Metric is δ_jk

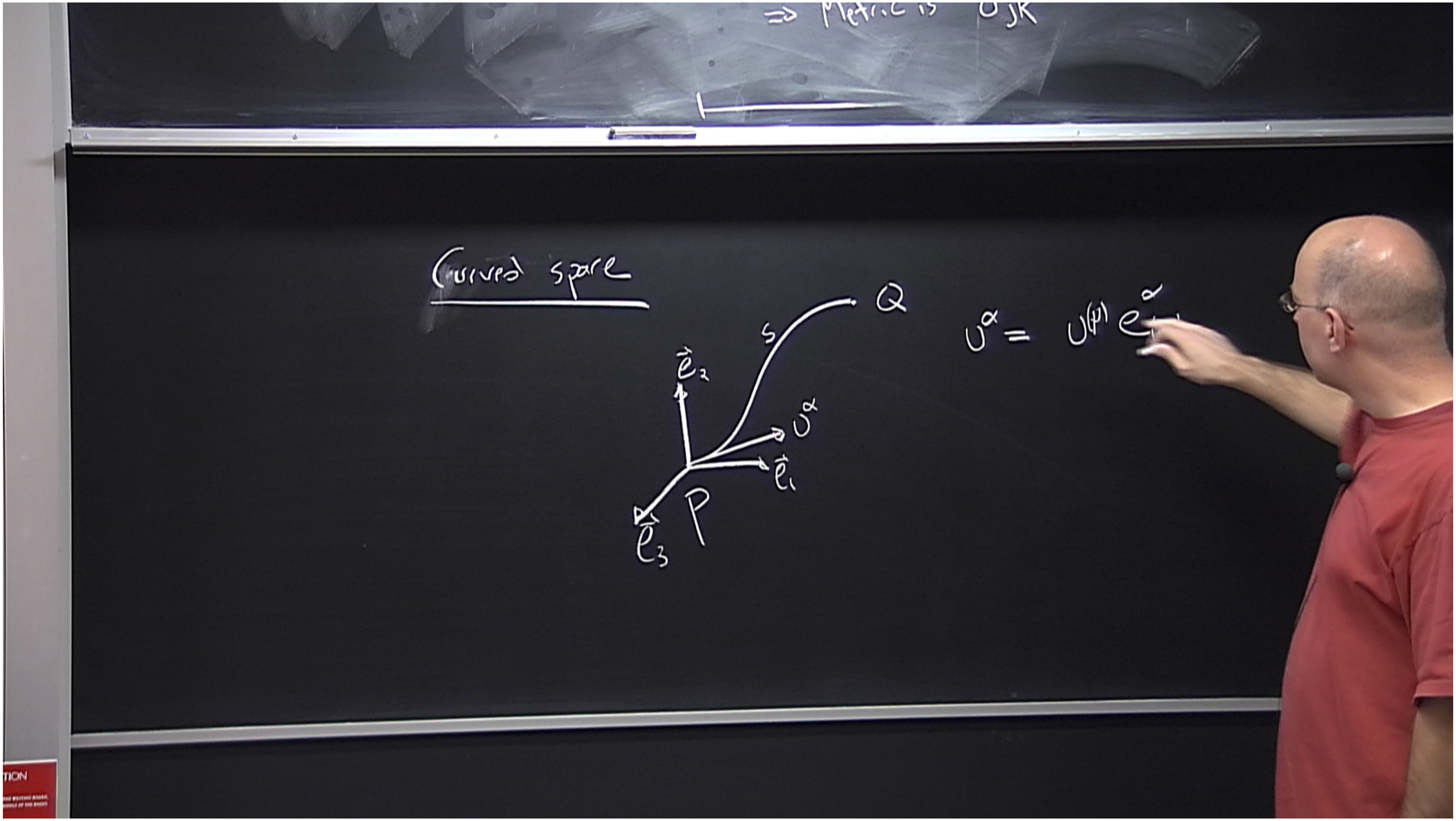

⇒ Metric is δJK
Curved space
p

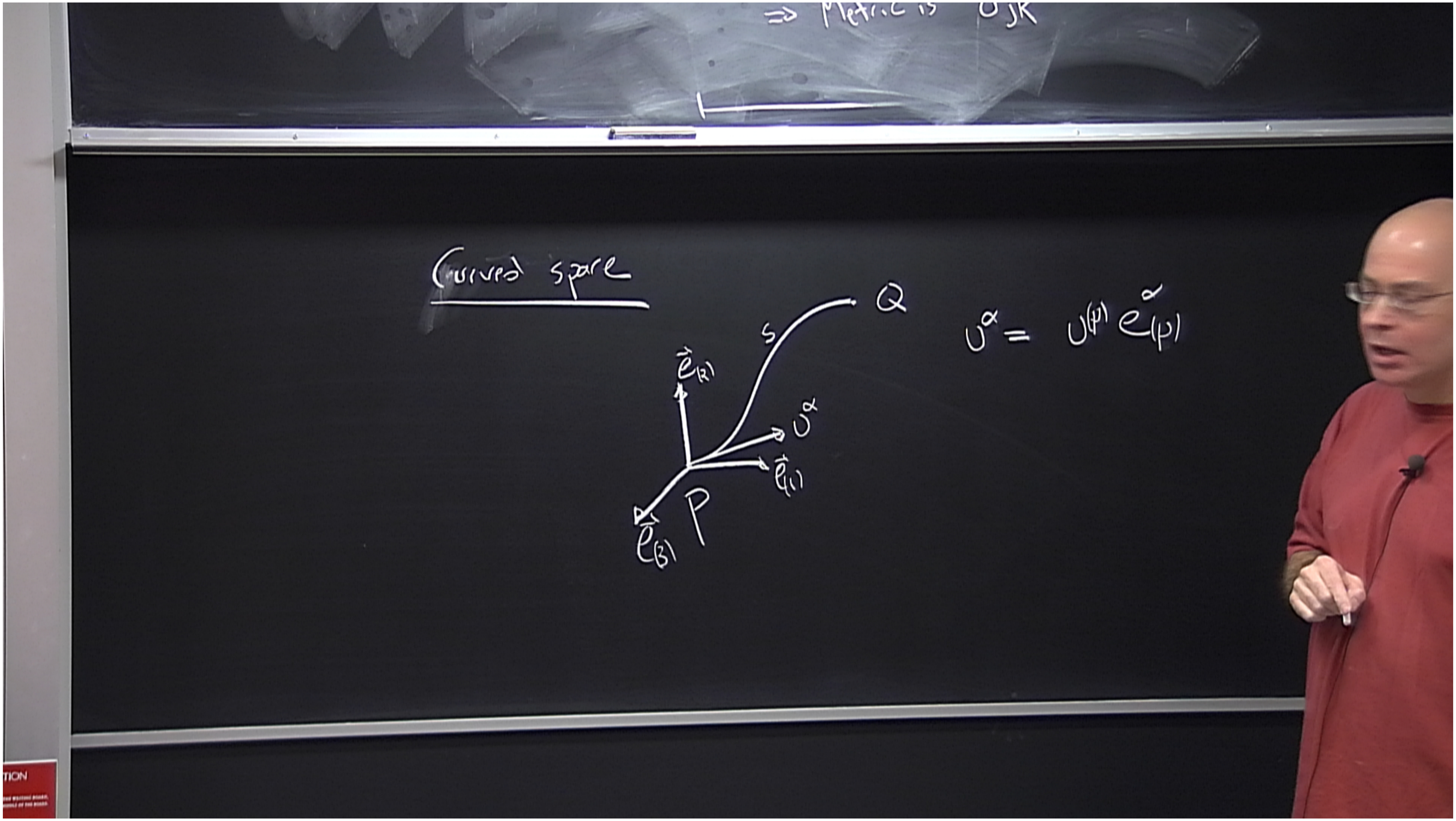

Curved space
$\vec{e}_2$
$\vec{e}_1$
$\vec{e}_3$
P


⇒ Metric is δjk
Curved space
Q
$\vec{e}_2$
$\vec{e}_1$
$\vec{e}_3$
P

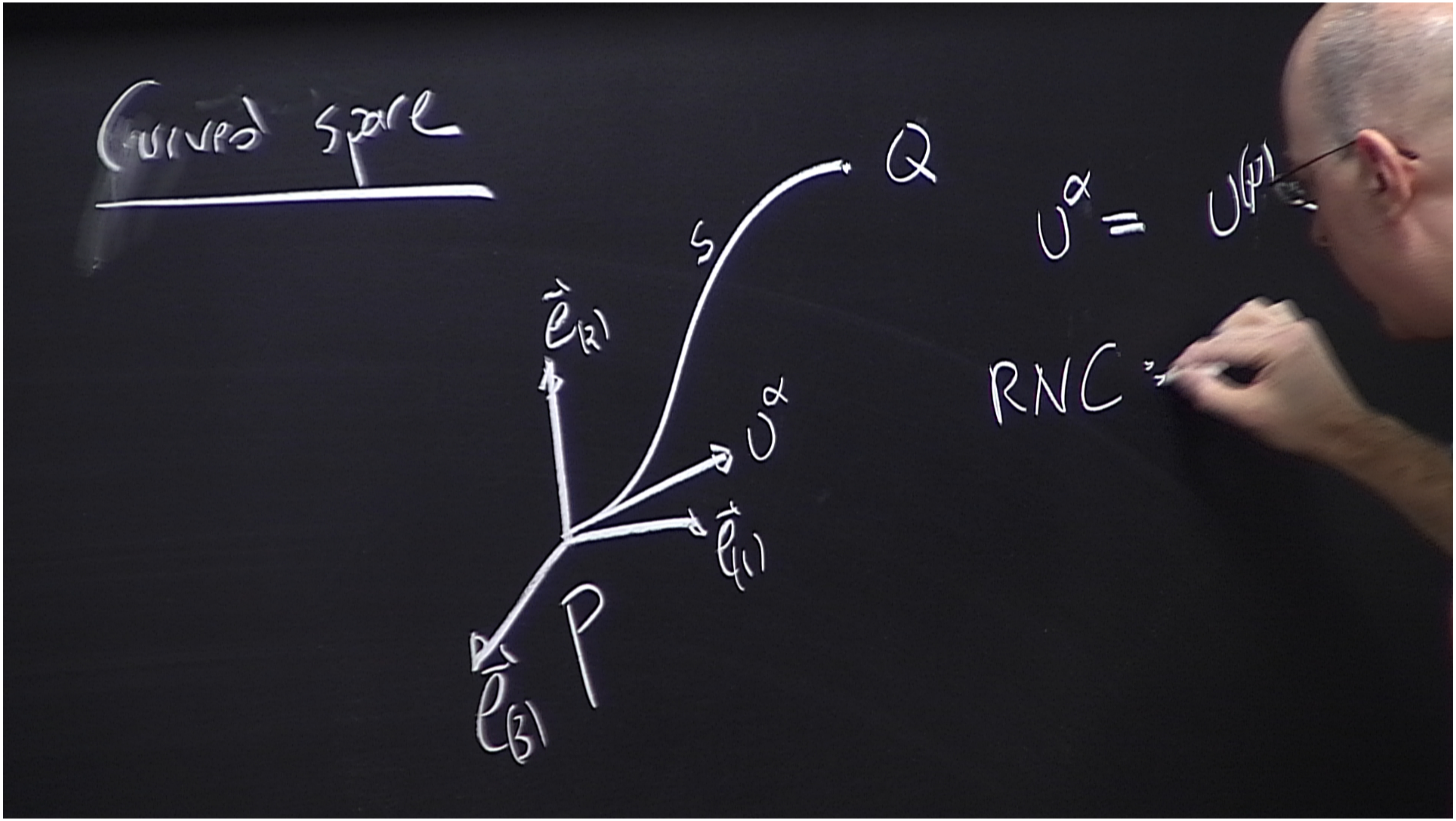


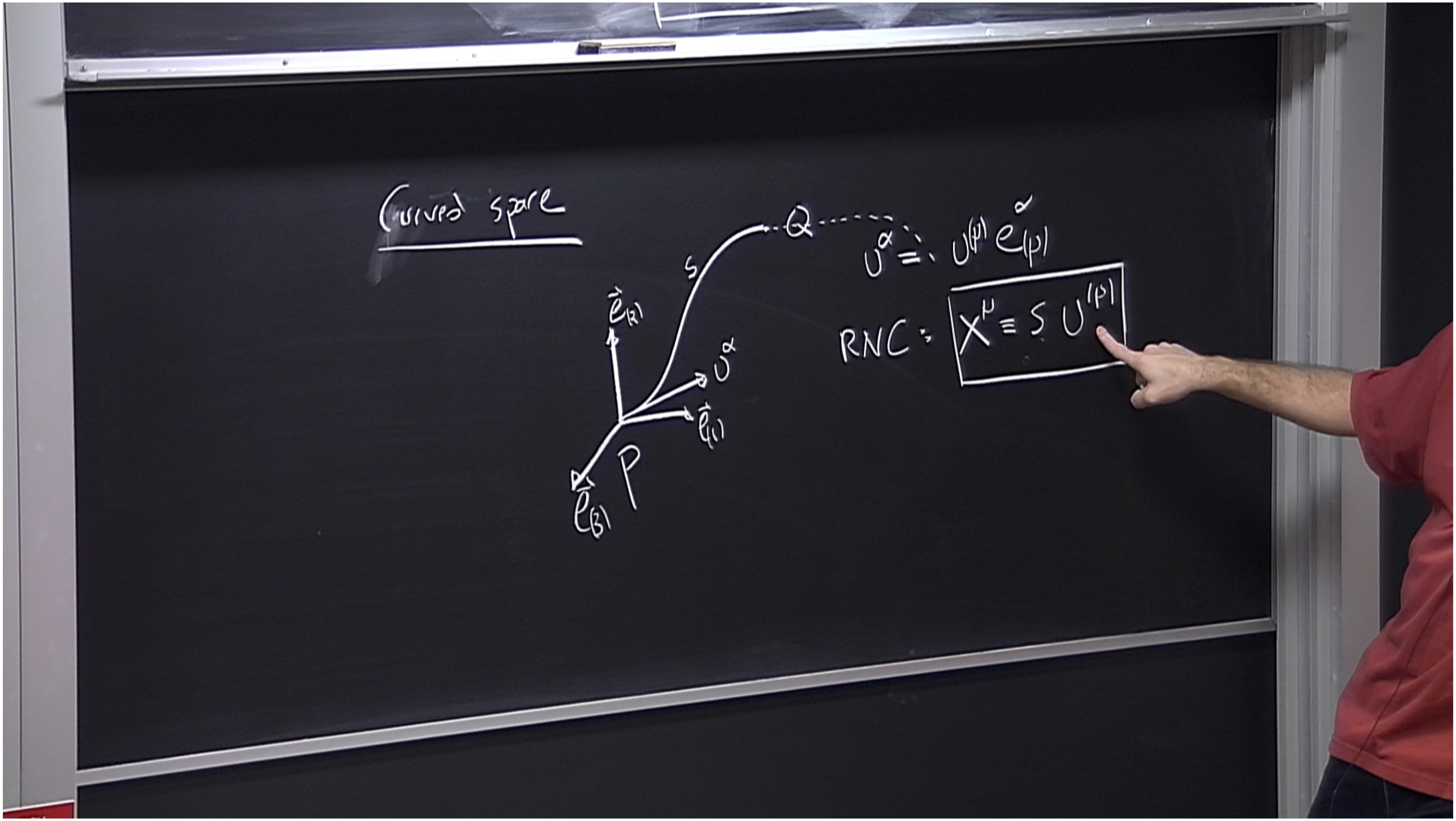
Curved space
Q
s
$\vec{e}_2$
u^α
$\vec{e}_1$
P
$\vec{e}_3$

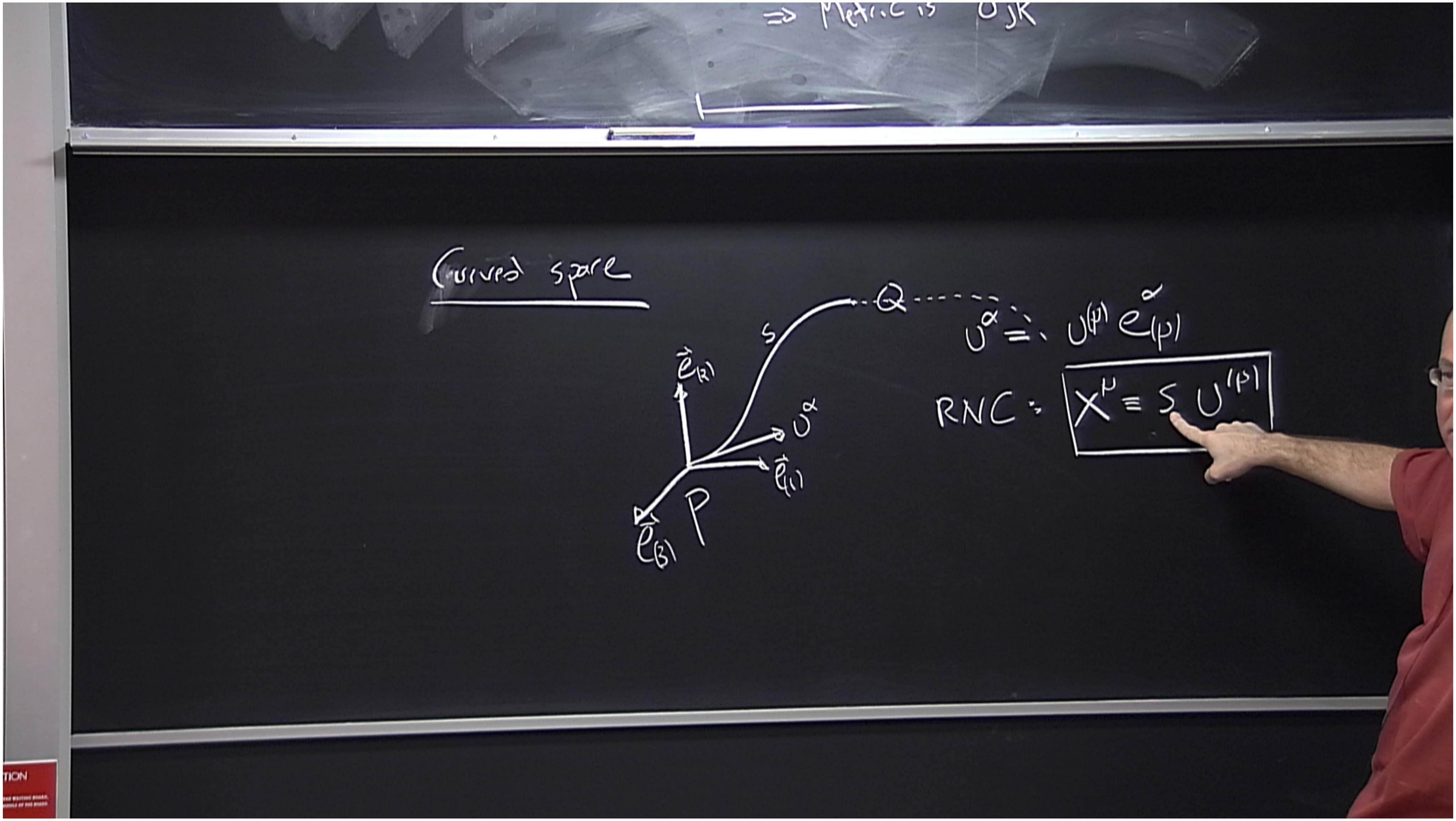

Curved space
Q
s
P
e1
e2
e3
u^α

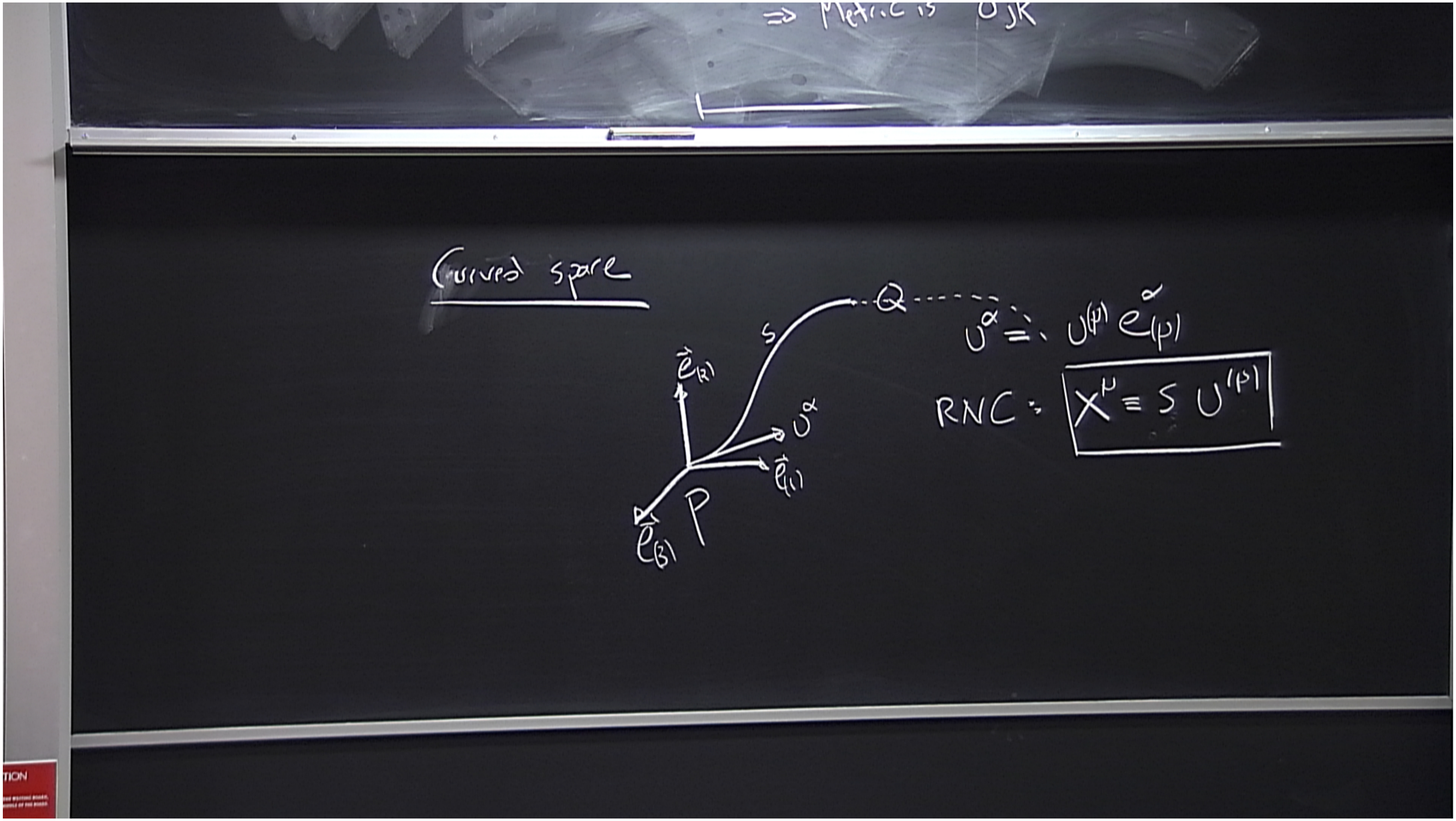


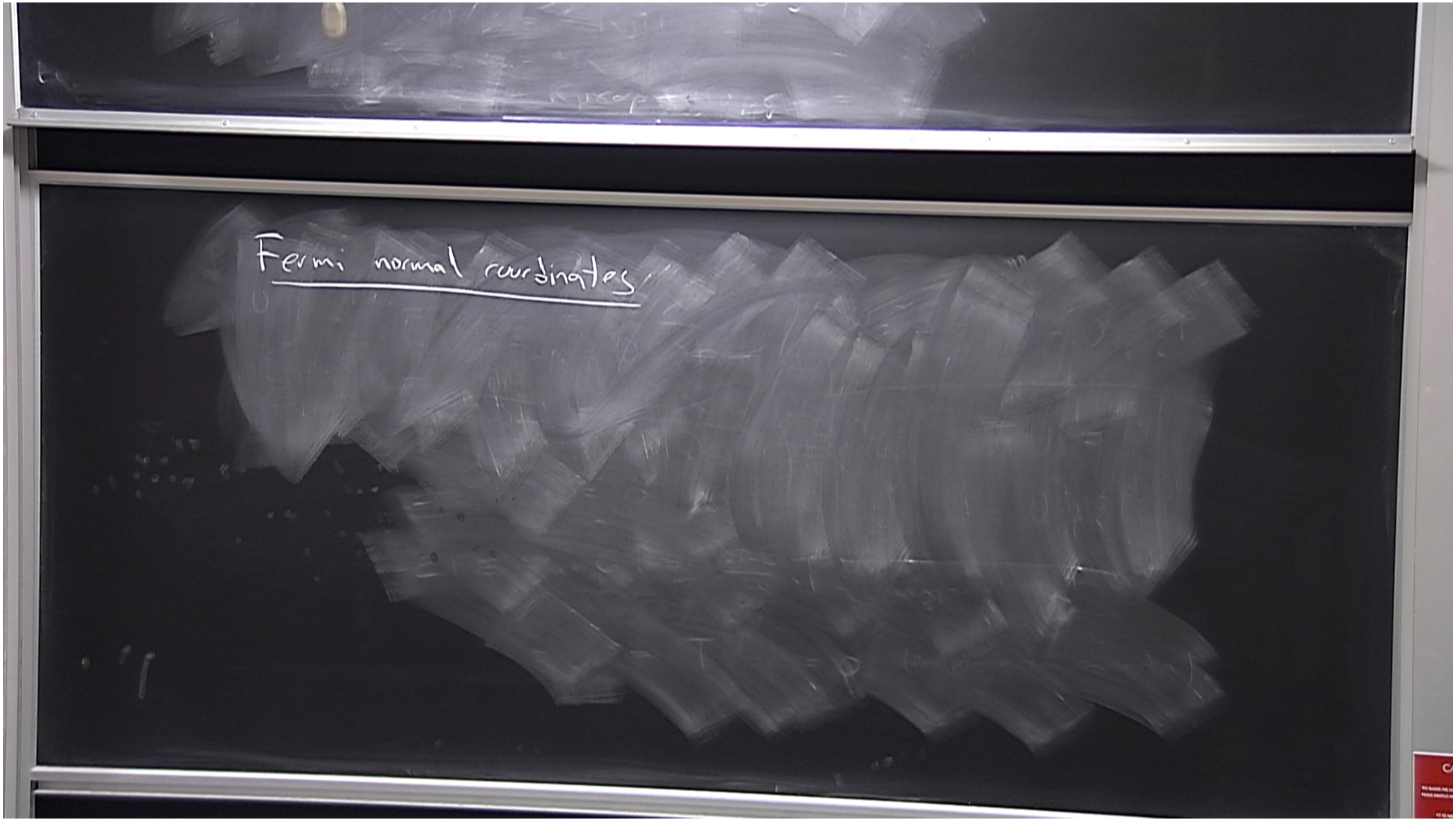
Curved space
Q
s
$\vec{e}_{(2)}$
u^α
$\vec{e}_{(1)}$
$\vec{e}_{(3)}$
P
$u^\alpha = u^{(\mu)} e^\alpha_{(\mu)}$

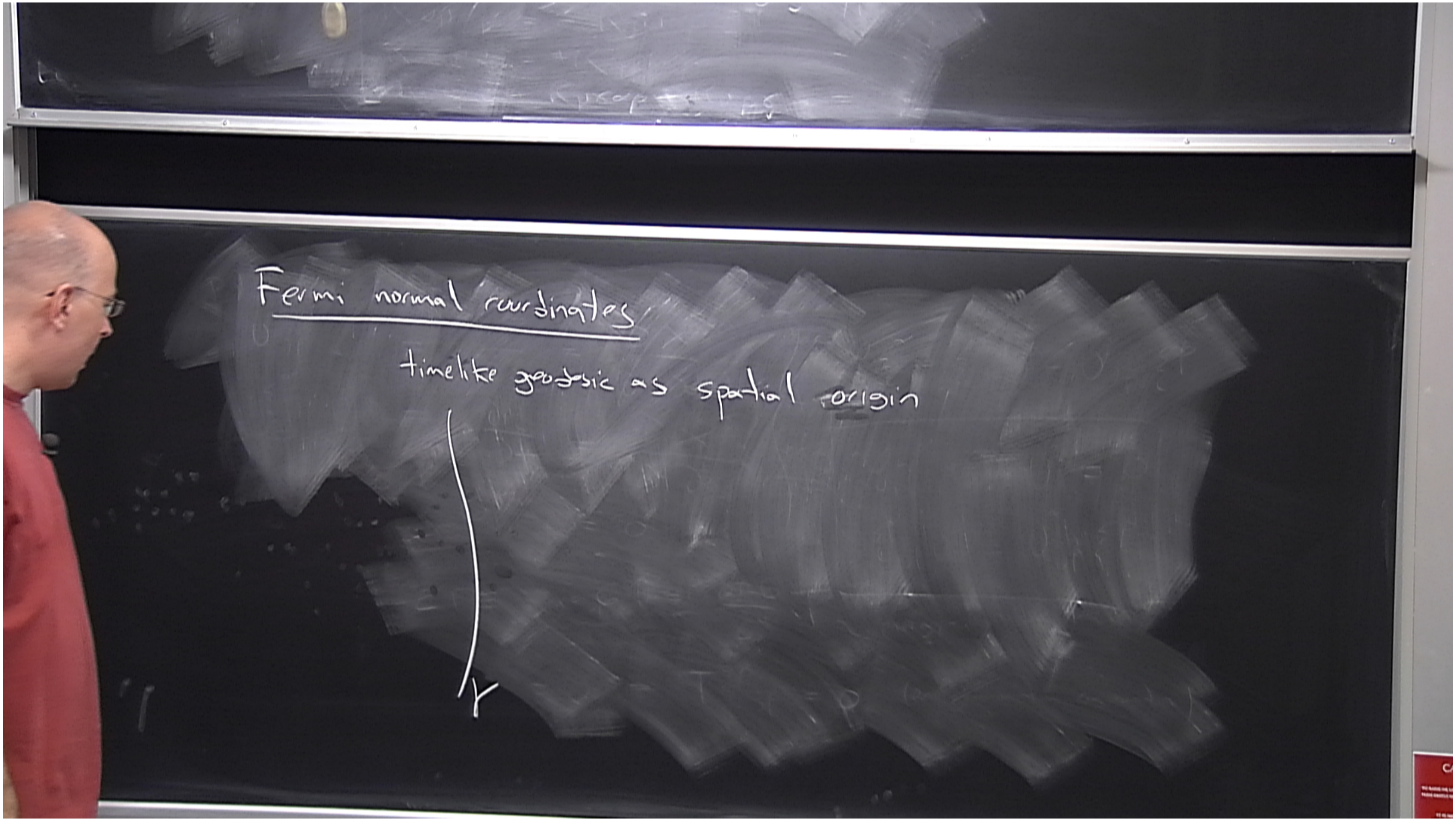

Curved space
Q
s
$\vec{e}_{(2)}$
u^α
$\vec{e}_{(1)}$
P
$\vec{e}_{(3)}$
$u^\alpha =$
RNC


Curved space
Q
s
$U^\alpha =: U^{(\mu)} e^\alpha_{(\mu)}$
RNC: $X^\mu \equiv s\, U^{(\mu)}$
$\vec{e}_{(2)}$
U^α
$\vec{e}_{(1)}$
$\vec{e}_{(3)}$
P


⇒ Metric is δ_{JK}
Curved space
$\vec{e}_{(2)}$
$\vec{e}_{(1)}$
$\vec{e}_{(3)}$
P
s
Q
U^α
$U^\alpha =: U^{(\mu)} e^\alpha_{(\mu)}$
RNC: $X^\mu \equiv s\, U^{(\mu)}$

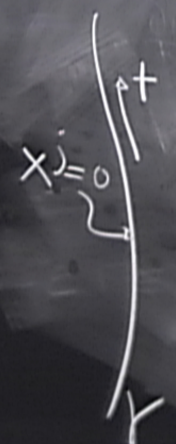

⇒ Metric is δJK
Curved space
Q
s
e(2)
uα
e(1)
P
e(3)
uα =: u(μ) eα(μ)
RNC: Xμ ≡ s u(μ)

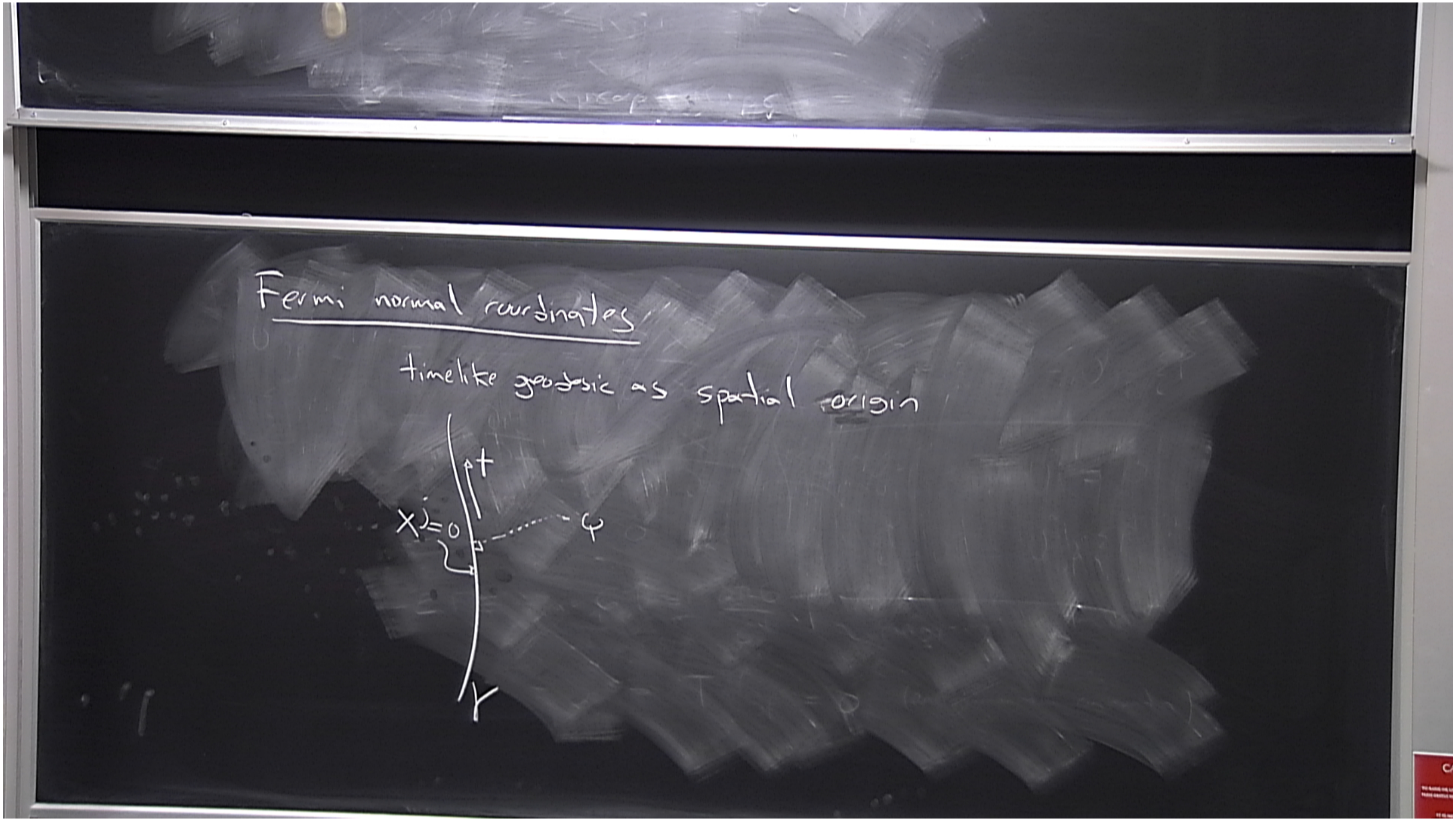

Fermi normal coordinates

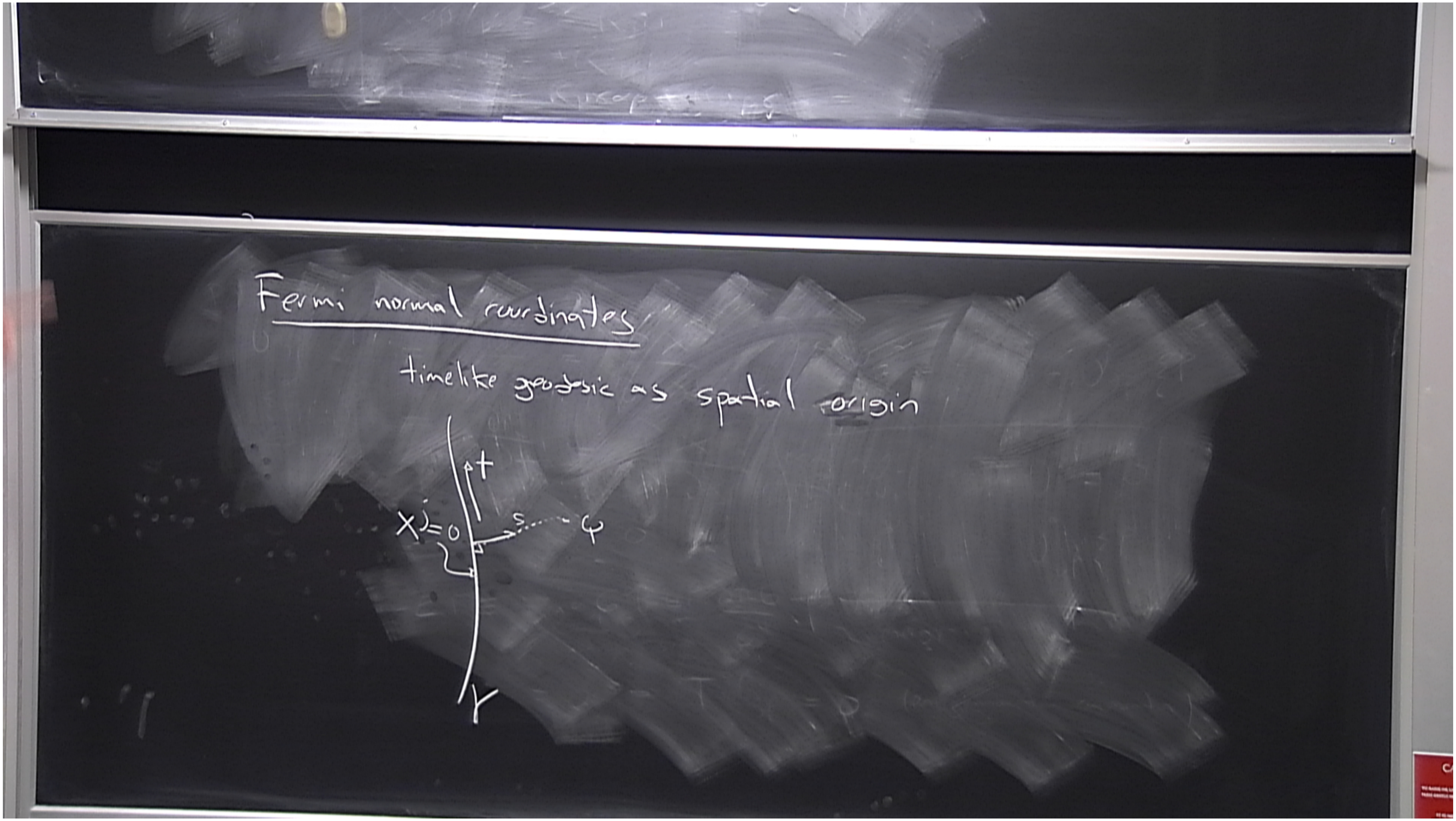

Fermi normal coordinates
timelike geodesic as spatial origin
γ

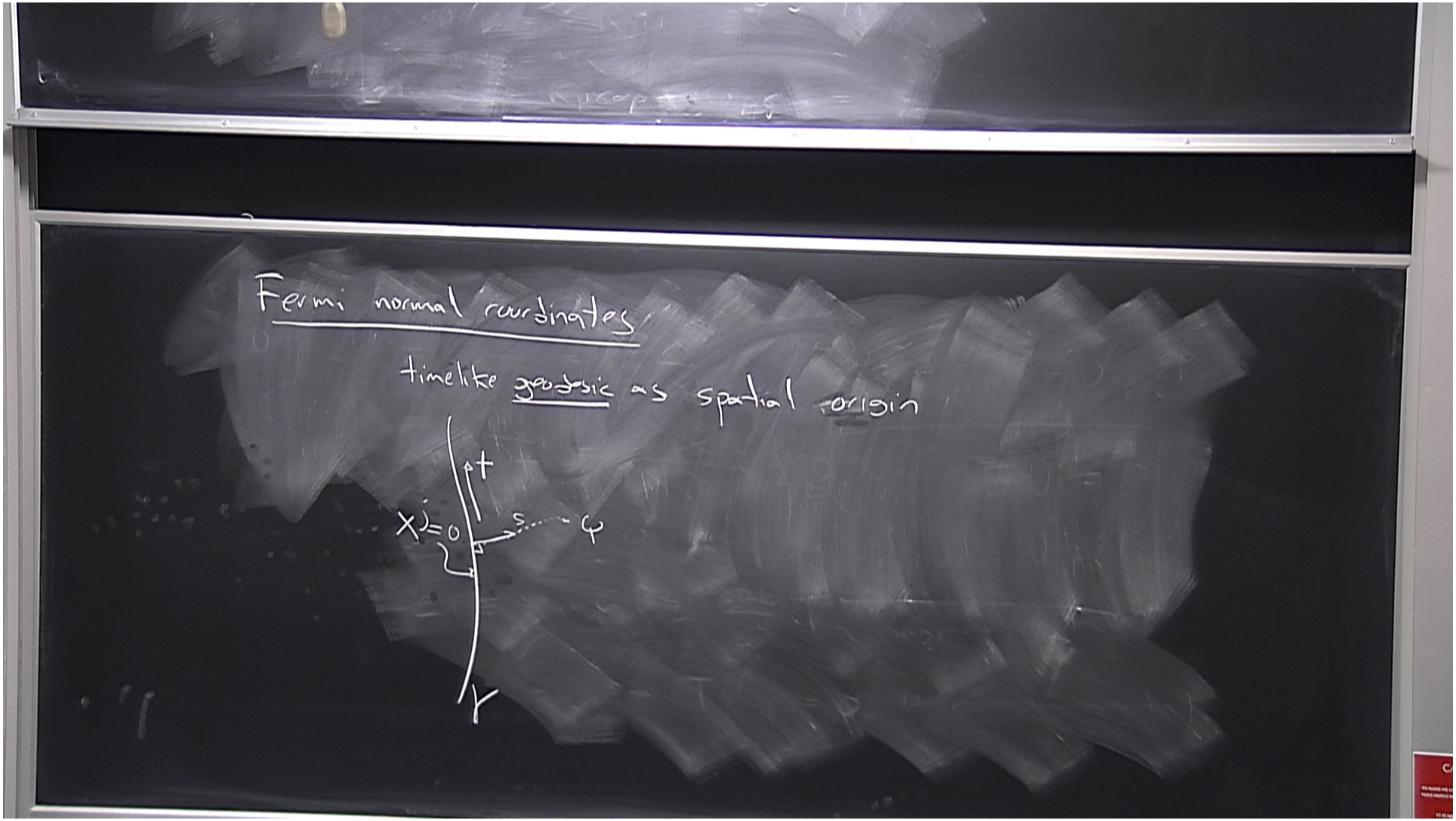
Fermi normal coordinates

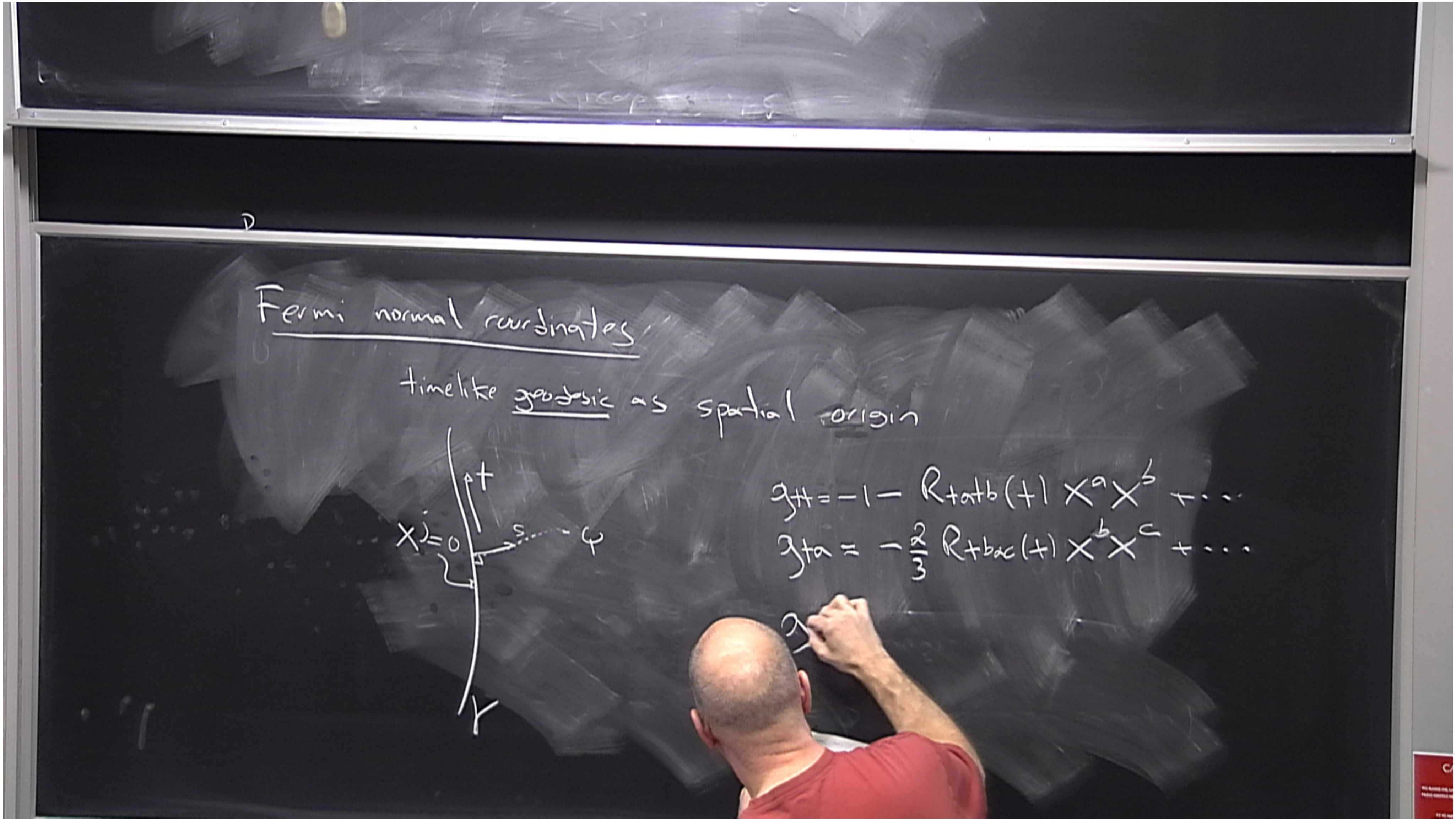
timelike geodesic as spatial origin

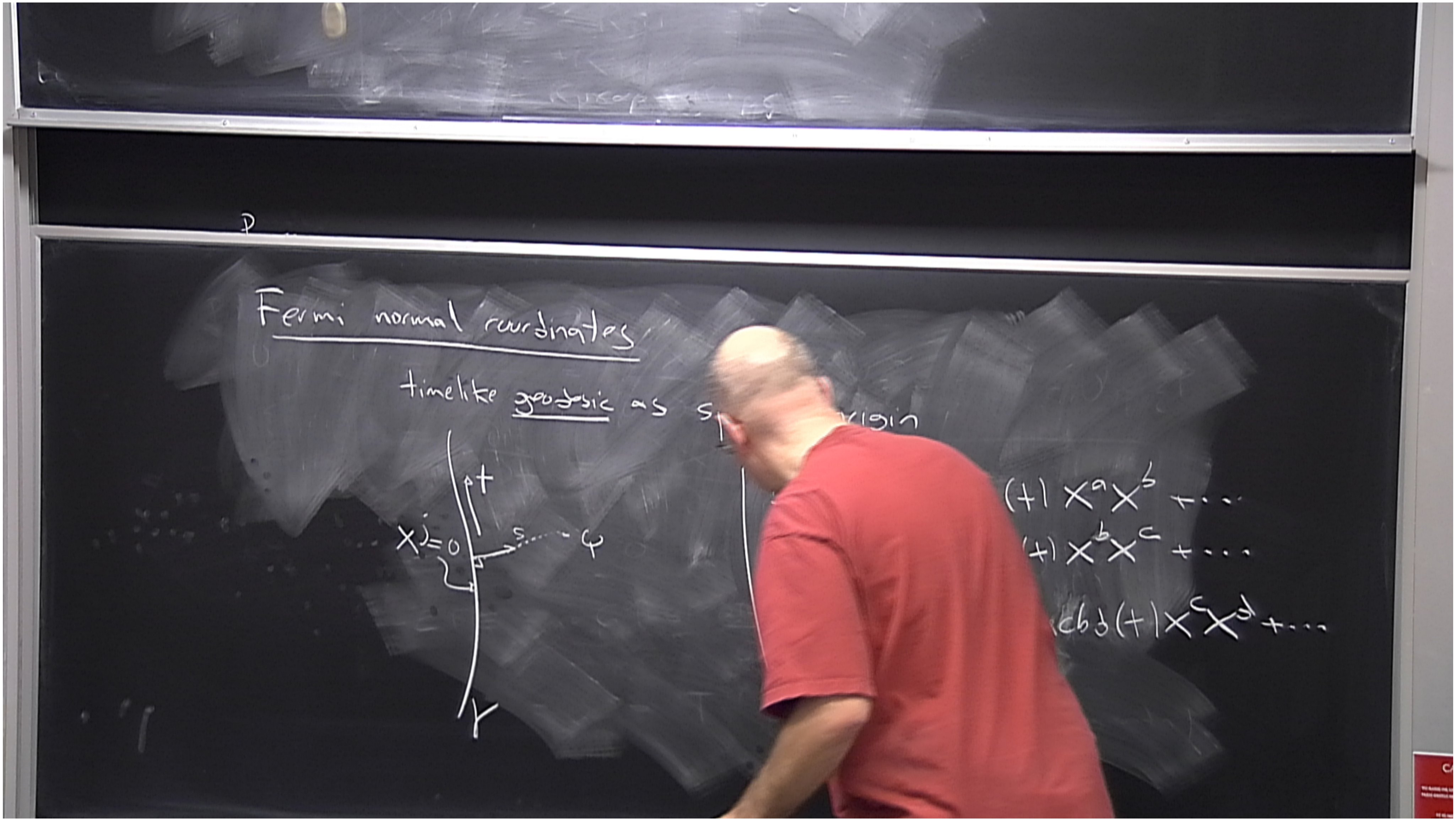



Fermi normal coordinates
timelike geodesic as spatial origin
t
x^j=0
φ
γ

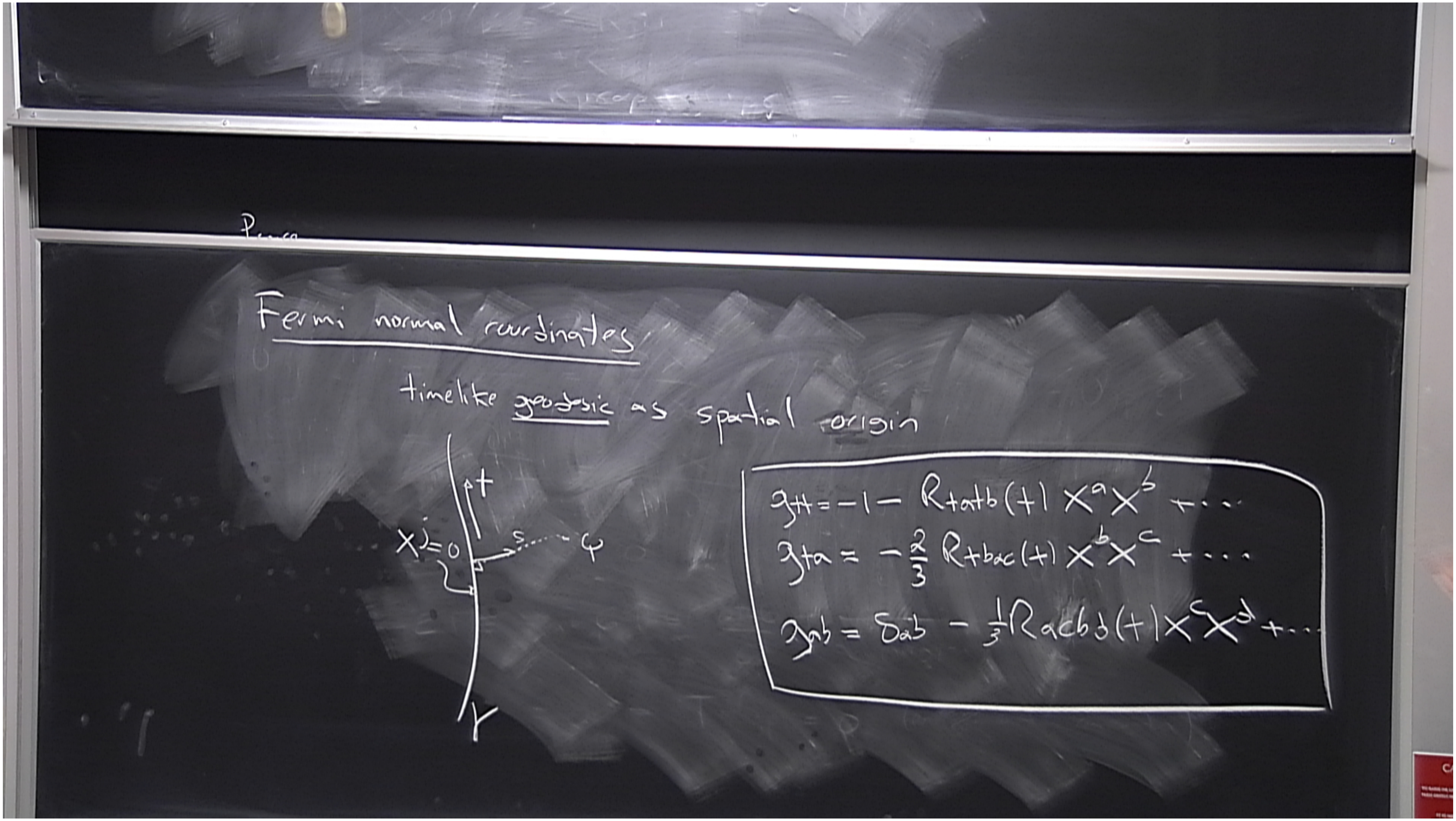

Fermi normal coordinates
timelike geodesic as spatial origin
$x^j = 0$
t
s
φ
γ


Fermi normal coordinates
timelike geodesic as spatial origin
t
$x^j=0$
s
p
γ


Fermi normal coordinates
timelike geodesic as spatial origin
t
x^j = 0
s
γ
g_tt = −1 − R_tatb(t) x^a x^b + ...
g_ta = −2/3 R_tbac(t) x^b x^c + ...



Fermi normal coordinates
timelike geodesic as s... origin
(t) x^a x^b + ...
(t) x^b x^c + ...
cbd (t) x^c x^d + ...


Fermi normal coordinates
timelike geodesic as spatial origin
$g_{tt} = -1 - R_{tatb}(t)\, x^a x^b + \cdots$
$g_{ta} = -\frac{2}{3} R_{tbac}(t)\, x^b x^c + \cdots$
$g_{ab} = \delta_{ab} - \frac{1}{3} R_{acbd}(t)\, x^c x^d + \cdots$
t
$x^j = 0$
s
φ
γ