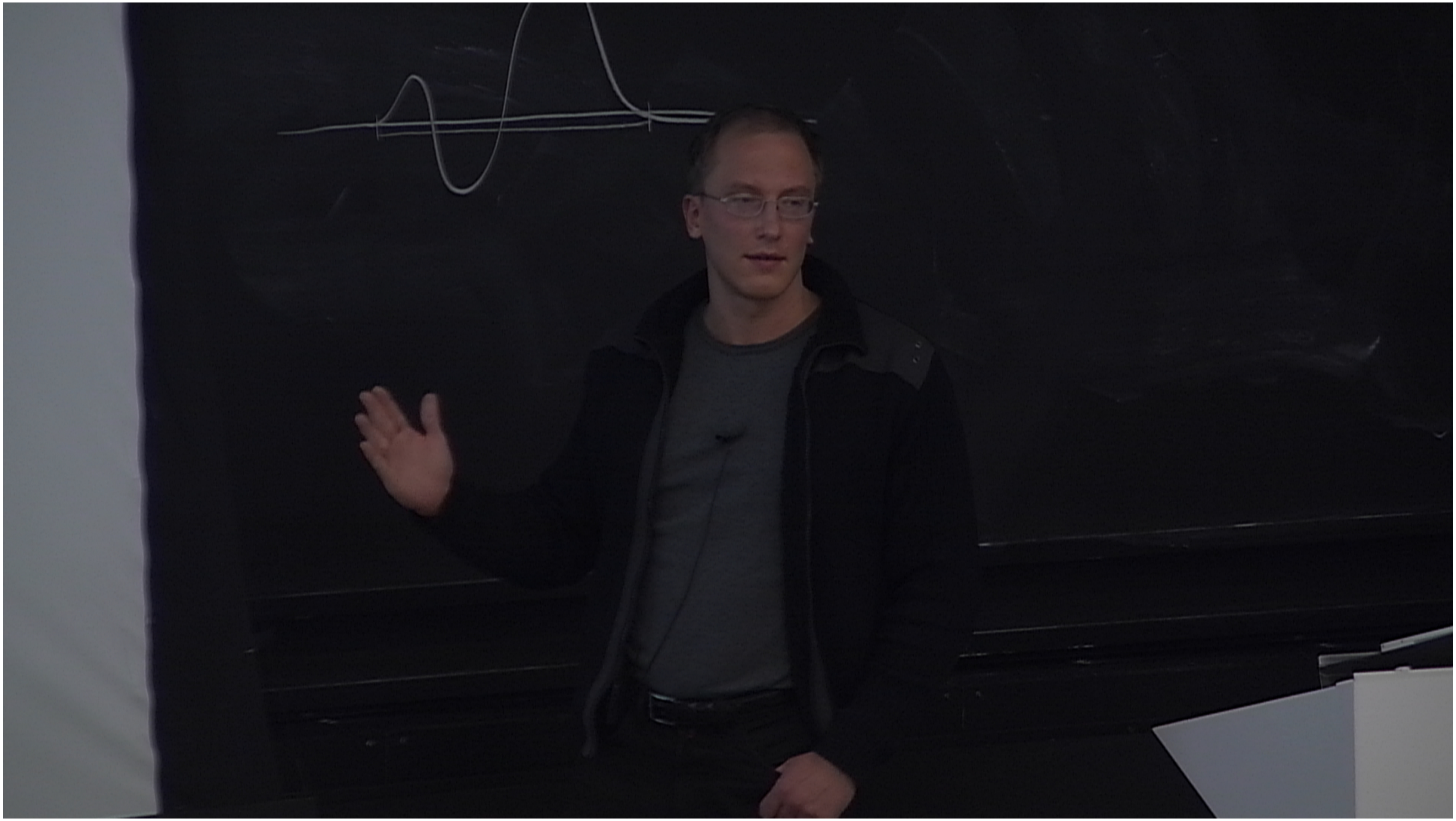
Title: Energy-Time Uncertainty Relation for Absorbing Detectors

Date: Jan 17, 2012 03:30 PM

URL: http://pirsa.org/12010145

Abstract: We prove an uncertainty relation for energy and arrival time, where the arrival of a particle at a detector is modeled by an absorbing term added to the Hamiltonian. In this well-known scheme the probability for the particle's arrival at the counter is identified with the loss of normalization for an initial wave packet. The result is obtained under the sole assumption that the absorbing term vanishes on the initial wave function. Nearly minimal uncertainty can be achieved in a two-level system.



ArXiv: 1109.5087

Overview

- Short introduction to energy-time UR
- Fixing the setting: absorbing detectors
- The URs: $\boxed{\Delta T \Delta E \geq \frac{\hbar}{2}\sqrt{p}\ , \quad \Delta H \cdot \langle T \rangle \geq 1.37\,\hbar\sqrt{p}}$
- Sketch of proof of the URs for fully absorbing detectors ($p = 1$)
- Understanding why $\sqrt{p}$ appears
- Minimum uncertainty
- Simplest example: two-level system

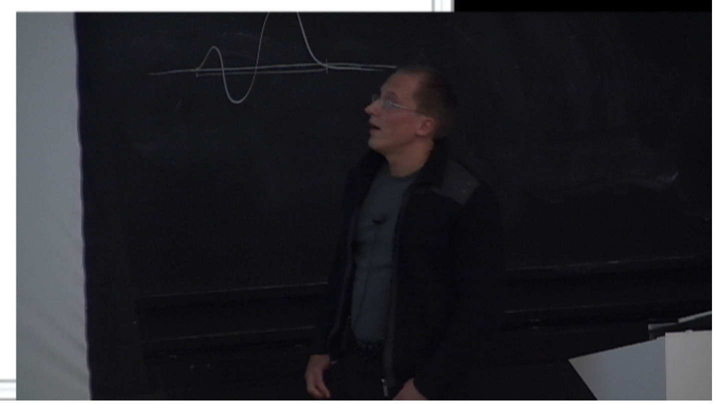
total absorption probability

$$\Delta T \Delta E \geq \text{const} \cdot \hbar$$

- Usually invoked in a handwaving fashion...
- We are looking for a preparation UR:

➡ ΔE and ΔT are standard deviations of outcome distributions for energy and time, measured separately on identically prepared systems (wave function ψ).

- What is the time distribution ?



What is time ?

- First we have to precisely define "time":
- We consider particle detection time, i.e. *time of arrival* (to the detector).
- Basic problem: no "time operator" having CCR with H exists (Pauli thm): analogy to QP fails.

➡ A solution: use some POVM

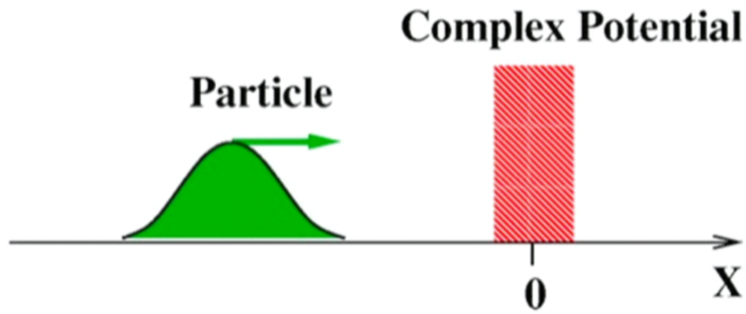
- Approaches: (1) covariant POVM

$$e^{itH/\hbar}F([t_1,t_2])\,e^{-itH/\hbar} = F([t_1+t,t_2+t])$$

only for systems with continuous energy spectrum

(2) absorbing detector (our choice here!)

Absorbing detector



G.R. Allcock, *Ann. Phys. (N.Y.)* **53** 253 (1969)

R.F. Werner, *Ann. Inst. H. Poincare Phys. Theor.* **47** 429 (1987).

- Add an imaginary potential $-iD$ to the Hamiltonian

➡ $K = H - iD$ generates a contraction semigroup $e^{-itK/\hbar}$

- Interpret $1 - \|e^{-itK}\psi\|^2$ as the probability of absorption before time t, of a particle prepared with wave function ψ

Absorbing detector

- Probability for absorption during $[t_1, t_2]$ can be written as $\langle\psi|F([t_1,t_2])\psi\rangle$, where

$$F([t_1,t_2]) := (e^{-it_1K/\hbar})^* e^{-it_1K/\hbar} - (e^{-it_2K/\hbar})^* e^{-it_2K/\hbar}$$

is the time observable (POVM).

- Total absorption probability: $p = 1 - \lim_{t\to\infty} \|e^{-itK/\hbar}\psi\|^2$

- Normalized probability density for absorption:

$$\mathbb{P}(t) = -\frac{1}{p}\frac{d}{dt}\|e^{-itK/\hbar}\psi\|^2$$ the time distribution

Can be determined experimentally from the "click" statistics of a real detector !

The uncertainty relation

$$\langle T \rangle := \int_0^\infty t\,\mathbb{P}(t)dt$$ expectation of absorption time

$$(\Delta T)^2 := \int_0^\infty t^2\,\mathbb{P}(t)dt - \langle T \rangle^2$$ variance of absorption time

$$(\Delta E)^2 := \langle \psi | H^2 \psi \rangle - \langle \psi | H \psi \rangle^2$$ variance of energy

Assumption: $D\psi = 0 \quad (K\psi = H\psi)$

Initial wave function has no overlap with the detector

$$\boxed{\begin{array}{l} \Delta T \Delta E \geq \frac{\hbar}{2}\sqrt{p} \\ \Delta H \cdot \langle T \rangle \geq 1.37\,\hbar\sqrt{p} \end{array}}$$

total absorption probability

Sketch of proof (for $p = 1$)

- Idea: transform the system so that time and energy become conjugate operators.

time distribution

- Find a wave function $J\psi$ such that $\mathbb{P}(t) = \|(J\psi)(t)\|^2$

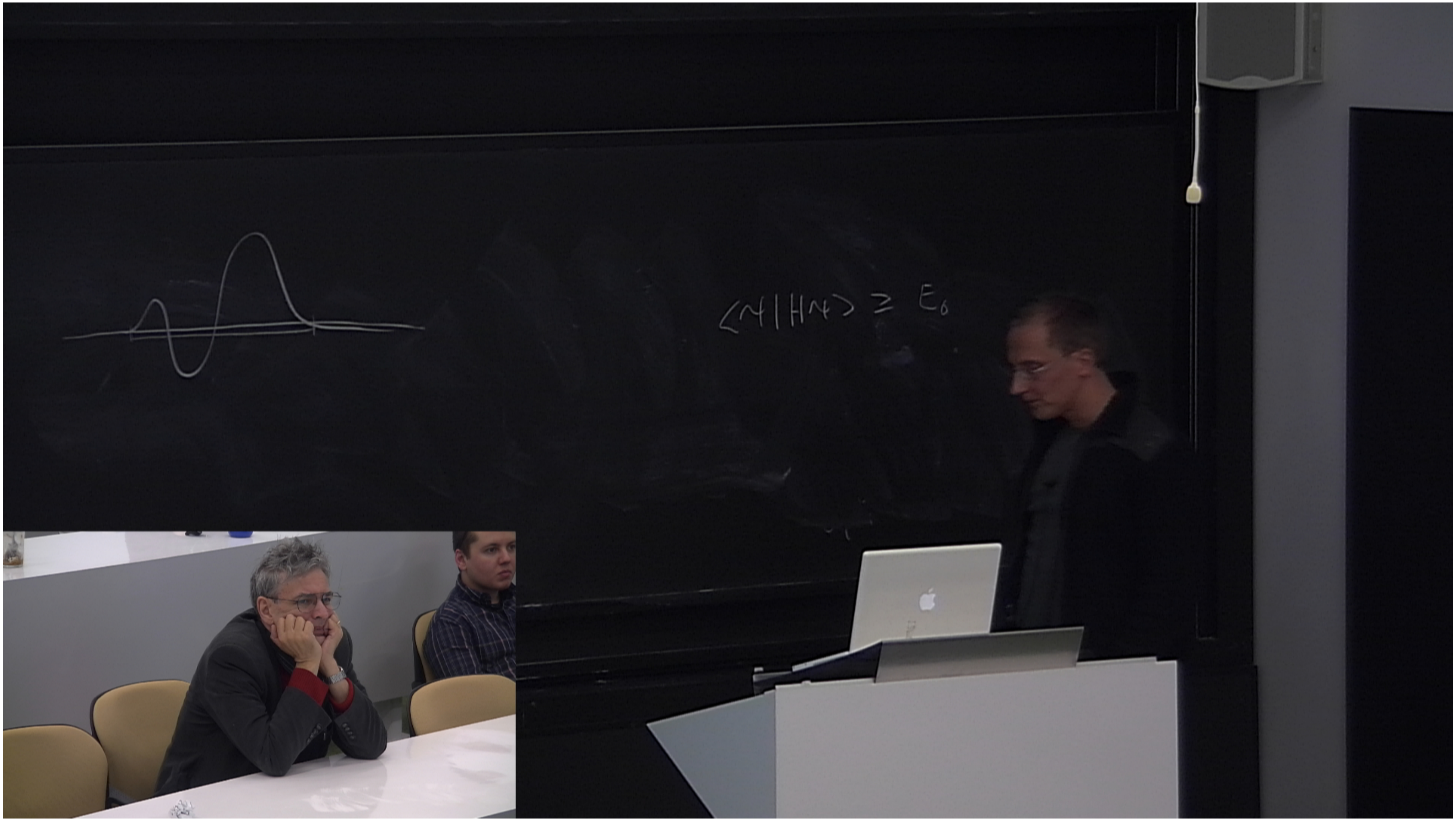
- We take $(J\psi)(t) = \begin{cases} \sqrt{2/\hbar}\,\sqrt{D}\, e^{-itK}\psi, & t \geq 0 \\ 0, & t < 0 \end{cases}$

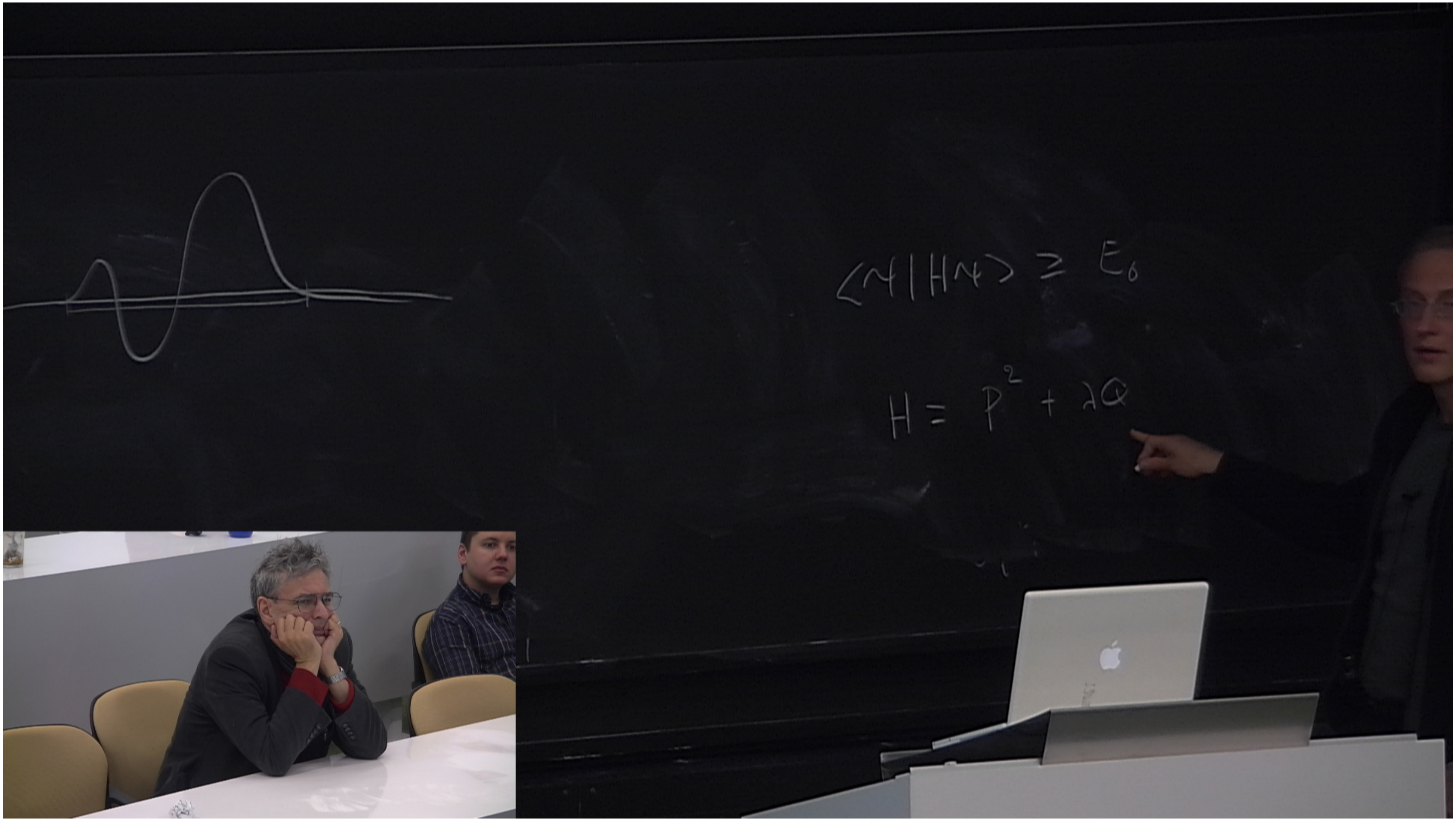
$$\|(J\psi)(t)\|^2 = \frac{2}{\hbar}\langle e^{-itK}\psi | D e^{-itK}\psi\rangle = -\frac{d}{dt}\langle e^{-itK}\psi | e^{-itK}\psi\rangle = \mathbb{P}(t)$$

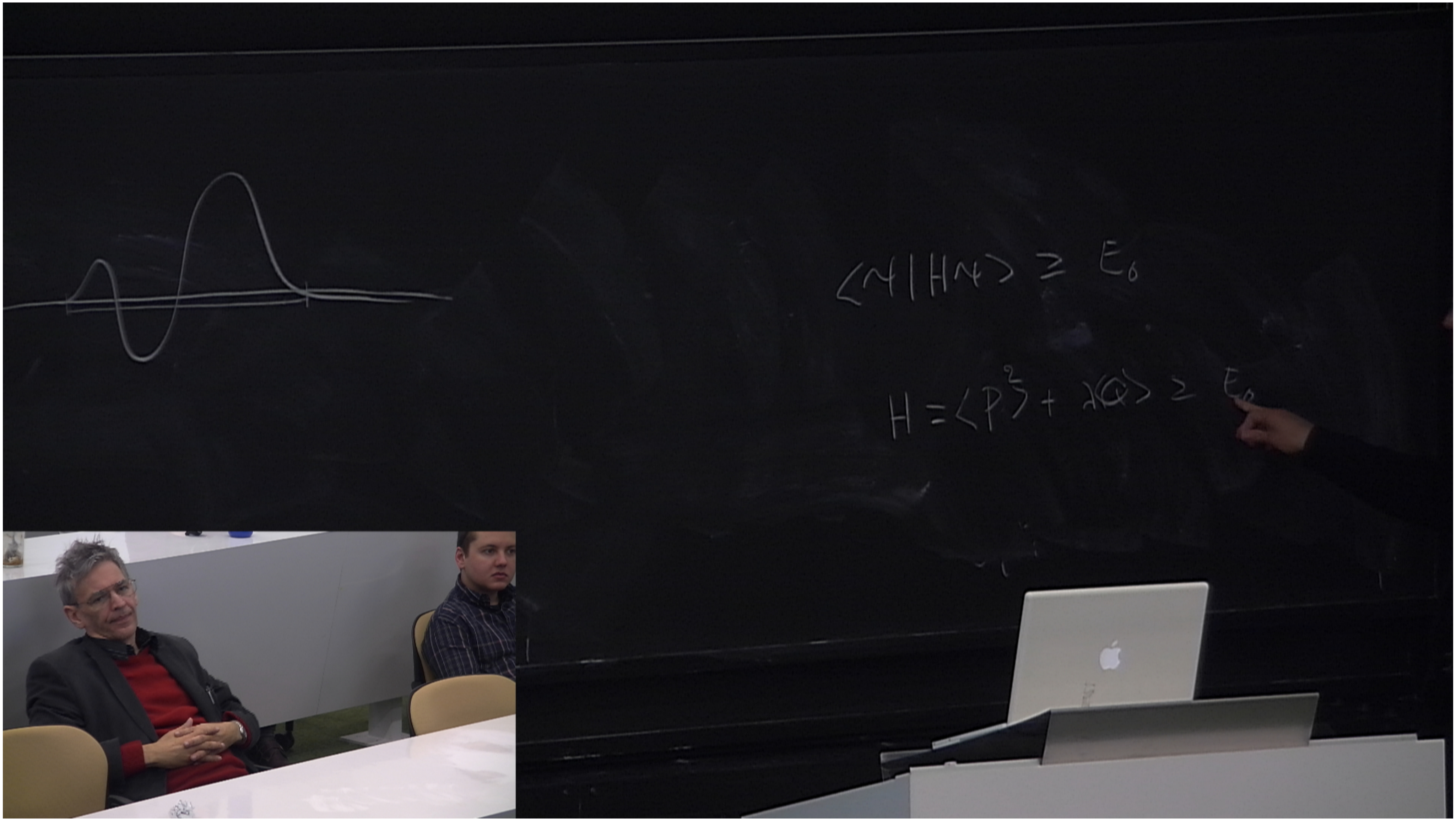
$$J^* J = 1$$

➡ $J\psi$ is the wave function in the "time space" $L^2(\mathbb{R}) \otimes \mathcal{H}$

- Assumption $D\psi = 0$ ⇨ no jump at zero


$\langle\psi|H\psi\rangle \geq E_0$


⟨ψ|Hψ⟩ ≥ E0
H = P2 + λQ


$\langle \psi | H \psi \rangle \geq E_0$
$H = \langle P^2 \rangle + \lambda \langle Q \rangle \geq E_0$

Sketch of proof ($p=1$)

Compute $\hat{H}J\psi$ (well-defined because $J\psi$ has no jump at zero):

$D\psi = 0$

$$(\hat{H}J\psi)(t) = i\hbar\sqrt{2/\hbar}\,\sqrt{D}\,e^{-itK/\hbar}(-iK/\hbar)\psi = (JK\psi)(t) = (JH\psi)(t)$$

$\Rightarrow$ $\hat{H}J\psi = JH\psi$

$J^*J = 1$

$$(\Delta\hat{H})^2 = \|\hat{H}J\psi\|^2 - \langle J\psi|\hat{H}J\psi\rangle^2 = \|H\psi\|^2 - \langle\psi|H\psi\rangle^2 = (\Delta E)^2$$

$$\langle\hat{T}\rangle = \int t\,\underbrace{\|(J\psi)(t)\|^2}_{\mathbb{P}(t)} = \langle T\rangle$$

$$(\Delta\hat{T})^2 = \int t^2\,\underbrace{\|(J\psi)(t)\|^2}_{\mathbb{P}(t)}\,dt - \langle\hat{T}\rangle^2 = (\Delta T)^2$$

$\Rightarrow$ $\boxed{\Delta E\Delta T \geq \frac{\hbar}{2} \qquad \Delta E\cdot\langle T\rangle \geq 1.37\hbar}$

Understanding case $p < 1$

- Why does the square root $\sqrt{p}$ of the absorption probability appear in the UR?

$$\boxed{p = 1 - \lim_{t\to\infty} \|e^{-itK}\psi\|^2}$$

- Start with a system $\mathcal{H}, H, D, \psi$ with $p = 1$
- A new system:

$$\boxed{\begin{aligned} \mathcal{H}' &= \mathcal{H} \oplus \mathcal{H}_0, \\ H' &= H \oplus \langle\psi|H\psi\rangle\, |\phi_0\rangle\langle\phi_0|, \\ D' &= D \oplus 0 \\ \psi' &= \sqrt{\lambda}\psi \oplus \sqrt{1-\lambda}\phi_0 \end{aligned}}$$

(Add to the Hilbert space a part not seen by the detector)

arbitrary parameter $\in [0, 1]$

semigroup: $e^{-itK'/\hbar}\psi' = \sqrt{\lambda}e^{-itK/\hbar}\psi \oplus \sqrt{1-\lambda}\, e^{-it\langle\psi|H\psi\rangle|\phi_0\rangle\langle\phi_0|/\hbar}\phi_0 \Rightarrow p' = \lambda$

time distribution: $\mathbb{P}'(t) = -\frac{1}{p'}\frac{d}{dt}\|e^{-itK'/\hbar}\psi'\|^2 = \mathbb{P}(t) \Rightarrow \langle T\rangle' = \langle T\rangle$, $\Delta T' = \Delta T$

new energy moments:

$\langle\psi'|H'\psi'\rangle = \langle\psi|H\psi\rangle$

$\|H'\psi'\|^2 = p'\|H\psi\|^2 + (1-p')\langle\psi|H\psi\rangle^2 \Rightarrow (\Delta E')^2 := \|H'\psi'\|^2 - \langle\psi'|H'\psi'\rangle^2 = p'(\Delta E)^2$

➡ For the new system: $\Delta E'\Delta T' \geq (\hbar/2)\sqrt{p}$

Minimum uncertainty

- For the standard QP UR we can find the distributions that saturate the UR.
- We can do the same for energy-time UR...
- The time distributions $\mathbb{P}_{\min}(t)$ saturating

  $\Delta\hat{H}\Delta\hat{T} \geq (\hbar/2)\sqrt{p}$ are just given by gaussians
- The corresponding ones for

  $\Delta\hat{H} \cdot \langle\hat{T}\rangle \geq C\hbar\sqrt{p}$ are given by the Airy function
- Open question: how does the wave function look in the position space?

Simplest example

- Hamiltonian: $H = \frac{\hbar}{2}\begin{pmatrix} 0 & \Omega \\ \Omega & 0 \end{pmatrix}$

For instance: Rabi oscillations for a laser driven atomic transition

- Initial state (lower level): $\psi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
- Absorption at the upper level $D = \frac{\hbar}{2}\begin{pmatrix} 0 & 0 \\ 0 & \gamma \end{pmatrix}$
- Compute $\mathbb{P}(t) = -\frac{d}{dt}\|e^{-itK/\hbar}\psi\|^2$ to get ΔT
- How close to the minimal uncertainty can we get?
- $\Delta E = \frac{\hbar}{2}\Omega$, ΔT smallest when

$\gamma = \sqrt{2}\Omega \Rightarrow$

$$\Delta E \Delta T = \frac{\hbar}{\sqrt{2}} \approx 0.707\hbar > 0.500\hbar$$

$$\Delta E \cdot \langle T \rangle = \sqrt{2}\,\hbar \approx 1.41\hbar > 1.37$$