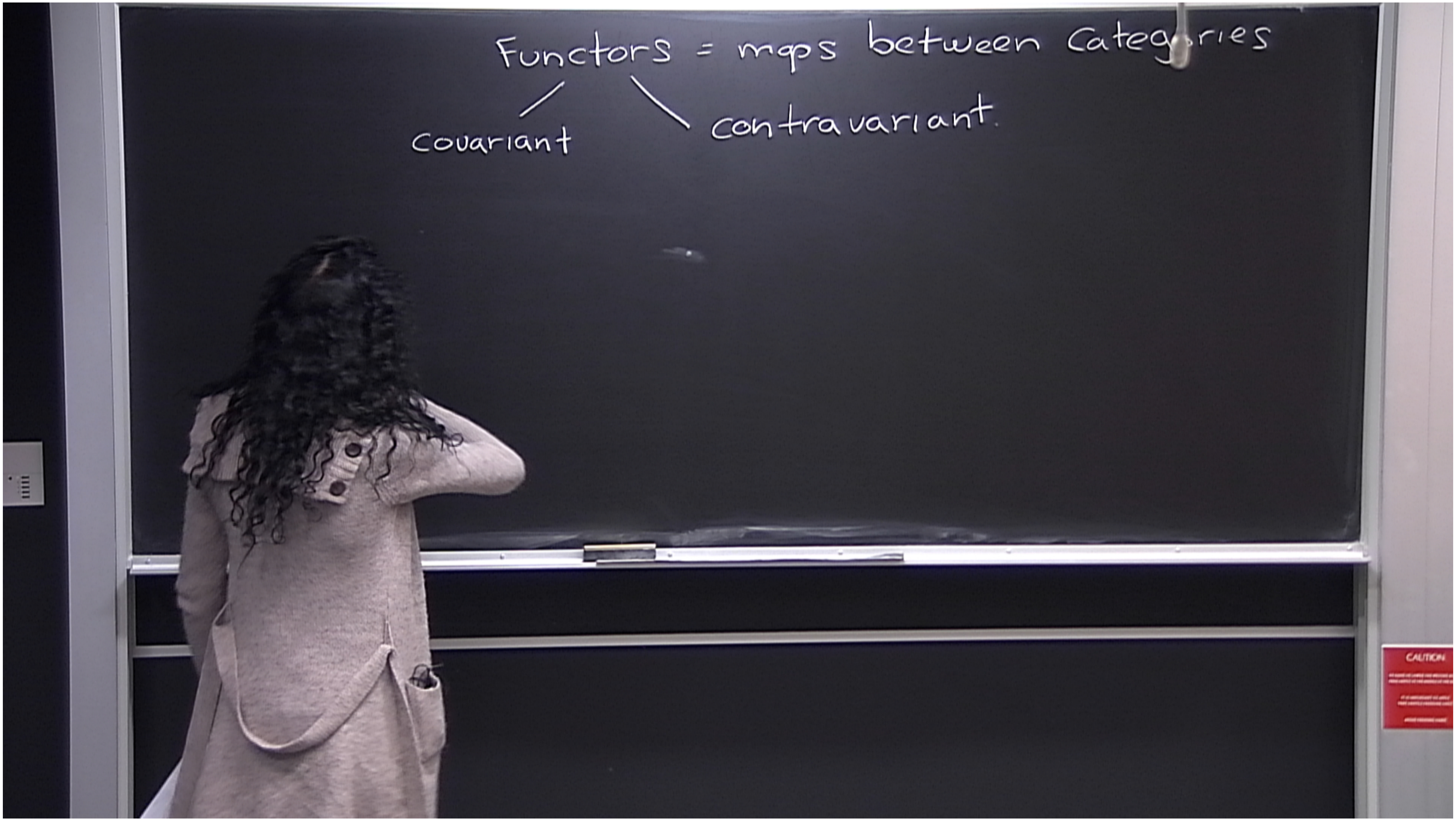
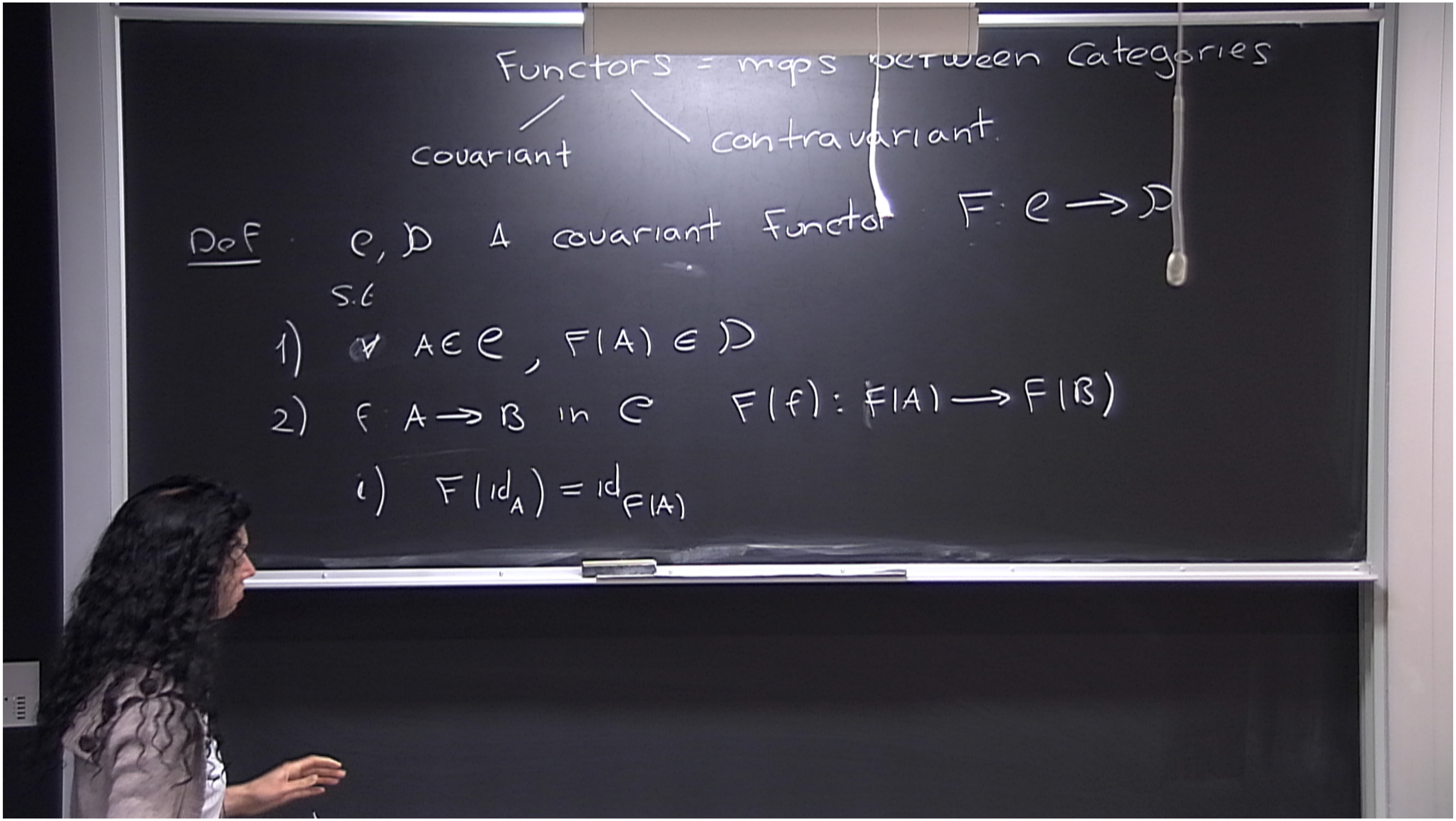
Title: Topos Quantum Physics - Lecture 4

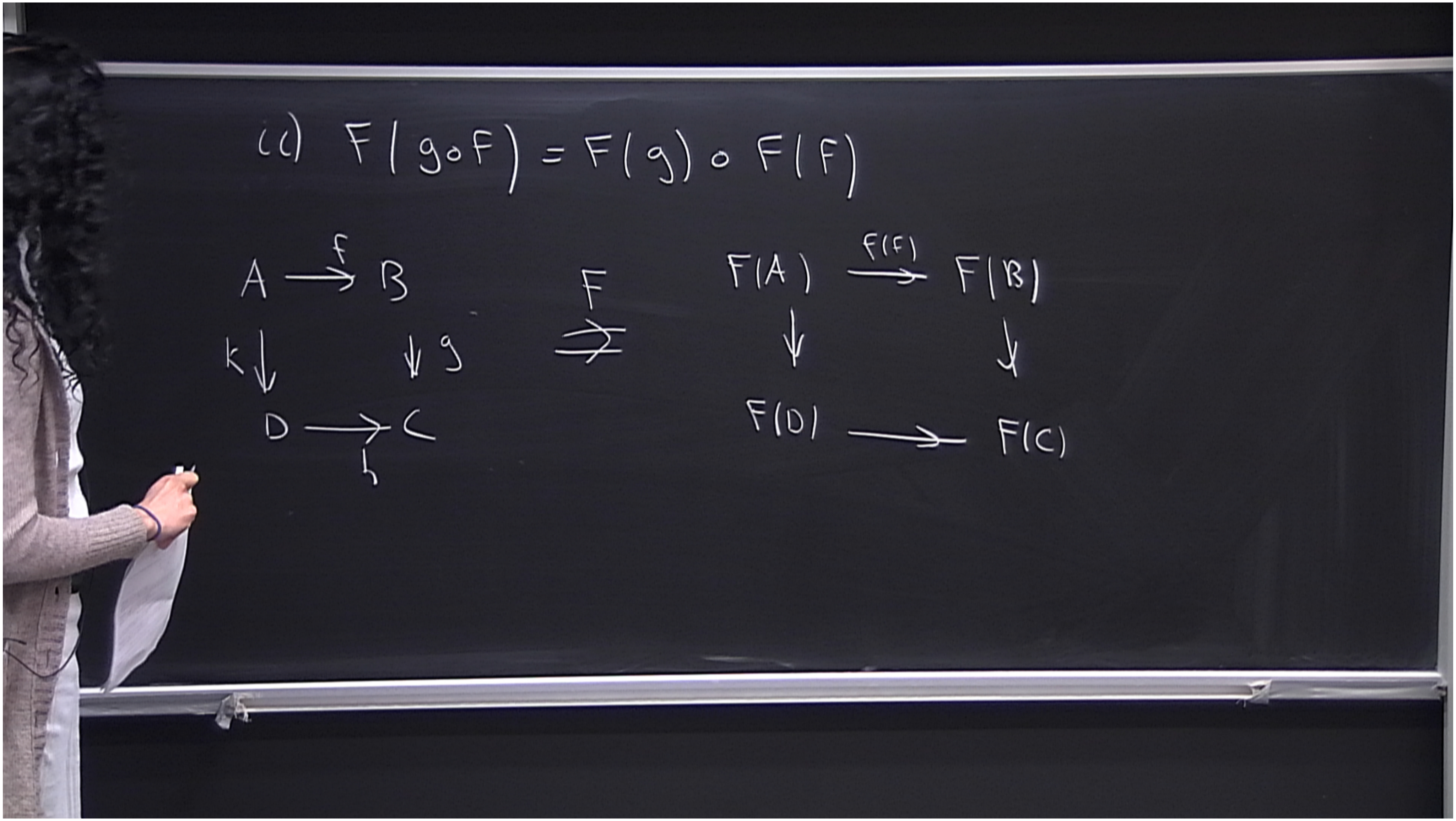
Date: Jan 16, 2012 01:30 PM

URL: http://pirsa.org/12010142

Abstract:


Functors = maps between Categ ries
covariant
contravariant
CAUTION

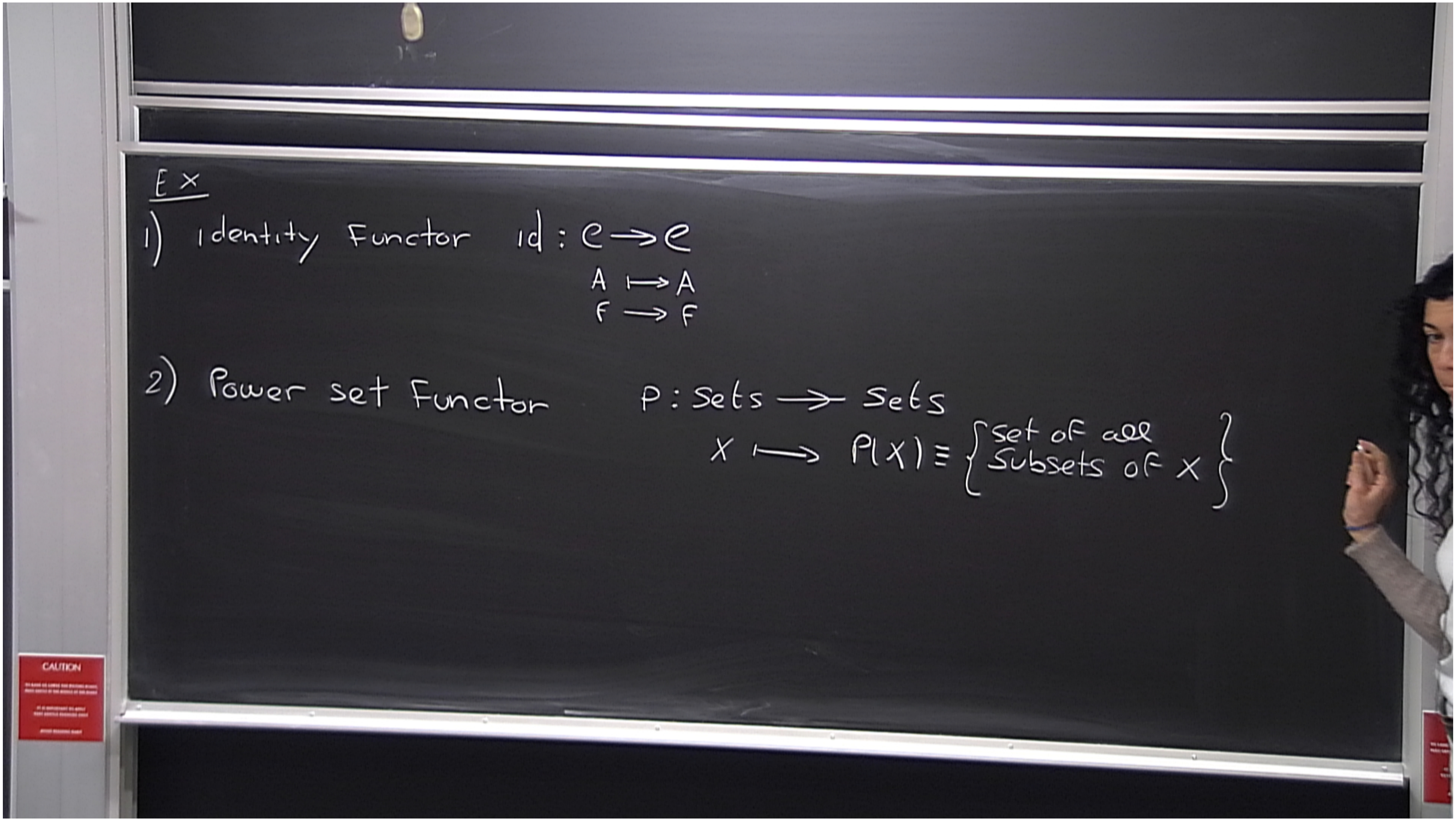

Functors = maps between Categories
covariant
contravariant.
Def: $\mathcal{C}, \mathcal{D}$ A covariant Functor $F: \mathcal{C} \longrightarrow \mathcal{D}$
s.t.
1) $\forall A \in \mathcal{C}, F(A) \in \mathcal{D}$
2) $f: A \to B$ in $\mathcal{C}$ $F(f): F(A) \longrightarrow F(B)$
i) $F(id_A) = id_{F(A)}$

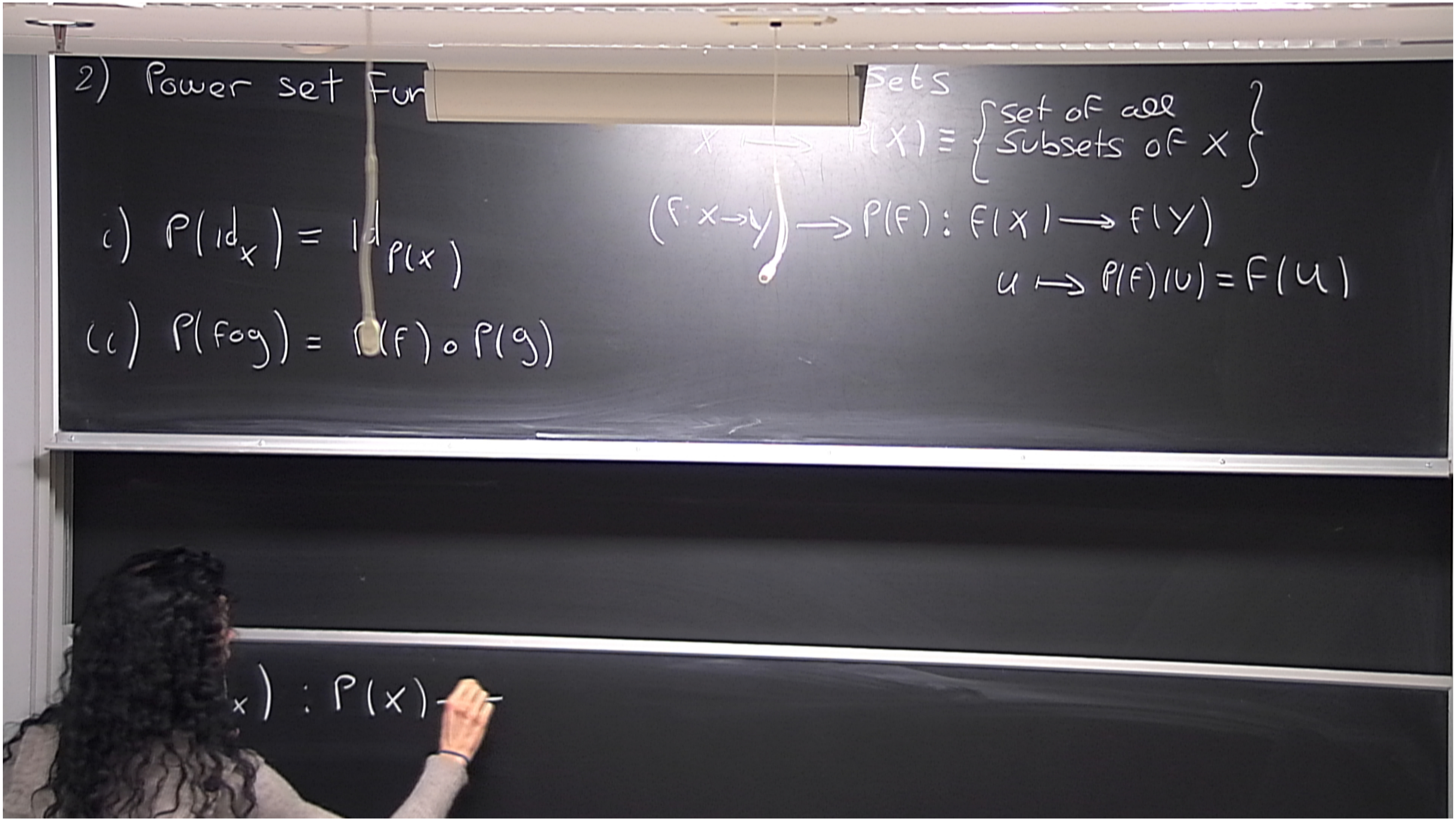

ii) F(g∘f) = F(g) ∘ F(f)
A →f B
k ↓ ↓ g
D →h C
F
F(A) →F(f) F(B)
F(D) → F(C)

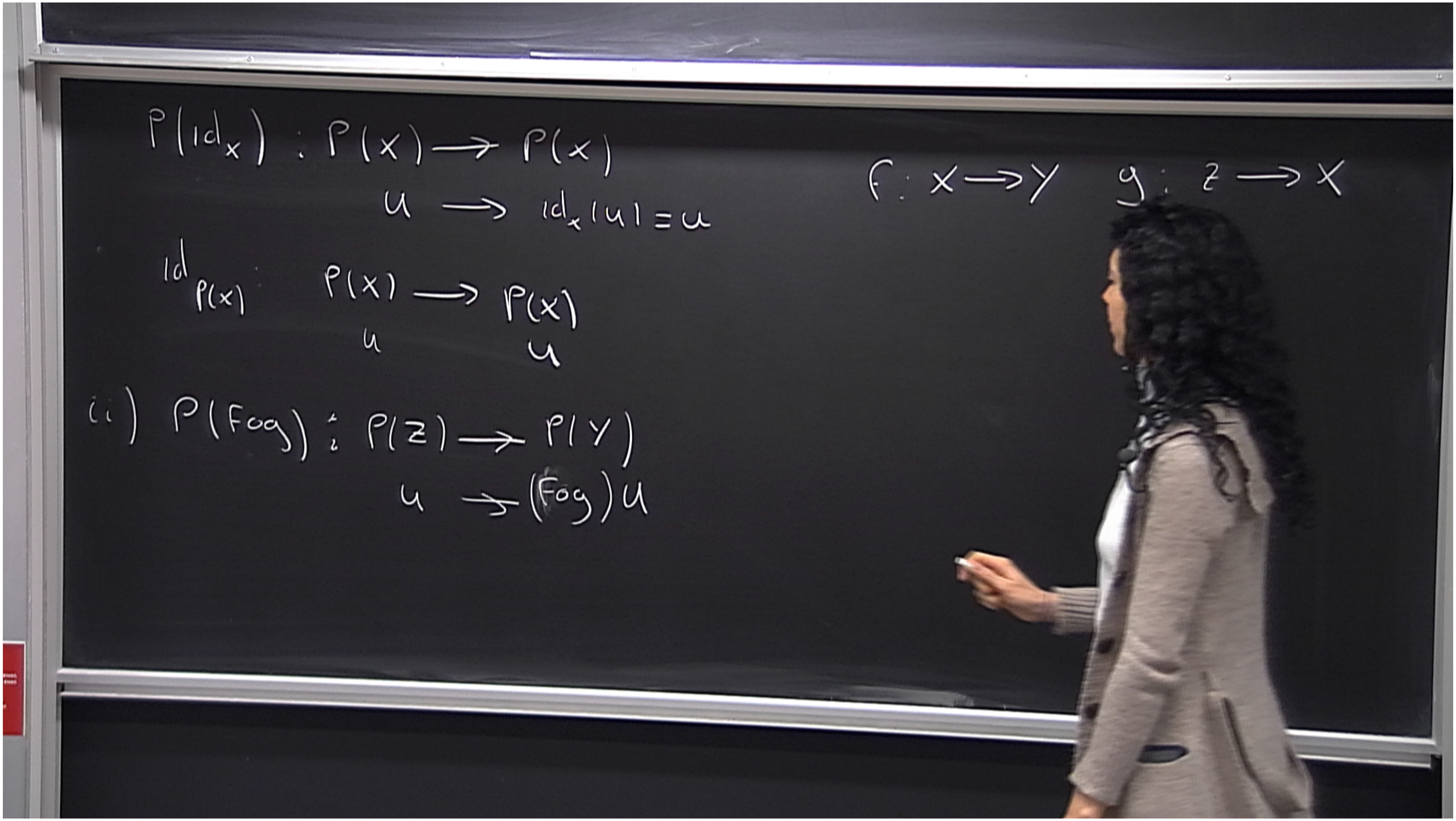
Ex

1) identity Functor $id : \mathcal{C} \to \mathcal{C}$

$$A \mapsto A$$
$$f \to f$$


Ex
1) identity Functor $id : \mathcal{C} \to \mathcal{C}$
$A \mapsto A$
$f \longrightarrow f$
2) Power set Functor $P : \text{Sets} \to \text{Sets}$
$X \longmapsto P(X) \equiv \{\text{set of all Subsets of } X\}$


2) Power set Fun
Sets
X ⟼ P(X) ≡ {set of all subsets of X}
(f: X→Y) ⟶ P(f): P(X) ⟶ P(Y)
U ⟼ P(f)(U) = f(U)
i) P(id_X) = id_P(X)
ii) P(f∘g) = P(f)∘P(g)
x) : P(X)


$P(id_X) : P(X) \longrightarrow P(X)$
$u \longrightarrow id_X(u) = u$
$f : X \longrightarrow Y$ $g : Z \longrightarrow X$
$id_{P(X)}$ $P(X) \longrightarrow P(X)$
u u
ii) $P(F \circ g) : P(Z) \longrightarrow P(Y)$
$u \longrightarrow (F \circ g)u$

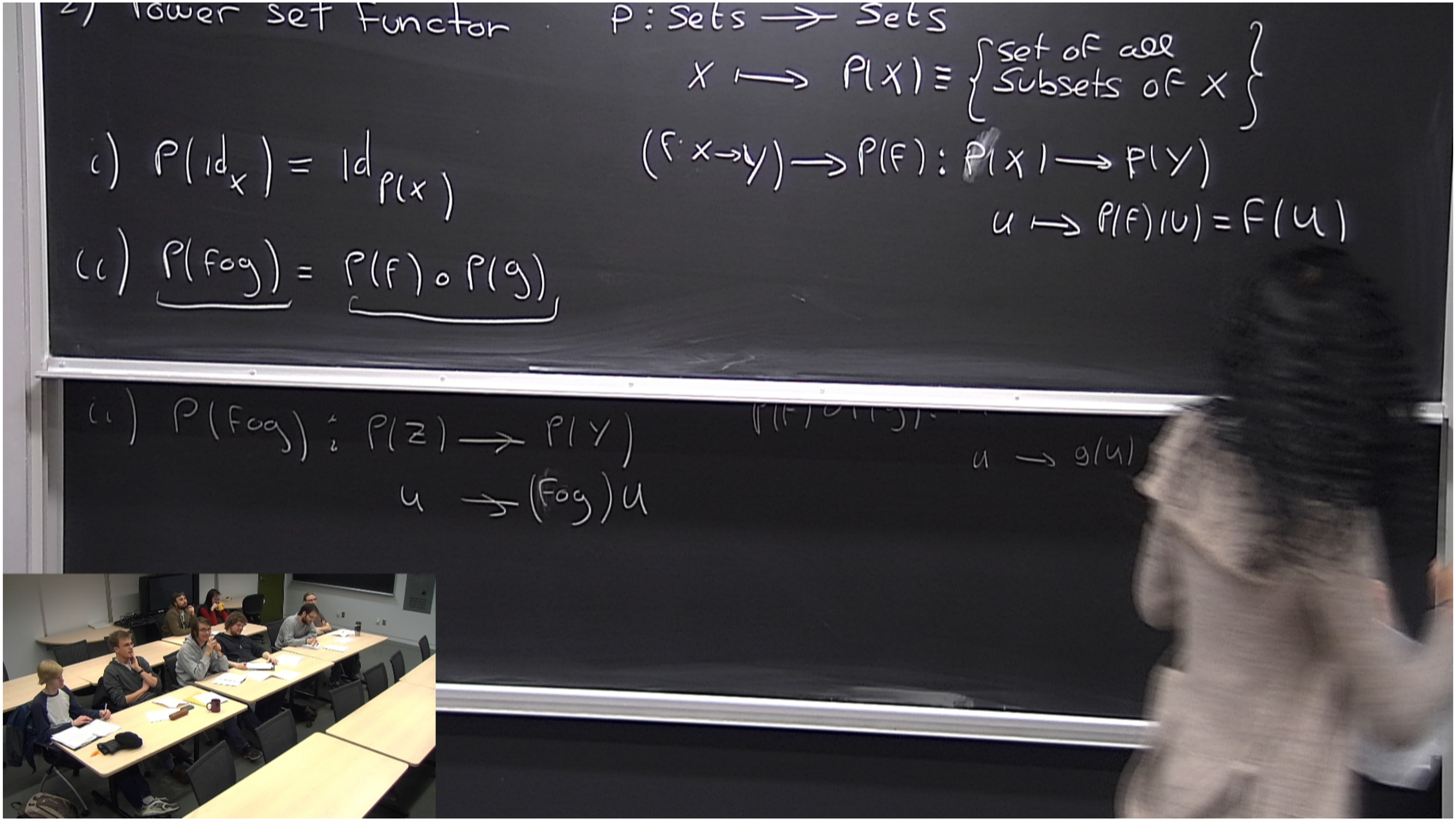
$$P(id_X) : P(X) \longrightarrow P(X)$$
$$u \longrightarrow id_X(u) = u$$

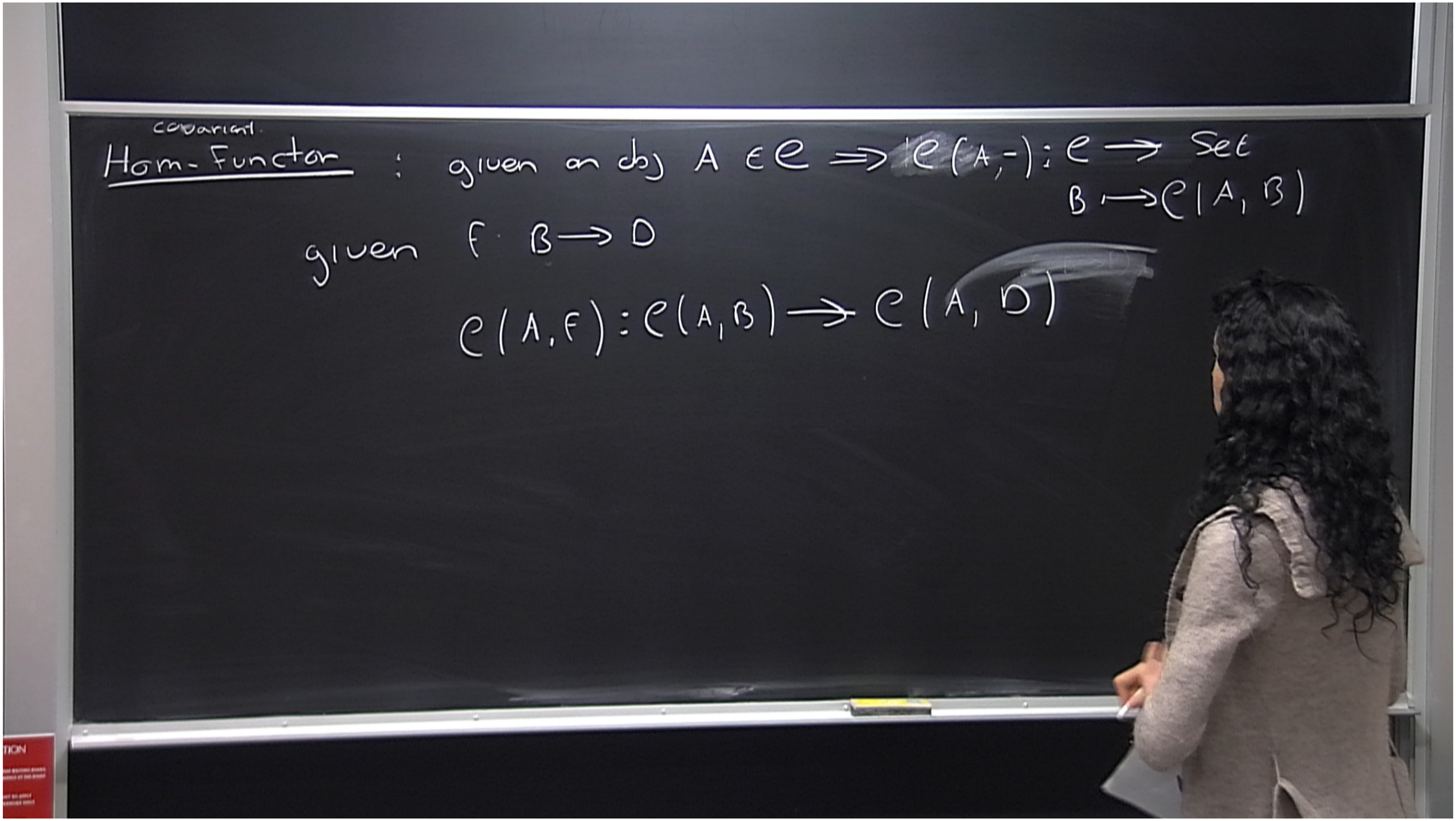
$$id_{P(X)} : P(X) \longrightarrow P(X)$$
$$u \qquad u$$

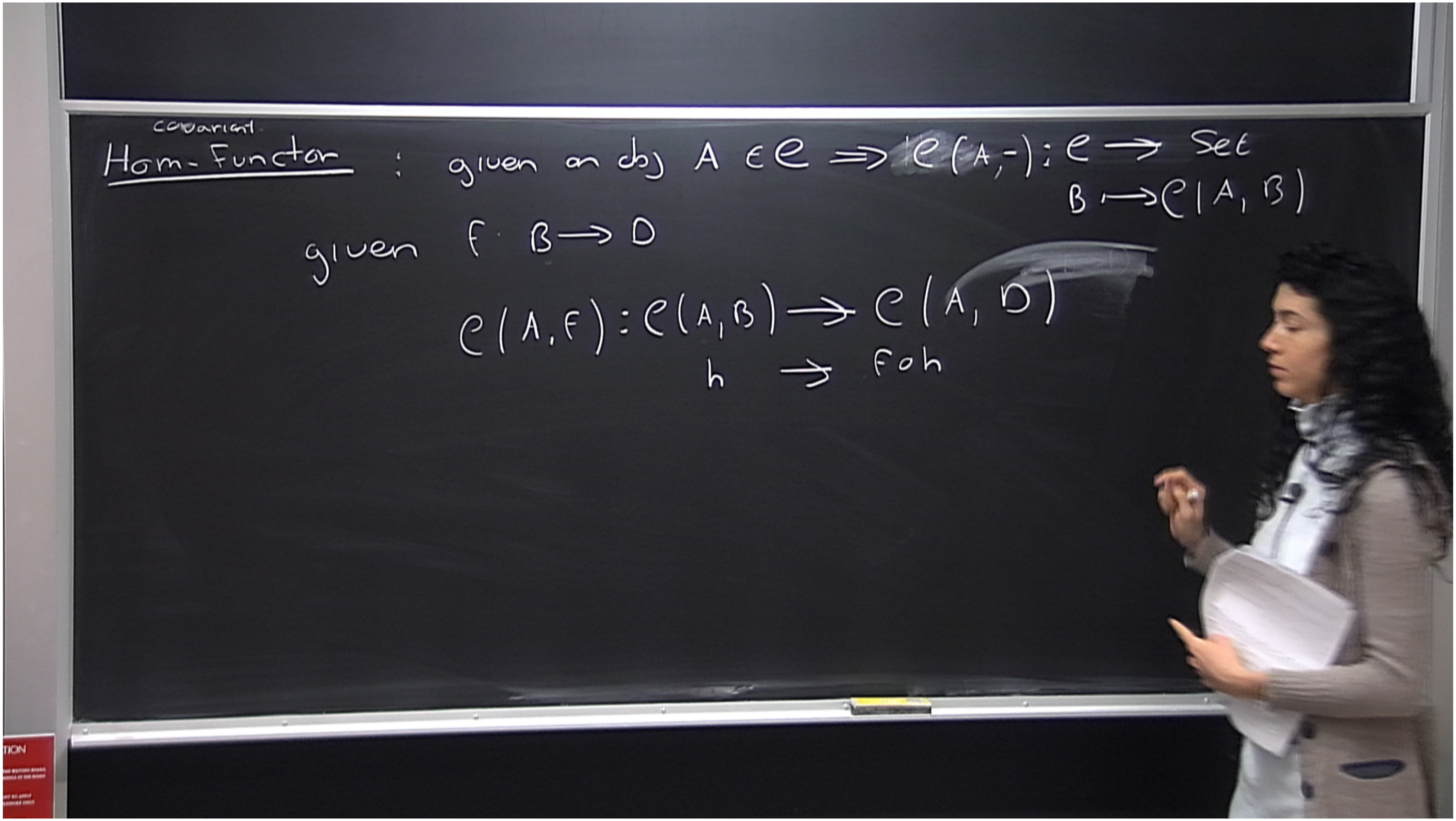
$$f : X \longrightarrow Y \qquad g : Z \longrightarrow X$$

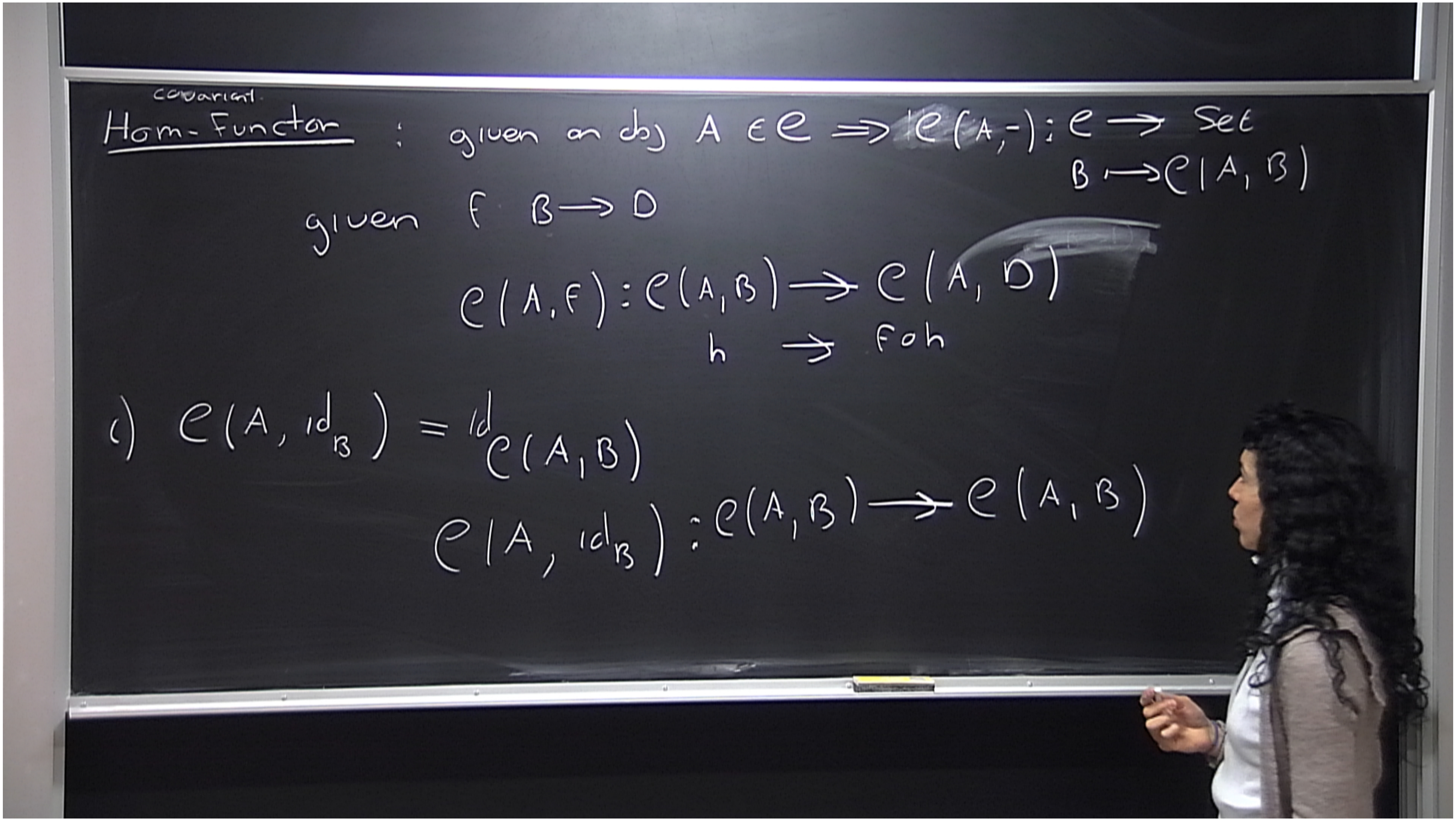
ii) $$P(f \circ g) : P(Z) \longrightarrow P(Y)$$
$$u \longrightarrow (f \circ g) u$$

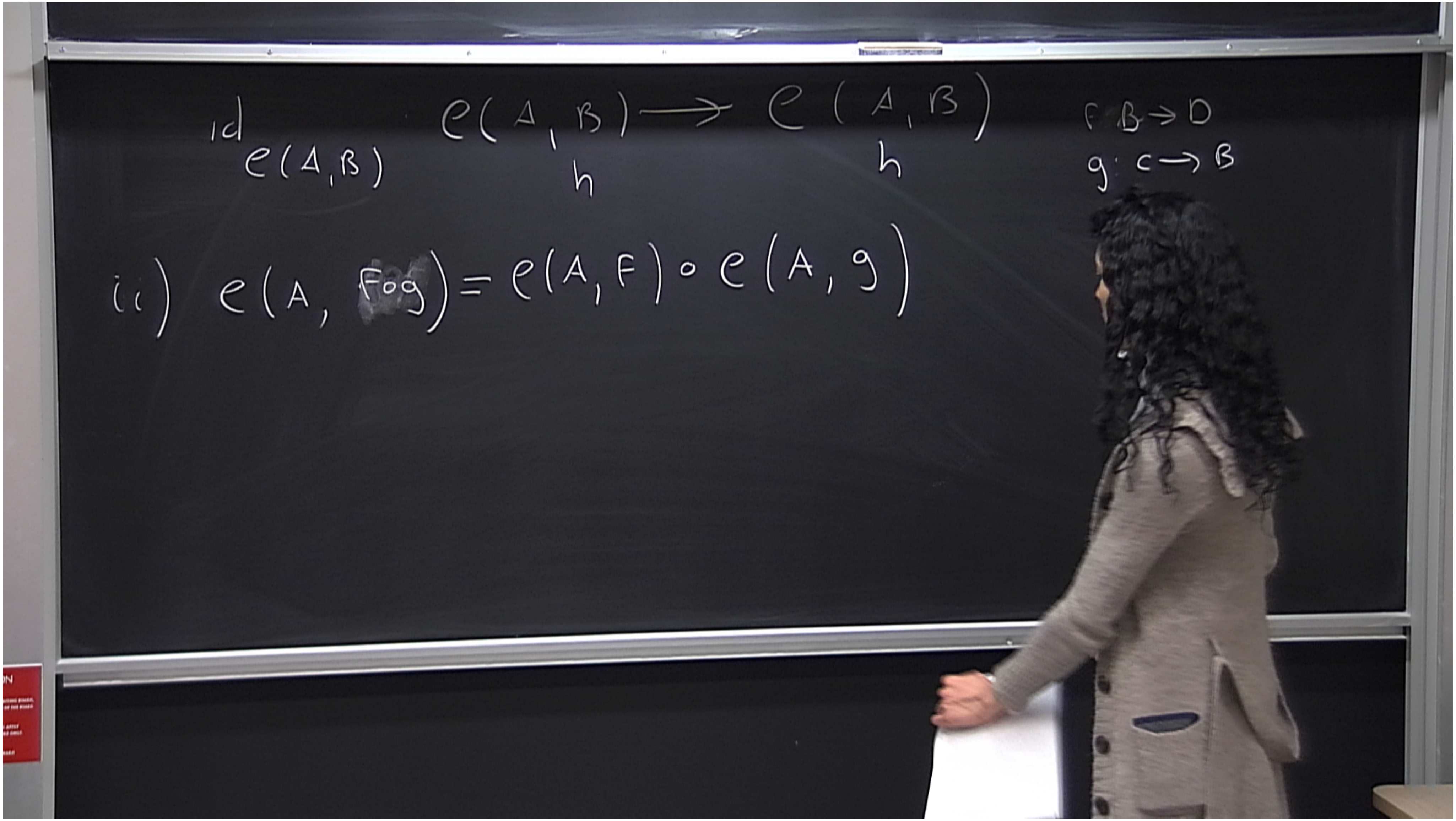
$$P(f) \circ P(g) : P(Z) \longrightarrow P(X) \longrightarrow P(Y)$$
$$u \longrightarrow g(u) \longrightarrow f(g(u))$$

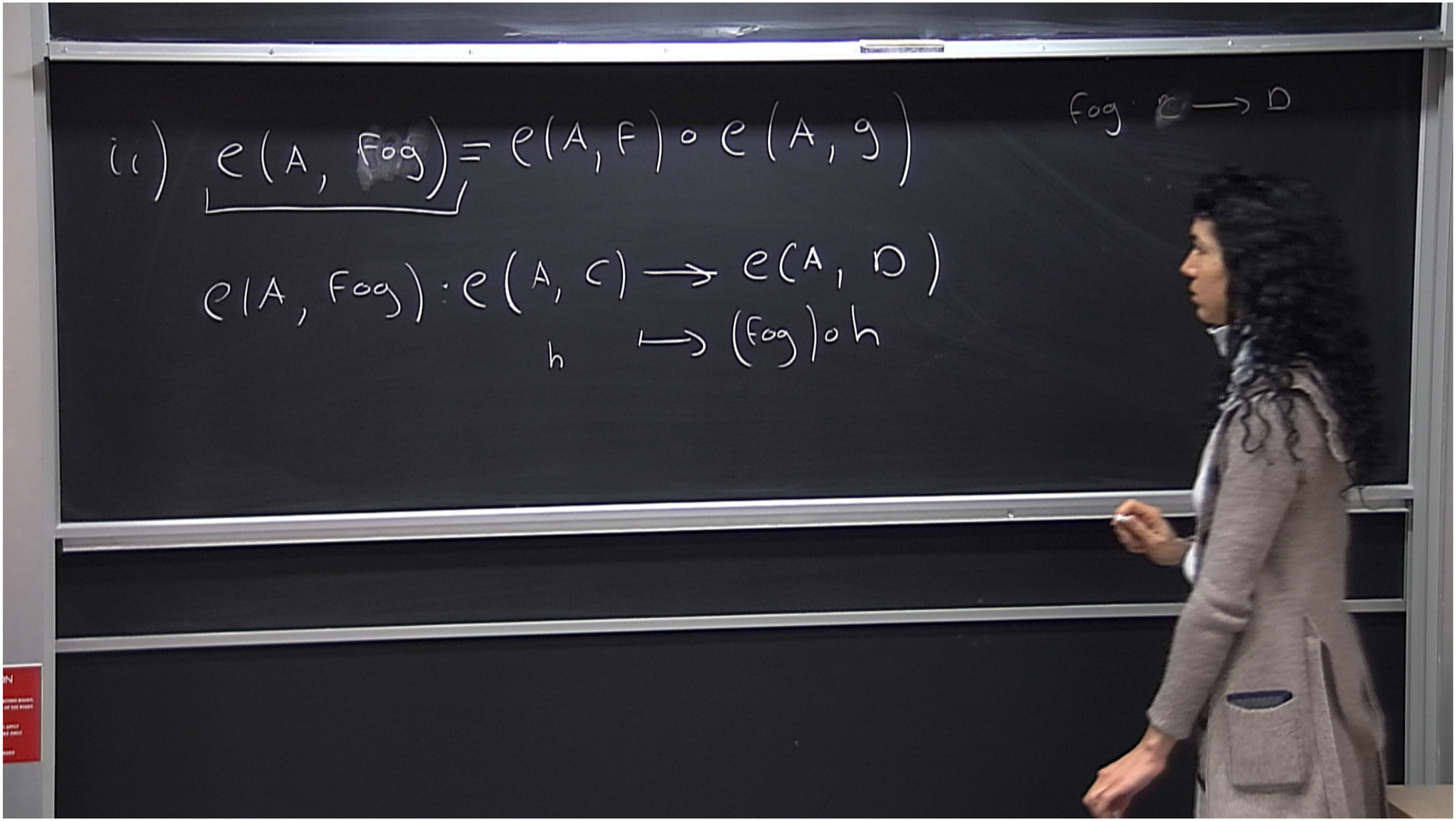

ower set functor
P : Sets → Sets
X ↦ P(X) ≡ {set of all subsets of X}
(f: X→Y) ↦ P(f) : P(X) → P(Y)
U ↦ P(f)(U) = f(U)
i) P(id_X) = id_P(X)
ii) P(f∘g) = P(f)∘P(g)
ii) P(f∘g) : P(Z) → P(Y)
U → (f∘g)U
U → g(U)


covariant
Hom-Functor : given an obj $A \in \mathcal{C} \Rightarrow \mathcal{C}(A,-) : \mathcal{C} \to \mathrm{Set}$
$B \mapsto \mathcal{C}(A,B)$
given $f : B \to D$
$\mathcal{C}(A,f) : \mathcal{C}(A,B) \to \mathcal{C}(A,D)$


covariant
Hom-Functor : given an obj A ∈ C ⇒ C(A,-) : C → Set
B ↦ C(A,B)
given f : B → D
C(A,f) : C(A,B) → C(A,D)
h → f∘h


covariant
Hom-Functor : given an obj $A \in \mathcal{C} \Rightarrow \mathcal{C}(A,-) : \mathcal{C} \to Set$
$B \mapsto \mathcal{C}(A,B)$
given $f: B \to D$
$\mathcal{C}(A,f) : \mathcal{C}(A,B) \to \mathcal{C}(A,D)$
$h \mapsto f \circ h$
i) $\mathcal{C}(A, id_B) = id_{\mathcal{C}(A,B)}$
$\mathcal{C}(A, id_B) : \mathcal{C}(A,B) \to \mathcal{C}(A,B)$


id e(A,B)
e(A, B) ⟶ e(A, B)
h h
f: B→D
g: C⟶B
ii) e(A, f∘g) = e(A, f) ∘ e(A, g)



$f \circ g : C \longrightarrow D$
ii) $e(A, f \circ g) = e(A, f) \circ e(A, g)$
$e(A, f \circ g) : e(A, C) \longrightarrow e(A, D)$
$h \longmapsto (f \circ g) \circ h$

$$\mathrm{id}_{\mathcal{C}(A,B)} \,:\, \mathcal{C}(A,B) \longrightarrow \mathcal{C}(A,B)$$

$$h \qquad\qquad h$$

$f : B \to D$

$g : C \longrightarrow B$

$f \circ g : C \longrightarrow D$

$$\text{ii)} \quad \underline{\mathcal{C}(A, f \circ g)} = \mathcal{C}(A, f) \circ \mathcal{C}(A, g)$$

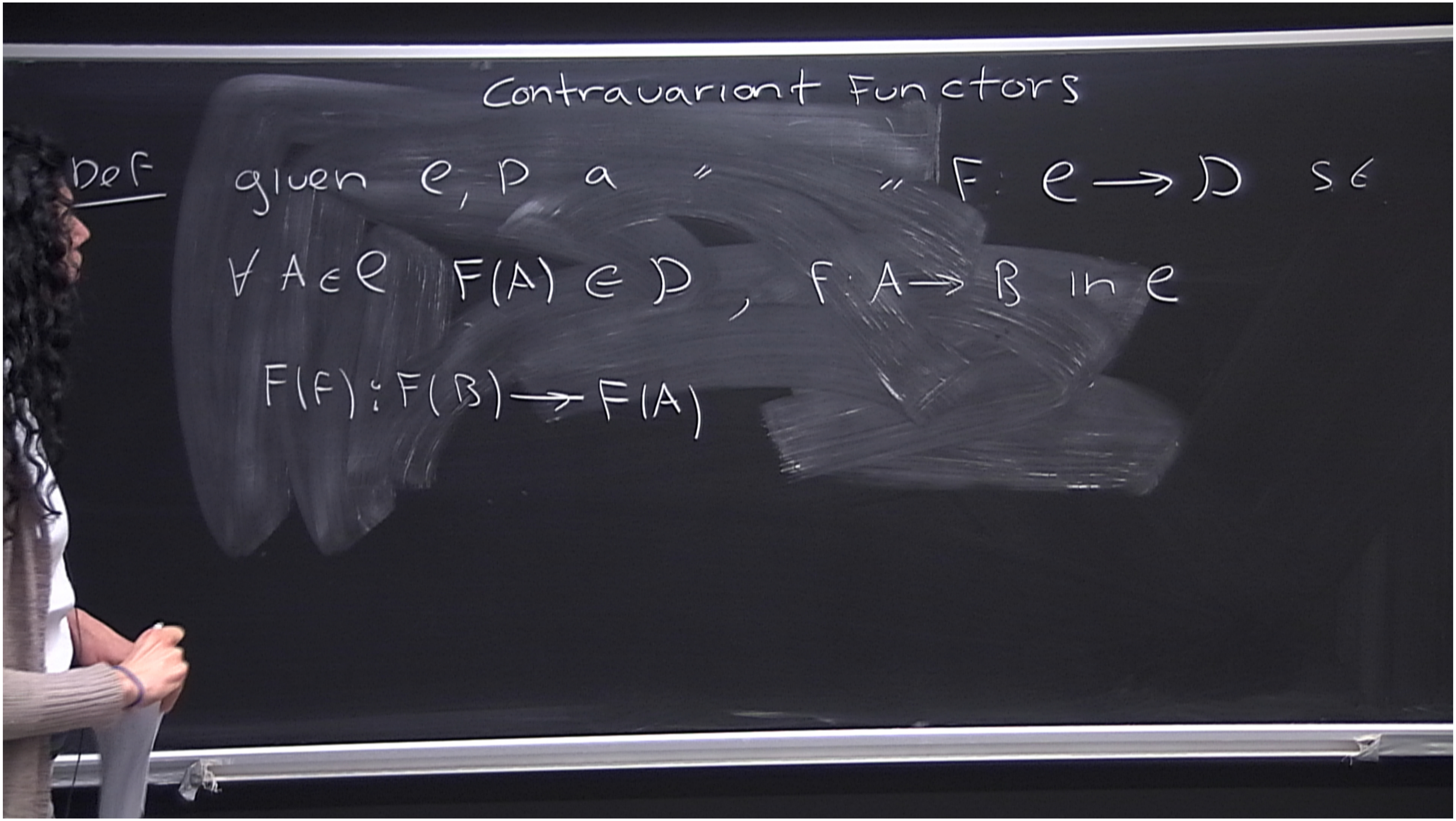
$$\mathcal{C}(A, f \circ g) : \mathcal{C}(A, C) \longrightarrow \mathcal{C}(A, D)$$

$$h \longmapsto (f \circ g) \circ h$$

$$e(A,f)\circ e(A,g) : e(A,C) \to e(A,B) \to e(A,D)$$

$$h \to (g\circ h) \to f\circ(g\circ h)$$

by associativity $f\circ(g\circ h) = (f\circ g)\circ h$


Contravariant Functors
Def given C, D a " " F: C → D s.t.
∀ A ∈ C F(A) ∈ D, f: A → B in C
F(f): F(B) → F(A)

Contravariant Functors

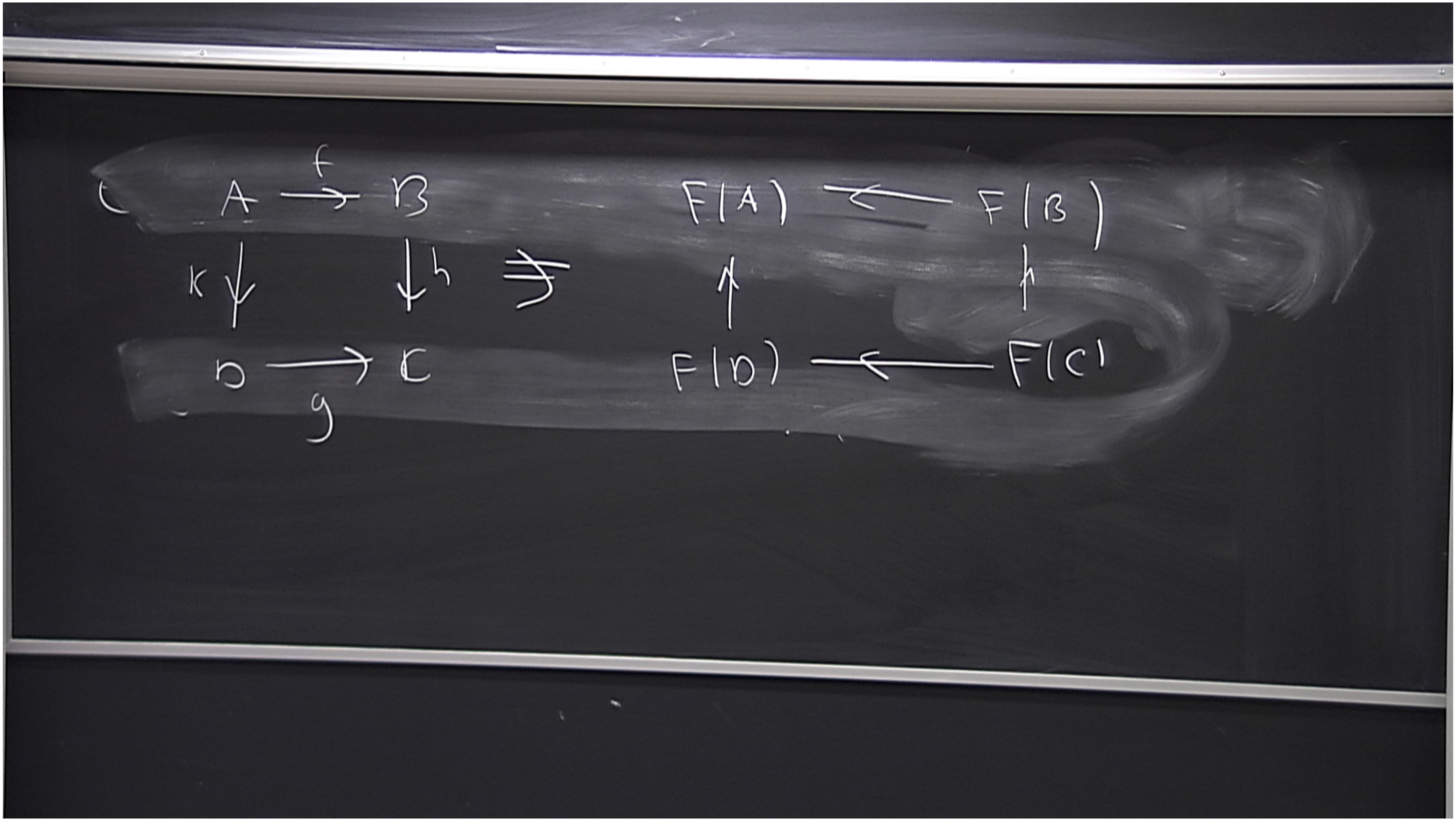
Def given $\mathcal{C}, \mathcal{D}$ a " " $F: \mathcal{C} \to \mathcal{D}$ s.t.

$\forall A \in \mathcal{C}$ $F(A) \in \mathcal{D}$, $f: A \to B$ in $\mathcal{C}$

$F(f): F(B) \to F(A)$

i) $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$

ii) $F(g \circ h) = F(h) \circ F(g)$


f
A B
k h
D C
g
F(A) F(B)
F(D) F(C)

<u>ex</u> Dual power set

$$\hat{P} : \text{Sets} \longrightarrow \text{Sets}$$

$$X \longmapsto \hat{P}(X) = P(X)$$

$$(f : X \to Y) \longrightarrow \hat{P}(f) : P(Y) \longrightarrow P(X)$$

$$U \longrightarrow \hat{P}(f)(U) := f^{-1}(U)$$

$$\hat{P}(\mathrm{id}_X) = \mathrm{id}_{\hat{P}(X)}$$

$$F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$$

$$X \longmapsto \tilde{P}($$

$$(f : X \to Y) \longmapsto \tilde{P}(f) : P(Y) \longrightarrow P(X)$$

$$U \longmapsto \tilde{P}(f)(U) := f^{-1}(U)$$

i) $\tilde{P}(\mathrm{id}_X) = \mathrm{id}_{\tilde{P}(X)}$

ii) $\tilde{P}(f \circ g) = \tilde{P}(g) \circ \tilde{P}(f)$

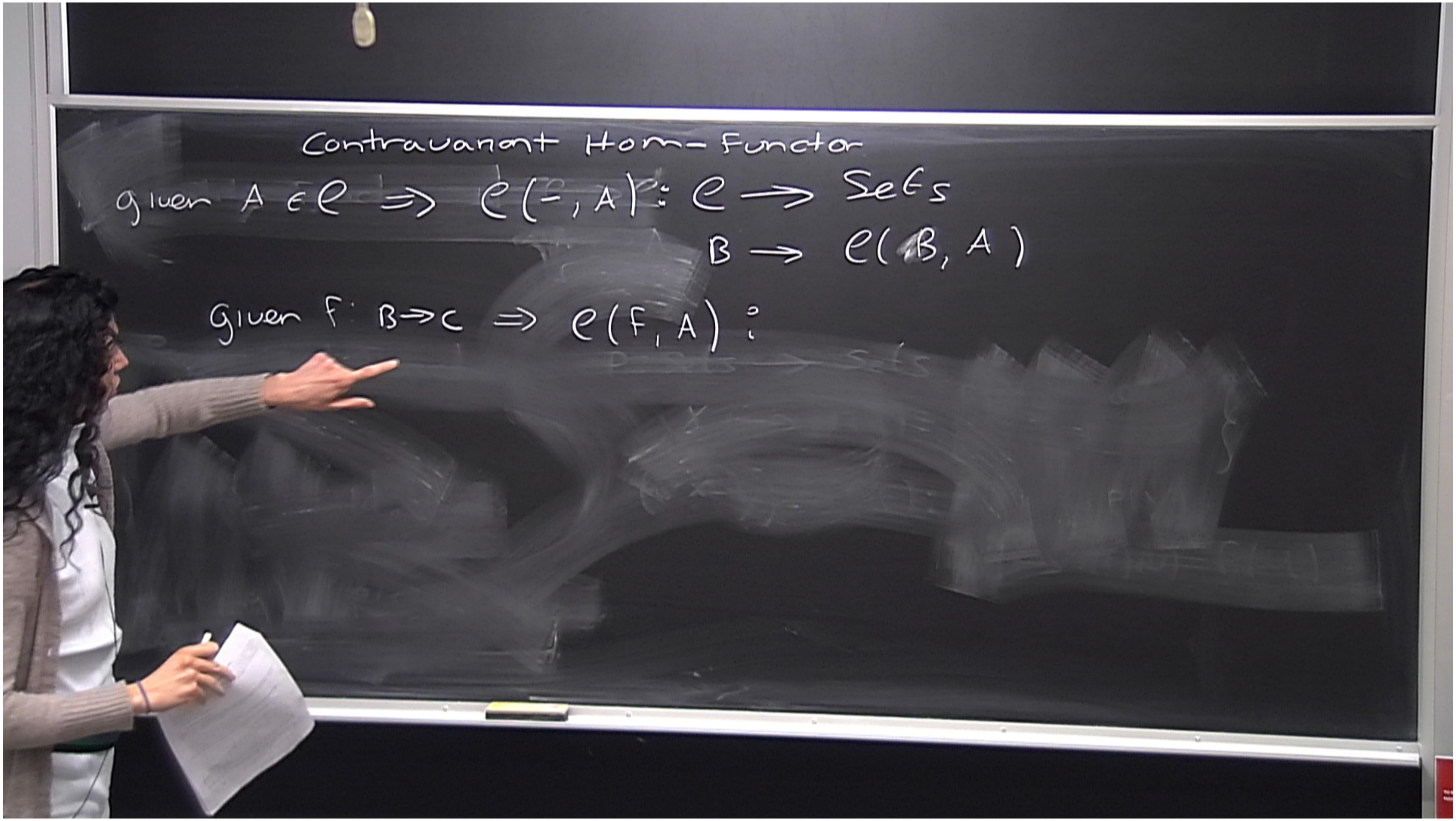
Contravariant Functors

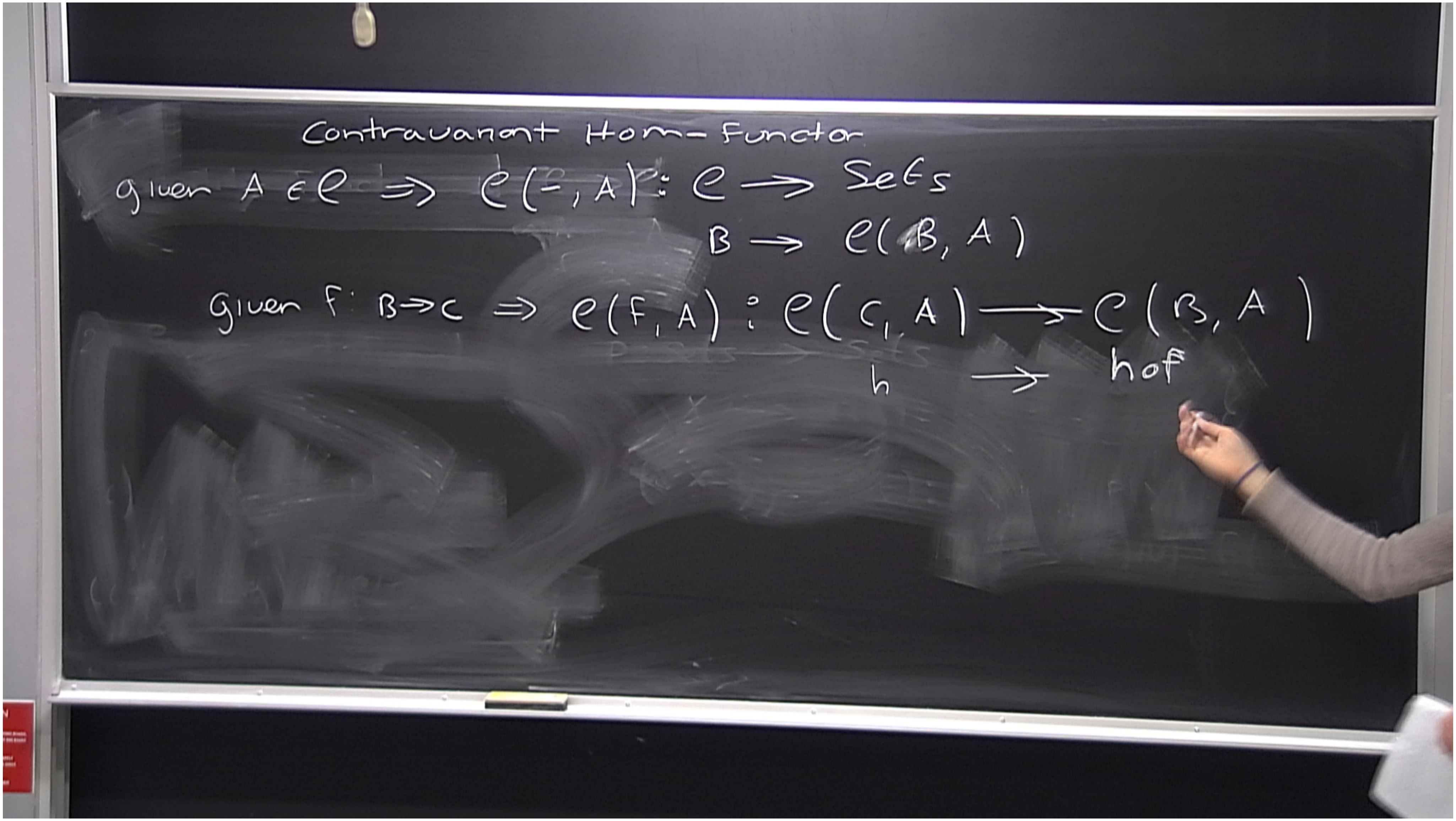
" $F : \mathcal{C} \longrightarrow \mathcal{D}$ s.t.

ii) $\underbrace{\tilde{P}(f \circ g)} = \underbrace{\tilde{P}(g) \circ \tilde{P}(f)}$ $\quad f: X \to Y \quad g: Z \to X$

$$\tilde{P}(f \circ g) : P(Y) \to P(Z)$$
$$u \to (f \circ g)^{-1} u =$$

$$\tilde{P}(g) \circ \tilde{P}(f) : P(Y) \to P(X) \to P(Z)$$
$$u \to f^{-1}(u) \to g^{-1}(f^{-1}(u))$$


Contravariant Hom-Functor
given A ∈ C ⇒ C(-, A): C → Sets
B → C(B, A)
given f: B→C ⇒ C(f, A):


Contravariant Hom-Functor
given $A \in \mathcal{C} \Rightarrow \mathcal{C}(-, A) : \mathcal{C} \to Sets$
$B \to \mathcal{C}(B, A)$
given $f : B \to C \Rightarrow \mathcal{C}(f, A) : \mathcal{C}(C, A) \to \mathcal{C}(B, A)$
$h \to h \circ f$

Contra

given $A \in \mathcal{C} \Rightarrow \mathcal{C}(-,A) : \mathcal{C} \to Sets$

$$B \mapsto \mathcal{C}(B,A)$$

given $f : B \to C \Rightarrow \mathcal{C}(f,A) : \mathcal{C}(C,A) \to \mathcal{C}(B,A)$

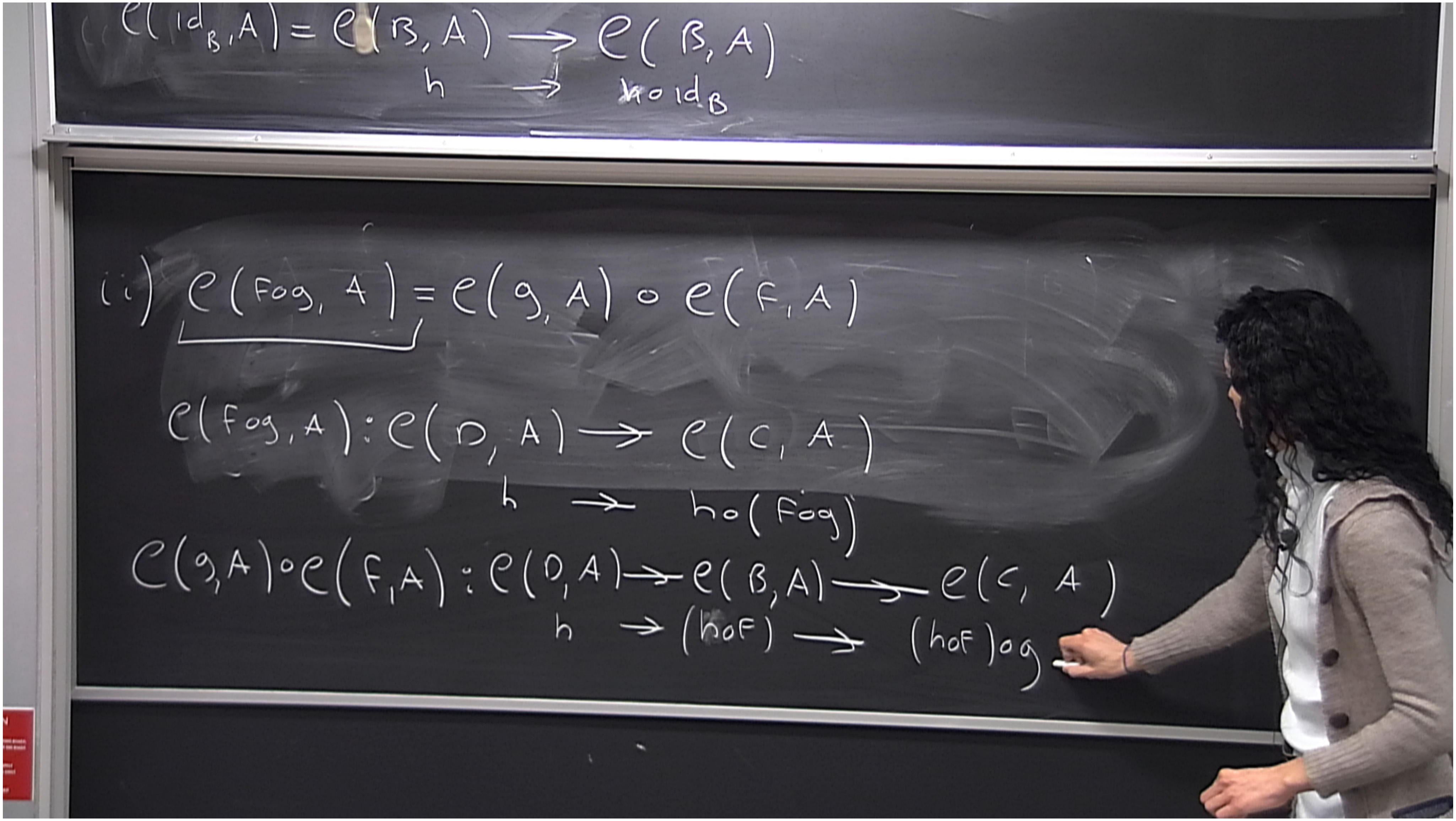
$$h \mapsto h \circ f$$

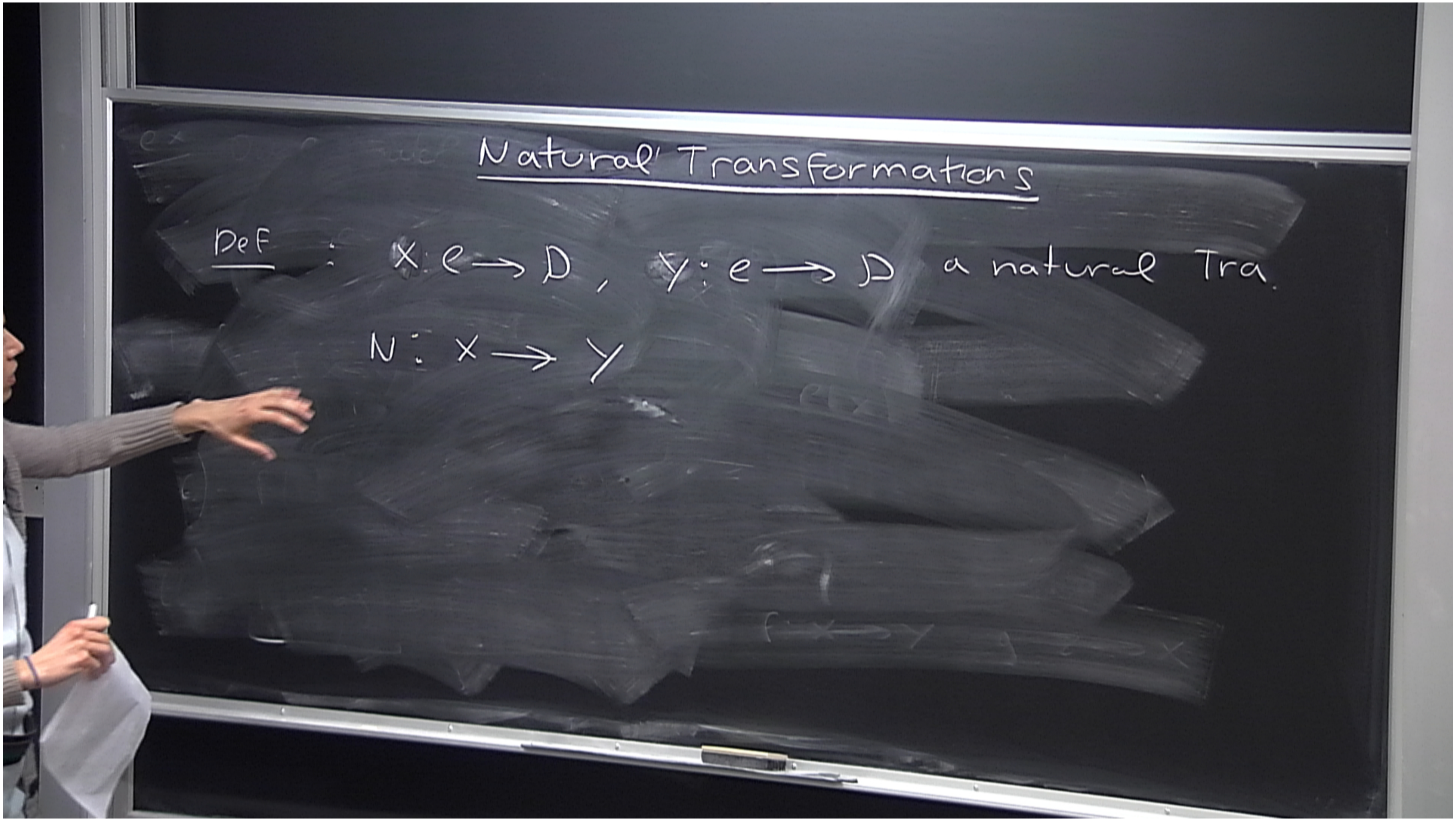
1) $\underbrace{\mathcal{C}(id_B, A)} = id_{\mathcal{C}(B,A)}$

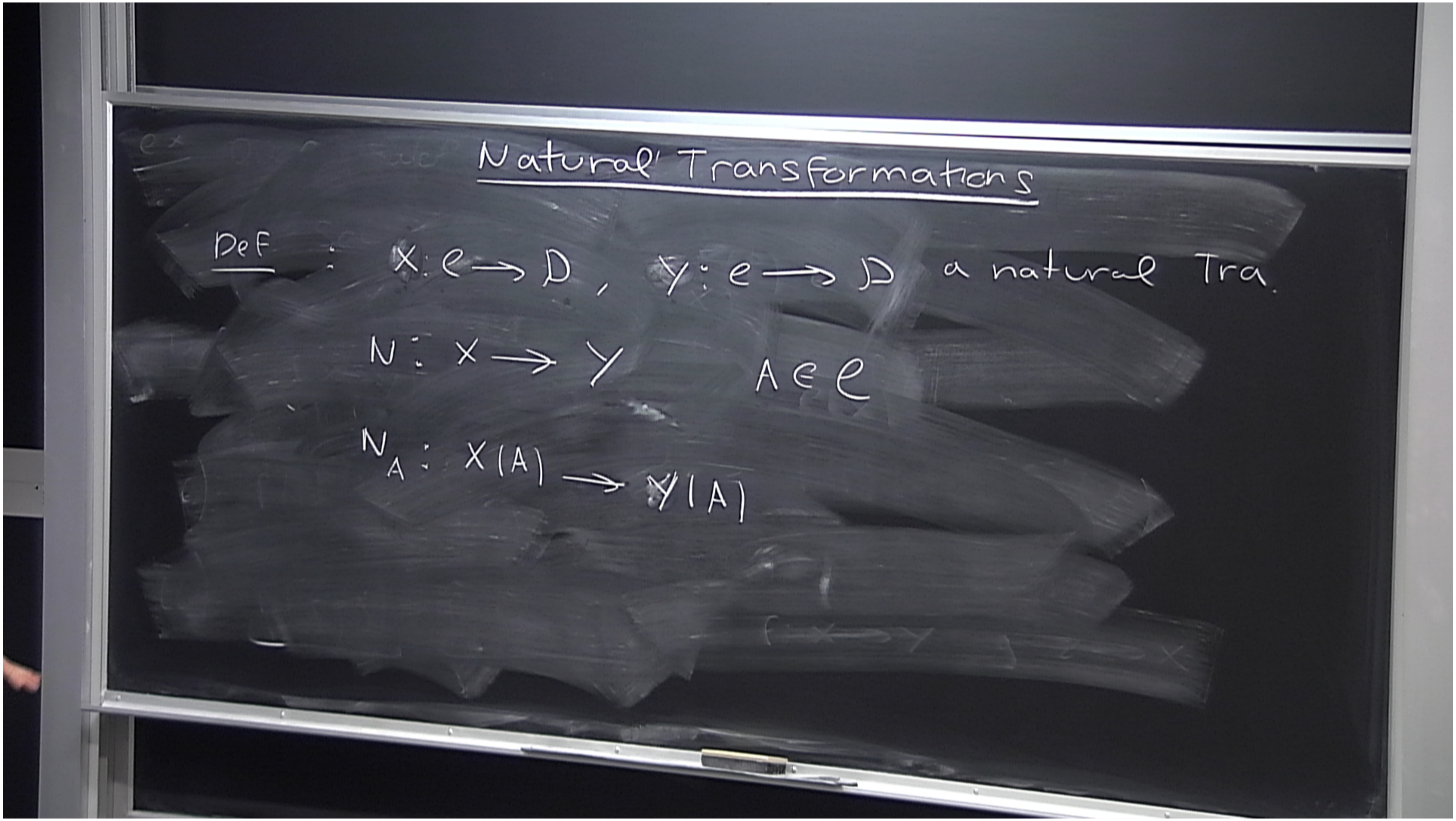
$$\mathcal{C}(id_B, A) = \mathcal{C}(B,A) \to \mathcal{C}(B,A)$$

$$h \mapsto h \circ id_B$$

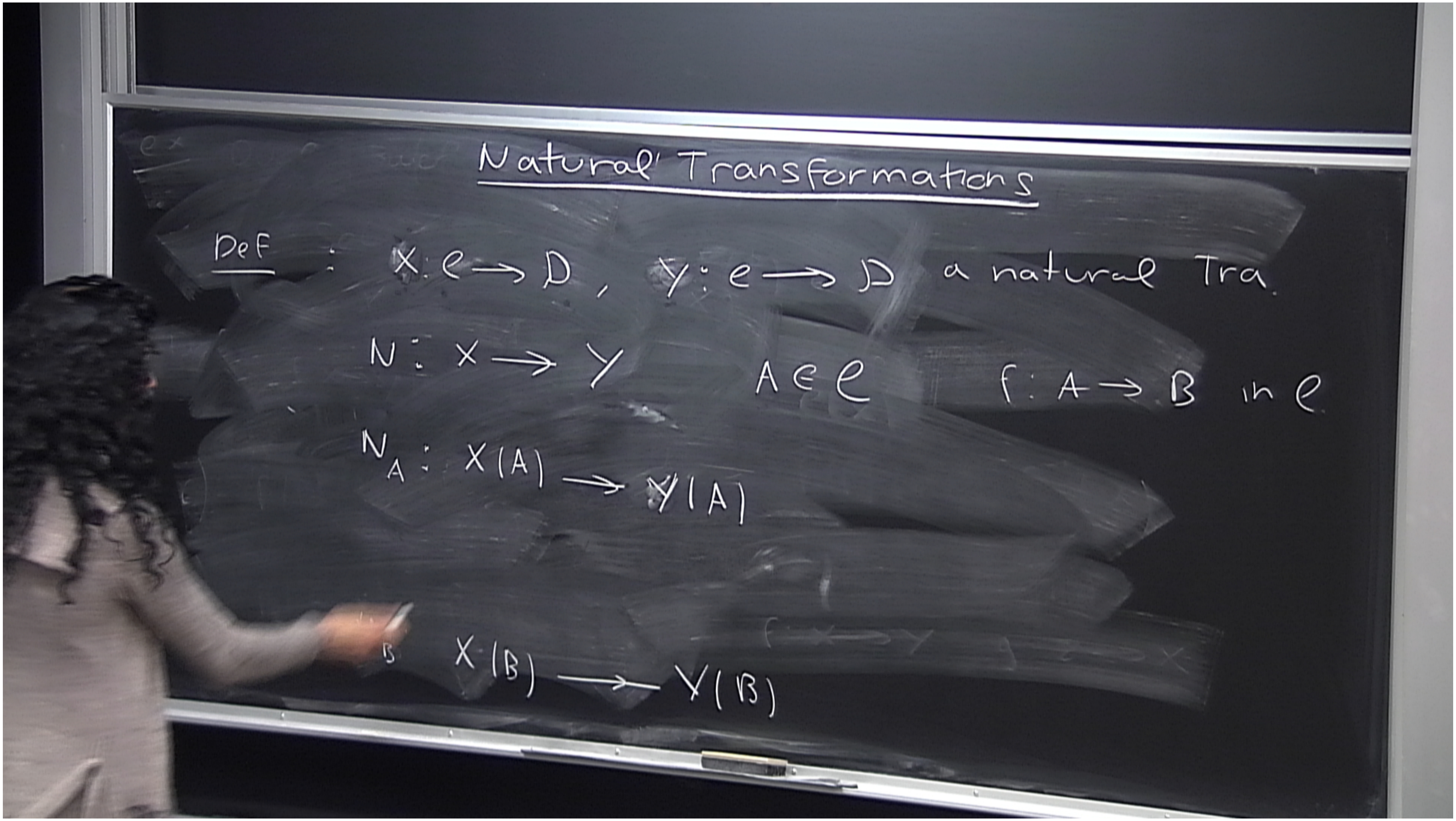
$D \xrightarrow{g} C$ $\qquad F(D) \leftarrow F(C)$

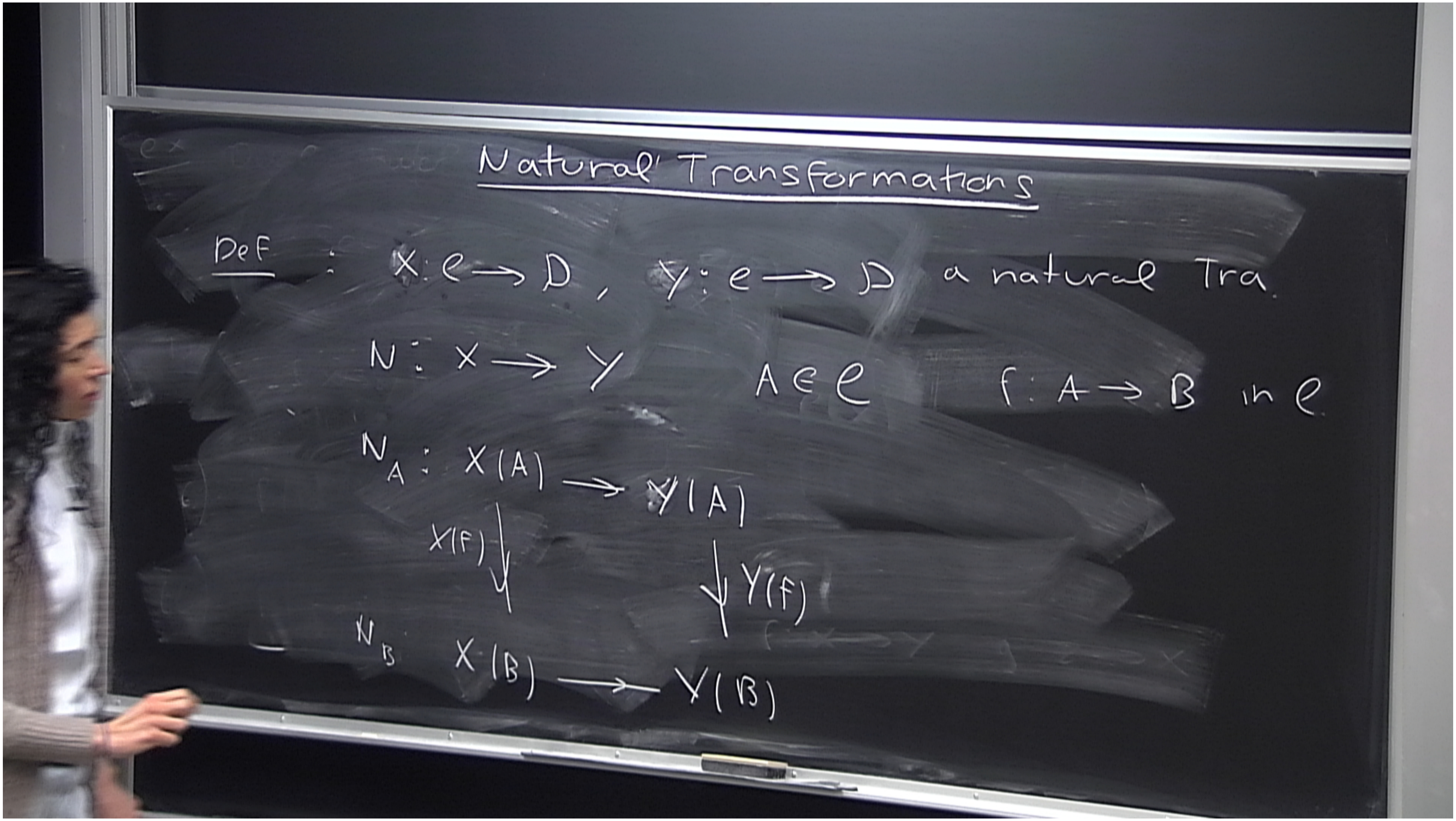

$\mathcal{C}(\mathrm{id}_B, A) = \mathcal{C}(B, A) \longrightarrow \mathcal{C}(B, A)$
$h \longrightarrow h \circ \mathrm{id}_B$
ii) $\mathcal{C}(f \circ g, A) = \mathcal{C}(g, A) \circ \mathcal{C}(f, A)$
$\mathcal{C}(f \circ g, A) : \mathcal{C}(D, A) \longrightarrow \mathcal{C}(C, A)$
$h \longrightarrow h \circ (f \circ g)$
$\mathcal{C}(g, A) \circ \mathcal{C}(f, A) : \mathcal{C}(D, A) \longrightarrow \mathcal{C}(B, A) \longrightarrow \mathcal{C}(C, A)$
$h \longrightarrow (h \circ f) \longrightarrow (h \circ f) \circ g$

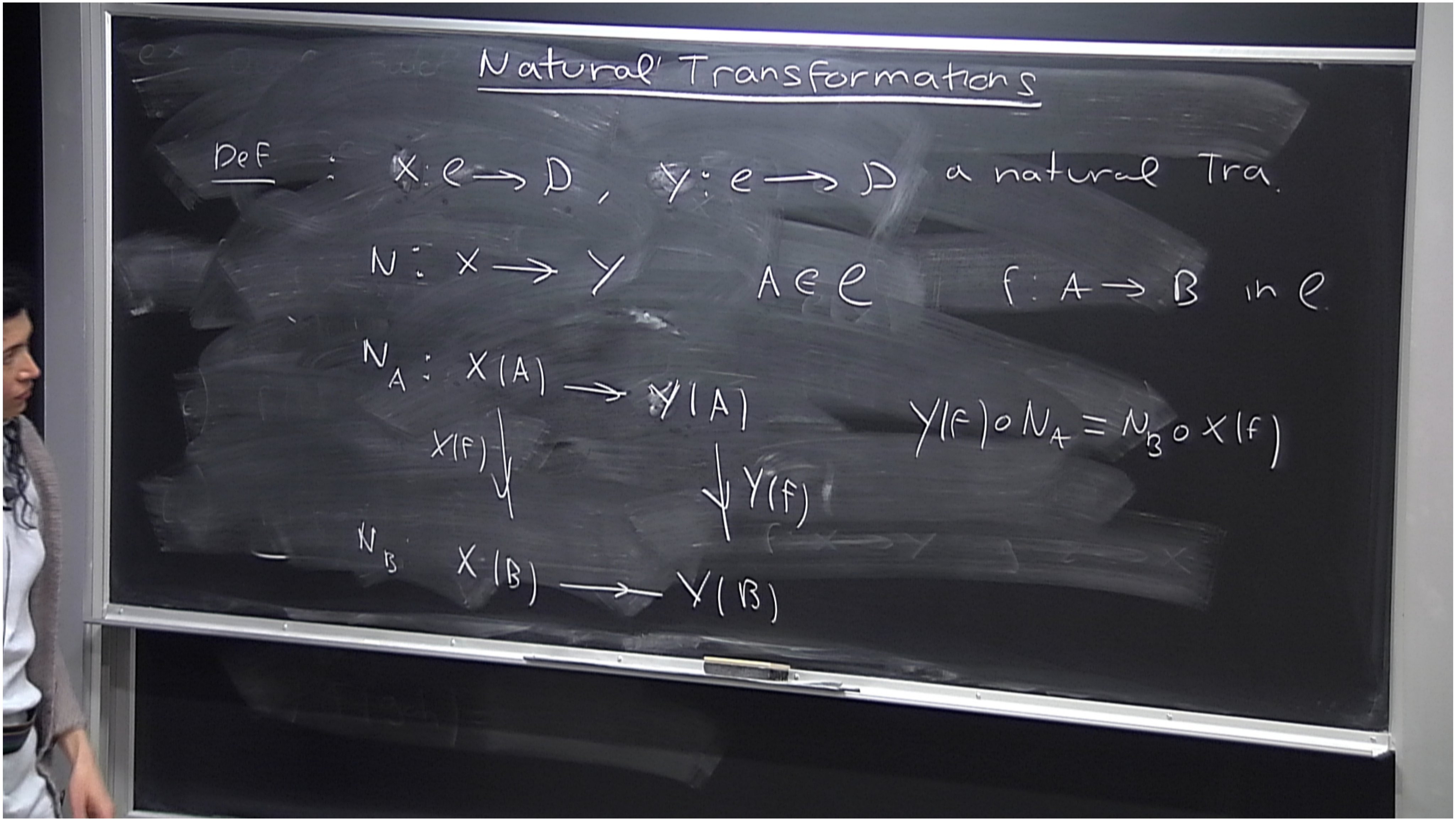

Natural Transformations
Def : X : C ⟶ D , Y : C ⟶ D a natural Tra.
N : X ⟶ Y

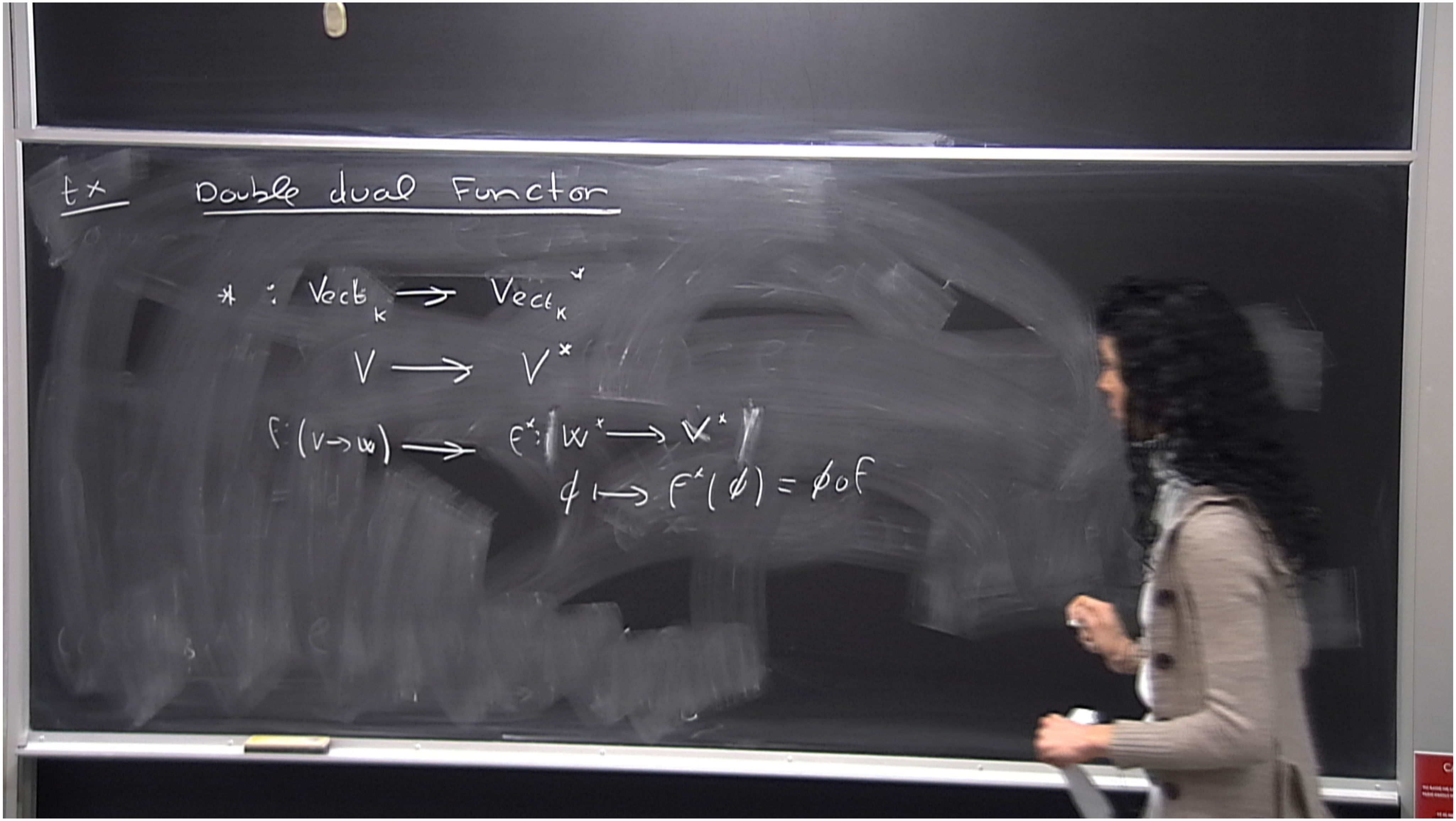


Natural Transformations
Def: $X: \mathcal{C} \to \mathcal{D}$, $Y: \mathcal{C} \to \mathcal{D}$ a natural Tra.
$N: X \to Y$
$A \in \mathcal{C}$
$N_A: X(A) \to Y(A)$


Natural Transformations
Def: $X: \mathcal{C} \to \mathcal{D}$, $Y: \mathcal{C} \to \mathcal{D}$ a natural Tra
$N: X \to Y$ $A \in \mathcal{C}$ $f: A \to B$ in $\mathcal{C}$
$N_A: X(A) \to Y(A)$
$X(B) \to Y(B)$


Natural Transformations
Def: X: C → D, Y: C → D a natural Tra
N: X → Y
A ∈ C
f: A → B in C
N_A: X(A) → Y(A)
X(f)
Y(f)
N_B X(B) → Y(B)


Natural Transformations
Def: $X: \mathcal{C} \to \mathcal{D}$, $Y: \mathcal{C} \to \mathcal{D}$ a natural Tra.
$N: X \to Y$
$A \in \mathcal{C}$
$f: A \to B$ in $\mathcal{C}$
$N_A: X(A) \to Y(A)$
$X(f)$
$Y(f)$
$N_B: X(B) \to Y(B)$
$Y(f) \circ N_A = N_B \circ X(f)$


Ex Double dual Functor
* : Vect_K → Vect_K^*
V → V*
f : (V → W) → f* : W* → V*
φ ↦ f*(φ) = φ∘f

$$P(f)\circ P(g): P(Z)\to P(X)\to P(Y)$$

Ex Double dual Functor

$$* : Vect_K \longrightarrow Vect_K^{*}$$

$$V \longrightarrow V^* = \mathrm{Hom}(V, K)$$

$$f : (V \to W) \longrightarrow f^* : (W^* \longrightarrow V^*)$$

$$\phi \longmapsto f^*(\phi) = \phi \circ f$$

$$** : Vect_K \longrightarrow Vect_K$$

$$V \longrightarrow V^{**}$$

$$* : \mathrm{Vect}_K \longrightarrow \mathrm{Vect}_K^{*}$$

$$V \longrightarrow V^{*} = \mathrm{Hom}(V, K)$$

$$f : (V \to W) \longrightarrow f^{*} : W^{*} \longrightarrow V^{*}$$

$$\phi \longmapsto f^{*}(\phi) = \phi \circ f$$

$$** : \mathrm{Vect}_K \longrightarrow \mathrm{Vect}_K$$

$$V \longrightarrow V^{**}$$

$$f : V \to W \Rightarrow f^{**}(v) = f(v)$$

$$\mathcal{C}(f \circ g, A) : \mathcal{C}(D, A) \longrightarrow \mathcal{C}(C, A)$$

$$h \longrightarrow h \circ (f \circ g)$$

$$\mathcal{C}(g, A) \circ \mathcal{C}(f, A) : \mathcal{C}(D, A) \longrightarrow \mathcal{C}(B, A) \longrightarrow \mathcal{C}(C, A)$$

$$V \longrightarrow V^{**}$$

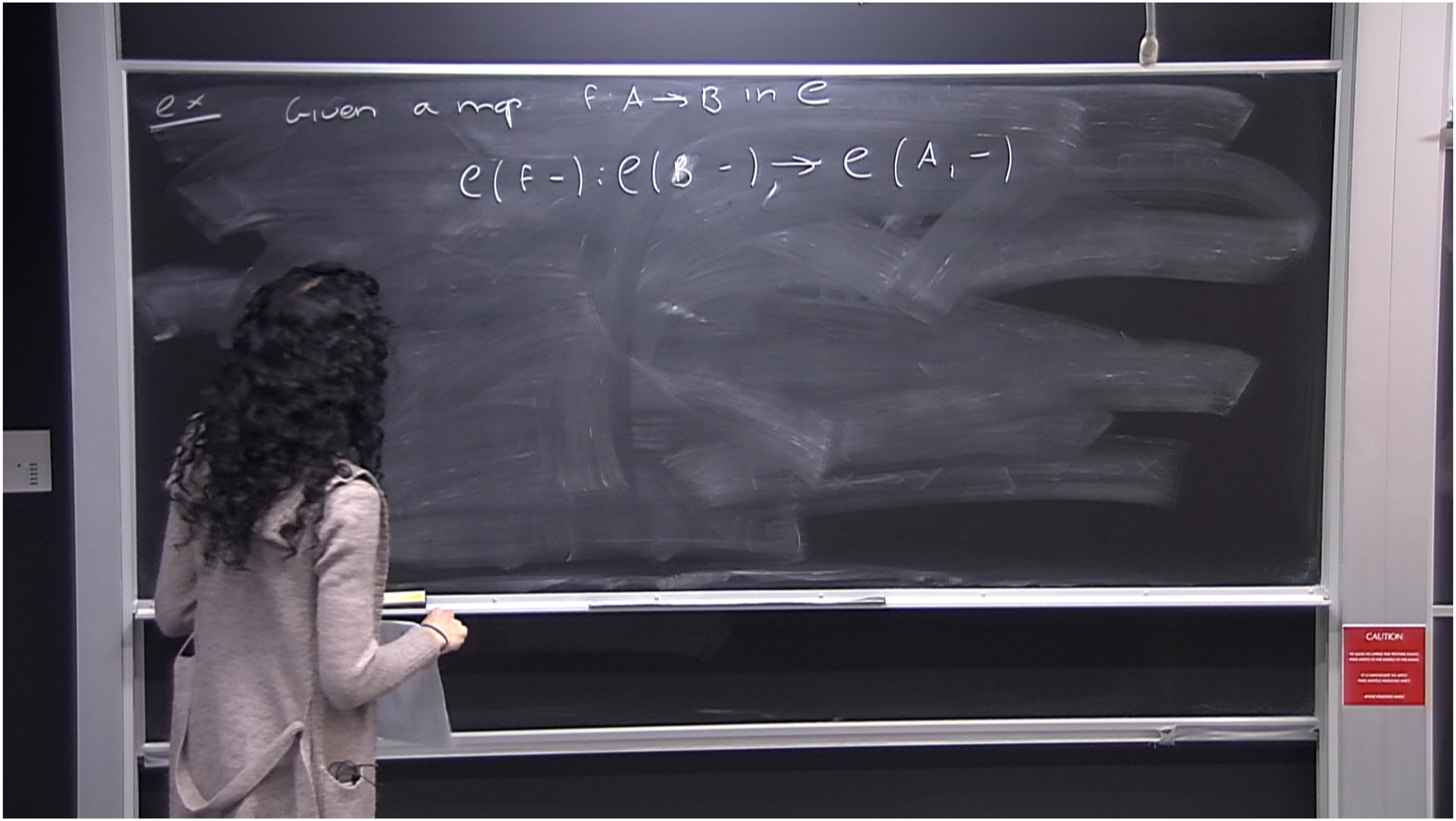
$$\mathrm{id}_{\mathrm{Vect}_k} : \mathrm{Vect}_k \to \mathrm{Vect}_k$$
$$V \longrightarrow V$$

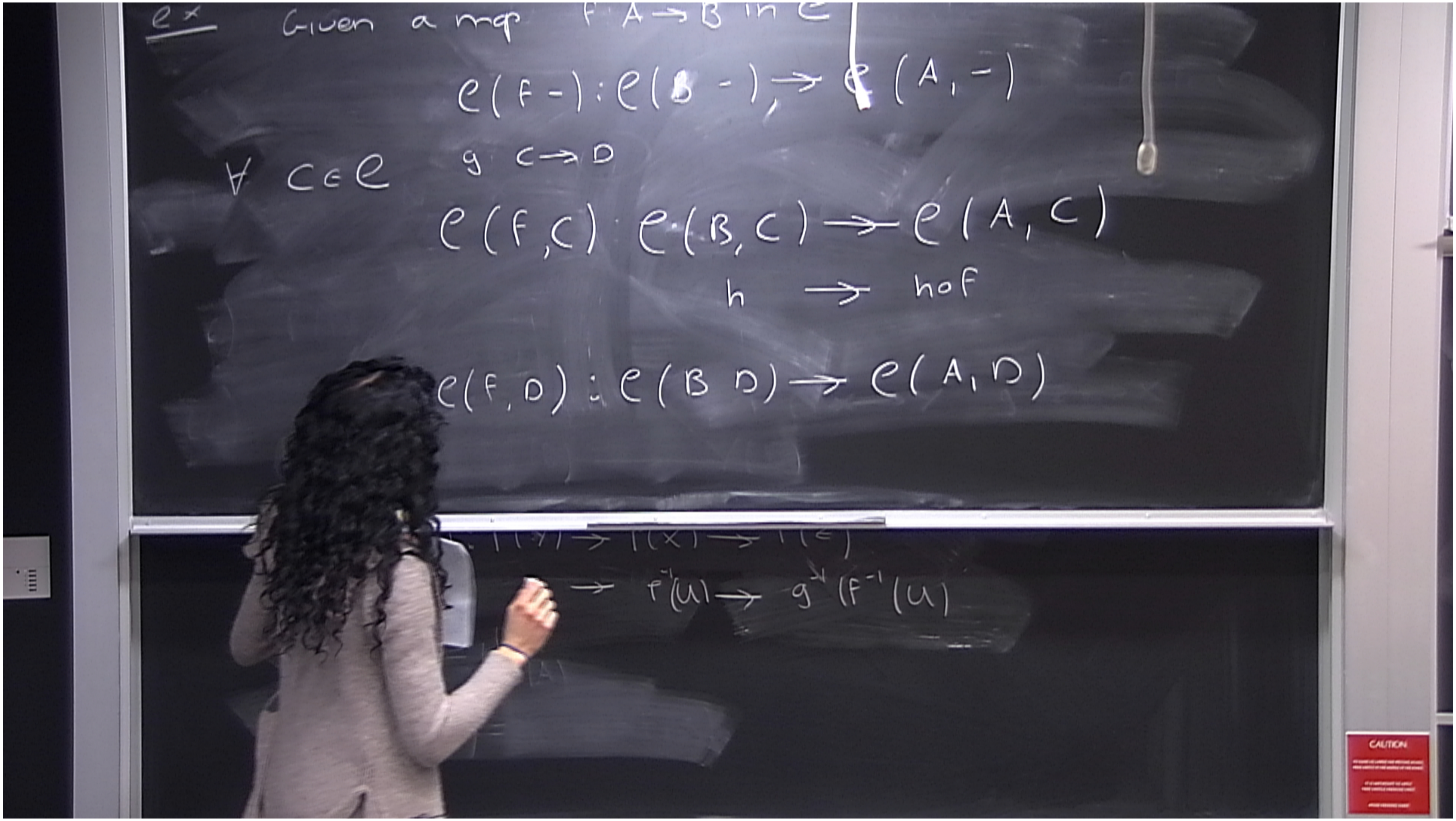
$$N : 1_{\mathrm{Vect}_k} \longrightarrow {**}$$

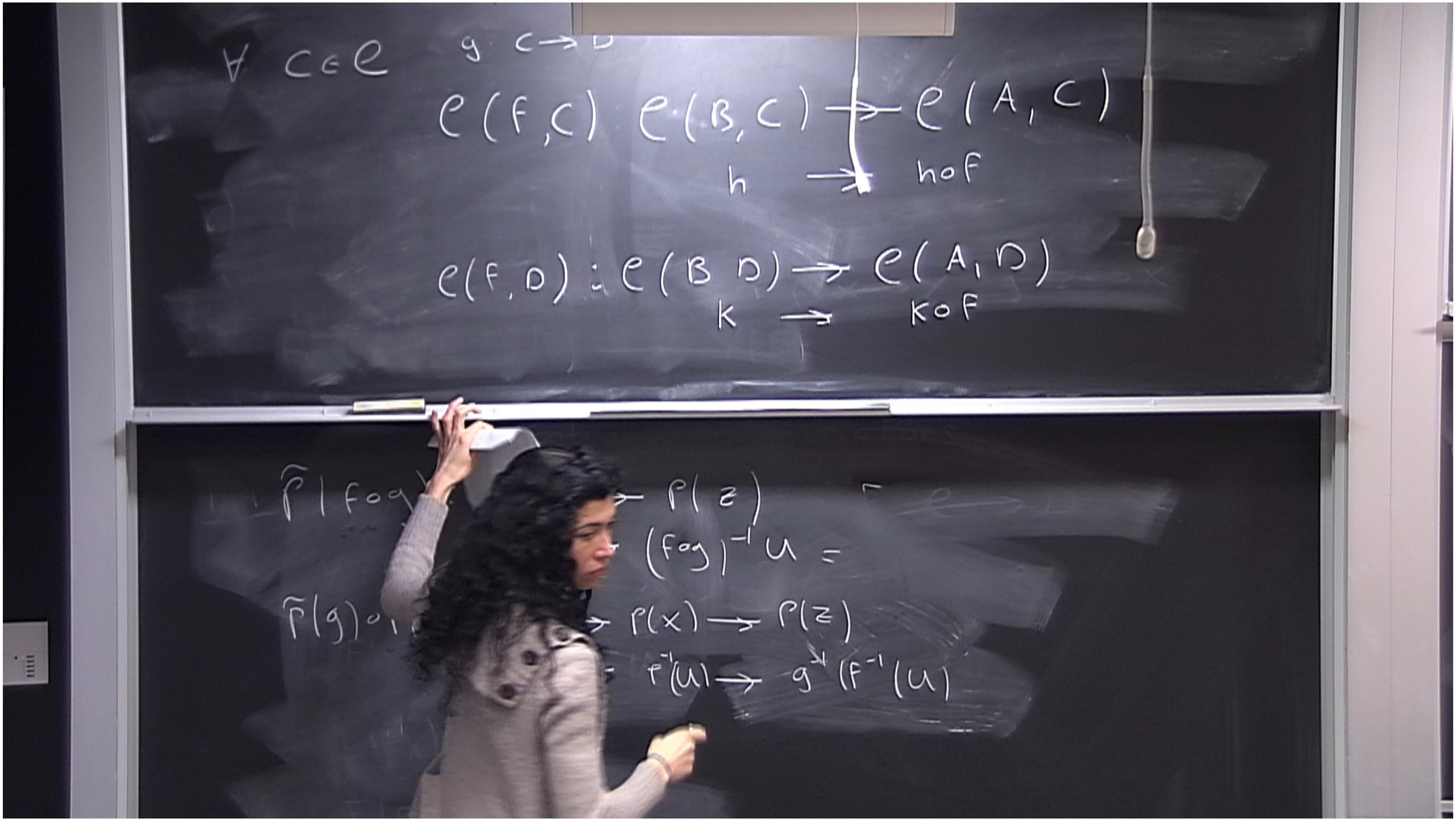
$$\begin{tikzcd}
1_{\mathrm{Vect}_k}(V) \arrow[r, "N_V"] \arrow[d, "f"'] & V^{**} \arrow[d, "f^{**}"] \\
1_{\mathrm{Vect}_k}(W) \arrow[r, "N_W"'] & W^{**}
\end{tikzcd}$$

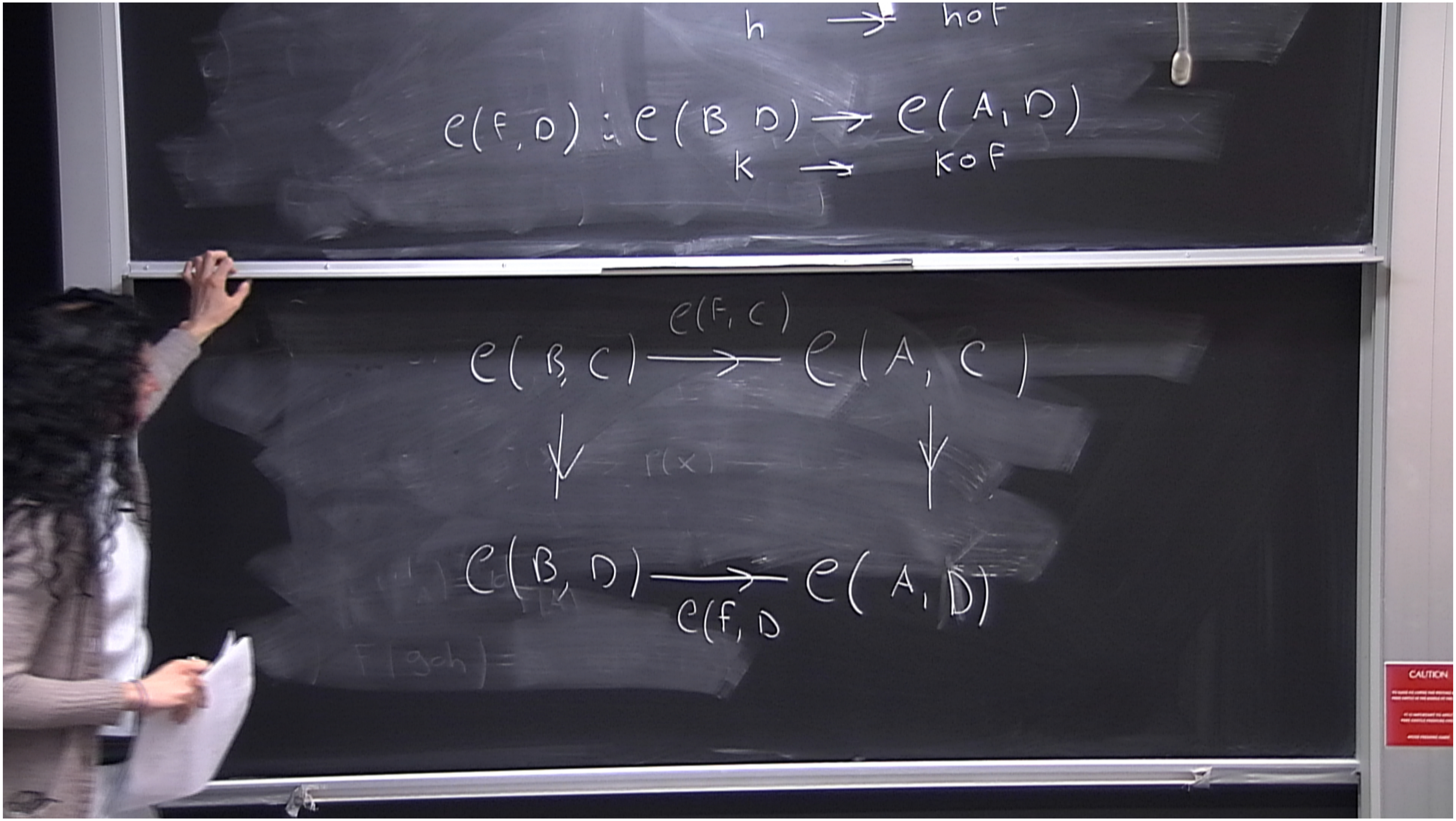
$$(f^{**} \circ N_V)(v) = f^{**}(v) = w$$

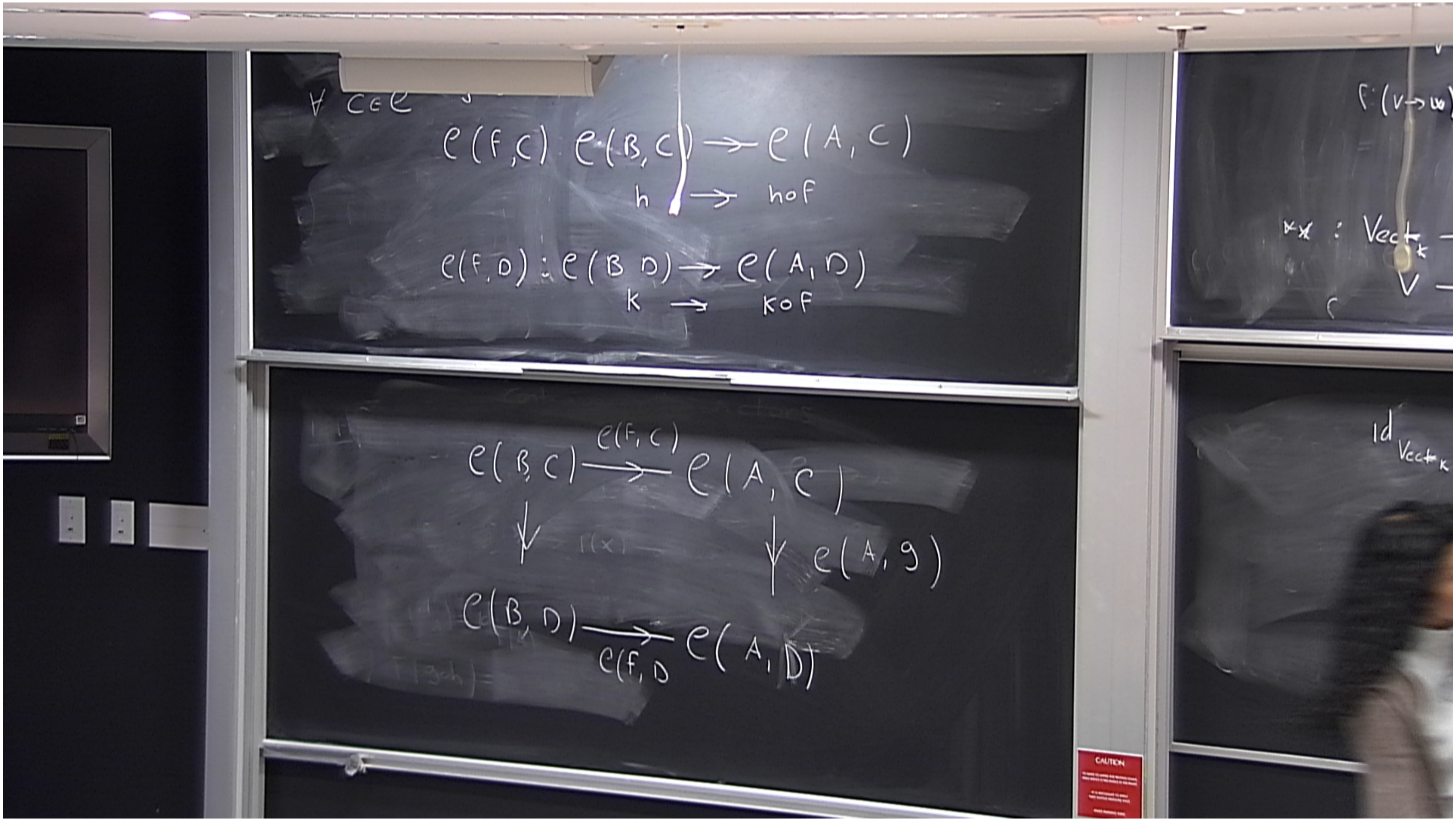
$$(N_W \circ f)(v) = N_W(w) = w$$

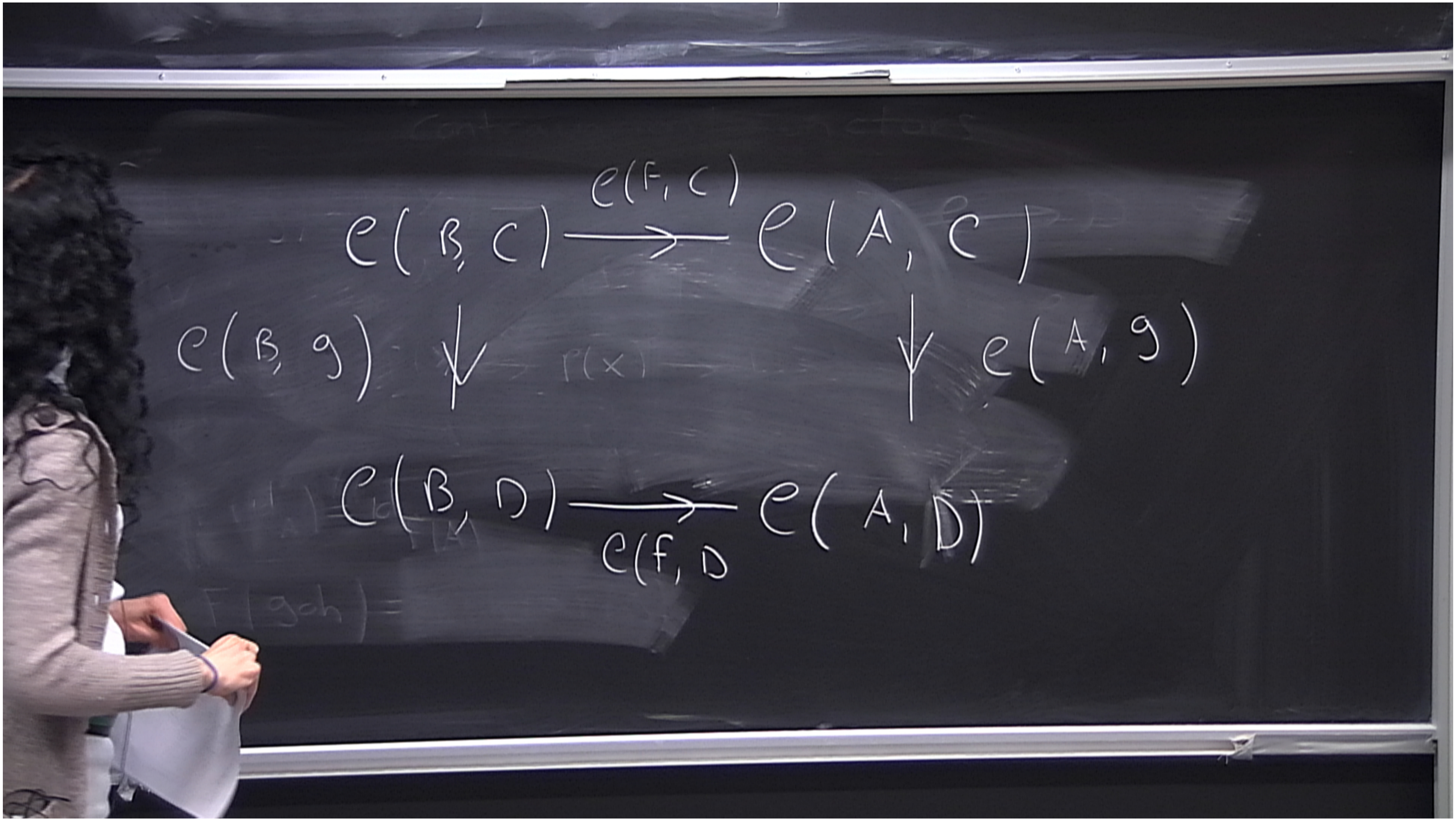

ex Given a map f: A → B in C
C(f, -) : C(B, -) → C(A, -)

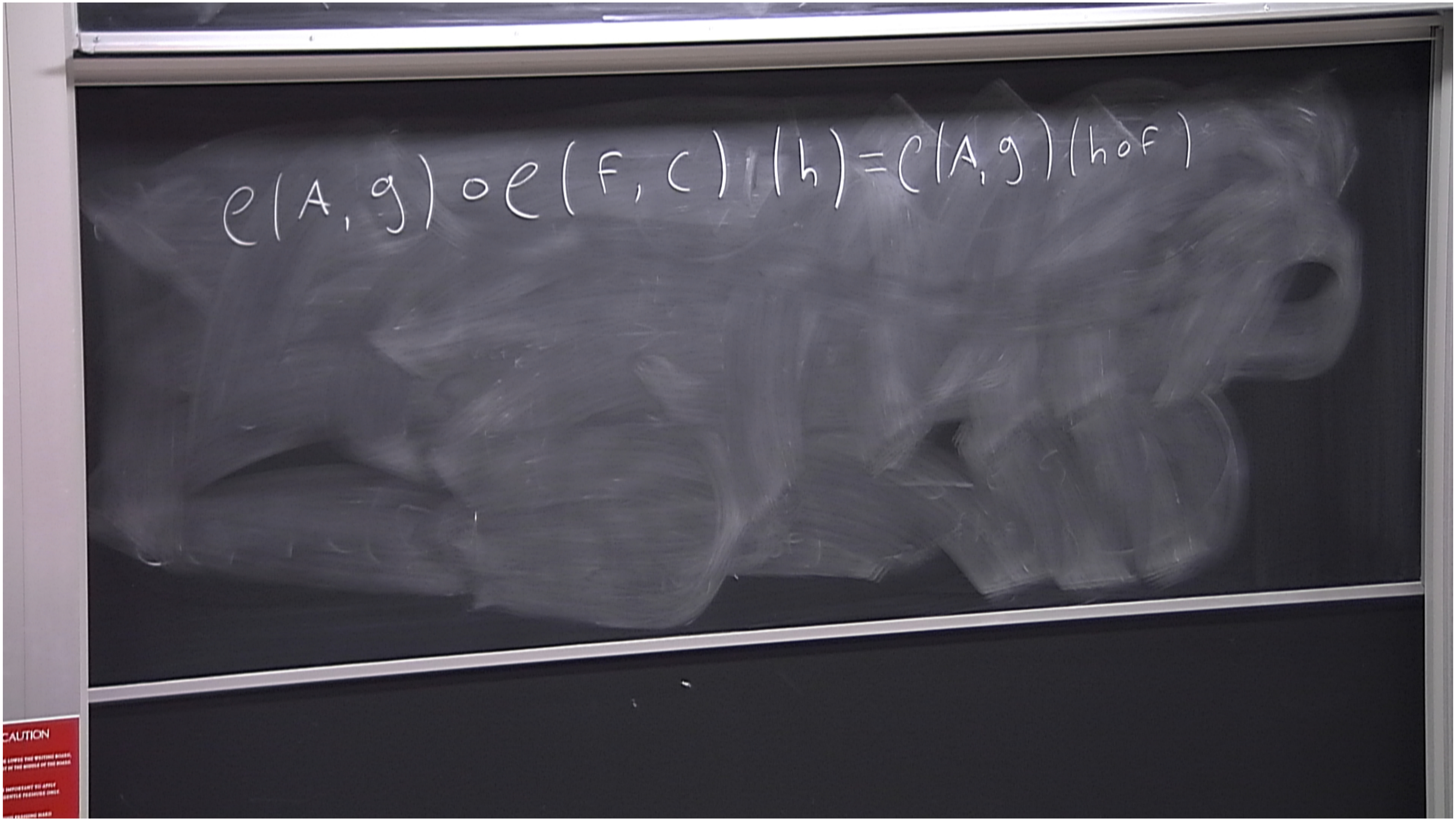

ex Given a map f: A → B in C
C(f,-): C(B,-) → C(A,-)
∀ c ∈ C   g: C → D
C(f,C): C(B,C) → C(A,C)
h → h∘f
C(f,D): C(B,D) → C(A,D)
→ f⁻¹(U) → g⁻¹(f⁻¹(U))

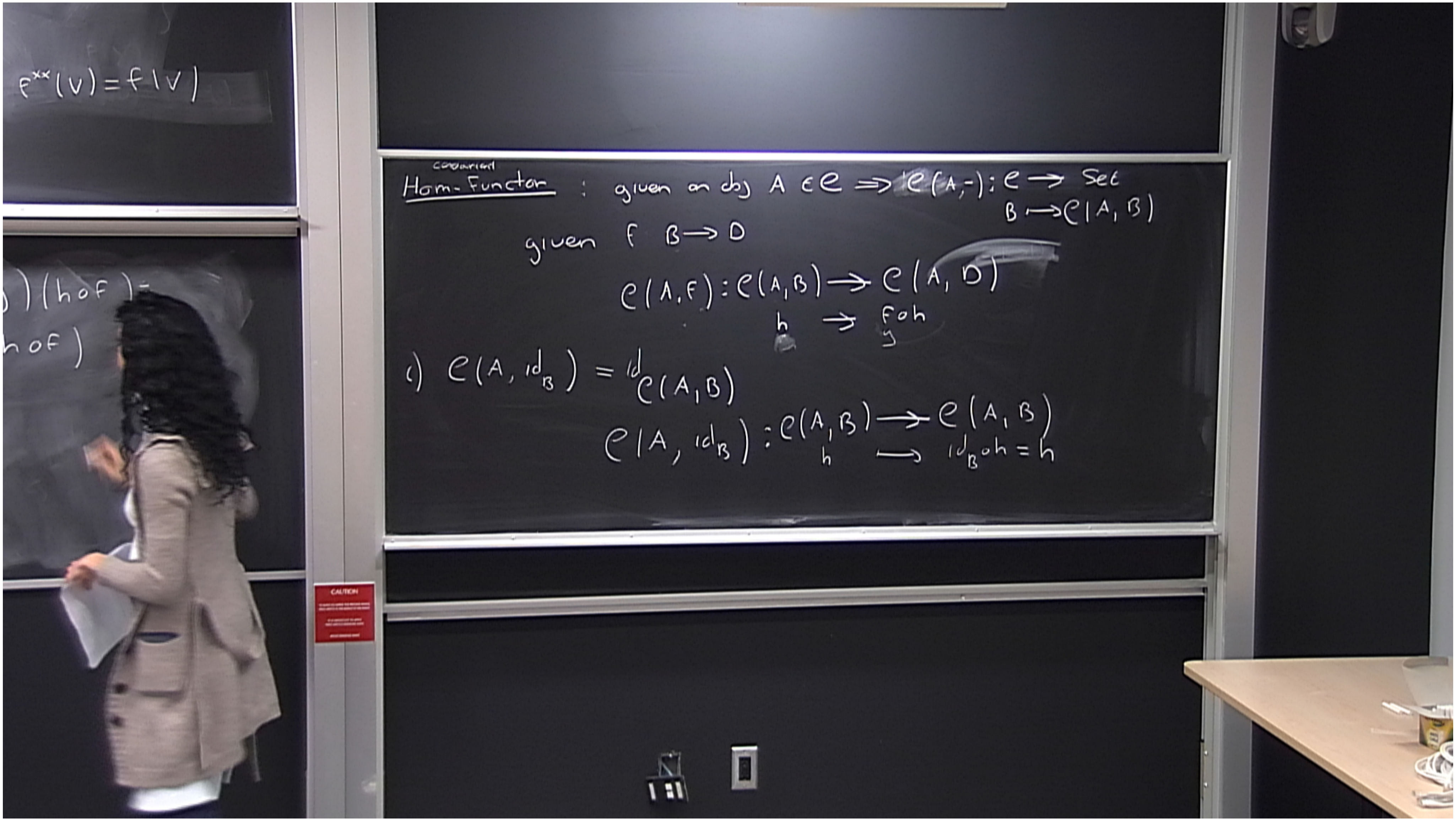

∀ C∈𝒞 g: C→D
𝒞(f,C) 𝒞(B,C) → 𝒞(A,C)
h → h∘f
𝒞(f,D): 𝒞(B,D) → 𝒞(A,D)
k → k∘f
P(z)
(f∘g)⁻¹U =
P(x) → P(z)
f⁻¹(U) → g⁻¹(f⁻¹(U))


h → h∘f
C(f, D) : C(B, D) → C(A, D)
k → k∘f
C(f, C)
C(B, C) → C(A, C)
C(B, D) → C(A, D)
C(f, D


∀ C∈𝒞
𝒞(f,C) : 𝒞(B,C) → 𝒞(A,C)
h ↦ h∘f
𝒞(f,D) : 𝒞(B,D) → 𝒞(A,D)
k ↦ k∘f
𝒞(B,C) → 𝒞(A,C)
𝒞(f,C)
𝒞(A,g)
𝒞(B,D) → 𝒞(A,D)
𝒞(f,D)
f:(V→W)
Vect_k
id_Vect_k
CAUTION

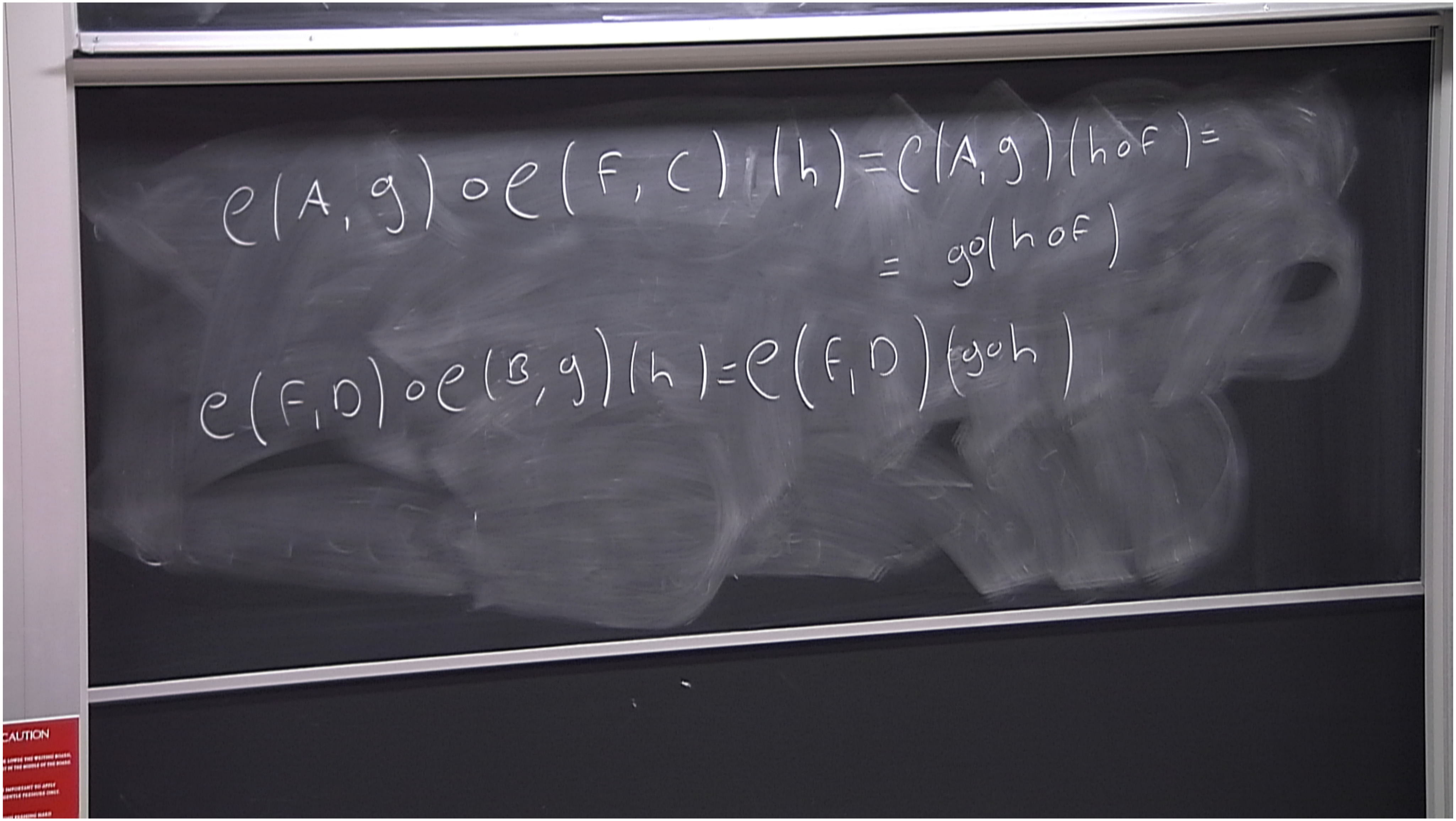

e(F, C)
e(B, C)
e(A, C)
e(B, g)
e(A, g)
e(B, D)
e(A, D)
e(f, D


$e(A,g)\circ e(f,C)(h)=e(A,g)(h\circ f)$
CAUTION

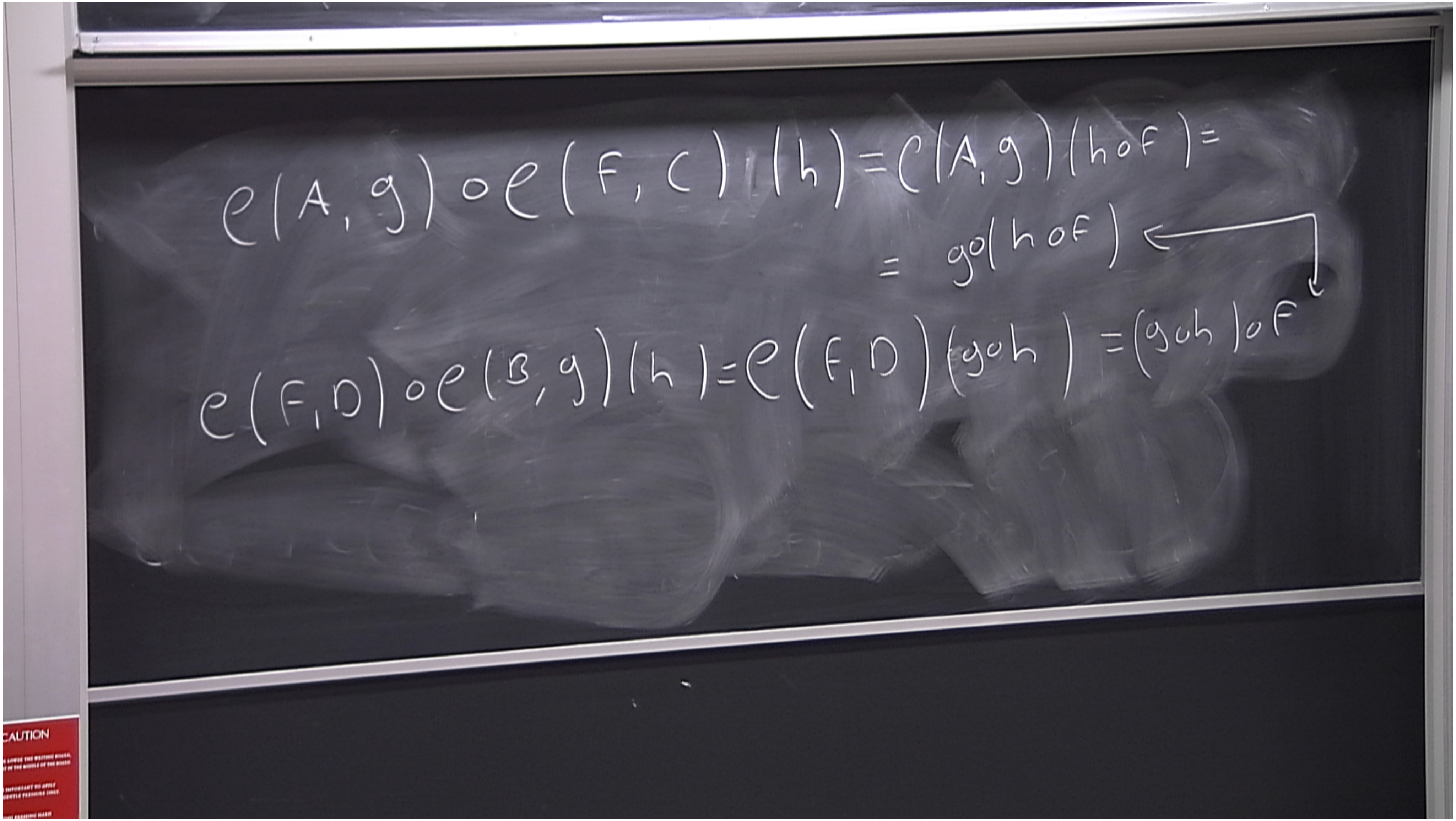

f**(v) = f(v)
covariant
Hom-Functor : given an obj A ∈ C ⇒ C(A,-) : C → Set
B ↦ C(A, B)
given f B → D
C(A,f) : C(A,B) → C(A, D)
h → f∘h
i) C(A, id_B) = id_C(A,B)
C(A, id_B) : C(A,B) → C(A, B)
h → id_B∘h = h
)(h∘f) =
h∘f)
CAUTION

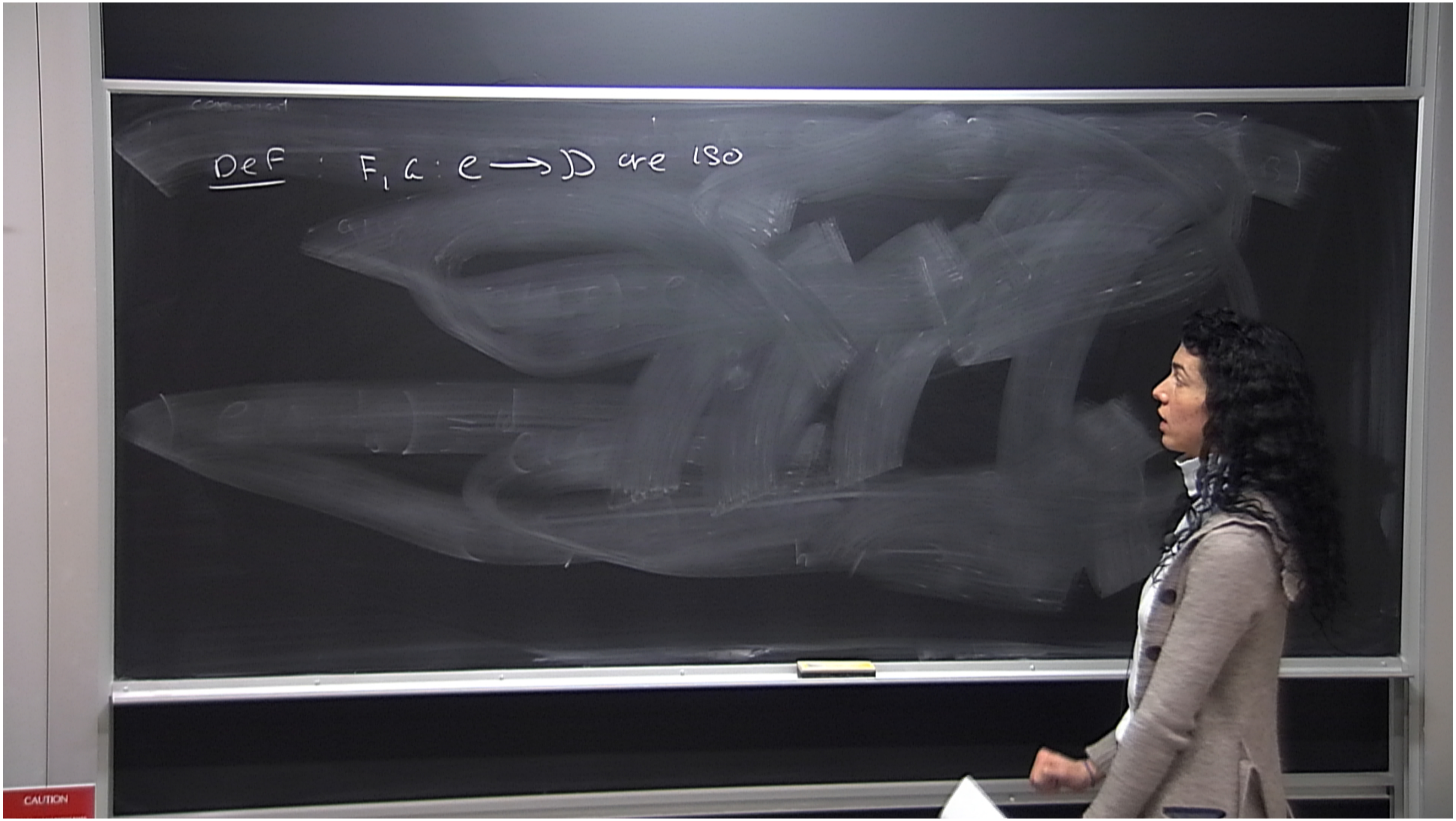
$$e(A,g)\circ e(f,C)\,(h) = e(A,g)(h\circ f) = g\circ(h\circ f)$$

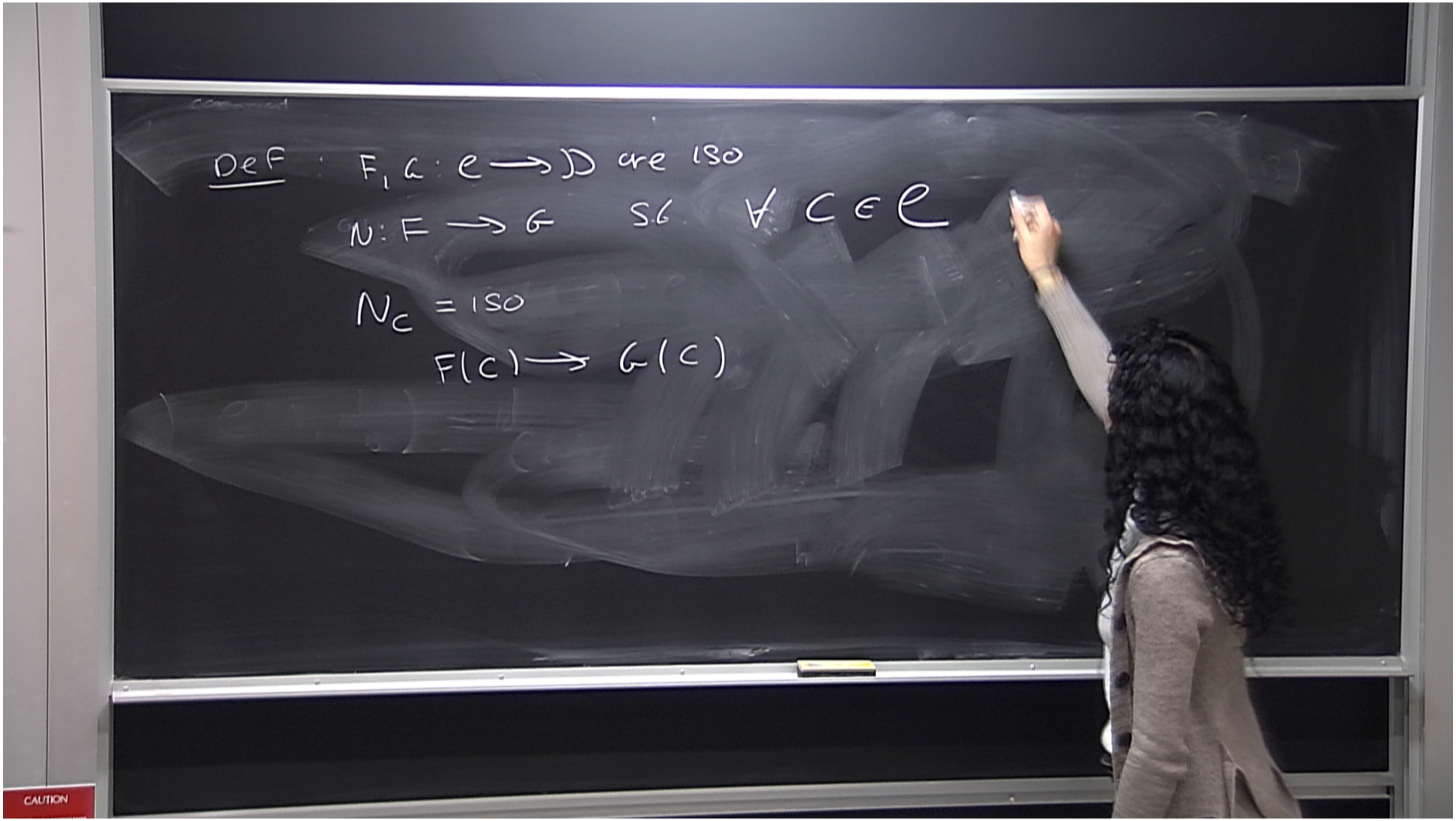
$$e(f,D)\circ e(B,g)(h) = e(f,D)\circ$$

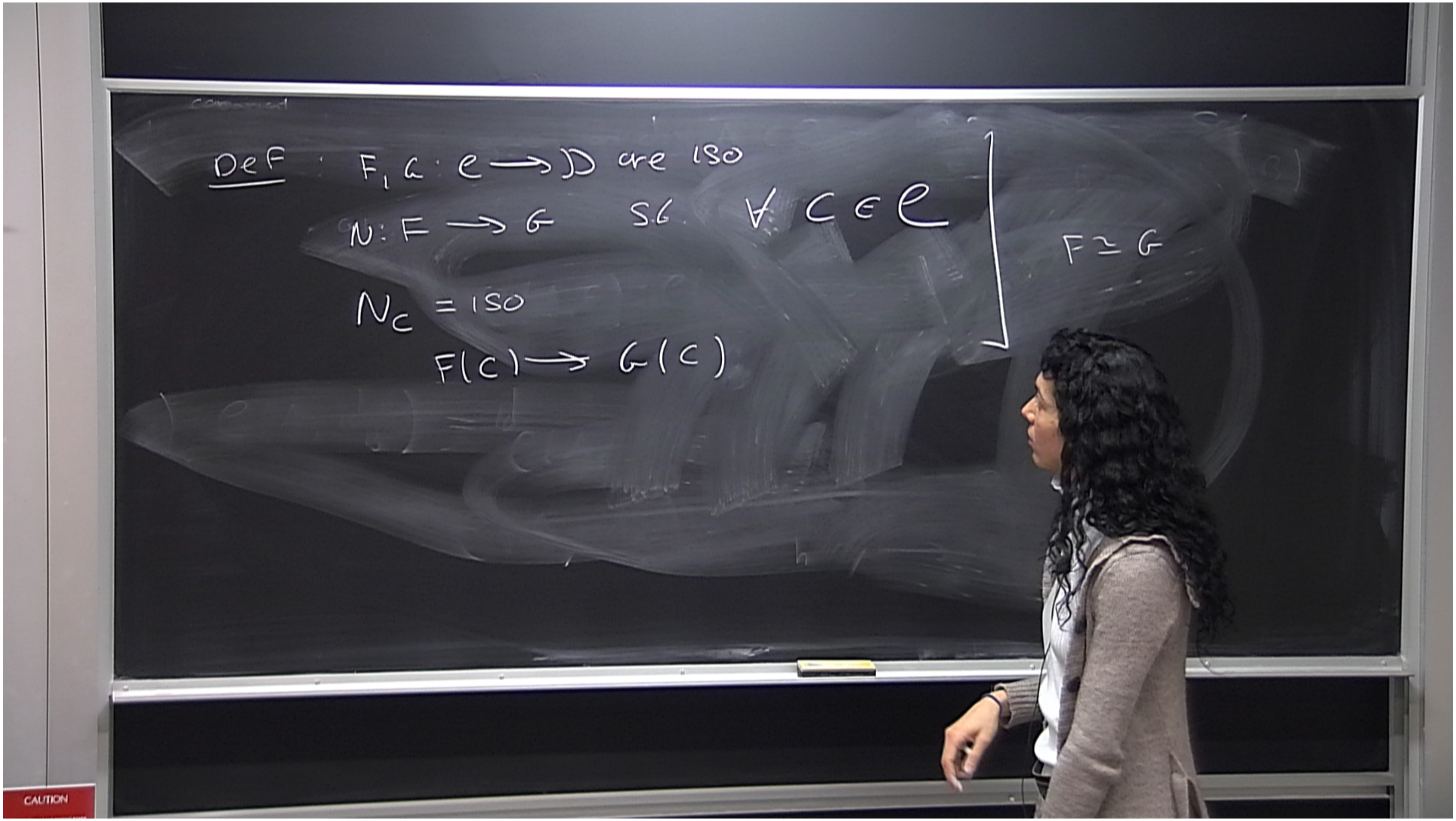


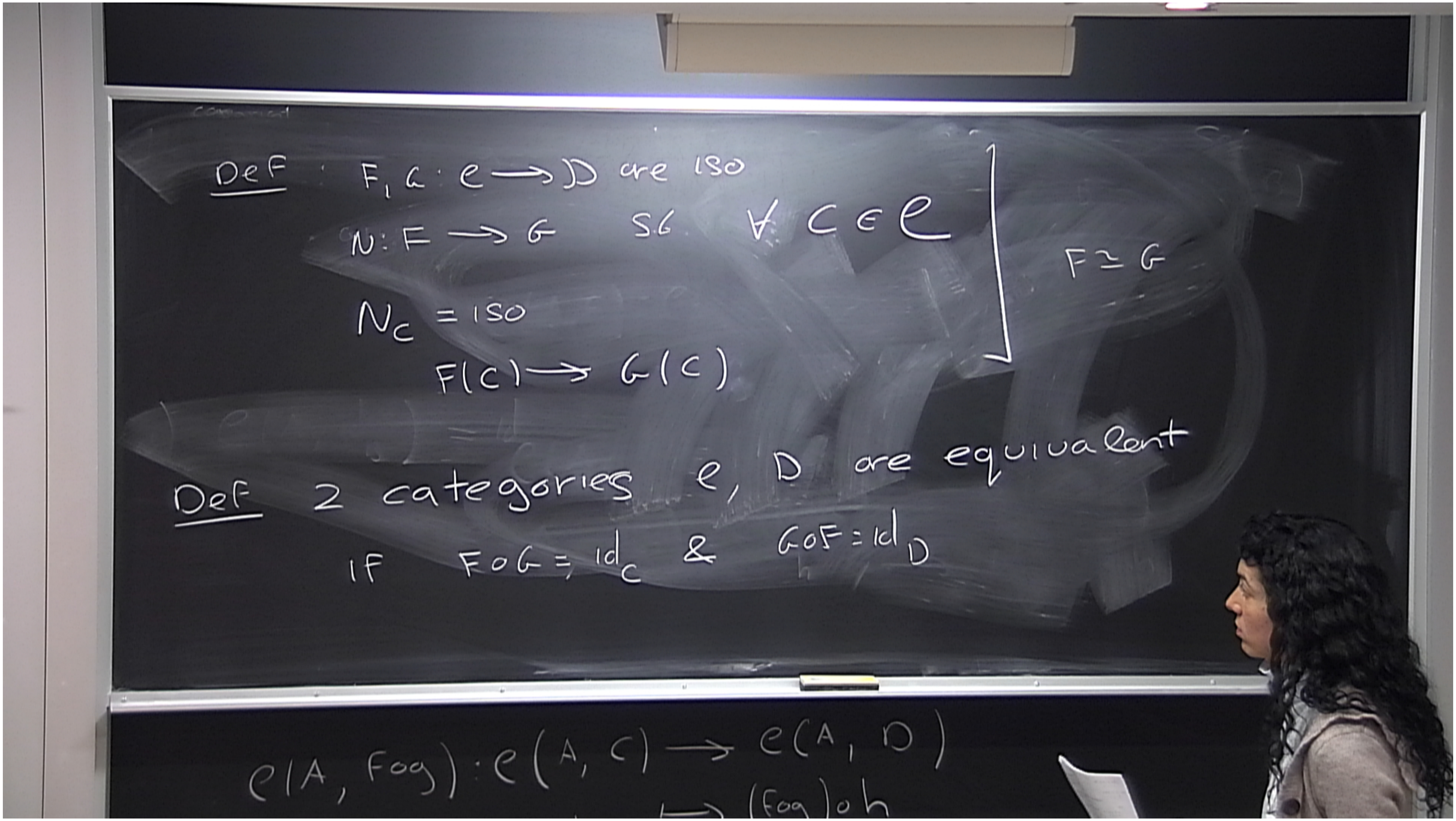
$\ell(A,g)\circ\ell(f,C)(h)=\ell(A,g)(h\circ f)=$
$=g\circ(h\circ f)$
$\ell(f,D)\circ\ell(B,g)(h)=\ell(f,D)(g\circ h)$
CAUTION

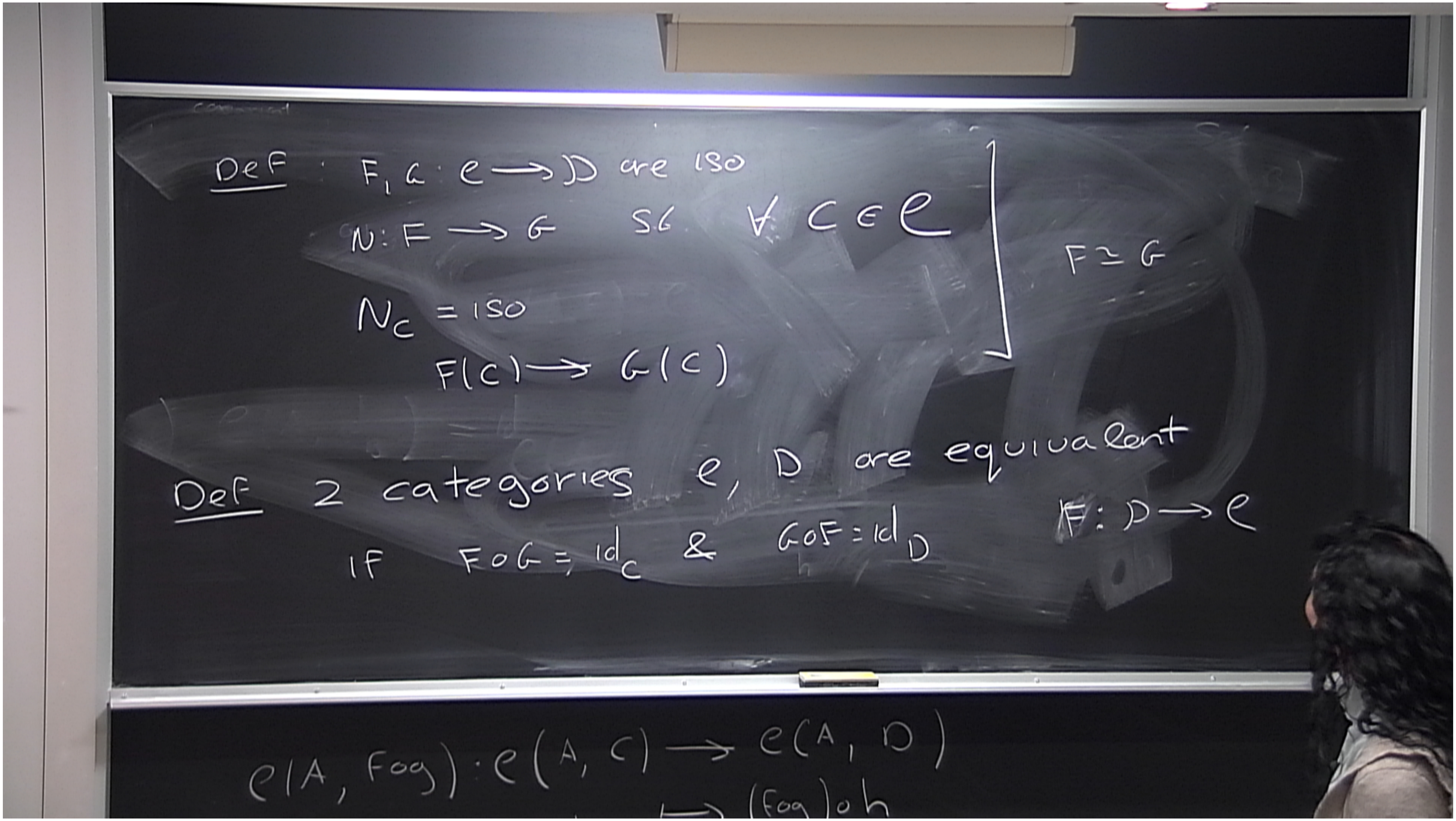

$e(A,g) \circ e(f,C)(h) = e(A,g)(h \circ f) =$
$= g \circ (h \circ f)$
$e(f,D) \circ e(B,g)(h) = e(f,D)(g \circ h) = (g \circ h) \circ f$
CAUTION


Def: F, G : C ⟶ D are iso
CAUTION


Def : F, G : C ⟶ D are iso
N : F ⟶ G st ∀ c ∈ C
N_c = iso
F(c) ⟶ G(c)
CAUTION


Def: F, G : C → D are iso
N: F → G s.t. ∀ c ∈ C
N_c = iso
F(c) → G(c)
F ≅ G
CAUTION


Def F, G : C → D are iso
N : F → G s.t. ∀ C ∈ C
N_C = iso
F(C) → G(C)
F ≅ G
Def 2 categories C, D are equivalent
if FoG = id_C & GoF = id_D
C(A, Fog) : C(A, C) → C(A, D)
↦ (Fog)oh


Def : F, G : C → D are iso
N : F → G st ∀ c ∈ C
N_c = iso
F(c) → G(c)
F ≅ G
Def 2 categories C, D are equivalent
if F∘G = id_C & G∘F = id_D
F : D → C
C(A, F∘g) : C(A, C) → C(A, D)
↦ (F∘g)∘h