Title: Fractional Chern and Topological Insulators

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Abstract: Recent years have seen a renewed interest, both theoretically and experimentally, in the search for topological states of matter. On the theoretical side, while much progress has been achieved in providing a general classification of non-interacting topological states, the fate of these phases in the presence of strong interactions remains an open question. The purpose of this talk is to describe recent developments on this front. In the first part of the talk, we will consider, in a scenario with time-reversal symmetry breaking, dispersionless electronic Bloch bands (flatbands) with non-zero Chern number and show results of exact diagonalization in a small system at 1/3 filling that support the existence of a fractional quantum Hall state in the absence of an external magnetic field. In the second part of the talk, we will discuss strongly interacting electronic phases with time-reversal symmetry in two dimensions and propose a candidate topological field theory with fractionalized excitations that describes the low energy properties of a class of time-reversal symmetric states.

Fractional Chern and Topological Insulators

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Claudio Chamon, BU
Christopher Mudry, PSI

PRL 106, 236804 (2011)
PRB 84, 165107 (2011)
PRB 84, 165138 (2011)

Perimeter Institute for Theoretical Physics

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Outline

This talk is divided into two parts:

Part 1 : We will discuss flat band models in 2-D with non-trivial topology as a route to obtain FQHE without an external magnetic field. We will present results of exact diagonalization in a small system showing supporting evidence for a $1/3$ filling fraction Quantum Hall state.

Part 2 : We will introduce a candidate time-reversal symmetric topological field theory as a way to describe an Interacting Topological Insulator in 2-D. We will discuss the constraints imposed by time-reversal symmetry on the structure of the bulk and edge states. We will also present results of exact diagonalization for a flat-band Hamiltonian that respects TRS and will discuss the various ground state orders that are obtained as interactions are considered.

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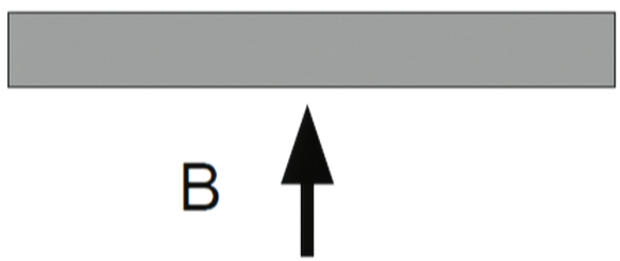
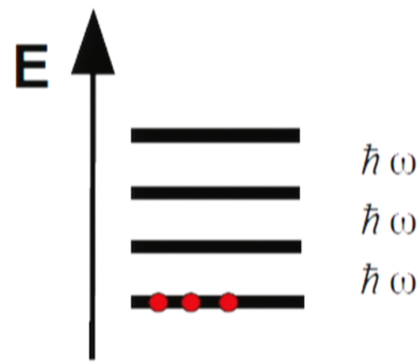
Landau Levels and the Quantum Hall Effect





Degeneracy: $N_\phi = \frac{B\,A}{\phi_0}$

Landau levels are interesting

Partially filled Landau levels $\Rightarrow$ FQHE

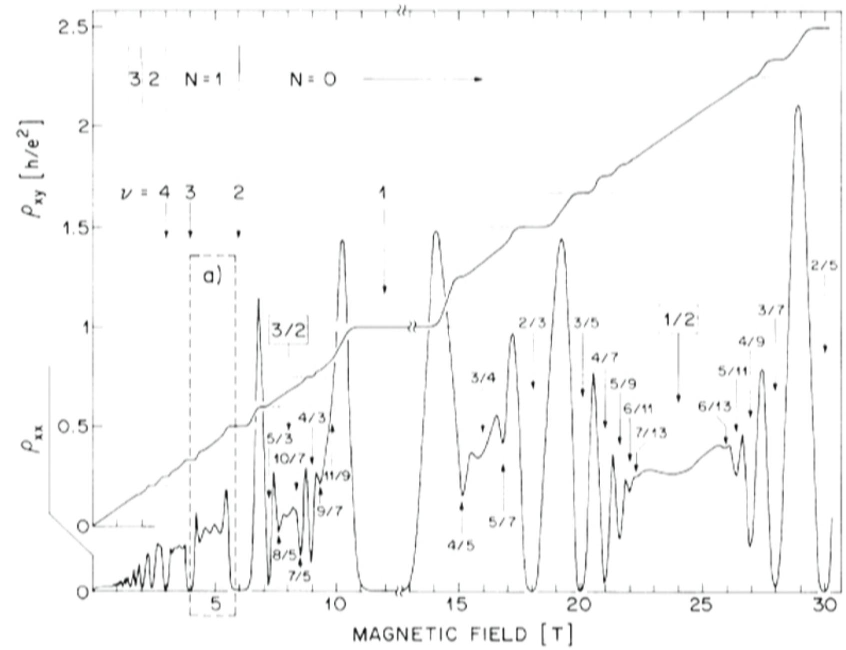


FIG. 1. Overview of diagonal resistivity ρ_{xx} and Hall resistance ρ_{xy} of sample described in text. The use of a hybrid magnet with fixed base field required composition of this figure from four different traces (breaks at $\simeq 12$ T). Temperatures were ≈ 150 mK except for the high-field Hall trace at $T = 85$ mK. The high-field ρ_{xx} trace is reduced in amplitude by a factor 2.5 for clarity. Filling factor ν and Landau levels N are indicated.

Source:
Willett et al, PRL 1987

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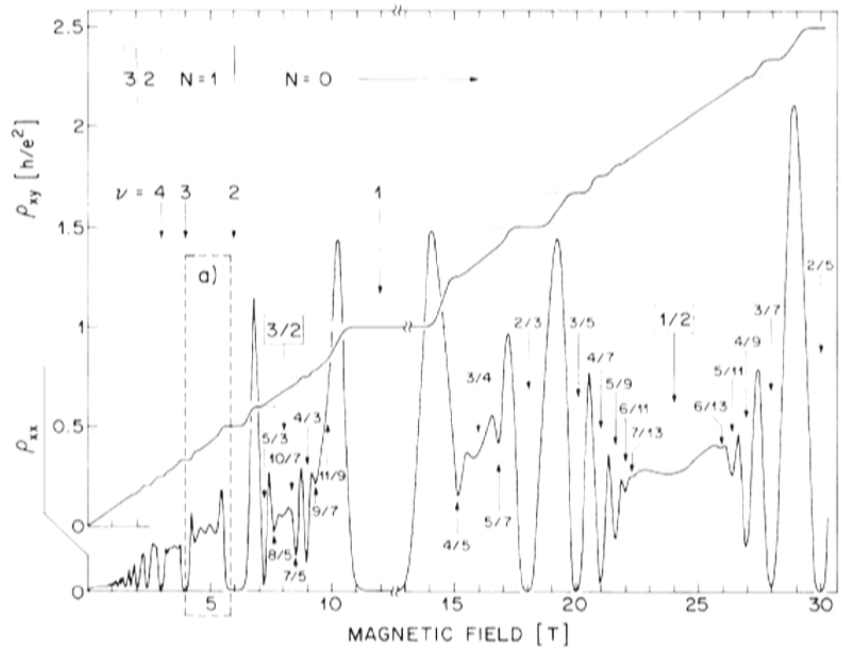

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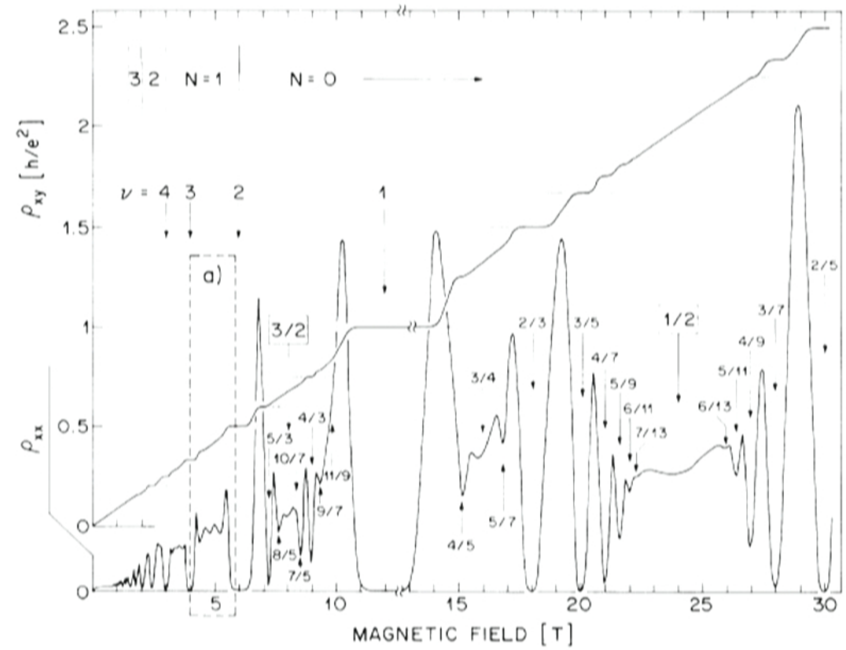

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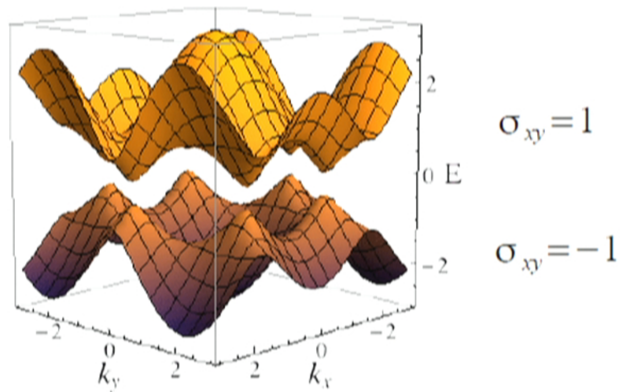
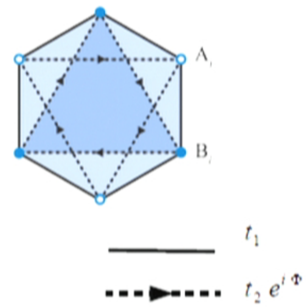
Are there other ways to get flatbands?

Are they as interesting as Landau levels?

Integer Quantum Hall effect without Landau levels

F. D. M. Haldane, Phys. Rev. Lett. 61, 2015 (1988).





$$S_{eff} = \frac{1}{2}\underbrace{(\frac{e^2}{h}C)}_{\sigma_{xy}}\int dtd^2\mathbf{x}\varepsilon^{\mu\nu\lambda}A_\mu(x)\partial_\nu A_\lambda(x) + \cdots$$

$$C = \int_{\mathrm{BZ}} \frac{d^2\mathbf{k}}{2\pi}\boldsymbol{\nabla}\times\mathbf{A}(\mathbf{k}), \qquad \mathbf{A}(\mathbf{k}) = -i\,\chi^\dagger(\mathbf{k})\cdot\boldsymbol{\nabla}\chi(\mathbf{k})$$

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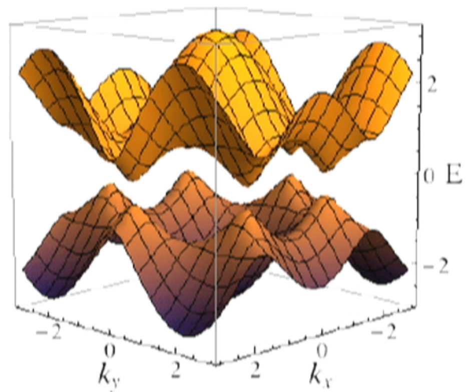
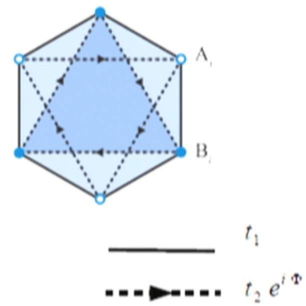


$\sigma_{xy} = 1$

$\sigma_{xy} = -1$



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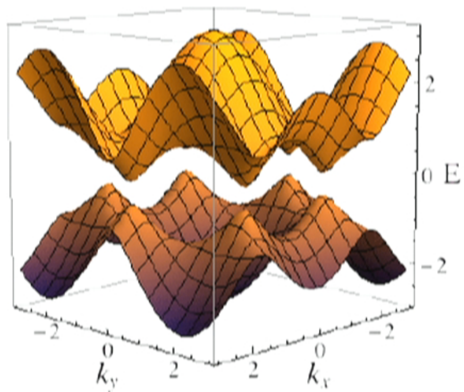

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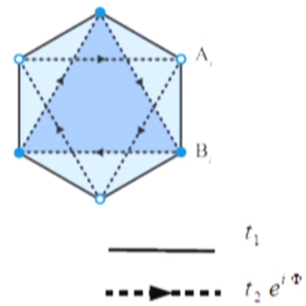

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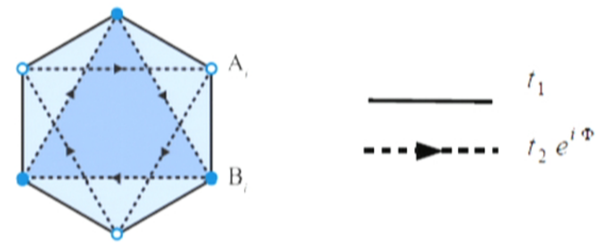
Two competing "masses"

t_2 : Haldane mass
μ_s : Staggered chemical potential



Low energy excitations have dispersion:

$$E^2(\mathbf{k}) = \mathbf{k}^2 + (t_2 - \mu_s)^2$$

$$|t_2/\mu_s| > 1 \Rightarrow |C| = 1 \qquad \text{(Chern Insulator at half-filling)}$$
$$|t_2/\mu_s| < 1 \Rightarrow C = 0 \qquad \text{(Trivial Insulator at half-filling)}$$

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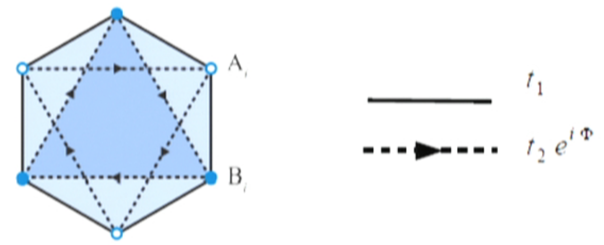

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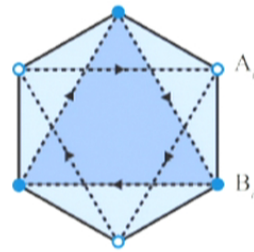
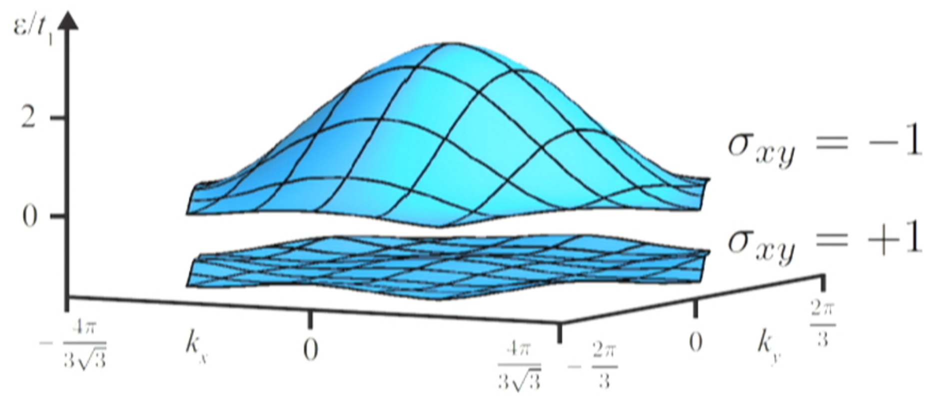
Integer Quantum Hall effect without Landau levels

$$t_2/t_1 = \frac{\sqrt{43}}{12\sqrt{3}} \approx 0.315495 \qquad \cos\Phi = \frac{1}{4}\,\frac{t_1}{t_2}$$

$$\delta_-/\Delta = 1/7$$

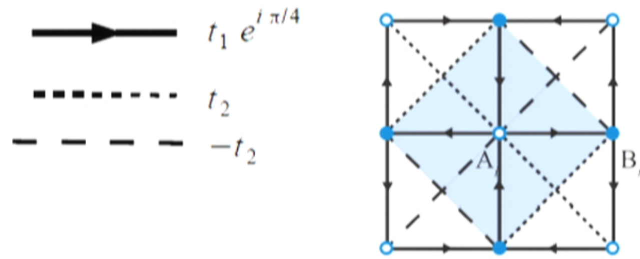


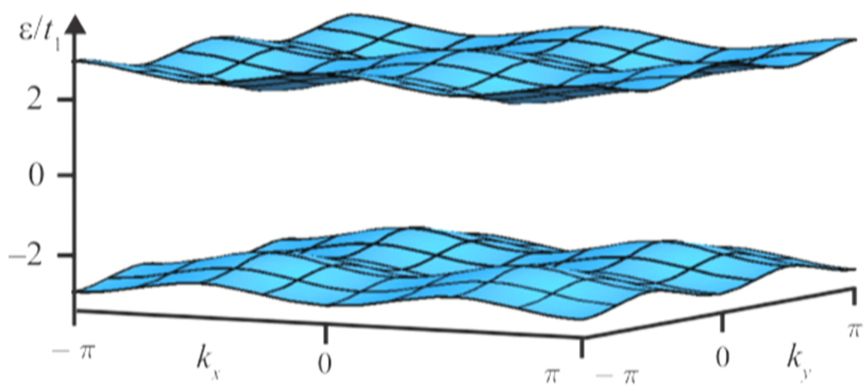


Integer Quantum Hall effect without Landau levels

$$t_2/t_1 = \frac{1}{\sqrt{2}}$$

$$\delta_-/\Delta \approx 1/5$$

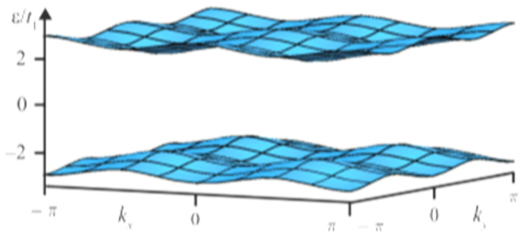


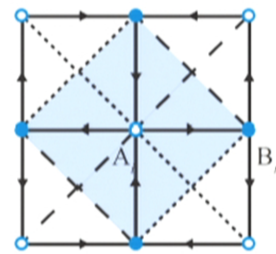


$$\sigma_{xy} = -1$$

$$\sigma_{xy} = +1$$

Can one get perfect flatbands?





$$H_0 := \sum_{\boldsymbol{k} \in \mathrm{BZ}} \psi^\dagger_{\boldsymbol{k}} \mathcal{H}_{\boldsymbol{k}} \psi_{\boldsymbol{k}}, \qquad \mathcal{H}_{\boldsymbol{k}} := B_{0,\boldsymbol{k}} \sigma_0 + \boldsymbol{B}_{\boldsymbol{k}} \cdot \boldsymbol{\sigma}$$

$$\mathcal{H}^{\mathrm{flat}}_{\boldsymbol{k}} := \frac{\mathcal{H}_{\boldsymbol{k}}}{\varepsilon_{-,\boldsymbol{k}}}$$

In particular $\quad B_{0,\boldsymbol{k}} \equiv 0 \quad \Rightarrow \quad \mathcal{H}^{\mathrm{flat}}_{\boldsymbol{k}} = \widehat{\boldsymbol{B}}_{\boldsymbol{k}} \cdot \boldsymbol{\sigma}$

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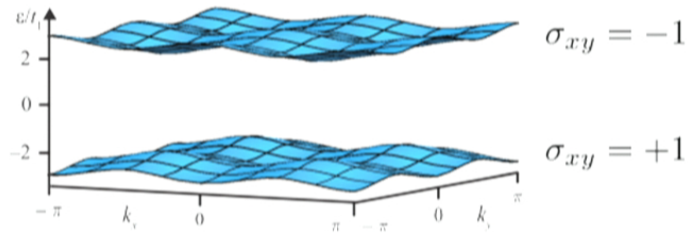

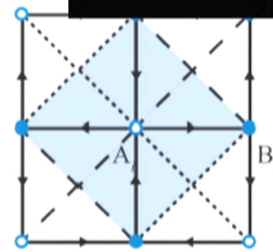

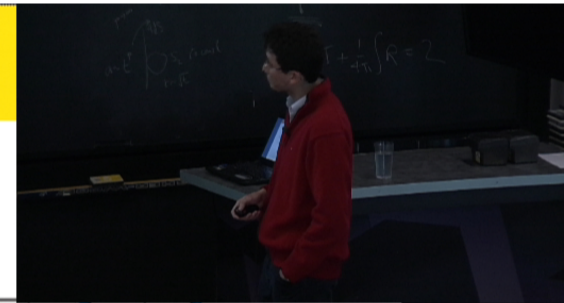

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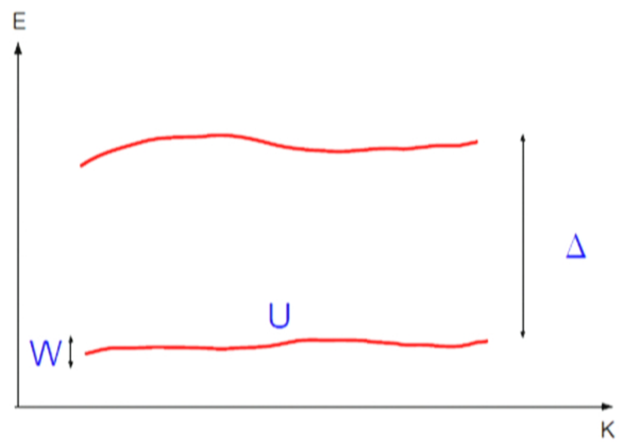
Flattening:

- Preserves the existence of the single particle gap
- Preserves the Chern number of the bands
- Does it preserve locality?

We are interested in the hierarchy of energy scales :

$$W << U << \Delta$$



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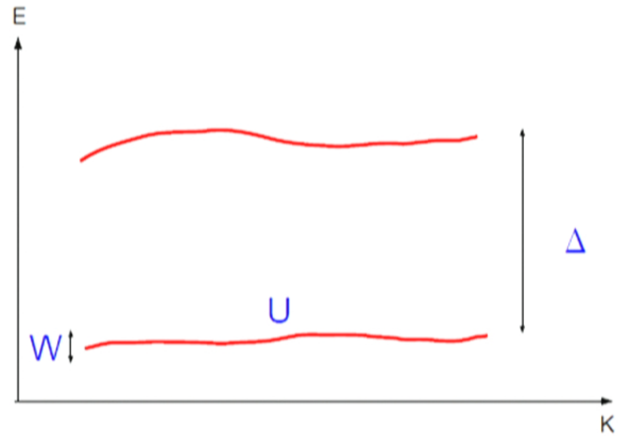

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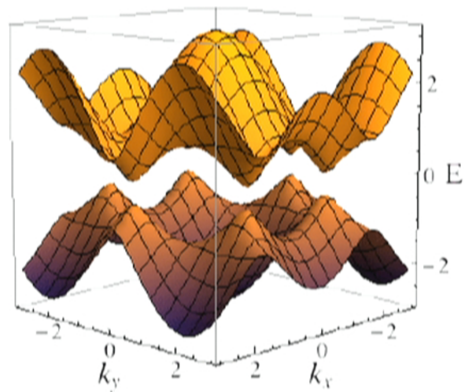

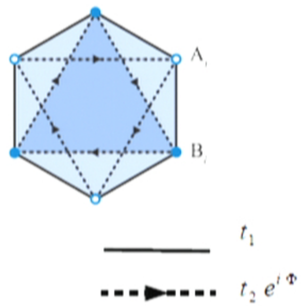

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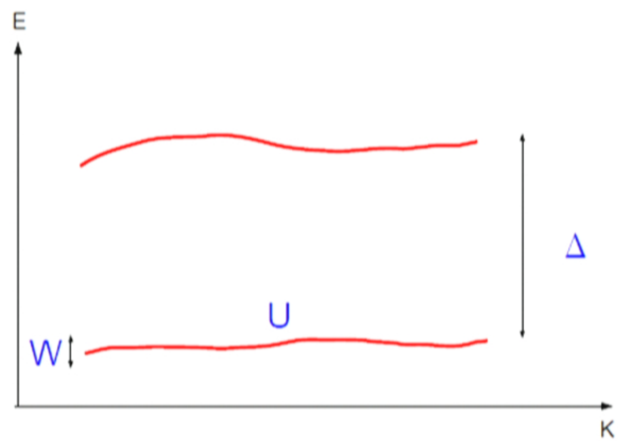

Can one get perfect flatbands in a local hoping Hamiltonian?

$$\mathcal{H}^{\mathrm{flat}}_{\mathbf{k}} := \frac{\mathcal{H}_{\mathbf{k}}}{\varepsilon_{-,\mathbf{k}}} \rightarrow \text{Hopping decays exponentially with distance}$$

$$\varepsilon_{\pm,\mathbf{k}} = (2t_1^2 - t_2^2)(2 + \cos k_+ + \cos k_-) + t_2^2(3 + \cos k_+ \cos k_-), \qquad k_\pm = k_x \pm k_y$$

$$\frac{1}{|\varepsilon_{+,\mathbf{k}}|} = \sum_{n,n'=0} C_{n,n'} \cos nk_+ \cos n'k_-$$

$C_{n,n'}$ decay exponentially with n and n'.

Can one get perfect flatbands?

Flattening:

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- Preserves locality

Are they as interesting as Landau levels?

Can one support a fractional Hall effect when flat bands with non-zero Chern number are partially filled?

Is there a fractional quantum Hall effect?

Two distinctive properties of a FQHE:

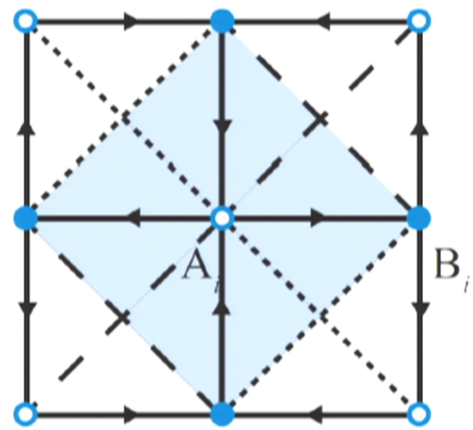
(i) Quantization of the Hall conductance

$$\sigma_{xy} = \nu \, \frac{e^2}{h}$$

(ii) Topological ground state degeneracy

$$N_{\mathrm{G}} = \nu^{-g}, \quad \text{for } \nu = 1/\mathrm{odd}$$

Add interactions



$$H_{\text{int}} := \frac{1}{2} \sum_{i,j} \rho_i V_{i,j} \rho_j \equiv V \sum_{\langle ij \rangle} \rho_i \rho_j, \qquad V > 0$$

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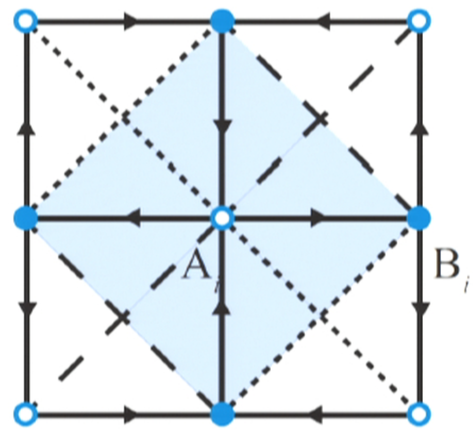

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Is there a FQHE?

Many-body Chern number as a response to twisted boundary conditions (Niu and Thouless):

$$|\Psi_\gamma(\boldsymbol{r}+N_x\boldsymbol{x})\rangle = e^{i\gamma_x}|\Psi_\gamma(\boldsymbol{r})\rangle$$
$$|\Psi_\gamma(\boldsymbol{r}+N_y\boldsymbol{y})\rangle = e^{i\gamma_y}|\Psi_\gamma(\boldsymbol{r})\rangle$$

$$\sigma_{xy} = \frac{e^2}{h}C \qquad C = \frac{1}{2\pi i}\int_{\gamma\in[0,2\pi]^2} \nabla_\gamma \times \langle\Psi_\gamma|\nabla_\gamma|\Psi_\gamma\rangle$$

3×6 *plaquettes* $(36$ *sites*$)$
$\nu = 1/3$ *lower band* $(6$ *particles*$)$

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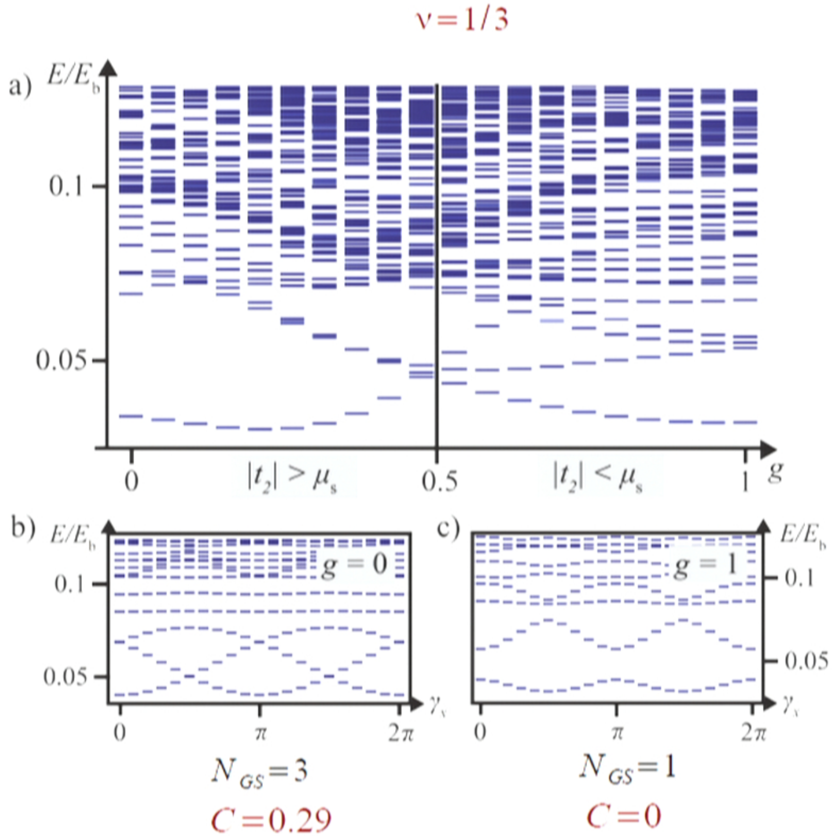
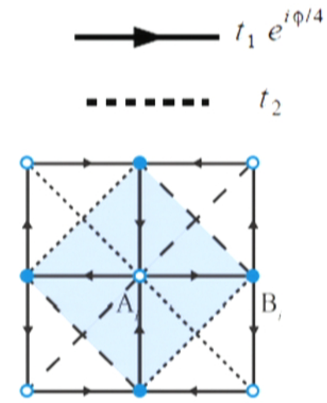
Gap and Ground state degeneracy





G. W. Semenoff, PRL 53, 2449 (1984)
F.D.M. Haldane, PRL 61, 2015 (1988)

μ_s vs. $|t_2|$

drives a topological phase transition

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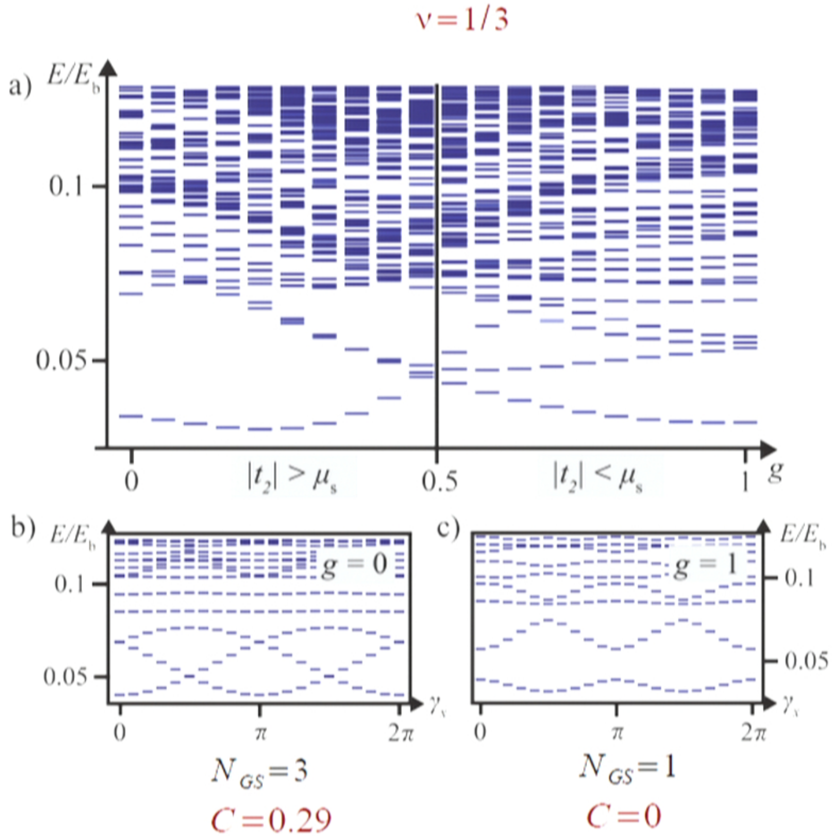

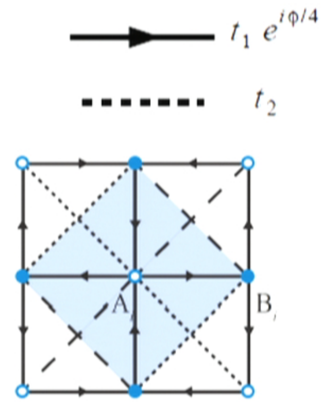

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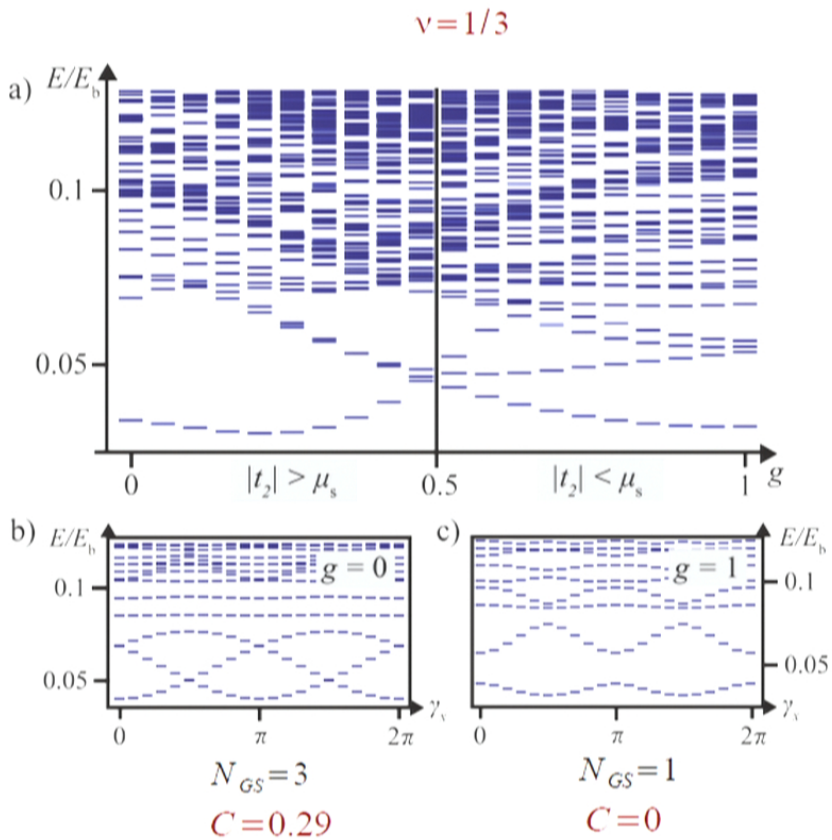

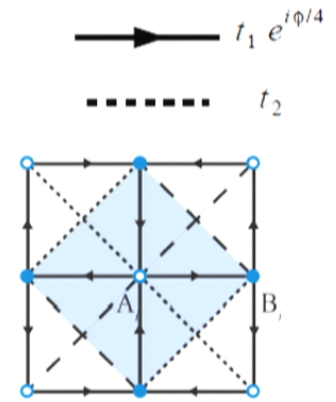

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"Effective" Girving-MacDonald-Platzman algebra for CI's

QHE (Uniform B field):

Density: $\rho_o(\mathbf{q}) = e^{i\mathbf{q}\cdot\mathbf{r}}$

Guiding-center Density: $\rho(\mathbf{q}) = e^{i\mathbf{q}\cdot\mathbf{R}}, \quad [R_\alpha, R_\beta] = i\varepsilon_{\alpha,\beta} l_B^2$

$$[\rho(\mathbf{q}), \rho(\mathbf{q}')] = 2i\sin(\frac{1}{2}\mathbf{q}\times\mathbf{q}' l_B^2)\rho(\mathbf{q}+\mathbf{q}')$$ Girving-Macdonald-Platzman (85)

"Chern" insulator:

$$\rho(\mathbf{q}) = \sum_{\mathbf{k}} c_\alpha^\dagger(\mathbf{k}) c_\alpha(\mathbf{k}+\mathbf{q}) = \sum_{\mathbf{k}} \underbrace{\chi_\alpha^{*\lambda_1}(\mathbf{k})\chi_\alpha^{\lambda_2}(\mathbf{k}+\mathbf{q})}_{\text{Eigenspinors}} \psi_{\lambda_1}^\dagger(\mathbf{k})\psi_{\lambda_2}(\mathbf{k}+\mathbf{q})$$

$$[\bar{\rho}(\mathbf{q}), \bar{\rho}(\mathbf{q}')] \approx i(\mathbf{q}\times\mathbf{q}')\cdot\hat{z}\frac{2\pi C}{(2\pi/a)^2}\bar{\rho}(\mathbf{q}+\mathbf{q}') \quad (\text{low } \mathbf{q}, \mathbf{q}' \text{ expansion})$$

$\bar{\rho}(\mathbf{q})$: Density projected onto a Chern band Parameswaran-Roy-Sondhi (11)

Derivation assumes that the "Berry" curvature $B(\mathbf{k})$ is uniform in $\mathbf{k}$-space, which is not a generic feature. In particular, for the 2-band models used in ED studies, $B(\mathbf{k})$ is not be uniform!

Open questions:

1) What is the many-body wave function of the Fractional Chern Insulator? This is a much harder problem than in the uniform B case since single particle wavefunctions in a Chern band do not have “nice” properties of LLL wavefunctions.

2) Landau levels are characterized by $C = 1$. The lattice problem opens the possibility of exploring strongly correlated states on $C > 1$ bands. (Unknown territory!)

3) Materials ?!

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Part 2

2D Fractional Topological Insulators

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Chern-Simons theory and the $\nu = \frac{1}{p}$ FQHE (odd integer p)

A_μ: external EM field, a_μ: statistical gauge field

$$\mathcal{L}_{\mathrm{CS}} = -\frac{p}{4\pi}\,\varepsilon^{\mu\nu\lambda}\,a_\mu\,\partial_\nu\,a_\lambda + \frac{e}{2\pi}\,\varepsilon^{\mu\nu\lambda}\,A_\mu\,\partial_\nu\,a_\lambda$$

$$J^\mu = \frac{e}{2\pi}\,\varepsilon^{\mu\nu\lambda}\,\partial_\nu\,a_\lambda = \frac{e^2}{2\pi\,p}\,\varepsilon^{\mu\nu\lambda}\,\partial_\nu\,A_\lambda \equiv \sigma_{\mathrm{H}}\,\varepsilon^{\mu\nu\lambda}\,\partial_\nu\,A_\lambda$$

Time-reversal transformation:

$$\begin{aligned}
&g^{\mu\nu} := (1, -1, -1) \\
&A_\mu(t, \mathbf{x}) \xrightarrow{\mathcal{T}} g^{\mu\nu} A_\nu(-t, \mathbf{x}) \\
&a_\mu(t, \mathbf{x}) \xrightarrow{\mathcal{T}} g^{\mu\nu} a_\nu(-t, \mathbf{x}) \\
&\mathcal{L}_{CS} \xrightarrow{\mathcal{T}} -\mathcal{L}_{CS} \quad \text{(Breaks TRS)}
\end{aligned}$$

TRS abelian fractional topological insulator

Simplest scenario: Two decoupled FQH states with opposite chiralities (Levin and Stern, 2009).

Pair of gauge fields $a_{1,\mu}$ and $a_{2,\mu}$ transforming under TR as

$$a_{1,\mu} \xrightarrow{\mathcal{T}} -g^{\mu\nu} a_{2,\mu}$$

$$\begin{aligned}\mathcal{L} = &- \frac{p_1}{4\pi} \varepsilon^{\mu\nu\lambda} a_{1,\mu} \partial_\nu a_{1,\lambda} + \frac{e}{2\pi} \varepsilon^{\mu\nu\lambda} A_\mu \partial_\nu a_{1,\lambda} \\ &+ \frac{p_1}{4\pi} \varepsilon^{\mu\nu\lambda} a_{2,\mu} \partial_\nu a_{2,\lambda} + \frac{e}{2\pi} \varepsilon^{\mu\nu\lambda} A_\mu \partial_\nu a_{2,\lambda}\end{aligned}$$

$$\mathcal{L} \xrightarrow{\mathcal{T}} \mathcal{L} \quad (\text{ TRS symmetric })$$

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$$+\frac{p_1}{4\pi}\varepsilon^{\mu\nu\lambda} a_{2,\mu}\partial_\nu a_{2,\lambda} + \frac{e}{2\pi}\varepsilon^{\mu\nu\lambda} A_\mu \partial_\nu a_{2,\lambda}$$

$$\mathcal{L} \xrightarrow{\mathcal{T}} \mathcal{L} \;\; (\text{ TRS symmetric })$$

2-D BF field theory

$$a_\mu^{(\pm)} := \frac{1}{2}\left(a_{1,\mu} \pm a_{2,\mu}\right)$$

$$\mathcal{L}_{\mathrm{BF}}^{\mathrm{TRS}} := -\frac{p}{\pi}\,\epsilon^{\mu\nu\lambda}\,a_\mu^{(+)}\,\partial_\nu\,a_\lambda^{(-)} + \frac{e}{\pi}\,\epsilon^{\mu\nu\lambda}\,A_\mu\,\partial_\nu\,a_\lambda^{(+)}$$

$$J_\pm^\mu := \frac{e}{\pi}\epsilon^{\mu\nu\lambda}\partial_\nu a_\lambda^{(\pm)}$$

Equations of motion:

$$J_+^\mu = 0 \qquad \text{(Charge Current)}$$

$$J_-^\mu = 2e \times \frac{e}{2\pi p}\,\epsilon^{\mu\nu\lambda}\,\partial_\nu\,A_\lambda \qquad \text{(“Spin Current”)}$$

Hierarchy of Time-Reversal Symmetric BF theories

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Let $\tilde{j}_{\pm}^{\mu}$ be the conserved currents of quasi-particle excitations. The hierarchy theory is achieved via the following flux attachment condition that preserves TRS:

$$\tilde{j}_{\pm}^{\mu} = \frac{\epsilon^{\mu\nu\lambda}}{\pi\,p_2}\,l^{(\pm)}\,\partial_{\nu}\,a_{\lambda}^{(\pm)}, \quad p_2, l^{\pm} \in \mathbb{Z}$$

The constraint means that any pair of flux quanta, arising when $a^{(+)}$ and $a^{(-)}$ each support a vortex, creates a quasi-particle with charge $2\,l^{(+)}/p_2$ and spin $2\,l^{(-)}/p_2$.

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Introduce two new gauge fields $\tilde{a}^{(\pm)}$ by

$$\tilde{j}^{\mu}_{\pm} := \frac{1}{\pi}\,\epsilon^{\mu\nu\lambda}\,\partial_{\nu}\,\tilde{a}^{(\pm)}_{\lambda}, \quad \partial_{\mu}\,\tilde{j}^{\mu}_{\pm} = 0$$

The second level of the hierarchy is described by the Lagrangian

$$\mathcal{L}^{\mathrm{TRS}}_{2} := -\,\frac{p}{\pi}\,\epsilon^{\mu\nu\lambda}\,a^{(+)}_{\mu}\,\partial_{\nu}\,a^{(-)}_{\lambda} + \frac{e}{\pi}\,\epsilon^{\mu\nu\lambda}\,A_{\mu}\,\partial_{\nu}\,a^{(+)}_{\lambda}$$
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$$\mathcal{L}_2^{\mathrm{TRS}} := -\sum_{\mathrm{i,j}=1}^{2} \frac{1}{\pi}\, \varkappa_{\mathrm{ij}}^{(2)}\, \epsilon^{\mu\nu\lambda}\, a_{\mathrm{i},\mu}^{(+)}\, \partial_\nu\, a_{\mathrm{j},\lambda}^{(-)} + \sum_{\mathrm{i}=1}^{2} \frac{e}{\pi}\, \varrho_{\mathrm{i}}^{(2)}\, \epsilon^{\mu\nu\lambda}\, A_\mu\, \partial_\nu\, a_{\mathrm{i},\lambda}^{(+)}$$

$$a_{\mathrm{i},\mu}^{(\pm)} = \left(a_{1,\mu}^{(\pm)},\ \tilde{a}_{1,\mu}^{(\pm)} \right), \qquad \varrho_{\mathrm{i}} = (1,0)$$

$$\varkappa^{(2)} = \begin{pmatrix} p & -l^{+} \\ -l^{-} & p_2 \end{pmatrix}$$

General hierarchy structure:

$$\varkappa^{(n+1)} = \begin{pmatrix} \varkappa^{(n)} & -l^{(+)} \\ -l^{(-)\mathrm{T}} & p_{n+1} \end{pmatrix}, \qquad \varrho^{(n+1)} = \left(\varrho^{(n)},\ 0 \right)$$

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TRS abelian 2-D field theory

$$a^{(\pm)}_{\mathrm{i},\mu} := \frac{1}{2}\left(a_{\mathrm{i},\mu} \pm a_{\mathrm{i}+N,\mu}\right), \qquad \mathrm{i} = 1, \dots, N$$

$$\mathcal{S} := \int \mathrm{d}t\,\mathrm{d}^2\mathbf{x}\,\epsilon^{\mu\nu\rho}\left(-\frac{1}{\pi}\varkappa_{\mathrm{ij}}\,a^{(+)}_{\mathrm{i},\mu}\,\partial_\nu\,a^{(-)}_{\mathrm{j},\rho} + \frac{e}{\pi}\,\rho_{\mathrm{i}}\,A_\mu\partial_\nu\,a^{(+)}_{\mathrm{i},\rho}\right)$$

$$\mathcal{S} := \int \mathrm{d}t\,\mathrm{d}^2\mathbf{x}\epsilon^{\mu\nu\rho}\left(-\frac{1}{4\pi}K_{ij}\,a_{i,\mu}\,\partial_\nu\,a_{j,\rho} + \frac{e}{2\pi}\,Q_i\,A_\mu\,\partial_\nu\,a_{i,\rho}\right)$$

$$K = \begin{pmatrix} \kappa & \Delta \\ \Delta^{\mathsf{T}} & -\kappa \end{pmatrix}, \quad Q = \begin{pmatrix} \varrho \\ \varrho \end{pmatrix}$$

$$\kappa = \kappa^{\mathsf{T}}, \quad \Delta = -\Delta^{\mathsf{T}}, \quad \varkappa = \kappa - \Delta$$

$$N_{\mathrm{GS}} = |\det(\mathrm{K})| = [\det(\varkappa)]^2 = (\text{integer})^2$$

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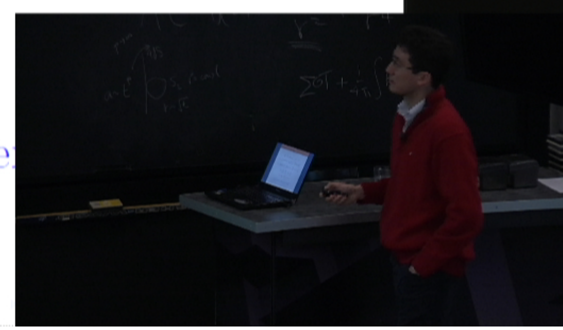

Stability of the edge modes

$$\widehat{\Psi}^{\dagger}_{T}(t,x) := \,:\, e^{-\mathrm{i}\, T_i\, K_{ij}\, \hat{\Phi}_j\,(t,x)} \,: \quad \text{Fermi-Bose operators}$$

$$\hat{\rho} = \frac{1}{2\pi}\, Q_i\ \partial_x \hat{\Phi}_i\ , \quad \text{charge density}$$

$$\hat{H} := \hat{H}_0 + \hat{H}_{\mathrm{int}}$$

$$\hat{H}_0 := \int_0^L \mathrm{d}x\, \frac{1}{4\pi} \partial_x \hat{\Phi}^{\mathsf{T}}\ V\ \partial_x \hat{\Phi}$$

Tunneling among the different edge branches:

$$\hat{H}_{\mathrm{int}} := -\int_0^L \mathrm{d}x \sum_{T,\ T^{\mathsf{T}}Q=0} h_T(x) : \cos\Big(T^{\mathsf{T}} K\, \hat{\Phi}(x) + \alpha_T(x)\Big) :$$

Bulk-Edge Correspondence

$$\mathcal{S}_K := \int \mathrm{d}t\, \mathrm{d}^2\mathbf{x}\, \epsilon^{\mu\nu\rho} \left(-\frac{1}{4\pi} K_{ij}\, a_{i,\mu}\, \partial_\nu\, a_{j,\rho}\right)$$

Gauge-fixing condition consistent with TRS:

$$a_0 = K^{-1}\, V\, a_1, \quad V = \Sigma_1\, V\, \Sigma_1, \quad K = -\Sigma_1\, K\, \Sigma_1$$

$$0 = \frac{\delta \mathcal{S}_K}{\delta a_0} \Longleftrightarrow \partial_1 a_2 - \partial_2 a_1 = 0, \quad \Rightarrow a_1 = \partial_1 \Phi, \quad a_2 = \partial_2 \Phi$$

Assume edge located on the x-axis at $y = 0$

$$\mathcal{S}_K = \frac{1}{4\pi} \int\limits_{-\infty}^{+\infty} \mathrm{d}t \int\limits_{-\infty}^{+\infty} \mathrm{d}x \left[(\partial_1 \Phi)^{\mathsf{T}}\, K\, \partial_0 \Phi - (\partial_1 \Phi)^{\mathsf{T}}\, V\, \partial_1 \Phi \right] (t, x, 0)$$

$$\Phi(t, x) \xrightarrow{\mathcal{T}} \Sigma_1\, \Phi(-t, x) + \pi K^{-1}\, \Sigma^{\downarrow}\, Q, \quad \Sigma^{\downarrow} = \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{I} \end{pmatrix}$$

Stability of the edge modes

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Stability of the edge modes

Restrict to TRS interaction:

$$\hat{H}_{\text{int}} := -\int_0^L \mathrm{d}x \sum_{T,\, T^{\mathsf{T}} Q = 0} h_T(x) : \cos\left(T^{\mathsf{T}} K\, \hat{\Phi}(x) + \alpha_T(x)\right) :$$

$$h_T(x) = h_{\Sigma_1 T}(x),$$

$$\alpha_T(x) = \left(-\alpha_{\Sigma_1 T}(x) + \pi T^{\mathsf{T}} \Sigma_{\downarrow} Q\right) \bmod 2\pi.$$

Edge modes unstable to processess in which:

$$\underbrace{T^{\mathsf{T}} K\, T' = 0}_{\text{Haldane's criterion}},$$

Stability of the edge modes

$$R := r\,\varrho^{\mathsf{T}}\,(\kappa - \Delta)^{-1}\,\varrho$$

r is the smallest integer such that all the N components of the vector $r\,(\kappa - \Delta)^{-1}\,\varrho$ are integers.

$\mathbb{Z}_2$ classification:

$R = \text{odd} \Rightarrow$ at least one edge branch remains gapless (stable)

$R = \text{even} \Rightarrow$ edge states are unstable to disorder

Even though the edge modes might be all gapped, the bulk theory is still interesting, contrary to the non-interacting case!

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Summary

- We have investigated electron bands with non-zero Chern number. In the limit where the band width becomes much smaller than the separation between bands and the particle interactions, we have found evidence for a FQHE at $1/3$ filling. .
- We have presented a hierarchical construction of an abelian toplogical field theory in 2-D that respects TRS, which can be a candidate theory for describing 2-D interacting TIs. The bulk theory describes fractionalized quasipaticles and edge excitations are made of counter-propagating pairs.
- We have numerically studied the spectrum of a TRS symmetric lattice hamiltonian. We have found three different ground states depending on the different parameters of the interactions. In special, we have found a case where TRS is spontenously broken and the ground state is a cohexistence of a ferromagnet and a FQHE.