Title: Group Field Theory and Simplicial Path Integrals

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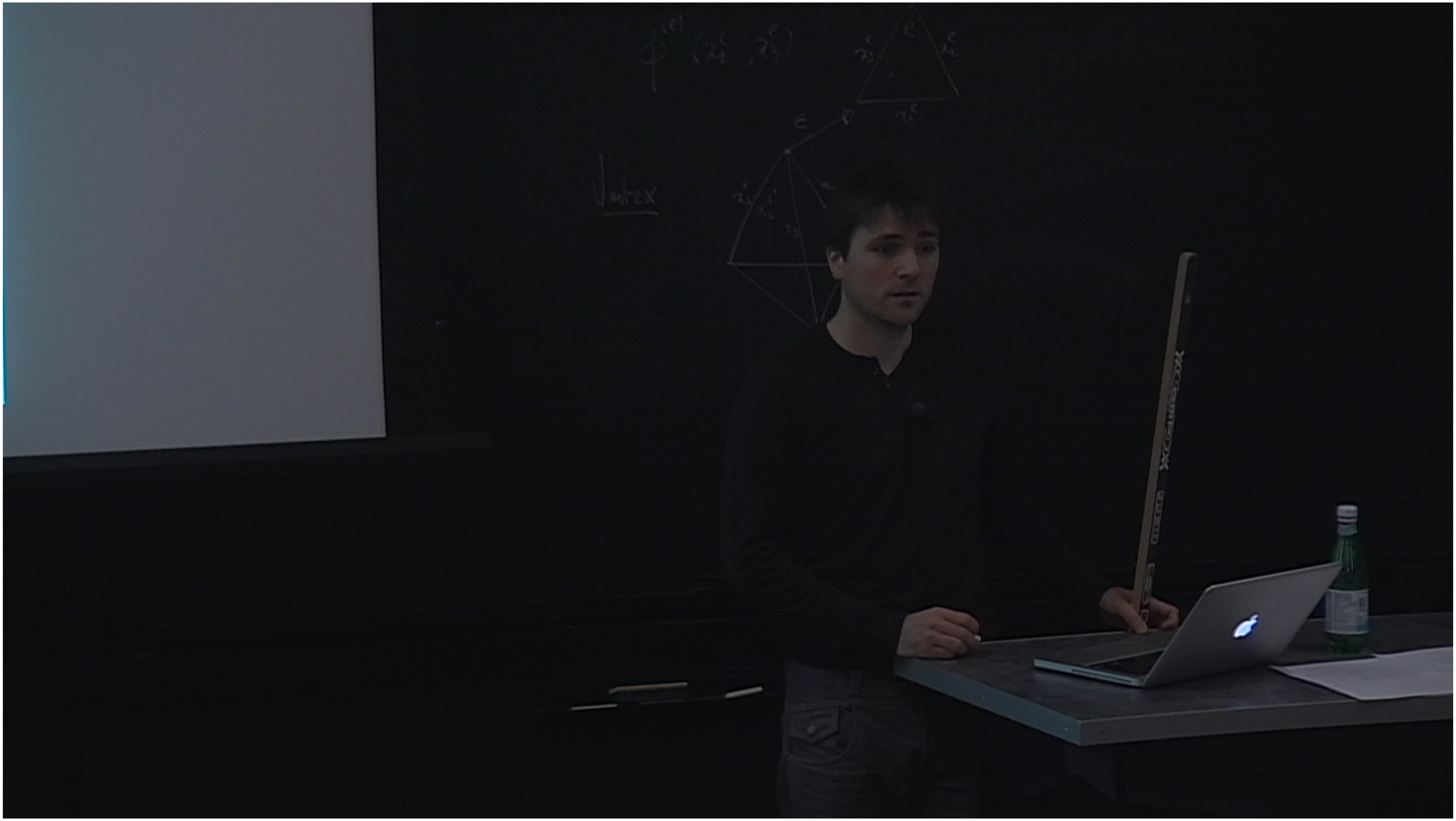
Abstract: Group field theories show up as a higher dimensional generalization of matrix models in background independent approaches to quantum gravity.
Their Feynman expansion generates simplicial complexes of all topologies weighted by spin foam amplitudes.
In this talk, we will present a dual formulation of these theories as non-commutative quantum fields theories, whose variables have a clear interpretation in terms of simplicial geometry. We will show that it gives a geometrically clear ways to define spin foam models for gravity which can be cast as


Vertex

Introduction

- Quantum gravity: the **basic problem** is to understand

$$\mathcal{Z}_M(\mathcal{O}) = \int [\mathcal{D}g][\mathcal{D}\phi] e^{iS(g,\phi)} \mathcal{O}[g,\phi]$$

with no background spacetime

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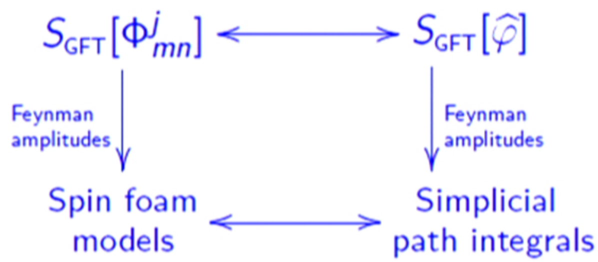
- **Key feature**: diffeomorphism invariance
- **Spin foam models:** convergence of various points of view:
  - Simplical gravity: Z_Δ discrete functional integral
  - Geometric quantization of simplicial structures
  - Loop quantum gravity: Spin foams = histories of spin networks. Projector onto physical states.
  - Matrix models: random geometries and topologies (group field theories).

Outline

- To present a dual formulation of GFT $\quad x_i \sim B_\Delta$

$$\widehat{\varphi}(x_1, \cdots x_4) := \int [\mathrm{d}g_i]^4 \, \varphi(g_1, \cdots g_4)\, \mathrm{e}^{i\mathrm{Tr} x_1 g_1} \ldots \mathrm{e}^{i\mathrm{Tr} x_4 g_4}$$



  - Spin foam models based on quantum BF = path integrals with explicit star product on space of B variables
  - Explicit measure on classical B, explicit action, star product induce corrections to E.O.M due to quantum geometry.
  - algebraic spin foam weights $\sim$ non-commutative geometry

- To discuss the encoding of (simplicial) difffeomorphism symmetry in GFT.

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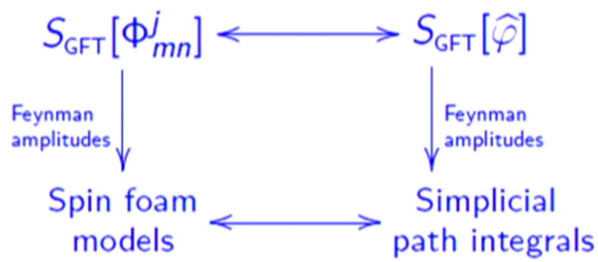

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GFT with non-commutative metric variables

Group Field Theory for 3d Riemannian gravity: $G = \mathrm{SO}(3)$ [Boulatov]

- Field $\varphi_{123} := \varphi(g_1, g_2, g_3)$ on G^3 with shift invariance:

$$\forall h \in G, \quad \varphi(hg_1, hg_2, hg_3) = \varphi(g_1, g_2, g_3)$$

- Dynamics governed by the action:

$$S = \frac{1}{2} \int [\mathrm{d}g]^3 \, \varphi_{123}^2 + \frac{\lambda}{4!} \int [\mathrm{d}g]^6 \, \varphi_{123} \varphi_{345} \varphi_{526} \varphi_{641}$$

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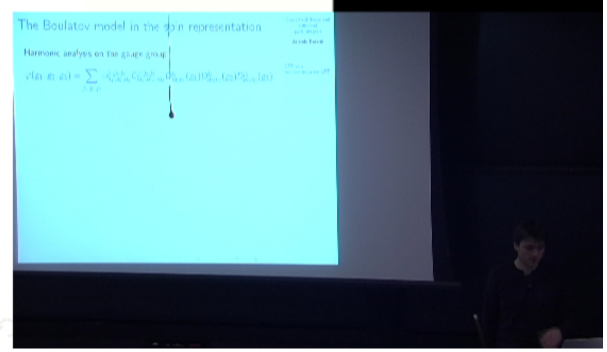
- Two Fourier dual representations: $\varphi(g_i) \mapsto \Phi^{j_i}_{m_i n_i}$ or $\widehat{\varphi}(\vec{x}_i)$

The Boulatov model in the spin representation

Harmonic analysis on the gauge group:

$$\varphi(g_1, g_2, g_3) = \sum_{j_1, j_2, j_3} \phi^{j_1, j_2, j_3}_{n_1, n_2, n_3} C^{j_1, j_2, j_3}_{m_1, m_2, m_3} D^{j_1}_{m_1 n_1}(g_1) D^{j_2}_{m_2 n_2}(g_2) D^{j_3}_{m_3 n_3}(g_3)$$



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- Field pictured as a 3-valent spin network vertex: quantum triangle.
- Vertex term written in terms of 6j symbols.

$$\mathcal{I}_{\mathcal{G}} = \sum_{\{j_e \in \frac{1}{2}\mathbb{N}\}} \prod_{\text{edges}} (2j_e + 1) \prod_{\text{tetrahedra}} \{6j\text{-symbol}\}_{SU(2)}$$

$\mathcal{I}_{\mathcal{G}}$ is the the Ponzano-Regge spin foam amplitude

The Boulatov model in metric space

Group Fourier transform of the Boulatov field:

$$\hat{\varphi}(x_1, x_2, x_3) = \int [\mathrm{d}g]^3 \, \varphi(g_1, g_2, g_3) \, e^{i \mathrm{Tr} x_1 g_1} e^{i \mathrm{Tr} x_2 g_2} e^{i \mathrm{Tr} x_3 g_3}$$

- plane waves: $e_g(x) = e^{i \mathrm{Tr} x g} = e^{i \vec{p}_g \cdot \vec{x}}$.
  $\vec{p}_g = \mathrm{tr} g \vec{\tau}$ coordinates on the group manifold.

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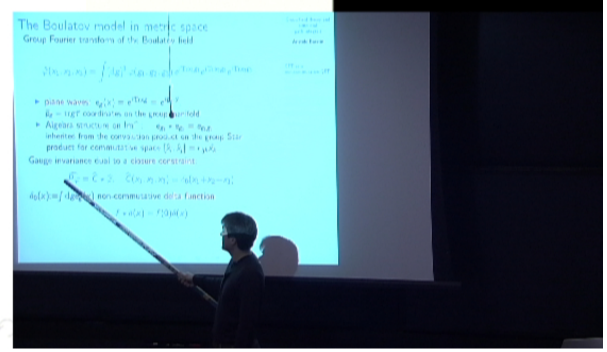
Gauge invariance dual to a closure constraint:

$$\widehat{P\varphi} = \widehat{C} \star \widehat{\varphi}, \quad \widehat{C}(x_1, x_2, x_3) = \delta_0(x_1 + x_2 + x_3)$$

$\delta_0(x) := \int \mathrm{d}g e_g(x)$ non-commutative delta function

$$f \star \delta(x) = f(0)\delta(x)$$



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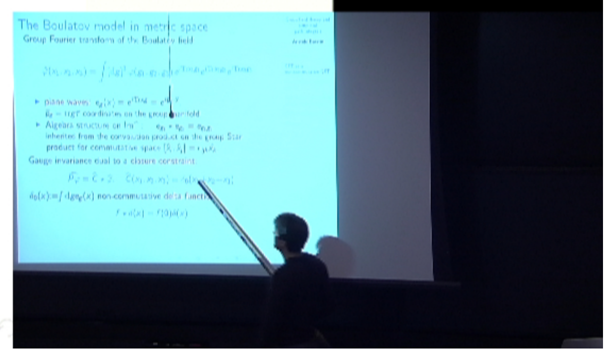

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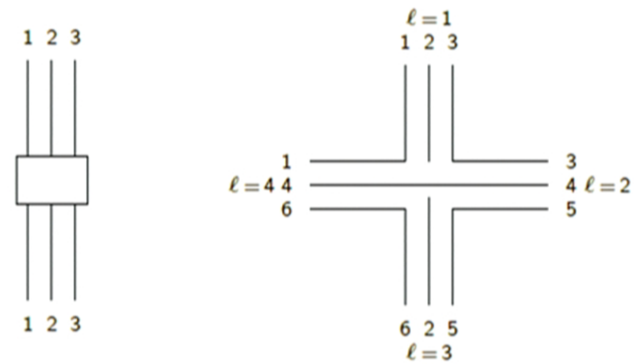
- Dual field as a (non-commutative) triangle

The Boulatov model in metric space

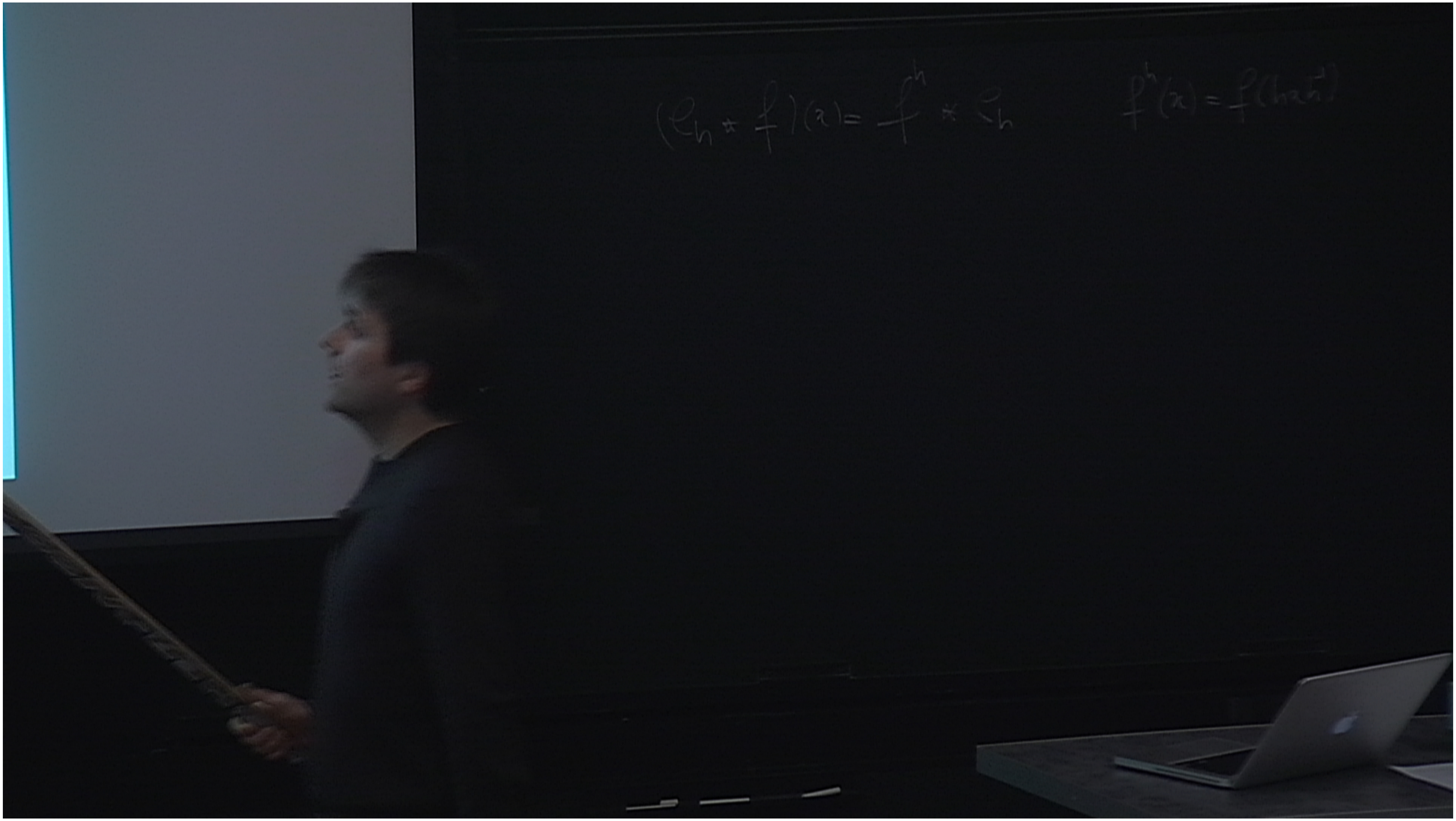
- The action reads:

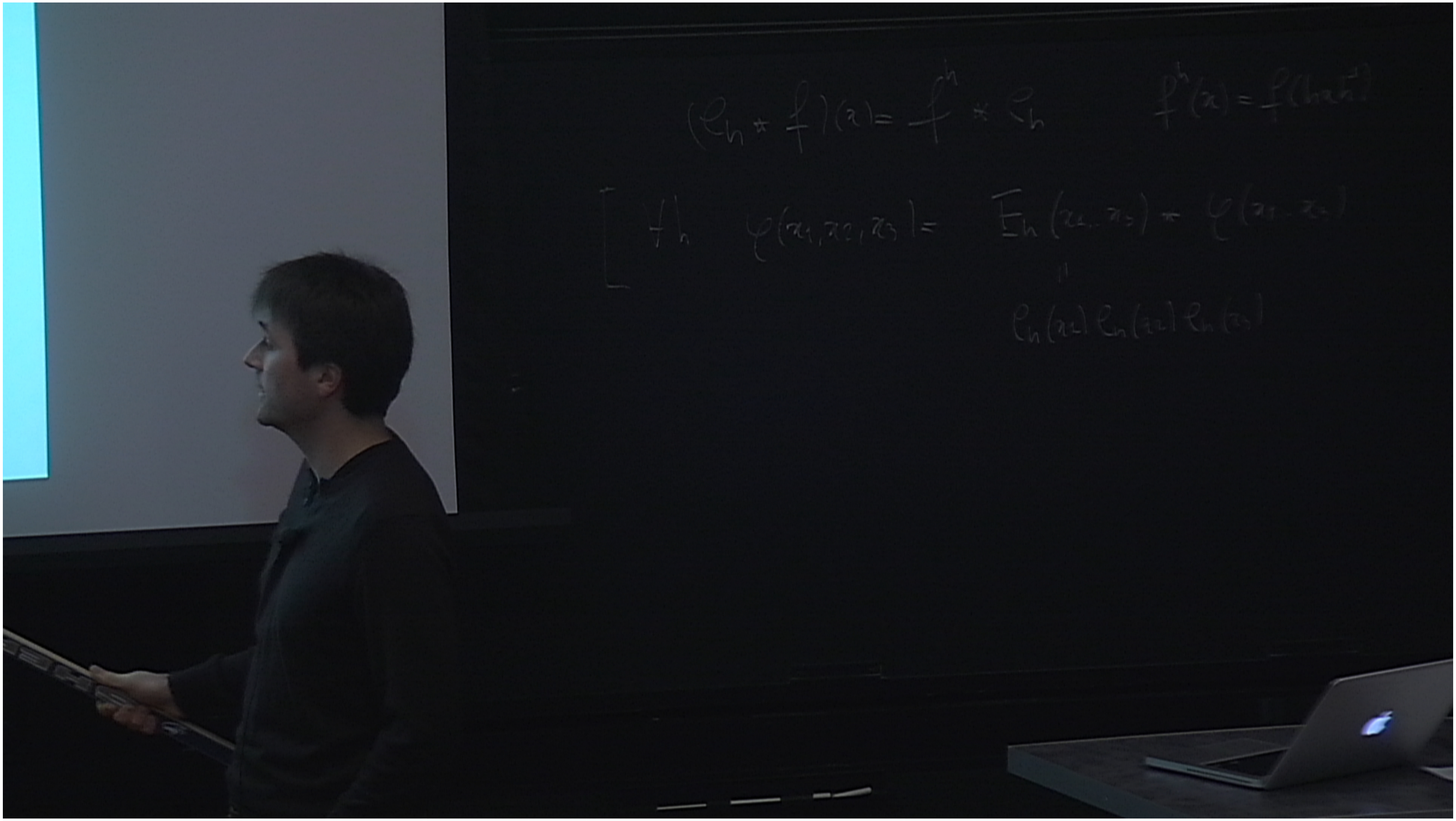
$$S = \frac{1}{2}\int [\mathrm{d}x]^3\, \widehat{\varphi}_{123}\star\widehat{\varphi}_{-1-2-3} - \frac{\lambda}{4!}\int [\mathrm{d}x]^6\, \widehat{\varphi}_{123}\star\widehat{\varphi}_{-345}\star\widehat{\varphi}_{-5-26}\star\widehat{\varphi}_{-6-4-1}$$

- Feynman rules:



$$\int \mathrm{d}h \prod_{i=1}^{3} (\delta_{-x_i} \star e_h)(x_i'), \qquad \int [\mathrm{d}h_\ell]^4 \prod_{i=1}^{6} (\delta_{-x_i^\ell} \star e_{h_\ell h_{\ell'}^{-1}})(x_i^{\ell'})$$


$(e_h * f)(x) = f^h * e_h$
$f^h(x) = f(hxh^{-1})$



The Boulatov model in metric space

- Under $\int \mathrm{d}h$, the amplitude factorizes into a product of face (= loop of strands) amplitude:

$$\mathcal{A}_{f_e}[h] = \int \prod_{j=0}^{N_e} [\mathrm{d}^3 x_j] \, \vec{\bigstar}_{j=0}^{N_e} \, (\delta_{x_j} \star \mathrm{e}_{h_{jj+1}})(x_{j+1})$$

- For each edge e, integrate over all x_j but one $x_e := x_0$:

$$\mathcal{I}_{\mathcal{G}} = \int \prod_l \mathrm{d}h_l \prod_e \mathrm{d}x_e \, e^{i \sum_e \mathrm{Tr}\, x_e H_e}$$

Discrete form of the 3d gravity action

$$S = \int \mathrm{Tr} e \wedge F(A) \quad \rightarrow \quad S_\Delta = \sum_e \mathrm{Tr}\, x_e H_e$$

$H_e := h_{t_1} \cdots h_{t_{N_e}}$ holonomy along the closed loop going across all the triangles sharing e.

$\mathcal{I}_{\mathcal{G}}$ is a simplicial path integral for 3d gravity

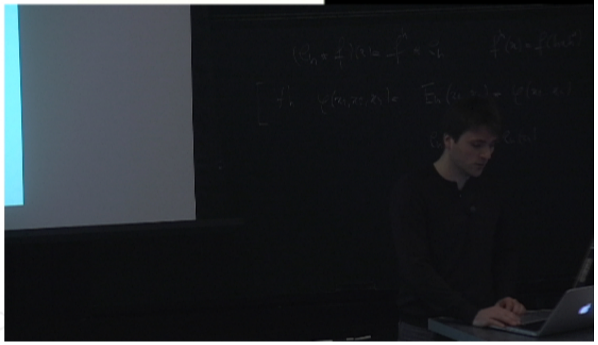
- boundary data inserted using star product

Going up dimension

Ooguri model for Spin(4) BF theory in the bivector representation:

$$\widehat{\varphi}(x_1, \cdots x_4) := \int [\mathrm{d}g_i]^4 \, \varphi(g_1, \cdots g_4) \, e^{i\mathrm{Tr} x_1 g_1} \ldots e^{i\mathrm{Tr} x_4 g_4}$$

$$\widehat{\varphi} = \delta(x_1 + .. + x_4) \star \widehat{\varphi}, \qquad \delta(x) := \int \mathrm{d}h E_h(x)$$



Constrained models for gravity

$$S = \frac{1}{2}\int \widehat{\varphi}^{\star 2}_{1234} - \frac{\lambda}{5!}\int \widehat{\varphi}_{1234} \star \widehat{\varphi}_{4567} \star \widehat{\varphi}_{7389} \star \widehat{\varphi}_{962\,10} \star \widehat{\varphi}_{10\,851}$$

- $\{x_j\}$ discrete B field, Feynman amplitudes = BF path integrals
- Idea: implement linear simplicity constraints in the GFT action: $\widehat{\varphi} \to (\mathcal{S} \star \widehat{\varphi})(x_j)$ for constraint functions $\mathcal{S}(x_j)$.

$$\forall t \subset \tau, \quad \exists k_\tau \in \mathrm{SU}(2), \quad k_\tau x_t^- k_\tau^{-1} + \beta x_t^+ = 0$$

$\beta = \frac{\gamma - 1}{\gamma + 1}$ [Alexandrov, Freidel Krasnov...]

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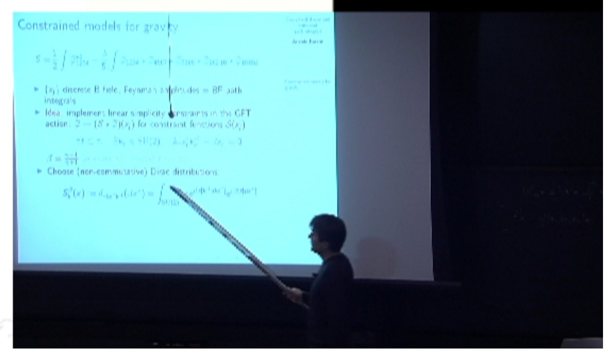

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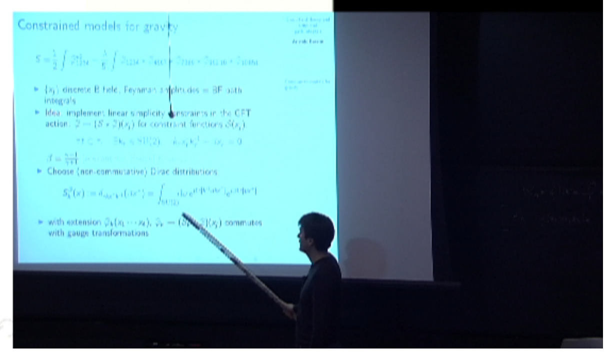

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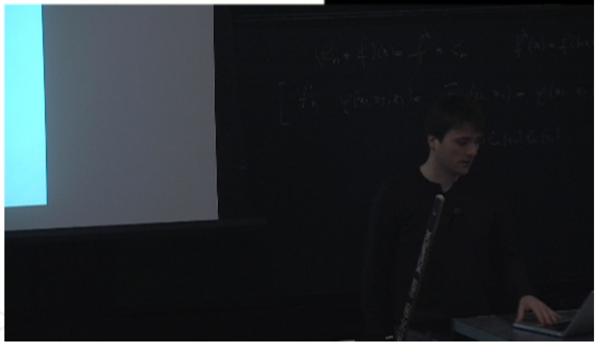
- GFT as theory of dynamical (non-commutative) geometric tetrahedra

Spin foams as simplicial gravity path integrals

- By construction: Feynman amplitudes are path integrals for BF theories with Holst-Plebanski constraints and non commutative B variables



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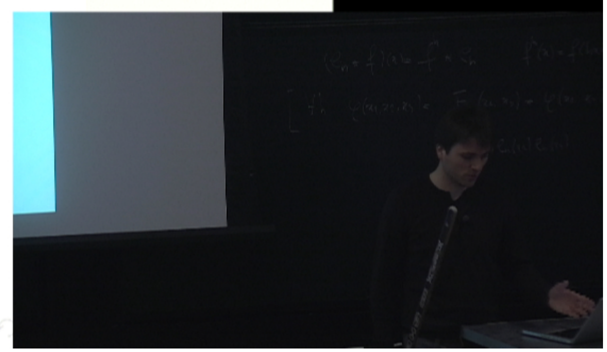

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- Dual formulations:

$$I_\Delta^\beta = \sum_{\{J_t, \iota_\tau\}} \prod_t d_{j_t^-} d_{j_t^+} \prod_\sigma \{15j\}_\sigma^- \{15j\}_\sigma^+ \prod_\tau f^{\iota_\tau}_{\iota^+, \iota^-}$$

$$= \int [\mathrm{d}h_{\tau\sigma}][\mathrm{d}^6 x_t] \left[\prod_t \bigstar_{j=0}^{N_t} \delta_{-\bar{h}_{0j} x_t^- \bar{h}_{0j}^{-1}}(\beta x_t^+) \right] \star e^{i \sum_t \mathrm{Tr}\, x_t H_t}$$

$$(e_h * f)(x) = f^h * e_h \qquad f^h(x) = f(hxh^{-1})$$

$$\forall h \quad \varphi(x_1, x_2, x_3) = E_h(x_1 \ldots x_3) * \varphi(x_1 \ldots x_3)$$

$$E_h(x_1 \ldots x_3) = e_h(x_1)\, e_h(x_2)\, e_h(x_3)$$

h_{0j} : holo ref $\tau_{j=0}$

$\longrightarrow \tau_j$

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Questions:

- Semi-classical limit, commutative limit?
- Compatible with canonical analysis of Holst -Plebanski?

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- Semi-classical limit, commutative limit?
- Compatible with canonical analysis of Holst -Plebanski?

Spin foams as simplicial gravity path integrals

- By construction: Feynman amplitudes are path integrals for BF theories with Holst-Plebanski constraints and non commutative B variables
- The strategy leads to a natural candidate for a model with γ, gives Barrett-Crane $\gamma = \infty$ but differs from existing model otherwise. No rationality condition for γ
- Dual formulations:

$$I_{\Delta}^{\beta} = \sum_{\{J_t, \iota_\tau\}} \prod_t d_{j_t^-} d_{j_t^+} \prod_\sigma \{15j\}_\sigma^- \{15j\}_\sigma^+ \prod_\tau f_{\iota^+, \iota^-}^{\iota_\tau}$$

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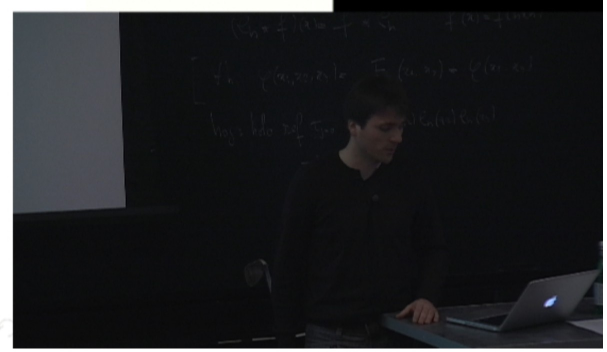
(Simplicial) diffeo and GFT

Question: can the various notions of diffeomorphisms invariance on discrete gravity be traced back to a symmetry of GFT?
(Colored) Boulatov model:

- Field transformation that ties together:
  - vertex translation invariance of discrete gravity
  - flatness constraint of canonical quantum gravity,
  - topological (coarse-graining) identities for the 6j-symbols.



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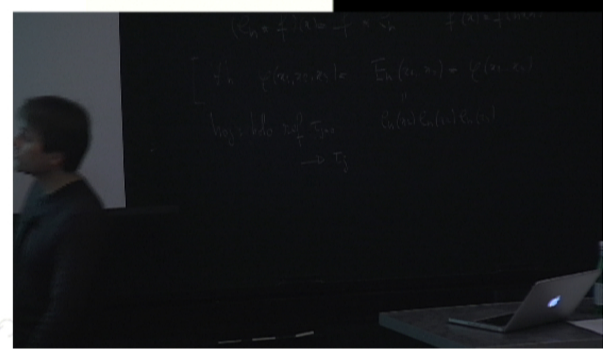
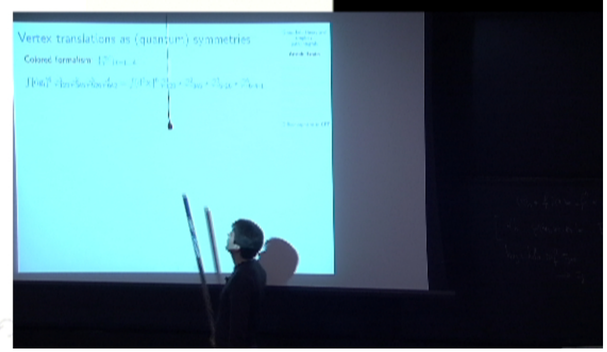
- Field transformation that ties together:
  - vertex translation invariance of discrete gravity
  - flatness constraint of canonical quantum gravity,
  - topological (coarse-graining) identities for the 6j-symbols.
- relates to gauge symmetries of discrete gravity action (Bianchi identities)
- *Deformed* translation symmetry which requires to embed GFT into framework of braided QFT.



Vertex translations as (quantum) symmetries

Colored formalism: $\{\widehat{\varphi}^\ell\}_{\ell=1\ldots4}$

$$\int[\mathrm{d}g_i]^6\, \varphi^1_{123}\varphi^2_{345}\varphi^3_{526}\varphi^4_{642} = \int[\mathrm{d}^3x_i]^6\, \widehat{\varphi}^1_{123}\star\widehat{\varphi}^2_{\text{-}345}\star\widehat{\varphi}^3_{\text{-}5\text{-}26}\star\widehat{\varphi}^4_{\text{-}6\text{-}4\text{-}1}$$



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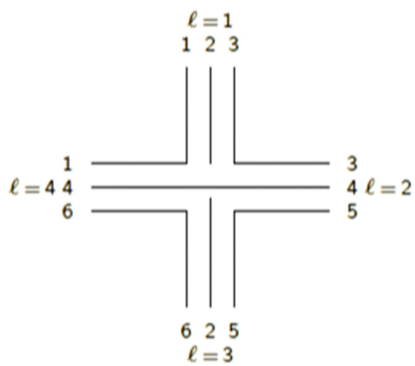
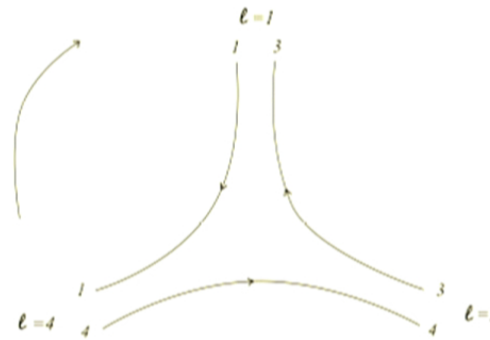
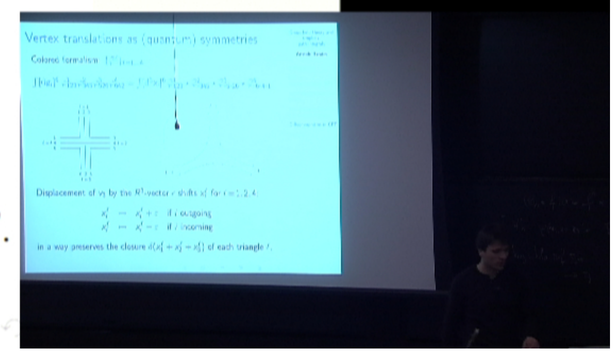


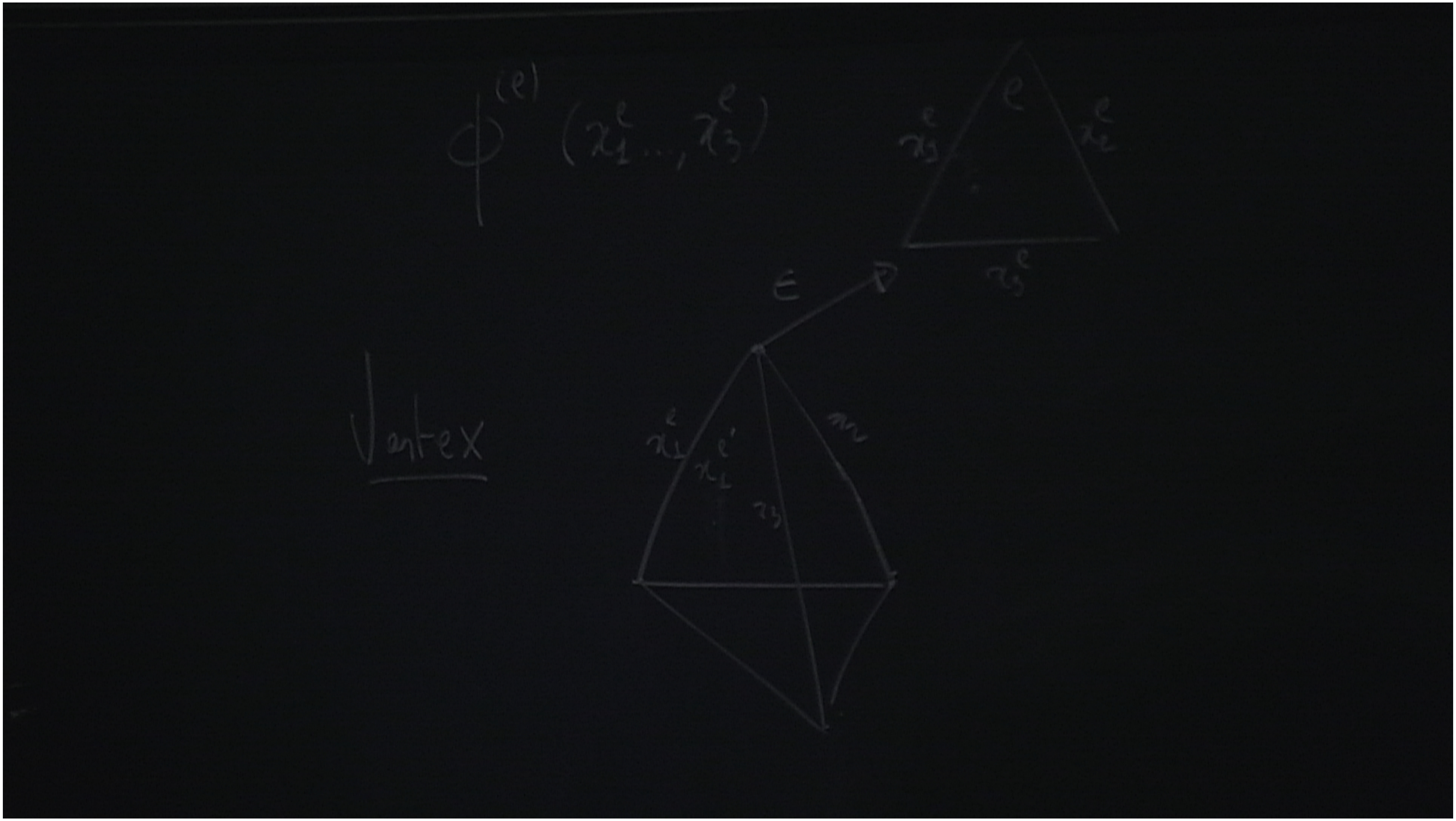


Displacement of v_3 by the R^3-vector ϵ shifts x_i^ℓ for $\ell = 1, 2, 4$:

$$\begin{aligned} x_i^\ell &\mapsto x_i^\ell + \varepsilon \quad \text{if } i \text{ outgoing} \\ x_i^\ell &\mapsto x_i^\ell - \varepsilon \quad \text{if } i \text{ incoming} \end{aligned}$$

in a way preserves the closure $\delta(x_1^\ell + x_2^\ell + x_3^\ell)$ of each triangle ℓ.




φ^(e) (x_1^e ..., x_3^e)
Vertex

Vertex translations as (quantum) symmetries

In the metric representation:

$$\begin{aligned}
\mathcal{T}_\varepsilon^3 \triangleright \widehat{\varphi}_1(x_1, x_2, x_3) &:= \bigstar_\varepsilon\, \widehat{\varphi}_1(x_1 - \varepsilon, x_2, x_3 + \varepsilon) \\
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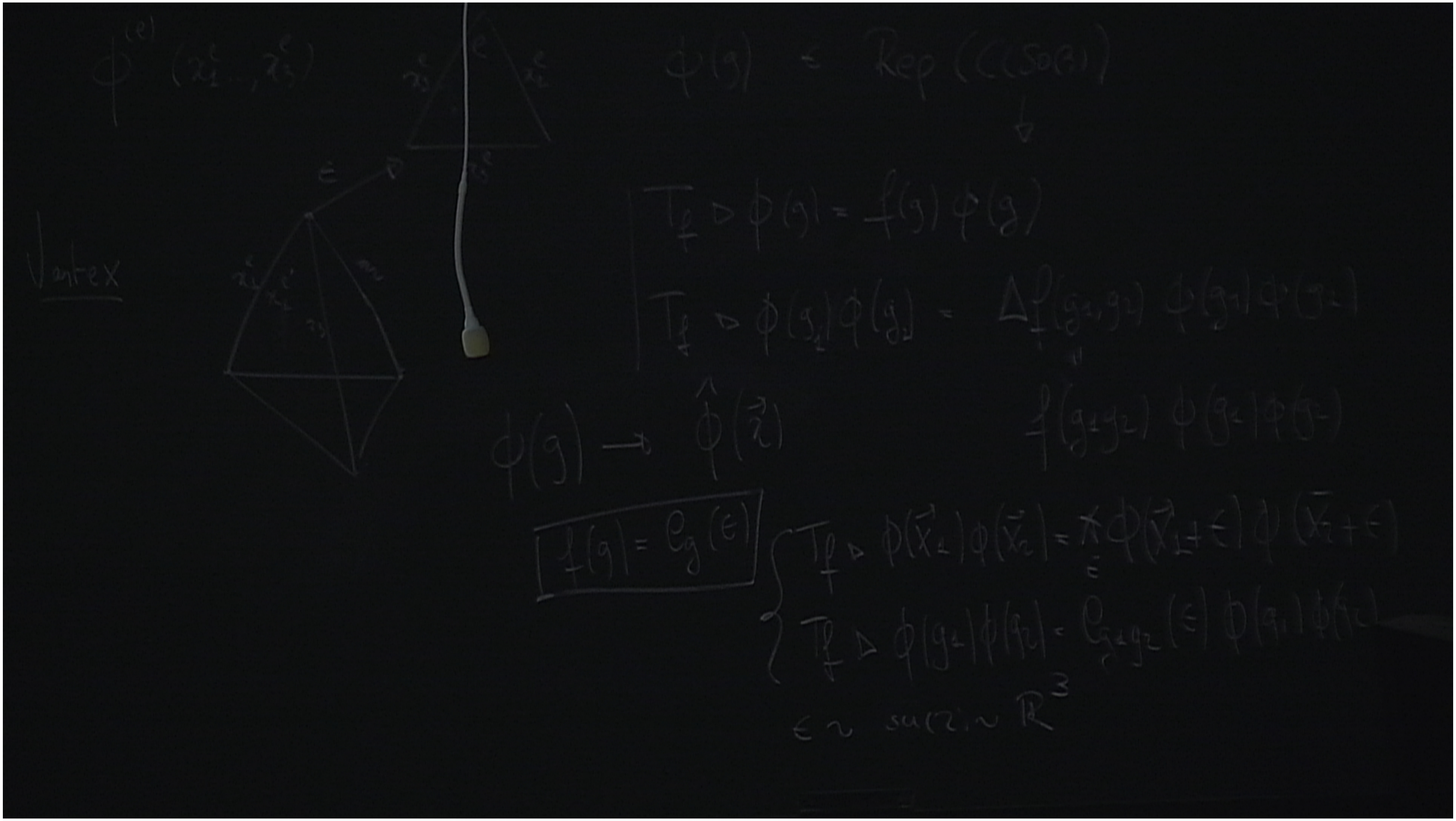
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Vertex translations as (quantum) symmetries

- due to non-commutativity of the metric space, translation symmetry gets deformed:
  Hopf algebra (quantum) symmetry, characterized by a non-trivial action on a tensor product of fields, due to a non-trivial coproduct.
- relevant quantum group: (translation part of the) Drinfel'd double $\mathcal{D}\mathrm{SO}(3) = \mathcal{C}(\mathrm{SO}(3)) \rtimes \mathbb{C}\mathrm{SO}(3)$
- requires to embed GFT into the framework of braided quantum field theories, $B_{\ell\ell'}^{v_{\ell_o}}$.

$$B_{11}^{(3)}[\phi^1(g_1, g_2, g_3)\phi^1(k_1, k_2, k_3)] =$$
$$\phi^1(k_1, k_2, k_3)\phi^1(k_3^{-1}k_1 \triangleright g_1, g_2, k_3^{-1}k_1 \triangleright g_3)$$

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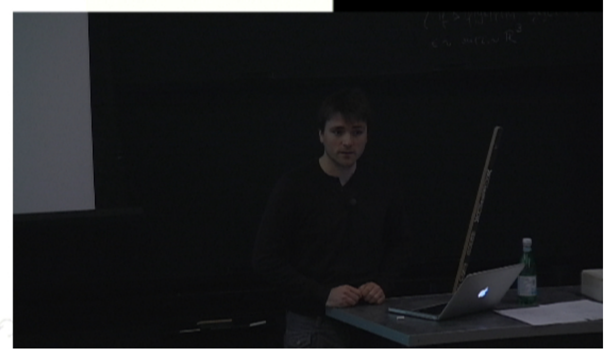
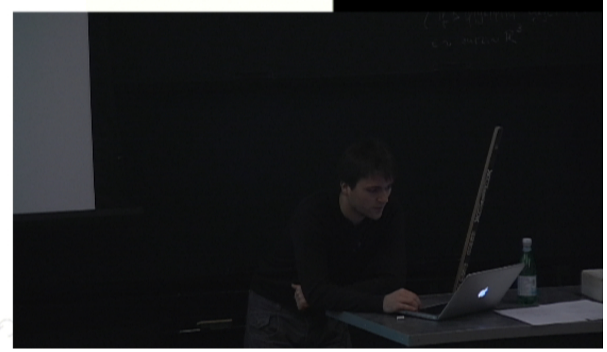


Invariance of the vertex function and diffeomorphisms

(i) **Metric representation.**

$$V(x_i^{\ell}, x_i^{\ell'}) = \int \prod_{\ell=1}^{4} [\mathrm{d}h_\ell] \prod_{i=1}^{6} (\delta_{-x_i^\ell} \star \mathrm{e}_{h_\ell h_{\ell'}^{-1}})(x_i^{\ell'})$$

- Symmetry expresses invariance of the matching condition under translation the vertices in an embedding of this tetrahedron in $\mathbb{R}^3$.
- also how action of discrete residual of diffeomorphisms is encoded in 3d Regge calculus.



Invariance of the vertex function and diffeomorphisms

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$$V(g_i^\ell, g_i^{\ell'}) = \int \prod_{\ell=1}^{4} \mathrm{d}h_\ell \prod_{i=1}^{6} \delta((g_i^\ell)^{-1} h_\ell h_{\ell'}^{-1} g_i^{\ell'})$$

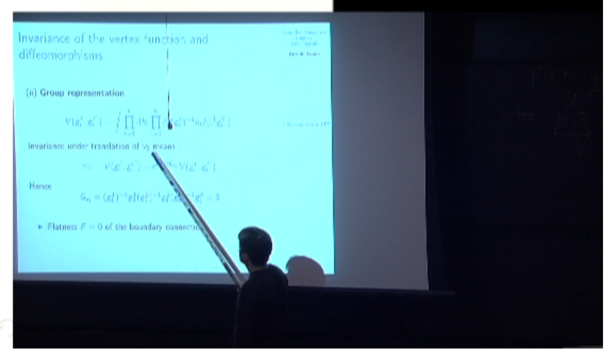
Invariance under translation of v_3 means :

$$\forall \varepsilon, \quad V(g_i^\ell, g_i^{\ell'}) = \mathrm{e}^{i \mathrm{Tr}\, \varepsilon G_{v_3}} V(g_i^\ell, g_i^{\ell'})$$

Hence

$$G_{v_3} = (g_1^1)^{-1} g_3^1 (g_3^2)^{-1} g_4^2 (g_4^4)^{-1} g_1^4 = 0$$

- Flatness $F = 0$ of the boundary connection.



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$$V(g_i^{\ell}, g_i^{\ell'}) = \int \mathrm{d}h\, e^{i\mathrm{Tr}\, h\varepsilon h^{-1} G_{v_3}}\, V(g_i^{\ell}, g_i^{\ell'})$$

Upon Plancherel decomposition:

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} = \sum_{k_i, j} d_{k_1} d_{k_3} d_{k_4} d_j\, \widehat{\chi}^j(\varepsilon)$$

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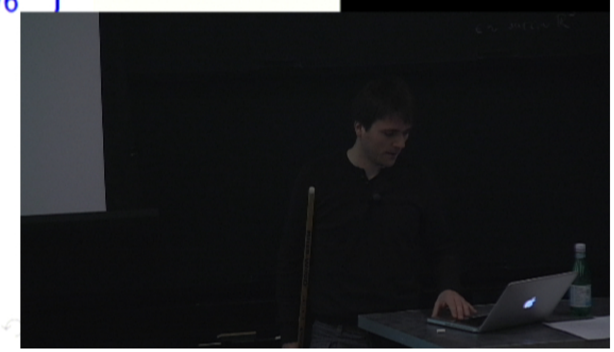
- (Gauge fixed) 1-4 Pachner move (coarse-graining) identity
- recursions relations, discrete Wheeler-De Witt equation.

Group field theory and simplicial path integrals

Aristide Baratin



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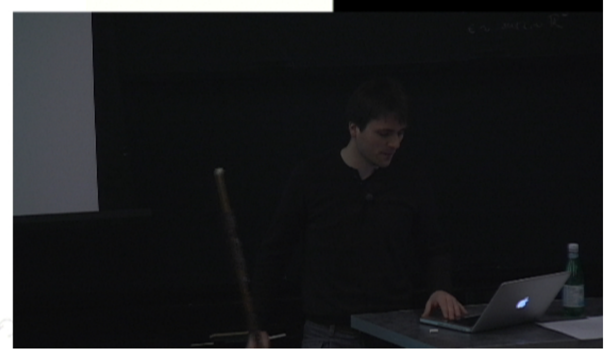

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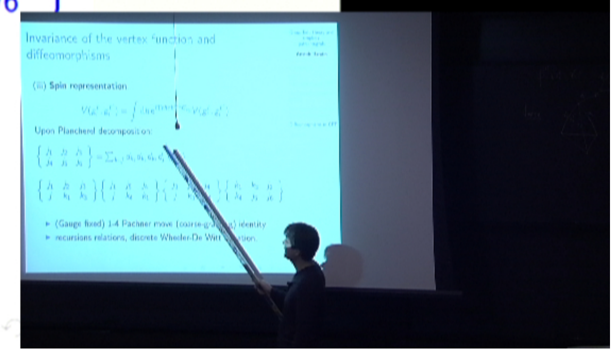

Spin foam symmetries revisited

GFT amplitudes:

$$\mathcal{I}_\Delta = \int \prod_l \mathrm{d}h_l \prod_e \mathrm{d}^3 x_e \; e^{iS_\Delta(x_e, h_l)},$$

with $S(x_e, h_l)$ discrete 3d gravity action $S(e, A) = \int \mathrm{Tr}\, e \wedge F$:

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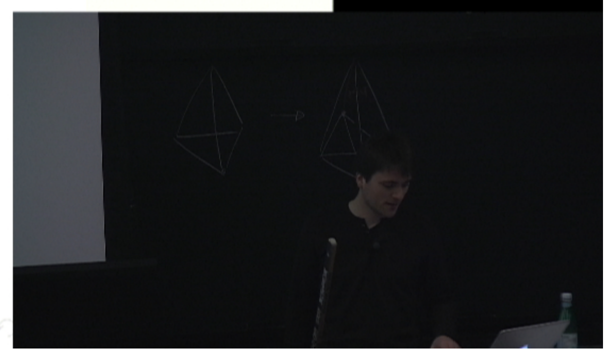

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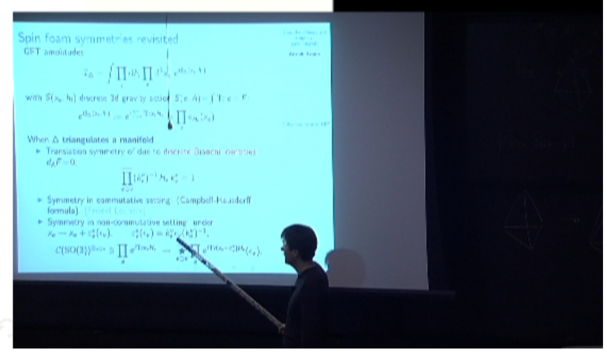


Spin foam symmetries revisited

- previous geometric considerations can be understood in purely algebraic way using braiding techniques

Simple example:

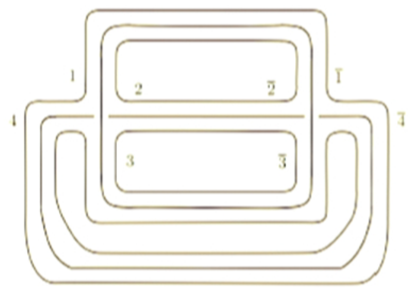


Figure: 2-vertex GFT diagram: triangulation of the sphere S^3.

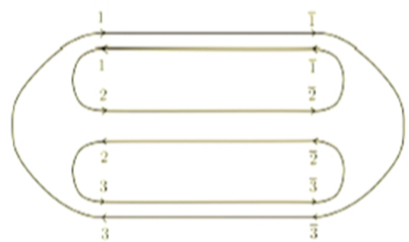
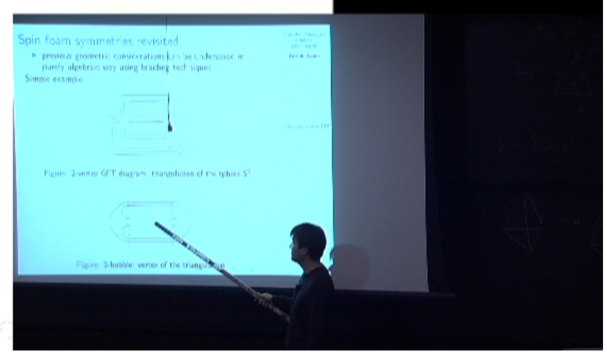


Figure: 3-bubble: vertex of the triangulation



Outlook

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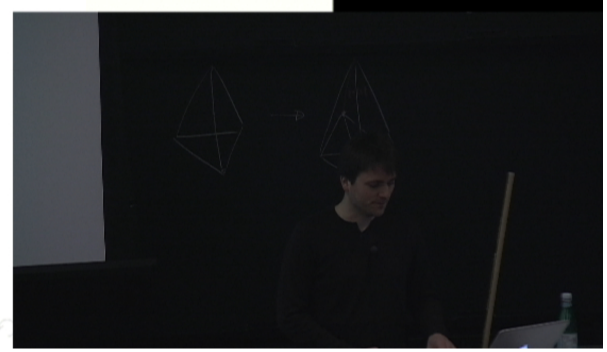
- Symmetries: GFT with braiding?
- GFT: renormalization, effective continuum physics



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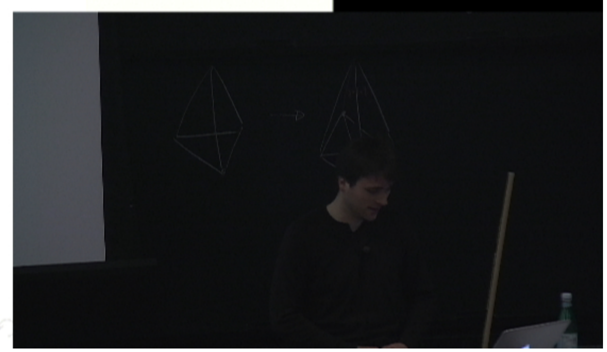

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