Title: Quantum Algorithms for Quantum Field Theories

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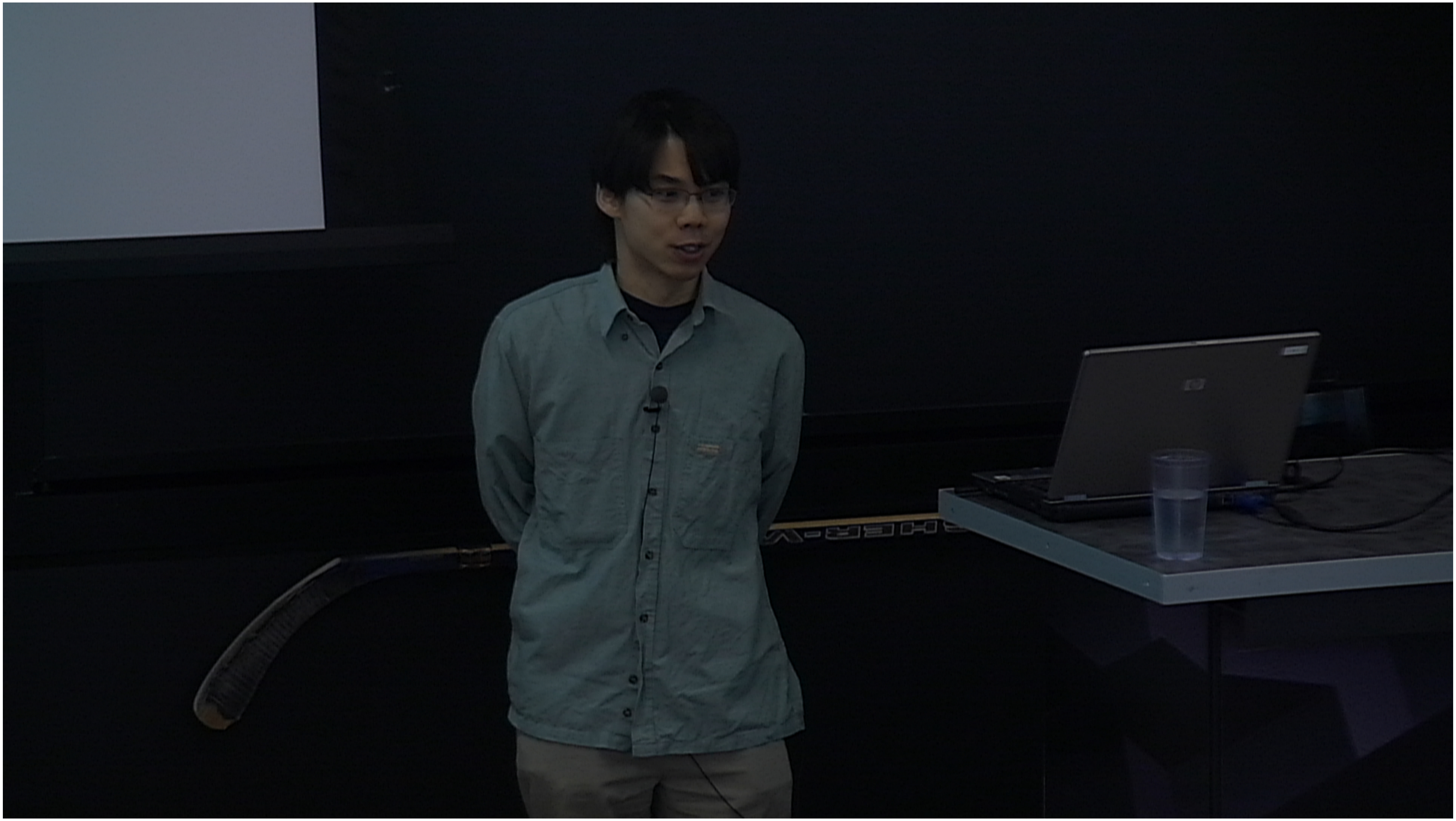
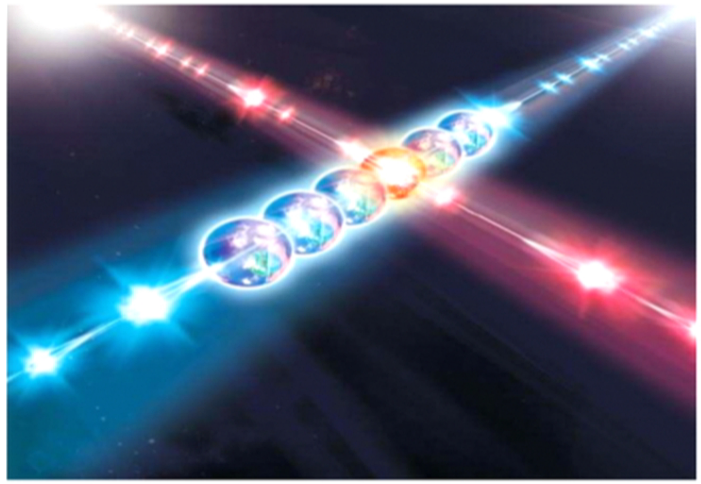
Abstract: Quantum field theory provides the framework for the Standard Model of particle physics and plays a key role in many areas of physics. However, calculations are generally computationally complex and limited to weak interaction strengths. I shall describe a polynomial-time algorithm for computing, on a quantum computer, relativistic scattering amplitudes in massive scalar quantum field theories. The quantum algorithm applies at both weak and strong coupling, achieving exponential speedup over known classical methods at high precision or strong coupling. The study of such quantum algorithms may also help us learn more about the nature and foundations of quantum field theory itself.

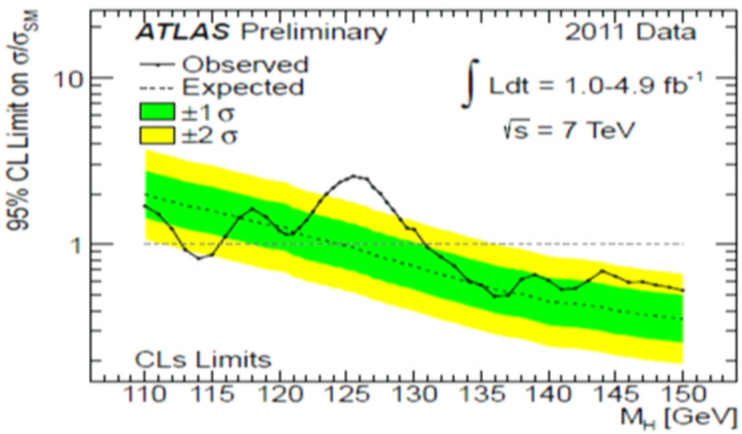


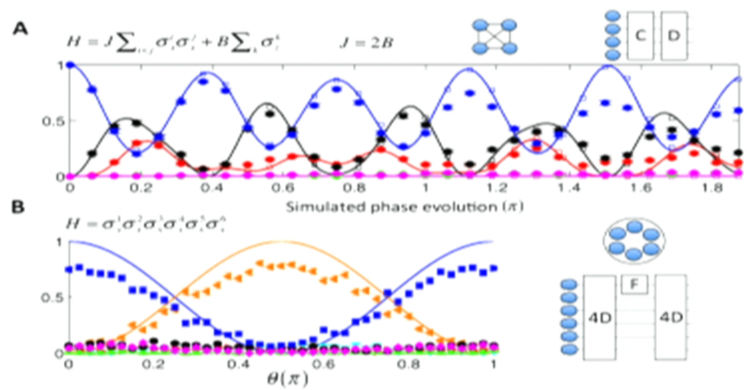
Quantum Algorithms for Quantum Field Theories

Keith S. M. Lee

6 January, 2012




ATLAS Preliminary
2011 Data
Observed
Expected
±1 σ
±2 σ
∫ Ldt = 1.0-4.9 fb⁻¹
√s = 7 TeV
CLs Limits
95% CL Limit on σ/σ_SM
10
1
110 115 120 125 130 135 140 145 150
M_H [GeV]


A
J = 2B
C
D
Simulated phase evolution (π)
B
F
4D
4D
θ(π)

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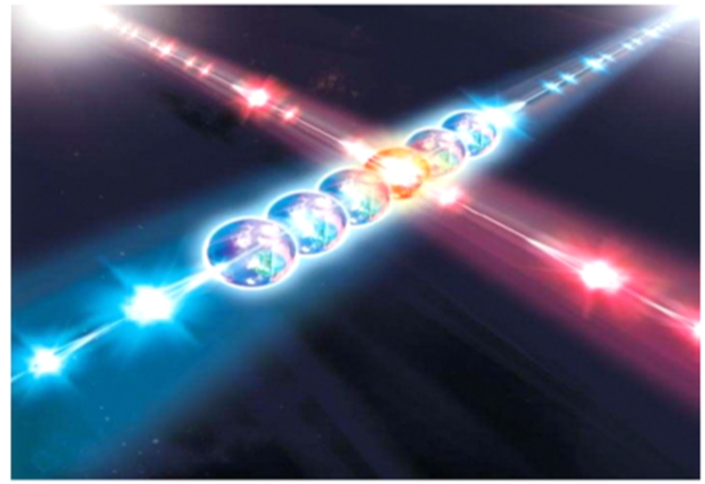
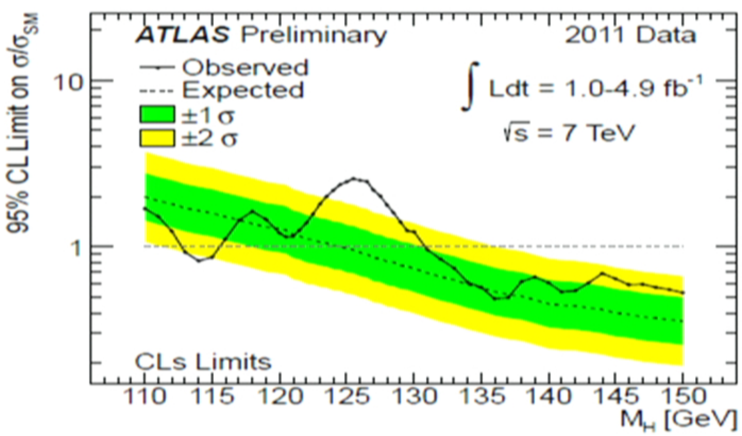
– Feynman (1982).

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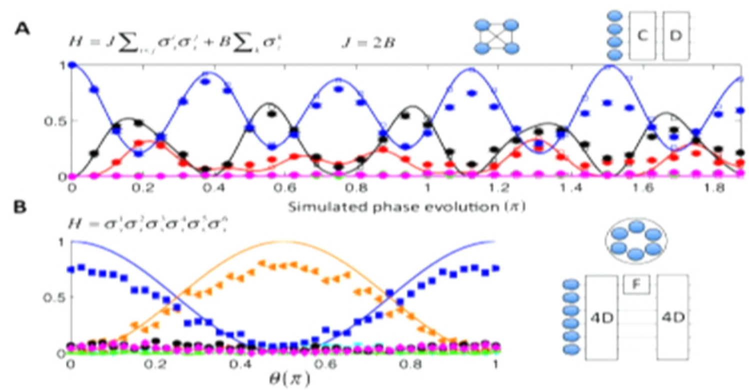
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Outline

A Quantum Field Theory Problem

Quantum Algorithm

Analysis of Efficiency

Motivations, Aspirations

S. Jordan, K. L., J. Preskill, arXiv:1111.3633
arXiv:1112.4833

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Quantum Field Theory

- Reconciles quantum mechanics & special relativity
- Fields $\phi(t, \mathbf{x})$

$$[\phi(t, \mathbf{x}), \pi(t, \mathbf{y})] = i\delta^d(\mathbf{x} - \mathbf{y})$$

- Particle creation & destruction
- Interactions are nonlinear
  (eqs. of motion are nonlinear)
- Infinitely many degrees of freedom,
  Divergences,
  Renormalization

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A QFT Problem

- Problem: compute scattering amplitudes
- Input: momenta of incoming particles
  Output: momenta of outgoing particles
  produced by physical scattering process
- Algorithm samples from probability distribution over possible outcomes

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ϕ^4 Theory

Lagrangian density & interaction picture

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{1}{4!}\lambda\phi^4$$

$$\phi(x) \in \mathbb{R}, \qquad D = d + 1 = 2, 3, 4.$$

Use Hamiltonian & Schrödinger picture

$$\begin{aligned}\mathcal{H} &= \pi\dot{\phi} - \mathcal{L} \\ &= \frac{1}{2}\pi(\mathbf{x})^2 + \frac{1}{2}(\nabla\phi)^2(\mathbf{x}) + \frac{1}{2}m^2\phi(\mathbf{x})^2 + \frac{\lambda}{4!}\phi(\mathbf{x})^4\end{aligned}$$

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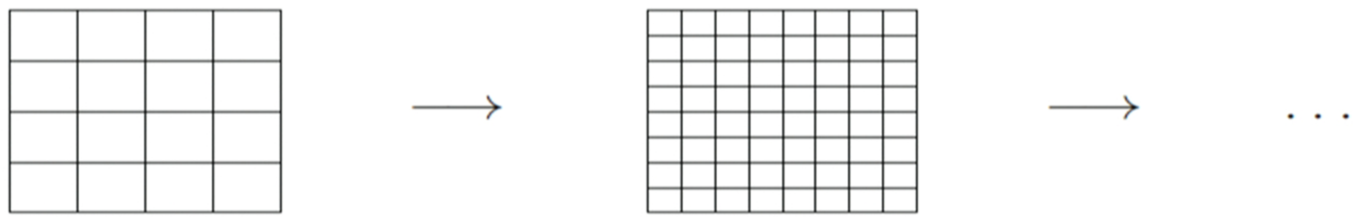
Representation by qubits $|0\rangle$, $|1\rangle$

- Cut off & discretize space, i.e. put theory on spatial lattice:

$$H = \sum_{\mathbf{x} \in \text{lattice}} a^d \left[\frac{1}{2}\pi(\mathbf{x})^2 + \frac{1}{2}(\nabla_a \phi)^2(\mathbf{x}) + \frac{1}{2}m^2\phi(\mathbf{x})^2 + \frac{\lambda}{4!}\phi(\mathbf{x})^4 \right]$$



- Parameters m^2 & λ are functions of a.

  Must address convergence to continuum theory as $a \to 0$:

  Does lattice Hamiltonian converge to continuum limit?

  How quickly?

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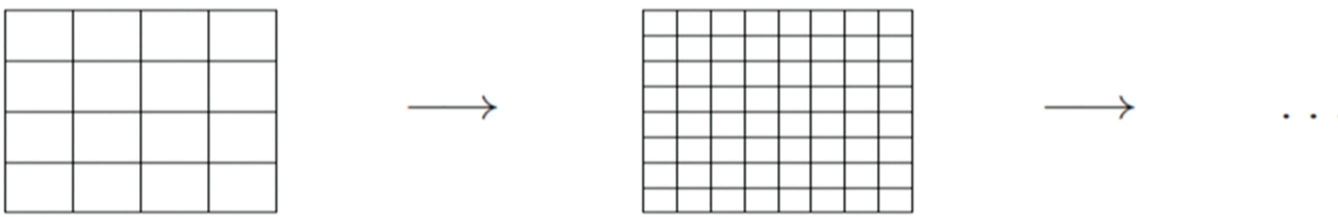

- Parameters m^2 & λ are functions of a.

  Must address convergence to continuum theory as $a \to 0$:

  Does lattice Hamiltonian converge to continuum limit?

  How quickly? Answer: as a^2

- Also cut off & discretize field $\phi(\mathbf{x})$:

  maximum magnitude ϕ_{max},

  increments of δ_ϕ
- Discretization of $\phi(\mathbf{x})$ equivalent to cutoff of canonically conjugate variable $\pi(\mathbf{x})$
- Result: for accuracy $\pm\epsilon$,
  suffices to use $n_b = O\left(\log\left(\frac{\mathcal{V}E}{m\epsilon}\right)\right)$ qubits per site.

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- Measure observables

Time evolution e^{-iHt}

- Straightforward to implement $e^{-iH_\phi t}$ and $e^{-iH_\pi t}$, $H = H_\phi + H_\pi$
- Can efficiently implement Trotter formula

$$e^{-i(H_\phi+H_\pi)t} = \left(e^{-iH_\phi t/n} e^{-iH_\pi/n}\right)^n + O(t^2/n)$$

- Can systematically construct higher-order Suzuki-Trotter formulae. e.g.

$$e^{-i(H_1+H_2)t} = \left(e^{-iH_2 t/2n} e^{-iH_1 t/n} e^{-iH_2 t/2n}\right)^n + O(t^3/n)$$

- Suzuki-Trotter formulae for large lattices:
  We derive linear scaling with number of lattice sites $\mathcal{V}$, provided Hamiltonian is local.
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State preparation

1. Prepare free vacuum
2. Excite wavepackets
3. Adiabatically turn on interaction

1. Preparation of Free Vacuum

- Free vacuum: ground state of $H^{(0)} = H(\lambda = 0)$
- Exactly solvable
  $|\mathrm{vac}(0)\rangle$ is multivariate Gaussian wavefunction
  in variables $\{\phi(\mathbf{x})|\mathbf{x} \in \mathrm{lattice}\}$
  $\Rightarrow$
  Can prepare with Kitaev-Webb method
  in $O(\mathcal{V}^{2.376})$ time

2. Wavepackets of Free Theory

- cf. harmonic oscillator
  $a_{\mathbf{p}}$, $a_{\mathbf{p}}^{\dagger}$ such that $H^{(0)} \sim \sum_{\mathbf{p}} \omega_{\mathbf{p}} a_{\mathbf{p}}^{\dagger} a_{\mathbf{p}}$

  $\rightarrow$ define operator $a_{\psi}^{\dagger}$: creates wavepacket with position-space wavefunction ψ
- Problem: $a_{\psi}^{\dagger}$ not unitary, cannot implement directly.

  $e^{i(a_{\psi}+a_{\psi}^{\dagger})}$ won't work either — uncontrolled particle number
- Solution: ancillary qubit

$$H_{\psi} = a_{\psi}^{\dagger} \otimes |1\rangle\langle 0| + a_{\psi} \otimes |0\rangle\langle 1|$$

$$e^{-iH_{\psi}\pi/2}|\mathrm{vac}(0)\rangle|0\rangle = -ia_{\psi}^{\dagger}|\mathrm{vac}(0)\rangle|1\rangle$$

$e^{-iH_{\psi}\pi/2}|\mathrm{vac}(0)\rangle$ gives desired state

3. Adiabatic Turn-on

- Slowly turn on interaction

$$\mathcal{H}(s) = s\lambda\phi^4 \,, \qquad 0 \leq s \leq 1$$

- For sufficiently large τ,

$$H(0) = H_{\text{init}} \quad \xrightarrow{H(t/\tau)} \quad H(1) = H_{\text{final}}$$

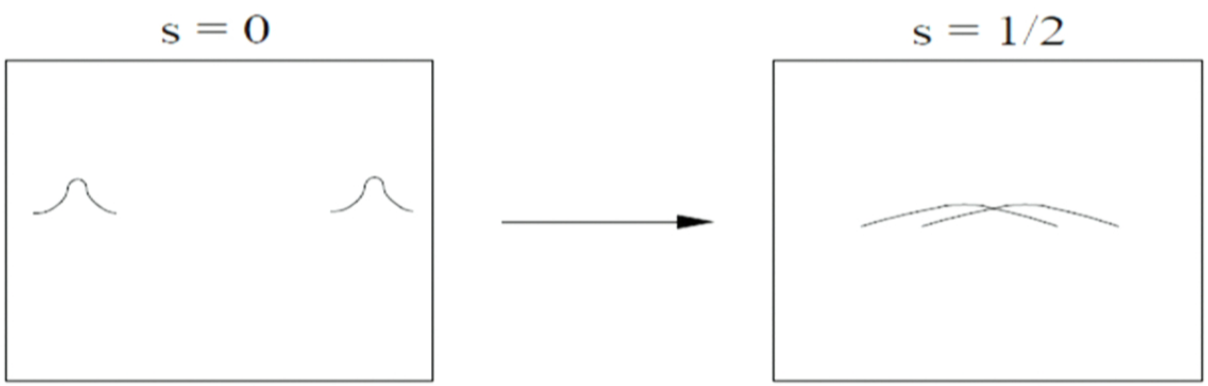
$$|\psi(0)\rangle \quad \longrightarrow \quad |\psi(1)\rangle$$

- Adiabatic theorem determines rate of turn-on: $\tau = O(1/\gamma^2)$, where γ = energy gap

- Problem: wavepackets propagate & broaden

Problem: Propagation & Broadening

- Due to different eigenstates acquiring different dynamical phases



- Problem, because don't want particles to collide & scatter before obtain desired coupling strength

Modified Adiabatic State Preparation

Problem: wavepackets propagate & broaden

Solution: undo dynamical phases by interspersing backwards time evolutions governed by time-independent Hamiltonians

Each of J steps consists of backward, forward, backward evolutions.
Dynamical phases $\theta \to 0$ as $J \to \infty$.

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Measurement

- Adiabatically turn off interaction
  Time-reversed version of turn-on
- Measure occupation numbers of momentum modes
  via phase estimation (Kitaev):

  Simulate $e^{ia_{\mathbf{p}}^{\dagger}a_{\mathbf{p}}t}$ for various t
  Fourier transform results
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  - Trotter approximations
  - Imperfect adiabaticity

Result: polynomial run time in number of particles, energy, and precision; thus exponential speedup at high precision or strong coupling.

Imperfect Adiabaticity

- Adiabatic Theorem [1]

$$|\mathcal{A}(\text{diabatic transition}(s))| \sim \int_0^s d\sigma \left| \frac{\langle \phi_k(\sigma)| \frac{dH}{ds} |\phi_l(\sigma)\rangle}{E_l(\sigma) - E_k(\sigma)} \right| \sim \frac{J}{\tau \gamma^2}$$

- Two types:

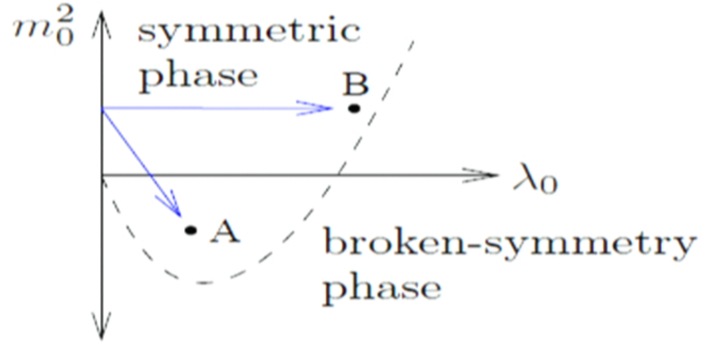
pair or quartet creation from vacuum $\gamma \sim m_{\text{phys}}$

particle splitting $(1 \rightarrow 3)$ $\gamma \sim \frac{m^2_{\text{phys}}}{p}$

[1] $J = \#\,\text{steps}$, $\tau = \text{period}$, $\gamma = \text{energy gap}$

Strong Coupling

- $\phi^4_{2,3}$ theory has $\phi \to -\phi$ symmetry-breaking phase transition.



- Strong coupling near phase transition
- Critical scaling & universality in phase transitions:

$$m_{\rm phys} \sim |\lambda_0 - \lambda_c|^\nu$$

$$\nu = \begin{cases} 1\,, & D = 2 \\ 0.63\ldots\,, & D = 3 \end{cases}$$

Discretization Errors via Effective Field Theory (EFT)

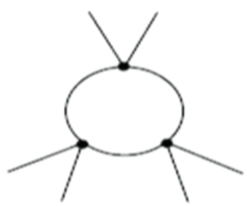
$$\mathcal{L}_{\text{eff}} = \mathcal{L}^{(0)} + \frac{c}{6!}\phi^6 + c'\phi^3\partial^2\phi + \frac{c''}{8!}\phi^8 + \cdots$$

e.g. $D = 4$:

$$c \sim a^2\,,\ c'' \sim a^4$$

General D:

$$[c] = 6 - 2D\,,\quad [c''] = 8 - 3D$$



$$\sim \quad \lambda^3 a^{6-D}$$

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Scaling with lattice spacing a given by scaling of coefficients.

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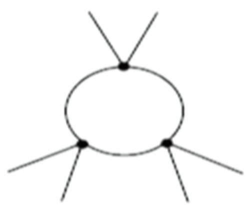

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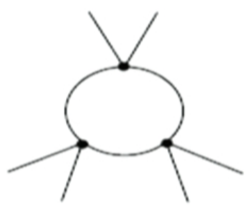

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Neglected EFT terms correspond to discretization errors.
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EFT Operators fall into three classes

Class	Operators	Scaling of coupling
I	ϕ^{2n} $(n \geq 3)$	$\lambda^n a^{2n-D}$
II	$\phi \partial_{\mathbf{x}}^{2l} \phi$ $(l \geq 2)$	a^{2l-2}
III	$\phi^{2j+1} \partial_{\mathbf{x}}^{2l} \phi$ $(j \geq 1, l \geq 2)$	$\lambda^{j+1} a^{2j+2l+2-D}$

- Dominant operators:

$\phi \partial_{\mathbf{x}}^4 \phi \equiv \phi \sum_{i=1}^{d} \partial_i^4 \phi \sim a^2$

Arise because difference operators in discretized theory only approximately equal to derivatives in continuum theory

$\phi^6 \sim a^{6-D} \sim a^2$ in $D = 4$

Weak Coupling

- Can efficiently calculate scattering to arbitrary precision $\pm\epsilon$

$$G_{\rm weak} \sim \begin{cases} \left(\frac{1}{\epsilon}\right)^{1.5+o(1)}, & d = 1 \\ \left(\frac{1}{\epsilon}\right)^{2.376+o(1)}, & d = 2 \\ \left(\frac{1}{\epsilon}\right)^{5.5+o(1)}, & d = 3 \end{cases}$$

$D = d + 1$

Strong Coupling

- Run time polynomial in number of particles, energy, & distance from critical point (where coupling strength maximum)

	$\lambda_c - \lambda_0$	p	n_{out}
$d = 1$	$\left(\frac{1}{\lambda_c - \lambda_0}\right)^{8+o(1)}$	$p^{4+o(1)}$	$\tilde{O}(n_{\text{out}}^{5})$
$d = 2$	$\left(\frac{1}{\lambda_c - \lambda_0}\right)^{5.04+o(1)}$	$p^{6+o(1)}$	$\tilde{O}(n_{\text{out}}^{7.128})$

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Computational Complexity Theory

What is the computational power of our universe?

- Complexity classes:

  P $\sim$ feasible classically (polynomial-time)
  NP $\sim$ solution easily checkable classically
  (nondeterministic polynomial-time)
  BQP $\sim$ feasible quantum mechanically
  (bounded-error quantum polynomial-time)
  $\vdots$
- Relationships between different classes?
  e.g. P vs. NP
- If QFT not in BQP, would raise possibility of quantum field computers

Foundations of Quantum Field Theory

- Wish to learn more about nature & foundations of quantum field theory itself.
- Precedent: Wilson & Symanzik discovered deep insights through work on classical computer simulation — renormalization group, etc.

"I'm trying to get . . . you people who think about computer-simulation possibilities to . . . digest . . . the real answers of quantum mechanics and see if you can't invent a different point of view than the physicists . . . I . . . was hoping that the computer-type thinking would give us some new ideas . . ."

– Feynman (1982).

Summary

- Quantum computers can efficiently calculate scattering amplitudes in ϕ^4 theory in $D \leq 4$
— at both weak & strong coupling
- Algorithm introduces several new techniques
- Leads way towards algorithm for Standard Model, which has new features

Some EFT Results I

D	Coefficient of $\phi^6/6!$
2	$-\frac{45}{64\pi^5}\lambda^3 a^4\left[1+\frac{20}{3}\frac{1}{\hat{L}^2}\right]$
3	$-\frac{5}{64\pi^5}\lambda^3 a^3\left[10\sqrt{2}+\frac{43\sqrt{2}}{\hat{L}^2}\right]$
4	$-\frac{15}{128\pi^5}\lambda^3 a^2\left[2(2\sqrt{3}+\pi)+\frac{4}{9}(26\sqrt{3}+9\pi)\frac{1}{\hat{L}^2}\right]$

D	$V(r\to\infty)$
2	$-\frac{\lambda^2}{32m^3}\frac{1}{\sqrt{\pi m r}}e^{-2mr}$
3	$-\frac{\lambda^2}{64\pi^{3/2}m}\frac{1}{(mr)^{3/2}}e^{-2mr}$
4	$-\frac{\lambda^2}{128\pi^{5/2}m^{3/2}}\frac{1}{r^{5/2}}e^{-2mr}$