Title: Gravitational Physics (Review) - Lecture 3

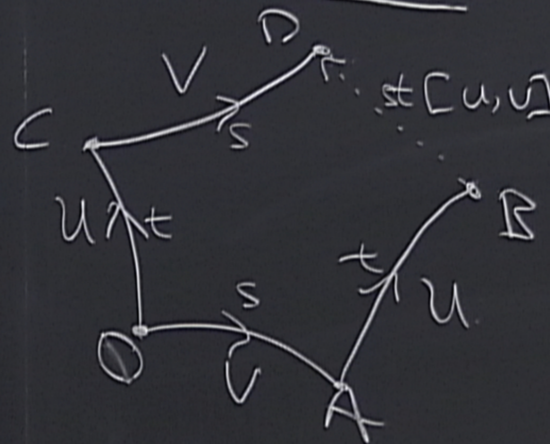
Date: Jan 25, 2012 09:00 AM

URL: http://pirsa.org/12010068

Abstract:

Geometrical Significance

- tells us how far a loop is from closing



$$X^{\mu}_{A} = X^{\mu}_{0} + s V^{\mu}_{0} + \frac{1}{2} s^{2} V^{\nu}_{0} V^{\mu}_{0,\nu}$$

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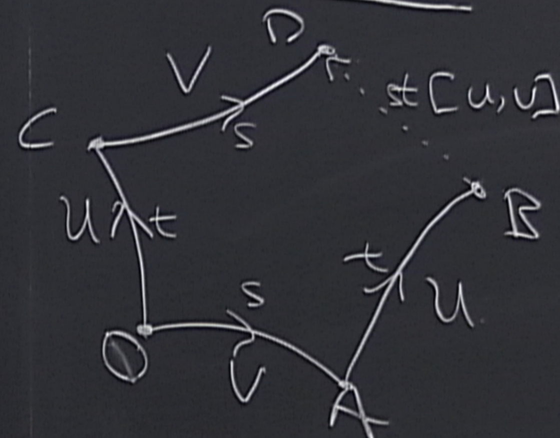
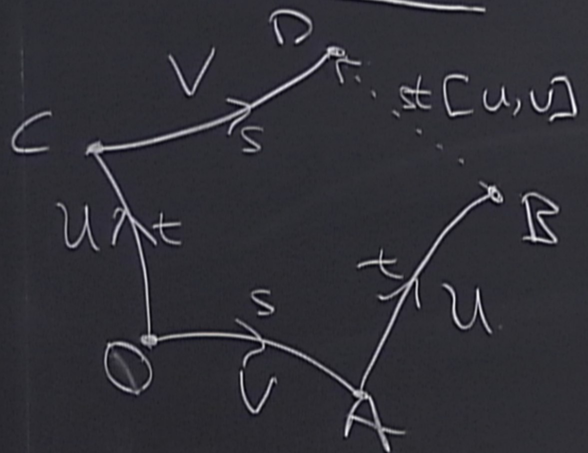


$$X^\mu_A = X^\mu_0 + s V^\mu_0 + \frac{1}{2} s^2 V^\nu_0 V^\mu_{0,\nu}$$

$$U^\mu_A = U^\mu_0 + s V^\nu_0 U^\mu_{0,\nu}$$

$$X_D^\mu = X_C^\mu +$$

...l Significance



$$+\frac{1}{2} s^2 V_0^\nu V^\mu_{0,\nu}$$

U^μ_0 ...

$$X_D^\mu = X_C^\mu + s\, V_C^\mu + \frac{1}{2} s^2 V_C^\nu V^\mu_{C,\nu} + \dots$$

$$= X_0^\mu + t\, U_0^\mu + \frac{1}{2} t^2 U_0^\nu U^\mu_{0,\nu}$$

$$+ s V_0^\mu + st\, U_0^\nu V^\mu_{0,\nu}$$

$$+ \frac{1}{2} s^2 V_0^\nu V^\mu_{0,\nu} + \dots$$

to get X_B^μ swap $s \leftrightarrow t$

$U \leftrightarrow V$

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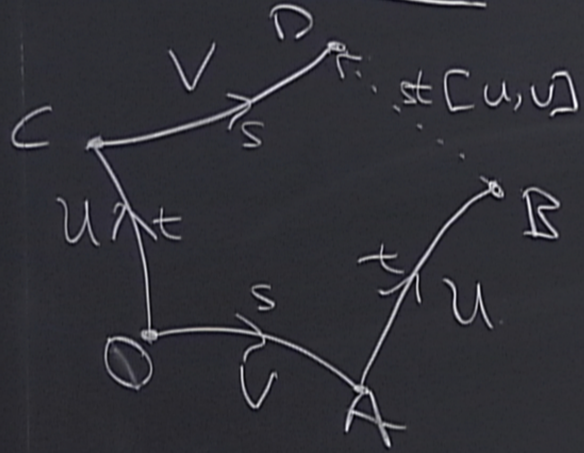

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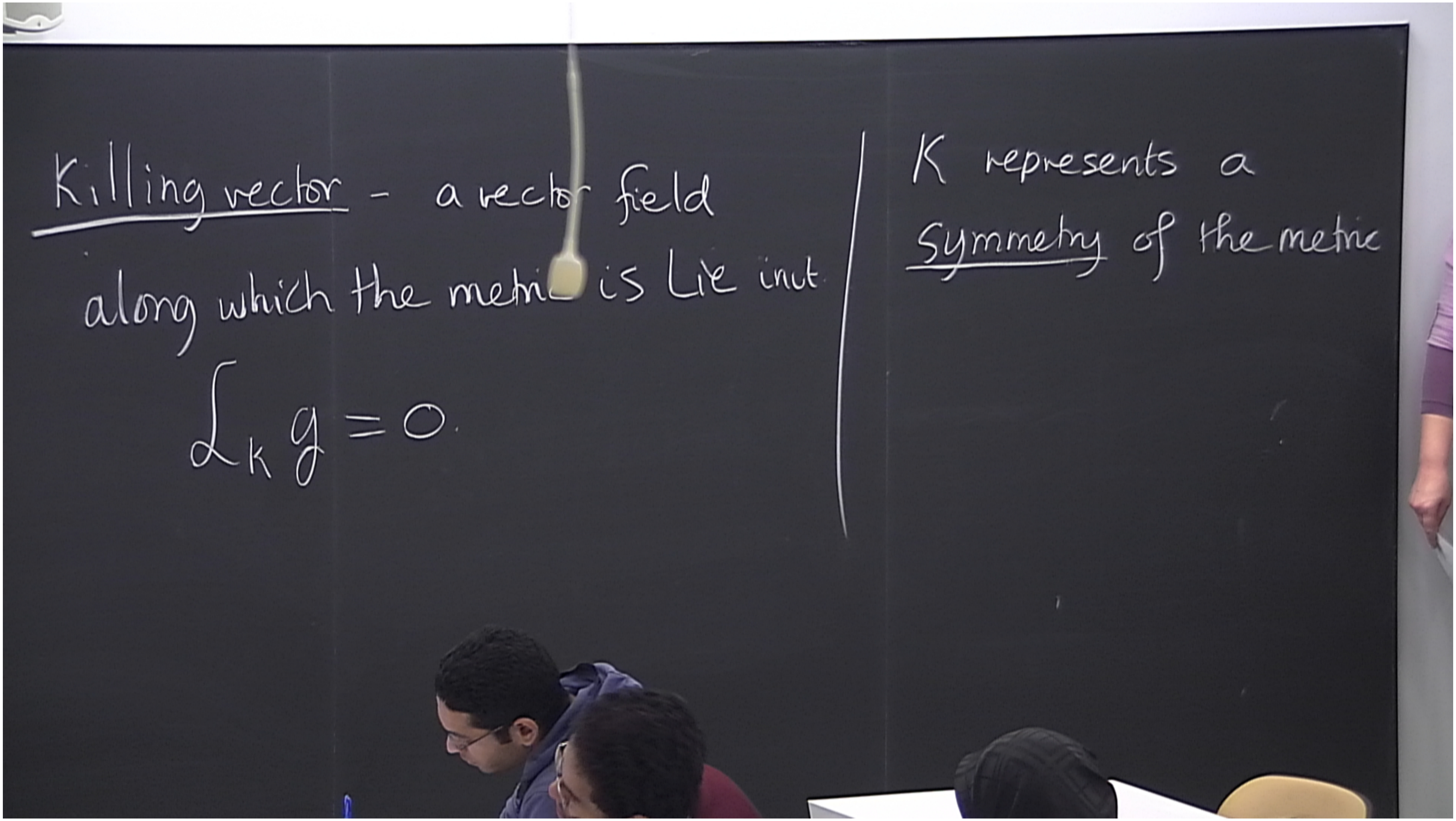
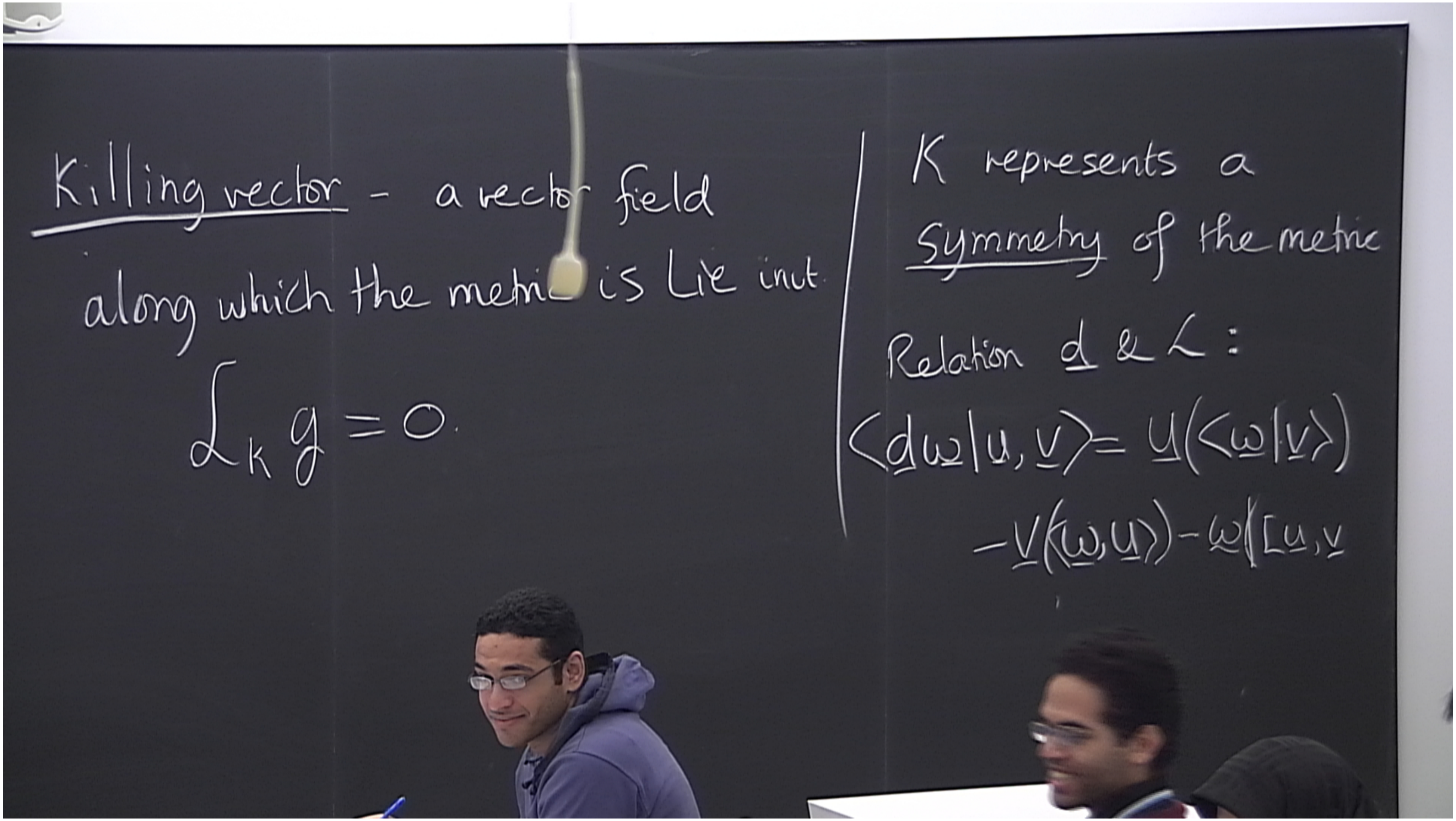
$$X_D^\mu - X_B^\mu = st\left(U_0^\nu V_{0,\nu}^\mu - V_0^\nu U_{0,\nu}^\mu\right)$$

$$= st\,[U,V]^\mu$$


Killing vector - a vector field
along which the metric is Lie invt.
$\mathcal{L}_K g = 0$.
K represents a
symmetry of the metric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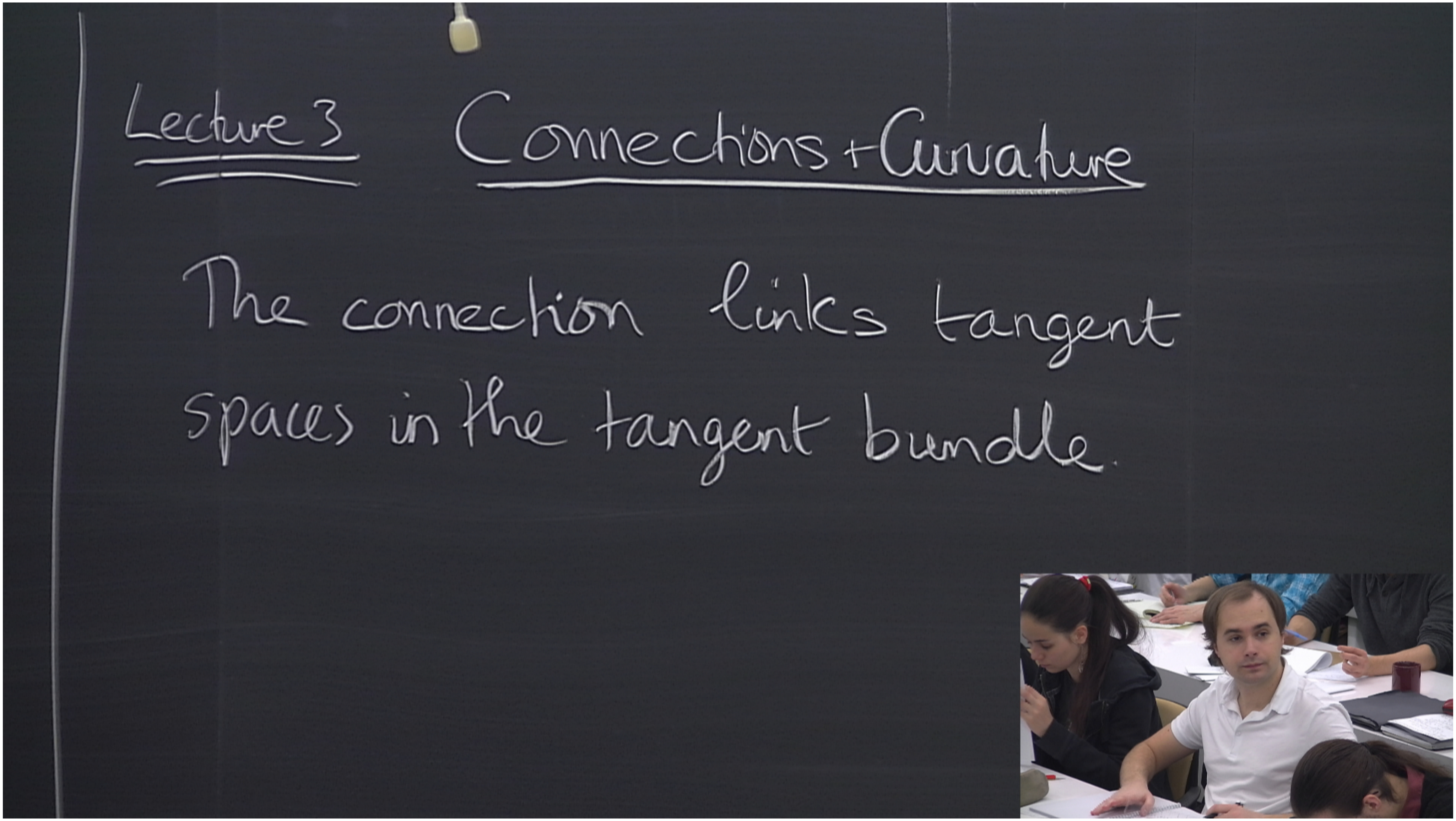
Relation d & L:
$\langle d\omega | u, v \rangle = u(\langle \omega | v \rangle)$
$- v(\langle \omega, u \rangle) - \omega | [u, v$

$$\langle \underline{d}\underline{\omega} \,|\, \underline{u}, \underline{v} \rangle = \underline{u}\langle \underline{\omega} | \underline{v} \rangle - \underline{v}\langle \underline{\omega}, \underline{u} \rangle - \langle \underline{\omega} \,|\, [\underline{u}, \underline{v}] \rangle$$


Lecture 3
Connections + Curvature
The connection links tangent spaces in the tangent bundle.

$|u\rangle$

Lecture 3 Connections + Curvature

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$$\underset{\sim}{\nabla}\, \underset{\sim}{e}_b = \Gamma^c{}_{ab}\, \underset{\sim}{e}_c \otimes \underset{\sim}{\omega}^a$$

"covector"

vector basis element

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or $\Gamma^a_{bc} = \langle \underset{\sim}{\omega}^a \,|\, \nabla_b \underset{\sim}{e}_c \rangle$

define the connection cpts.

$\underset{\sim}{\nabla}$ is a derivation which

(i) Commutes with contractions

(ii) Leibnizian

(iii) Reduces to $\underline{d}$ on fns.

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$$= \left(\underset{\sim}{\nabla} V^a\right) \underset{\sim}{e}_a + V^a \left(\underset{\sim}{\nabla}\, \underset{\sim}{e}_a\right)$$

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$$= \underset{\sim}{d}\,V^a\,\underset{\sim}{e}_a + V^a\,\Gamma^c_{b\,a}\,\underset{\sim}{e}_c\,\underset{\sim}{\omega}^b$$

$$= \underset{\sim}{\omega}^b\left(V^a_{,b} + V^c\,\Gamma^a_{b\,c}\right)\underset{\sim}{e}_a$$

In GR we choose a metric connection

$$\underline{\nabla}\, \underset{\sim}{g} = 0$$

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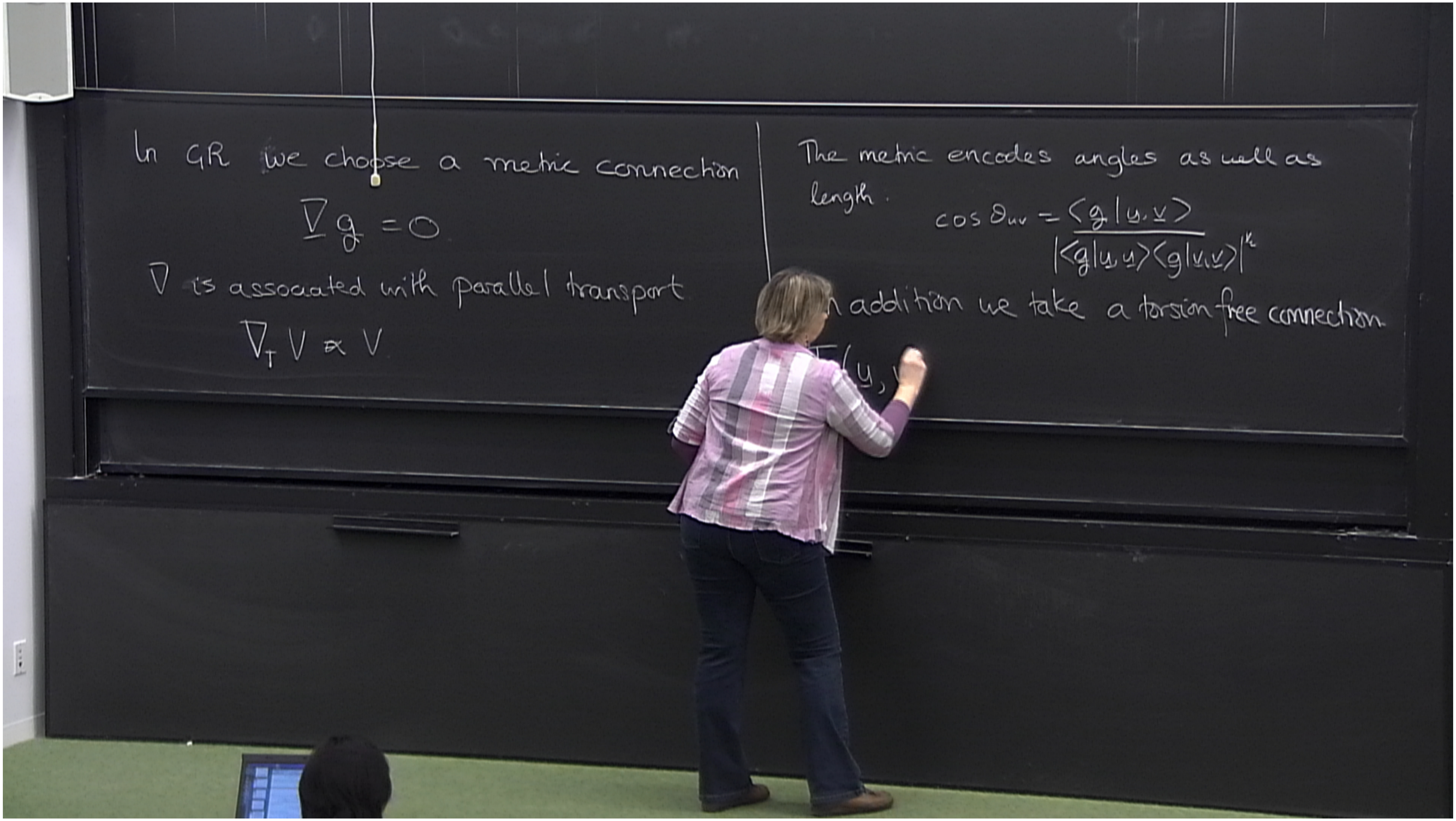
The metric encodes angles as well as length:

$$\cos\theta_{uv} = \frac{\langle \underline{g} | \underline{u}, \underline{v} \rangle}{|\langle g | \underline{u}, \underline{u} \rangle \langle g | \underline{v}, \underline{v} \rangle|^{1/2}}$$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In addition we take a torsion free connection.

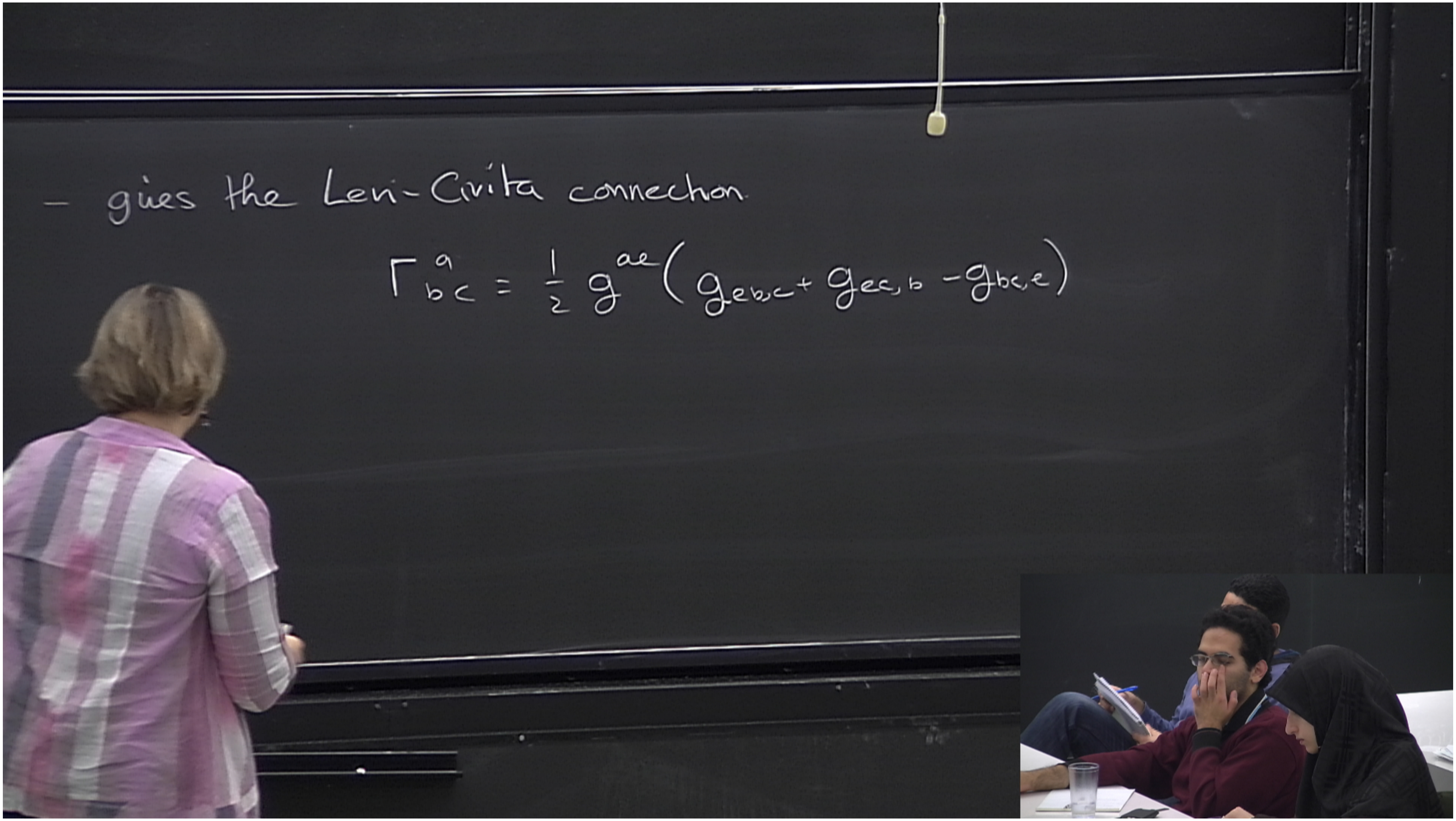
$$\underline{T}(\underline{u}, \underline{v}) = \nabla_u \underline{v} - \nabla_v \underline{u} - [\underline{u}, \underline{v}] = 0$$

(almost) says Γ is symmetric

$$T^a_{bc} = \Gamma^a_{bc} - \Gamma^a_{cb} - C^a_{bc}$$

where $C^a_{bc} = \langle \underline{\omega}^a \,|\, [\underline{e}_b, \underline{e}_c] \rangle$

are the structure constants of $\{\underline{e}_a\}$.


– gives the Levi-Civita connection
$\Gamma^a_{bc} = \frac{1}{2} g^{ae}(g_{eb,c} + g_{ec,b} - g_{bc,e})$

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$$\Gamma^a{}_{bc} = \frac{1}{2} g^{ae}\left(g_{eb,c} + g_{ec,b} - g_{bc,e}\right)$$

Now define the connection 1-forms:

$$\underset{\sim}{\Theta}{}^a{}_b = \Gamma^a{}_{cb}\,\underline{\omega}^c$$

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spin connection

Then $\underset{\sim}{\Omega}_{ac} + \underset{\sim}{\Omega}_{ca} = \underline{d}\, g_{ac}$

Proof

scalars

$$\underline{d}\, g_{ab} = \underline{\nabla}\, g_{ab} = \underline{\nabla}\left(\langle \underline{g} \,|\, \underline{e}_a, \underline{e}_b \rangle\right)$$

$$= \langle \underline{g} \,|\, \underline{\nabla}\underline{e}_a, \underline{e}_b \rangle + \langle \underline{g} \,|\, \underline{e}_a, \underline{\nabla}\underline{e}_b \rangle$$

spin connection

$$= \langle g \,|\, \Gamma_c{}^d{}_a \underline{e}_d \underline{\omega}^c, \underline{e}_b \rangle + \langle g \,|\, \underline{e}_a, \Gamma_c{}^d{}_b \underline{\omega}^c \underline{e}_d \rangle$$

$$= g_{db} \underline{\Theta}^d{}_a + g_{ad} \underline{\Theta}^d{}_c$$

$$= \underline{\Theta}_{ba} + \underline{\Theta}_{ab}.$$

(almost) says Γ is symmetric

$$T^a_{\ bc} = \Gamma^a_{\ bc} - \Gamma^a_{\ cb} - C^a_{\ bc}$$

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Then $\underline{\Theta}_{ac} + \underline{\Theta}_{ca} = \underline{d}\, g_{ac}$

Proof

$$\underline{d}\, g_{ab} = \underline{\nabla}\, g_{ab} = \nabla(\langle g | \underline{e}_a, \underline{e}_b \rangle)$$

scalars

$$= \langle g | \nabla \underline{e}_a, \underline{e}_b \rangle + \langle g | \underline{e}_a, \nabla \underline{e}_b \rangle$$

$$= \langle g | \Gamma^d_{\ ca} \underline{e}_d \underline{\omega}^c, \underline{e}_b \rangle + \langle g | \underline{e}_a, \Gamma^d_{\ cb} \underline{\omega}^c \underline{e}_d \rangle$$

$$= g_{db}\, \underline{\Theta}^d_{\ a} + g_{ad}\, \underline{\Theta}^d_{\ c}$$

$$= \underline{\Theta}_{ba} + \underline{\Theta}_{ab}.$$

Finally:

$$\underset{\sim}{\Theta}^{a}{}_{c} \wedge \underline{\omega}^{c} = \Gamma_{b}{}^{a}{}_{c}\, \underline{\omega}^{b} \wedge \underline{\omega}^{c}$$

$$= \frac{1}{2}\left(\Gamma_{b}{}^{a}{}_{c} - \Gamma_{c}{}^{a}{}_{b}\right) \underline{\omega}^{b} \wedge \underline{\omega}^{c}$$

$$= \frac{1}{2}\left(T_{b}{}^{a}{}_{c} + C_{bc}{}^{a}\right) \underline{\omega}^{b} \wedge \underline{\omega}^{c}$$

$$= \underset{\sim}{T}$$

where $\underline{T}^a = \frac{1}{2} T^a{}_{bc}\, \underline{\omega}^b \wedge \underline{\omega}^c$

is torsion 2-form.

Finally:

$$\underset{\sim}{\Theta}^{a}{}_{c} \wedge \underline{\omega}^{c} = \Gamma^{a}{}_{bc}\, \underline{\omega}^{b} \wedge \underline{\omega}^{c}$$

$$= \frac{1}{2}\left(\Gamma^{a}{}_{bc} - \Gamma^{a}{}_{cb}\right) \underline{\omega}^{b} \wedge \underline{\omega}^{c}$$

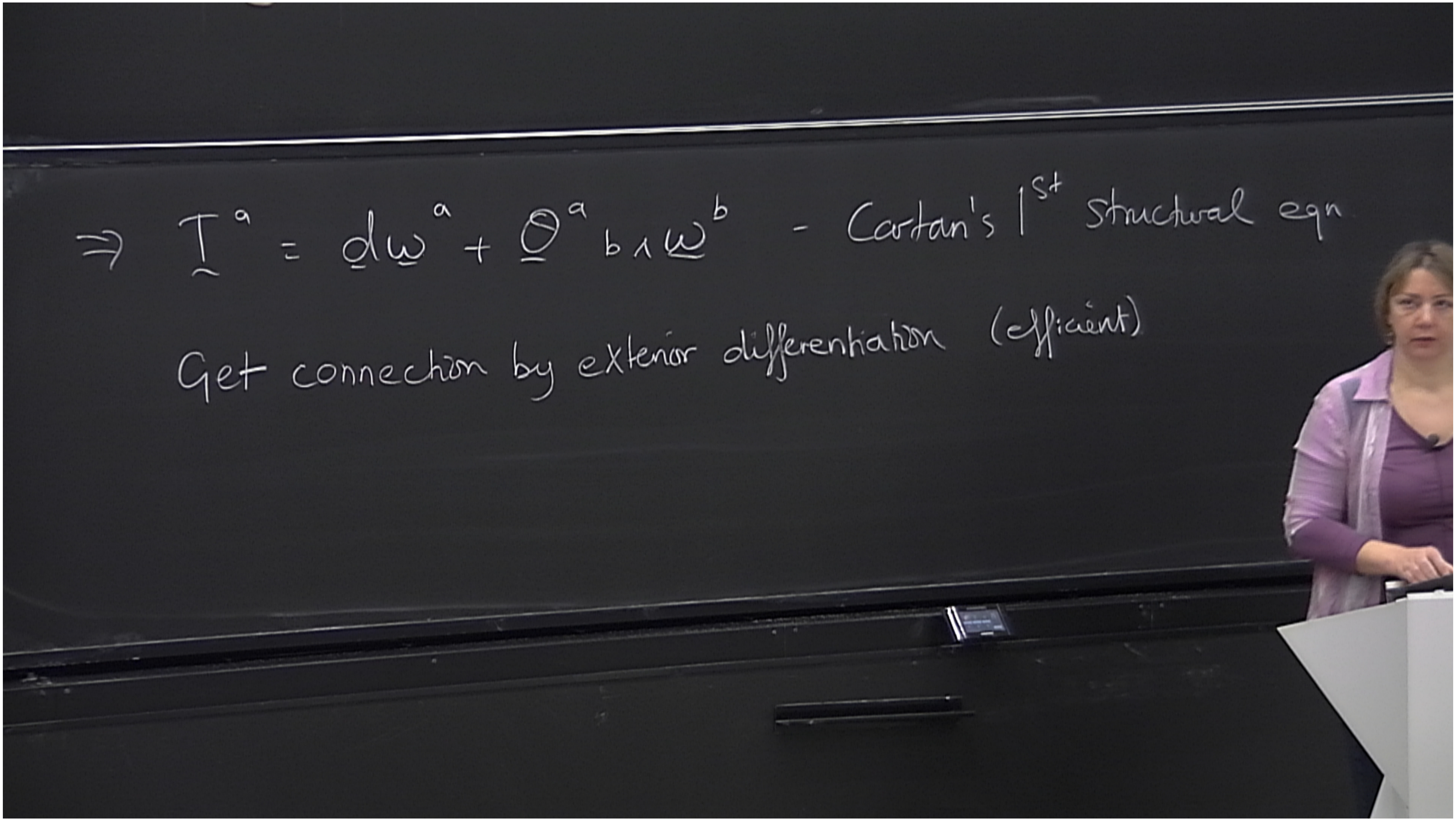
$$= \frac{1}{2}\left(T^{a}{}_{bc} + C^{a}{}_{bc}\right) \underline{\omega}^{b} \wedge \underline{\omega}^{c}$$

$$= \underset{\sim}{T} + \frac{1}{2} \left\langle \underline{\omega}^{a} \middle| [\underline{e}_{b}, \underline{e}_{c}] \right\rangle \underline{\omega}^{b} \wedge \underline{\omega}^{c}$$

where $\underset{-}{T}^a = \frac{1}{2} T^a{}_{bc}\, \underset{-}{\omega}^b \wedge \underset{-}{\omega}^c$

is torsion 2-form.

$\underset{\sim}{\omega}^b \wedge \underset{\sim}{\omega}^c = \underset{\sim}{T}^a - \frac{1}{2} \underbrace{\langle \underset{-}{d}\underset{-}{\omega}^a \,|\, \underset{-}{e}_b, \underset{-}{e}_c \rangle\, \underset{-}{\omega}^b \wedge \underset{-}{\omega}^c}_{\underset{-}{d}\underset{-}{\omega}^a}$


$\Rightarrow \underset{\sim}{T}^a = \underset{\sim}{d}\underset{\sim}{\omega}^a + \underset{\sim}{\theta}^a{}_b \wedge \underset{\sim}{\omega}^b$ - Cartan's 1st structural eqn
Get connection by exterior differentiation (efficient)