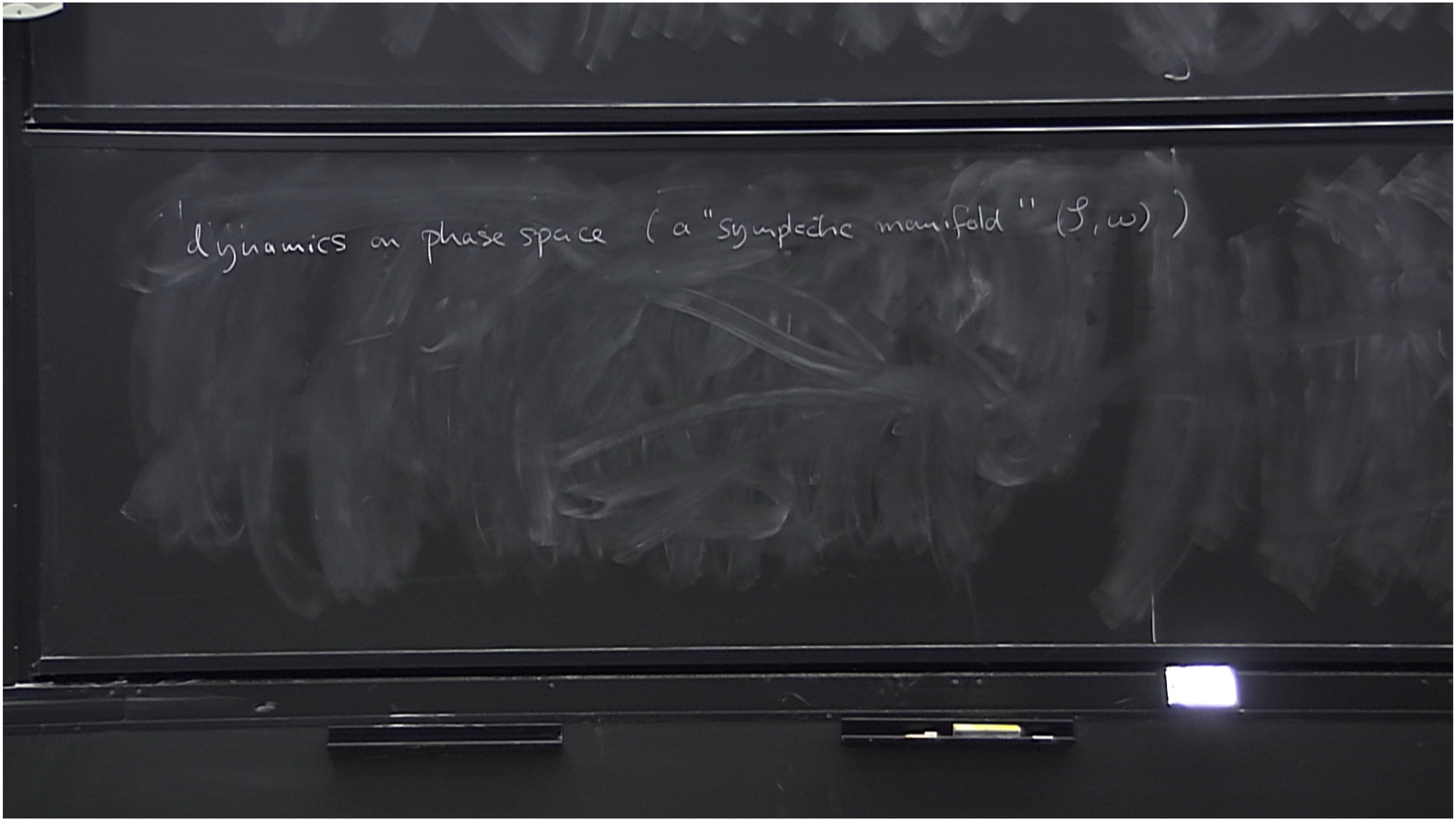
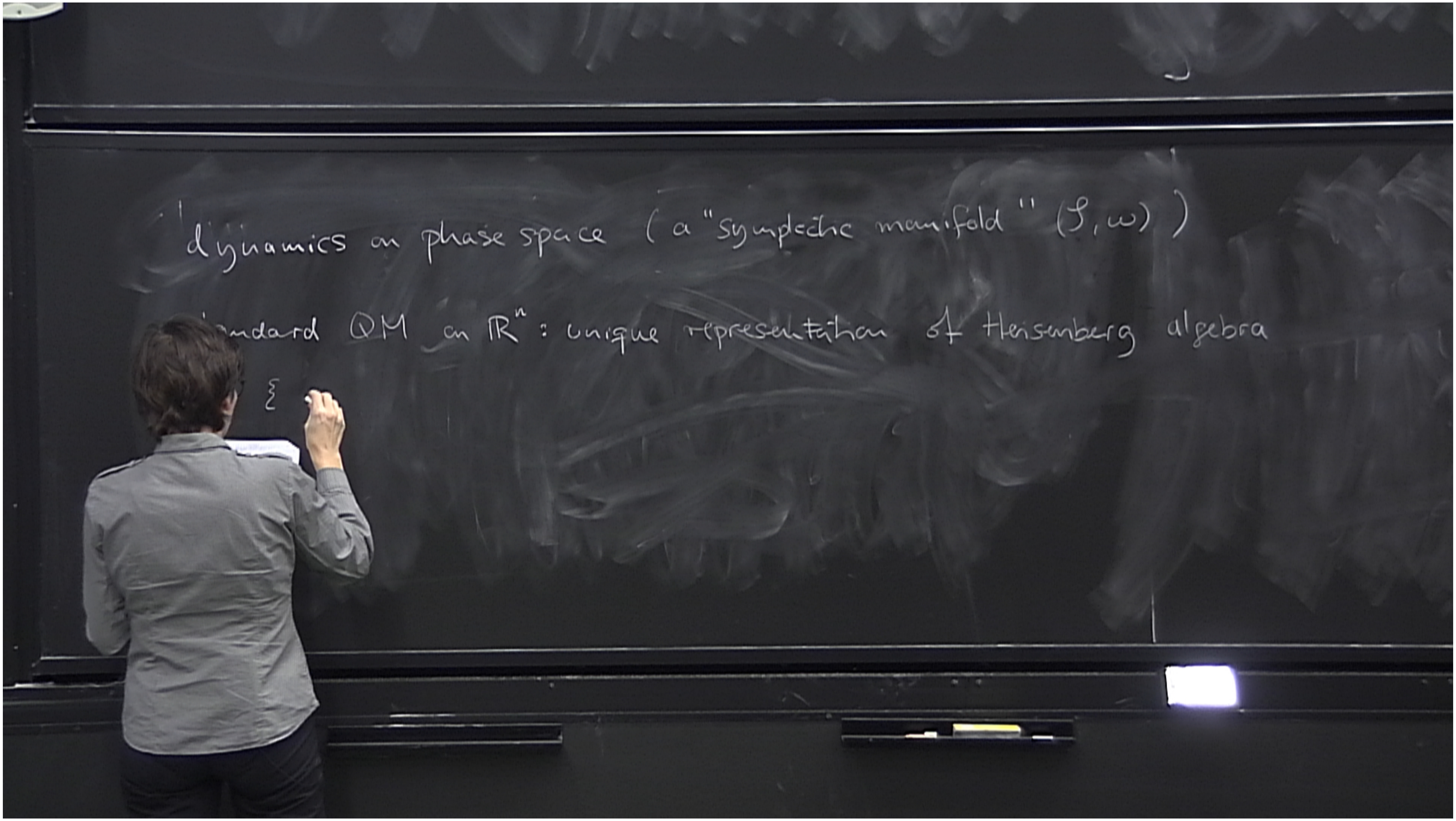
Title: Quantum Gravity (Review) - Lecture 7

Date: Jan 31, 2012 10:15 AM

URL: http://pirsa.org/12010065

Abstract:

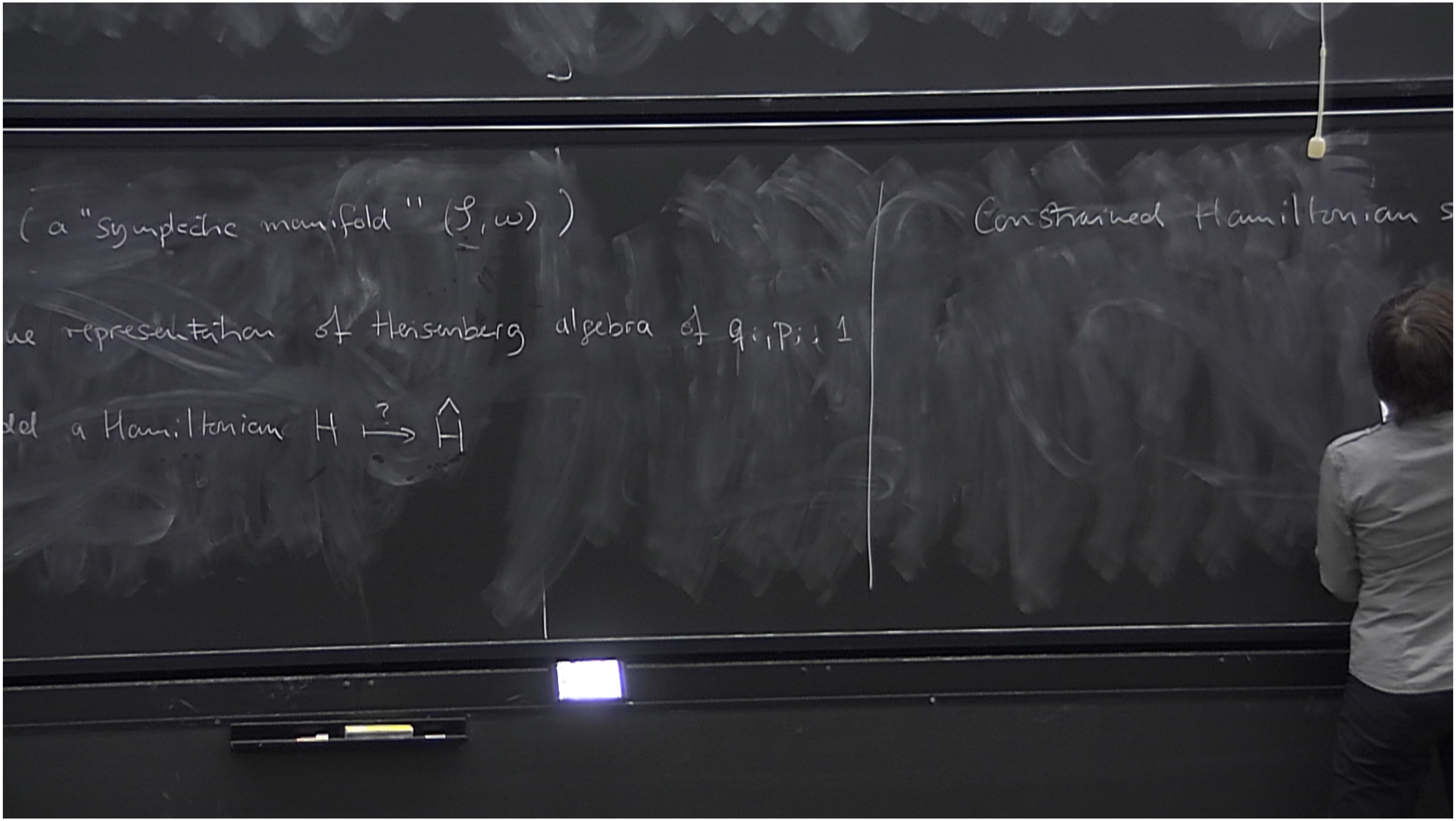

dynamics on phase space (a "symplectic manifold" $(\mathcal{S}, \omega)$)

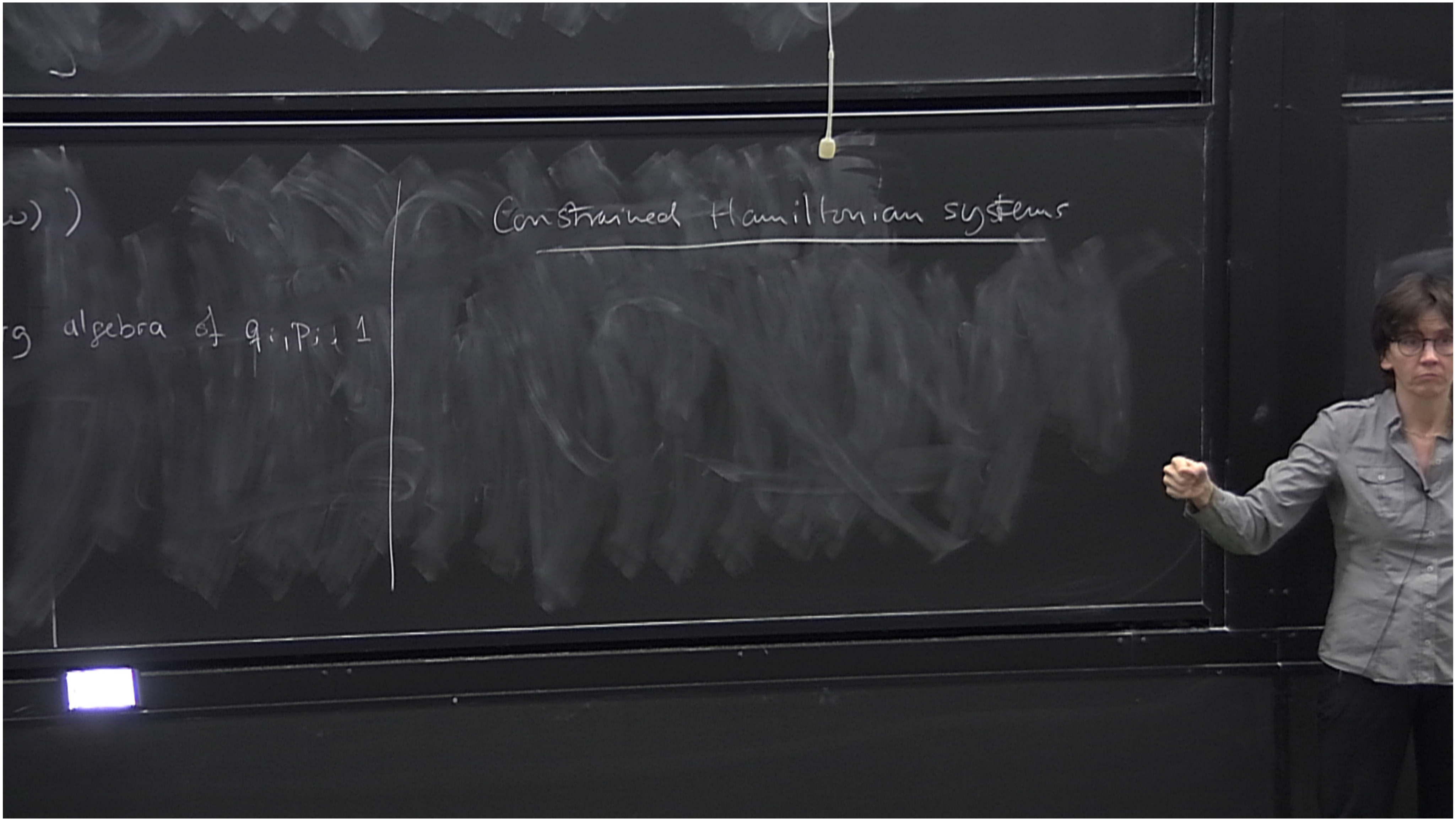

dynamics on phase space (a "symplectic manifold" $(\mathcal{S}, \omega)$)
ndard QM on $\mathbb{R}^n$: unique representation of Heisenberg algebra

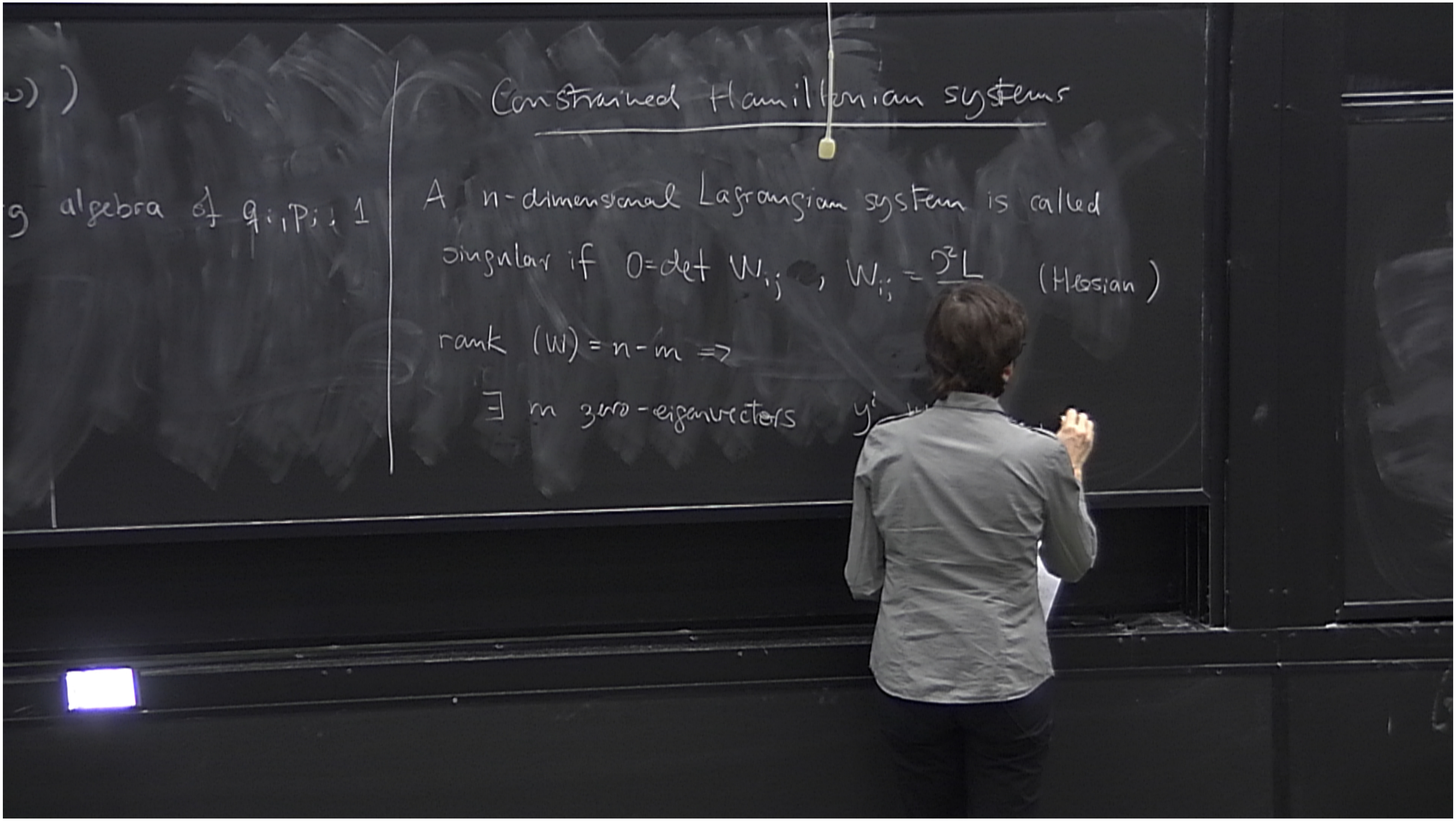
dynamics on phase space (a "symplectic manifold" $(\mathcal{S}, \omega)$)

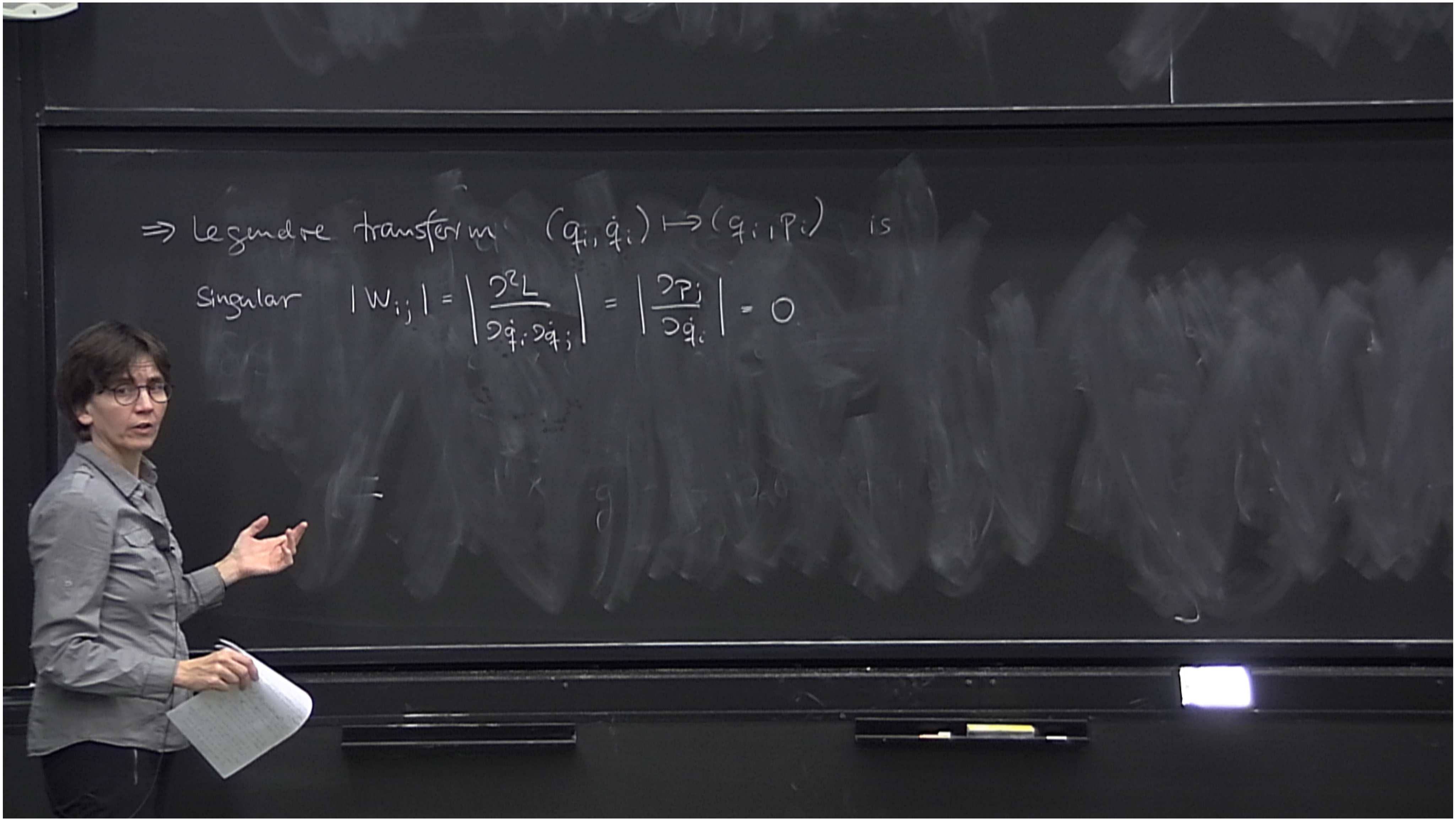
standard QM on $\mathbb{R}^n$: unique representation of Heisenberg algebra of $q_i, p_i, 1$

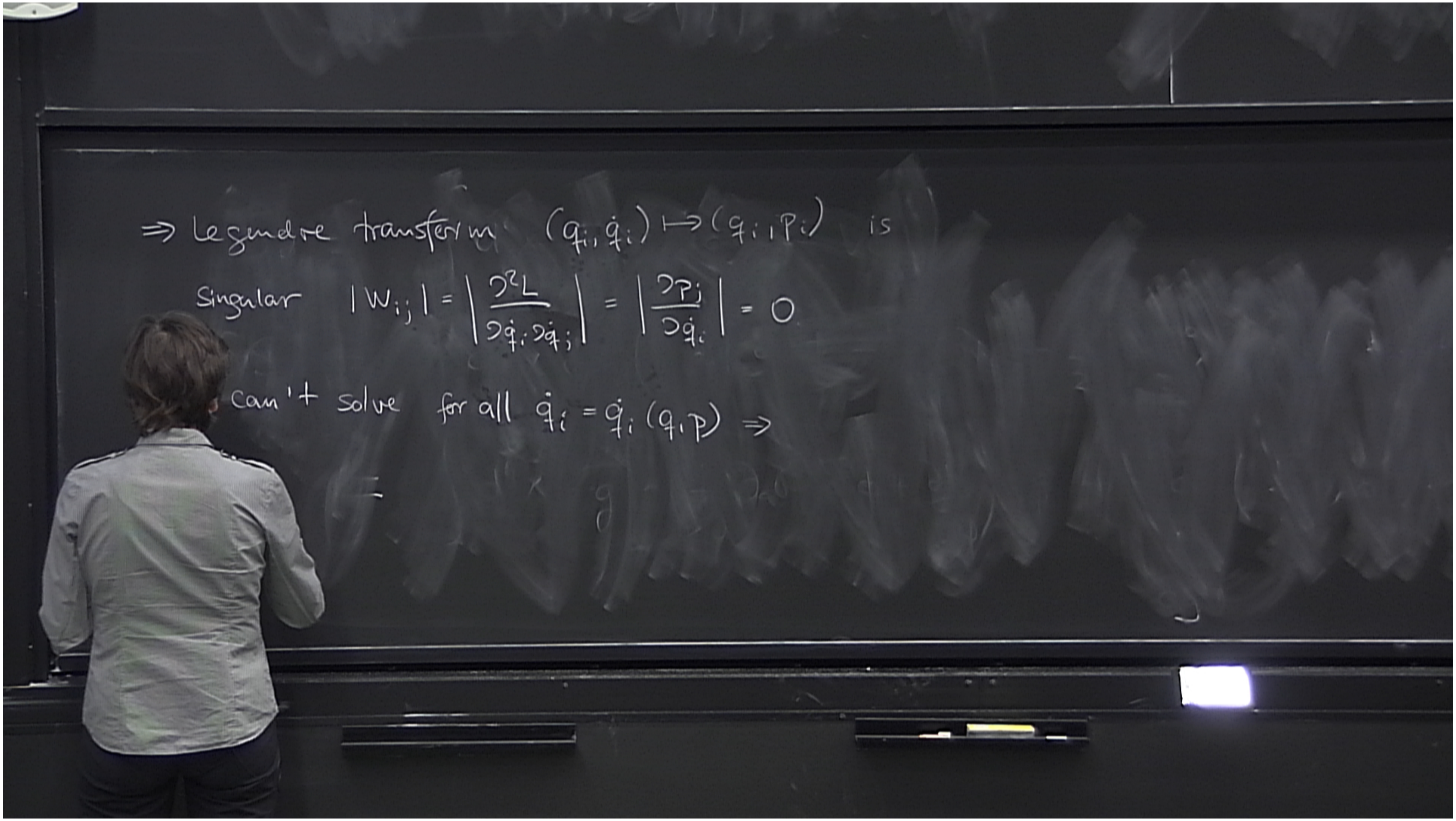
$\{q_i, p_j\} = \delta_{ij}$, and add a Hamiltonian $H \xrightarrow{?} \hat{H}$

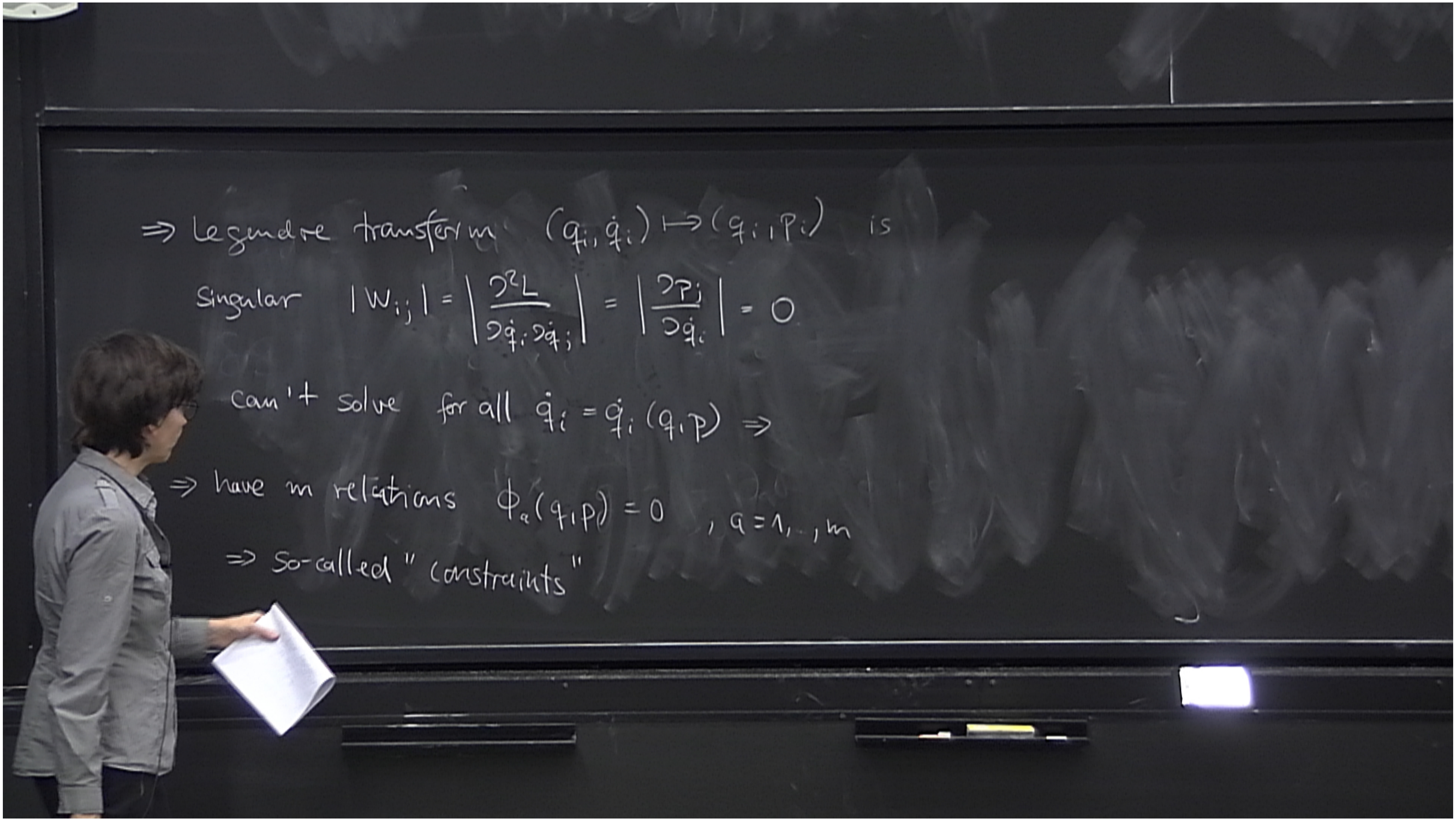

(a "symplectic manifold" $(\mathcal{S}, \omega)$)
ue representation of Heisenberg algebra of $q_i, p_i, 1$
del a Hamiltonian $H \xrightarrow{?} \hat{H}$
Constrained Hamiltonian s

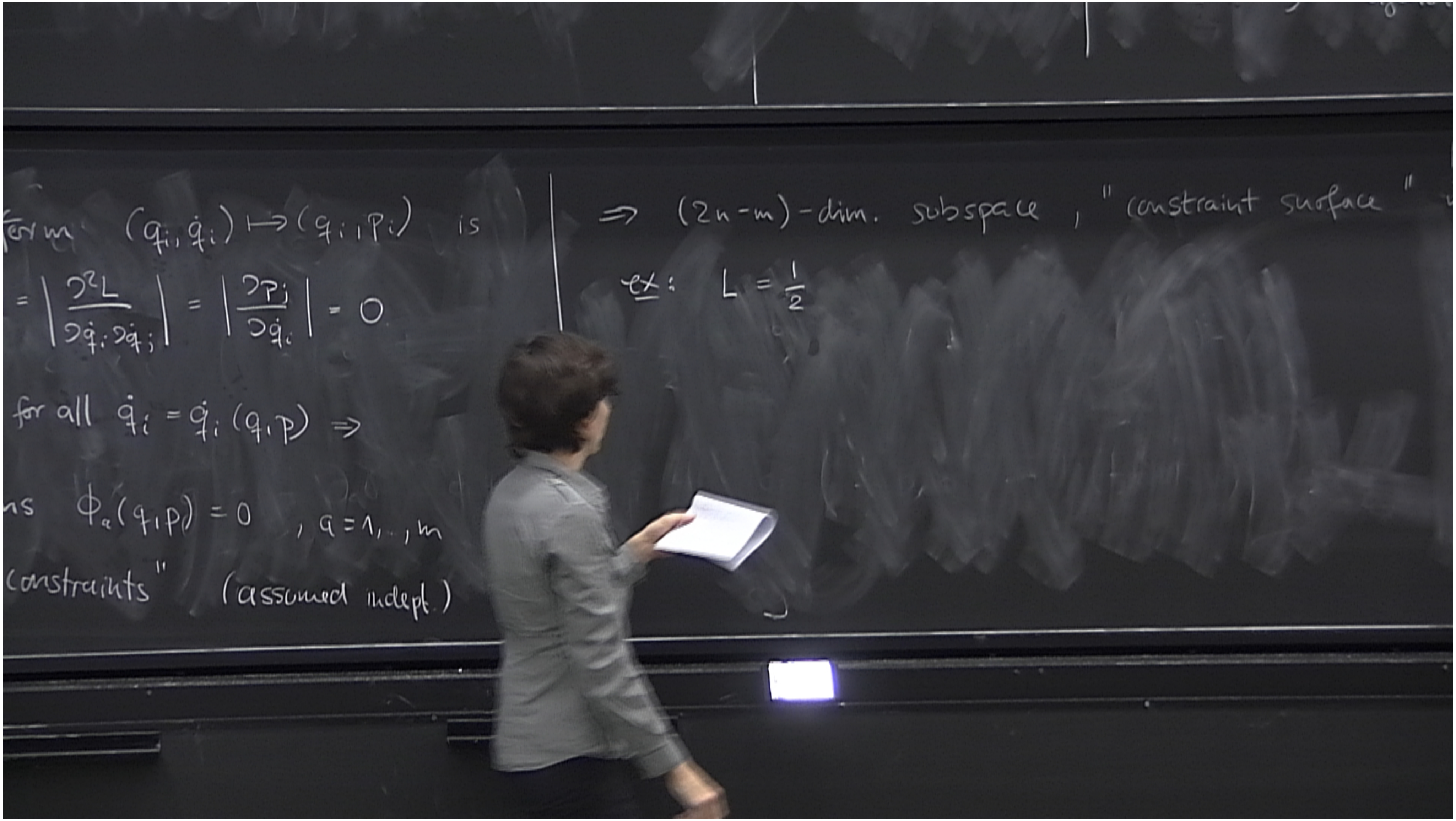

ω))
g algebra of $q_i, p_i, 1$
Constrained Hamiltonian systems

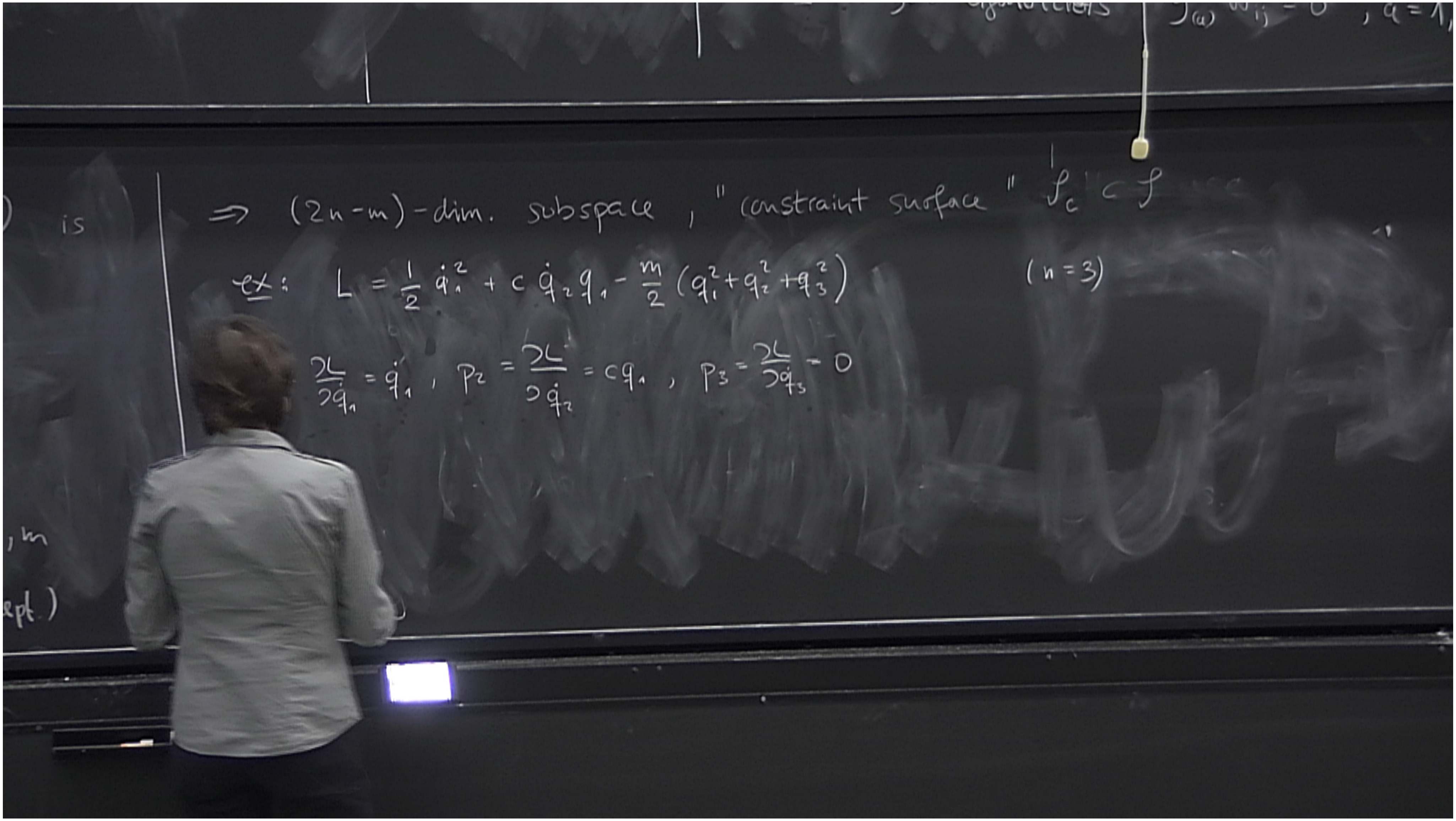

Constrained Hamiltonian systems
A n-dimensional Lagrangian system is called
singular if 0=det W_ij , W_ij = ∂²L (Hessian)
rank (W) = n-m =>
∃ m zero-eigenvectors
algebra of q_i, p_i, 1


⇒ Legendre transform $(q_i, \dot{q}_i) \mapsto (q_i, p_i)$ is
singular $|W_{ij}| = \left|\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}\right| = \left|\frac{\partial p_j}{\partial \dot{q}_i}\right| = 0$


⇒ Legendre transform $(q_i, \dot{q}_i) \mapsto (q_i, p_i)$ is
singular $|W_{ij}| = \left| \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j} \right| = \left| \frac{\partial p_j}{\partial \dot{q}_i} \right| = 0$
can't solve for all $\dot{q}_i = \dot{q}_i(q, p)$ ⇒


⇒ Legendre transform $(q_i, \dot{q}_i) \mapsto (q_i, p_i)$ is
singular $|W_{ij}| = \left| \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j} \right| = \left| \frac{\partial p_j}{\partial \dot{q}_i} \right| = 0$
can't solve for all $\dot{q}_i = \dot{q}_i(q, p)$ ⇒
⇒ have m relations $\phi_a(q, p) = 0$, $a = 1, .., m$
⇒ so-called "constraints"


$(q_i, \dot q_i) \mapsto (q_i, p_i)$ is
$= \left|\frac{\partial^2 L}{\partial \dot q_i \partial \dot q_j}\right| = \left|\frac{\partial p_j}{\partial \dot q_i}\right| = 0$
for all $\dot q_i = \dot q_i(q,p) \Rightarrow$
$\phi_a(q,p) = 0$, $a = 1, .., m$
constraints" (assumed indept.)
$\Rightarrow$ $(2n-m)$-dim. subspace , "constraint surface"
ex: $L = \frac{1}{2}$


is
$\Rightarrow$ $(2n-m)$-dim. subspace, "constraint surface" $\Gamma_c \subset \Gamma$
ex: $L = \frac{1}{2}\dot{q}_1^2 + c\,\dot{q}_2 q_1 - \frac{m}{2}(q_1^2+q_2^2+q_3^2)$
$(n=3)$
$\frac{\partial L}{\partial \dot{q}_1} = \dot{q}_1$, $p_2 = \frac{\partial L}{\partial \dot{q}_2} = c\,q_1$, $p_3 = \frac{\partial L}{\partial \dot{q}_3} = 0$
, m
ept.)

$\Rightarrow$ $(2n-m)$-dim. subspace, "constraint surface" $\Gamma_c \subset \Gamma$

ex: $L = \frac{1}{2}\dot{q}_1^2 + c\,\dot{q}_2 q_1 - \frac{m}{2}(q_1^2 + q_2^2 + q_3^2)$ $(n=3)$

$p_1 = \frac{\partial L}{\partial \dot{q}_1} = \dot{q}_1$, $p_2 = \frac{\partial L}{\partial \dot{q}_2} = c q_1$, $p_3 = \frac{\partial L}{\partial \dot{q}_3} = 0$

$W_{ij} = \Big($

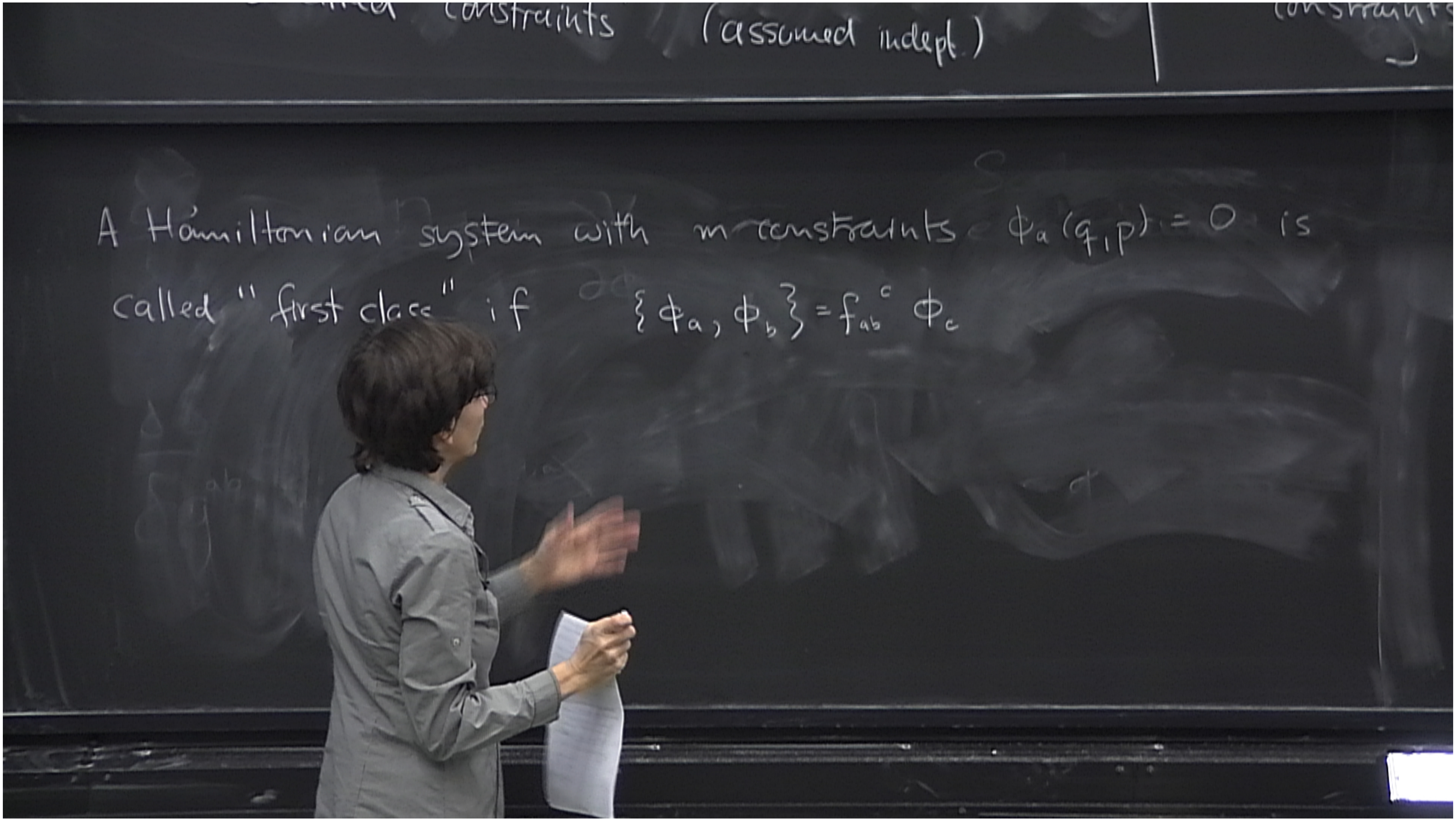
$\Rightarrow$ $(2n-m)$-dim. subspace, "constraint surface" $\Gamma_c \subset \Gamma$

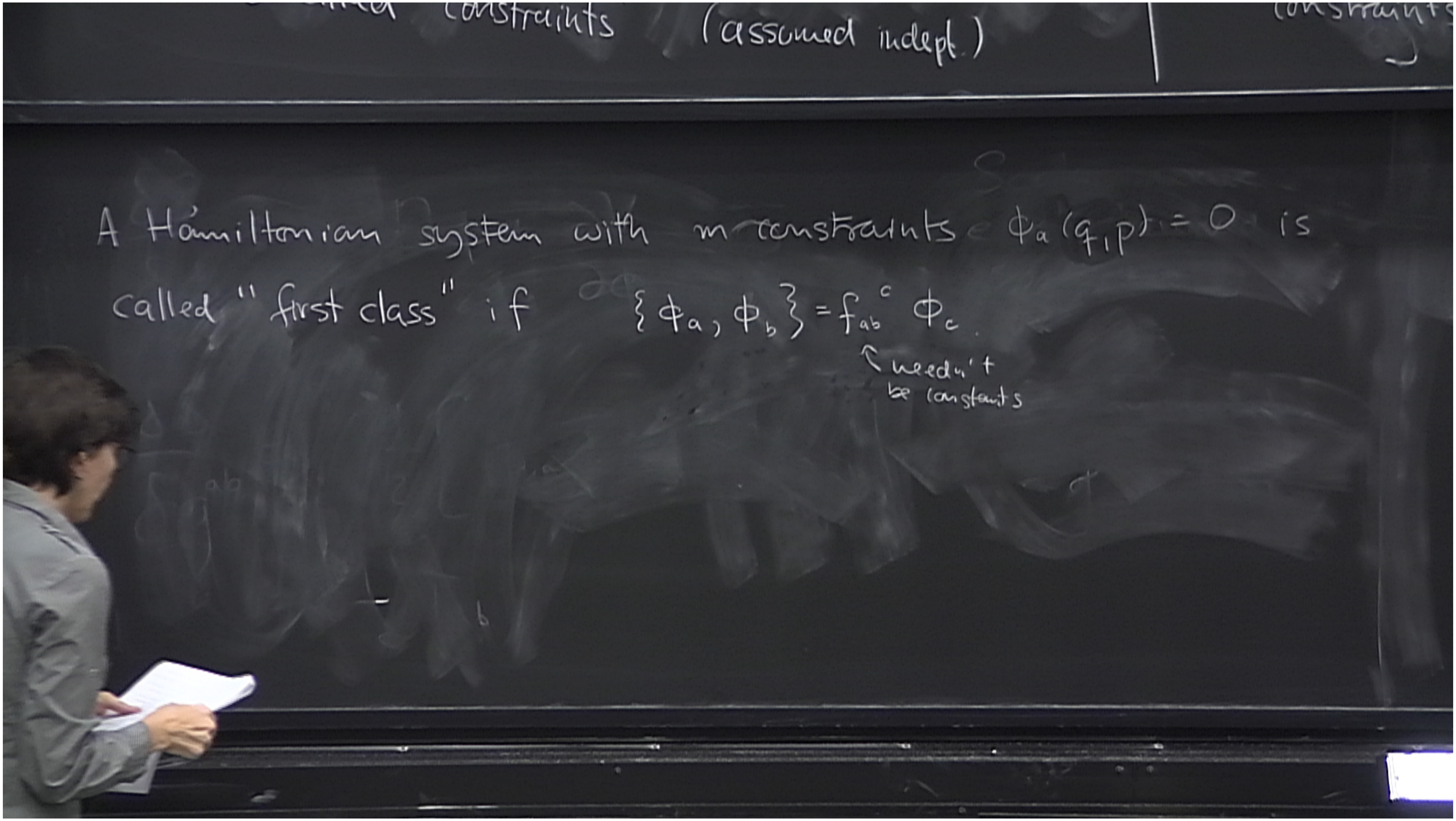
ex: $L = \frac{1}{2}\dot{q}_1^2 + c\,\dot{q}_2 q_1 - \frac{m}{2}(q_1^2+q_2^2+q_3^2)$ $(n=3)$

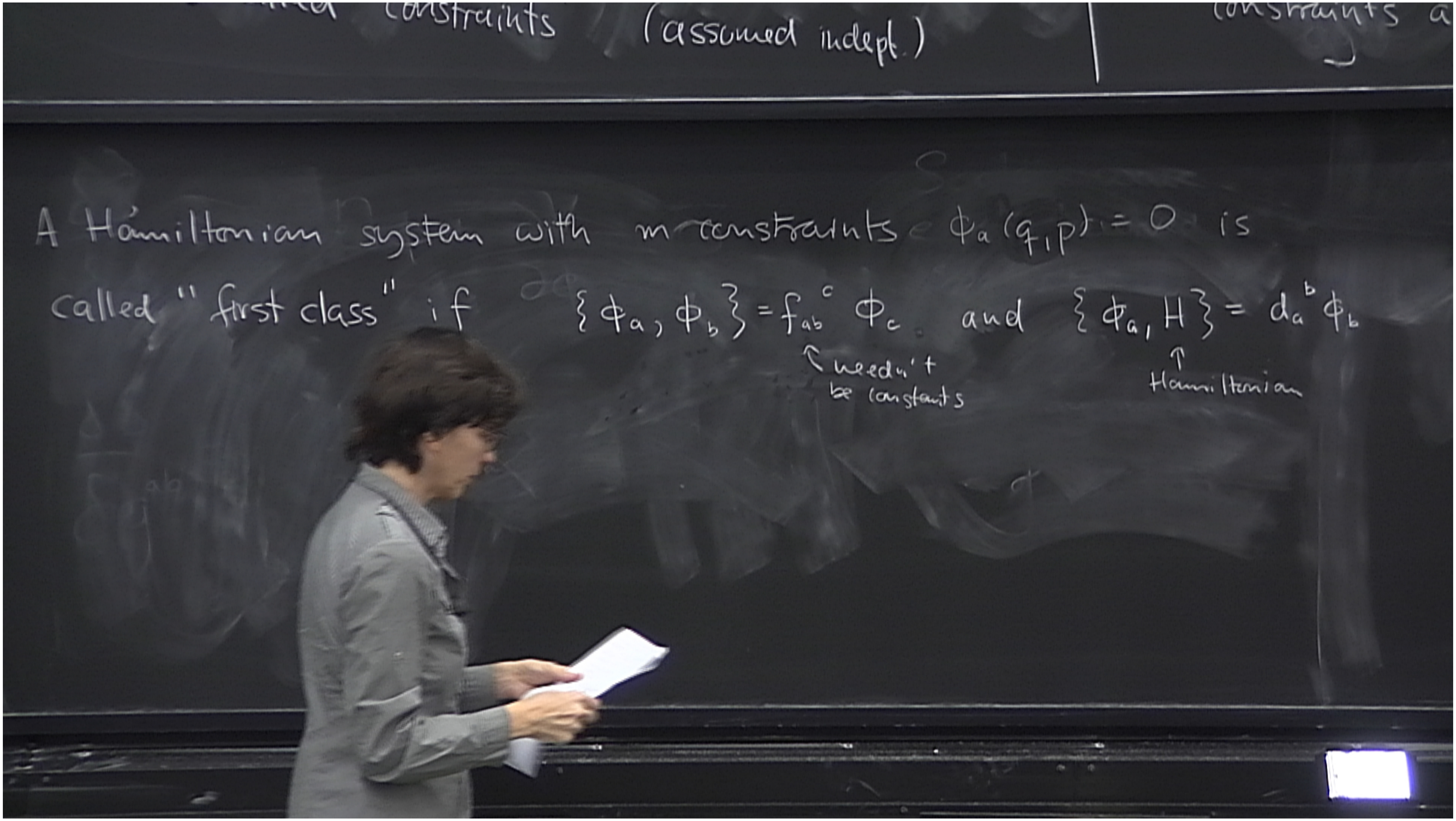
$p_1 = \frac{\partial L}{\partial \dot{q}_1} = \dot{q}_1$, $p_2 = \frac{\partial L}{\partial \dot{q}_2} = c q_1$, $p_3 = \frac{\partial L}{\partial \dot{q}_3} = 0$

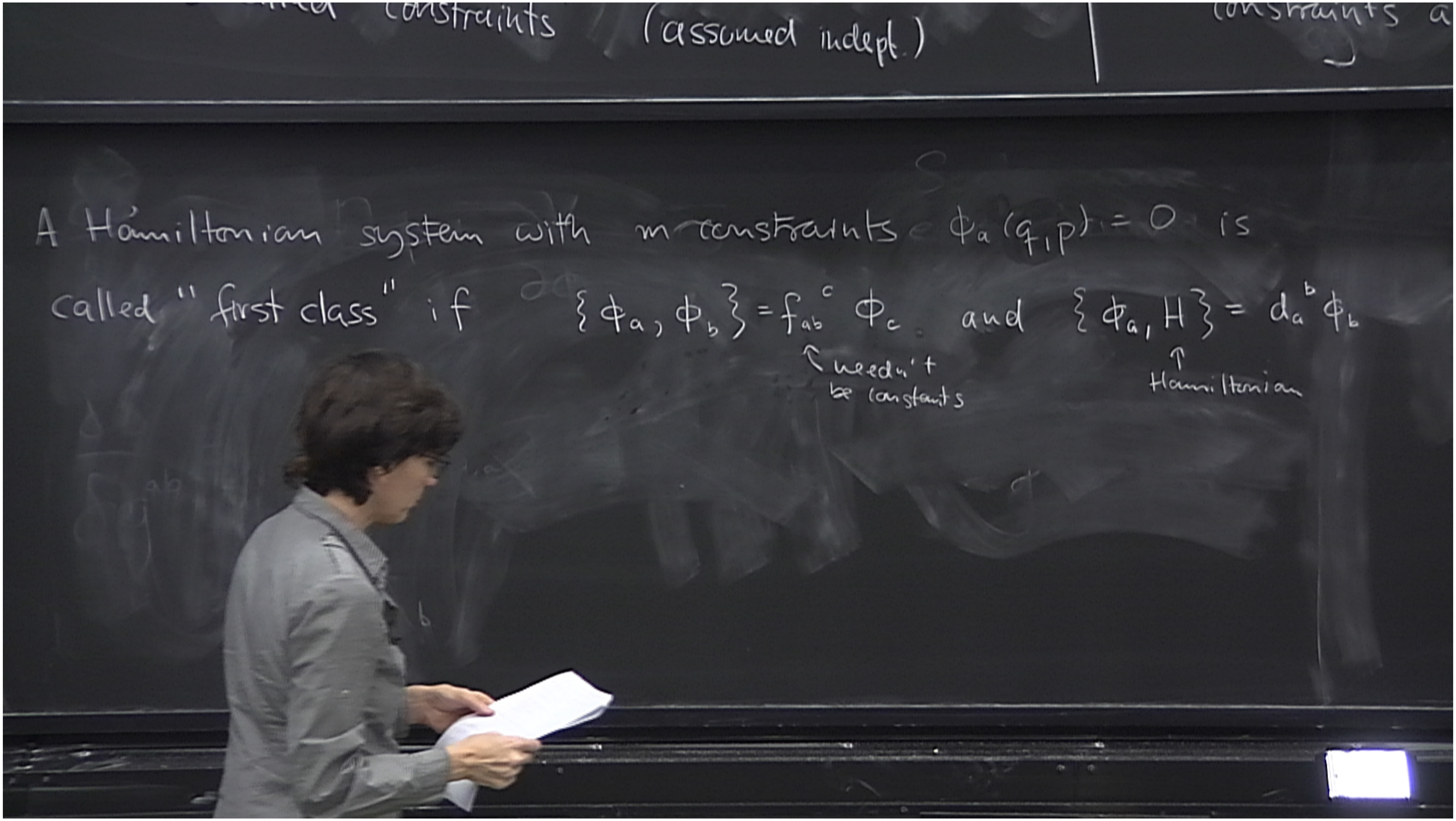
$W_{ij} = \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix}$, rank $W_{ij} = 1$

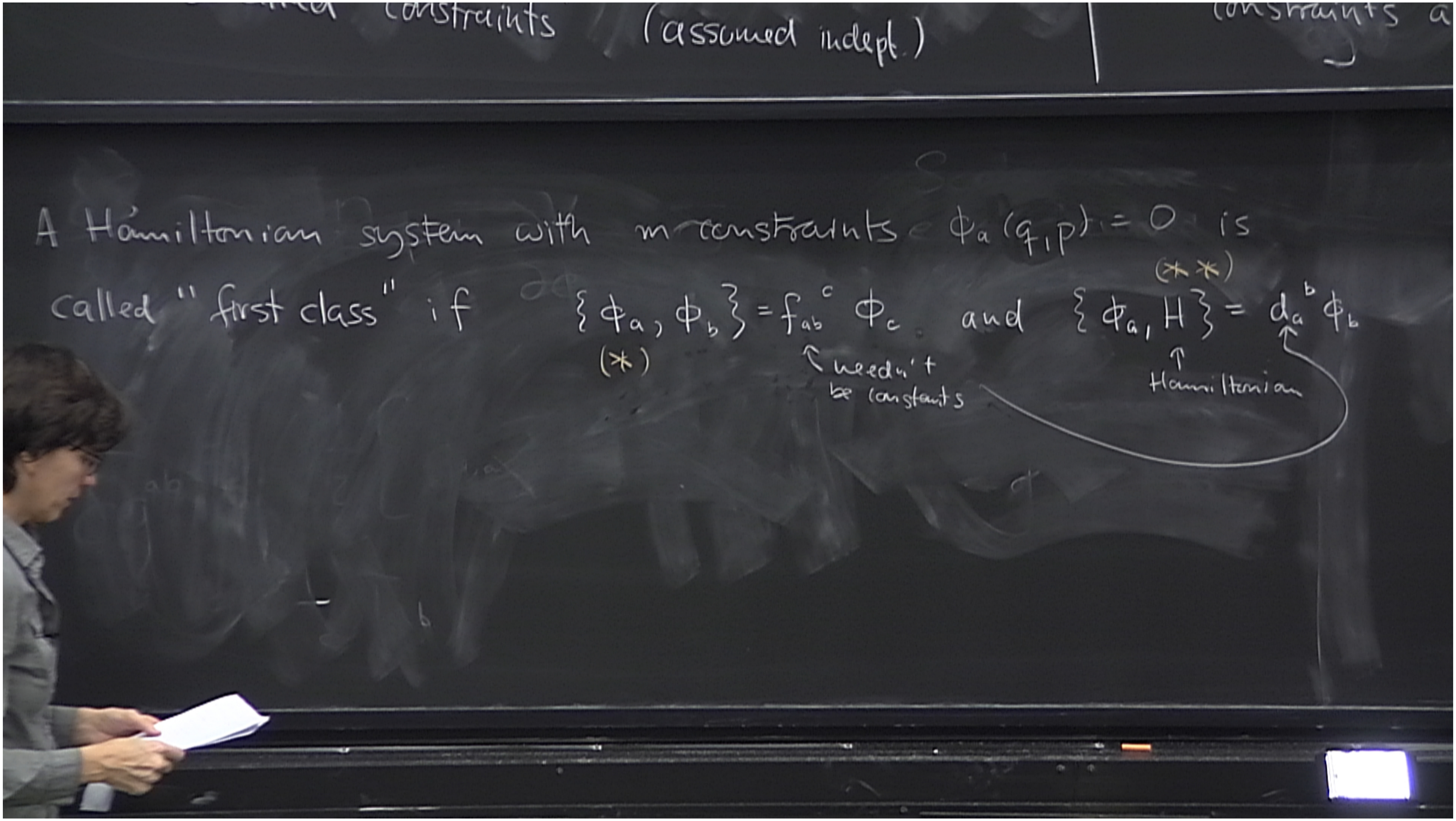
constraints are $\phi_1 := p_2 - c q_1 = 0$, $\phi_2 := p_3 = 0$ $\Rightarrow$ Γ_c is 4-dim.

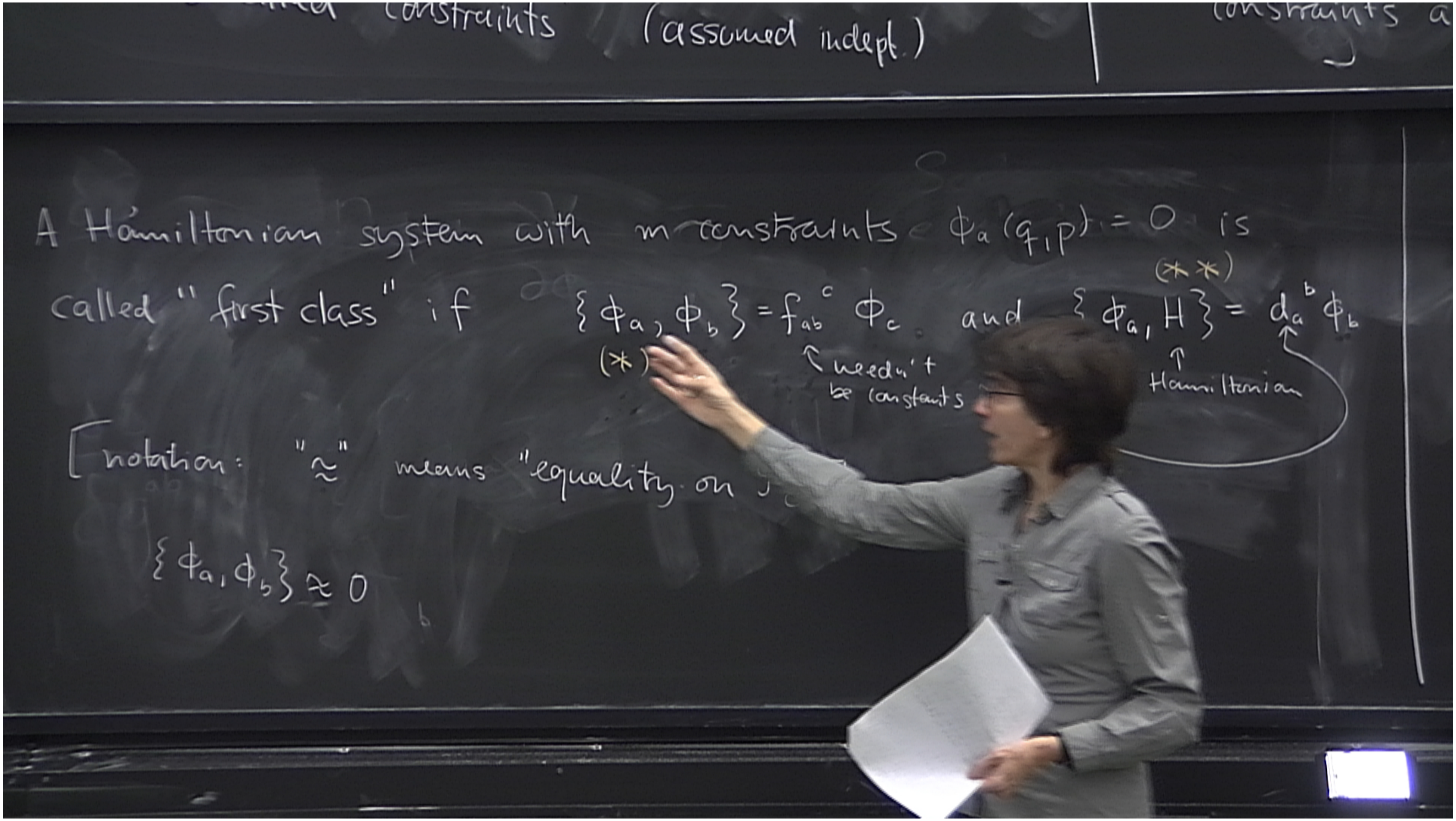

constraints (assumed indept.)
A Hamiltonian system with m constraints $\phi_a(q,p) = 0$ is
called "first class" if $\{\phi_a, \phi_b\} = f_{ab}{}^{c}\, \phi_c$


constraints (assumed indept.)
constraints
A Hamiltonian system with m constraints $\phi_a(q,p) = 0$ is
called "first class" if $\{\phi_a, \phi_b\} = f_{ab}{}^{c}\,\phi_c$.
needn't be constants


constraints (assumed indep.)
constraints a
A Hamiltonian system with m constraints $\phi_a(q,p) = 0$ is
called "first class" if $\{\phi_a, \phi_b\} = f_{ab}{}^{c}\,\phi_c$ and $\{\phi_a, H\} = d_a{}^{b}\,\phi_b$
needn't be constants
Hamiltonian


constraints (assumed indept.)
A Hamiltonian system with m constraints $\phi_a(q,p)=0$ is called "first class" if $\{\phi_a,\phi_b\}=f_{ab}{}^{c}\,\phi_c$ and $\{\phi_a,H\}=d_a{}^{b}\,\phi_b$
needn't be constants
Hamiltonian


constraints (assumed indept.)
A Hamiltonian system with m constraints $\phi_a(q,p) = 0$ is
called "first class" if $\{\phi_a, \phi_b\} = f_{ab}{}^c\,\phi_c$ and $\{\phi_a, H\} = d_a{}^b\,\phi_b$
(*)
(**)
needn't be constants
Hamiltonian


constraints (assumed indept.)
A Hamiltonian system with m constraints $\phi_a(q,p) = 0$ is
called "first class" if $\{\phi_a, \phi_b\} = f_{ab}{}^c \phi_c$ and $\{\phi_a, H\} = d_a{}^b \phi_b$
(*)
(**)
needn't be constants
Hamiltonian
[notation: "$\approx$" means "equality on
$\{\phi_a, \phi_b\} \approx 0$

... constraints (assumed indept.) | constraints a ...

A Hamiltonian system with m constraints $\phi_a(q,p) = 0$ is called "first class" if

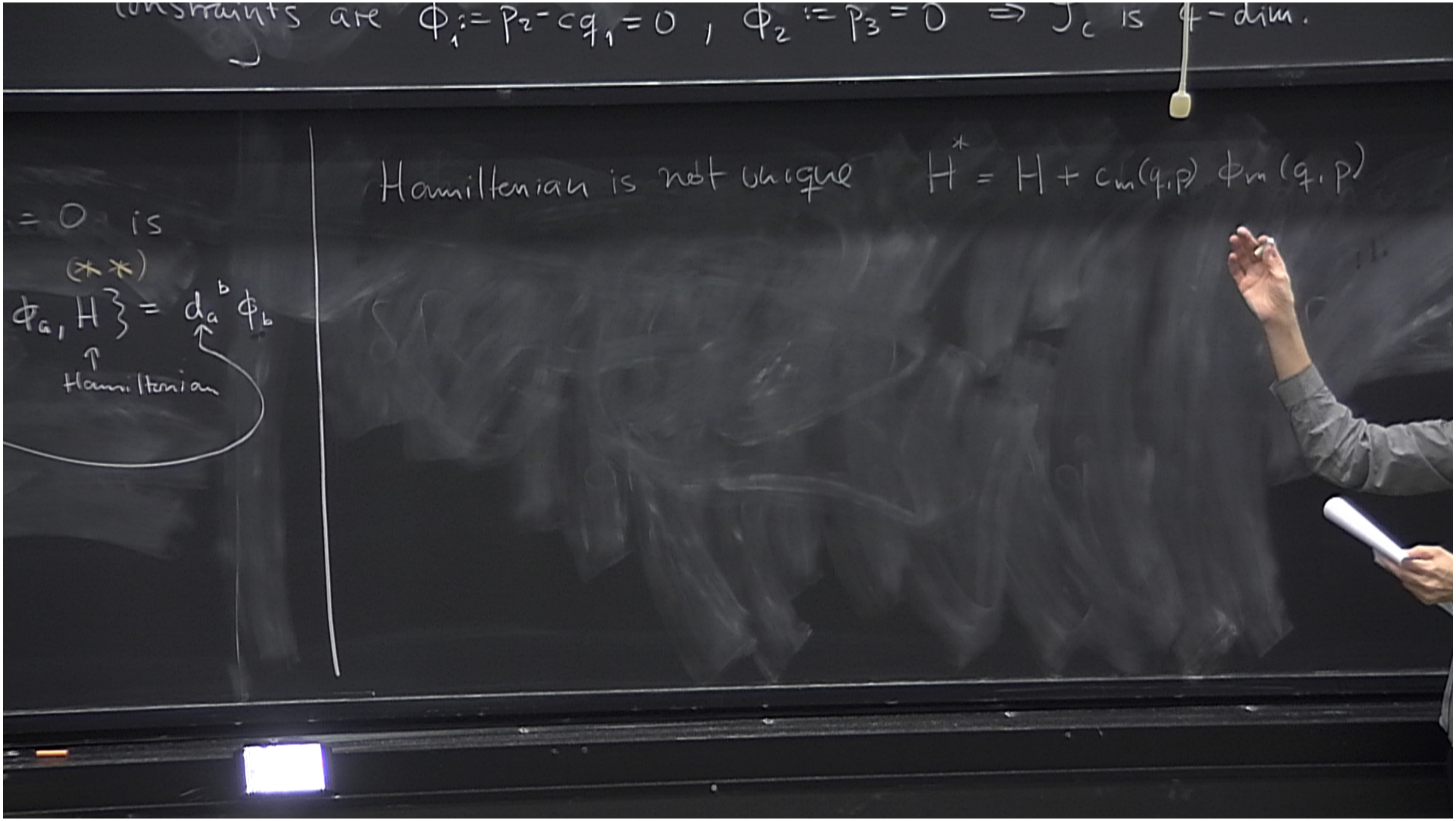
$$\underset{(*)}{\{\phi_a, \phi_b\}} = f_{ab}{}^{c}\, \phi_c \quad \text{and} \quad \{\phi_a, H\} \overset{(**)}{=} d_a{}^{b}\, \phi_b$$

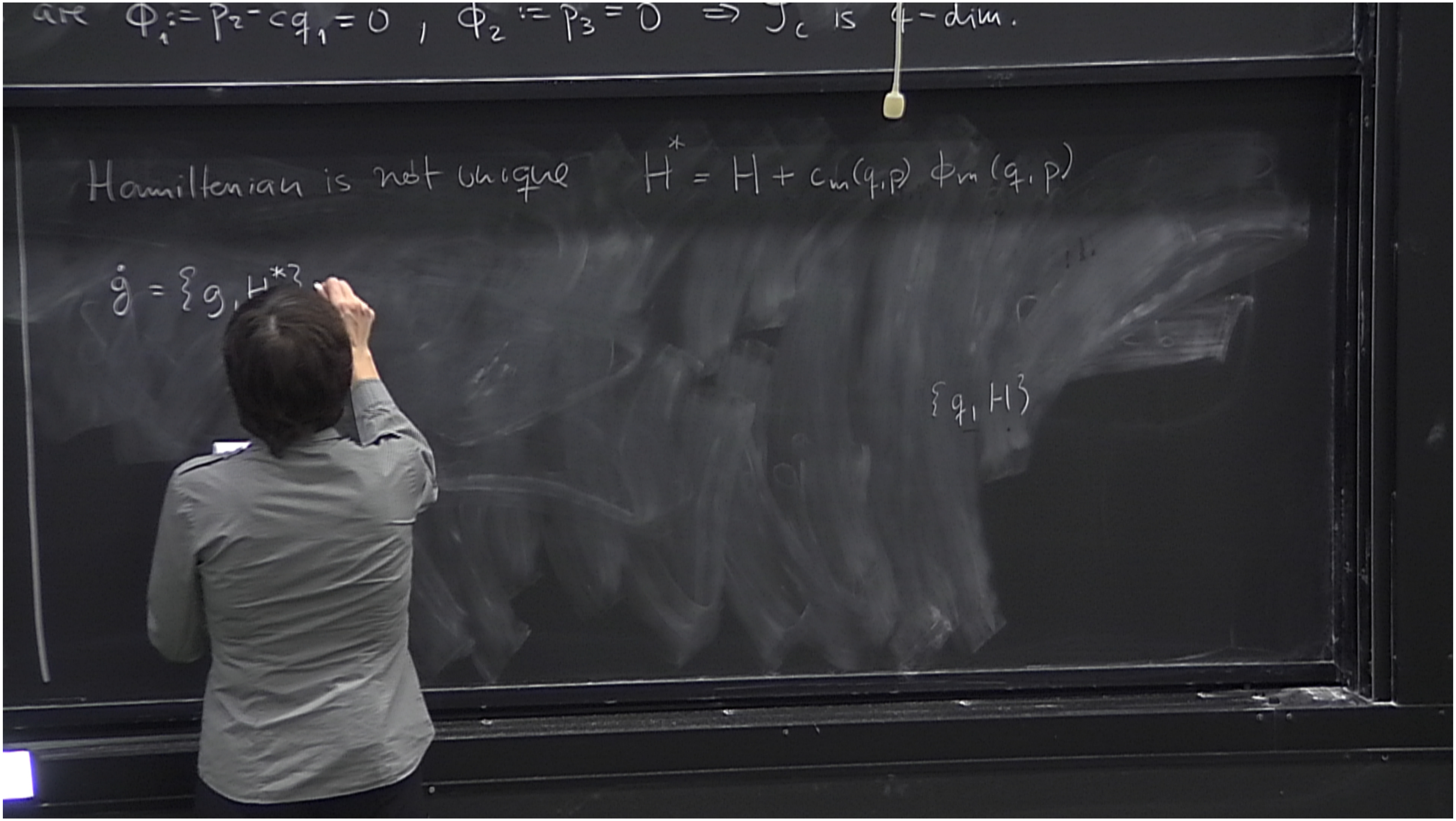
needn't be constants

Hamiltonian

[notation: "$\approx$" means "equality on Σ_c"]

$$\{\phi_a, \phi_b\} \approx 0$$


are $\Phi_1 := p_2 - c q_1 = 0$, $\Phi_2 := p_3 = 0$ $\Rightarrow$ Γ_c is 4-dim.
Hamiltonian is not unique $H^* = H + c_m(q,p)\,\phi_m(q,p)$
$= 0$ is
$(**)$
$\{\phi_a, H\} = d_a{}^b \phi_b$
Hamiltonian


are $\Phi_1 := p_2 - c q_1 = 0$, $\Phi_2 := p_3 = 0$ $\Rightarrow$ Γ_c is 4-dim.
Hamiltonian is not unique $H^* = H + c_m(q,p)\,\phi_m(q,p)$
$\dot{g} = \{g, H^*\}$
$\{q, H\}$

are $\Phi_1 := p_2 - c q_1 = 0$, $\Phi_2 := p_3 = 0$ $\Rightarrow$ Γ_c is 4-dim.

Hamiltonian is not unique $H^* = \underline{H} + c_m(q,p)\,\phi_m(q,p)$

$$\dot g = \{g, H^*\} = \{g, H\} + c_m\{g,\phi_m\} + \underbrace{\phi_m\{g, c_m\}}_{\approx 0}$$

$\dot q_i$

are $\Phi_1 := p_2 - c q_1 = 0$, $\Phi_2 := p_3 = 0$ $\Rightarrow$ Γ_c is 4-dim.

Hamiltonian is not unique $\quad H^* = \underline{H} + c_m(q,p)\,\phi_m(q,p)$

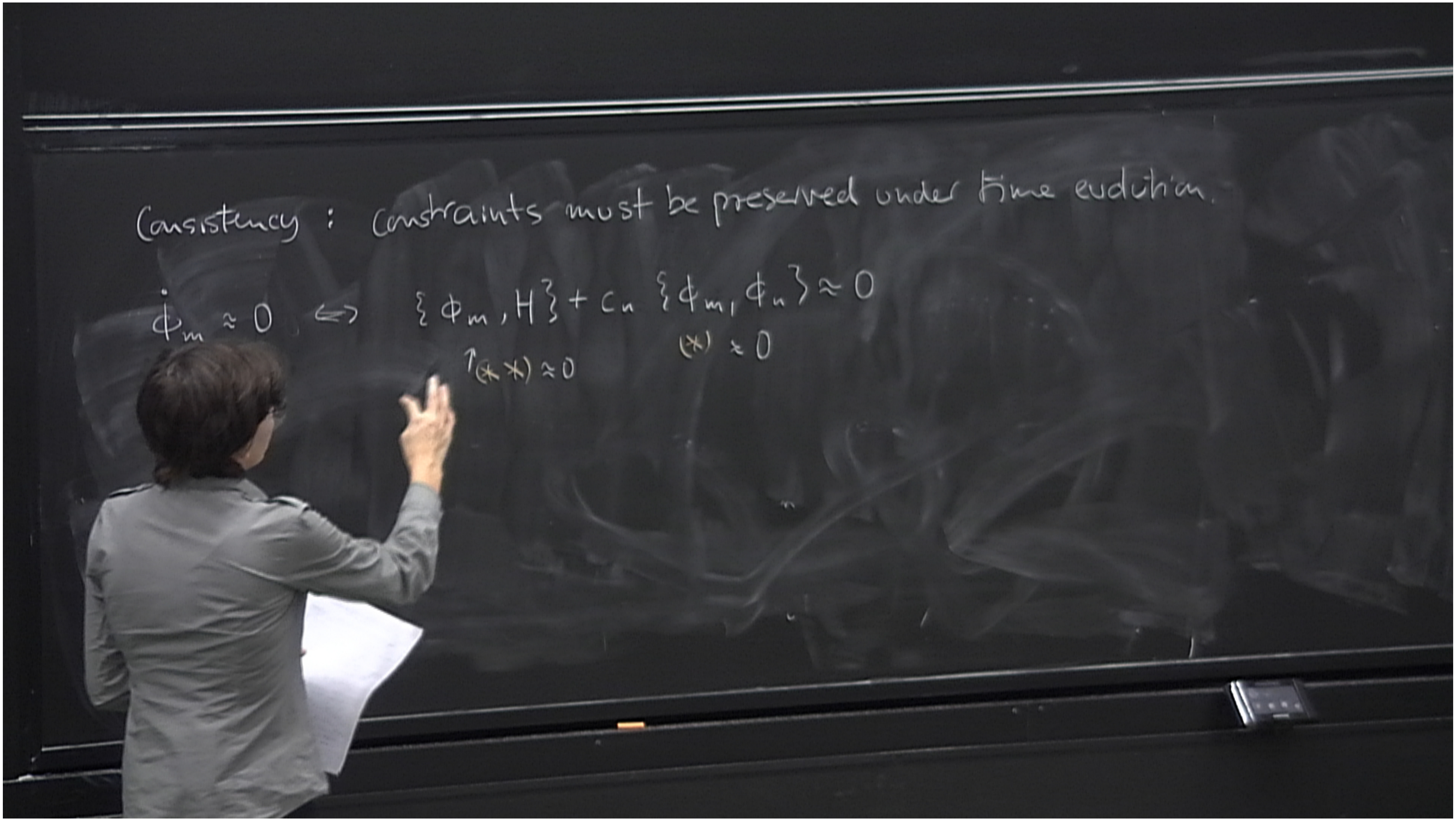
$$\dot{g} = \{g, H^*\} = \{g, H\} + c_m \{g, \phi_m\} + \underbrace{\phi_m \{g, c_m\}}_{\approx 0}$$

$$\dot{q}_i \approx \frac{\partial H}{\partial p_i} + c_m \frac{\partial \phi_m}{\partial p_i} \; , \quad \dot{p}_i \approx -\frac{\partial H}{\partial q_i} - c_m \frac{\partial \phi_m}{\partial q_i}$$

$\Rightarrow$ e.o.m. contain m arbitrary phase space functions c_m.

Consistency : constraints must be preserved under time evolution.

$$\dot{\phi}_m \approx 0 \iff \{\phi_m, H\} +$$


Consistency : constraints must be preserved under time evolution.
$\dot{\phi}_m \approx 0 \Leftrightarrow \{\phi_m, H\} + c_n \{\phi_m, \phi_n\} \approx 0$
$(**) \approx 0$
$(*) \approx 0$

Consistency : constraints must be preserved under time evolution.

$$\dot{\phi}_m \approx 0 \iff \{\phi_m, H\} + c_n \{\phi_m, \phi_n\} \approx 0$$

$$(**) \approx 0 \qquad (*) \approx 0$$

$$\{\phi_m, H\} \approx 0$$

(i)

Consistency : constraints must be preserved under time evolution.

$$\dot{\phi}_m \approx 0 \iff \{\phi_m, H\} + c_n \{\phi_m, \phi_n\} \approx 0$$

$(**) \approx 0$ $\qquad (*) \approx 0$

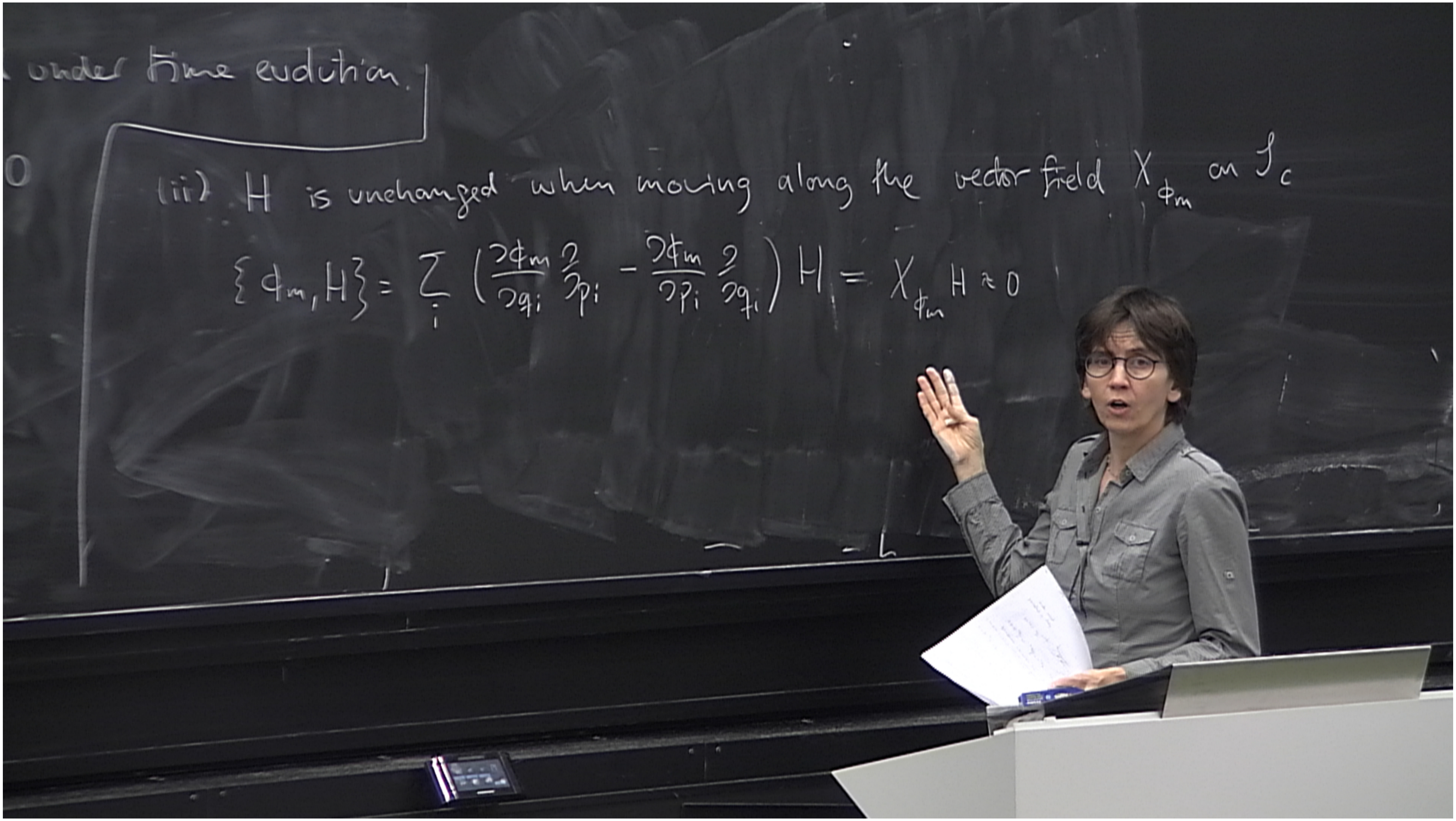
$$\{\phi_m, H\} \approx 0$$

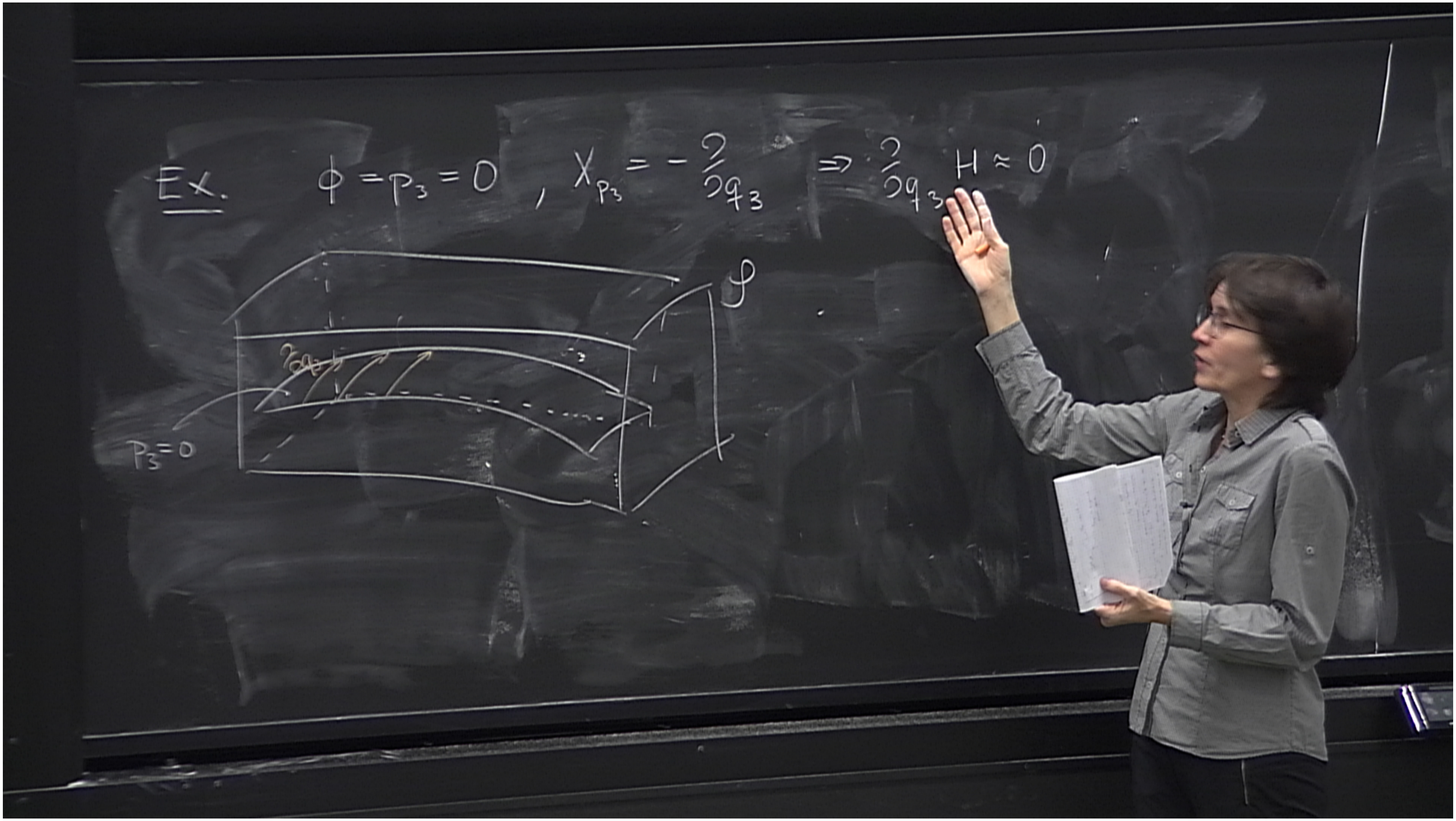
(i) flow of Hamiltonian stays on $\mathcal{S}_c$

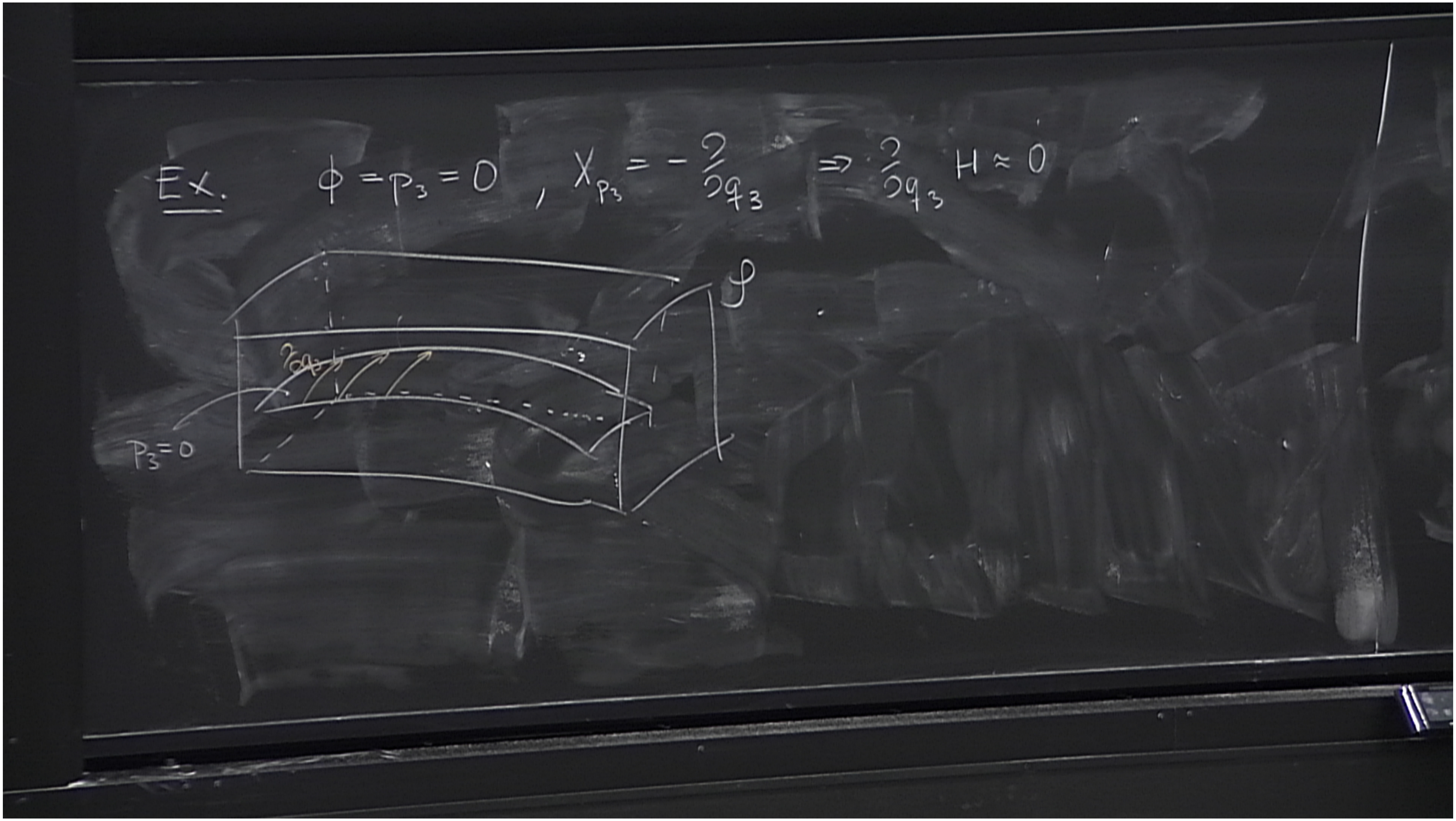
$$\{\phi_m, H\} = \sum_i \left(-\frac{\partial H}{\partial q_i}\frac{\partial}{\partial p_i} + \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q_i} \right) \phi_m$$

$$=: X_H \phi_m \approx 0$$

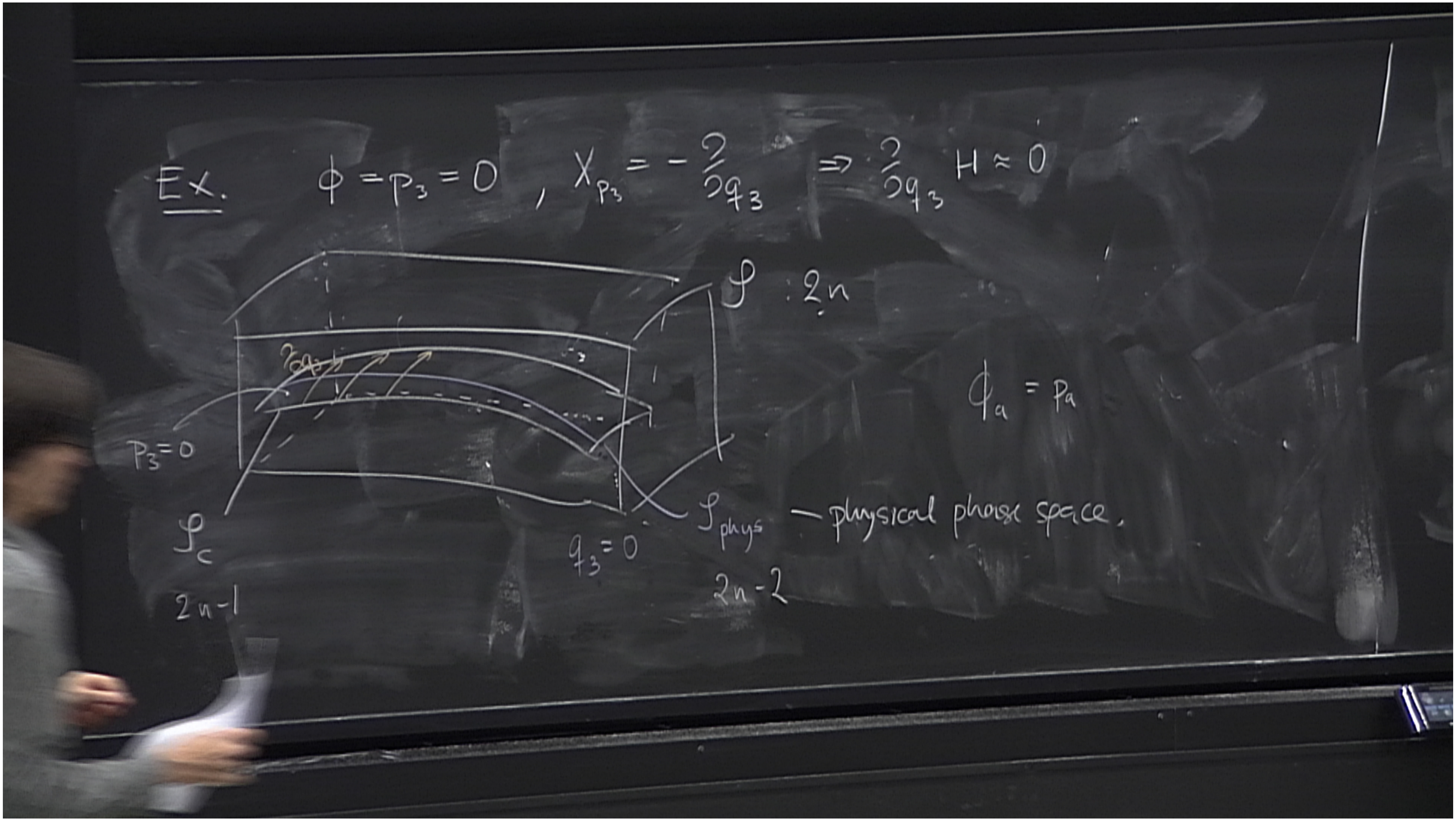
(ii) H is unchanged when m


under time evolution.
(ii) H is unchanged when moving along the vector field X_{ϕ_m} on S_c
$\{\phi_m, H\} = \sum_i \left(\frac{\partial \phi_m}{\partial q_i} \frac{\partial}{\partial p_i} - \frac{\partial \phi_m}{\partial p_i} \frac{\partial}{\partial q_i} \right) H = X_{\phi_m} H \approx 0$


Ex. $\phi = p_3 = 0$, $X_{p_3} = -\frac{\partial}{\partial q_3}$ $\Rightarrow$ $\frac{\partial}{\partial q_3} H \approx 0$
$\frac{\partial}{\partial q_3}$
$p_3 = 0$
φ



Ex. $\phi = p_3 = 0$, $X_{p_3} = -\frac{\partial}{\partial q_3}$ $\Rightarrow$ $\frac{\partial}{\partial q_3} H \approx 0$
φ
∂_{q_3}
$p_3 = 0$


Ex. $\phi = p_3 = 0$, $X_{p_3} = -\frac{\partial}{\partial q_3}$ $\Rightarrow$ $\frac{\partial}{\partial q_3} H \approx 0$
Γ : $2n$
$\phi_a = p_a$
$p_3 = 0$
Γ_c
$2n-1$
$q_3 = 0$
Γ_{phys} — physical phase space.
$2n-2$