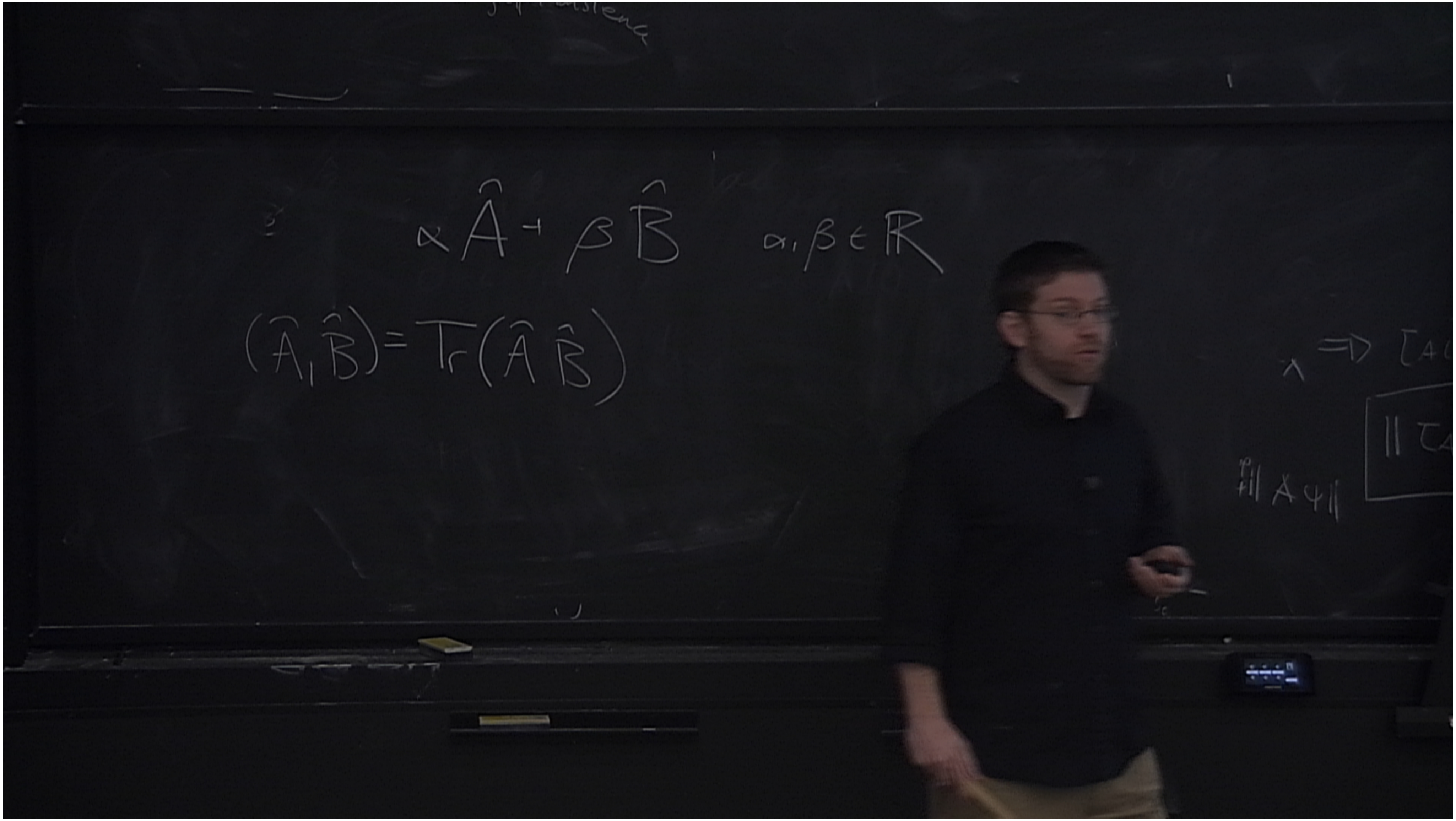
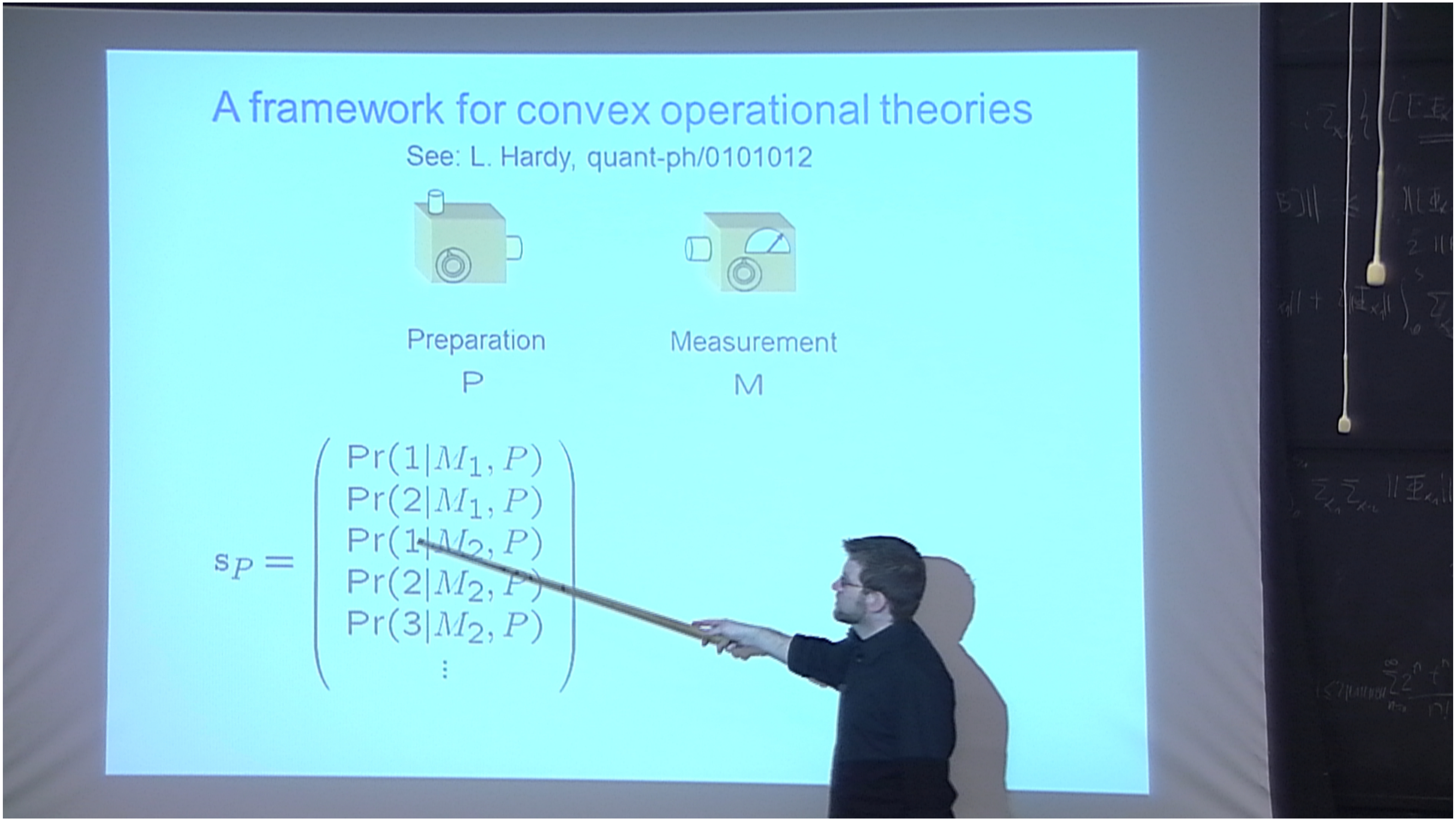
Title: Foundations of Quantum Mechanics - Lecture 5

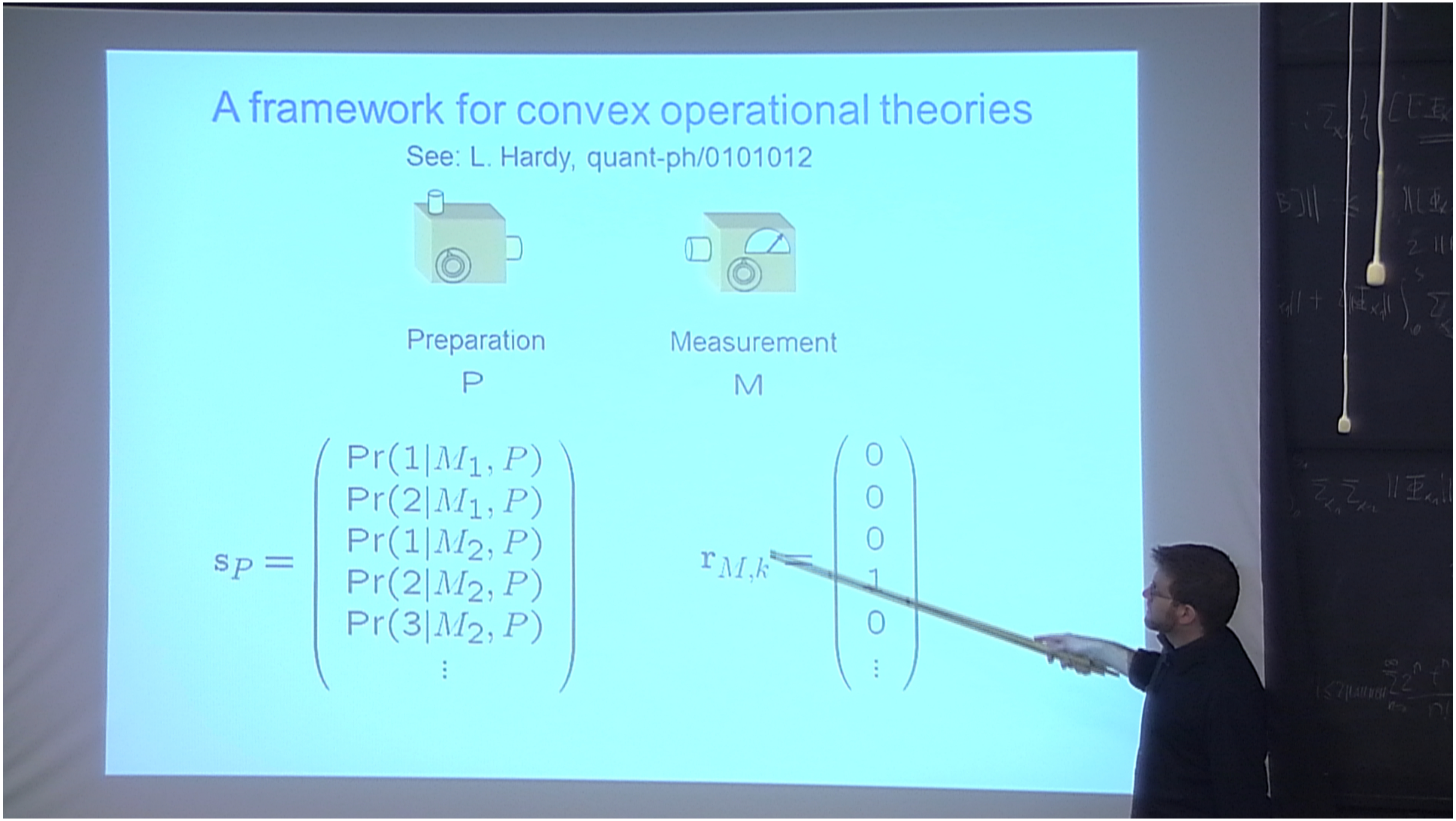
Date: Jan 06, 2012 11:30 AM

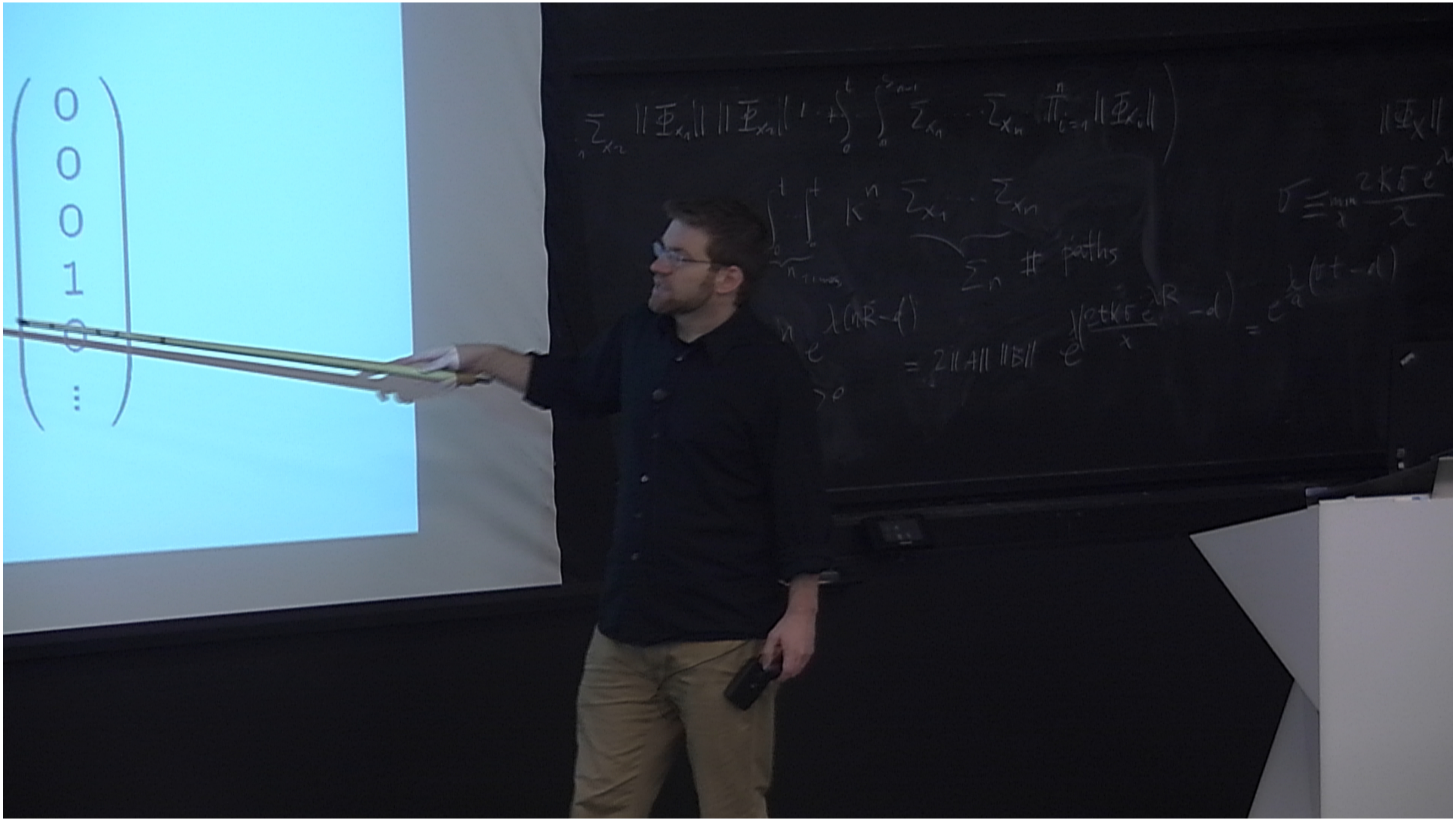
URL: http://pirsa.org/12010042

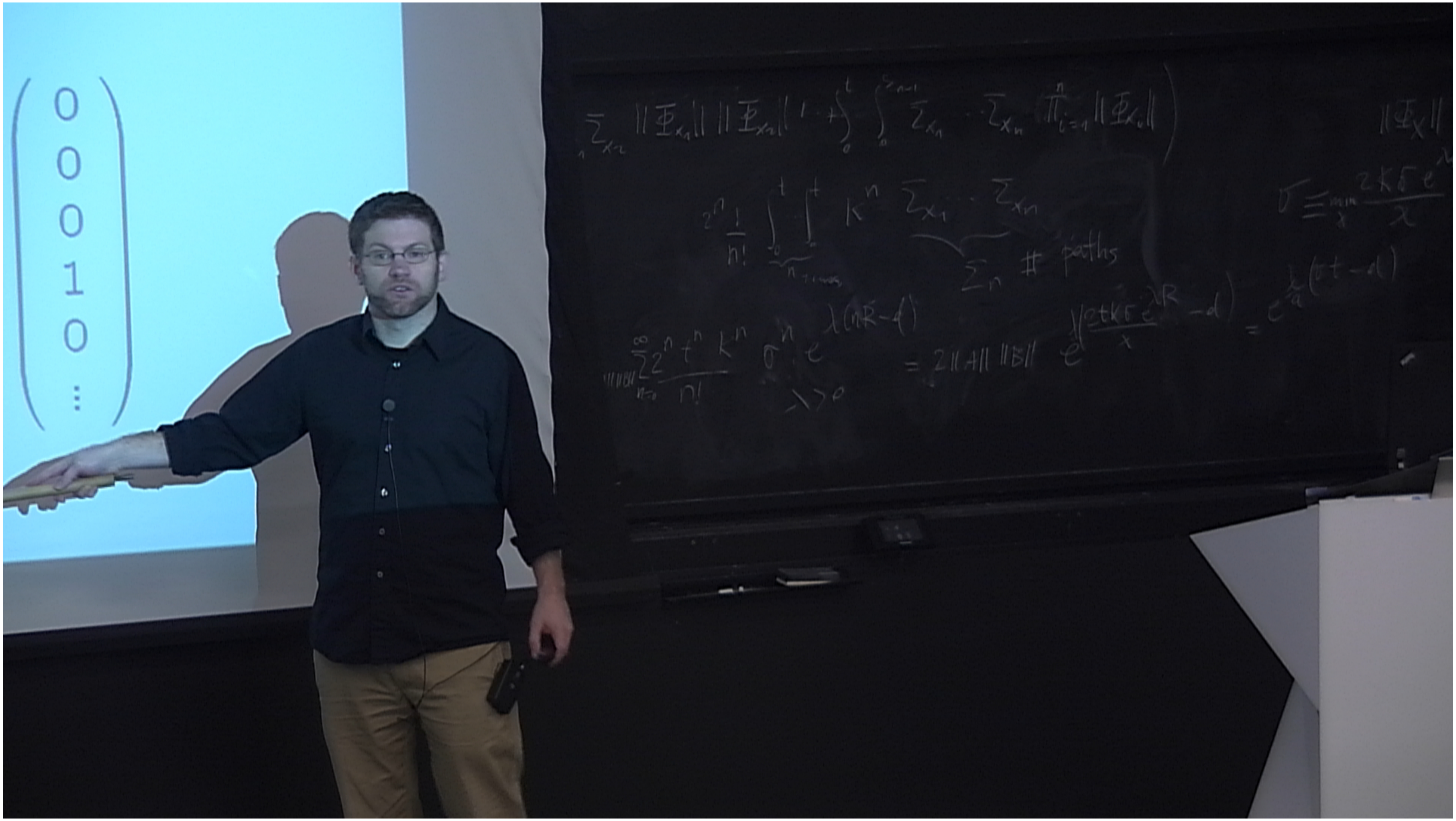
Abstract:

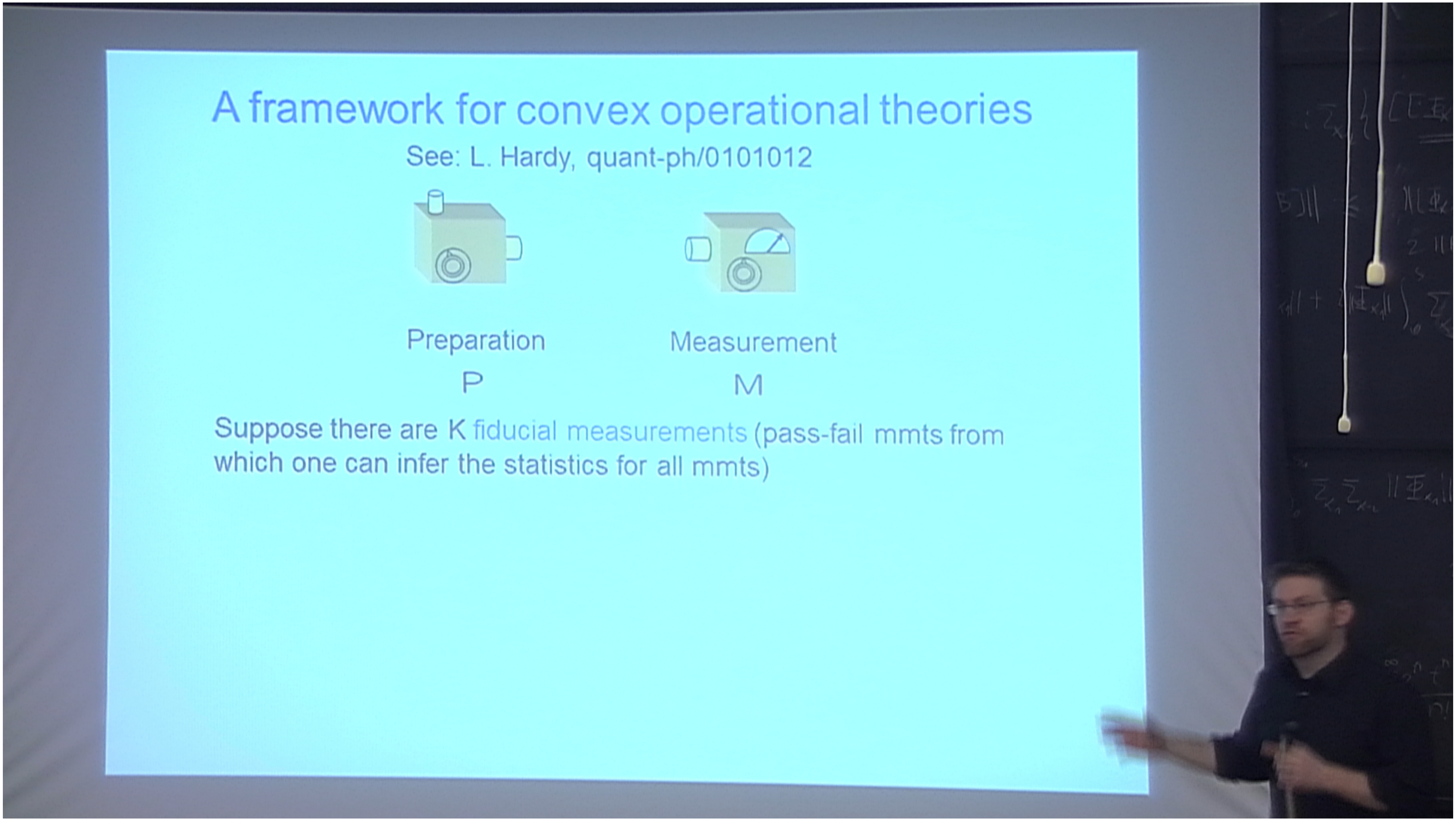

α Â + β B̂   α, β ∈ ℝ
(Â, B̂) = Tr(Â B̂)

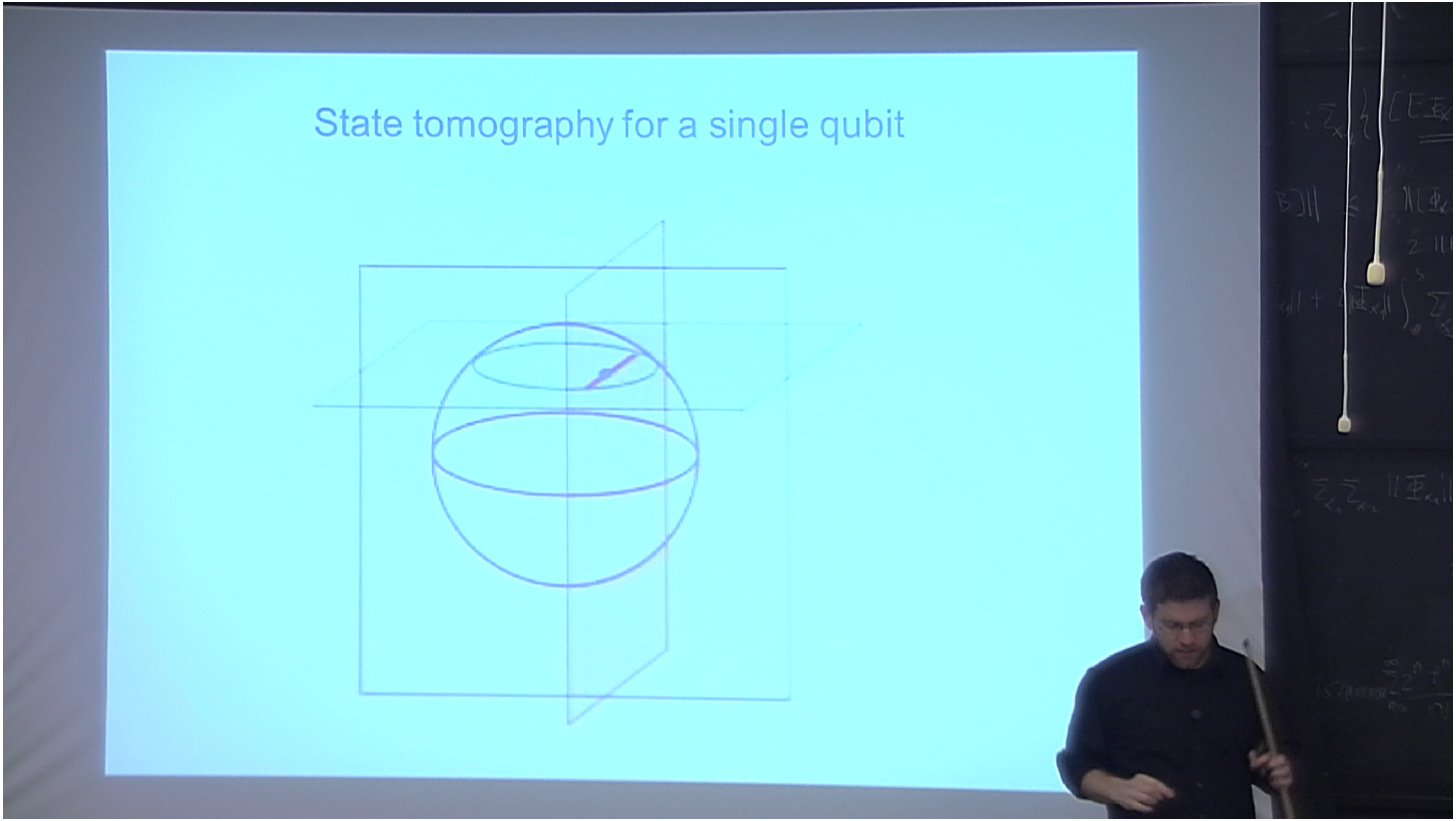

A framework for convex operational theories
See: L. Hardy, quant-ph/0101012
Preparation
P
Measurement
M
$s_P = \begin{pmatrix} \Pr(1|M_1,P) \\ \Pr(2|M_1,P) \\ \Pr(1|M_2,P) \\ \Pr(2|M_2,P) \\ \Pr(3|M_2,P) \\ \vdots \end{pmatrix}$

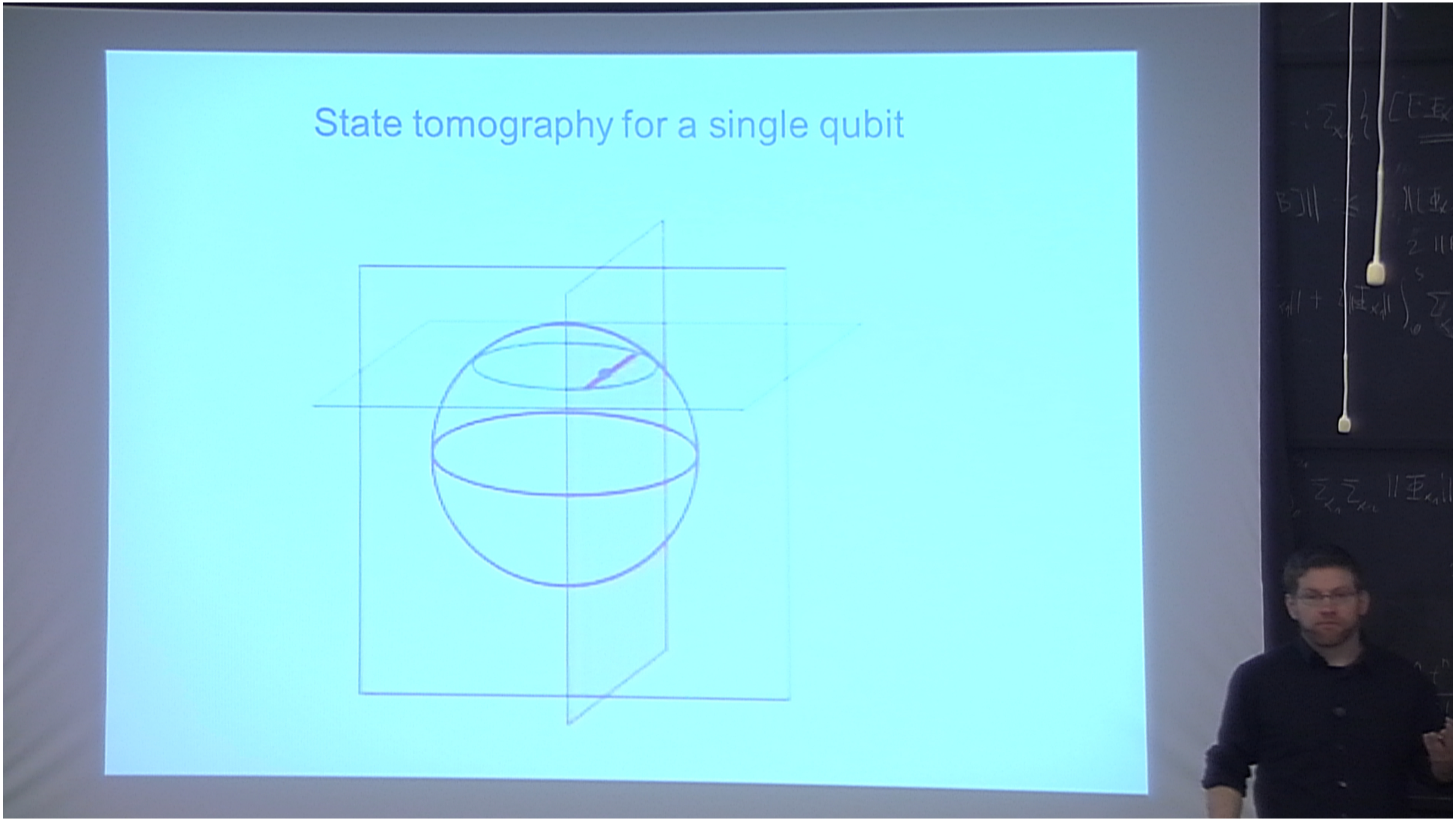

A framework for convex operational theories
See: L. Hardy, quant-ph/0101012
Preparation
P
Measurement
M
$\mathbf{s}_P = \begin{pmatrix} \Pr(1|M_1, P) \\ \Pr(2|M_1, P) \\ \Pr(1|M_2, P) \\ \Pr(2|M_2, P) \\ \Pr(3|M_2, P) \\ \vdots \end{pmatrix}$
$\mathbf{r}_{M,k} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix}$

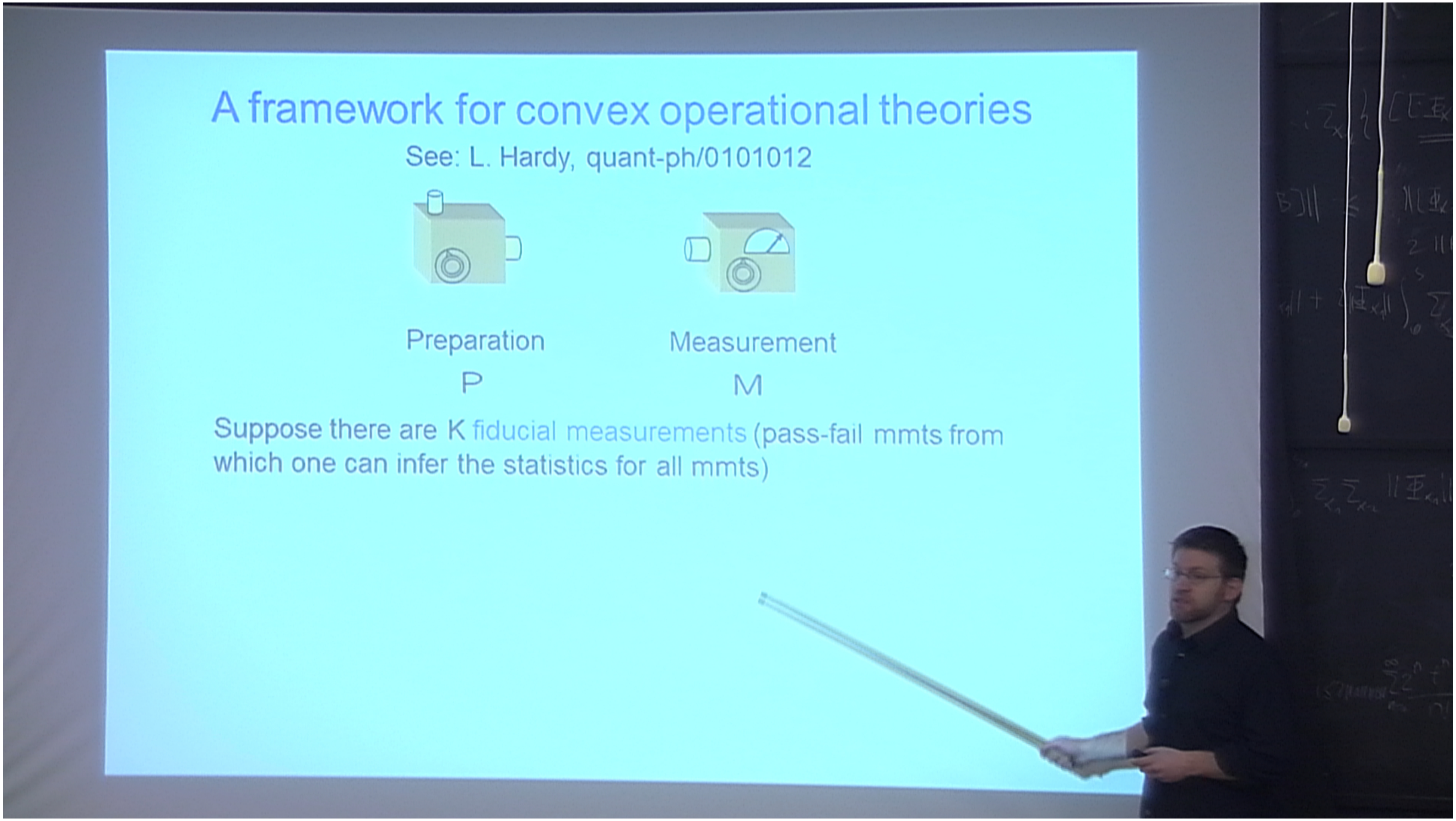

0
0
0
1
0
⋮

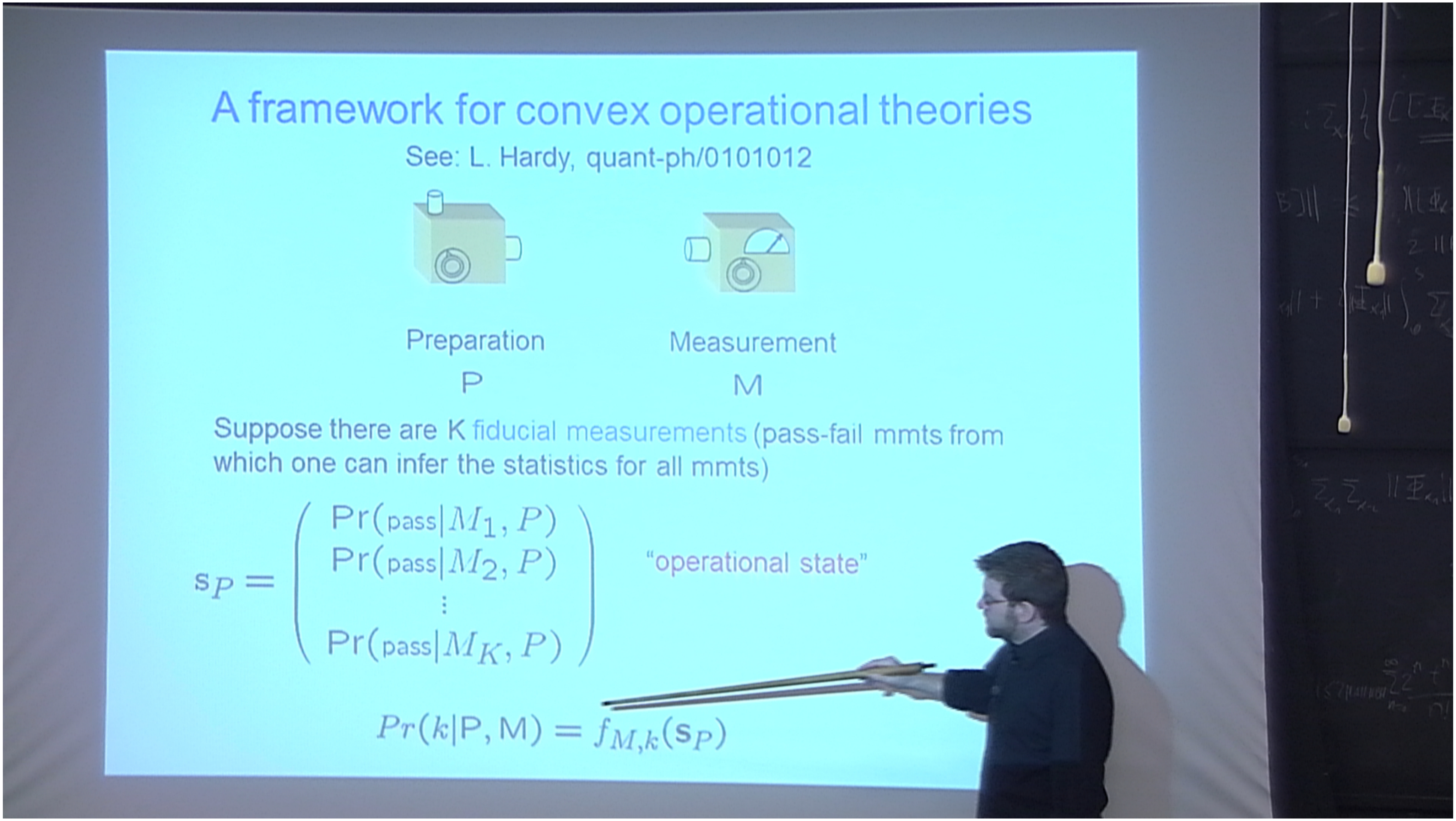


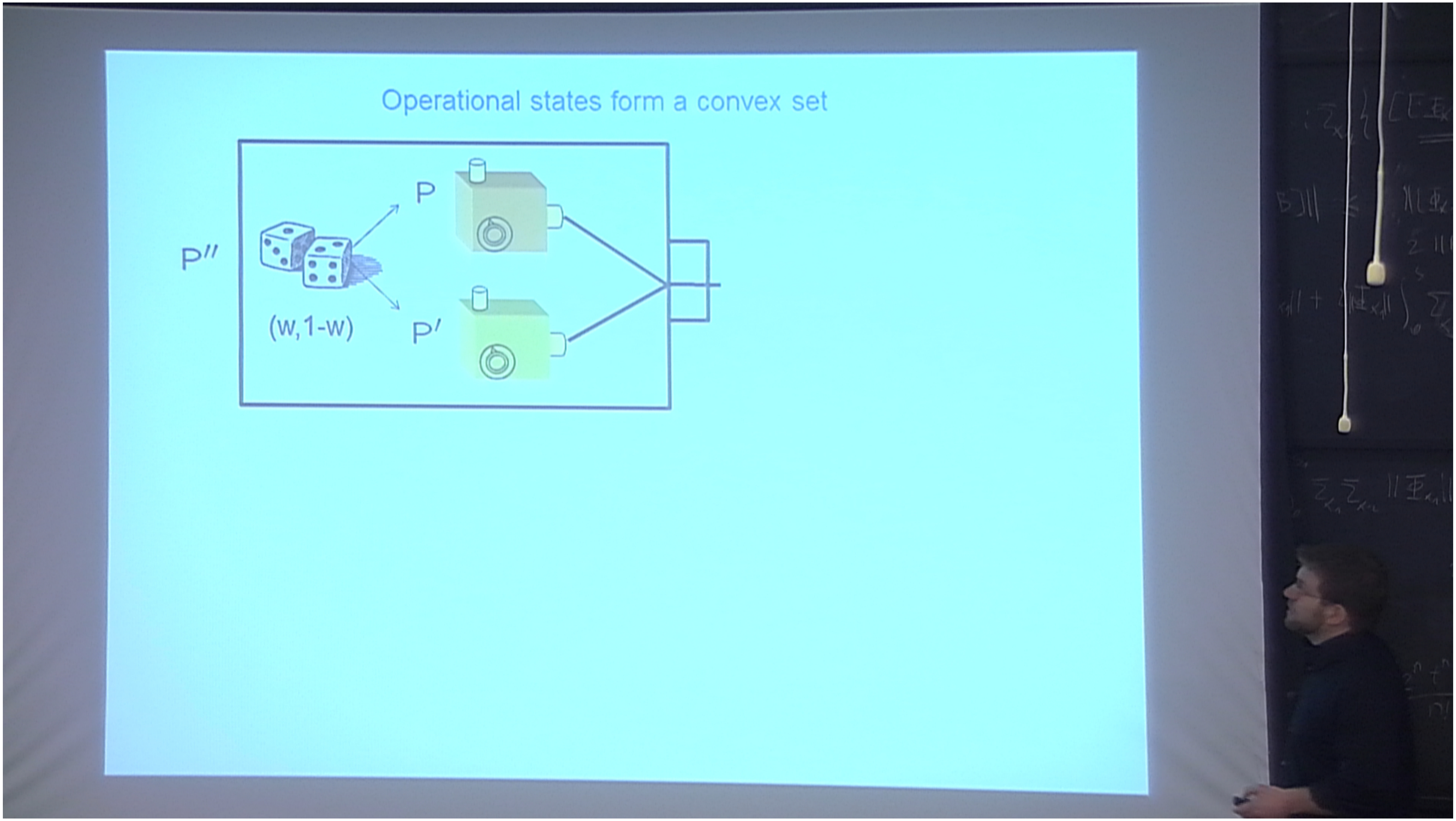

A framework for convex operational theories
See: L. Hardy, quant-ph/0101012
Preparation
P
Measurement
M
Suppose there are K fiducial measurements (pass-fail mmts from which one can infer the statistics for all mmts)

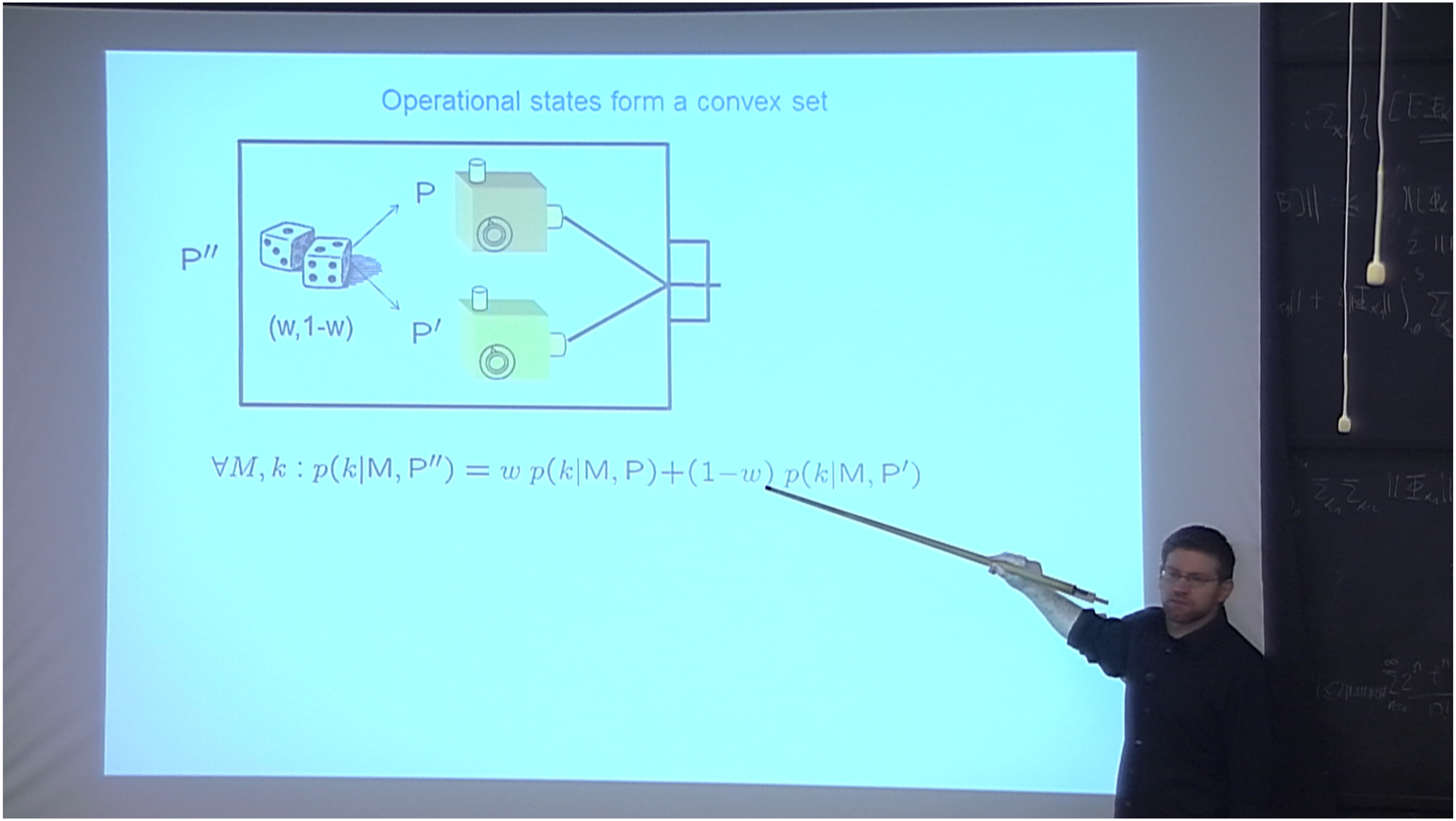

State tomography for a single qubit

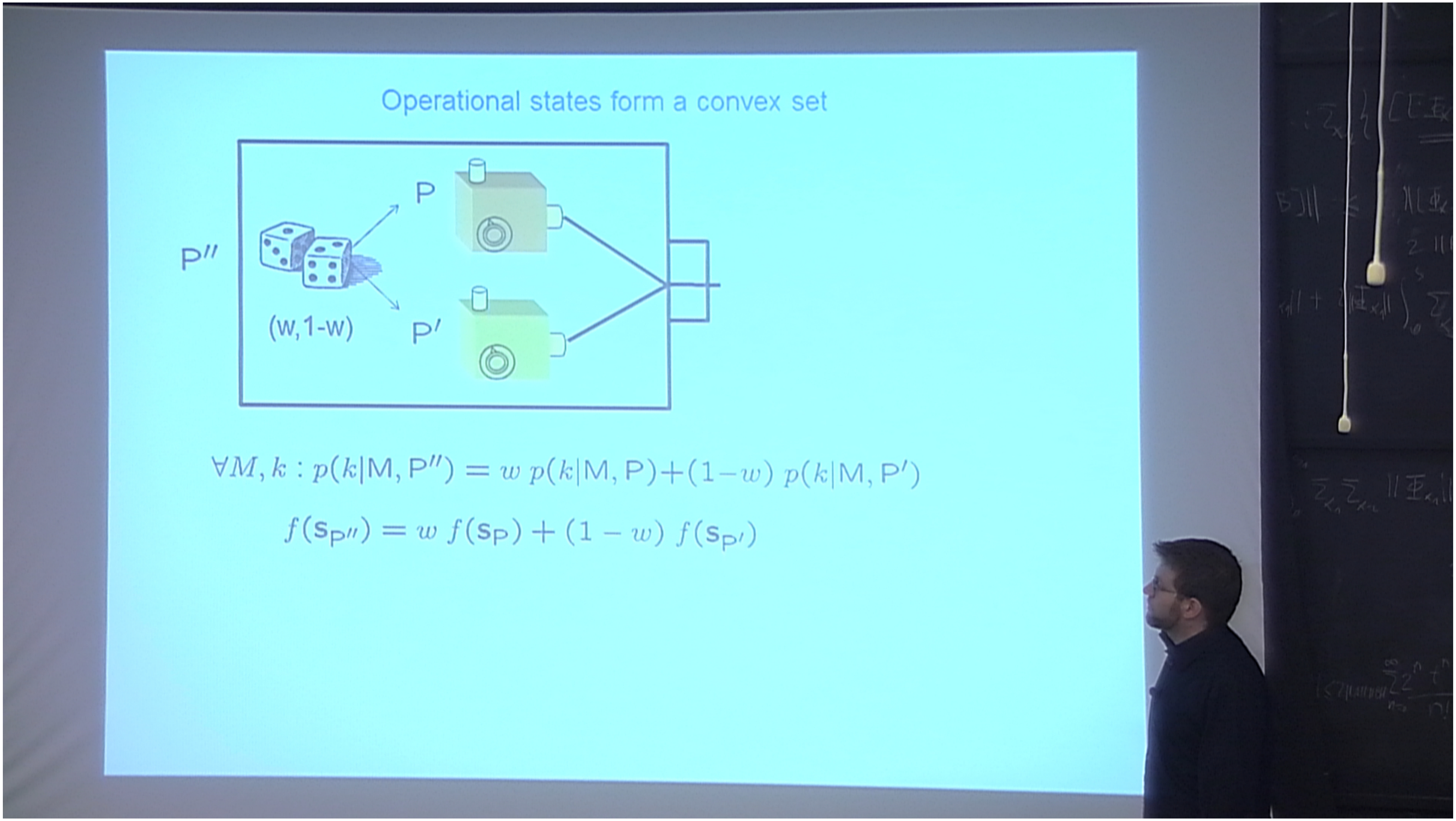

State tomography for a single qubit

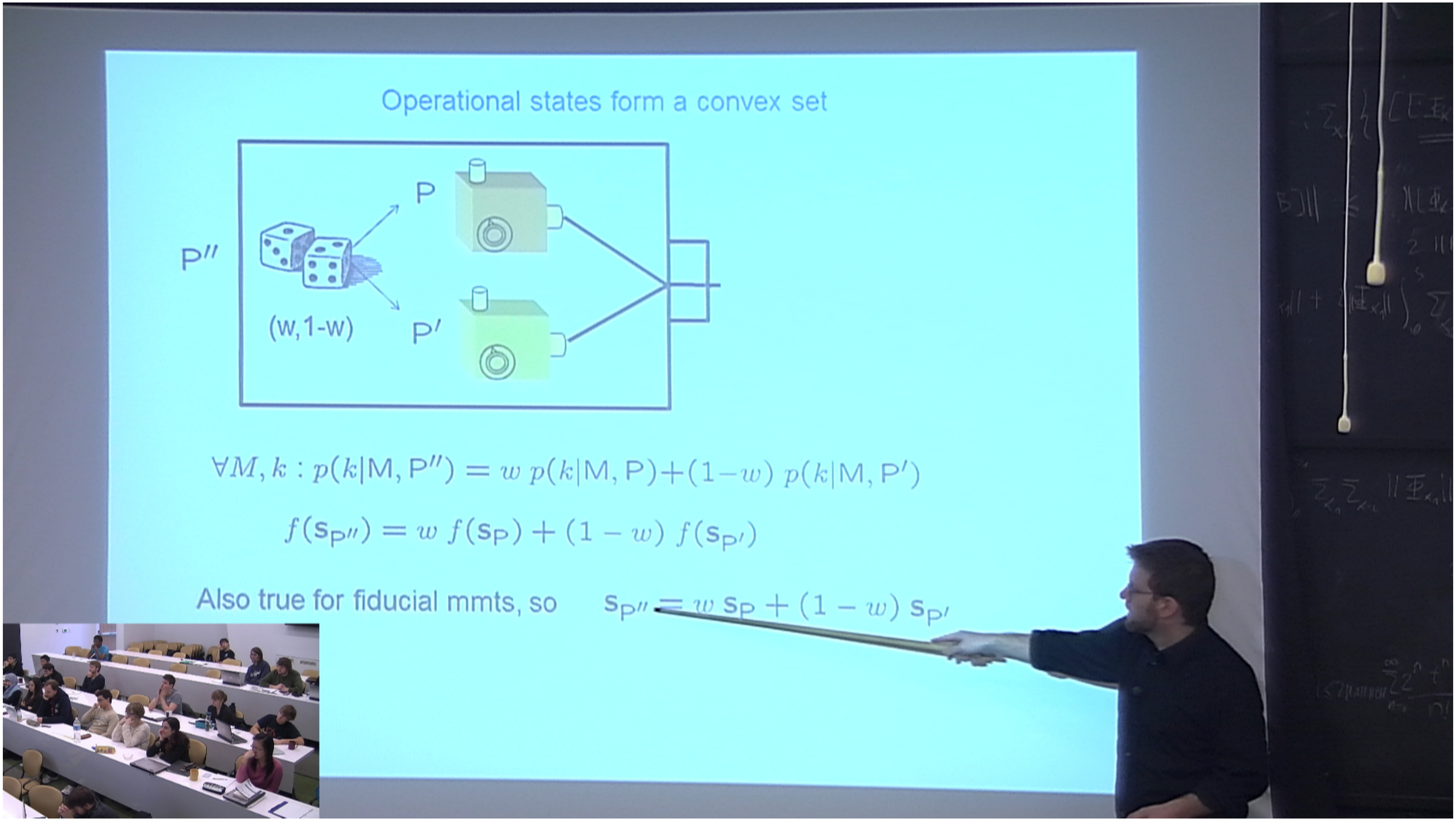

A framework for convex operational theories
See: L. Hardy, quant-ph/0101012
Preparation
P
Measurement
M
Suppose there are K fiducial measurements (pass-fail mmts from which one can infer the statistics for all mmts)

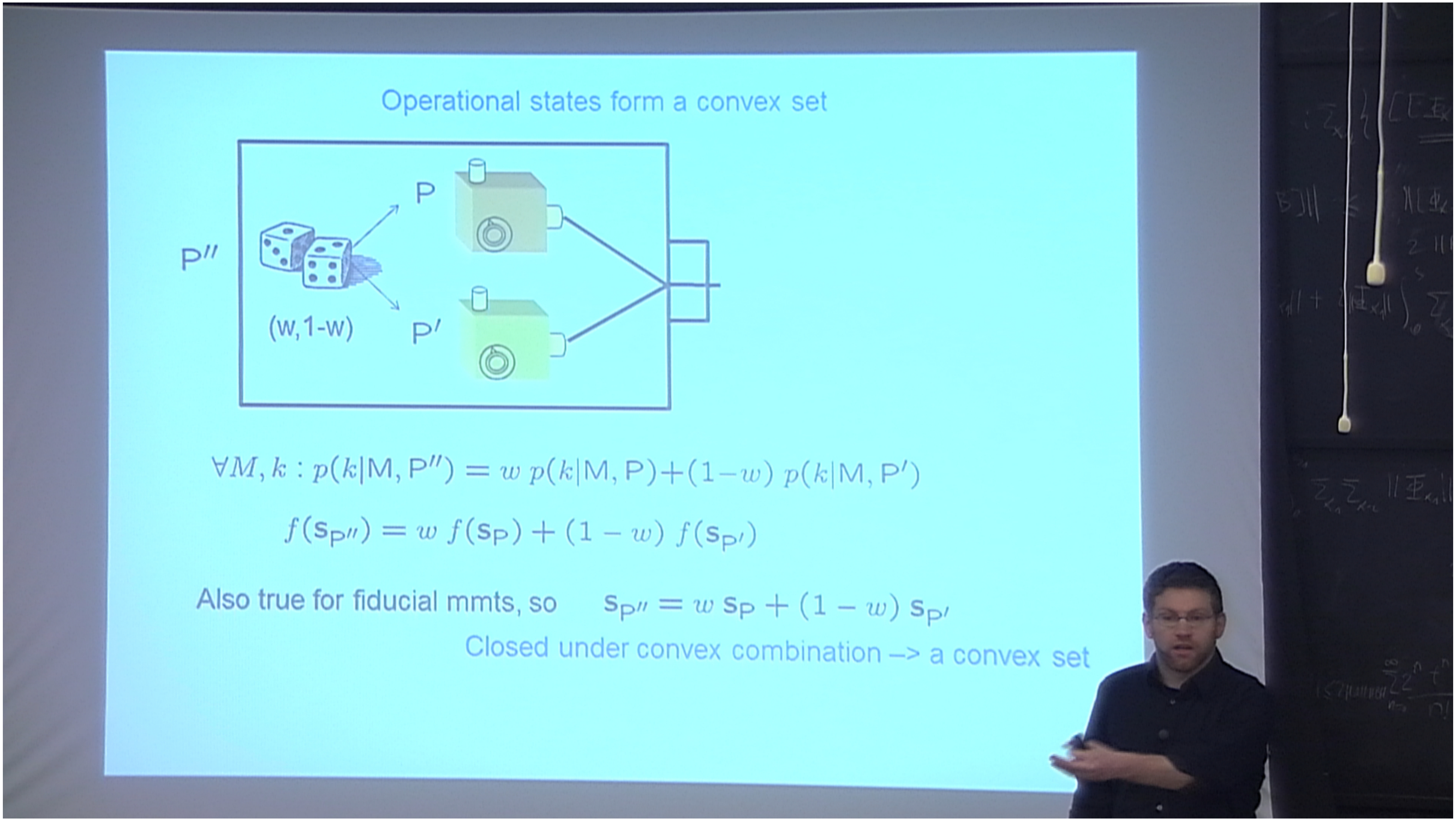

A framework for convex operational theories
See: L. Hardy, quant-ph/0101012
Preparation
P
Measurement
M
Suppose there are K fiducial measurements (pass-fail mmts from which one can infer the statistics for all mmts)
$\mathbf{s}_P = \begin{pmatrix} \mathrm{Pr}(\text{pass}|M_1, P) \\ \mathrm{Pr}(\text{pass}|M_2, P) \\ \vdots \\ \mathrm{Pr}(\text{pass}|M_K, P) \end{pmatrix}$
"operational state"
$Pr(k|\mathrm{P},\mathrm{M}) = f_{M,k}(\mathbf{s}_P)$

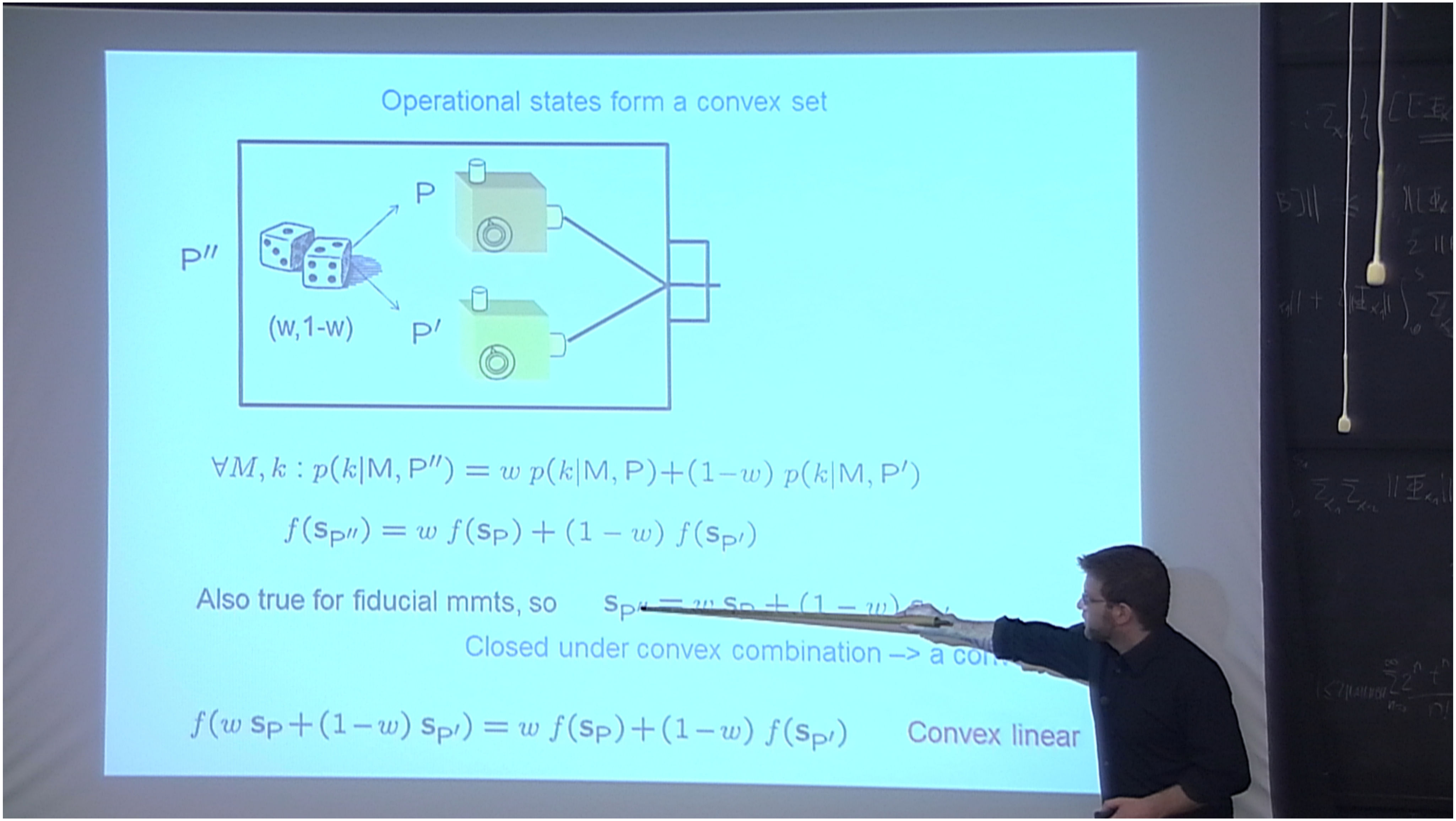

Operational states form a convex set
P''
P
P'
(w,1-w)


Operational states form a convex set
P
P″
(w,1-w)
P′
∀M, k : p(k|M, P″) = w p(k|M, P)+(1−w) p(k|M, P′)

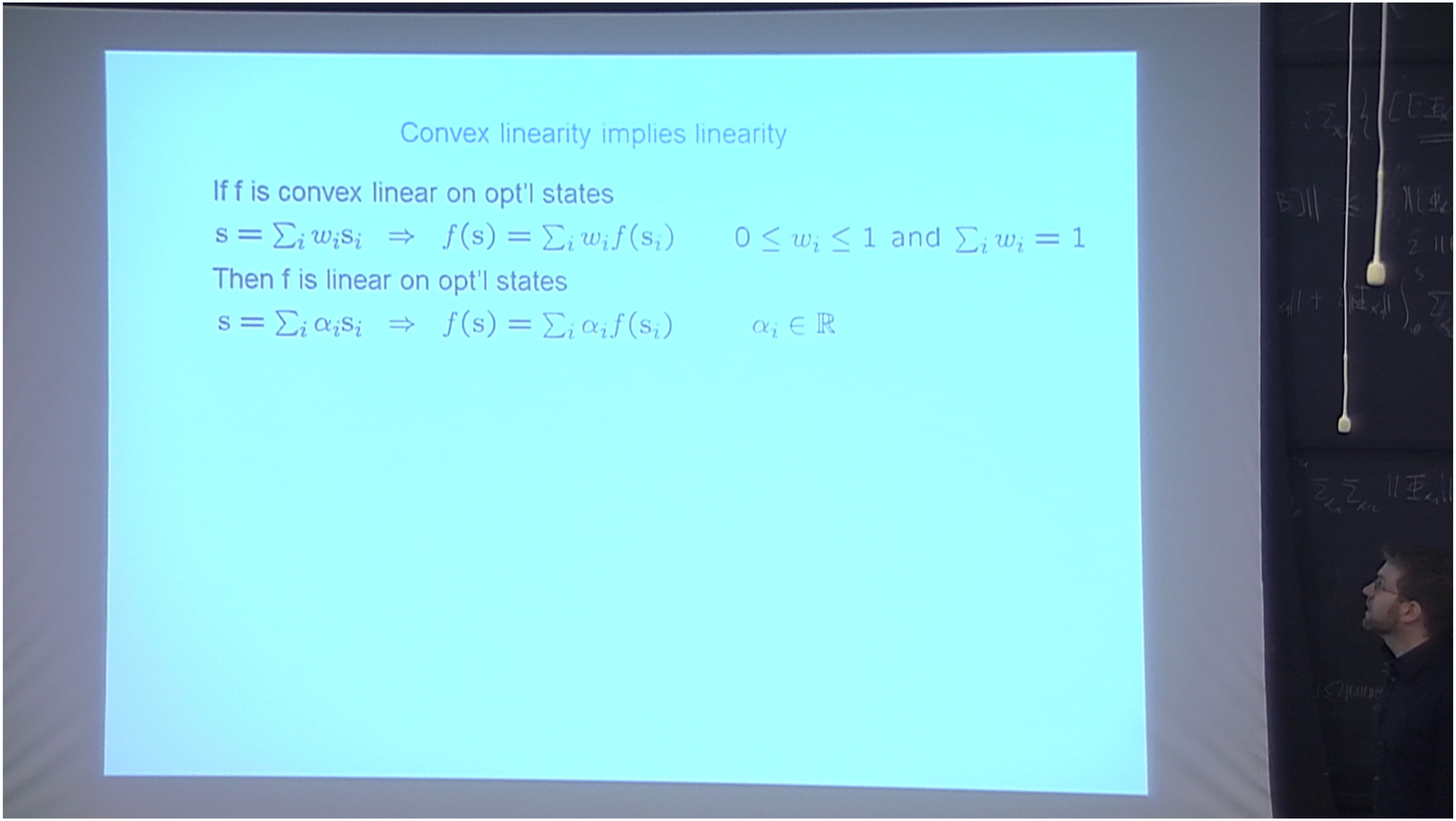

Operational states form a convex set
P''
P
P'
(w,1-w)
$\forall M, k : p(k|\mathsf{M}, \mathsf{P}'') = w\, p(k|\mathsf{M}, \mathsf{P}) + (1-w)\, p(k|\mathsf{M}, \mathsf{P}')$
$f(\mathsf{s}_{\mathsf{P}''}) = w\, f(\mathsf{s}_{\mathsf{P}}) + (1-w)\, f(\mathsf{s}_{\mathsf{P}'})$

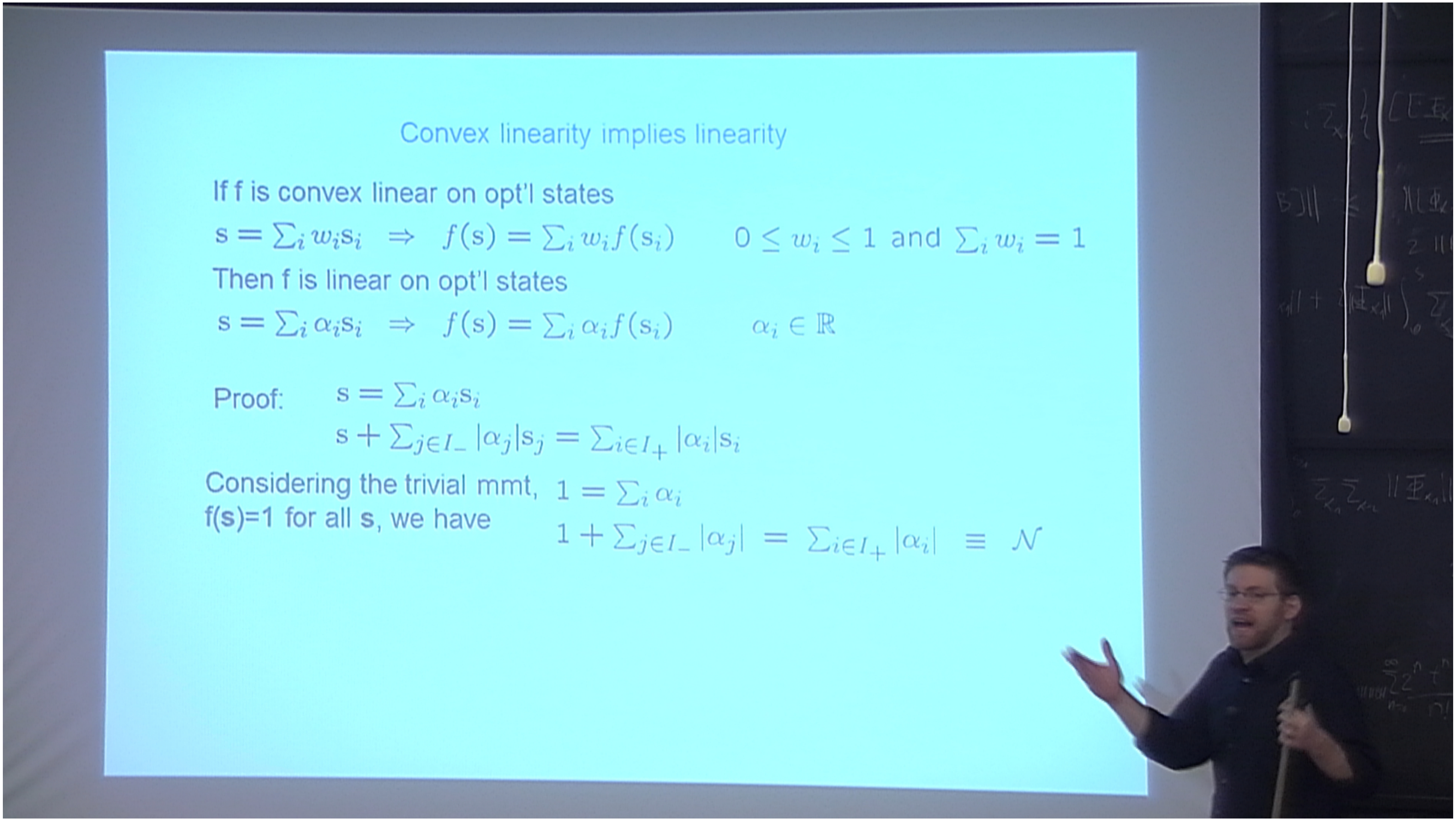

Operational states form a convex set
P
P''
(w,1-w)
P'
$\forall M, k : p(k|\mathrm{M}, \mathrm{P}'') = w\, p(k|\mathrm{M}, \mathrm{P}) + (1-w)\, p(k|\mathrm{M}, \mathrm{P}')$
$f(\mathsf{s}_{\mathrm{P}''}) = w\, f(\mathsf{s}_{\mathrm{P}}) + (1-w)\, f(\mathsf{s}_{\mathrm{P}'})$
Also true for fiducial mmts, so $\mathsf{s}_{\mathrm{P}''} = w\, \mathsf{s}_{\mathrm{P}} + (1-w)\, \mathsf{s}_{\mathrm{P}'}$

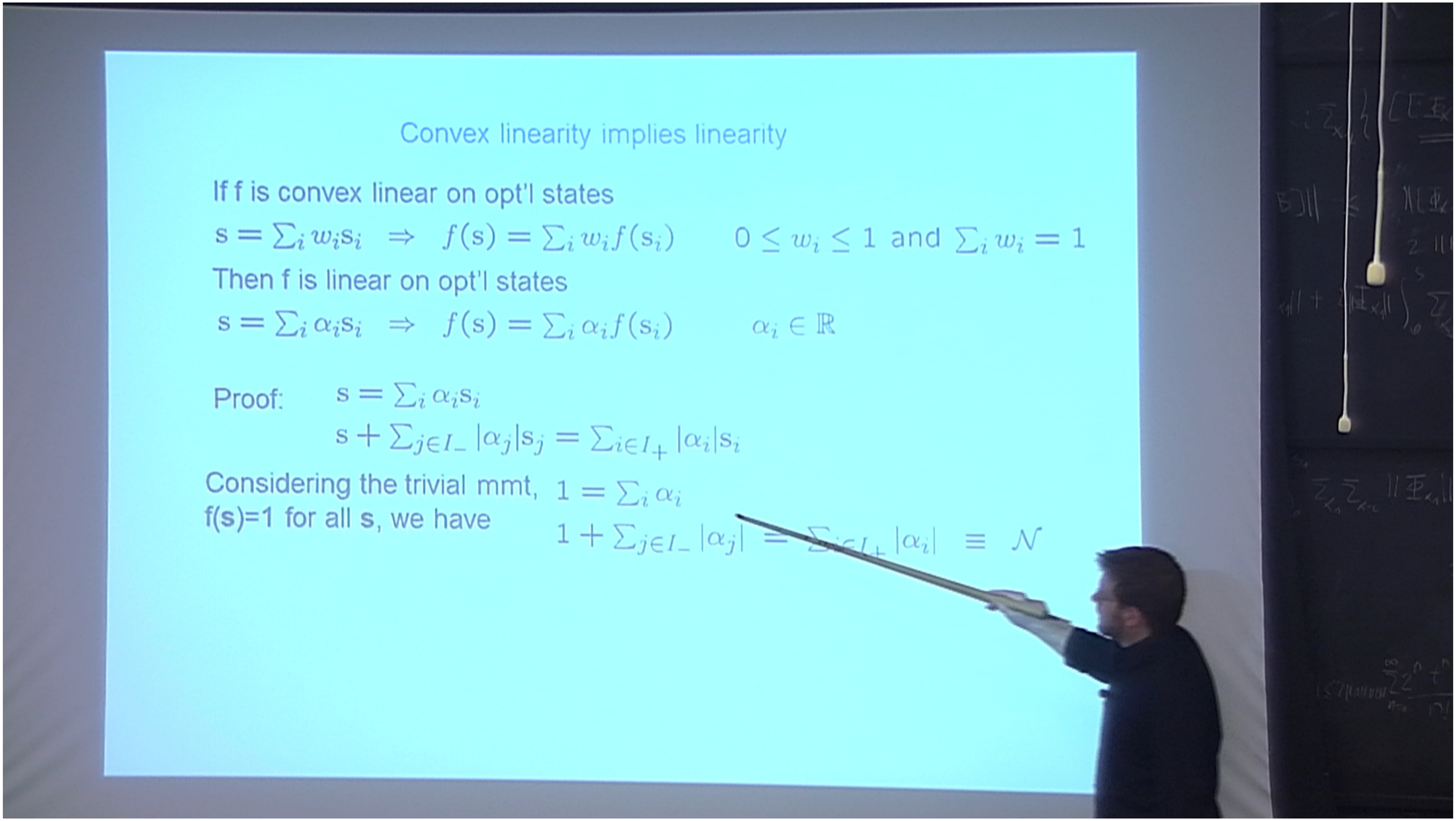

Operational states form a convex set
P
P''
(w,1-w)
P'
$\forall M, k : p(k|\mathsf{M},\mathsf{P}'') = w\, p(k|\mathsf{M},\mathsf{P}) + (1-w)\, p(k|\mathsf{M},\mathsf{P}')$
$f(\mathbf{s}_{\mathsf{P}''}) = w\, f(\mathbf{s}_{\mathsf{P}}) + (1-w)\, f(\mathbf{s}_{\mathsf{P}'})$
Also true for fiducial mmts, so $\mathbf{s}_{\mathsf{P}''} = w\, \mathbf{s}_{\mathsf{P}} + (1-w)\, \mathbf{s}_{\mathsf{P}'}$
Closed under convex combination –> a convex set

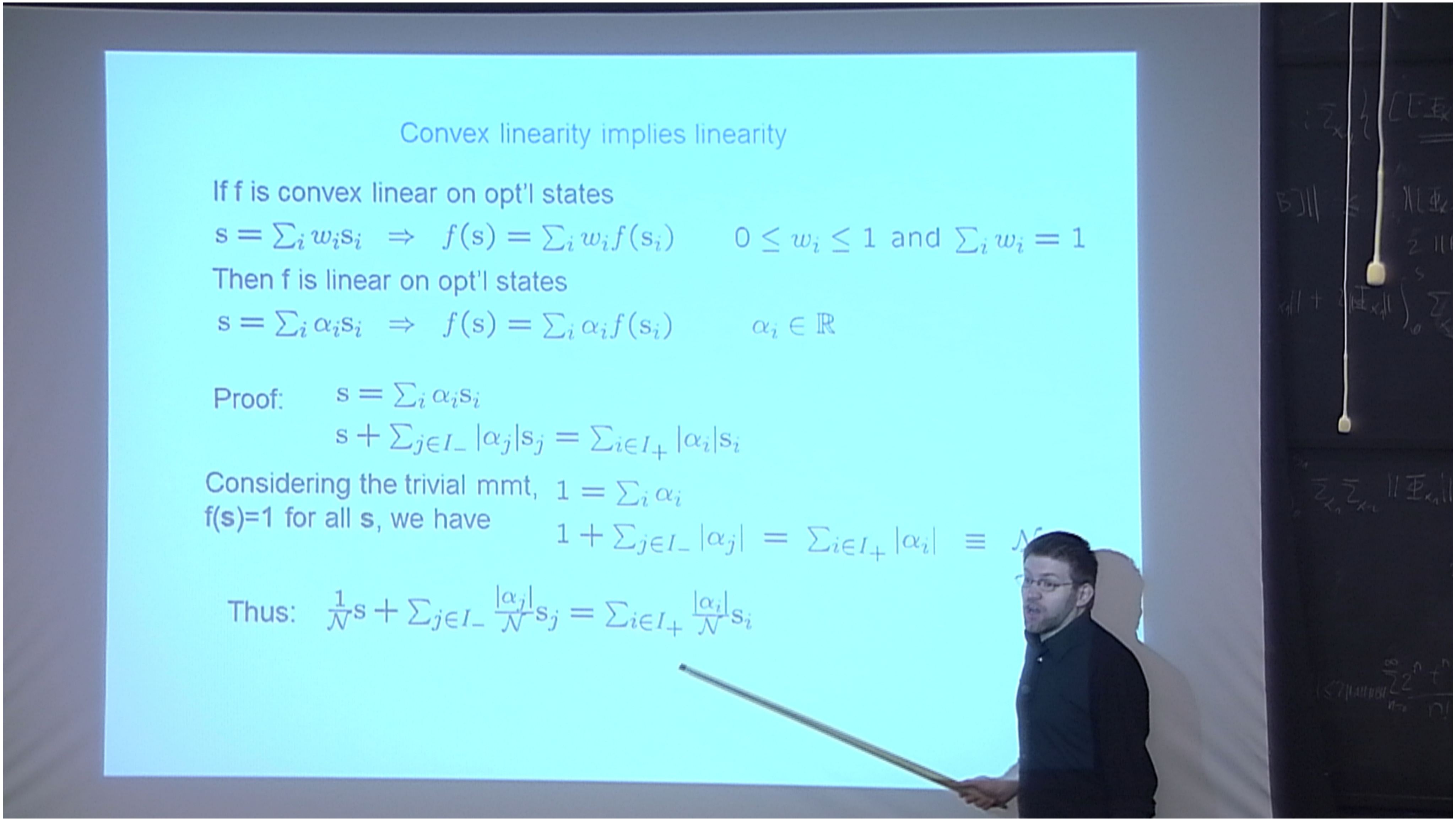


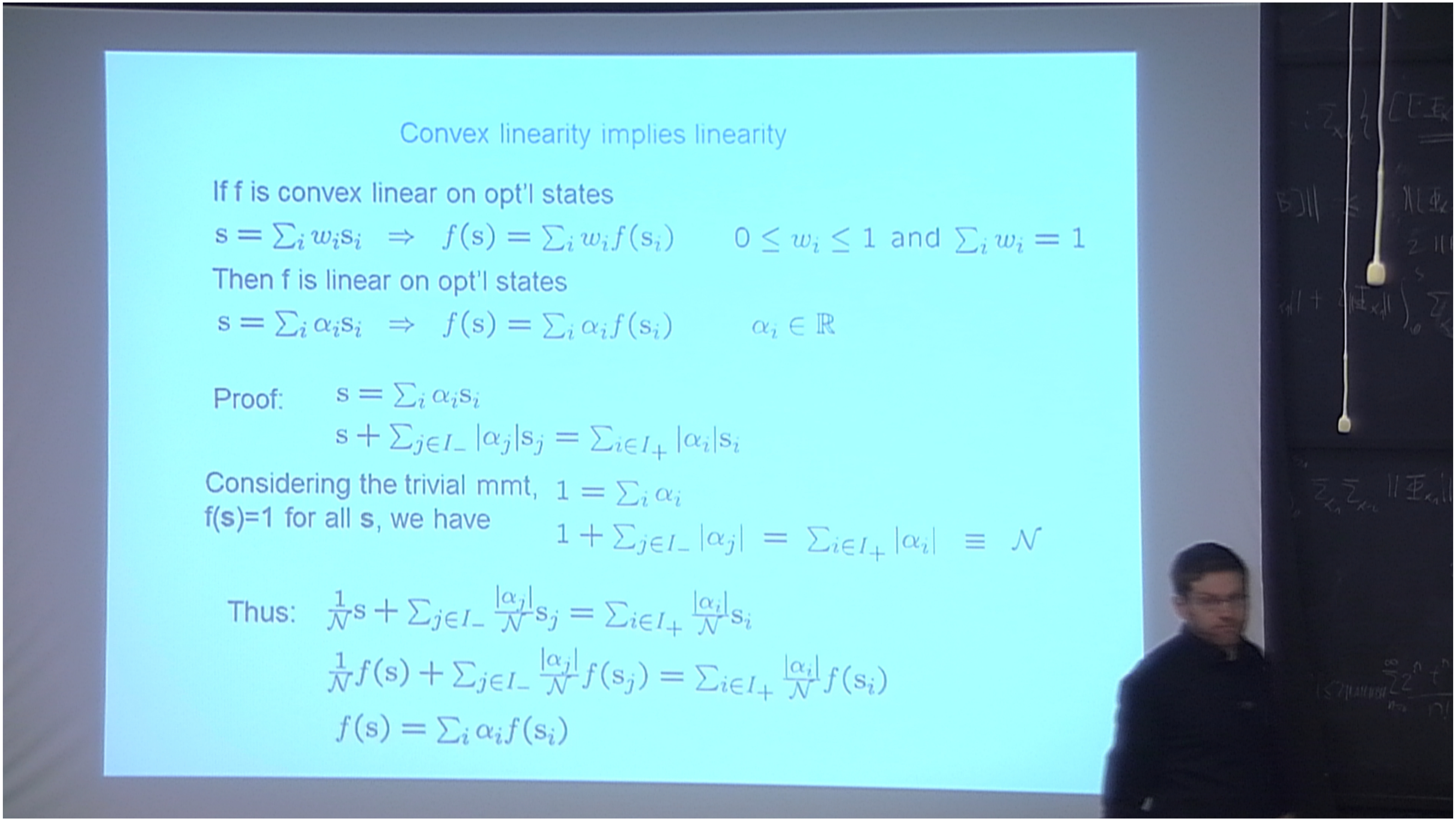
Operational states form a convex set
P''
P
P'
(w,1-w)
$\forall M, k : p(k|\mathsf{M},\mathsf{P}'') = w\, p(k|\mathsf{M},\mathsf{P}) + (1-w)\, p(k|\mathsf{M},\mathsf{P}')$
$f(\mathbf{s}_{\mathsf{P}''}) = w\, f(\mathbf{s}_{\mathsf{P}}) + (1-w)\, f(\mathbf{s}_{\mathsf{P}'})$
Also true for fiducial mmts, so
Closed under convex combination –> a
$f(w\, \mathbf{s}_{\mathsf{P}} + (1-w)\, \mathbf{s}_{\mathsf{P}'}) = w\, f(\mathbf{s}_{\mathsf{P}}) + (1-w)\, f(\mathbf{s}_{\mathsf{P}'})$
Convex linear

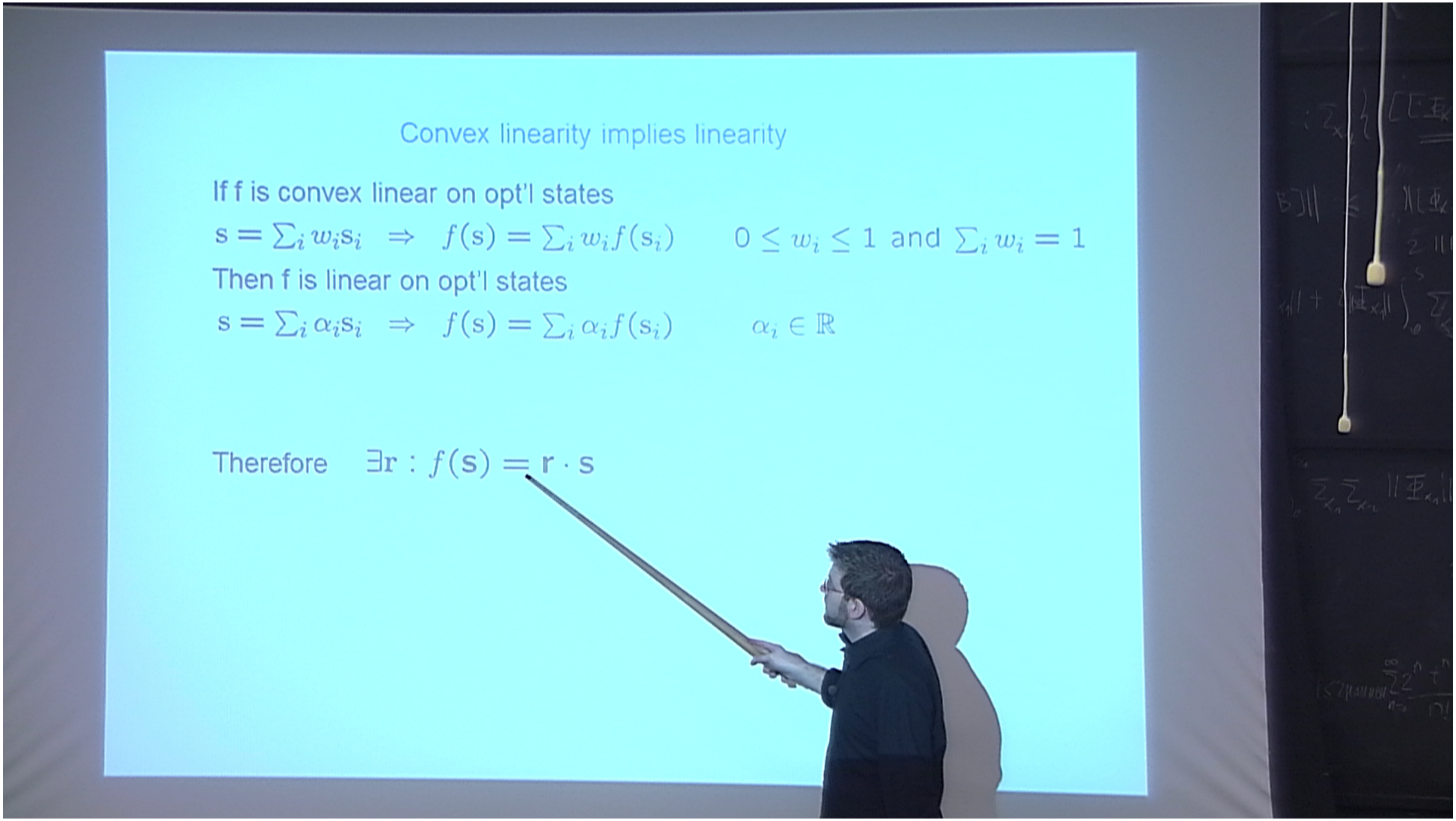

Convex linearity implies linearity
If f is convex linear on opt'l states
$\mathrm{s} = \sum_i w_i \mathrm{s}_i \;\Rightarrow\; f(\mathrm{s}) = \sum_i w_i f(\mathrm{s}_i) \qquad 0 \leq w_i \leq 1$ and $\sum_i w_i = 1$
Then f is linear on opt'l states
$\mathrm{s} = \sum_i \alpha_i \mathrm{s}_i \;\Rightarrow\; f(\mathrm{s}) = \sum_i \alpha_i f(\mathrm{s}_i) \qquad \alpha_i \in \mathbb{R}$


Convex linearity implies linearity
If f is convex linear on opt'l states
$\mathrm{s} = \sum_i w_i \mathrm{s}_i \;\Rightarrow\; f(\mathrm{s}) = \sum_i w_i f(\mathrm{s}_i)$ $0 \leq w_i \leq 1$ and $\sum_i w_i = 1$
Then f is linear on opt'l states
$\mathrm{s} = \sum_i \alpha_i \mathrm{s}_i \;\Rightarrow\; f(\mathrm{s}) = \sum_i \alpha_i f(\mathrm{s}_i)$ $\alpha_i \in \mathbb{R}$
Proof: $\mathrm{s} = \sum_i \alpha_i \mathrm{s}_i$
$\mathrm{s} + \sum_{j \in I_-} |\alpha_j| \mathrm{s}_j = \sum_{i \in I_+} |\alpha_i| \mathrm{s}_i$
Considering the trivial mmt, f(**s**)=1 for all **s**, we have
$1 = \sum_i \alpha_i$
$1 + \sum_{j \in I_-} |\alpha_j| = \sum_{i \in I_+} |\alpha_i| \equiv \mathcal{N}$


Convex linearity implies linearity
If f is convex linear on opt'l states
$\mathrm{s} = \sum_i w_i \mathrm{s}_i \;\Rightarrow\; f(\mathrm{s}) = \sum_i w_i f(\mathrm{s}_i)$ $\quad 0 \leq w_i \leq 1$ and $\sum_i w_i = 1$
Then f is linear on opt'l states
$\mathrm{s} = \sum_i \alpha_i \mathrm{s}_i \;\Rightarrow\; f(\mathrm{s}) = \sum_i \alpha_i f(\mathrm{s}_i)$ $\quad \alpha_i \in \mathbb{R}$
Proof: $\mathrm{s} = \sum_i \alpha_i \mathrm{s}_i$
$\mathrm{s} + \sum_{j \in I_-} |\alpha_j| \mathrm{s}_j = \sum_{i \in I_+} |\alpha_i| \mathrm{s}_i$
Considering the trivial mmt, f(**s**)=1 for all **s**, we have
$1 = \sum_i \alpha_i$
$1 + \sum_{j \in I_-} |\alpha_j| = \sum_{i \in I_+} |\alpha_i| \equiv \mathcal{N}$


Convex linearity implies linearity
If f is convex linear on opt'l states
$\mathrm{s} = \sum_i w_i \mathrm{s}_i \;\Rightarrow\; f(\mathrm{s}) = \sum_i w_i f(\mathrm{s}_i)$ $0 \leq w_i \leq 1$ and $\sum_i w_i = 1$
Then f is linear on opt'l states
$\mathrm{s} = \sum_i \alpha_i \mathrm{s}_i \;\Rightarrow\; f(\mathrm{s}) = \sum_i \alpha_i f(\mathrm{s}_i)$ $\alpha_i \in \mathbb{R}$
Proof: $\mathrm{s} = \sum_i \alpha_i \mathrm{s}_i$
$\mathrm{s} + \sum_{j \in I_-} |\alpha_j| \mathrm{s}_j = \sum_{i \in I_+} |\alpha_i| \mathrm{s}_i$
Considering the trivial mmt, f(**s**)=1 for all **s**, we have
$1 = \sum_i \alpha_i$
$1 + \sum_{j \in I_-} |\alpha_j| = \sum_{i \in I_+} |\alpha_i| \equiv$
Thus: $\frac{1}{\mathcal{N}}\mathrm{s} + \sum_{j \in I_-} \frac{|\alpha_j|}{\mathcal{N}} \mathrm{s}_j = \sum_{i \in I_+} \frac{|\alpha_i|}{\mathcal{N}} \mathrm{s}_i$


Convex linearity implies linearity
If f is convex linear on opt'l states
$\mathrm{s} = \sum_i w_i \mathrm{s}_i \;\Rightarrow\; f(\mathrm{s}) = \sum_i w_i f(\mathrm{s}_i)$ $\quad 0 \le w_i \le 1$ and $\sum_i w_i = 1$
Then f is linear on opt'l states
$\mathrm{s} = \sum_i \alpha_i \mathrm{s}_i \;\Rightarrow\; f(\mathrm{s}) = \sum_i \alpha_i f(\mathrm{s}_i)$ $\quad \alpha_i \in \mathbb{R}$
Proof: $\mathrm{s} = \sum_i \alpha_i \mathrm{s}_i$
$\mathrm{s} + \sum_{j \in I_-} |\alpha_j| \mathrm{s}_j = \sum_{i \in I_+} |\alpha_i| \mathrm{s}_i$
Considering the trivial mmt, f(s)=1 for all s, we have
$1 = \sum_i \alpha_i$
$1 + \sum_{j \in I_-} |\alpha_j| = \sum_{i \in I_+} |\alpha_i| \equiv \mathcal{N}$
Thus: $\frac{1}{\mathcal{N}}\mathrm{s} + \sum_{j \in I_-} \frac{|\alpha_j|}{\mathcal{N}} \mathrm{s}_j = \sum_{i \in I_+} \frac{|\alpha_i|}{\mathcal{N}} \mathrm{s}_i$
$\frac{1}{\mathcal{N}} f(\mathrm{s}) + \sum_{j \in I_-} \frac{|\alpha_j|}{\mathcal{N}} f(\mathrm{s}_j) = \sum_{i \in I_+} \frac{|\alpha_i|}{\mathcal{N}} f(\mathrm{s}_i)$
$f(\mathrm{s}) = \sum_i \alpha_i f(\mathrm{s}_i)$


Convex linearity implies linearity
If f is convex linear on opt'l states
$\mathrm{s} = \sum_i w_i \mathrm{s}_i \;\Rightarrow\; f(\mathrm{s}) = \sum_i w_i f(\mathrm{s}_i) \qquad 0 \leq w_i \leq 1$ and $\sum_i w_i = 1$
Then f is linear on opt'l states
$\mathrm{s} = \sum_i \alpha_i \mathrm{s}_i \;\Rightarrow\; f(\mathrm{s}) = \sum_i \alpha_i f(\mathrm{s}_i) \qquad \alpha_i \in \mathbb{R}$
Therefore $\exists \mathbf{r} : f(\mathrm{s}) = \mathbf{r} \cdot \mathrm{s}$

Convex linearity implies linearity

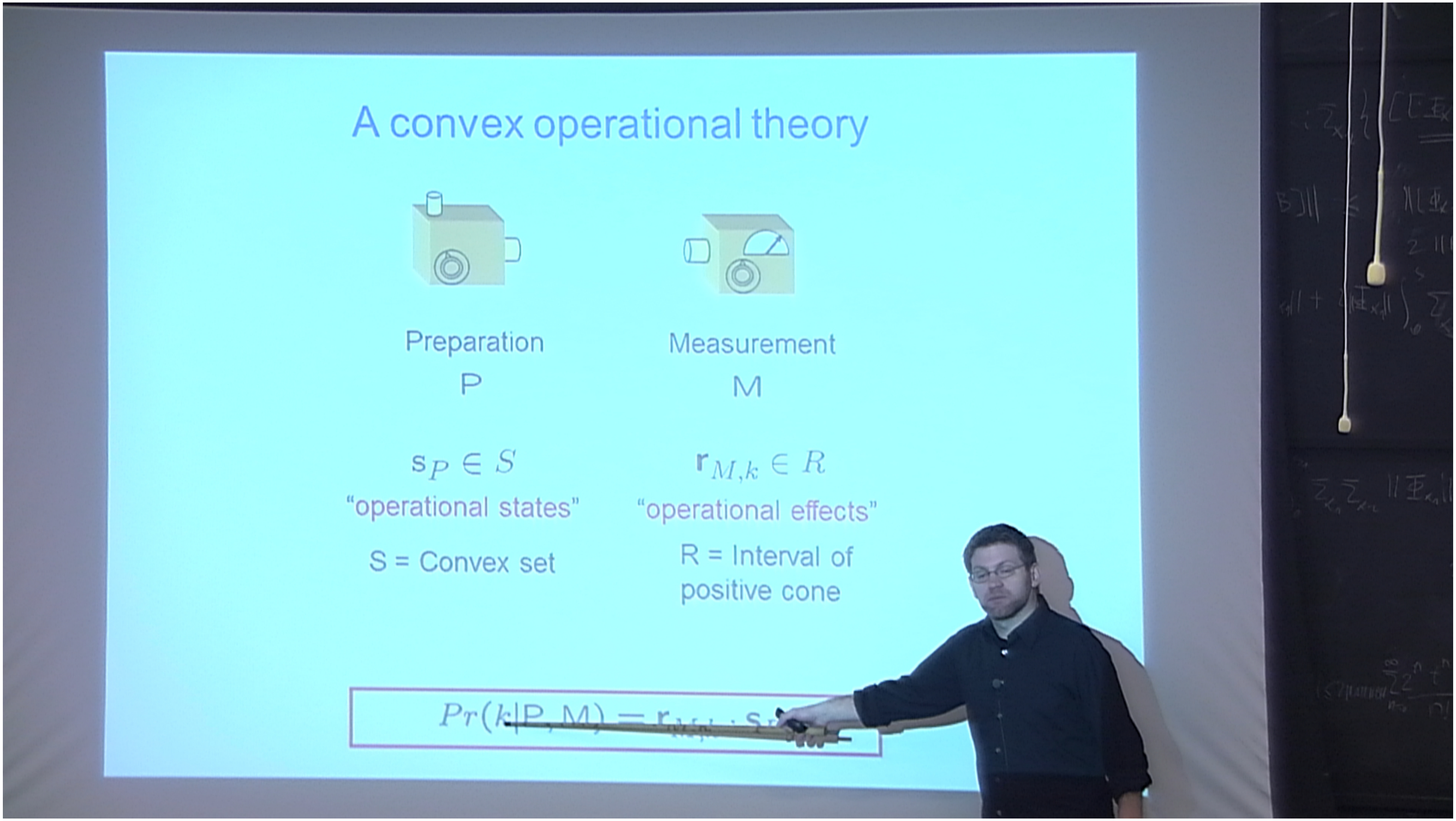
If f is convex linear on opt'l states

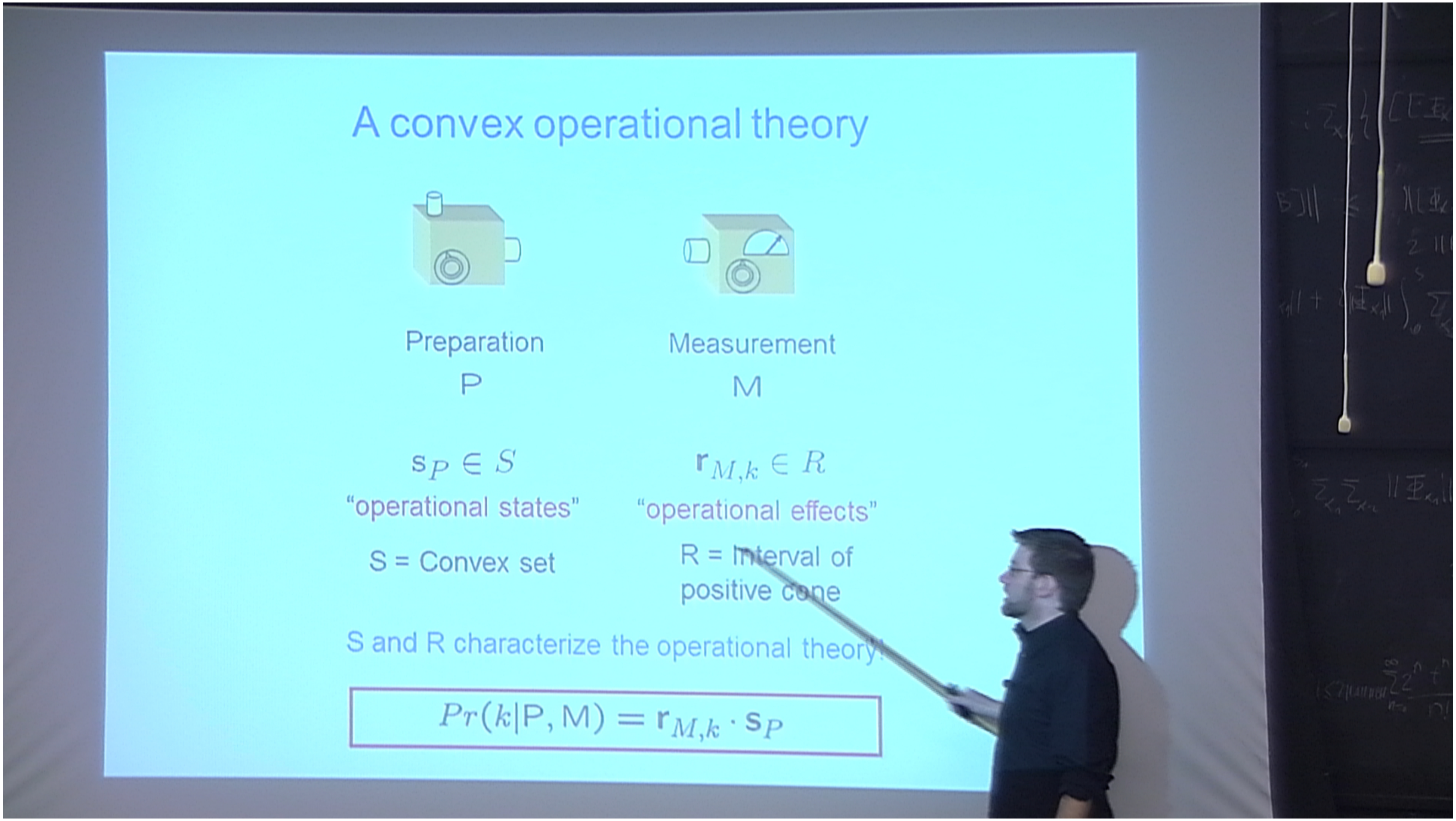
$\mathrm{s} = \sum_i w_i \mathrm{s}_i \;\; \Rightarrow \;\; f(\mathrm{s}) = \sum_i w_i f(\mathrm{s}_i) \qquad 0 \leq w_i \leq 1$ and $\sum_i w_i = 1$

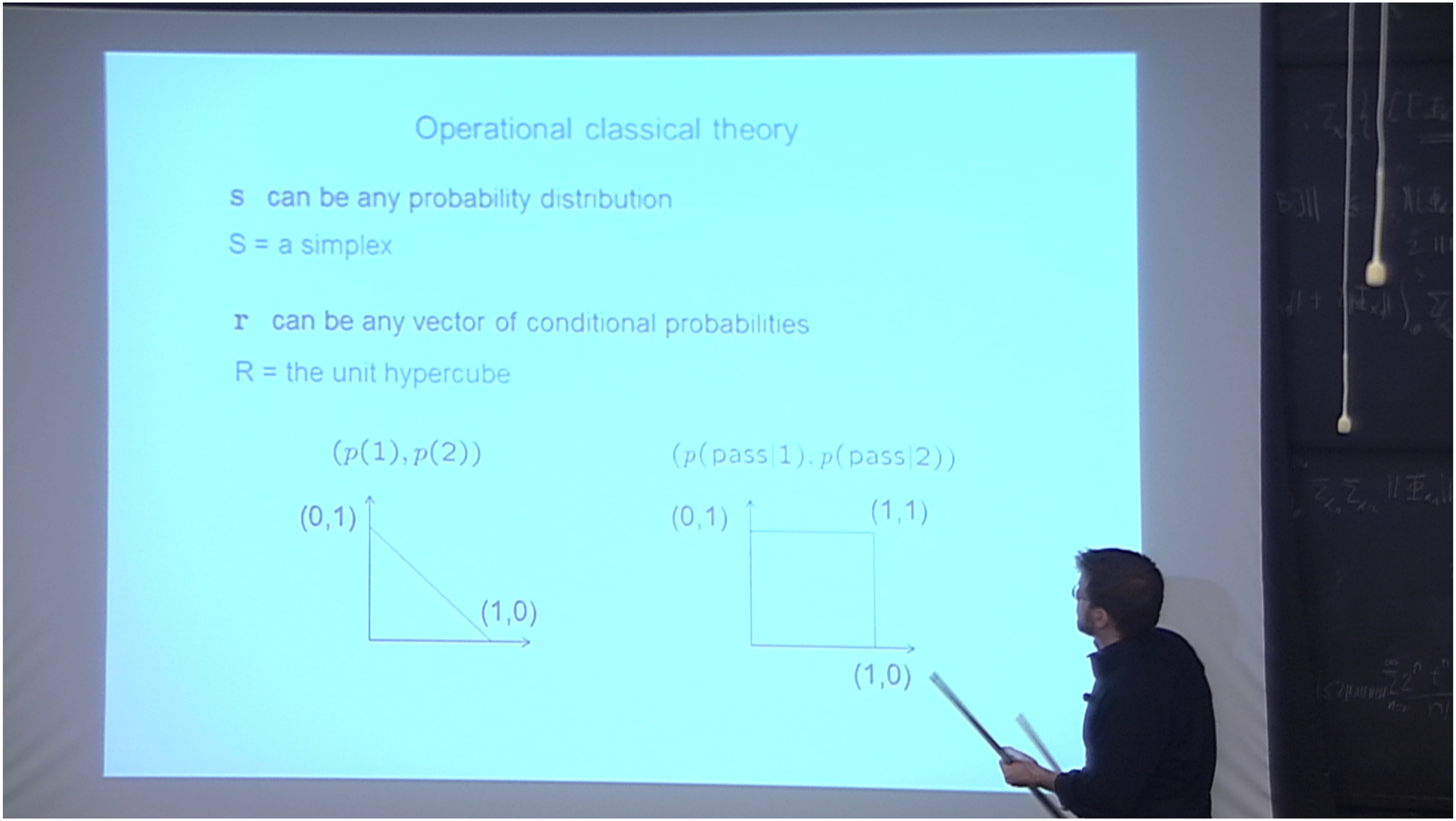
Then f is linear on opt'l states

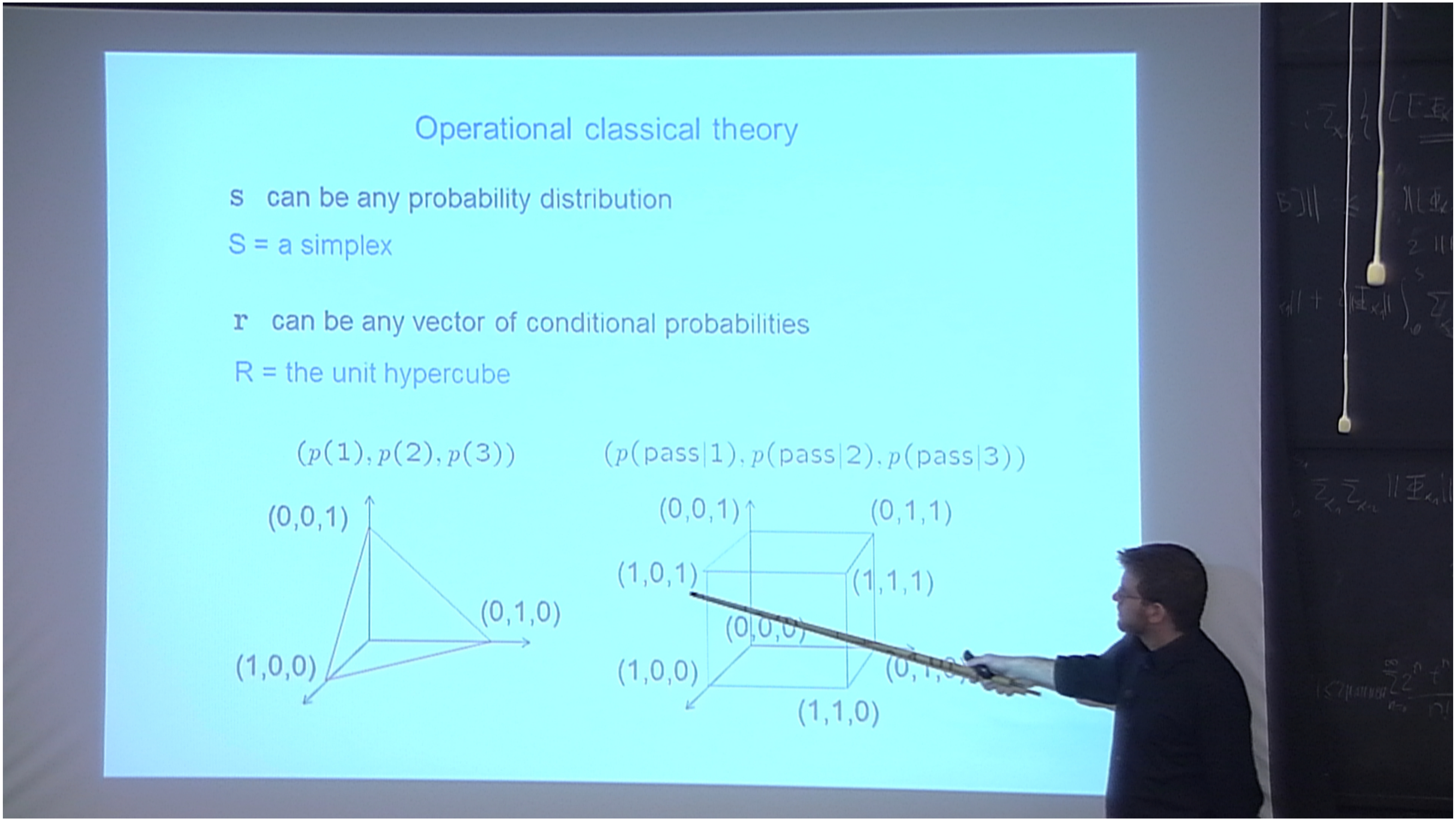
$\mathrm{s} = \sum_i \alpha_i \mathrm{s}_i \;\; \Rightarrow \;\; f(\mathrm{s}) = \sum_i \alpha_i f(\mathrm{s}_i) \qquad \alpha_i \in \mathbb{R}$

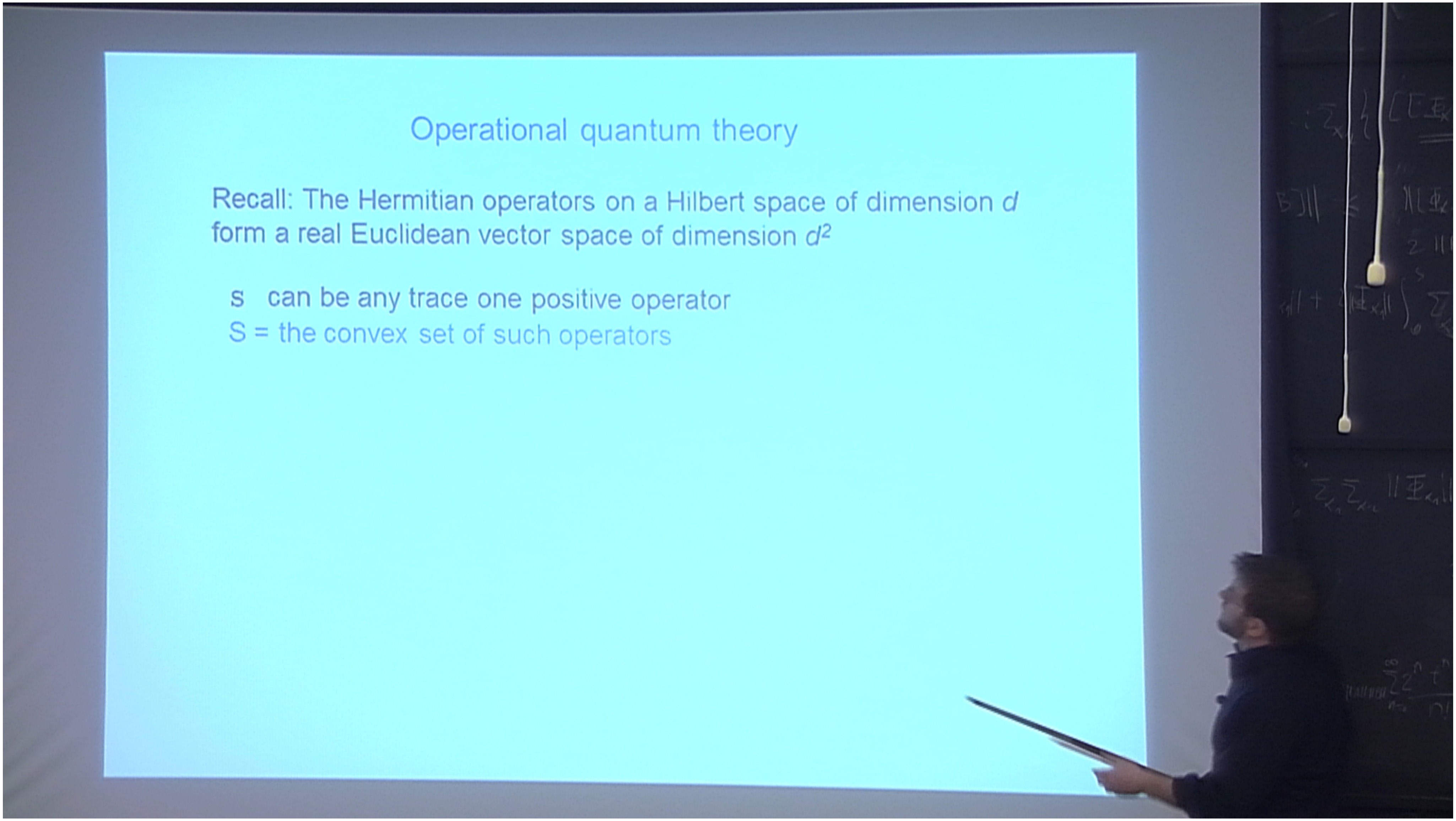
Therefore $\quad \exists \mathbf{r} : f(\mathbf{s}) = \mathbf{r} \cdot \mathbf{s}$

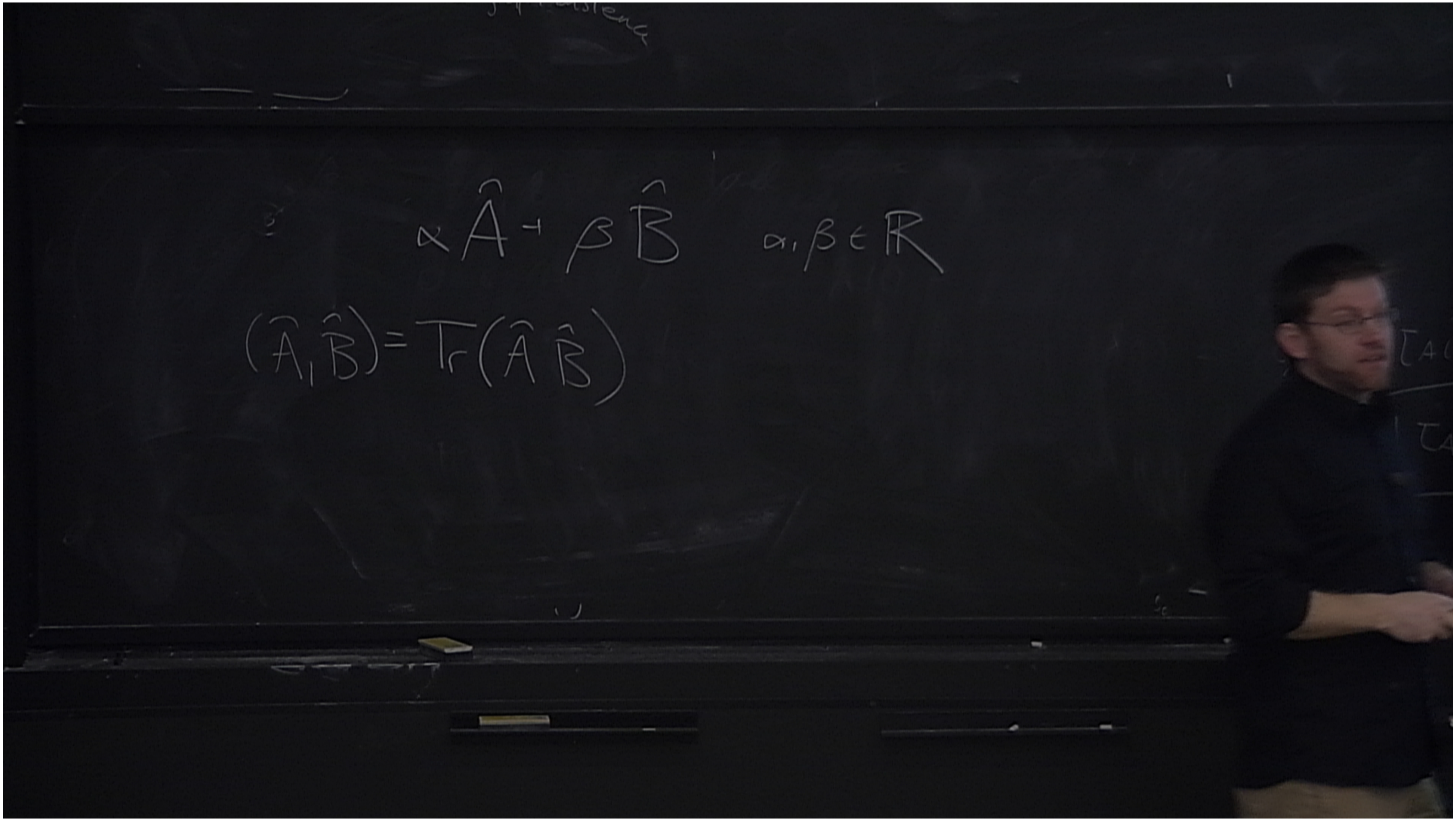

A convex operational theory
Preparation
P
Measurement
M
$\mathbf{s}_P \in S$
"operational states"
$\mathbf{r}_{M,k} \in R$
"operational effects"
S = Convex set
R = Interval of positive cone
$Pr(k|\text{P}, \text{M}) = \mathbf{r}_{M,k} \cdot \mathbf{s}_P$


A convex operational theory
Preparation
P
Measurement
M
$\mathbf{s}_P \in S$
"operational states"
$\mathbf{r}_{M,k} \in R$
"operational effects"
S = Convex set
R = Interval of positive cone
S and R characterize the operational theory!
$Pr(k|\mathrm{P},\mathrm{M}) = \mathbf{r}_{M,k} \cdot \mathbf{s}_P$


Operational classical theory
s can be any probability distribution
S = a simplex
r can be any vector of conditional probabilities
R = the unit hypercube
$(p(1), p(2))$
(0,1)
(1,0)
$(p(\text{pass}|1), p(\text{pass}|2))$
(0,1)
(1,1)
(1,0)


Operational classical theory
s can be any probability distribution
S = a simplex
r can be any vector of conditional probabilities
R = the unit hypercube
(p(1), p(2), p(3))
(p(pass|1), p(pass|2), p(pass|3))
(0,0,1)
(0,1,0)
(1,0,0)
(0,0,1)
(0,1,1)
(1,0,1)
(1,1,1)
(0,0,0)
(1,0,0)
(1,1,0)


Operational quantum theory
Recall: The Hermitian operators on a Hilbert space of dimension d form a real Euclidean vector space of dimension d2
s can be any trace one positive operator
S = the convex set of such operators

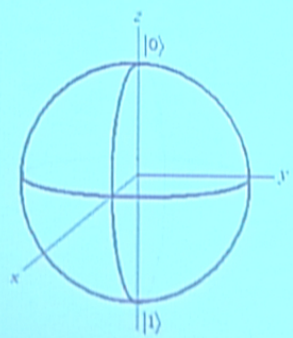

α Â + β B̂   α, β ∈ R
(Â, B̂) = Tr(Â B̂)

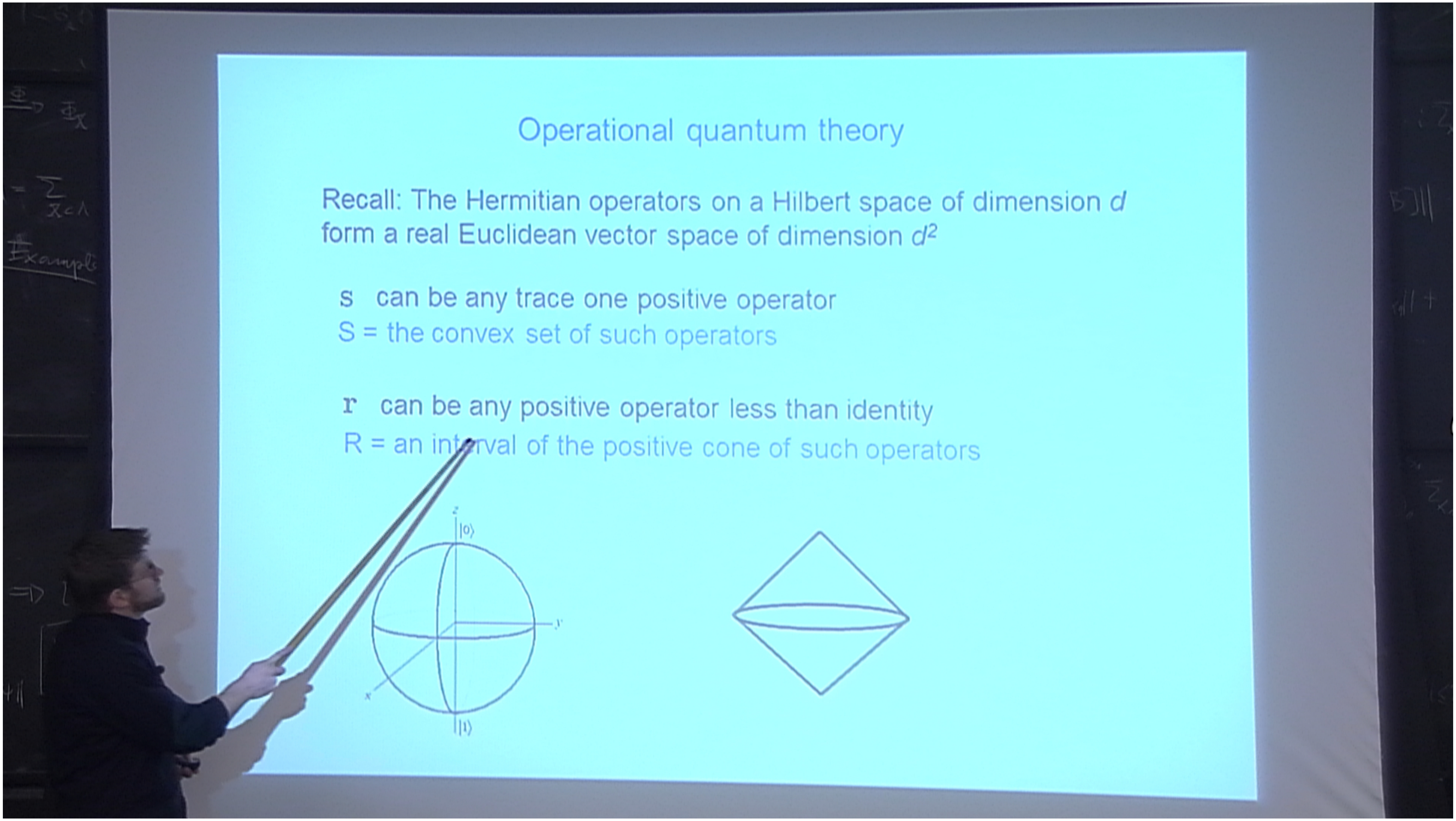
Operational quantum theory

Recall: The Hermitian operators on a Hilbert space of dimension d form a real Euclidean vector space of dimension d^2

s can be any trace one positive operator

S = the convex set of such operators




Operational quantum theory
Recall: The Hermitian operators on a Hilbert space of dimension d form a real Euclidean vector space of dimension d²
s can be any trace one positive operator
S = the convex set of such operators
r can be any positive operator less than identity
R = an interval of the positive cone of such operators
|0⟩
|1⟩

Operational quantum theory

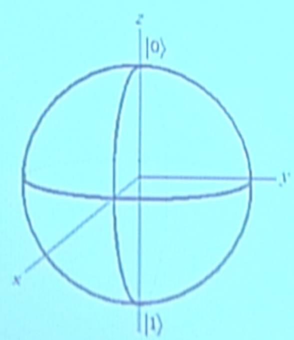
Recall: The Hermitian operators on a Hilbert space of dimension *d* form a real Euclidean vector space of dimension d^2

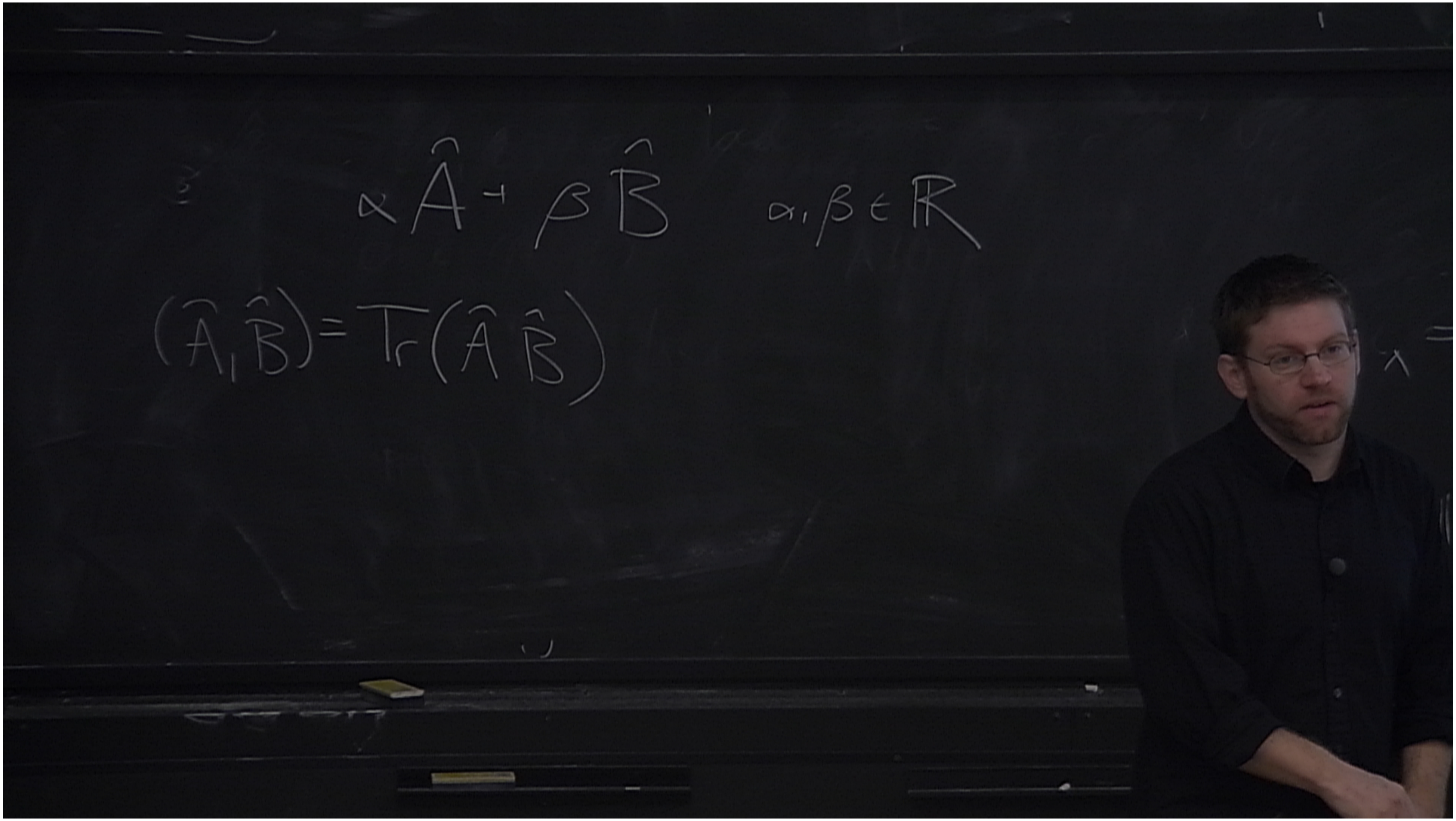
s can be any trace one positive operator

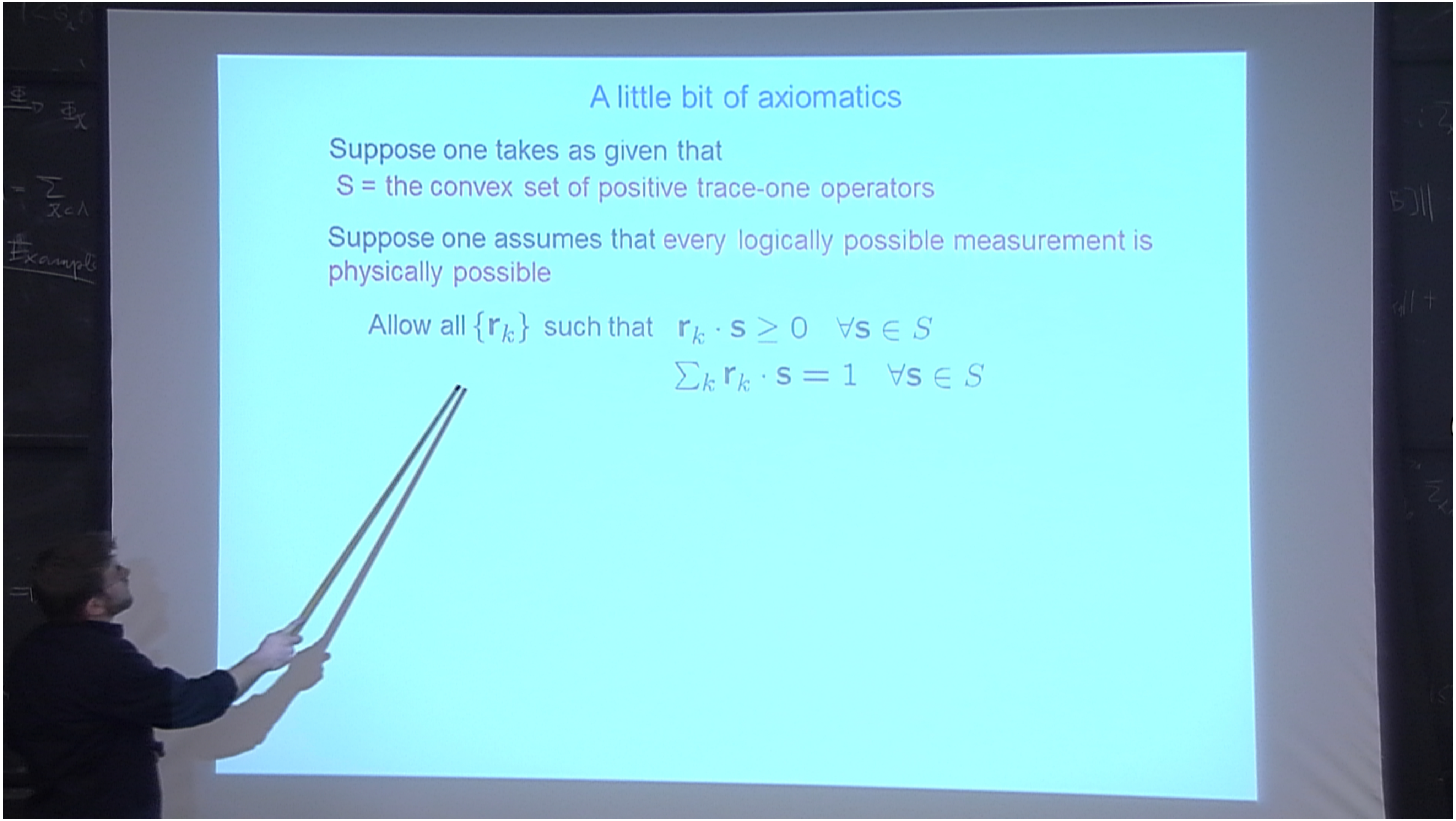
S = the convex set of such operators

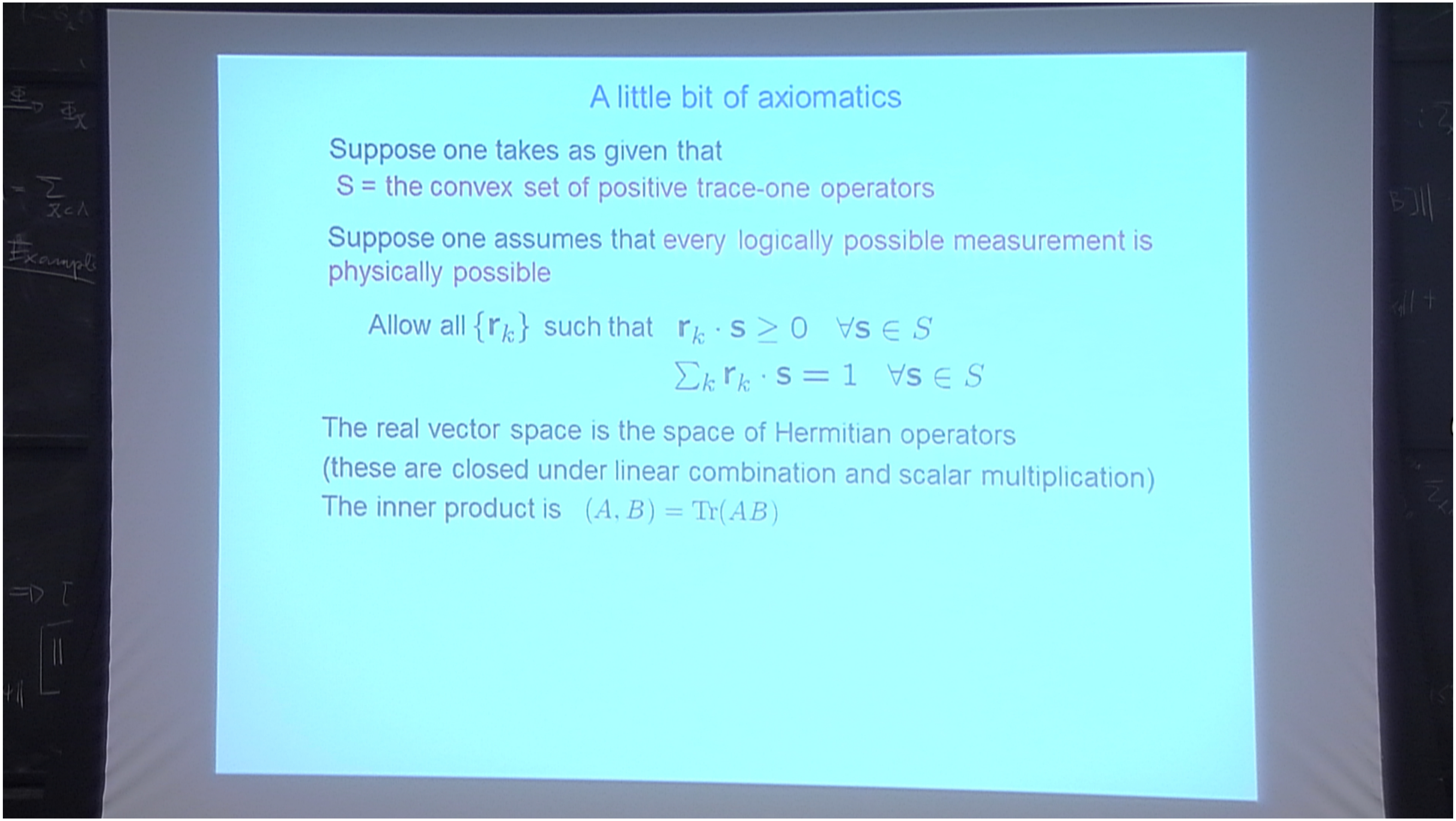
r can be any positive operator less than identity

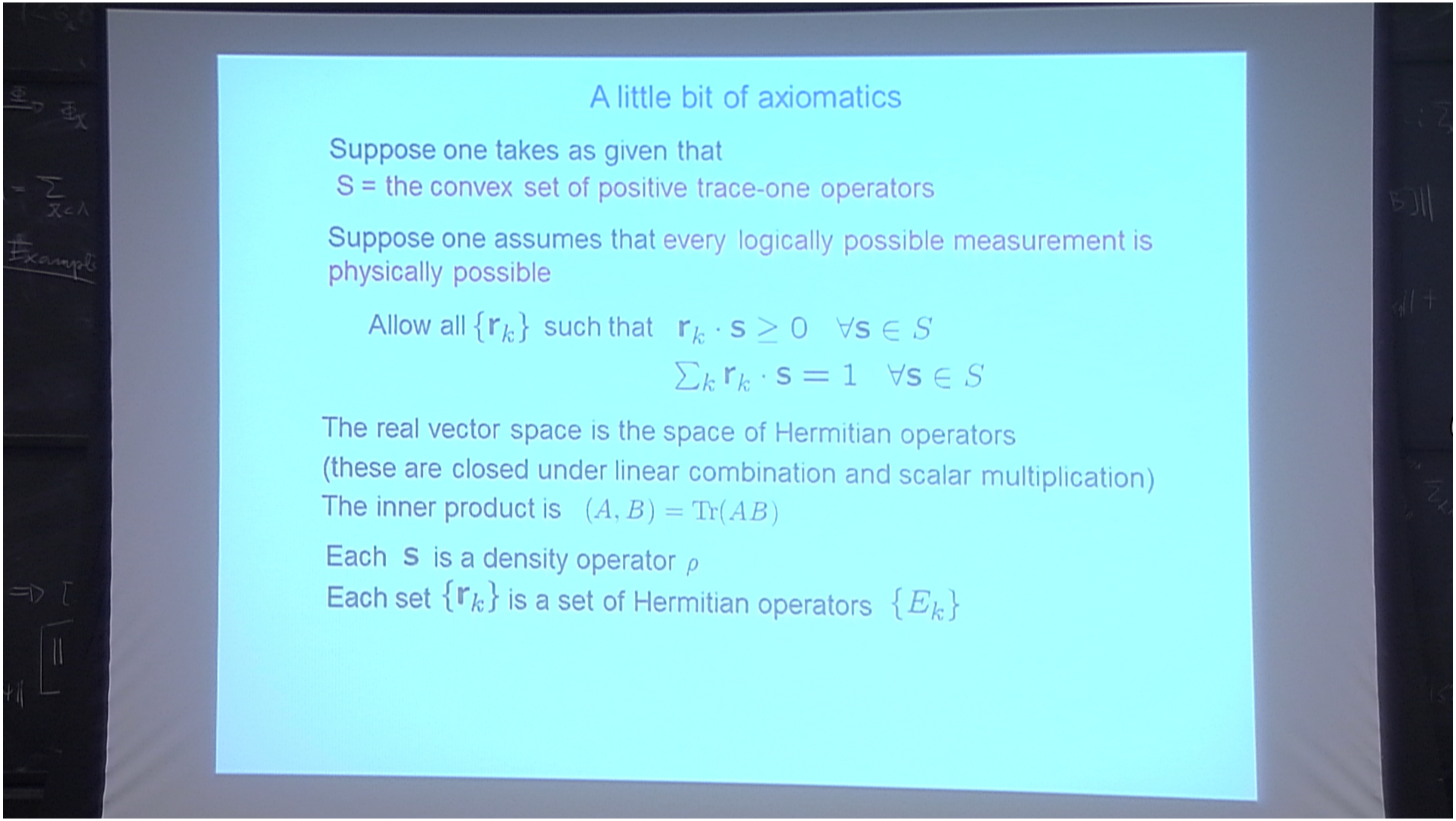
R = an interval of the positive cone of such operators




$\alpha \hat{A} + \beta \hat{B}$ $\alpha, \beta \in \mathbb{R}$
$(\hat{A}, \hat{B}) = \mathrm{Tr}(\hat{A}\hat{B})$


A little bit of axiomatics
Suppose one takes as given that
S = the convex set of positive trace-one operators
Suppose one assumes that every logically possible measurement is physically possible
Allow all $\{\mathbf{r}_k\}$ such that $\mathbf{r}_k \cdot \mathbf{s} \geq 0 \quad \forall \mathbf{s} \in S$
$\sum_k \mathbf{r}_k \cdot \mathbf{s} = 1 \quad \forall \mathbf{s} \in S$


A little bit of axiomatics
Suppose one takes as given that
S = the convex set of positive trace-one operators
Suppose one assumes that every logically possible measurement is physically possible
Allow all $\{\mathbf{r}_k\}$ such that $\mathbf{r}_k \cdot \mathbf{s} \geq 0 \quad \forall \mathbf{s} \in S$
$\sum_k \mathbf{r}_k \cdot \mathbf{s} = 1 \quad \forall \mathbf{s} \in S$
The real vector space is the space of Hermitian operators
(these are closed under linear combination and scalar multiplication)
The inner product is $(A, B) = \mathrm{Tr}(AB)$


A little bit of axiomatics
Suppose one takes as given that
S = the convex set of positive trace-one operators
Suppose one assumes that every logically possible measurement is physically possible
Allow all $\{\mathbf{r}_k\}$ such that $\mathbf{r}_k \cdot \mathbf{s} \geq 0 \quad \forall \mathbf{s} \in S$
$\sum_k \mathbf{r}_k \cdot \mathbf{s} = 1 \quad \forall \mathbf{s} \in S$
The real vector space is the space of Hermitian operators
(these are closed under linear combination and scalar multiplication)
The inner product is $(A, B) = \mathrm{Tr}(AB)$
Each $\mathbf{s}$ is a density operator ρ
Each set $\{\mathbf{r}_k\}$ is a set of Hermitian operators $\{E_k\}$

A little bit of axiomatics

Suppose one takes as given that
S = the convex set of positive trace-one operators

Suppose one assumes that every logically possible measurement is physically possible

Allow all $\{\mathbf{r}_k\}$ such that $\mathbf{r}_k \cdot \mathbf{s} \geq 0 \quad \forall \mathbf{s} \in S$

$$\sum_k \mathbf{r}_k \cdot \mathbf{s} = 1 \quad \forall \mathbf{s} \in S$$

The real vector space is the space of Hermitian operators
(these are closed under linear combination and scalar multiplication)
The inner product is $(A, B) = \mathrm{Tr}(AB)$

Each $\mathbf{s}$ is a density operator ρ
Each set $\{\mathbf{r}_k\}$ is a set of Hermitian operators $\{E_k\}$

$$\mathbf{r}_k \cdot \mathbf{s} = (E_k, \rho) = \mathrm{Tr}(E_k \rho)$$

A little bit of axiomatics

Suppose one takes as given that
S = the convex set of positive trace-one operators

Suppose one assumes that every logically possible measurement is physically possible

Allow all $\{\mathbf{r}_k\}$ such that $\mathbf{r}_k \cdot \mathbf{s} \geq 0 \quad \forall \mathbf{s} \in S$

$$\sum_k \mathbf{r}_k \cdot \mathbf{s} = 1 \quad \forall \mathbf{s} \in S$$

The real vector space is the space of Hermitian operators
(these are closed under linear combination and scalar multiplication)
The inner product is $(A, B) = \mathrm{Tr}(AB)$

Each $\mathbf{s}$ is a density operator ρ
Each set $\{\mathbf{r}_k\}$ is a set of Hermitian operators $\{E_k\}$
$\mathbf{r}_k \cdot \mathbf{s} = (E_k, \rho) = \mathrm{Tr}(E_k \rho)$ ← the form of the Born rule

A little bit of axiomatics

Suppose one takes as given that
S = the convex set of positive trace-one operators

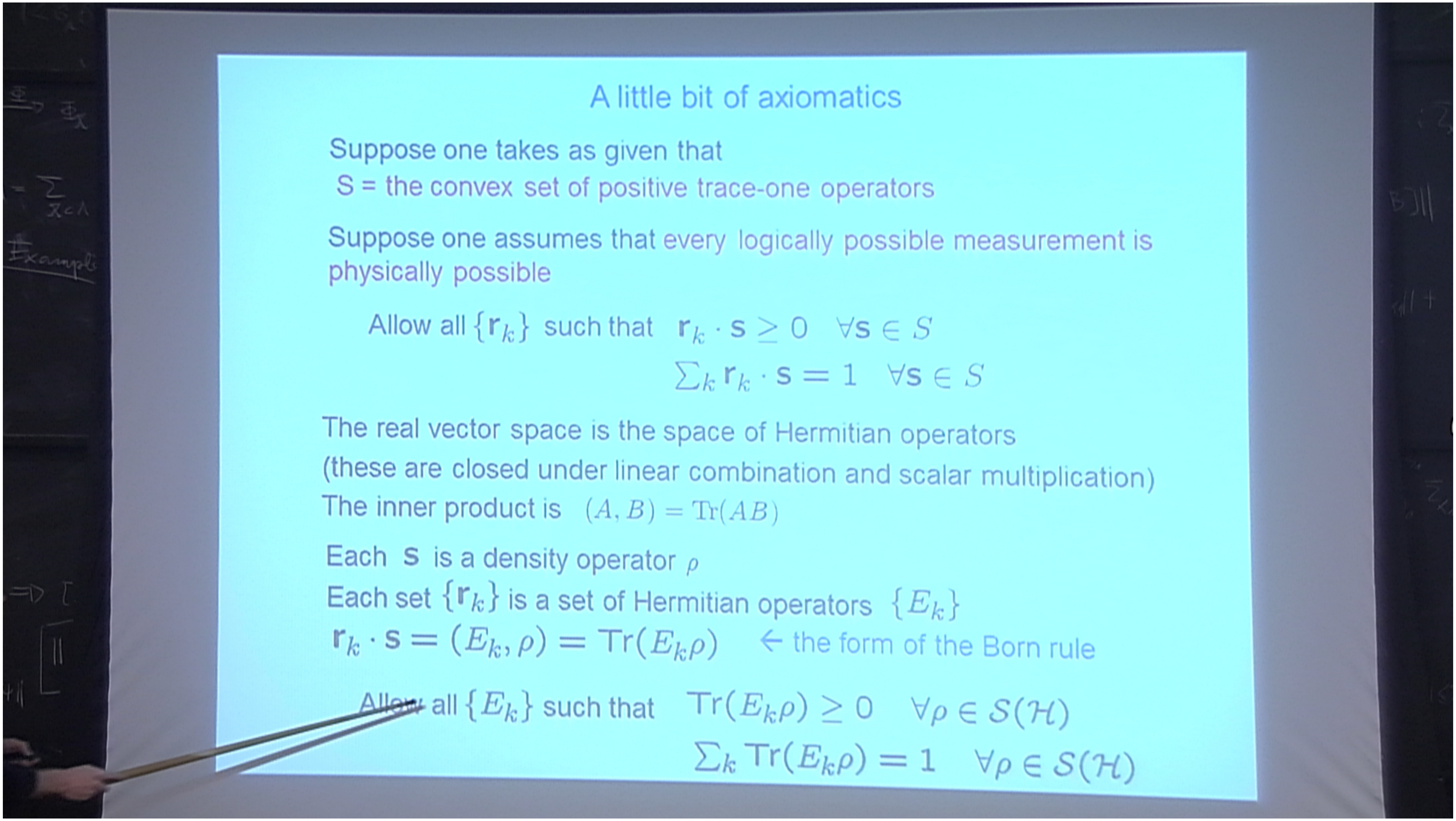
Suppose one assumes that every logically possible measurement is physically possible

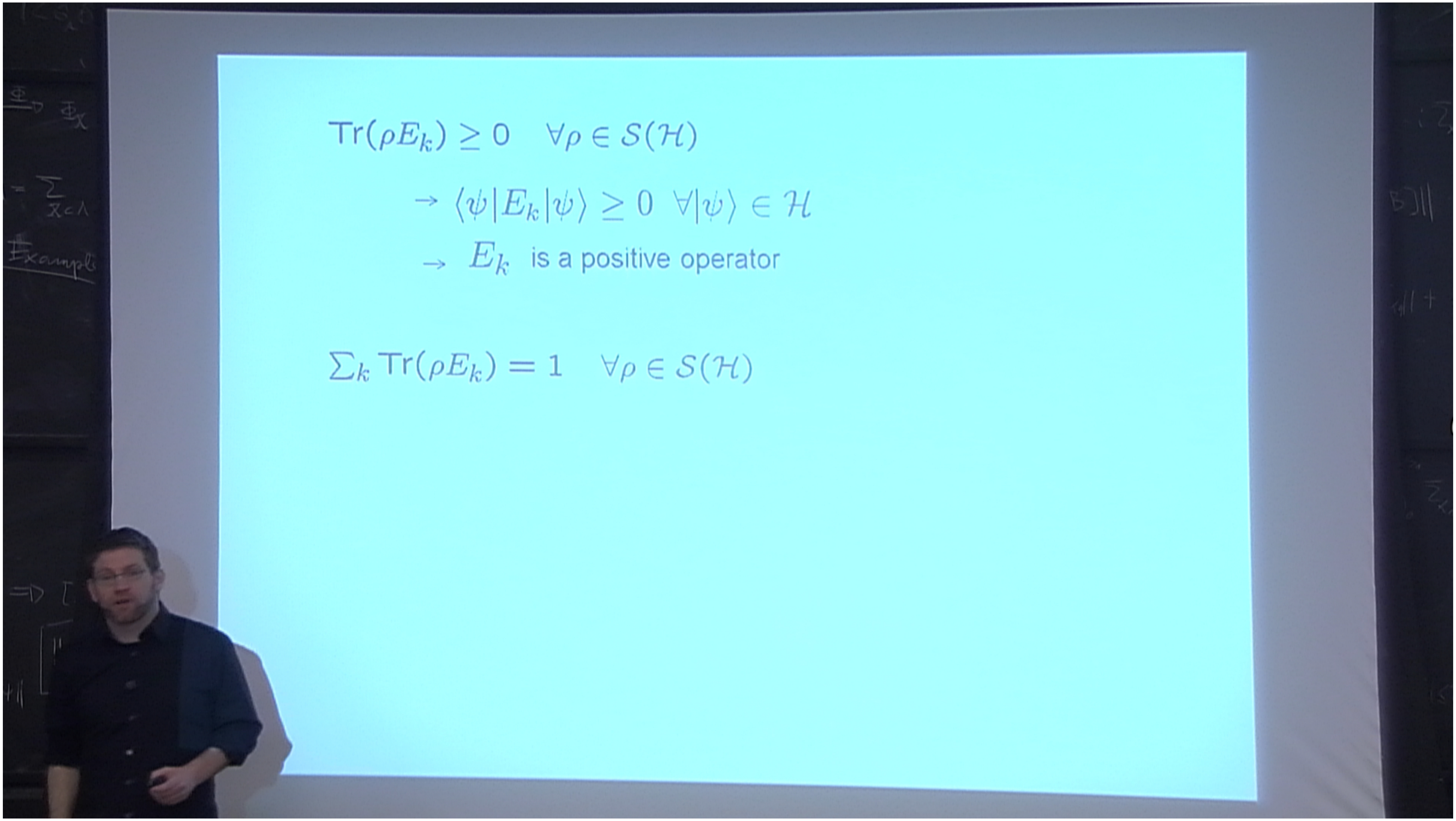
Allow all $\{\mathbf{r}_k\}$ such that $\mathbf{r}_k \cdot \mathbf{s} \geq 0 \quad \forall \mathbf{s} \in S$

$$\sum_k \mathbf{r}_k \cdot \mathbf{s} = 1 \quad \forall \mathbf{s} \in S$$

The real vector space is the space of Hermitian operators
(these are closed under linear combination and scalar multiplication)
The inner product is $(A, B) = \mathrm{Tr}(AB)$

Each $\mathbf{s}$ is a density operator ρ
Each set $\{\mathbf{r}_k\}$ is a set of Hermitian operators $\{E_k\}$
$\mathbf{r}_k \cdot \mathbf{s} = (E_k, \rho) = \mathrm{Tr}(E_k \rho)$ ← the form of the Born rule


A little bit of axiomatics
Suppose one takes as given that
S = the convex set of positive trace-one operators
Suppose one assumes that every logically possible measurement is physically possible
Allow all $\{\mathbf{r}_k\}$ such that $\mathbf{r}_k \cdot \mathbf{s} \geq 0 \quad \forall \mathbf{s} \in S$
$\sum_k \mathbf{r}_k \cdot \mathbf{s} = 1 \quad \forall \mathbf{s} \in S$
The real vector space is the space of Hermitian operators
(these are closed under linear combination and scalar multiplication)
The inner product is $(A, B) = \mathrm{Tr}(AB)$
Each $\mathbf{s}$ is a density operator ρ
Each set $\{\mathbf{r}_k\}$ is a set of Hermitian operators $\{E_k\}$
$\mathbf{r}_k \cdot \mathbf{s} = (E_k, \rho) = \mathrm{Tr}(E_k \rho)$ ← the form of the Born rule
Allow all $\{E_k\}$ such that $\mathrm{Tr}(E_k \rho) \geq 0 \quad \forall \rho \in \mathcal{S}(\mathcal{H})$
$\sum_k \mathrm{Tr}(E_k \rho) = 1 \quad \forall \rho \in \mathcal{S}(\mathcal{H})$


$\mathrm{Tr}(\rho E_k) \geq 0 \quad \forall \rho \in \mathcal{S}(\mathcal{H})$
$\rightarrow \langle \psi | E_k | \psi \rangle \geq 0 \ \forall | \psi \rangle \in \mathcal{H}$
$\rightarrow$ E_k is a positive operator
$\sum_k \mathrm{Tr}(\rho E_k) = 1 \quad \forall \rho \in \mathcal{S}(\mathcal{H})$

$\mathrm{Tr}(\rho E_k) \geq 0 \quad \forall \rho \in \mathcal{S}(\mathcal{H})$

$\rightarrow \langle \psi | E_k | \psi \rangle \geq 0 \;\; \forall |\psi\rangle \in \mathcal{H}$

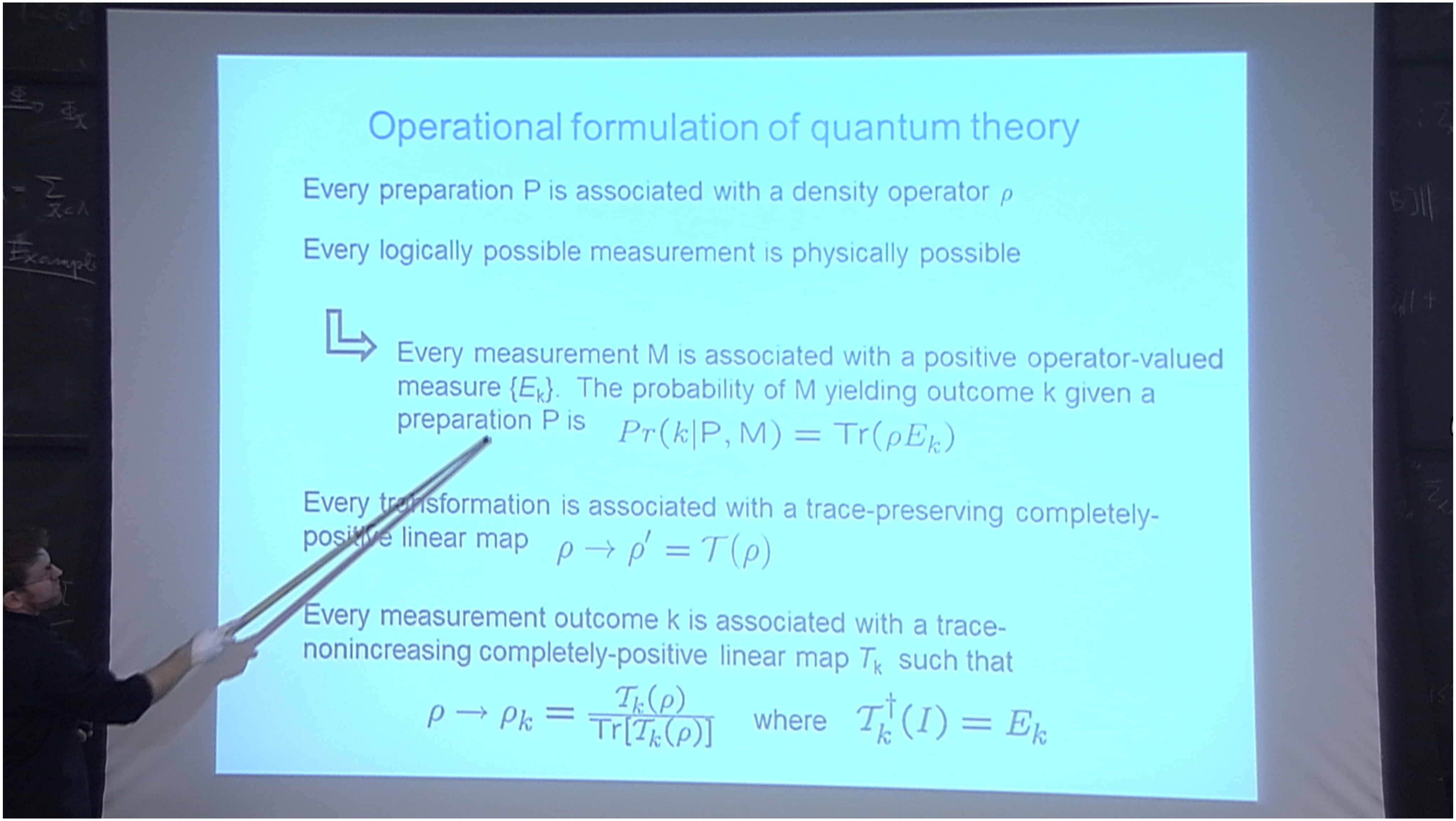
$\rightarrow$ E_k is a positive operator

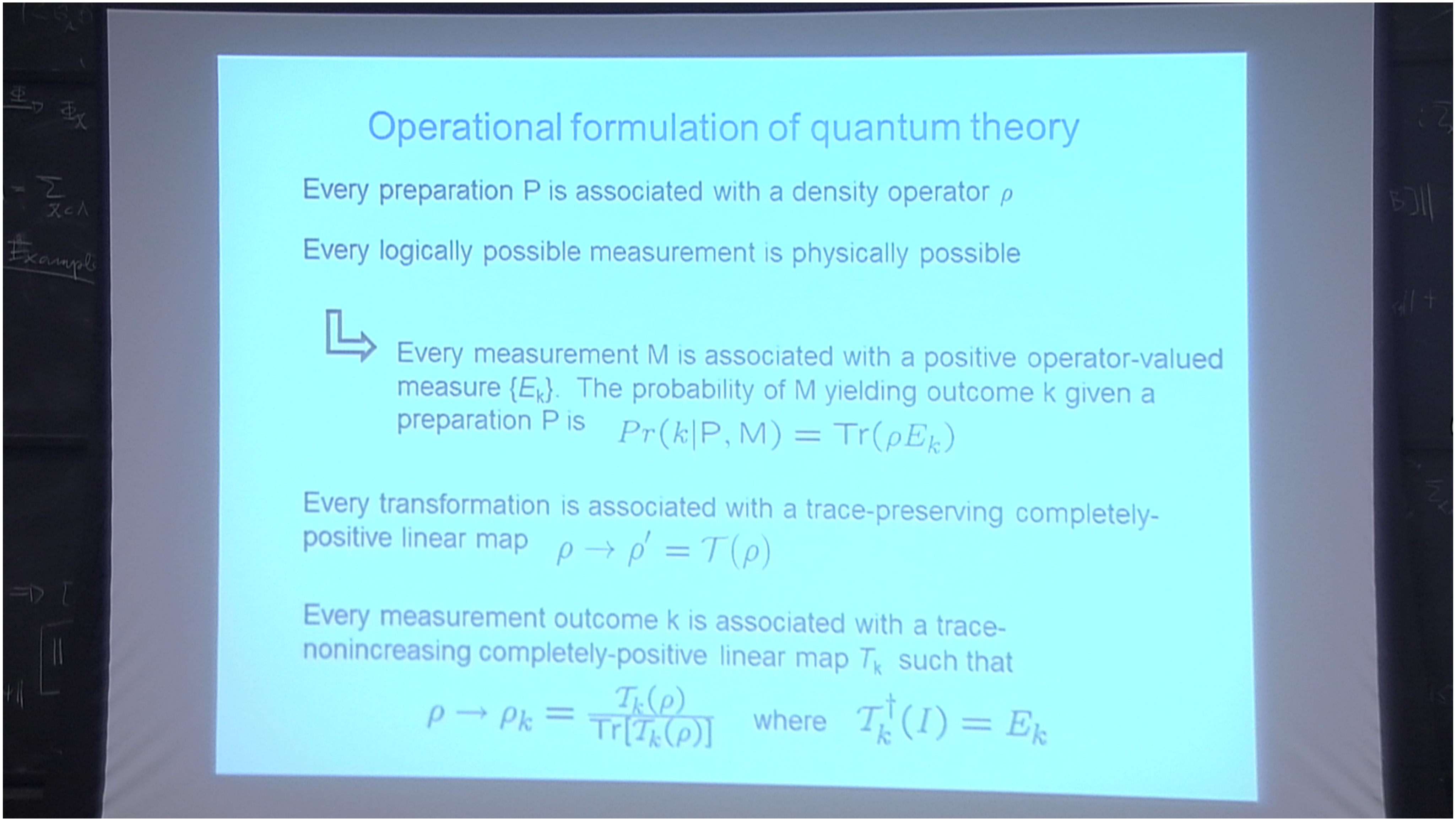
$\sum_k \mathrm{Tr}(\rho E_k) = 1 \quad \forall \rho \in \mathcal{S}(\mathcal{H})$

$\rightarrow \langle \psi | \left(\sum_k E_k \right) | \psi \rangle = 1 \;\; \forall |\psi\rangle \in \mathcal{H}$

$\rightarrow \sum_k E_k = I$

The logically possible measurements correspond to the POVMs!


Operational formulation of quantum theory
Every preparation P is associated with a density operator ρ
Every logically possible measurement is physically possible
Every measurement M is associated with a positive operator-valued measure $\{E_k\}$. The probability of M yielding outcome k given a preparation P is $Pr(k|\mathrm{P},\mathrm{M}) = \mathrm{Tr}(\rho E_k)$
Every transformation is associated with a trace-preserving completely-positive linear map $\rho \to \rho' = \mathcal{T}(\rho)$
Every measurement outcome k is associated with a trace-nonincreasing completely-positive linear map T_k such that
$\rho \to \rho_k = \frac{\mathcal{T}_k(\rho)}{\mathrm{Tr}[\mathcal{T}_k(\rho)]}$ where $\mathcal{T}_k^\dagger(I) = E_k$


Operational formulation of quantum theory
Every preparation P is associated with a density operator ρ
Every logically possible measurement is physically possible
Every measurement M is associated with a positive operator-valued measure $\{E_k\}$. The probability of M yielding outcome k given a preparation P is $Pr(k|\mathrm{P},\mathrm{M}) = \mathrm{Tr}(\rho E_k)$
Every transformation is associated with a trace-preserving completely-positive linear map $\rho \to \rho' = \mathcal{T}(\rho)$
Every measurement outcome k is associated with a trace-nonincreasing completely-positive linear map T_k such that
$\rho \to \rho_k = \frac{\mathcal{T}_k(\rho)}{\mathrm{Tr}[\mathcal{T}_k(\rho)]}$ where $\mathcal{T}_k^\dagger(I) = E_k$

A little bit of axiomatics

Suppose one takes as given that
S = the convex set of positive trace-one operators

Suppose one assumes that every logically possible measurement is physically possible

Allow all $\{\mathbf{r}_k\}$ such that $\mathbf{r}_k \cdot \mathbf{s} \geq 0 \quad \forall \mathbf{s} \in S$

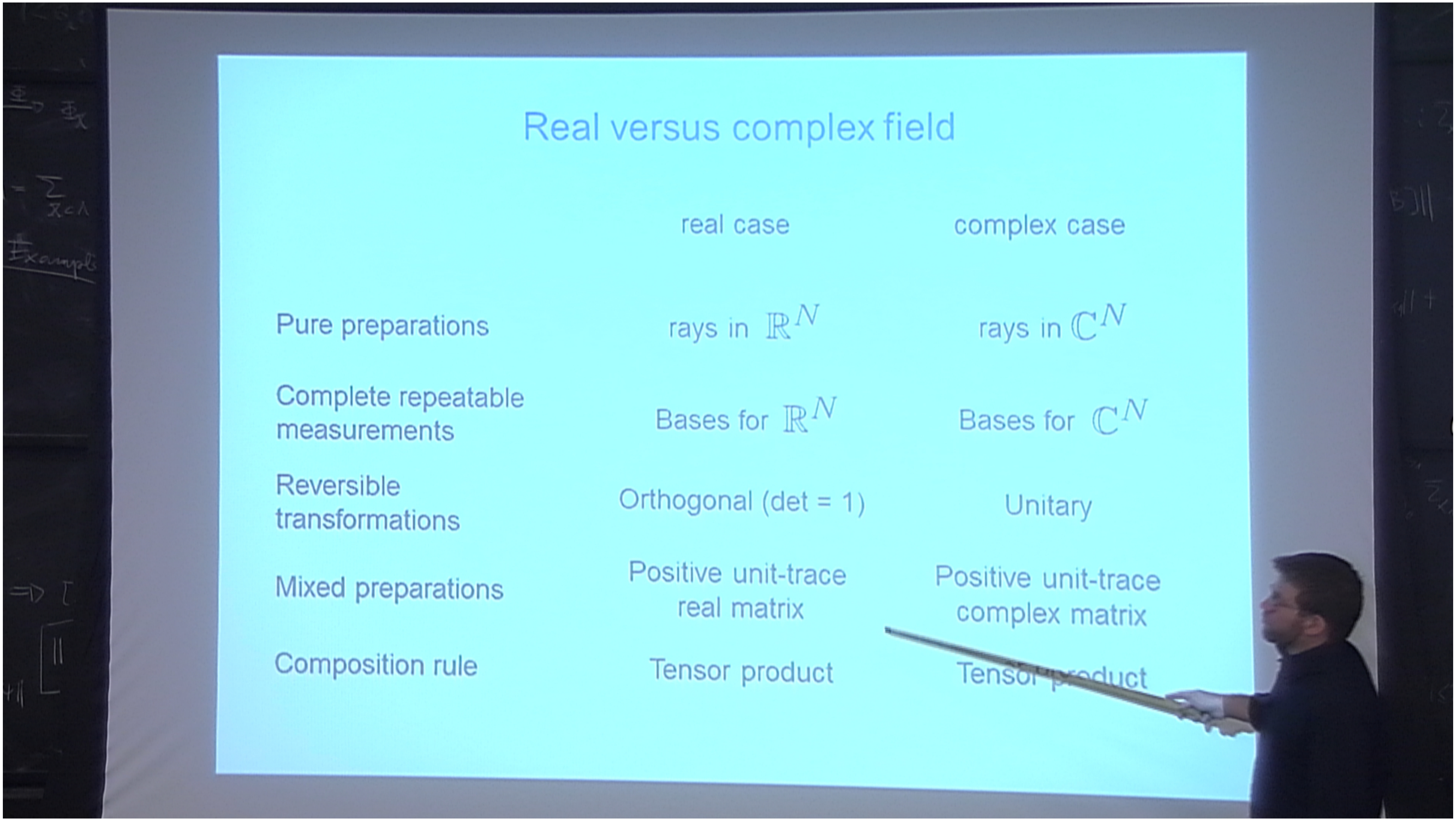
$$\sum_k \mathbf{r}_k \cdot \mathbf{s} = 1 \quad \forall \mathbf{s} \in S$$

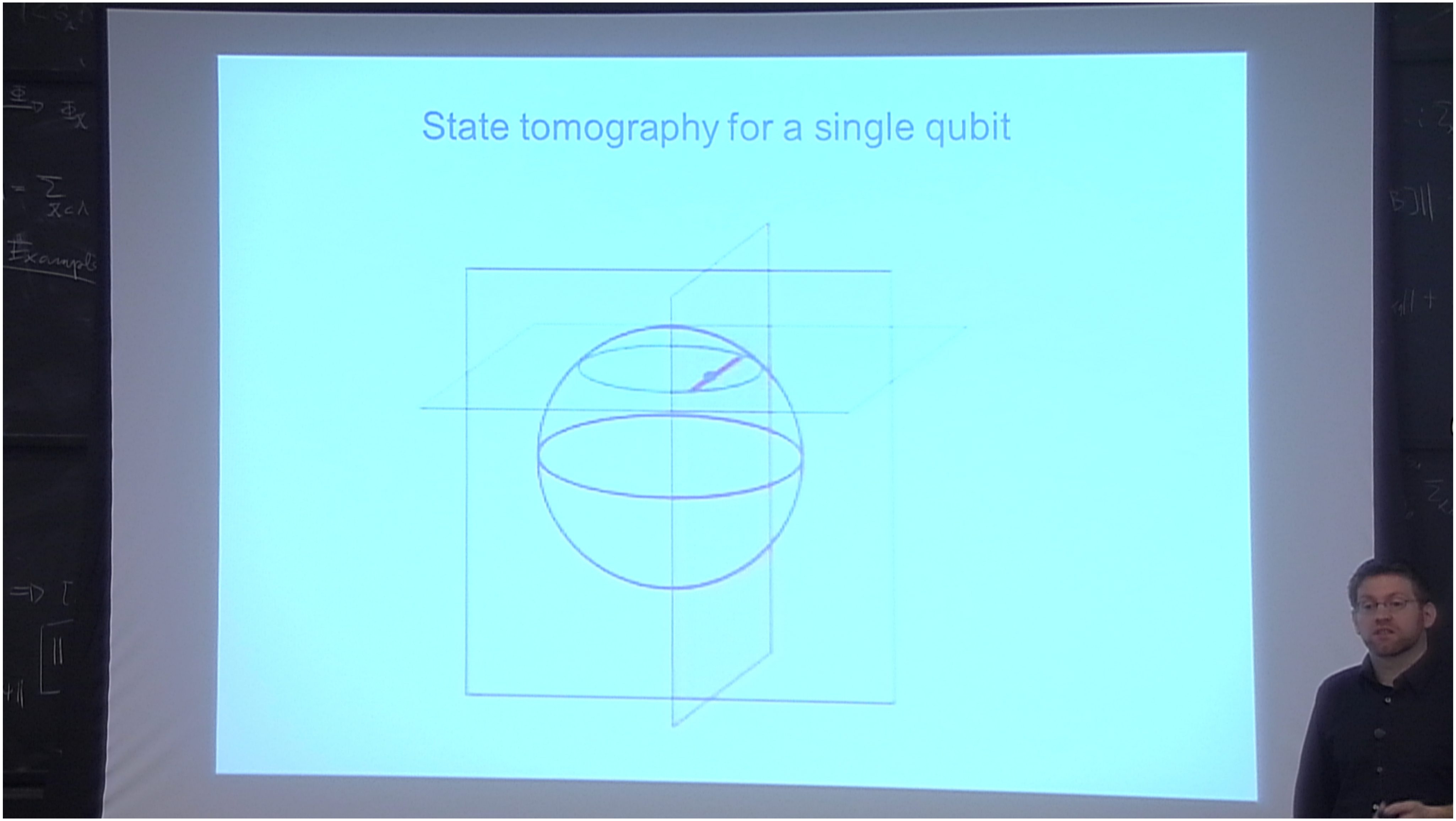
The real vector space is the space of Hermitian operators
(these are closed under linear combination and scalar multiplication)
The inner product is $(A, B) = \mathrm{Tr}(AB)$

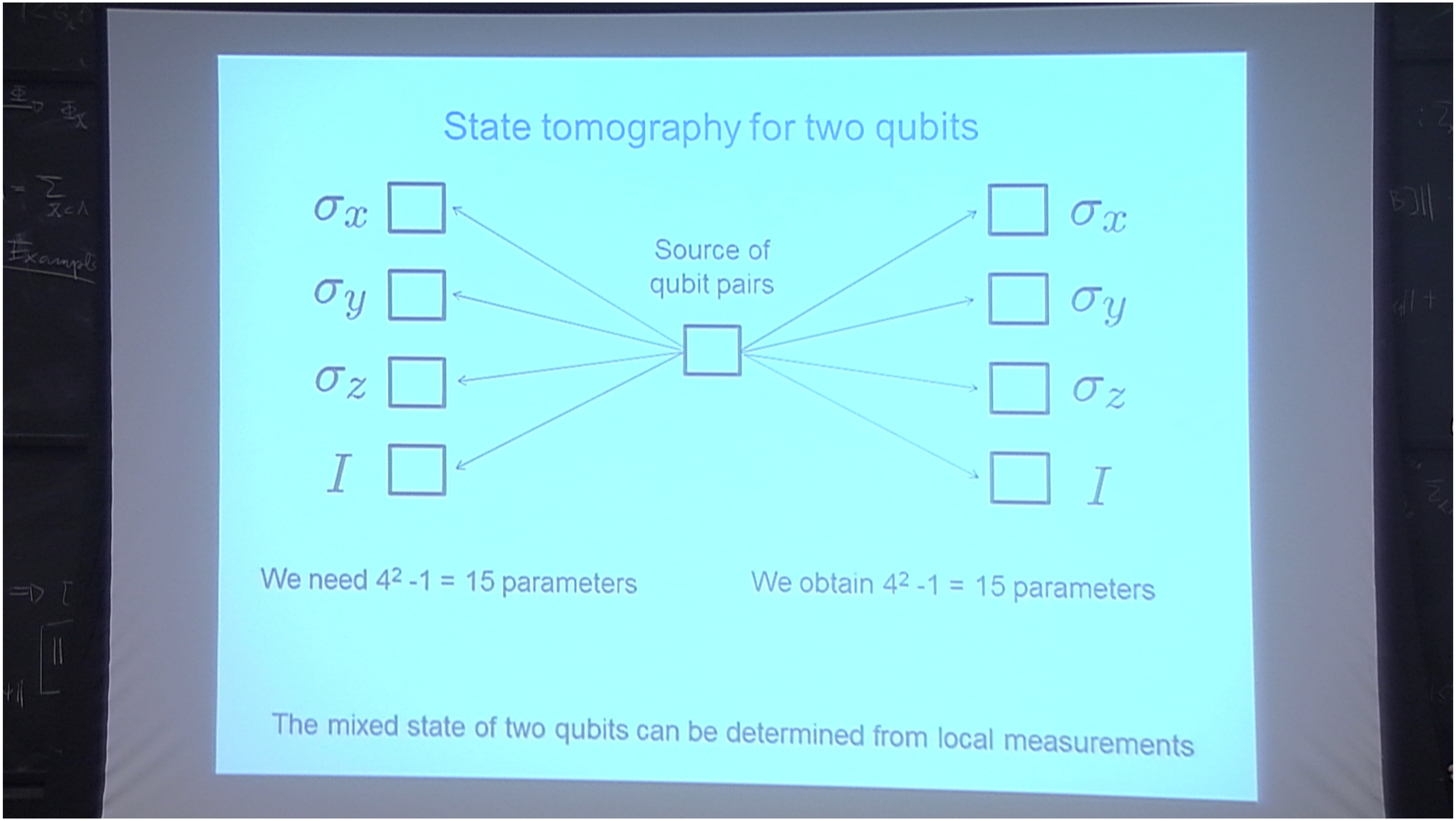
Each $\mathbf{s}$ is a density operator ρ
Each set $\{\mathbf{r}_k\}$ is a set of Hermitian operators $\{E_k\}$
$\mathbf{r}_k \cdot \mathbf{s} = (E_k, \rho) = \mathrm{Tr}(E_k \rho)$ ← the form of the Born rule

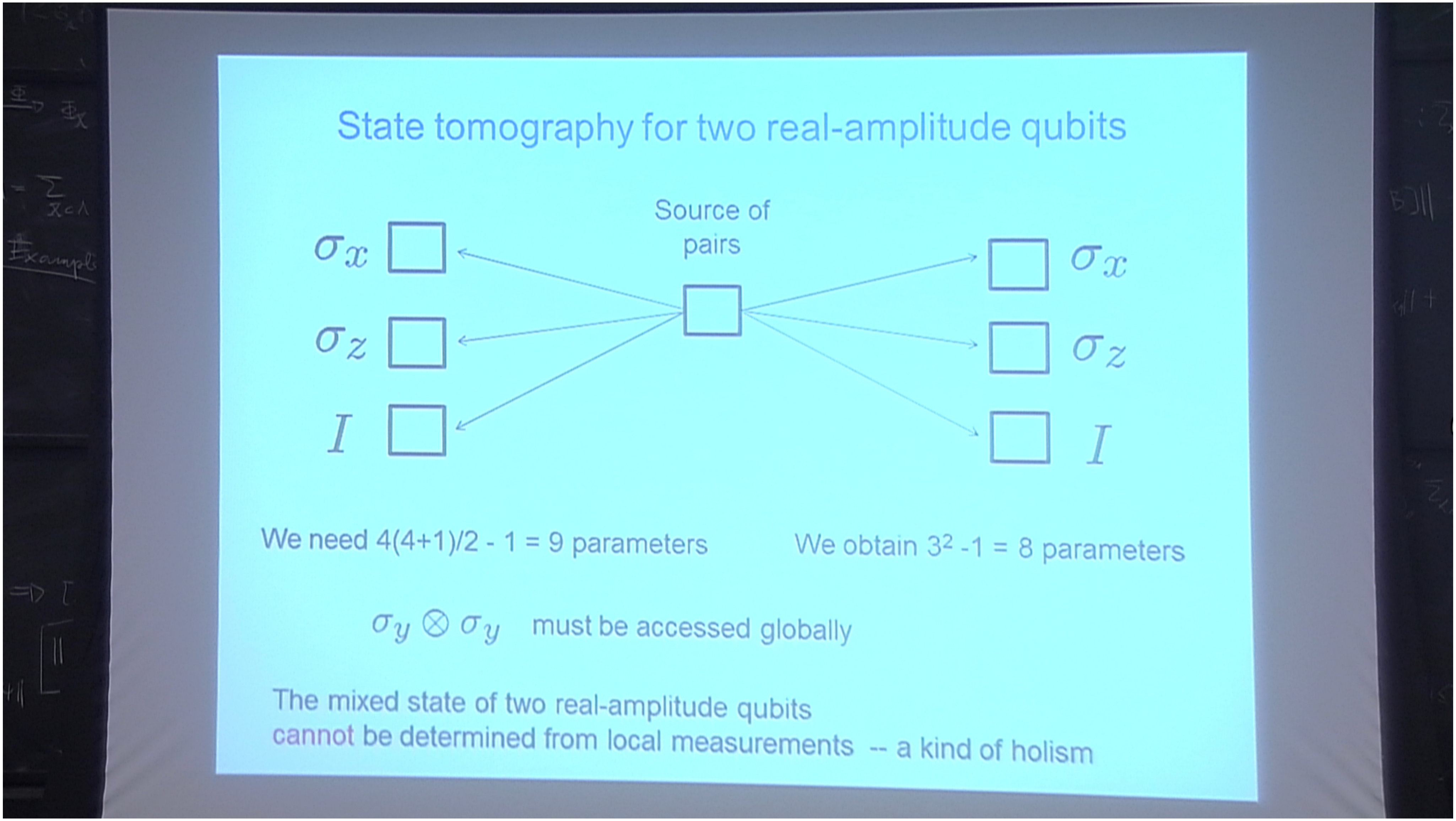
Allow all $\{E_k\}$ such that $\mathrm{Tr}(E_k\rho) \geq 0 \quad \forall \rho \in \mathcal{S}(\mathcal{H})$

$$\sum_k \mathrm{Tr}(E_k \rho) = 1 \quad \forall \rho \in \mathcal{S}(\mathcal{H})$$


Real versus complex field
real case
complex case
Pure preparations
rays in $\mathbb{R}^N$
rays in $\mathbb{C}^N$
Complete repeatable measurements
Bases for $\mathbb{R}^N$
Bases for $\mathbb{C}^N$
Reversible transformations
Orthogonal (det = 1)
Unitary
Mixed preparations
Positive unit-trace real matrix
Positive unit-trace complex matrix
Composition rule
Tensor product
Tensor


State tomography for a single qubit


State tomography for two qubits
σ_x
σ_y
σ_z
I
Source of qubit pairs
σ_x
σ_y
σ_z
I
We need $4^2 - 1 = 15$ parameters
We obtain $4^2 - 1 = 15$ parameters
The mixed state of two qubits can be determined from local measurements


State tomography for two real-amplitude qubits
Source of pairs
σ_x
σ_z
I
σ_x
σ_z
I
We need 4(4+1)/2 - 1 = 9 parameters
We obtain 3^2 -1 = 8 parameters
$\sigma_y \otimes \sigma_y$ must be accessed globally
The mixed state of two real-amplitude qubits
cannot be determined from local measurements -- a kind of holism