Title: Bipartite Fluctuations as a Probe of Many-Body Entanglement

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Abstract: The scaling of entanglement entropy, and more recently the full entanglement spectrum, have become useful tools for characterizing certain universal features of quantum many-body systems.
Although entanglement entropy is difficult to measure experimentally, we show that for systems that can be mapped to non-interacting fermions both the von Neumann entanglement entropy and generalized Renyi entropies can be related exactly to the cumulants of number fluctuations, which are accessible experimentally. Such systems include free fermions in all dimensions, the integer quantum Hall states and topological insulators in two dimensions, strongly repulsive bosons in one-dimensional optical lattices, and the spin-1/2 XX chain, both pure and strongly disordered.
The same formalism can be used for analyzing entanglement entropy generation in quantum point contacts with non-interacting electron reservoirs.
Beyond the non-interacting case, we show that the scaling of fluctuations in one-dimensional critical systems behaves quite similarly to the entanglement entropy, and in analogy to the full counting statistics used in mesoscopic transport, give important information about the system. The behavior of fluctuations, which are the essential feature of quantum systems, are explained in a general framework and analyzed in a variety of specific situations.

Bipartite Fluctuations as a Probe of Many-Body Entanglement

Francis Song

Work done in collaboration with Karyn Le Hur, Stephan Rachel, Christian Flindt, Israel Klich, and Nicolas Laflorencie

Department of Physics, Yale University

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Quantum Entanglement

"Spukhafte Fernwirkung"

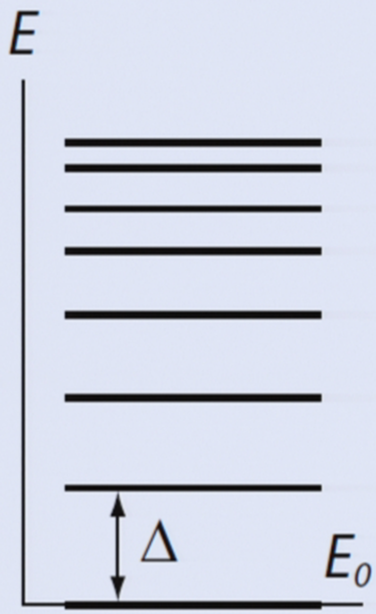
Entanglement, the "spooky" non-local correlations inherent to quantum mechanics, has long been understood to be the fundamentally novel feature of quantum systems as compared to their classical counterparts. Important first in ontological status of quantum mechanics, then in quantum information, now in many-body physics.

Example: Bell pair

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle_A \otimes |\downarrow\rangle_B - |\downarrow\rangle_A \otimes |\uparrow\rangle_B).$$

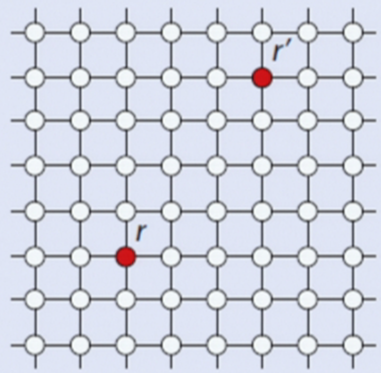
"Traditional" Many-Body Physics (e.g., Mahan)

Ground state and excited state energies



Gap, spectrum

Correlation functions

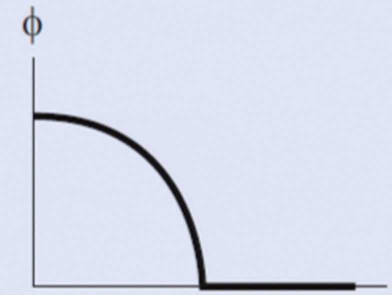


$$\langle \hat{\psi}^\dagger(r,t)\hat{\psi}(r',t')\rangle$$

$$\langle \hat{\rho}(r,t)\hat{\rho}(r',t')\rangle$$

Long-range order, response to perturbations

Symmetry-breaking order parameters



$$\langle \phi \rangle \neq 0$$

Many-Body Physics and Entanglement

There are states of matter that are not fully characterized by the above formalism, e.g., topological states of matter. So one reason for studying the structure of entanglement is to see if entanglement can illuminate such states of matter.

Many-Body Physics from the Quantum Information Perspective: Bipartite Entanglement (DMRG, MPS)

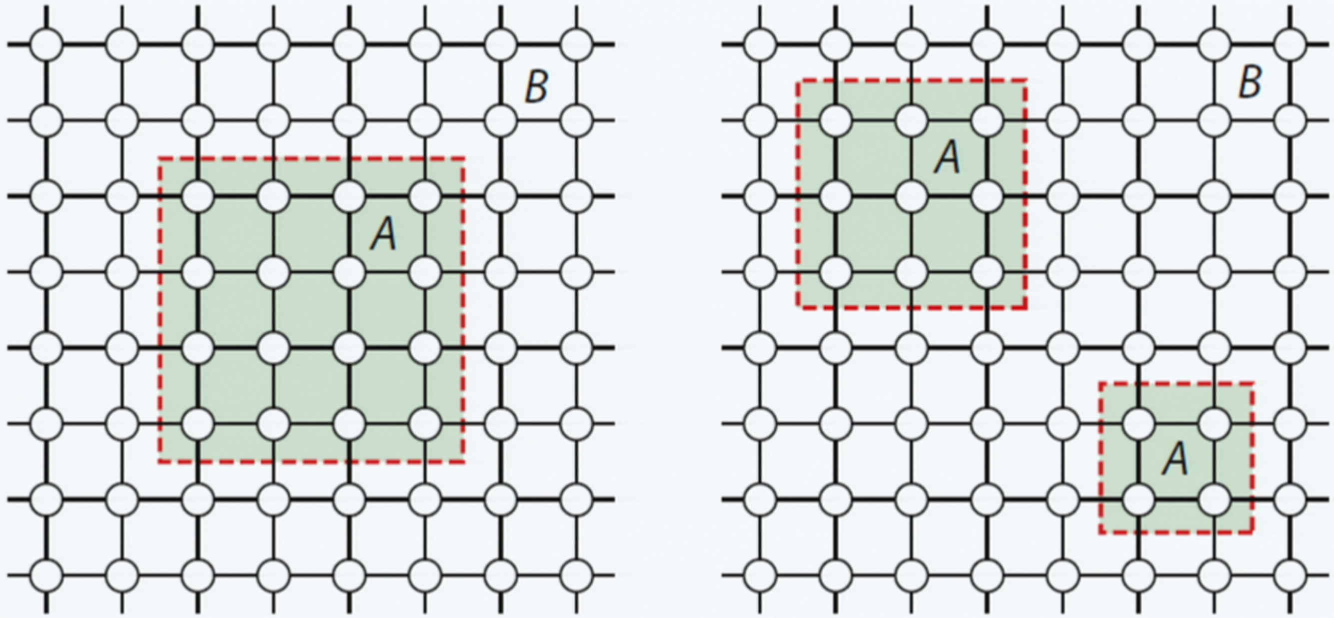


Figure: Divide the system into two parts A and $\bar{A} = B$, focus on A. A can be a single connected region (left) or multiple disconnected regions (right). Note the formation of a boundary. In the context of black holes we are actually interested in the "exterior" region B rather than the "interior" region A.

The Quantum Information Perspective (Cont.)

Reduced density matrix

Given a pure state $|\Psi\rangle$, the reduced density matrix for subsystem A is

$$\hat{\rho}_A = \mathrm{Tr}_B |\Psi\rangle\langle\Psi|.$$

- **Approach 1**: Quantify the amount of entanglement. Entanglement entropy, bipartite fidelity, valence bond entanglement entropy, ...
- **Approach 2**: Study the entanglement spectrum. Analyze the full set of eigenvalues of the reduced density matrix $\hat{\rho}_A$, usually in terms of the energy levels of the "entanglement Hamiltonian" $\hat{H}_A$ defined by

$$\hat{\rho}_A = \frac{e^{-\beta \hat{H}_A}}{\mathrm{Tr}(e^{-\beta \hat{H}_A})}, \qquad \beta = 1.$$

"Low temperature," or "low energy," limit corresponds to dominant eigenvalues of $\hat{\rho}_A$. This is only useful if $\hat{H}_A$ has an interesting structure.

The Quantum Information Perspective (Cont.)

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Motivation: Topological Insulators

Chern insulator

As a simple example, consider

$$\hat{H} = \frac{1}{2}\sum_{xy}\left[\hat{c}^{\dagger}_{xy}(\sigma^z - i\sigma^x)\hat{c}_{x+1,y} + \hat{c}^{\dagger}_{xy}(\sigma^z - i\sigma^y)\hat{c}_{x,y+1} + \text{H.c.}\right] + m\sum_{xy}\hat{c}^{\dagger}_{xy}\sigma^z\hat{c}_{xy}.$$

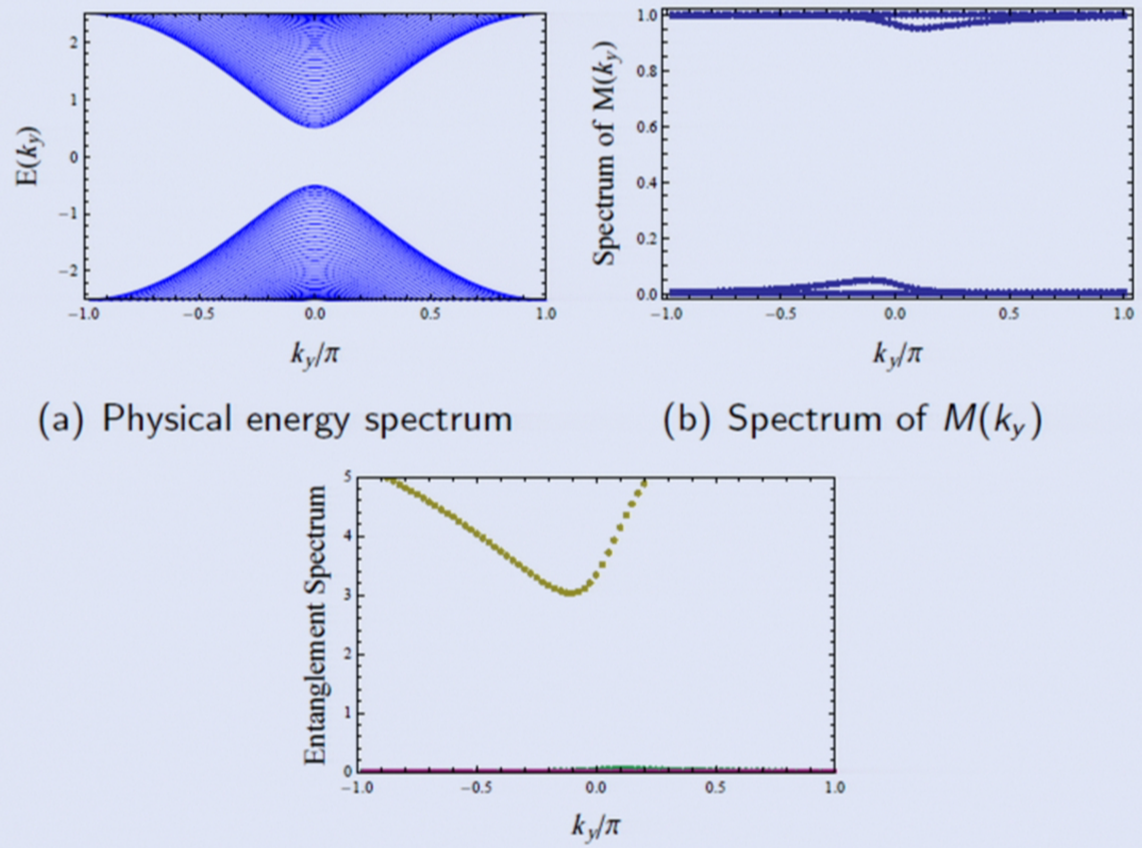
The Chern number for this system is

$$C_1 = \begin{cases} -1 & \text{for } 0 < m < 2, \\ 1 & \text{for } -2 < m < 0, \\ 0 & \text{otherwise.} \end{cases}$$

Consequence: **edge modes**.

Motivation: Topological Insulators (Cont.)

System with edge, trivial phase $m = -2.5$ (Alexandradinata et *al.*)



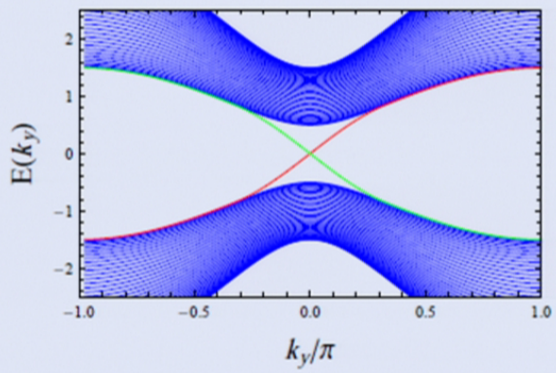
(a) Physical energy spectrum

(b) Spectrum of $M(k_y)$

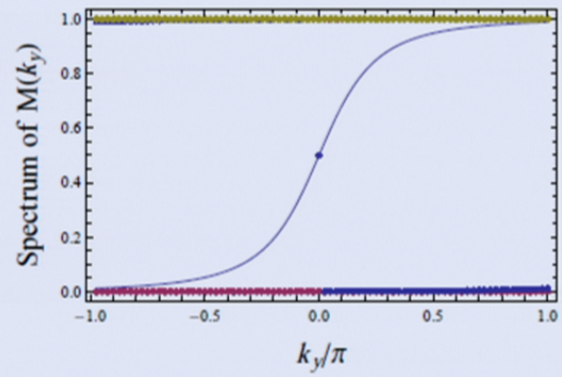
(c) Entanglement energy levels

Motivation: Topological Insulators (Cont.)

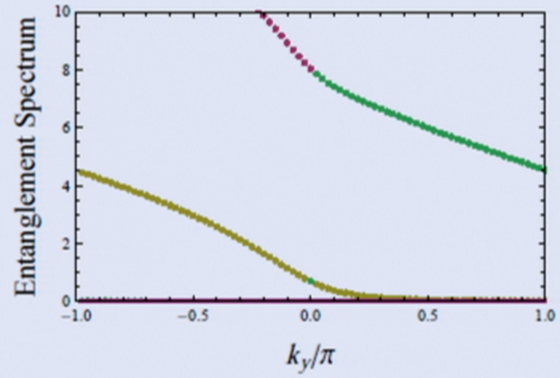
System with edge, topological phase $m = -1.5$



(d) Physical energy spectrum



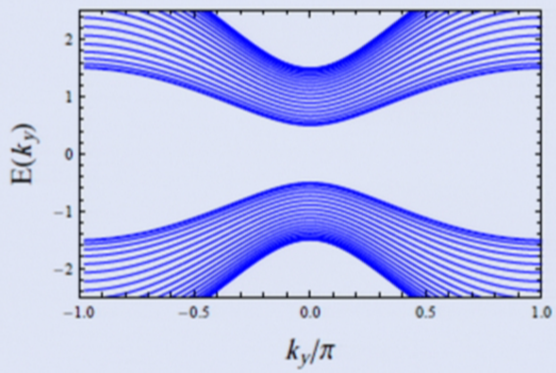
(e) Spectrum of $M(k_y)$



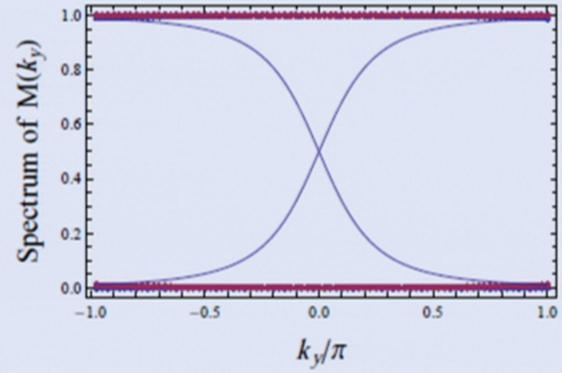
(f) Entanglement energy levels

Motivation: Topological Insulators (Cont.)

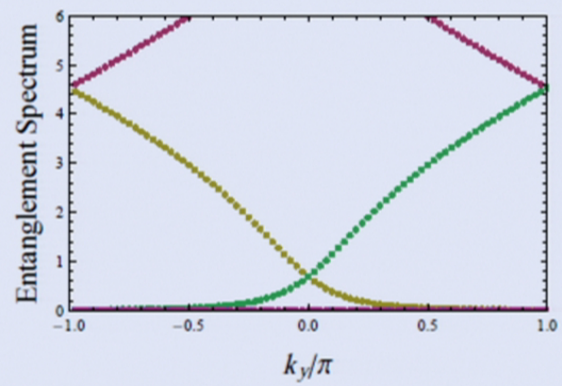
System **without** edge, topological phase $m = -1.5$



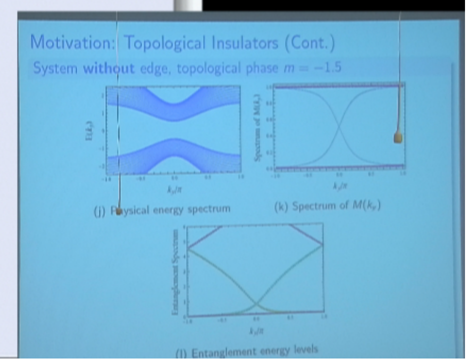
(j) Physical energy spectrum



(k) Spectrum of $M(k_y)$



(l) Entanglement energy levels



Motivation: Topological Insulators

Virtual "edge"

The topological nature of the insulating phase is a property of the bulk wave function, present whether there is an edge or not (cf. computation of the Chern number). The entanglement Hamiltonian contains information about the edge state, even though there is no physical edge!

Proofs of bulk-boundary correspondence–in lofty terms: holographic principle.

Entanglement Entropy

The entanglement entropy is the von Neumann entropy

$$\mathcal{S}_A = -\mathrm{Tr}(\hat{\rho}_A \ln \hat{\rho}_A).$$

More generally, the Rényi entanglement entropy of order α is defined as

$$\mathcal{S}_A^{(\alpha)} = \frac{1}{1-\alpha} \ln[\mathrm{Tr}(\hat{\rho}_A^{\alpha})], \qquad \lim_{\alpha \to 1} \mathcal{S}_A^{(\alpha)} = \mathcal{S}_A.$$

Relation to Schmidt decomposition

Can always write

$$|\Psi\rangle = \sum_i \sqrt{\lambda_i} |i\rangle_A |i\rangle_B,$$

so

$$\mathrm{Tr}(\hat{\rho}_A^{\alpha}) = \sum_i \lambda_i^{\alpha}.$$

Symmetry between A and B is evident.

Entanglement Entropy (Cont.)

Important properties of the entanglement entropy at zero temperature

1. Zero if and only if $|\Psi\rangle$ is a product state

$$|\Psi\rangle = |\Psi_A\rangle \otimes |\Psi_B\rangle.$$

2. Symmetry between subsystems

$$\mathcal{S}_A^{(\alpha)} = \mathcal{S}_B^{(\alpha)} \equiv \mathcal{S}^{(\alpha)}.$$

This suggests the importance of the boundary.

3. Subadditivity

$$\mathcal{S}_A + \mathcal{S}_B \geq \mathcal{S}_{A\cup B} = 0.$$

Entanglement Entropy (Cont.)

Some famous results

- At a conformally invariant critical point in 1D,

$$\mathcal{S}(\ell) = \frac{c}{3} \ln \left[\frac{L}{\pi a} \sin \frac{\pi \ell}{L} \right],$$

  where c is the central charge of the underlying conformal field theory (CFT), ℓ is the length of the subsystem, L is the length of the total system, and a is a non-universal short-distance cutoff; for finite temperature replace L with $i\beta$ (Holzhey '04, Calabrese & Cardy '04).
- Area law for harmonic lattices (Bombelli, Srednicki, Plenio, ...), ground states of realistic Hamiltonians are rather unusual—Hilbert space is "gratuitously" large.

Entanglement Entropy (Cont.)

Some famous results, cont.

- For free fermions in d dimensions,

$$\mathcal{S}(L) \sim L^{d-1} \ln L + \text{sub-leading terms.}$$

  This is the prototypical **violation** of the area law, with multiplicative logarithmic correction (Wolf '06, Gioev & Klich '06, Swingle '10). Fermions have a smaller Hilbert space but more entanglement due to non-local nature!
- Topological entanglement entropy (Levin & Wen '06, Kitaev & Preskill '06)
- Entanglement spectrum of fractional quantum Hall states, "entanglement gap" (Li & Haldane '08)
- DMRG can be understood as a variational algorithm over Matrix Product States, which explicitly incorporate an area-law scaling for the entanglement entropy. Generalizations to PEPS, MERA, and classical simulation of quantum systems (Vidal).

Entanglement Entropy (Cont.)

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Can We Measure Entanglement Entropy?

One possible way to measure Rényi entropies (Cardy '11)

Observe that when there are n copies of the Hilbert space $\mathcal{H} = \bigotimes_{j=1}^{n}(\mathcal{H}_{A,j} \otimes \mathcal{H}_{B,j})$ and n copies of the ground state $|0\rangle = \bigotimes_{j=1}^{n} |0\rangle_j$

$$\mathrm{Tr}(\hat{\rho}_A^n) = \langle \Pi_n \rangle$$

where Π_n cyclically permutes the A part of Hilbert space

$$\Pi_n : \ \mathcal{H}_{A,j} \to \mathcal{H}_{A,(j+1) \bmod n}.$$

Already used to measure entanglement entropy in quantum Monte Carlo (QMC) [Hastings et *al.* '10].

Same idea used to derive famous CFT result.

Can We Measure Entanglement Entropy? (Cont.)

Our claim

"Easier" way, for systems of non-interacting fermions: measure particle-number fluctuations

$$C_n = (-i\partial_\lambda)^n \ln \chi(\lambda)|_{\lambda=0}, \qquad \chi(\lambda) = \langle e^{i\lambda \hat{N}_A} \rangle.$$

$\chi(\lambda)$ is the cumulant generating, or characteristic, function.

Applications: Free fermions in any dimension, spin-1/2 XX chain $\equiv$ hard-core bosons in an optical lattice, integer quantum Hall effect, topological insulators (note eigenvalues of M associated with zeros of the generating function), ...

Further claim

Number fluctuations are interesting to study in their own right, e.g., Full Counting Statistics (FCS) in Quantum Point Contacts (QPC) and applications to detecting quantum phase transitions.

Number Fluctuations

Important properties of $\mathcal{F} = C_2 = \langle \hat{N}_A^2 \rangle - \langle \hat{N}_A \rangle^2$ with total number conservation

1. Zero if $|\Psi\rangle$ is a product state

$$|\Psi\rangle = |\Psi_A\rangle \otimes |\Psi_B\rangle.$$

   But converse not true (Furukawa '09).

2. Symmetry between subsystems

$$\mathcal{F}_A = \mathcal{F}_B \equiv \mathcal{F}.$$

   This suggests the importance of the boundary.

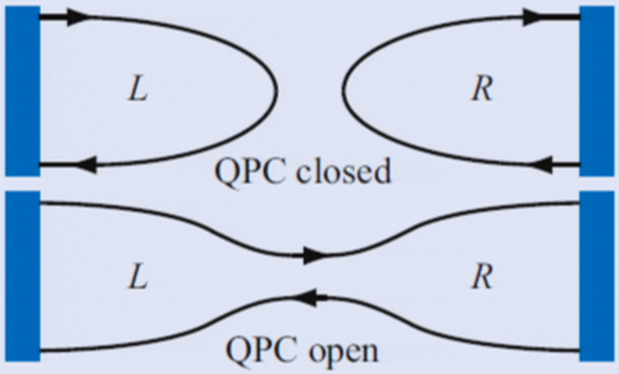
3. Subadditivity

$$\mathcal{F}_A + \mathcal{F}_B \geq \mathcal{F}_{A \cup B} = 0.$$

Number Fluctuations (Cont.)

More generally, the **even** cumulants C_{2n} also satisfy properties (1) and (2).

There is already an incredibly active field in this direction: FCS (Levitov & Lesovik '93) in mesoscopic transport.



In a QPC, $\chi(\lambda, t) = \sum_n P_n(t) e^{i\lambda n}$ where $P_n(t)$ is the probability that n charges were transferred during the span $[0, t]$, say from L ("source") to R ("drain").

Number Fluctuations in Numerics

Simple observation

In the Density Matrix Renormalization Group (DMRG) the reduced density matrix is already block-diagonal in U(1) numbers like particle number $\hat{N}$ and spin $\hat{S}^z$. Therefore cumulants are trivial to compute.

Also simple to compute in QMC.

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Entanglement Entropy and Fluctuations

Entanglement entropy of free fermions

$$\mathcal{S} = \lim_{K\to\infty} \sum_{n=1}^{K+1} \alpha_n(K) C_n,$$

where

$$\alpha_n(K) = \begin{cases} 2\sum_{k=n-1}^{K} \frac{S_1(k,n-1)}{k!k} & \text{for } n \text{ even,} \\ 0 & \text{for } n \text{ odd.} \end{cases}$$

Here $S_1(n,m)$ are **unsigned Stirling numbers of the first kind**. Practically, K is the number of available cumulants and should be taken to be even. Increasingly better **lower bound**.

Order of limit is important

$$\lim_{K\to\infty}\left[\sum_{n=1}^{K+1} \alpha_n(K) C_n\right] \neq \sum_{n=1}^{\infty}\left[\lim_{K\to\infty} \alpha_n(K)\right] C_n.$$

The RHS is usually not convergent.

Brief Interlude: Stirling Numbers

Combinatoric interpretation

The Stirling numbers of the first kind $S_1(n, m)$ are the number of ways n objects can be arranged into m cycles. They satisfy the recursion relation

$$S_1(n+1, m) = S_1(n, m-1) + nS_1(n, m).$$

Properties of the Series

Only even cumulants contribute

The formula works for pure states, and only even cumulants are symmetric between A and B.

What happens when we bring the limit inside?

For even n (Klich & Levitov '09),

$$\alpha_n(\infty) = 2\zeta(n) = \frac{(2\pi)^n |B_n|}{n!},$$

where $\zeta(n)$ is the Riemann zeta function and B_n are Bernoulli numbers. So

$$\alpha_n(K) C_n \sim 2C_n \text{ for large } n,$$

but

$$\lim_{n\to\infty} |C_n| = \infty$$

Not convergent! More on this later.

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Derivation

Important formulas

$$\mathcal{S} = -\mathrm{Tr}[M \ln M + (1-M)\ln(1-M)],$$

$$\mathcal{S}^{(\alpha)} = \frac{1}{1-\alpha}\mathrm{Tr}\{\ln[M^{\alpha} + (1-M)^{\alpha}]\},$$

$$\chi(\lambda) = \det\{[1 + (e^{i\lambda} - 1)M]e^{-i\lambda q}\}.$$

- $M_{ij} = \langle \hat{a}_i^{\dagger} \hat{a}_j \rangle$ is essentially the Green's function.
- q is an irrelevant phase factor related to the background charge.
- These formulas can be derived easily using the reduced density matrix obtained by Peschel '03.

The Clever, but (Partially) Wrong, Derivation

Here "wrong" means "unless the generating function is gaussian an infinite number of cumulants plus a divergent resummation is required to get the entanglement entropy." (Generalizes Klich & Levitov '09.)

The spectral density function

Define the spectral density function

$$\mu(z) = \mathrm{Tr}[\delta(M - z)] = \frac{1}{\pi}\mathrm{Im}[\partial_z \ln \chi(\lambda(z - i0^+))],$$

where

$$\lambda(z) = -\pi - i \ln\left(\frac{1}{z} - 1\right).$$

Then the Rényi entanglement entropy is

$$\mathcal{S}^{(\alpha)} = \frac{1}{1-\alpha}\int_0^1 dz\ \mu(z) \ln[z^\alpha + (1-z)^\alpha].$$

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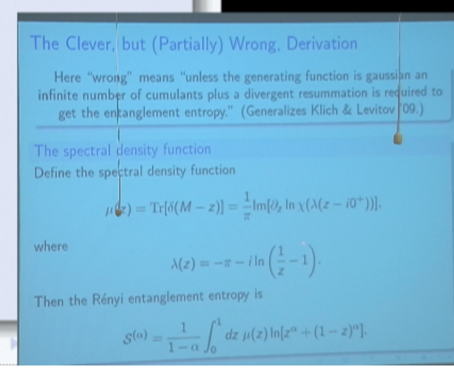


The Clever, but (Partially) Wrong, Derivation (Cont.)

Rényi entanglement entropy

$$\mathcal{S}^{(\alpha)} = \frac{\alpha}{\pi}\int_{-\infty}^{\infty} du\ \frac{\tanh(\alpha u) - \tanh u}{\alpha - 1}\mathrm{Im}[\ln \chi(\pi - 2iu)].$$

Expand the generating function $\chi(\lambda) = \sum_{k=1}^{\infty}[(i\lambda)^k/k!]C_k$ to get

$$\mathcal{S}^{(\alpha)} = \sum_{k=1}^{\infty} \beta_k^{(\alpha)} C_k,$$

$$\beta_k^{(\alpha)} = \frac{\alpha(2\pi)^k}{(1-\alpha)k!}\mathrm{Im}\left\{\int_{-\infty}^{\infty} du\ [\tanh(\alpha\pi u) - \tanh(\pi u)]\left(u - \frac{i}{2}\right)^k\right\}.$$

Evaluate for integer n and analytically continue to non-integer α to get

$$\beta_k^{(\alpha)} = \frac{2}{\alpha - 1}\frac{1}{k!}\left(\frac{2\pi i}{\alpha}\right)^k \zeta\left(-k, \frac{\alpha+1}{2}\right),$$

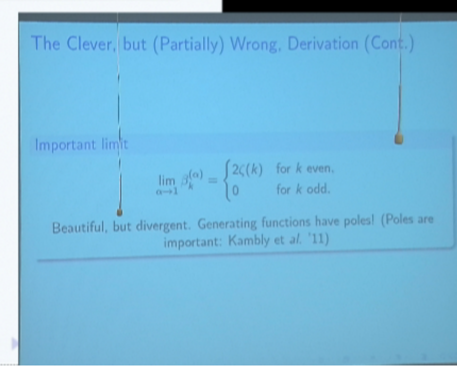
where $\zeta(s, a)$ is the Hurwitz zeta function.

The Clever, but (Partially) Wrong, Derivation (Cont.)

Important limit

$$\lim_{\alpha \to 1} \beta_k^{(\alpha)} = \begin{cases} 2\zeta(k) & \text{for } k \text{ even,} \\ 0 & \text{for } k \text{ odd.} \end{cases}$$

Beautiful, but divergent. Generating functions have poles! (Poles are important: Kambly et *al.* '11)



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The Clever, but (Partially) Wrong, Derivation (Cont.)

Example

Suppose $M = 1/2$, e.g., a single fermion at a QPC.

$$\chi(\lambda) = \cos\frac{\lambda}{2}$$

and

$$C_k = \frac{2^k - 1}{k} B_k, \qquad |C_k| \sim \frac{2(k-1)!}{\pi^k} \text{ for large } k.$$

Each term in the series is therefore **factorially** divergent.

This is very typical (Flindt et *al.* '09): Any non-gaussian generating function $\chi(\lambda)$ will have an infinite number of cumulants, and these cumulants will in general diverge factorially due to singularities in the complex plane of the generating function. *If* the generating function is gaussian, however, note

$$\beta_2^{(\alpha)} = \frac{\pi^2}{6}\left(1 + \frac{1}{\alpha}\right), \qquad \beta_2^{(1)} = \frac{\pi^2}{3}.$$

The Still Clever, but Correct, Derivation

Here "correct" means "convergent, plus the approximation is an increasingly sharper **lower bound** to the exact entanglement entropy."

Basic idea

Expand logarithms to get

$$\mathcal{S} = \sum_{n=1}^{\infty} \frac{\mathrm{Tr}[M(1-M)^n + M^n(1-M)]}{n},$$

and notice that the factorial cumulants $F_n = \partial_\lambda^n \ln \chi(-i \ln \lambda)|_{\lambda=1}$ are given by

$$F_k = (-1)^{k-1}(k-1)![\mathrm{Tr}(M^k) - q], \qquad k \geq 1$$

to get

$$\mathcal{S} = \sum_{n=1}^{\infty} \left\{ \frac{(-1)^{n-1}}{n} \left[\frac{F_n}{(n-1)!} + \frac{F_{n+1}}{n!} \right] + \sum_{k=0}^{n} \binom{n}{k} \frac{F_{k+1}}{k!n} \right\}$$

The Still Clever, but Correct, Derivation (Cont.)

To rewrite $\mathcal{S}$ in terms of cumulants, introduce cutoff K and switch the order of sums. After some algebra we get (HFS et *al.* '11)

$$\mathcal{S} = \lim_{K\to\infty} \sum_{n=1}^{K+1} \alpha_n(K) C_n$$

with **cutoff-dependent** coefficients

$$\alpha_n(K) = \begin{cases} 2\sum_{k=n-1}^{K} \frac{S_1(k,n-1)}{k!k} & \text{for } n \text{ even,} \\ 0 & \text{for } n \text{ odd.} \end{cases}$$

Similar (but more complicated) expressions exists for the Rényi entanglement entropies, but appear to be unnecessary: the simpler series converges for $\alpha >= 2$.

The Still Clever, but Correct, Derivation (Cont.)

To rewrite $\mathcal{S}$ in terms of cumulants, introduce cutoff K and switch the order of sums. After some algebra we get (HFS et *al.* '11)

$$\mathcal{S} = \lim_{K\to\infty} \sum_{n=1}^{K+1} \alpha_n(K) C_n$$

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Entanglement Spectrum from Rényi Entropies

Recall the Rényi entanglement entropies

$$\mathcal{S}_\alpha = \frac{1}{1-\alpha} \ln[\mathrm{Tr}(\rho^\alpha)].$$

Define

$$R_\alpha = \mathrm{Tr}(\rho^\alpha) = e^{(1-\alpha)S_\alpha}.$$

Note that $R_1 = 1$ for a properly normalized density matrix.

Entanglement Spectrum from Rényi Entropies (Cont.)

Define the $D \times D$ matrix

$$E = \begin{pmatrix} 1 & 1 & 0 & \cdots & \\ R_2 & 1 & 2 & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ R_{D-1} & R_{D-2} & \cdots & 1 & D-1 \\ R_D & R_{D-1} & \cdots & R_2 & 1 \end{pmatrix},$$

i.e., a quasi-lower triangular matrix with $R_1 = 1$ on the main diagonal, R_2 on the sub-diagonal, R_n on the $(n-1)$-th sub-diagonal, $1, 2, 3, \ldots, D-1$ on the super-diagonal, and zero everywhere else.

Entanglement Spectrum from Rényi Entropies (Cont.)

Newton-Girard formulas

The zeros of the polynomial

$$P(x) = \sum_{n=0}^{D} \frac{(-1)^n}{n!} (\det E_n) x^{D-n},$$

with the understanding that $\det E_0 = 1$, are the entanglement spectrum.

Entanglement Spectrum from Rényi Entropies (Cont.)

Define the $D \times D$ matrix

$$E = \begin{pmatrix} 1 & 1 & 0 & \cdots & \\ R_2 & 1 & 2 & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ R_{D-1} & R_{D-2} & \cdots & 1 & D-1 \\ R_D & R_{D-1} & \cdots & R_2 & 1 \end{pmatrix},$$

i.e., a quasi-lower triangular matrix with $R_1 = 1$ on the main diagonal, R_2 on the sub-diagonal, R_n on the $(n-1)$-th sub-diagonal, $1, 2, 3, \ldots, D-1$ on the super-diagonal, and zero everywhere else.

Entanglement Spectrum from Rényi Entropies (Cont.)

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Entanglement Spectrum from Rényi Entropies (Cont.)

Simple examples

For a pure state $R_2 = R_3 = \cdots = R_D = 1$ so

$$E = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 \\ 1 & 1 & 1 & 3 & 0 \\ 1 & 1 & 1 & 1 & 4 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \qquad P(x) = x^{D-1}(x-1).$$

For the fully mixed state where ρ has $1/D$ on the diagonal and zero everywhere else

$$R_n = D^{1-n} \implies P(x) = \left(x - \frac{1}{D}\right)^D.$$

Application: Spin-1/2 XX Chain

Hamiltonian

$$\hat{H}_{XX} = \sum_i J_i(\hat{S}_i^x \hat{S}_{i+1}^x + \hat{S}_i^y \hat{S}_{i+1}^y).$$

Becomes a model of non-interacting fermions through the Jordan-Wigner transformation.

- The pure case $J_i = J$ is the standard spin-1/2 XX chain.
- Can also describe hard-core bosons in an optical lattice; experimentally relevant (Bakr '09).
- The random singlet phase for J_i random.

Application: Spin-1/2 XX Chain (Cont.)

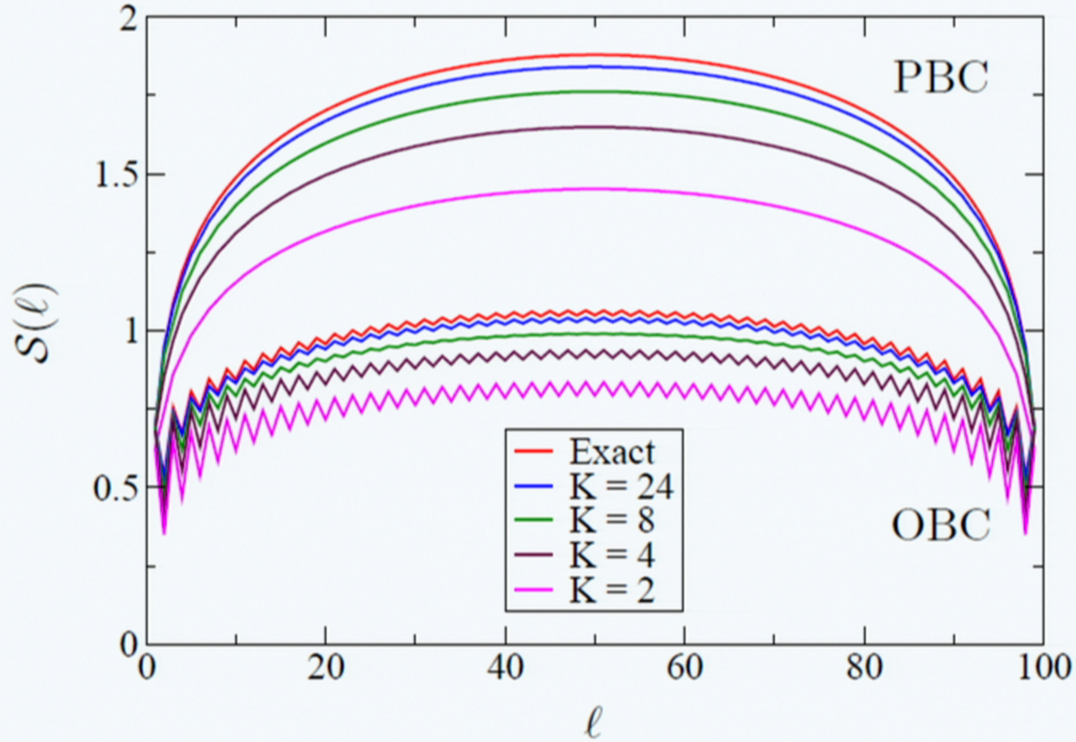

Figure: Entanglement entropy of the spin-1/2 XX chain, $L = 100$, as a function of subsystem size ℓ, with periodic boundary conditions (PBCs) and open boundary conditions (OBCs).

Application: Spin-1/2 XX Chain (Cont.)

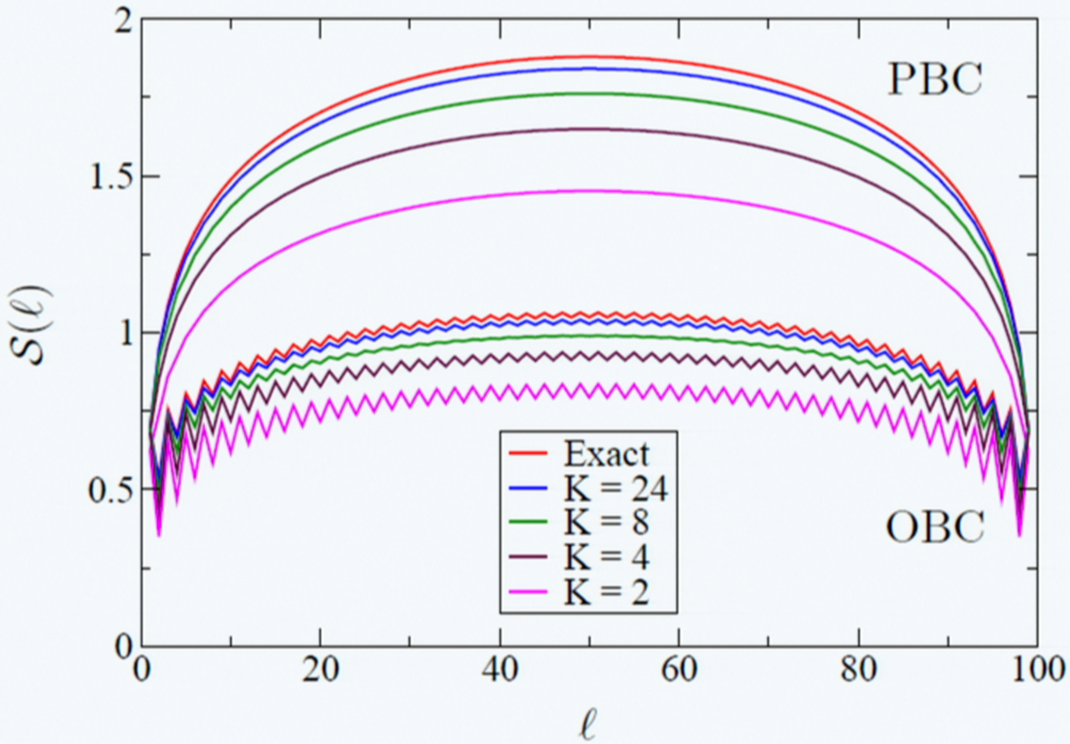

Figure: Entanglement entropy of the spin-1/2 XX chain, $L = 100$, as a function of subsystem size ℓ, with periodic boundary conditions (PBCs) and open boundary conditions (OBCs).

Application: Spin-1/2 XX Chain (Cont.)

Analytical formula for entanglement entropy

For PBCs the problem can be formulated in terms of Toeplitz matrices (Jin & Korepin '04). For OBCs form only conjectured (Calabrese et *al.* '10):

$$\mathcal{S}_{\mathrm{PBC}}(\ell) = \frac{1}{3}\log_2 \ell + s_1,$$

$$\mathcal{S}_{\mathrm{OBC}}(\ell) = \frac{1}{2}\mathcal{S}_{\mathrm{PBC}}(\ell) + a_1 \frac{1}{(2\ell)} - a_2 \frac{(-1)^\ell}{(2\ell)}.$$

Here $s_1 \simeq 1.047$. Let $\ell \to (L/\pi)\sin(\pi\ell/L)$ for finite size L.

Application: Spin-1/2 XX Chain (Cont.)

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Here $s_1 \simeq 1.047$. Let $\ell \to (L/\pi)\sin(\pi\ell/L)$ for finite size L.

Application: Spin-1/2 XX Chain (Cont.)

Analytical formulas for the fluctuations $\mathcal{F} = C_2$ (HFS et *al.* '10)

For PBCs the problem can be formulated in terms of Toeplitz matrices. For OBCs (and PBCs) the problem turns into the summation over the spin-spin correlation function:

$$\mathcal{F}_A = \sum_{i,j \in A} (\langle \hat{S}_i^z \hat{S}_j^z \rangle - \langle \hat{S}_i^z \rangle \langle \hat{S}_j^z \rangle).$$

The result is

$$\pi^2 \mathcal{F}_{\text{PBC}}(\ell) = \ln \ell + f_1 + O(\ell^{-2}),$$

$$\pi^2 \mathcal{F}_{\text{OBC}}(\ell) = \frac{1}{2}\pi^2 \mathcal{F}_{\text{PBC}}(2\ell) + \frac{1}{2}\frac{1}{(2\ell)} - [\ln(2\ell) + \gamma + \ln 2]\frac{(-1)^\ell}{(2\ell)} + \frac{(-1)^\ell}{(2\ell)^2}\ln(2\ell) + O(\ell^{-2}).$$

Here $f_1 = 1 + \gamma + \ln 2 \simeq 2.270$. Let $\ell \to (L/\pi)\sin(\pi\ell/L)$ for finite size L.

Application: Spin-1/2 XX Chain (Cont.)

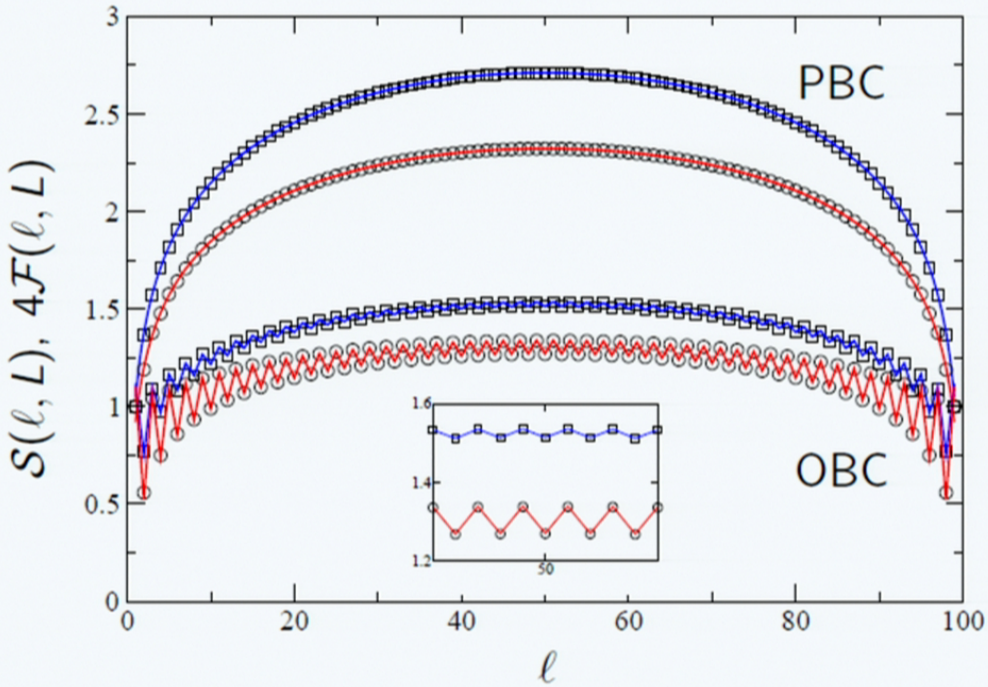


Figure: Entanglement entropy (squares) and fluctuations $\mathcal{F} = C_2$ (circles) of the spin-1/2 XX chain, $L = 100$, as a function of subsystem size ℓ, with periodic boundary conditions (PBCs) and open boundary conditions (OBCs).

Application: Random Singlet Phase (RSP)

The spin-1/2 XX chain with J_i drawn from almost any probability distritution flows to the infinite randomness fixed point where the ground state is a pure valence bond state called the random singlet phase. The entanglement entropy was computed to be (Refael & Moore '04)

$$\mathcal{S}_{\text{RSP}}(\ell) = \bar{n} \ln 2, \qquad \bar{n} \sim \frac{1}{3} \ln \ell.$$

Here $\bar{n}$ is the number of singlets that cross the boundary, averaged over realizations of the disorder.

The corresponding cumulant generating function is

$$\overline{\ln \chi(\lambda)} = \bar{n} \ln \cos \frac{\lambda}{2},$$

leading to the same result through cumulants. **The relation between $\mathcal{S}$ and the C_n is linear, so it can be averaged over disorder.**

Application: QPC (Cont.)

Zero bias

No voltage bias, imperfect transmission D:

$$\chi(\lambda) = e^{-\lambda_*^2 G/(2\pi^2)}, \qquad \sin\frac{\lambda_*}{2} = \sqrt{D}\sin\frac{\lambda}{2},$$

with $G = \ln(t/\tau)$, τ is a short-time cutoff. Gaussian for $D = 1$.

Finite bias

Voltage bias V, imperfect transmission D:

$$\chi(\lambda) = [1 + D(e^{i\lambda} - 1)]^{eVt/h}$$

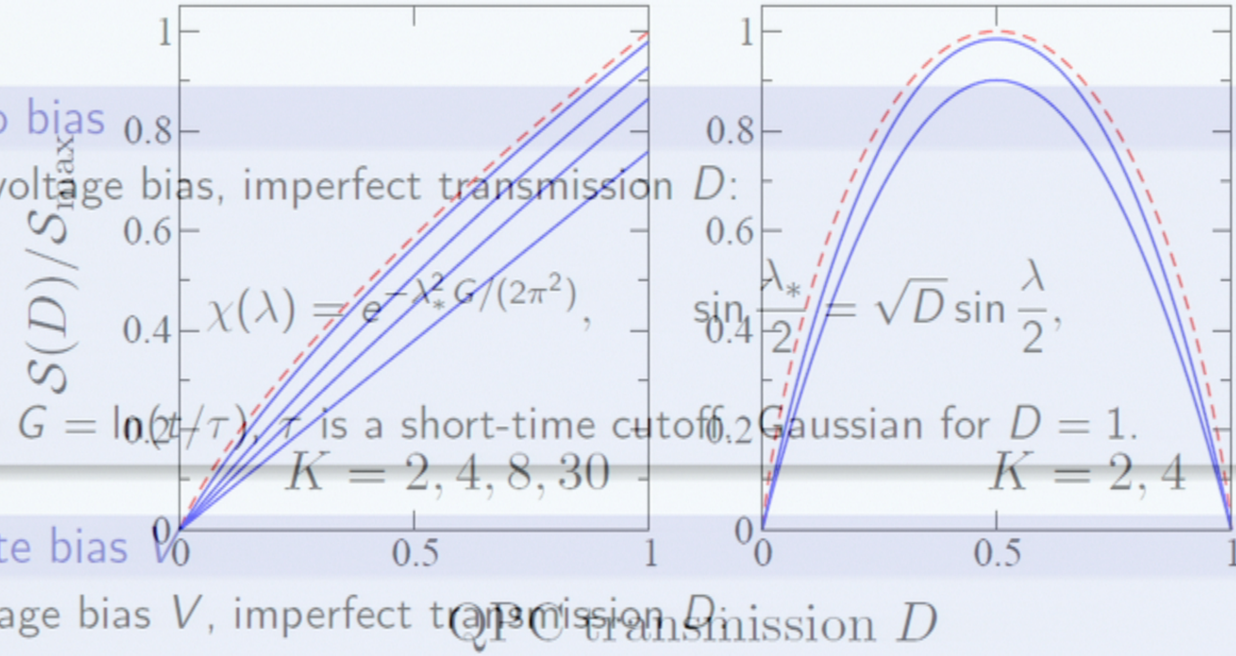


Figure: Entanglement entropy in a QPC with imperfect transmission D at zero bias voltage (left) and bias voltage V (right), scaled to the maximum value at $D = 1$ and $D = 0.5$, respectively.

Application: QPC (Cont.)

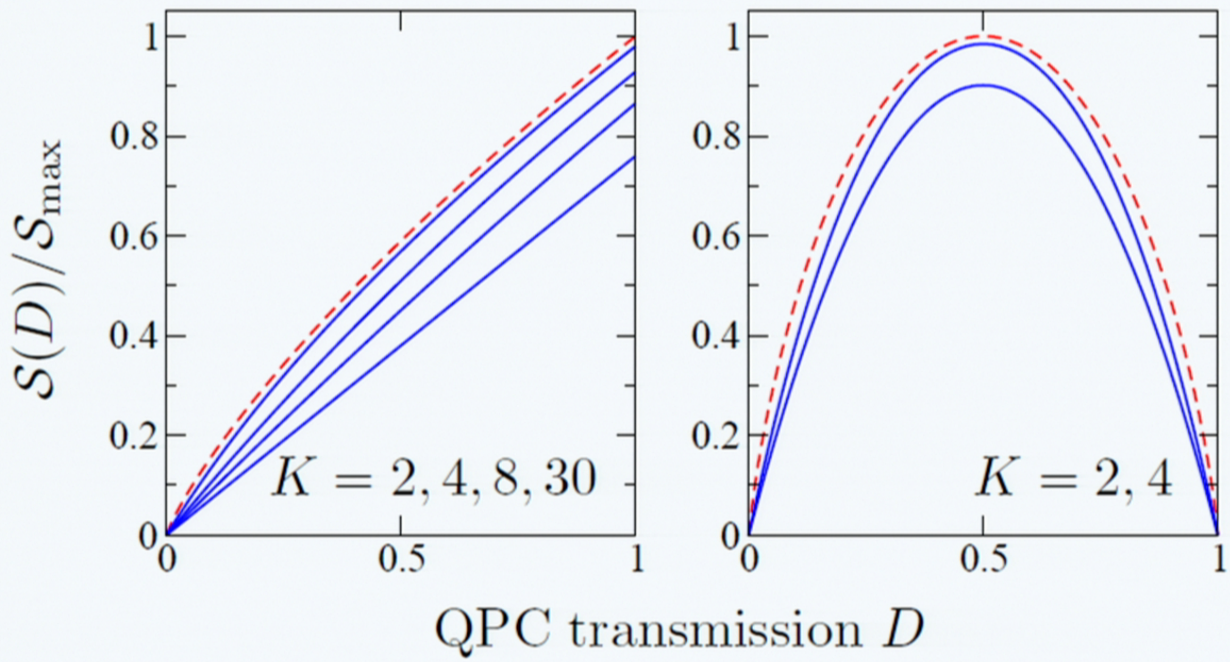


Figure: Entanglement entropy in a QPC with imperfect transmission D at zero bias voltage (left) and bias voltage V (right), scaled to the maximum value at $D = 1$ and $D = 0.5$, respectively.

Application: Free Fermions in 2D

Cumulants can reproduce the $\mathcal{S} \sim L^{d-1} \ln L$ scaling of the entanglement entropy in d-dimensions, but this is already well-documented (see below, however). How about a system that obeys a strict area law?

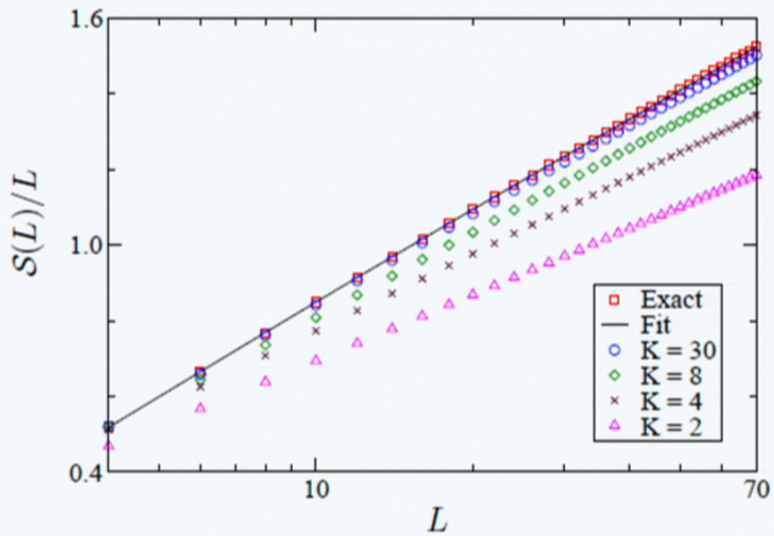


Figure: Entanglement entropy for free fermions in two dimensions.

Application: Integer Quantum Hall Effect (IQHE)

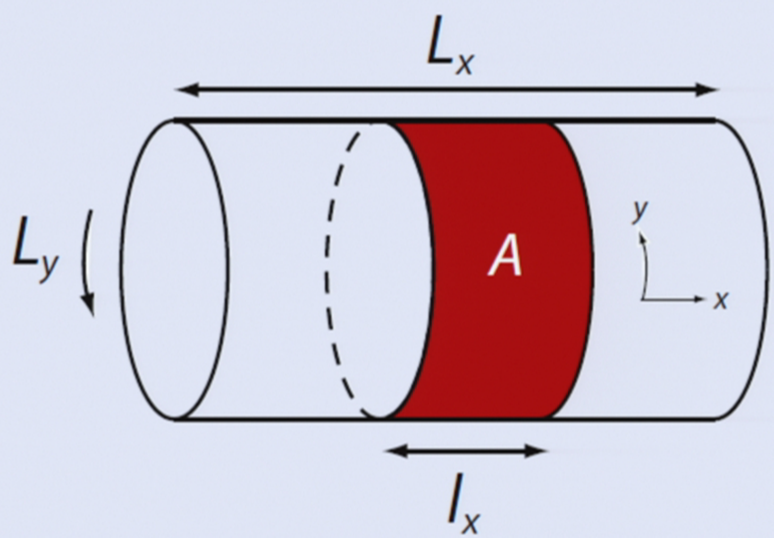
Setup (Rodríguez & Sierra '09)

Cylinder of size $L_x \times L_y$, periodic in y-direction, with vector potential $\mathbf{A} = B(0, x)$. For unit filling $\nu = 1$ the ground state correlation matrix is (assume $L_x, L_y \gg 1$, set magnetic length $\ell_B = 1$)

$$M_{\mathbf{rr}'} = \frac{1}{2\pi} \exp\left[-\frac{1}{4}(x - x')^2 - \frac{1}{4}(y - y')^2 - \frac{i}{2}(x + x')(y - y') \right].$$

Define region A as

$$-\frac{\ell_x}{2} \leq x \leq \frac{\ell_x}{2}$$
$$0 \leq y \leq L_y$$



Application: IQHE (Cont.)

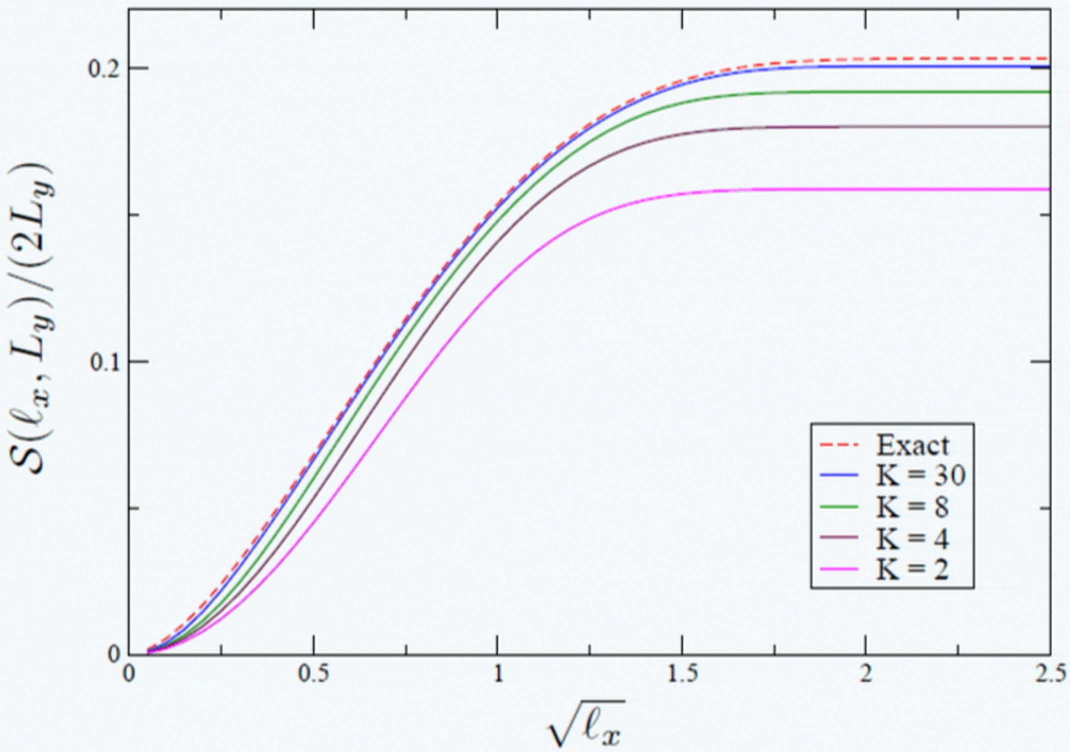


Figure: Von Neumann entanglement entropy of the IQHE in the cylinder geometry at filling factor $\nu = 1$.

Application: IQHE (Cont.)

Recall the close relation between the IQHE to topological insulators.

In principle, anything the entanglement entropy/spectrum can do, the cumulants can also do!

Beyond Free Fermions

For a single fermion the real-space entanglement entropy is just the probability of finding the particle in region A.

This is no longer true for interacting systems, but some interesting parallels exist between the entanglement entropy and particle number fluctuations in one-dimensional systems. We will focus on

$$\mathcal{F}_A = C_2 = \langle \hat{N}_A^2 \rangle - \langle \hat{N}_A \rangle^2.$$

Luttinger liquids (LLs)

LLs describe the low-energy physics of many one-dimensional systems: interacting fermions and bosons, the spin-1/2 XXZ chain, the edge theory of the $\nu = 1/(2p+1)$ fractional quantum Hall effect, ...

The LL Hamiltonian is a Gaussian model:

$$H_{\mathrm{LL}} = \frac{v}{2\pi} \int dx \left[K(\partial_x \theta)^2 + \frac{1}{K} (\partial_x \phi)^2 \right].$$

Luttinger Liquids (Cont.)

Fluctuations

The long-wavelength density fluctuations are given by

$$\rho(x) = \rho_0 + \frac{1}{\pi}\partial_x \phi(x),$$

so for a block of length ℓ extending from $x = 0$ to $x = \ell$

$$\hat{N}_A - \langle \hat{N}_A \rangle = \frac{1}{\pi}[\phi(\ell) - \phi(0)].$$

Standard LL calculation (e.g., in Giamarchi '04) gives

$$\pi^2 \mathcal{F}(\ell) = K \ln \frac{\ell}{a}, \qquad \ell \gg a,$$

where a is a short-distance cutoff.

Luttinger Liquids (Cont.)

Oscillating correction

We can also account for oscillating corrections. For the gapless phsae of the spin-1/2 XXZ chain

$$\hat{H}_{\mathrm{XXZ}} = \sum_i (\hat{S}_i^x \hat{S}_{i+1}^x + \hat{S}_i^y \hat{S}_{i+1}^y + \Delta \hat{S}_i^z \hat{S}_{i+1}^z).$$

Focus on $0 \leq \Delta \leq 1$.

$$\pi^2 \mathcal{F}_{\mathrm{XXZ}}(\ell) = \underbrace{K \ln \ell}_{\text{from } 1/r^2 \text{ term}} + \underbrace{f_2}_{\text{from all terms}} - \underbrace{A_2 \frac{(-1)^\ell}{\ell^{2K}}}_{\text{from } (-1)^r/r^{2K} \text{ term}} + O(\ell^{-2}).$$

The long-distance behavior of the correlation function is dominated by the oscillating term ($1/2 \leq K \leq 1$), but the logarithmic divergence originates from the $1/r^2$ term—importance of **short-distance correlations**.

Luttinger Liquids (Cont.)

This is a useful way (quick and doesn't require computing the correlation function) to extract K in DMRG.

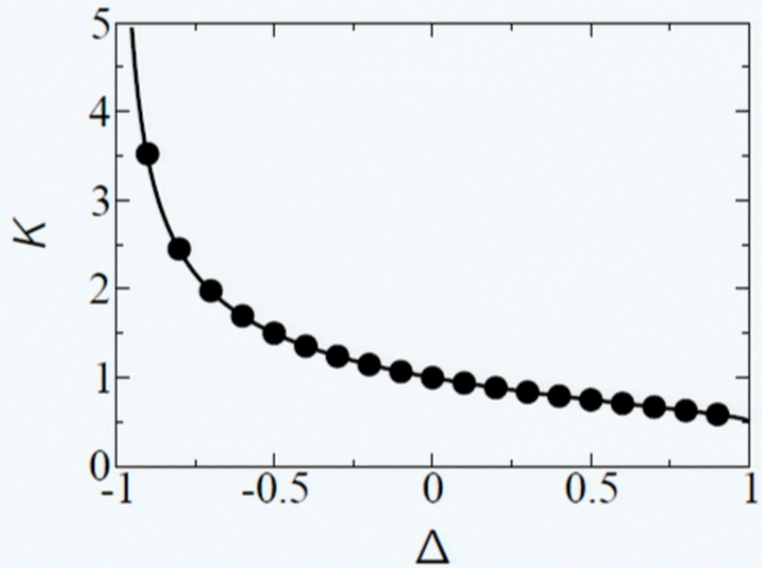


Figure: Luttinger parameter K as a function of anisotropy Δ, extending into the ferromagnetic regime. Solid line is the Bethe Ansatz curve $K = (1/2)[1 - (\cos^{-1}\Delta)/\pi]^{-1}$.

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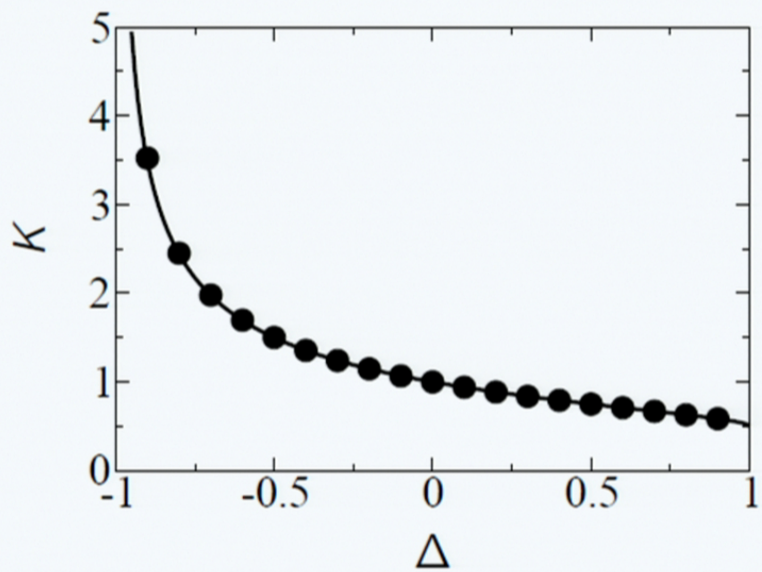

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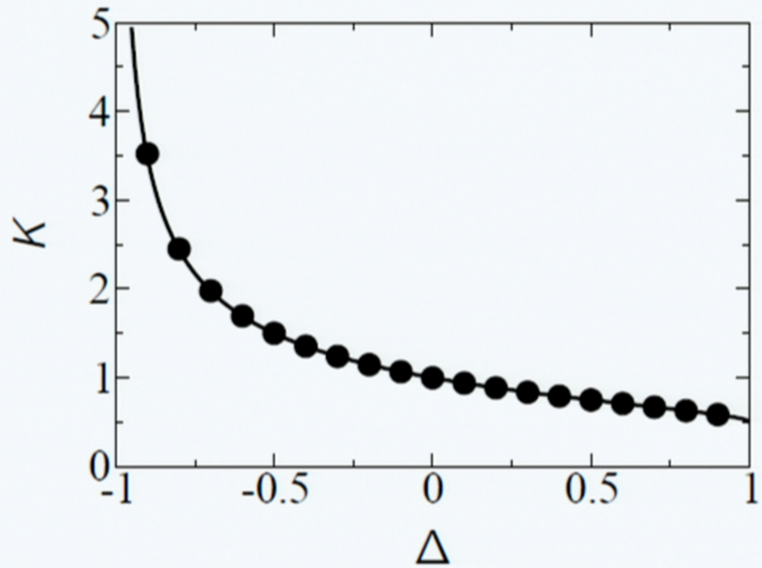

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Haldane-Shastry (HS) Model

Actually, for $K = 1$ (isotropic Heisenberg point) the oscillating correction acquires a multiplicative logarithmic correction: Study the HS chain (same universality class = $SU(2)_1$ Wess-Zumino-Witten nonlinear σ-model) instead:

$$\hat{H}_{\mathrm{HS}} = \sum_{i<j} \frac{1}{d(i-j)^2} \hat{\mathbf{S}}_i \cdot \hat{\mathbf{S}}_j, \qquad d(x) = \frac{L}{\pi} \left| \sin \frac{\pi x}{L} \right| .$$

The exact spin-spin correlation function is

$$\langle \hat{S}^z_{i+r} \hat{S}^z_i \rangle - \langle \hat{S}^z_{i+r} \rangle \langle \hat{S}^z_i \rangle = \frac{1}{4} (-1)^r \frac{\mathrm{Si}(\pi r)}{\pi r}, \qquad \mathrm{Si}(x) = \int_0^x dt\, \frac{\sin t}{t},$$

so

$$\pi^2 \mathcal{F}_{\mathrm{HS}}(\ell) = \frac{1}{2} \ln \ell + f_{\mathrm{HS}} - \frac{\pi^2}{16} \frac{(-1)^\ell}{\ell} + O(\ell^{-2}),$$

where $f_{\mathrm{HS}}/\pi^2 \simeq 0.197$.

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Haldane-Shastry Model (Cont.)

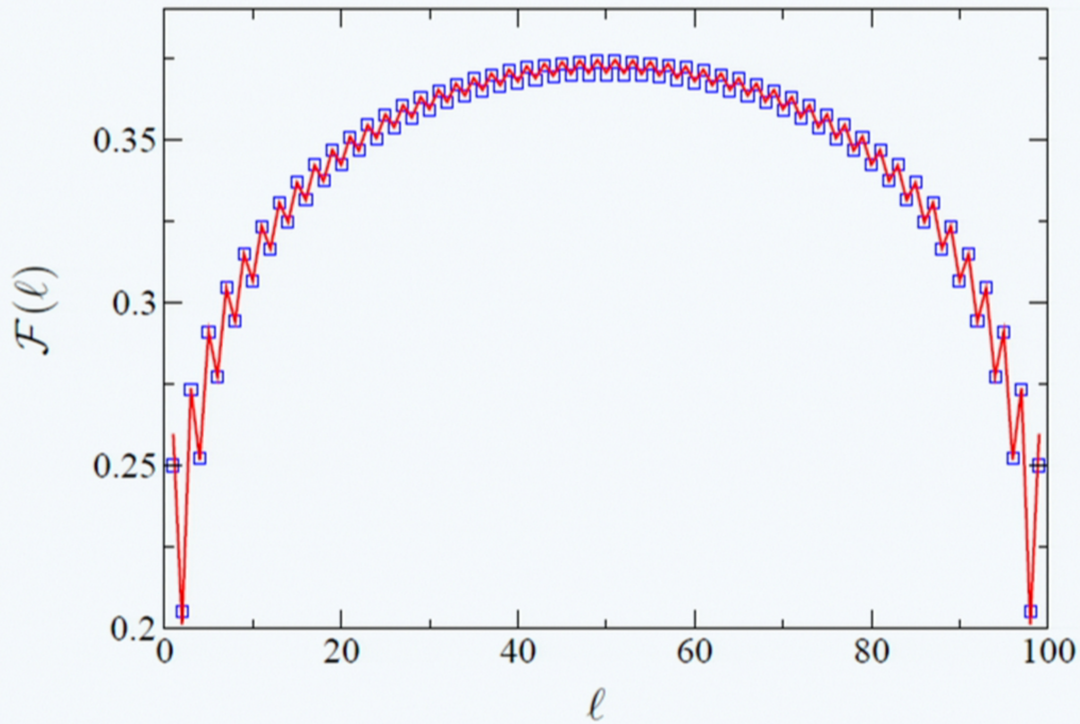


Figure: Fluctuations of the HS model, $L = 100$, as a function of subsystem size.

Bose-Hubbard Model (Cont.)

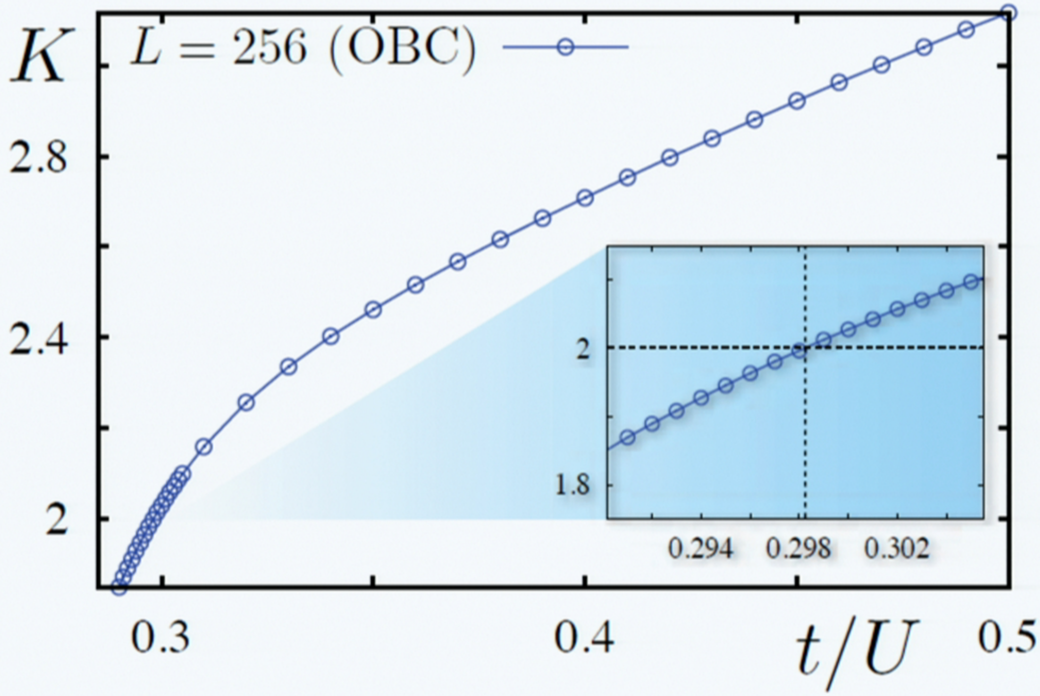
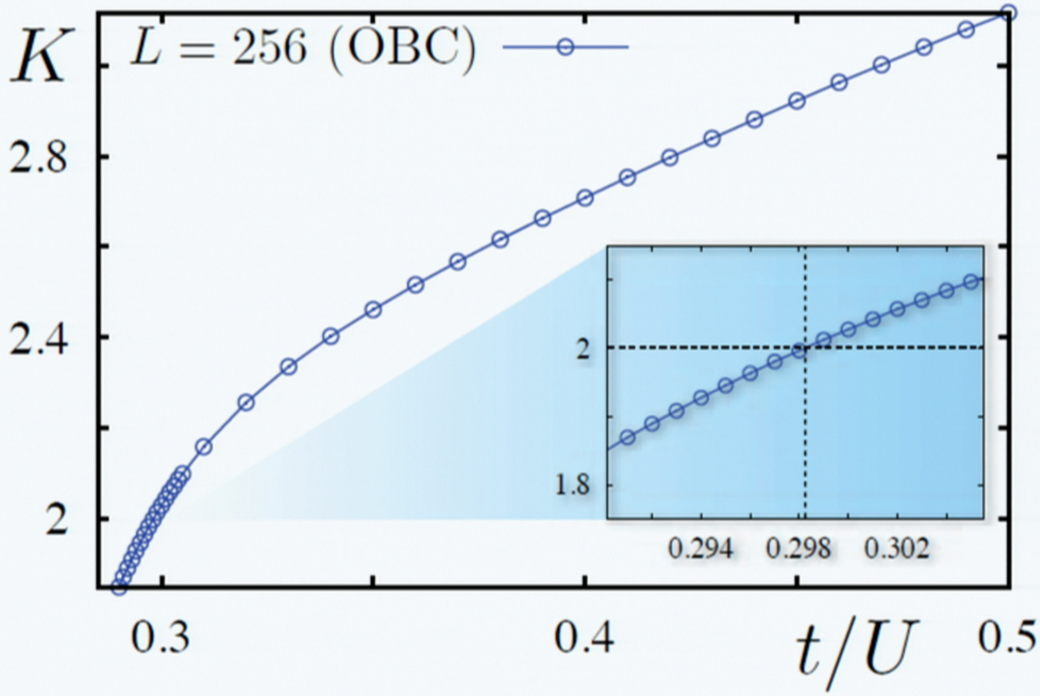


Figure: Using the Luttinger parameter K extracted from the fluctuations to locate the phase transition at $t/U \simeq 0.298$, cf. previous estimate. Thanks to Stephan Rachel for the nice figure.

Bose-Hubbard Model (Cont.)



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LLs and CFT

CFT argument

A more general way to look at the LL result is to consider a conserved U(1) charge in a CFT, which is generated by a free boson. The generating function is then known as a "vertex operator" and given by

$$\chi(\lambda) = \langle e^{i\lambda \hat{N}_A} \rangle = \left(\frac{x}{a}\right)^{-g\lambda^2/(2\pi^2)} .$$

Therefore

$$\pi^2 \mathcal{F}(\ell) = \pi^2 (-i\partial_\lambda)^2 \ln \chi(\lambda)|_{\lambda=0} = g \ln \frac{\ell}{a}.$$

What is g?

LLs and CFT (Cont.)

CFT argument: Fixing the prefactor

g is fixed by the specific meaning of the U(1) charge, but there is a simple, heuristic way to fix it. Thanks to CFT, at finite temperature $1/\beta$

$$\pi^2 \mathcal{F}(x,\beta) = g \ln\left(\frac{\beta}{\pi a} \sinh \frac{\pi x}{\beta}\right).$$

For $x \gg \beta$ we have standard thermodynamic relation (replace compressibility κ with susceptibility $\chi = \partial m/\partial B$ for spins)

$$\mathcal{F}(x,\beta) \sim \frac{\kappa x}{\beta}, \qquad \kappa = \frac{\partial n}{\partial \mu}.$$

Matching for $x \gg \beta, a$ gives (v now inserted for dimensional correctness)

$$g = \pi v \kappa.$$

Compare to LL result $K = \pi v \kappa$ (Haldane '81).

LLs and CFT (Cont.)

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Compare to LL result $K = \pi v \kappa$ (Haldane '81).

LLs and CFT (Cont.)

CFT argument

So,

$$\frac{\mathcal{S}(\ell)}{\mathcal{F}(\ell)} \sim \frac{c}{\pi v \kappa} \frac{\pi^2}{3}, \qquad x \gg a,$$

which "generalizes" the non-interacting fermion result in the Gaussian limit. But, not clear whether the fluctuations fully account for the entanglement entropy.

Disjoint Intervals

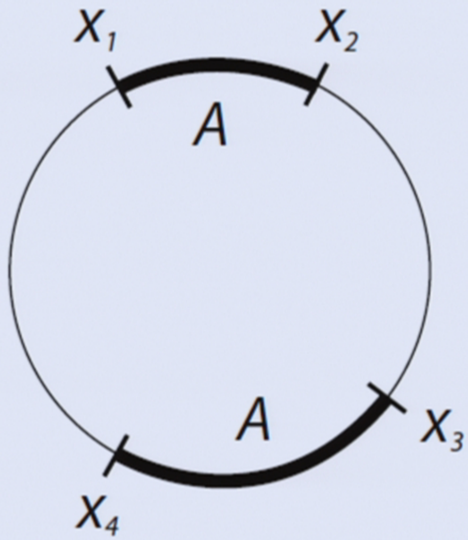
Entanglement entropy of a free boson

For two intervals it was initially believed (Cardy & Calabrese '04)

$$\mathcal{S}_A(x_1, x_2, x_3, x_4) = \frac{1}{3} \ln \frac{x_{12} x_{34} x_{14} x_{23}}{x_{13} x_{24} a^2}.$$

The corresponding result for fluctuations is

$$\pi^2 \mathcal{F}(x_1, x_2, x_3, x_4) = K \ln \frac{x_{12} x_{34} x_{14} x_{23}}{x_{13} x_{24} a^2}.$$



Nice, but the entanglement entropy is wrong! The relevant Riemann surface is non-trivial and cannot be treated simply within CFT. The correct answer is quite complicated (Calabrese et *al.* '09).

Higher Dimensions: The Spin-1/2 Heisenberg Antiferromagnet on the Square Lattice

Hamiltonian

$$\hat{H}_{\mathrm{AFHM}}(h) = \frac{1}{2}\sum_{ij} J_{ij}\hat{\mathbf{S}}_i \cdot \hat{\mathbf{S}}_j - h\sum_i (-1)^{|i|}\hat{S}_i^z,$$

where J_{ij} is the symmetric matrix with $J > 0$ if sites i, j are nearest neighbors, zero otherwise. $(-1)^{|i|} = 1$ on one sublattice (A) and -1 on the other sublattice (B); h is a staggered magnetic field needed to regularize the zero mode, but we are ultimately interested in $h \to 0$.

Goldstone boson $\Longrightarrow$ Area law for entanglement entropy

Higher Dimensions: The Spin-1/2 Heisenberg Antiferromagnet on the Square Lattice (Cont.)

The entanglement entropy and fluctuations can be computed within modified spin-wave theory with staggered field.

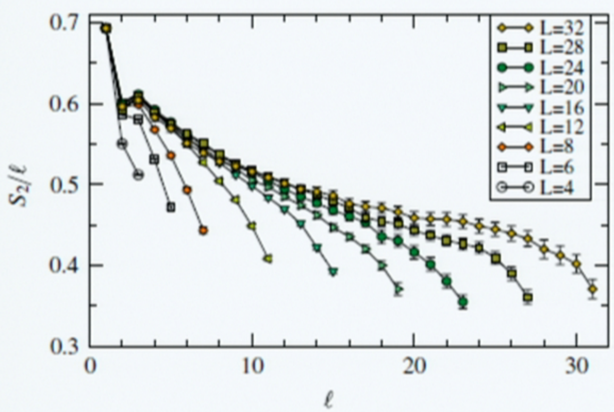


Figure: Rényi entropy $\mathcal{S}_2$ from QMC (Hastings et *al.* '10)

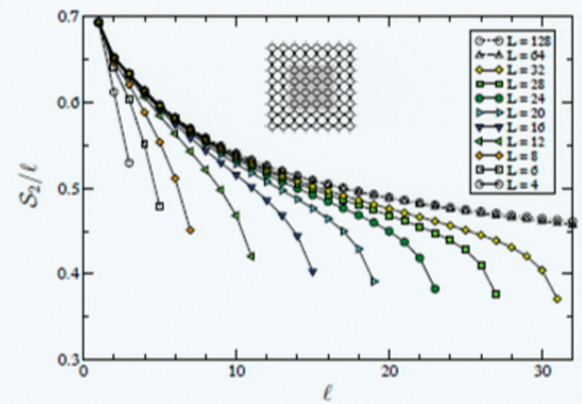


Figure: Rényi entropy $\mathcal{S}_2$ from spin-wave theory (HFS et *al.* '11)

Higher Dimensions: The Spin-1/2 Heisenberg Antiferromagnet on the Square Lattice (Cont.)

The fluctuations have a **multiplicative** logarithmic correction.

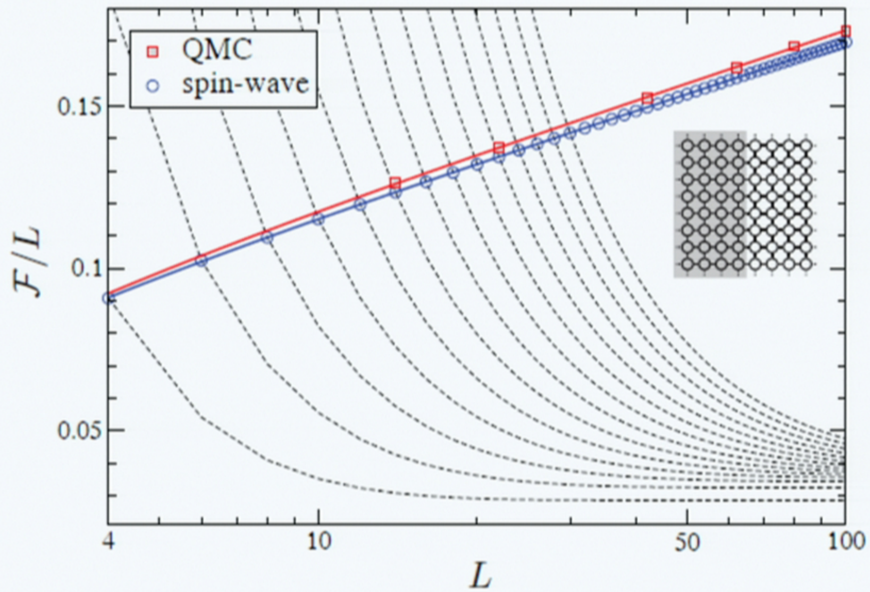
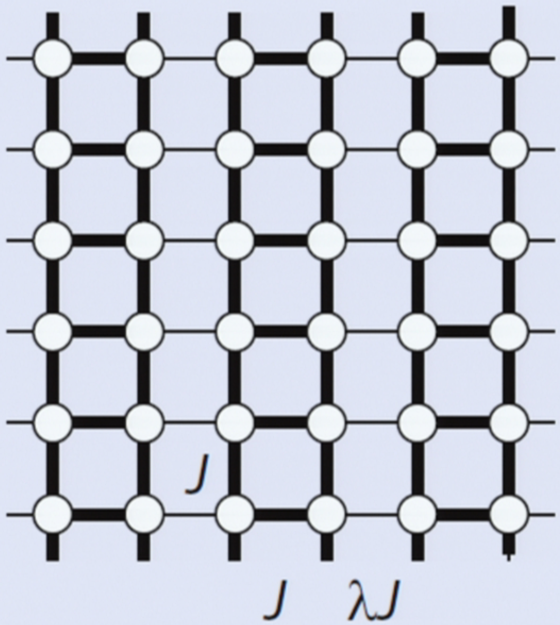


Figure: Fluctuations $\mathcal{F}$, with QMC result superimposed. Dashed lines show calculation at fixed staggered field.

Higher Dimensions: Coupled Ladders

Coupled Ladders

Consider the slightly generalized case where some of the couplings are λJ, $0 \leq \lambda \leq 1$ so that we effectively have a system of coupled ladders:



Higher Dimensions: Coupled Ladders (Cont.)

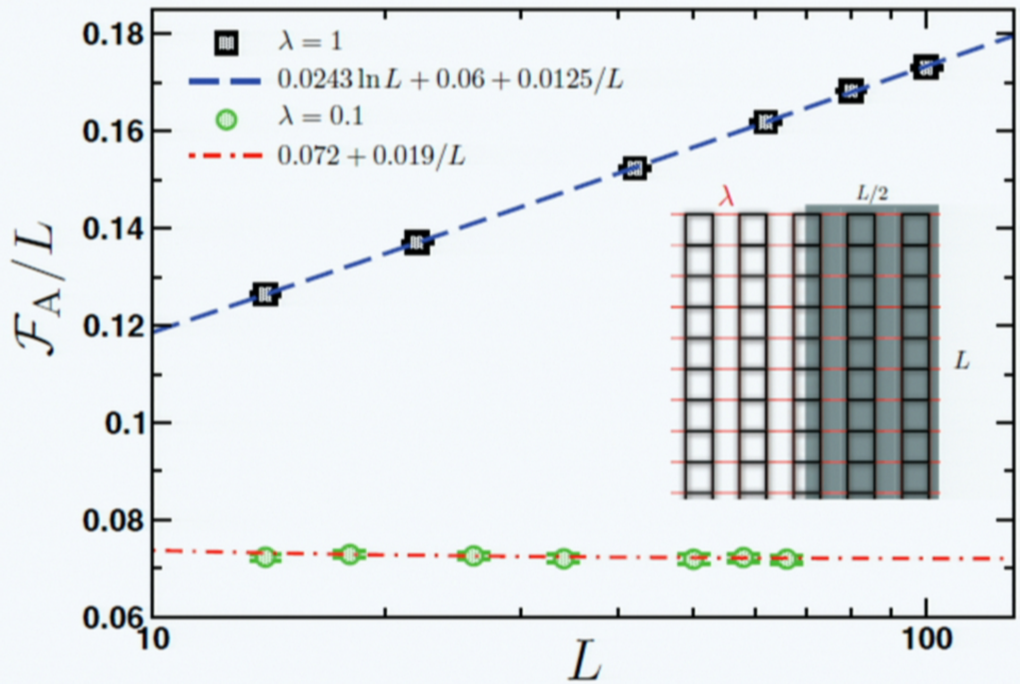


Figure: Comparison of the scaling of fluctuations in the dimerized and Néel phases. Thanks to Nicolas Laflorencie for the nice figure.

Higher Dimensions: Coupled Ladders (Cont.)

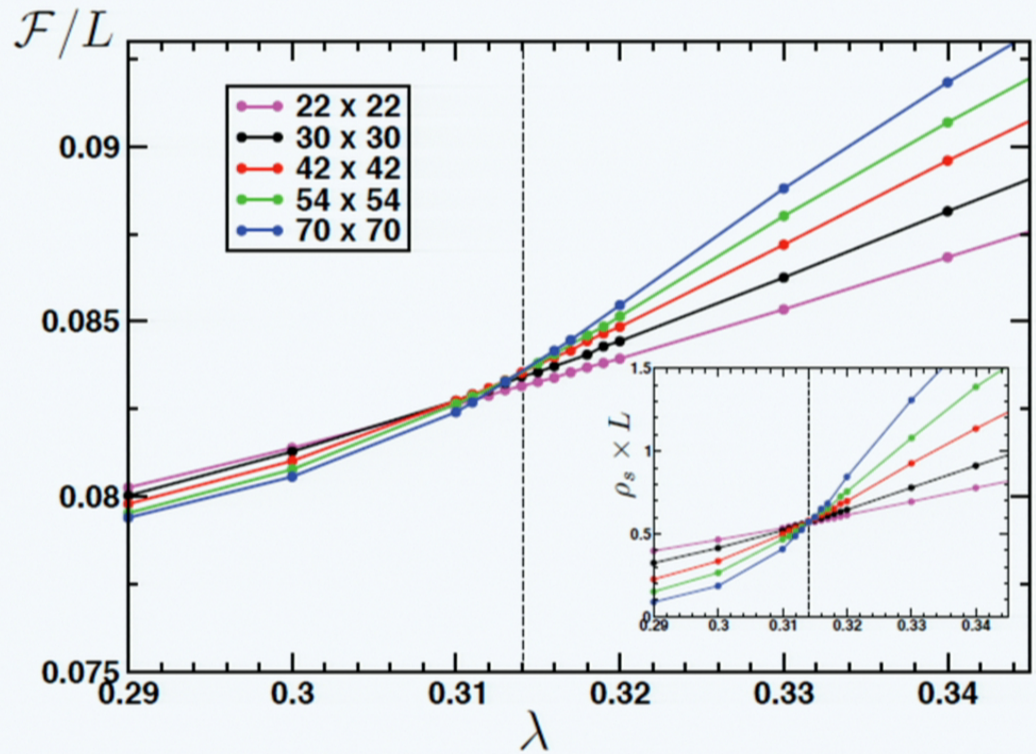


Figure: Using the scaling of fluctuations to locate the Néel-dimerized transition in coupled ladders. Thanks to Nicolas Laflorencie for these preliminary results.

Gapped systems

Haven't really talked much about gapped systems, but the fluctuations are generally expected to obey a strict area law due to exponentially decaying correlations—it would be nice to prove this rigorously. The prototypical system in 1D is the spin-1 Affleck-Kennedy-Lieb-Tasaki (AKLT) chain.

Conclusions

- Studying many-body physics from the quantum information perspective has yielded many interesting results.
- We can relate the new results to the more conventional idea of fluctuations, especially for non-interacting fermions. This also provides a way to measure entanglement entropy.
- Fluctuations are interesting to study in their own right. FCS already appears in a natural way, but fluctuations in the ground state of many-body Hamiltonians yields useful information.