Title: Canonical Time Evolution in Simplicial Gravity

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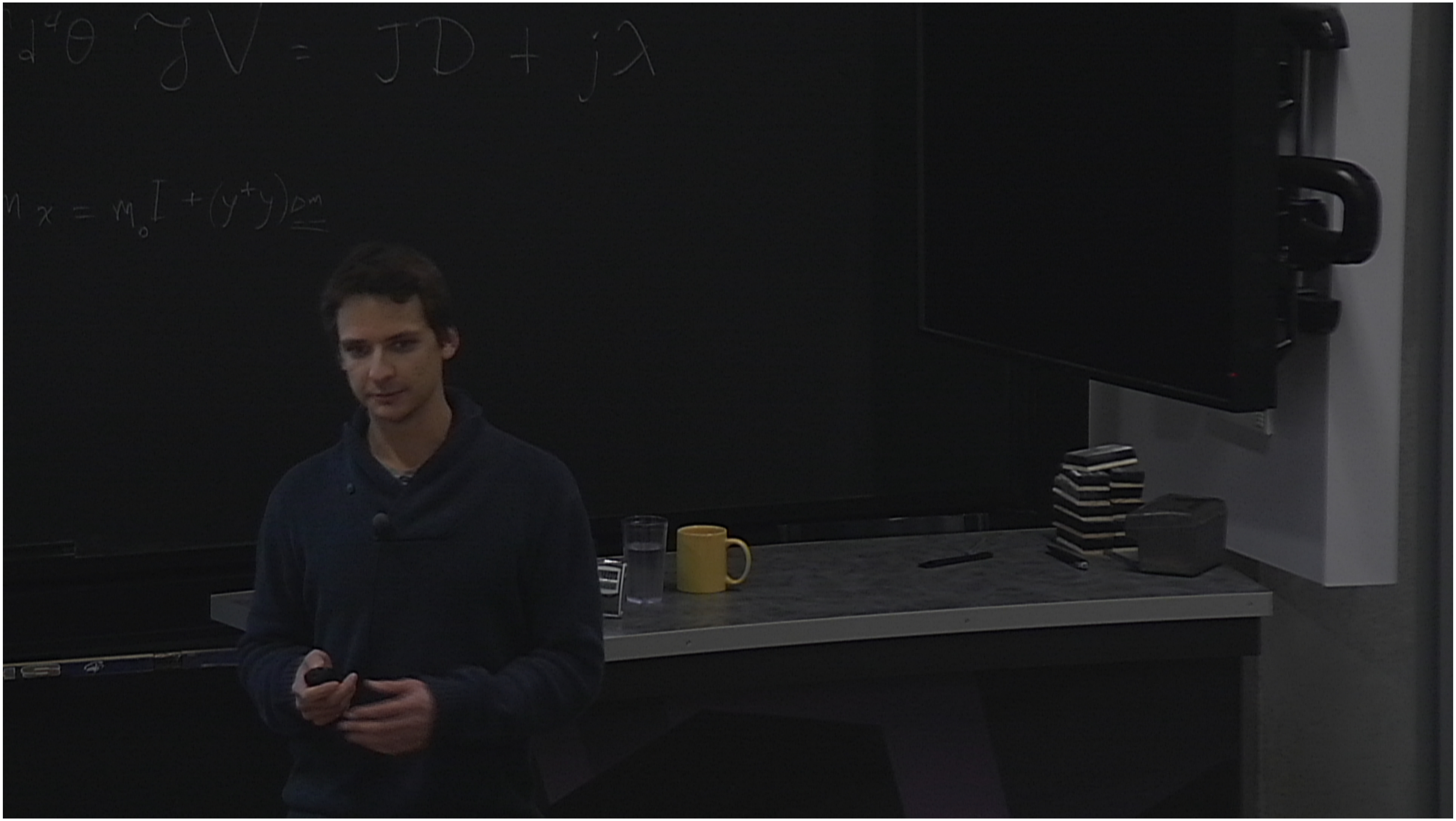
Abstract:


d$^4\theta$ JV = JD + jλ
m x = m$_0$ I + (y$^+$y)$_{DM}$

Canonical time evolution in simplicial gravity

Philipp Höhn

ITF, Universiteit Utrecht

Perimeter Institute Seminar
December 7th, 2011

based on B. Dittrich, PH arXiv:1108.1974, 0912.1817 and *wip*
(summary PH 1110.3947)

Plan of the talk

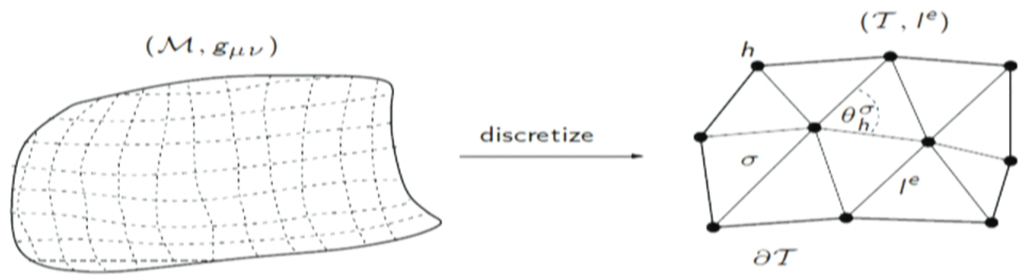
1. Motivation for canonical dynamics
2. Discrete evolution in simplicial gravity
3. Canonical discrete dynamics
4. Constraints and symmetries
5. Conclusions and challenges

Discretizing General Relativity: Recap of Regge Calculus

- Regge Calculus [Regge '61]: replace smooth D–dim. spacetime $(\mathcal{M}, g_{\mu\nu})$ by piecewise-linear flat metric living on triangulation $\mathcal{T}$, comprised of D–simplices σ

h: 'hinge' ($(D-2)$–subsimplex)
θ_h^σ: interior dihedral angle at h in σ
A_h: volume of h
$\epsilon_h := 2\pi - \sum_{\sigma \supset h} \theta_h^\sigma$: deficit angle

$\psi_h := \pi - \sum_{\sigma \supset h} \theta_h^\sigma$: exterior angle



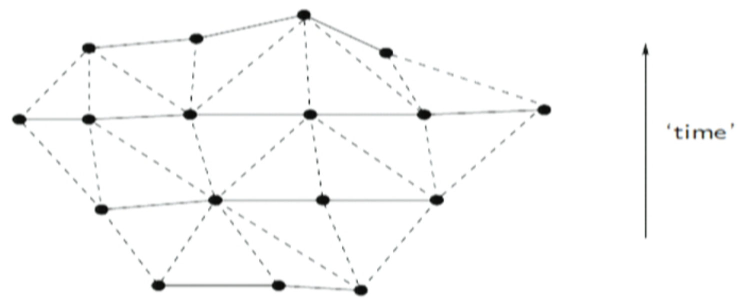
- configuration variables: edge lengths $\{l^e\}_{e\in\mathcal{T}}$, encode complete geometry
- (Euclidean) action $S_{EH} = -\frac{1}{2}\int_{\mathcal{M}} \sqrt{g} R d^4x \underset{\text{discretize}}{\longrightarrow} S_R$

$$S_R(\{l^e\}) = -\sum_{h\subset\mathcal{T}\setminus\partial\mathcal{T}} A_h \epsilon_h - \sum_{h\subset\partial\mathcal{T}} A_h \psi_h \qquad \Rightarrow \qquad S_R \text{ additive}$$

Goal

Canonical time evolution for triangulations

- reproduce covariant solutions (get dynamics)
- challenge: problem of foliations
  - hypersurfaces of different numbers of σ's which carry variables $\Rightarrow$ need mapping between phase spaces of different dimension



- How to treat general situation where lattice evolves/changes? (as in LQG)
  - Numbers of degrees of freedom may vary

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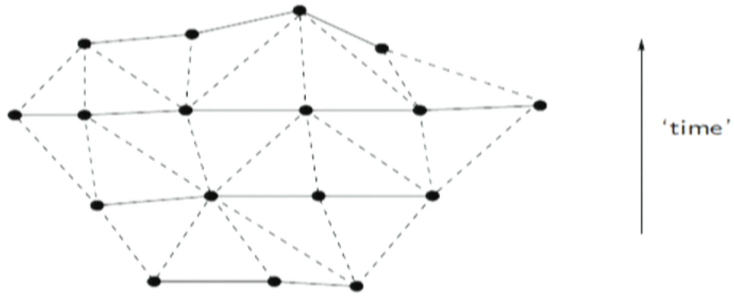

- How to treat general situation where lattice evolves/changes? (as in LQG)
  - Numbers of degrees of freedom may vary

Why bother to construct general canonical formalism?

1. classically: understand discrete dynamics and capture full space of (discrete) solutions
2. possibly advantageous for several physical situations (e.g. expanding/contracting universe) and their numerical implementation
3. in Quantum gravity
   - linking covariant and canonical approaches to quantum gravity (LQG vs. Spin Foams, etc....)
   - understand Hamiltonian dynamics (geometrically) in regularized theory
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Continuum vs. discrete

requirement: equivalence of canonical and covariant formalism

⇒ discrete time evolution

⇒ generated by set of evolution moves, not constraints

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discrete

- at the outset $\mathcal{T} = \mathcal{I} \times \Sigma$
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- evolution not necessarily hyperbolic

Algorithm to generate D–solutions

(i) choose canonical initial data on $(D-1)$–triangulated hypersurface
(ii) choose set of evolution moves
(iii) choose sequence of chosen evolution moves (and perform)
(iv) make sure constraints always satisfied (if constraints violated, choose different sequence...)
$\Rightarrow$ end result: D–Regge solution

nontrivial task:

- constraints always satisfied (these implement equations of motion)

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Further choices

1. fix evolution steps (ii) and sequences (iii) at the outset

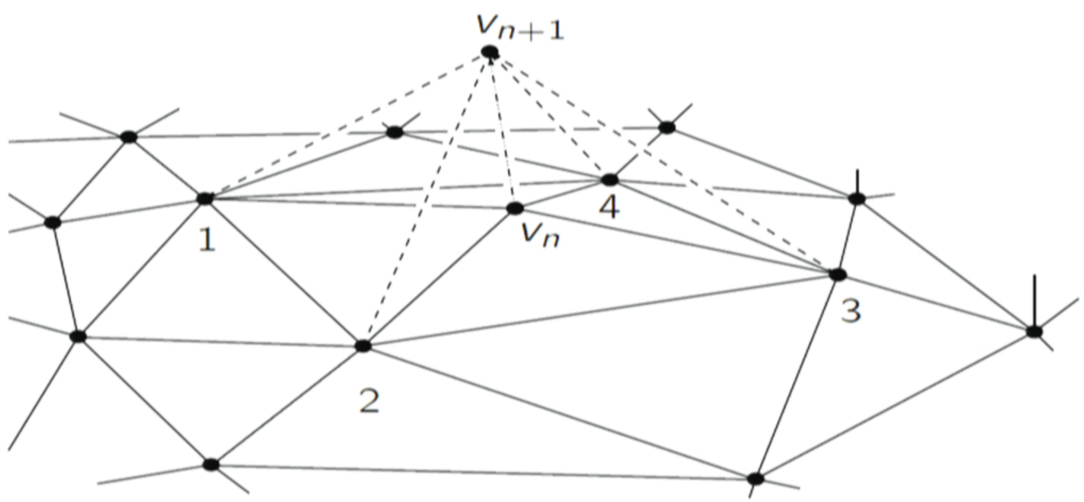
   $\Rightarrow$ restrict space of solutions arising from initial data (e.g. tent moves....)

2. choose general set of evolution moves under (ii) and leave sequence (iii) open

   $\Rightarrow$ generate all triangulations arising from given initial data

   $\Rightarrow$ allow for full freedom in evolution

Example option 1: tent moves [Barrett, Galassi, Miller, Sorkin, Tuckey, Williams '97; Bahr, Dittrich '09, Dittrich, PH '09]



$\Rightarrow$ recover standard phase space picture (preserved spatial triangulations)

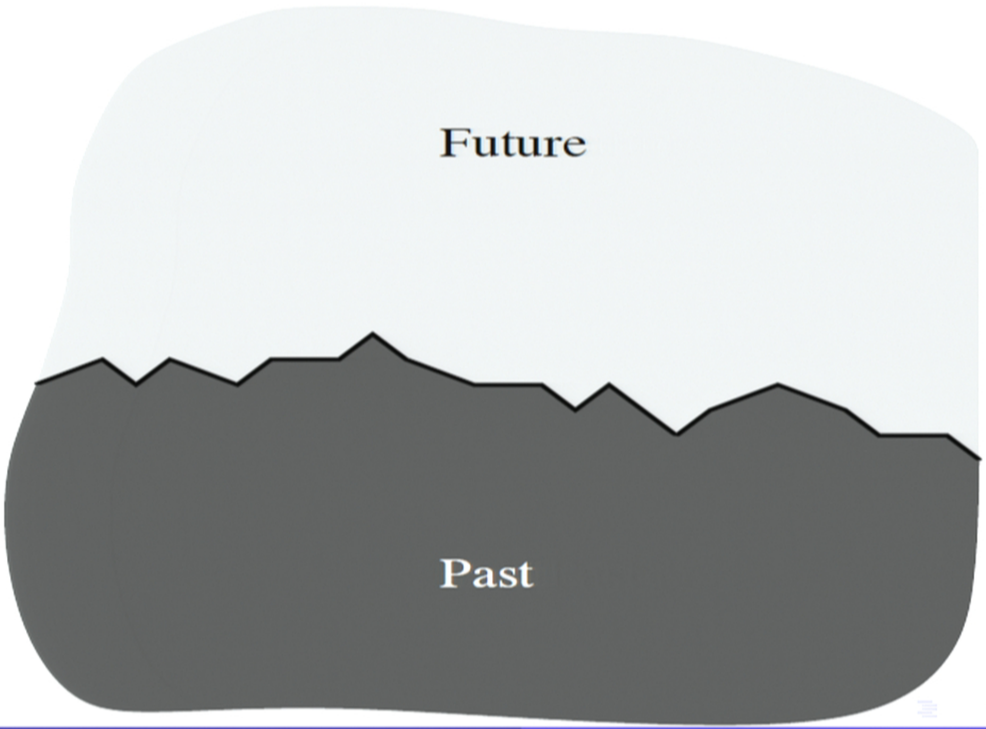
[Dittrich, PH '09]

Option 2: evolution in discrete 'multi-fingered'

step n

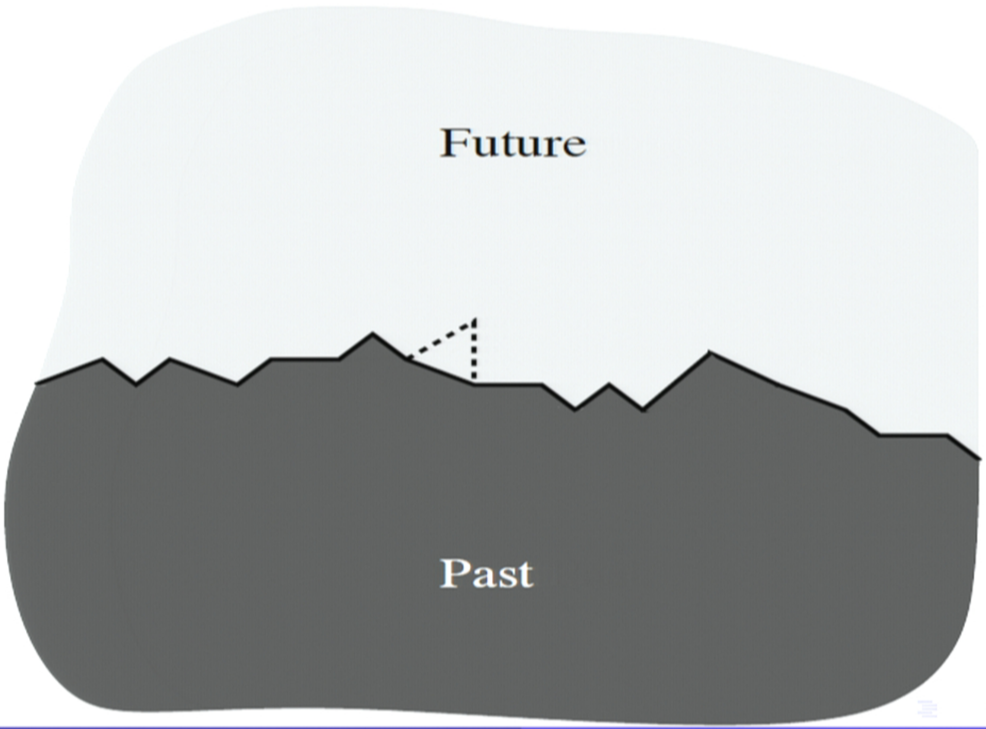
Idea:
glue single D-simplex, to $D-1$-dimensional triangulated hypersurface Σ_n at each elementary step counted by $n \in \mathbb{Z}$

$\Rightarrow$ requires action to be additive



Option 2: evolution in discrete 'multi-fingered'

step $n+4$



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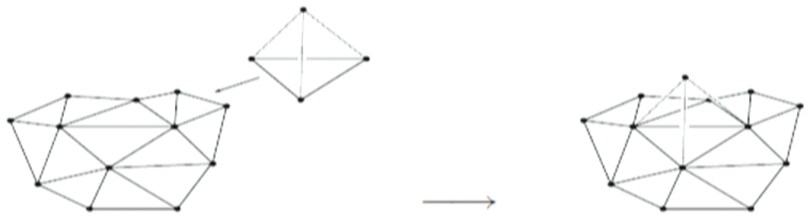
Interpretation within $D-1$ hypersurface: $D-1$ Pachner moves

3D Example: gluing of tetrahedron onto single triangle

3D perspective:



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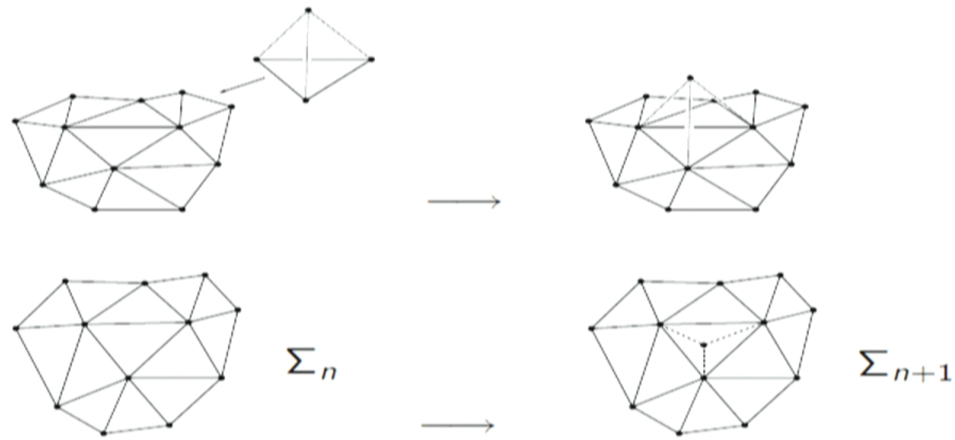
3D perspective:

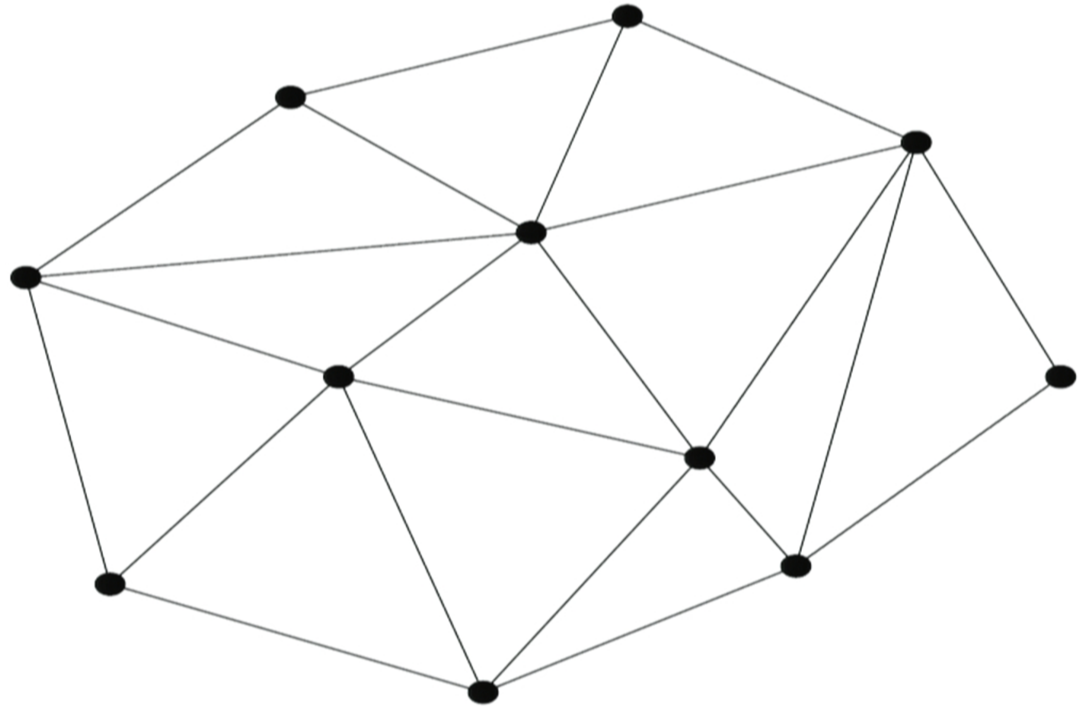
2D perspective:



$\Rightarrow$ 1–3 Pachner move (other Pachner moves in 3D and 4D similarly)
$\Rightarrow$ Pachner moves [Pachner '86] are local, ergodic and topology preserving
$\Rightarrow$ $\mathcal{T} = \mathcal{I} \times \Sigma$ (as in canon. GR)

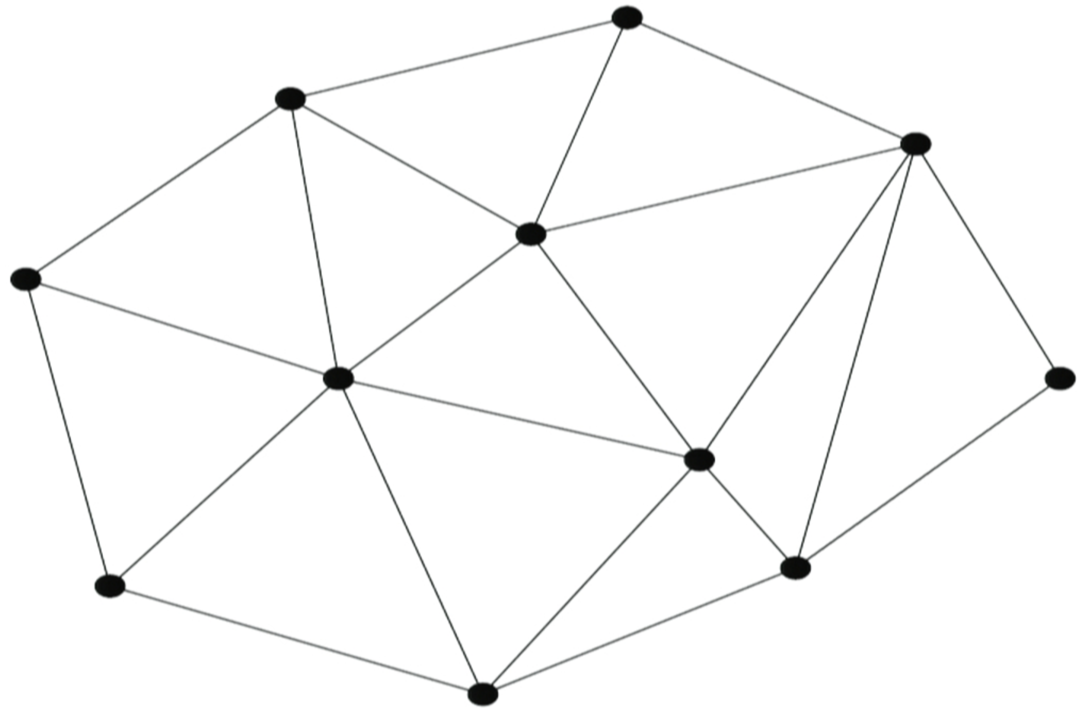
Local evolution of the hypersurface with Pachner moves

step n



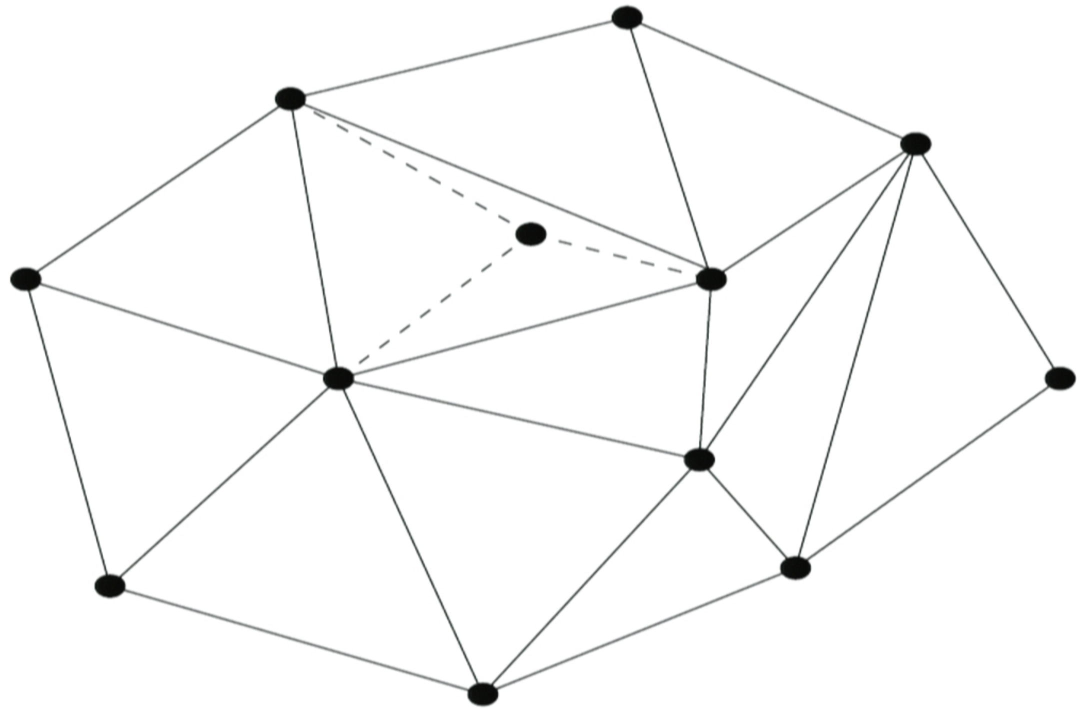
Local evolution of the hypersurface with Pachner moves

step $n+1$

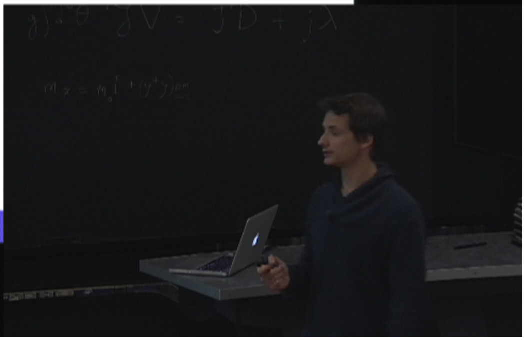


Local evolution of the hypersurface with Pachner moves

step $n + 6$



How can one implement such evolution moves in a canonical language?



P. Höhn (ITF Utrecht) Canonical simplicial gravity

Canonical discrete dynamics

- central idea: use Hamilton's principal fct. $\tilde{S}(x_{ini}, x_{fin})$ as generating fct. of canonical time evolution
- assume additivity of action (A, B some 'regions')

$$S(\{x\}_{A\cup B}) = S(\{x\}_A) + S(\{x\}_B)$$

$\Rightarrow$ convolution

$$\tilde{S}(x_{ini}, x_{fin}) = \text{extr}_{x_{inter}} \left[\tilde{S}(x_{ini}, x_{inter}) + \tilde{S}(x_{inter}, x_{fin})\right]$$

$\{x_{ini}, x_{inter}\}$, $\{x_{inter}, x_{fin}\}$ boundary variables of A, B

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Canonical momenta [Marsden, West '01; Gambini, Pullin '03; Dittrich, PH '09, '11]

- discrete action $S = \sum_{n=1}^{N} S_n(x_{n-1}, x_n)$

  $S_n : \mathcal{Q}_{n-1} \times \mathcal{Q}_n \longrightarrow \mathbb{R}$

- in cont. $L : T\mathcal{Q} \longrightarrow \mathbb{R}$, $L(q, \dot{q}) \in \mathbb{R}$

- S_n as generating fct.

$$^{-}p^{n-1} := -\frac{\partial S_n(x_{n-1}, x_n)}{\partial x_{n-1}} \quad , \quad ^{+}p^{n} := \frac{\partial S_n(x_{n-1}, x_n)}{\partial x_n}$$

  ^{-}p: pre–momenta, ^{+}p: post–momenta

- defines time evolution map

$$\mathcal{H}_n : (x_{n-1}, {}^{-}p^{n-1}) \longmapsto (x_n, {}^{+}p^{n})$$

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Momentum matching [Marsden, West '01; Gambini, Pullin '03; Dittrich, PH '09, '11]

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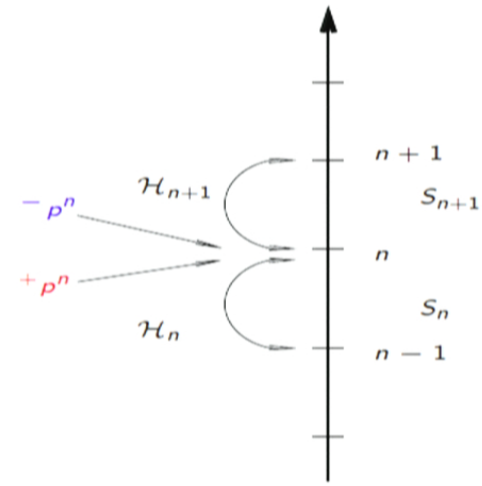

- similarly, use $S_{n+1}(x_n, x_{n+1})$ as gen. fct.

$$^{-}p^{n} = -\frac{\partial S_{n+1}}{\partial x_n}$$

- eom $\frac{\partial S_n}{\partial x_n} + \frac{\partial S_{n+1}}{\partial x_n} = 0 \Leftrightarrow {}^{+}p^{n} = {}^{-}p^{n}$
  momentum matching

Discrete Legendre transformations [Marsden, West '01; Dittrich, PH '11]

- in cont. $\mathbb{F} : T\mathcal{Q} \longrightarrow T^*\mathcal{Q}, \quad (q, \dot{q}) \mapsto (q, \frac{\partial L(q,\dot{q})}{\partial \dot{q}})$
- Now in discrete: using action S_n, define <u>two</u> Legendre transf. at each n:

pre– and post–Legendre transf.

$$\begin{aligned}
\mathbb{F}^+ S_n : \mathcal{Q}_{n-1} \times \mathcal{Q}_n &\longrightarrow T^*\mathcal{Q}_n \\
(x_{n-1}, x_n) &\mapsto (x_n, {}^+p^n) = \left(x_n, \frac{\partial S_n}{\partial x_n}\right) \\
\mathbb{F}^- S_n : \mathcal{Q}_{n-1} \times \mathcal{Q}_n &\longrightarrow T^*\mathcal{Q}_{n-1} \\
(x_{n-1}, x_n) &\mapsto (x_{n-1}, {}^-p^{n-1}) = \left(x_{n-1}, -\frac{\partial S_n}{\partial x_{n-1}}\right)
\end{aligned}$$

Constraints [Dittrich, PH '11]

- in cont. $\mathbb{F}$ not isomor. $\Leftrightarrow \det\left(\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}\right) = 0 \quad \Rightarrow$ get constraints
- Now in discrete have two Leg. transf.:

  $\mathbb{F}^{\pm} S_n$ fail to be isomorphisms $\Leftrightarrow \det\left(\frac{\partial^2 S_n}{\partial x^i_{n-1} \partial x^j_n}\right) = 0$
- Obtain two constraint surfaces:

  $\mathcal{C}^+_n := \mathrm{Im}(\mathbb{F}^+ S_n) \subset T^*\mathcal{Q}_n$: post–constraint surface

  $\mathcal{C}^-_{n-1} := \mathrm{Im}(\mathbb{F}^- S_n) \subset T^*\mathcal{Q}_{n-1}$: pre–constraint surface
- clearly, $\dim \mathcal{C}^+_n = \dim \mathcal{C}^-_{n-1}$

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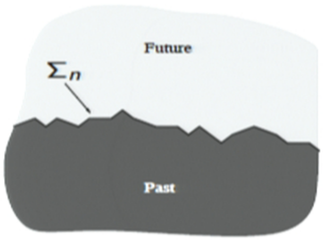
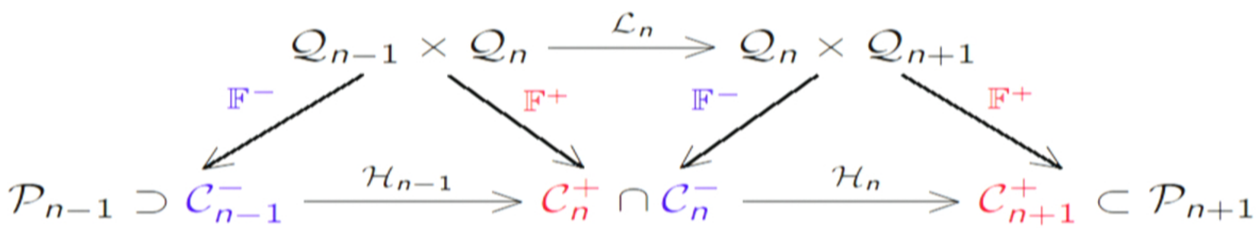
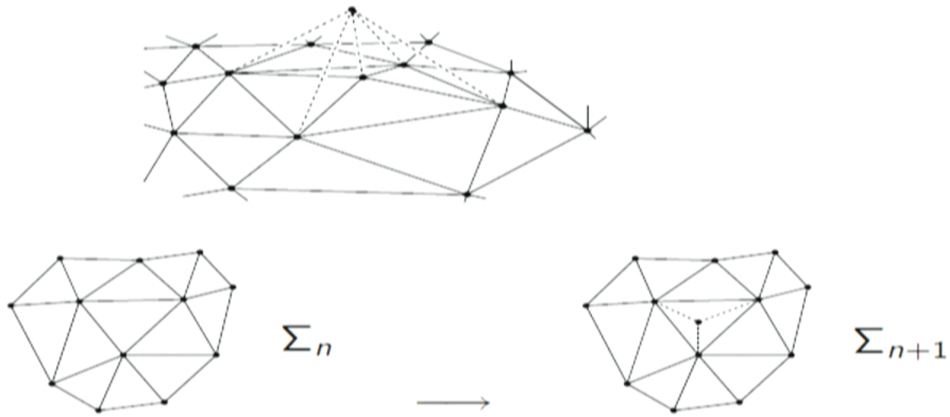
- clearly, $\dim \mathcal{C}^+_n = \dim \mathcal{C}^-_{n-1}$
- post–constraints/pre–constraints automatically satisfied by 'past'/'future' region (but not vice versa)



Diagrammatically



'Problems'



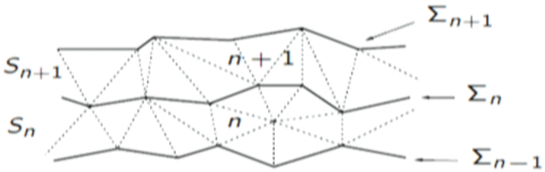
In general, face 'problems':

(a) evolution step may involve bulk variables

(b) subsets of variables coincide at different steps, i.e. $\Sigma_{n+1} \cap \Sigma_n \neq \emptyset$

(c) numbers of variables differ (phase space dim. varies) from step to step

Solve 'problem' (a)



x_n^i bulk variables at step n

- $S_n(x_n^e, x_n^i, x_{n-1}^{e'})$ as 'generating function'

$$^+p_e^n := \frac{\partial S_n}{\partial x_n^e} \qquad {}^-p_e^{n-1} := -\frac{\partial S_n}{\partial x_{n-1}^e}$$

$$^+p_i^n := \frac{\partial S_n}{\partial x_n^i} \qquad {}^-p_i^{n-1} := -\frac{\partial S_n}{\partial x_{n-1}^i} = 0$$

- momentum matching for internal variables x_n^i

$$p_i^n = 0 = {}^-p_i^n = {}^+p_i^n = \frac{\partial S_n}{\partial x_n^i}$$

constraints as equations of motion $\Rightarrow$ obtain $\tilde{S}_n(x_n^e, x_{n-1}^{e'})$

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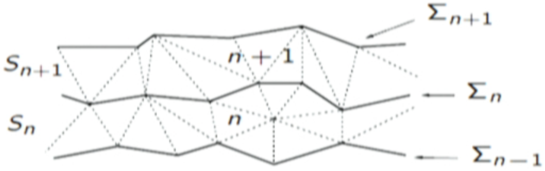

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$$^+p^n_i := \frac{\partial S_n}{\partial x^i_n} \qquad {}^-p^{n-1}_i := -\frac{\partial S_n}{\partial x^i_{n-1}} = 0$$

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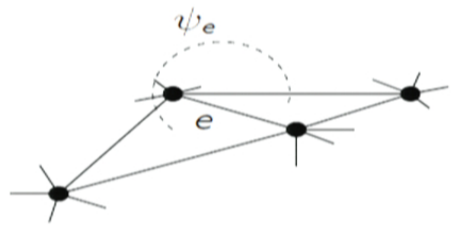
constraints as equations of motion $\Rightarrow$ obtain $\tilde{S}_n(x^e_n, x^{e'}_{n-1})$

Solve 'problem' (b): momentum updating



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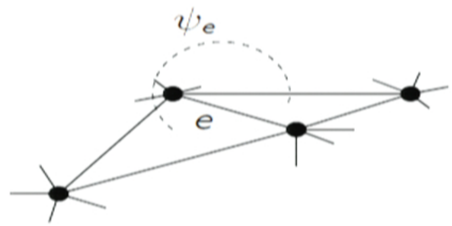

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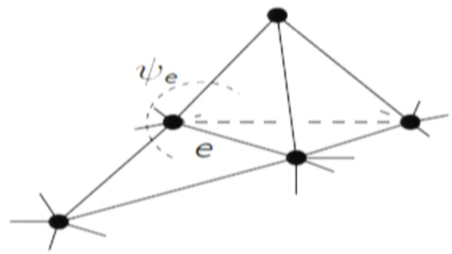

- make use of additivity of action:
  $S_n \longrightarrow S_n + S_{n+1} \Rightarrow$ *momentum updating*

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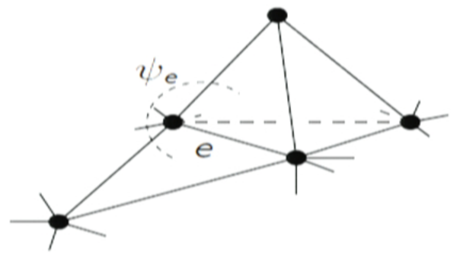

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Phase space(s) picture for Regge Calculus

$$\mathcal{P}_{sup}$$
$$\downarrow ?$$
$$\mathcal{P}_{tot} := T^*\mathcal{Q}_{tot}$$
$$\downarrow \text{`drop'}(x^e_n, p^n_e = 0)$$
$$\mathcal{P}_n := T^*\mathcal{Q}_n$$

- $\mathcal{P}_{sup}$: ‘covariant phase space’, contains information about all Regge triangulations
- $\mathcal{P}_{tot}$: ‘totally extended phase space’ ($\mathcal{Q}_{tot}$: configuration manifold of entire (fixed) 4D triangulation)
- $\mathcal{P}_n$: phase space at evolution step n

'Canonical transformations' [Dittrich, PH '11]

- time evol. map only defined between constraint surfaces $\mathcal{H}_n : \mathcal{C}^-_{n-1} \longrightarrow \mathcal{C}^+_n$

$$ {}^-p^{n-1} := -\frac{\partial S_n(x_{n-1}, x_n)}{\partial x_{n-1}} \quad , \quad {}^+p^n := \frac{\partial S_n(x_{n-1}, x_n)}{\partial x_n} $$

- Theorem : sympl. form restricted to constraint surfaces preserved under $\mathcal{H}_n$ and momentum updating

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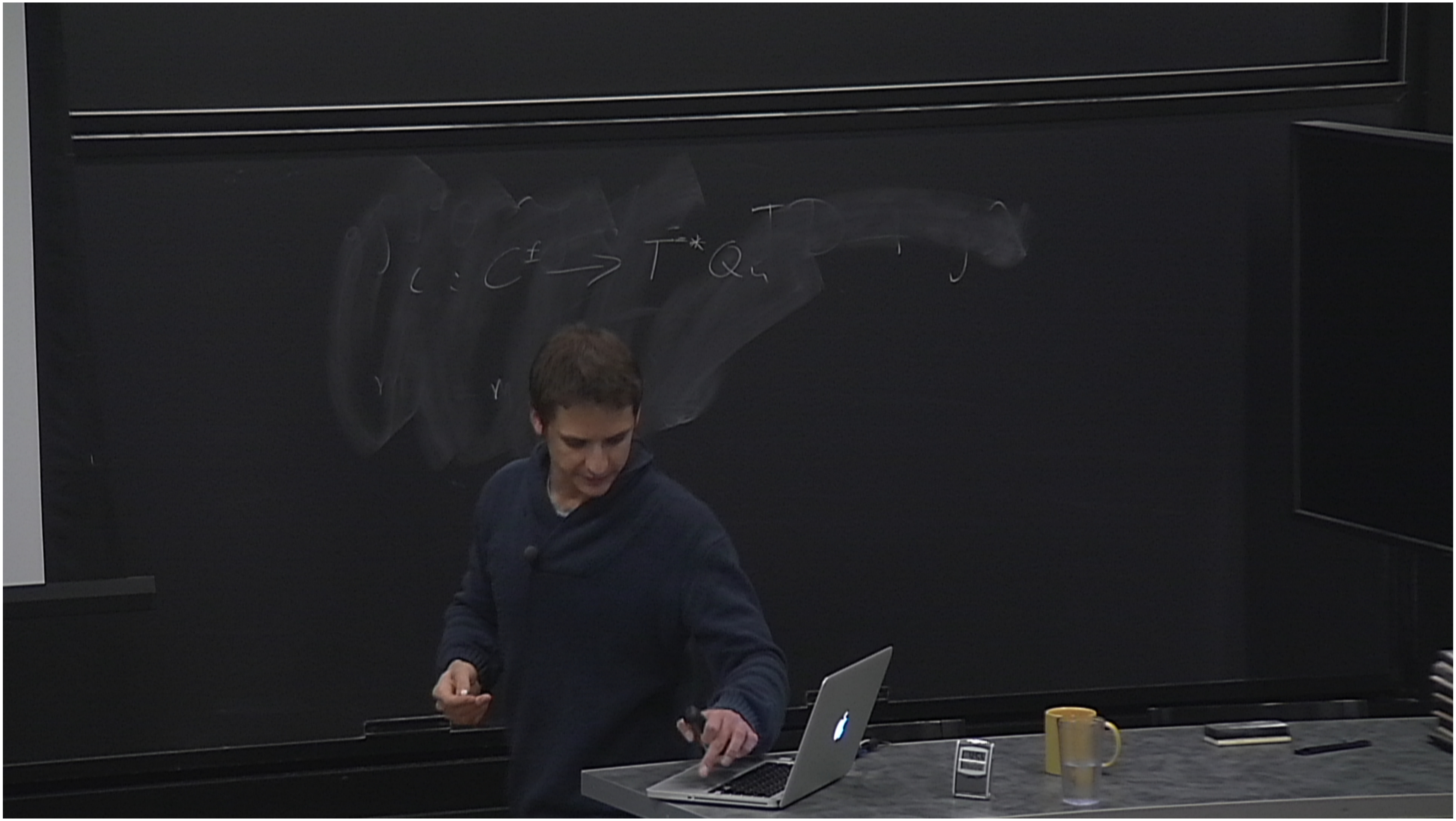

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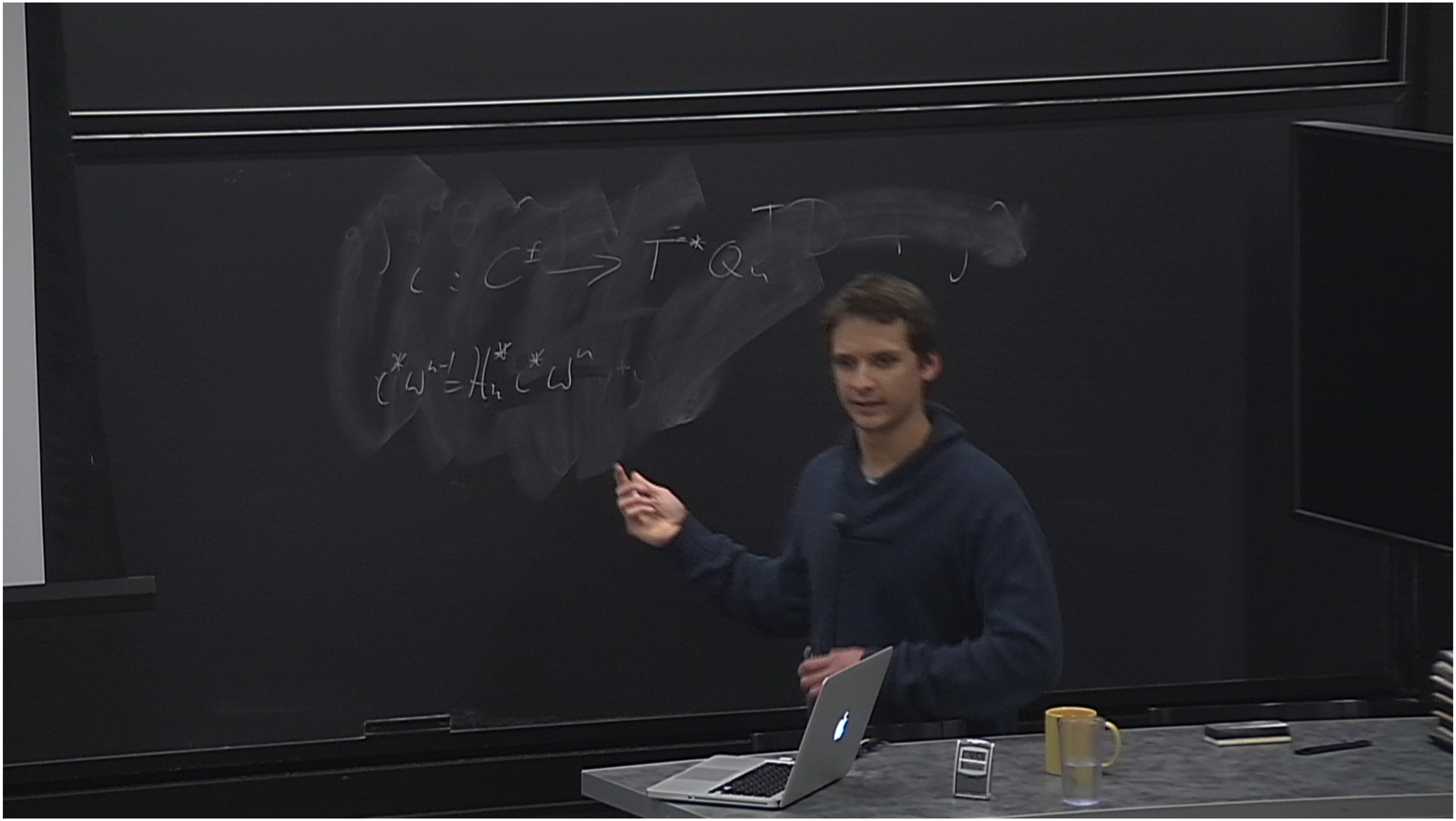
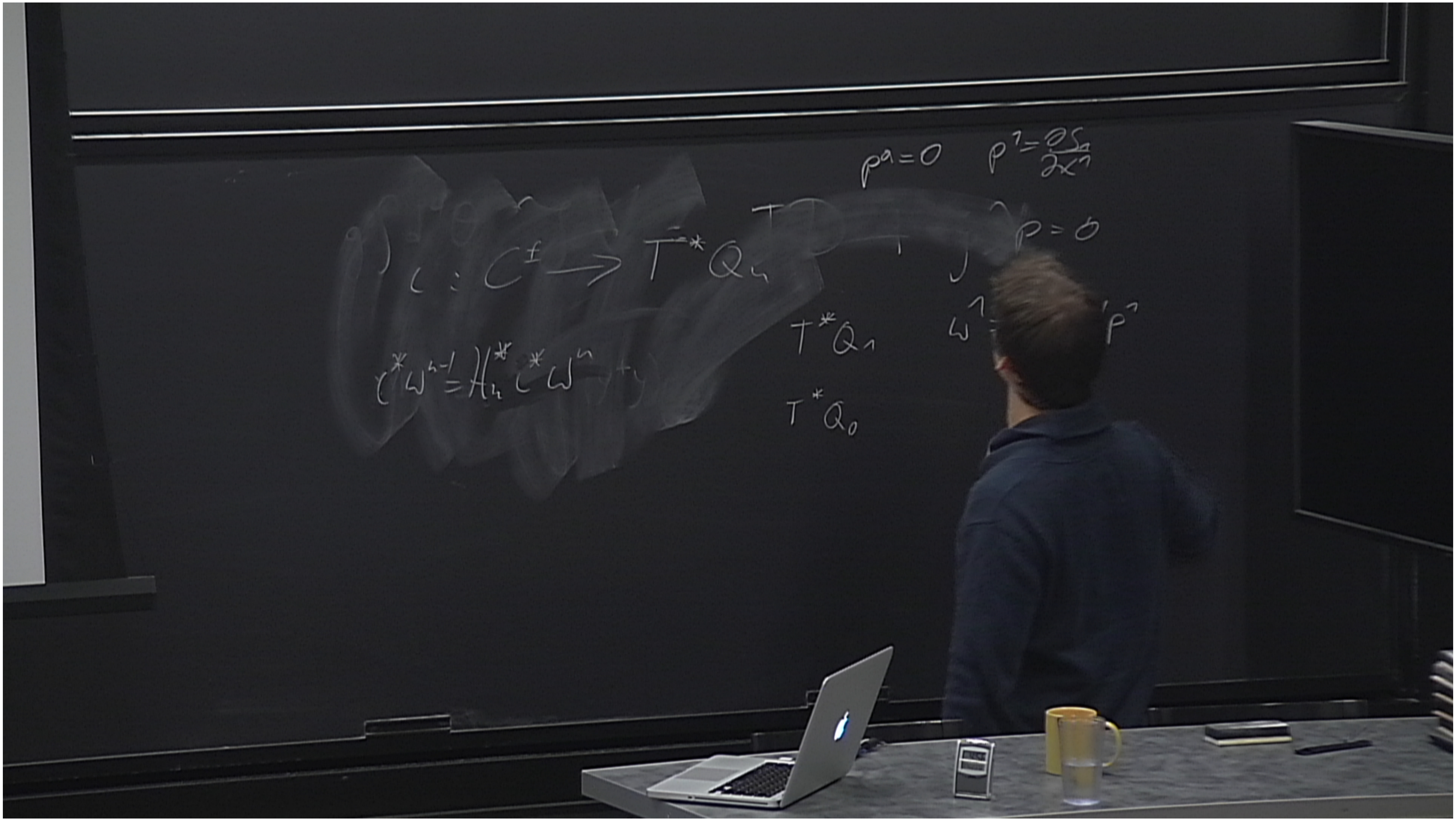
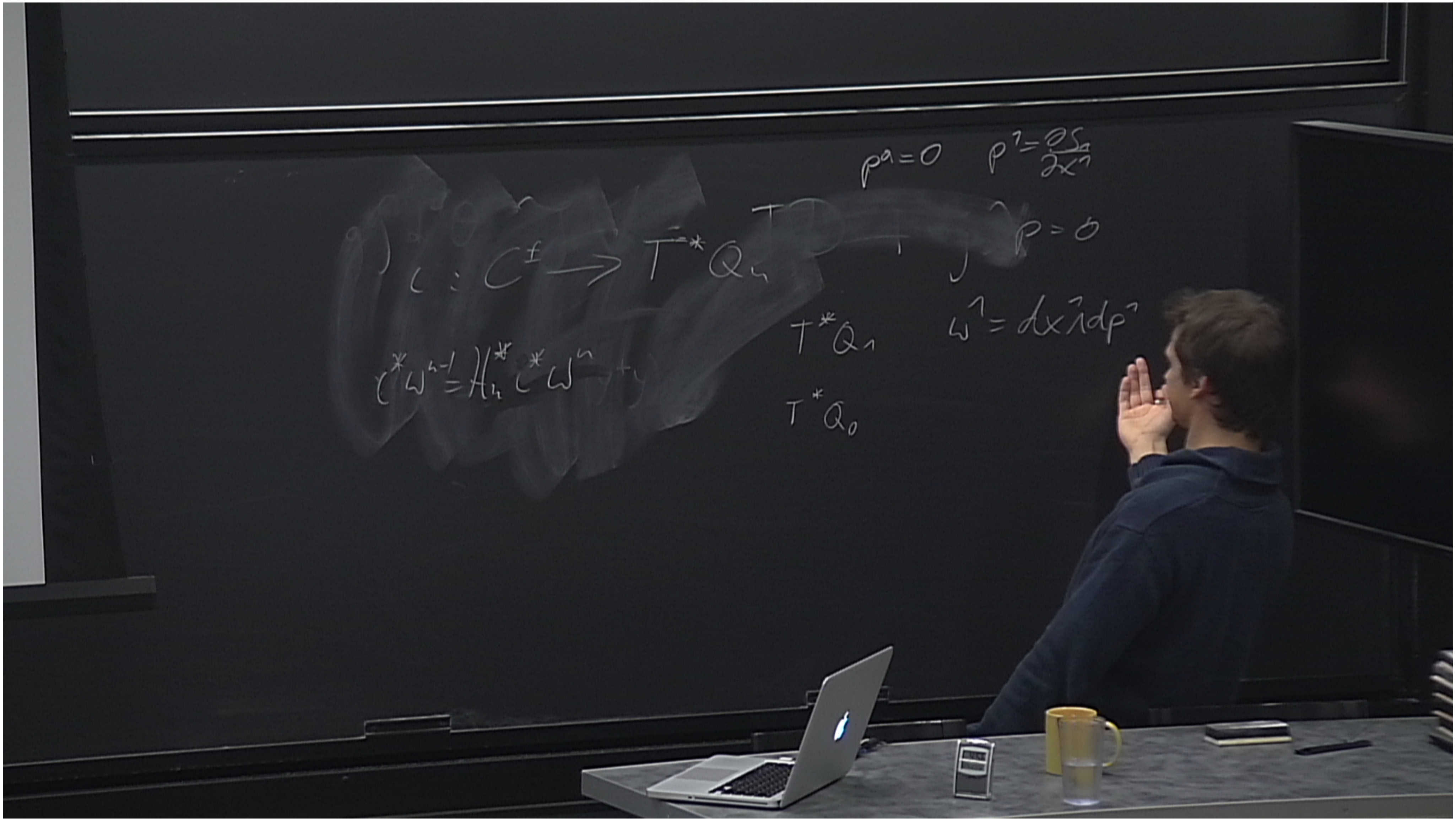
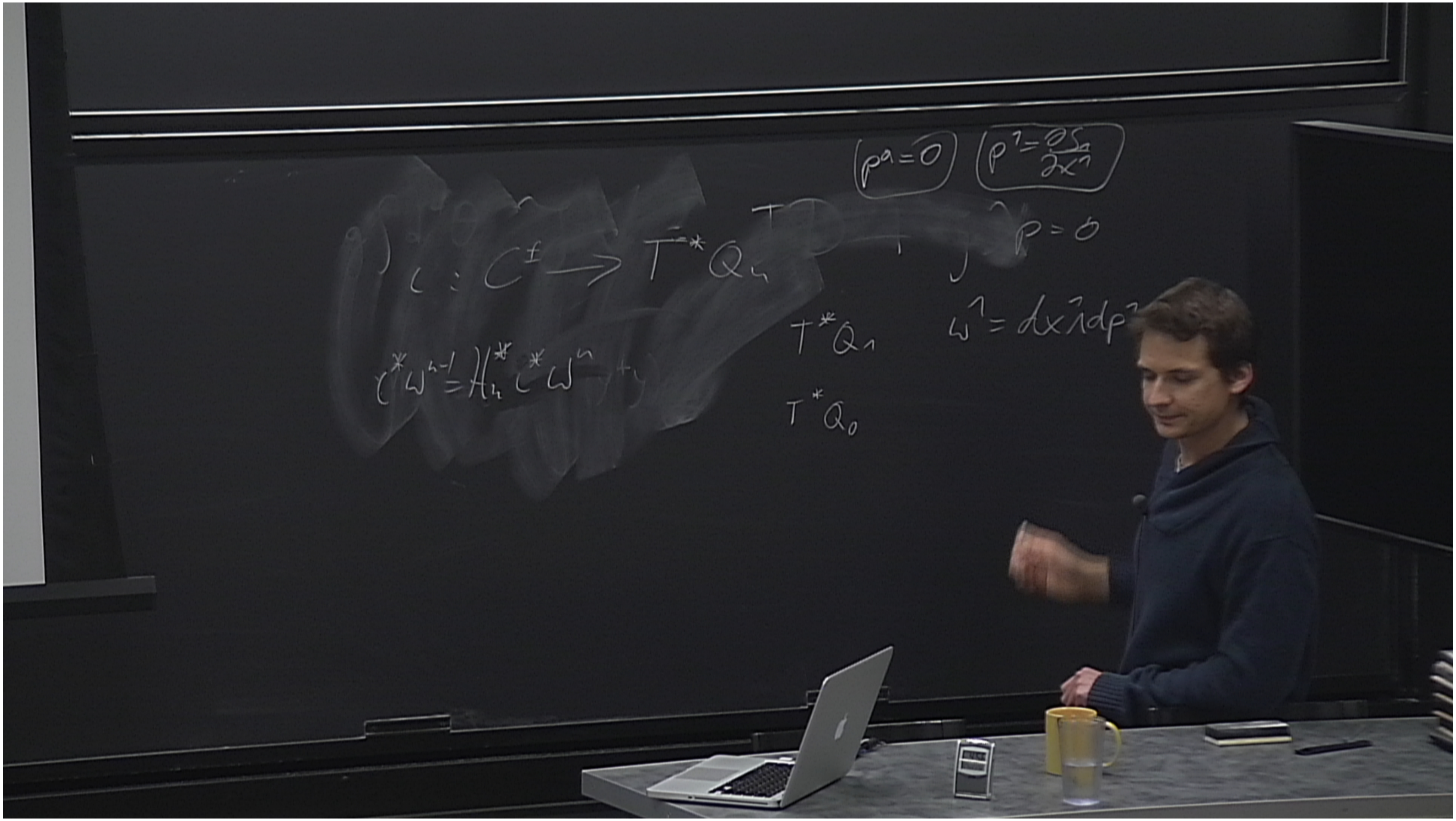
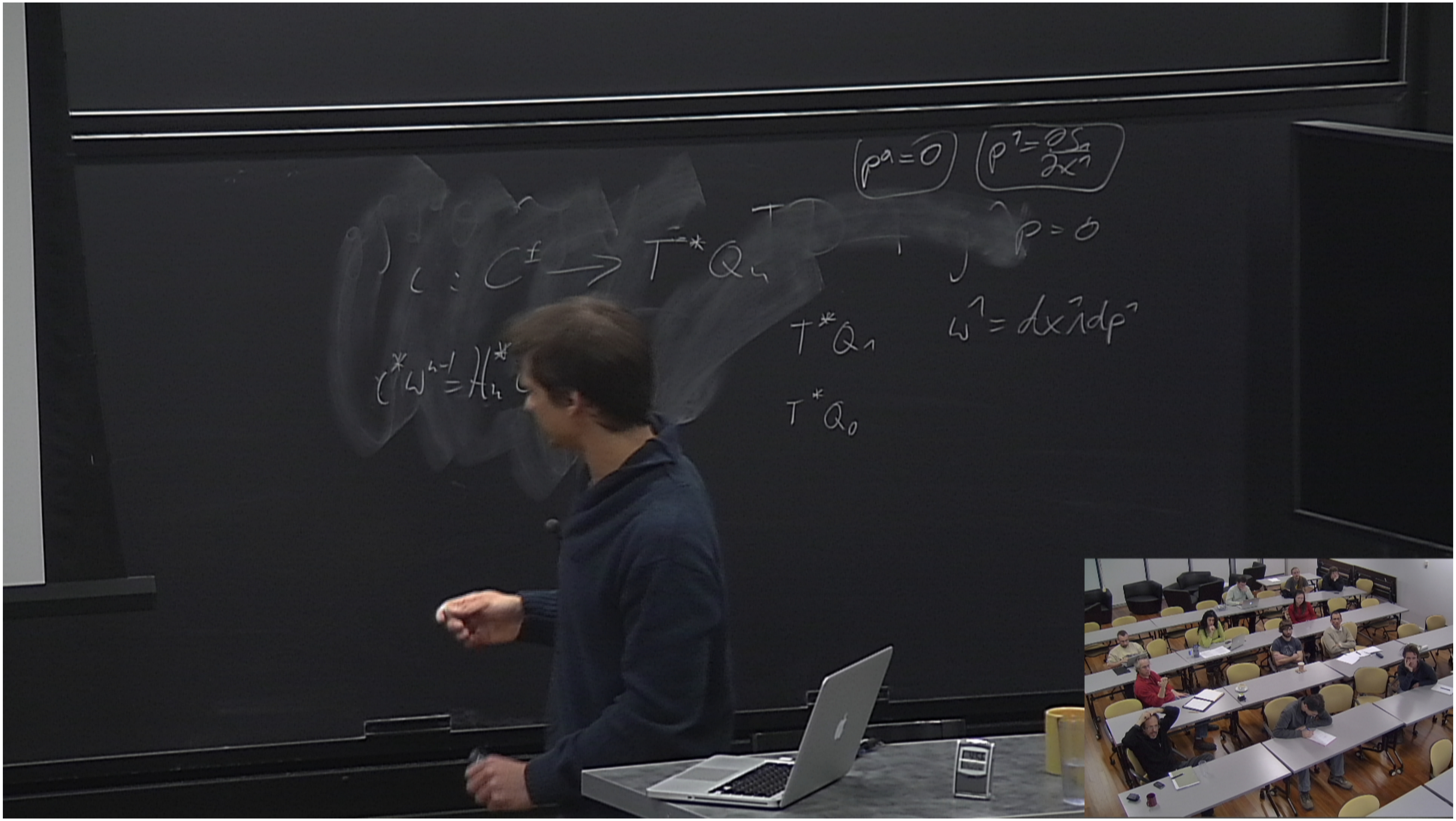
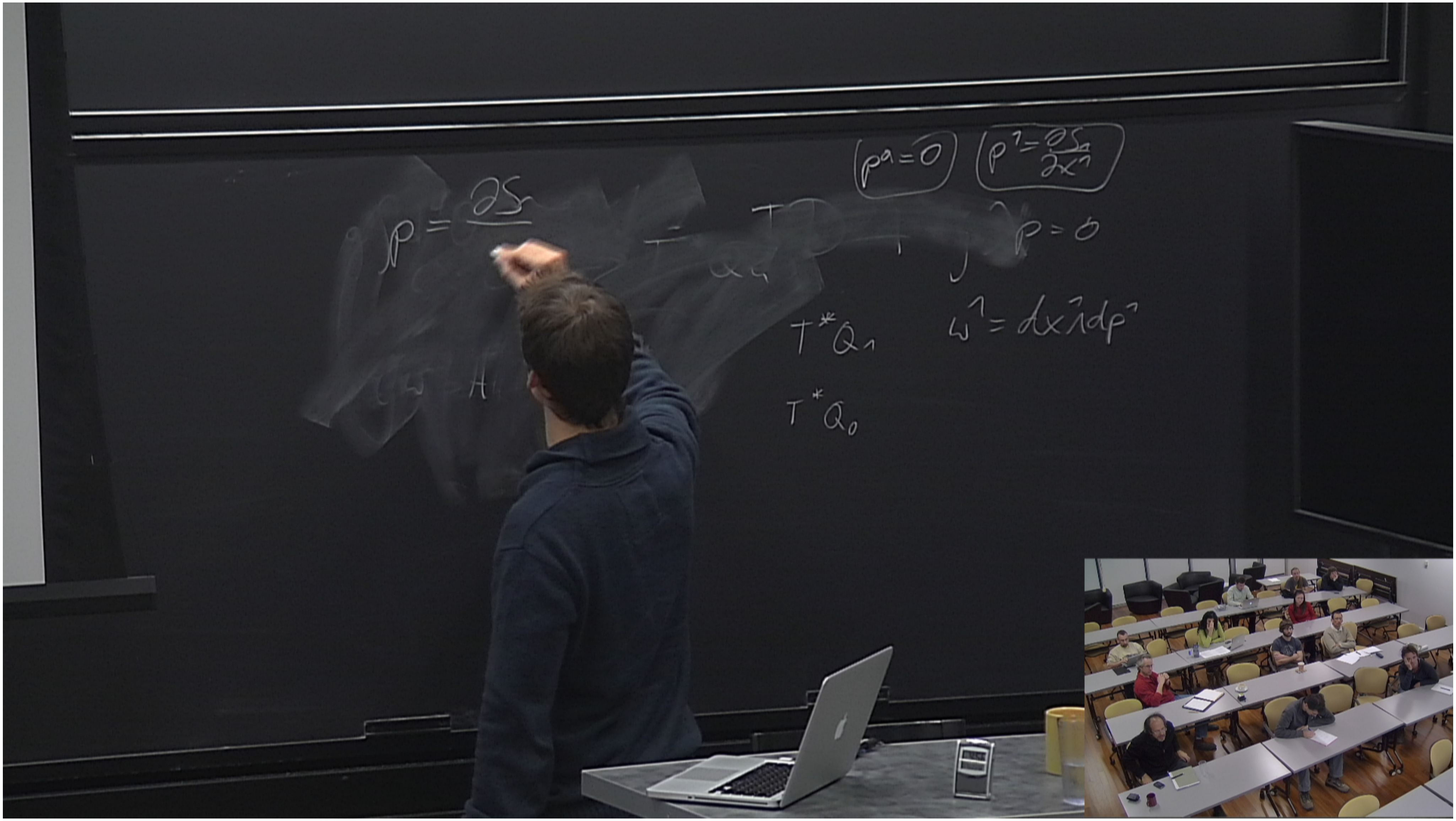

ι : C^ε → T*Q_n

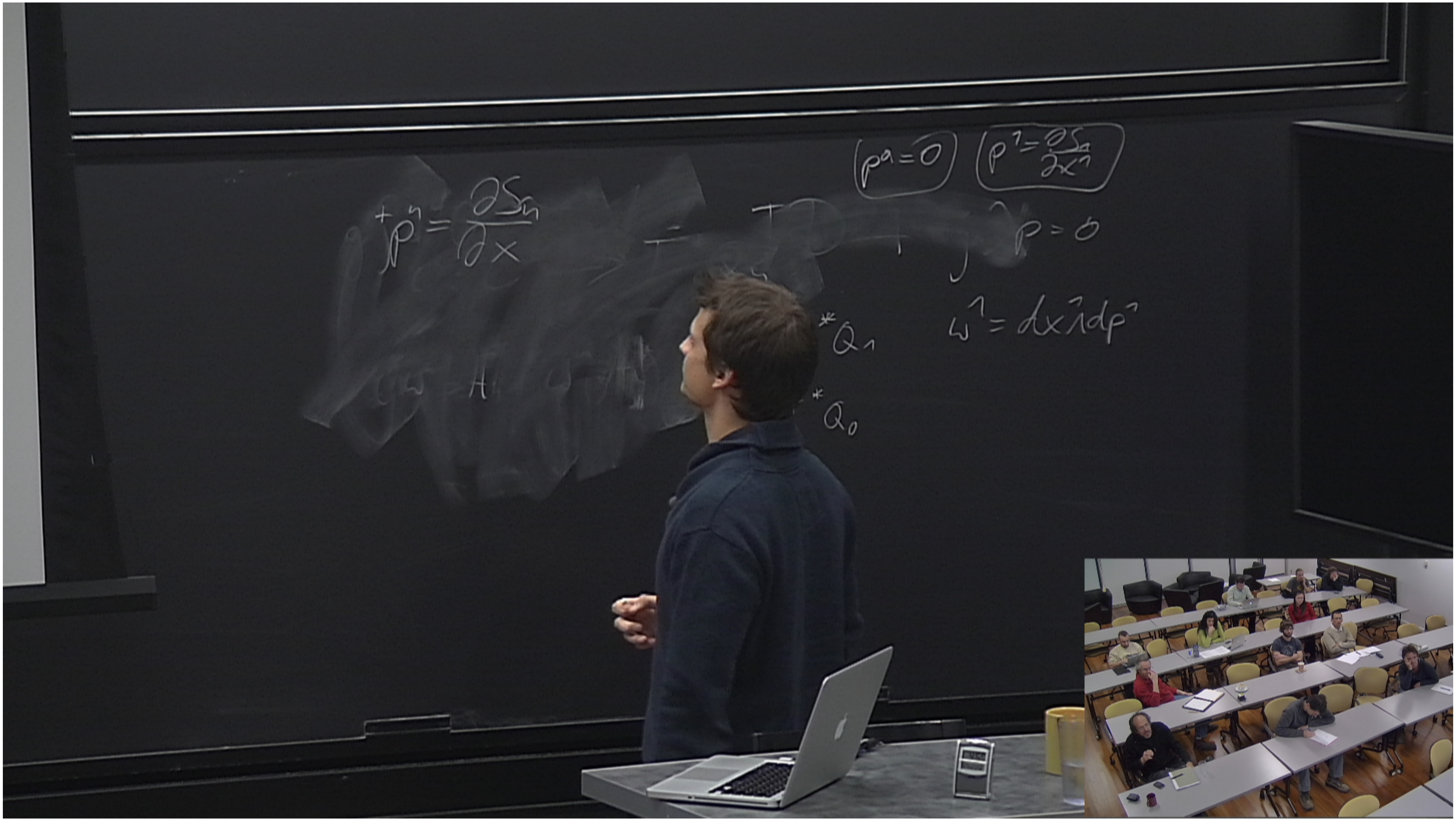

pᵃ = 0
T*Q₁
T*Q₀


p^a = 0
T*Q_1
T*Q_0
p = 0


T*Q_1
T*Q_0
p = 0


p^a = 0
T*Q_n
T*Q_1
T*Q_0
p = 0
ω^1 = dx^1 ∧ dp^1


p^a = 0
p^1 = ∂S_1/∂x^1
p = 0
T*Q_1
ω^1 = dx^1 ∧ dp^1
T*Q_0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p^a = ∂S_a/∂x^a
p^a = ∂S_a/∂x
p = 0
*Q_1
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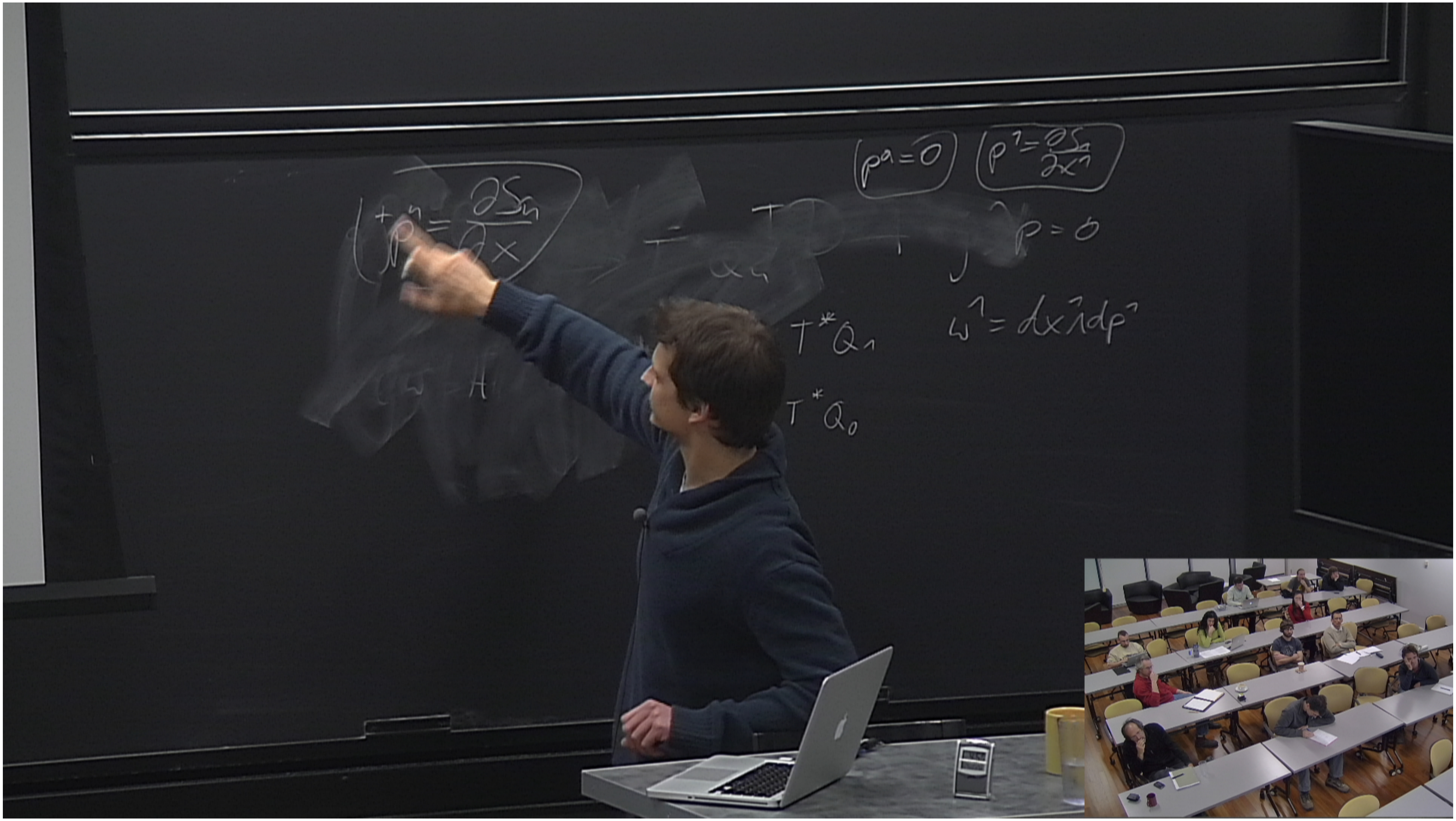
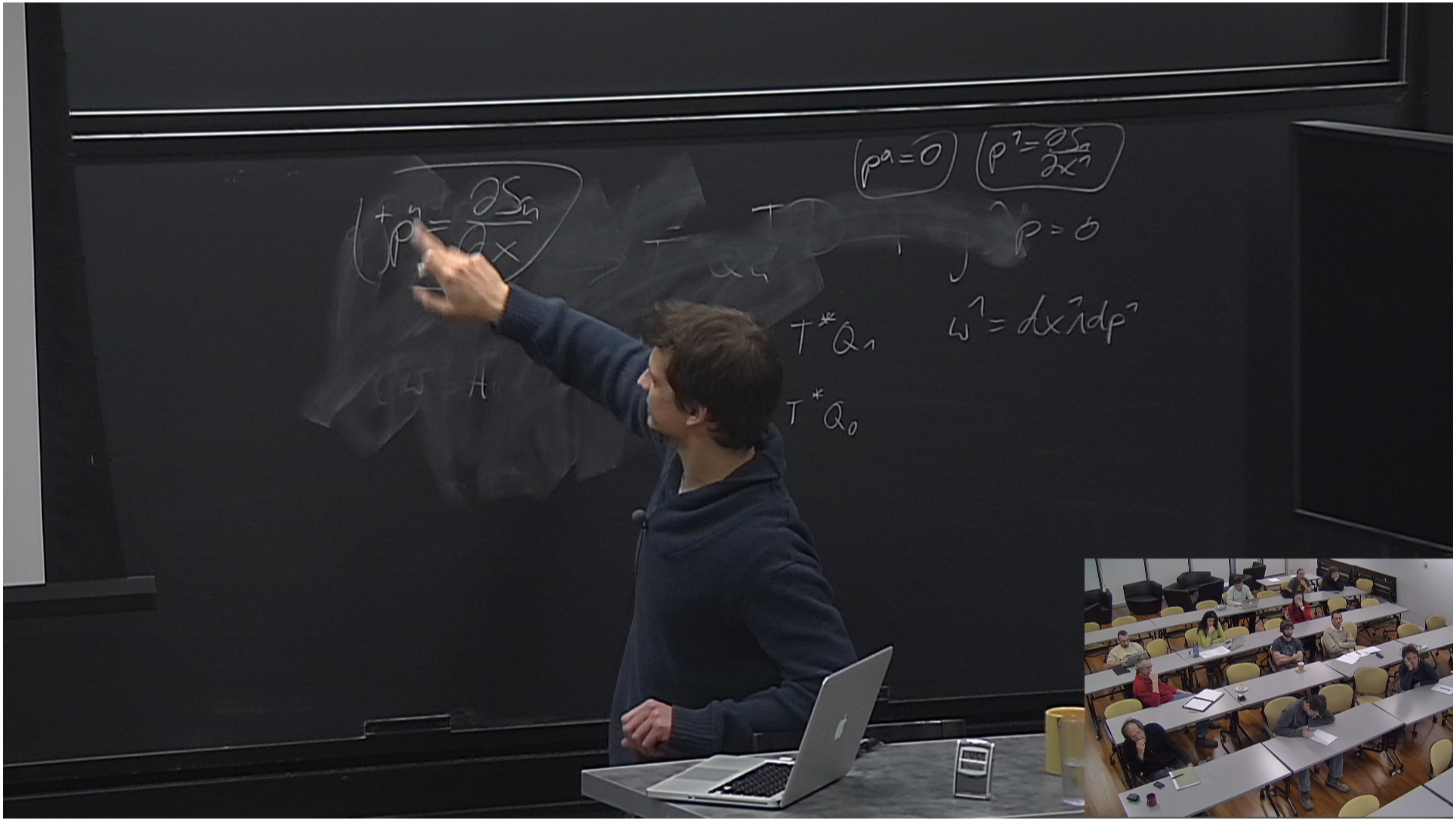
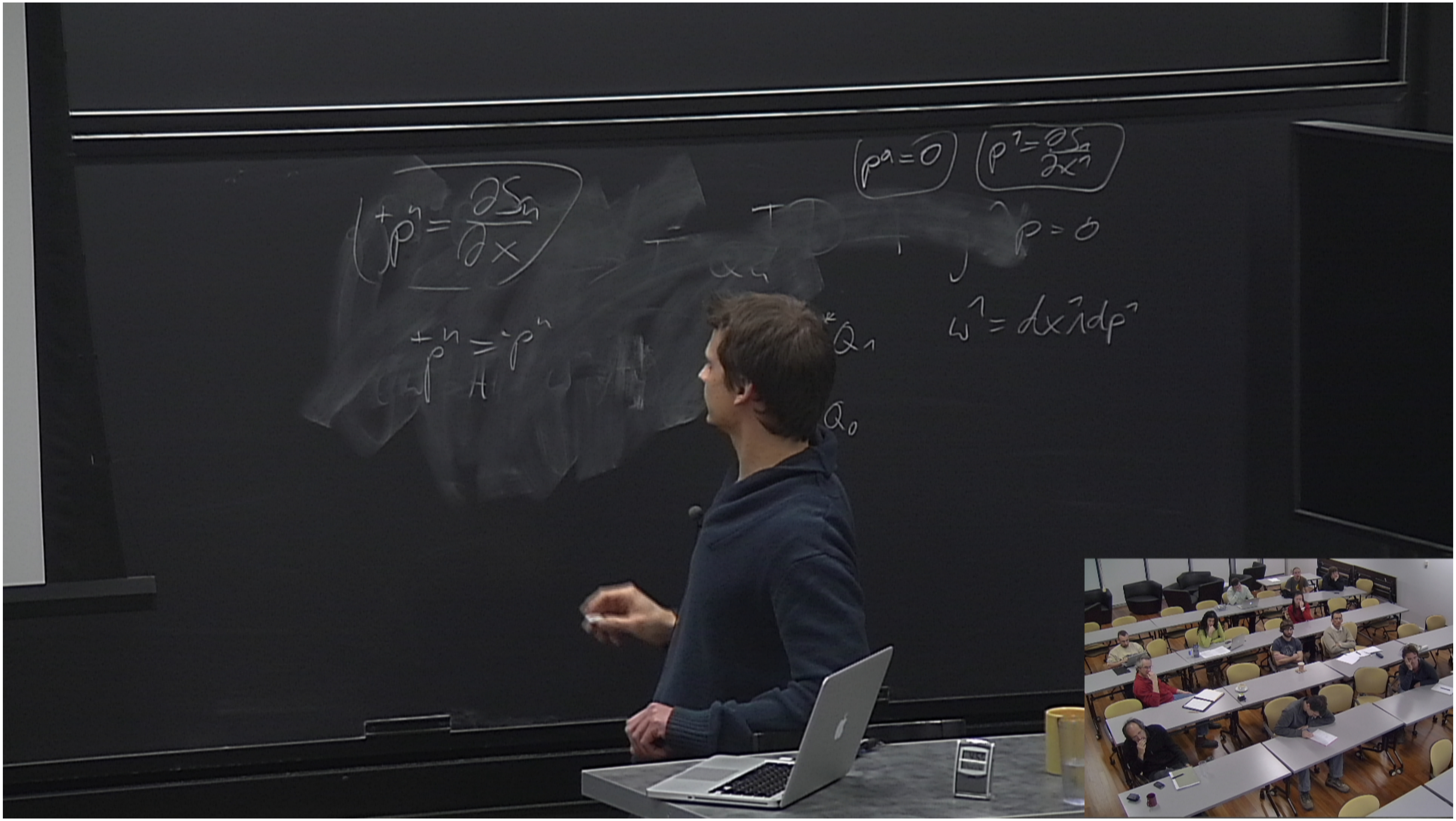

pa=0
p1=∂S1/∂x1
T*Q1
ω1=dx1∧dp1
T*Q0





pᵃ = 0
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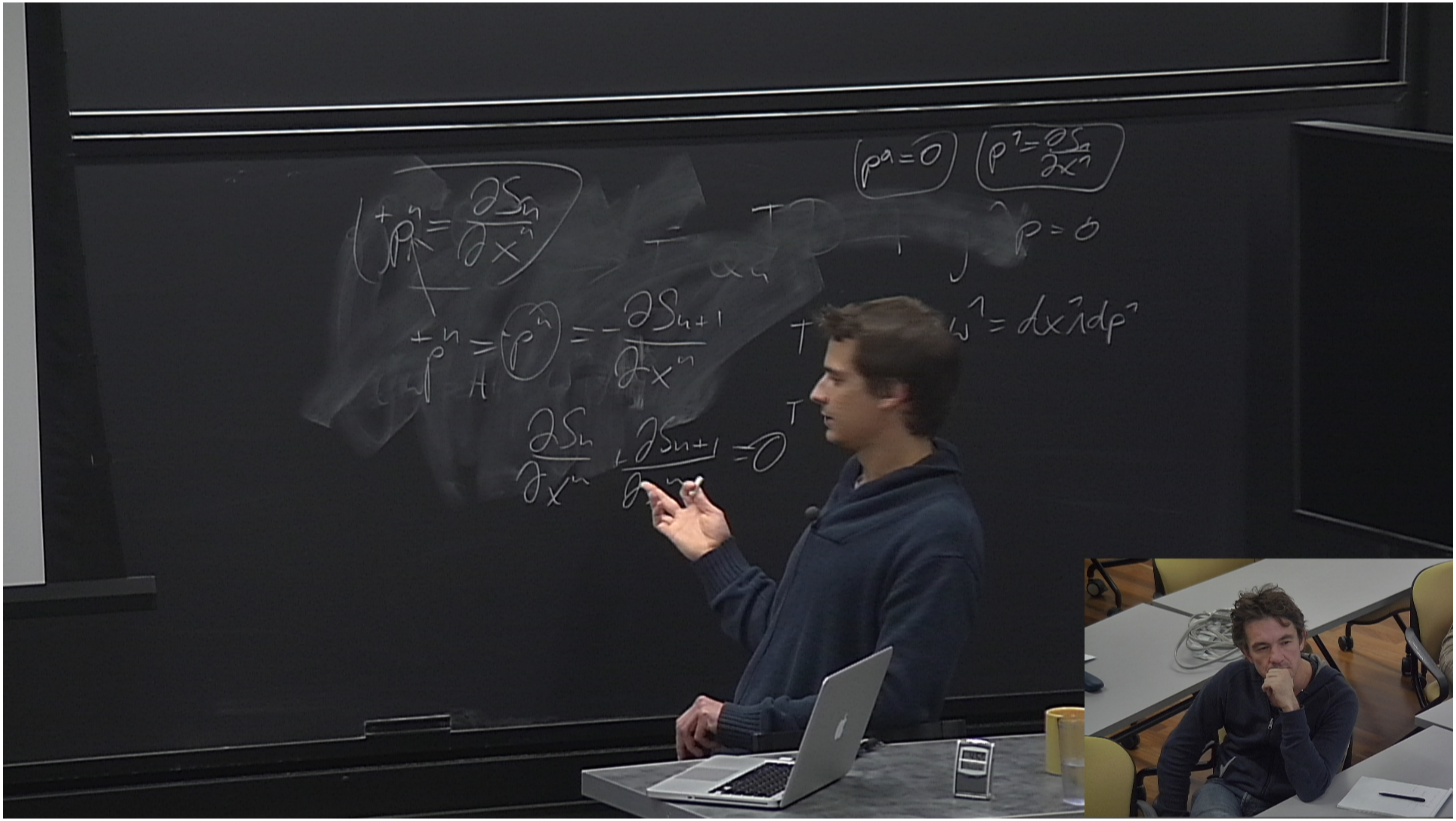

$p^a = 0$
$p^a = \frac{\partial S_n}{\partial x^a}$
$\omega^a = dx^a \wedge dp^a$
$\frac{\partial S_n}{\partial x^n} + \frac{\partial S_{n+1}}{\partial x^n} = 0$

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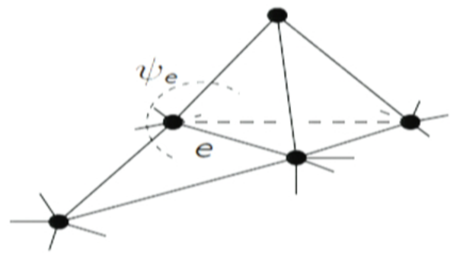

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Arbitrariness in evolution

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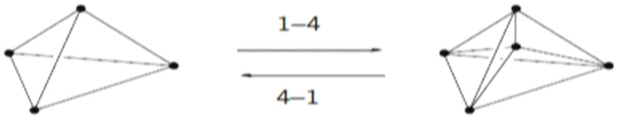
Synopsis: Pachner moves for 4D simplicial gravity



- 1–4 move: introduces 4 new edges, momenta satisfy $C^{n+1}_{new} = p^{n+1}_{new} - \frac{\partial S_\sigma}{\partial x^{new}_{n+1}} = 0$ (post-constraints)

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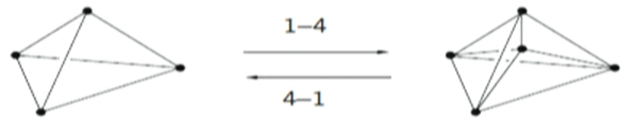

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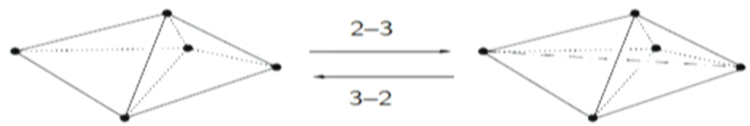

- 2–3 move: introduces 1 edge, renders 1 triangle internal, no new internal edge $\Rightarrow$ freely choosable curvature generated, new momentum $C^{n+1}_{new} = p^{n+1}_{new} - \frac{\partial S_\sigma}{\partial x^{new}_{n+1}} = 0$ (post-constraint)
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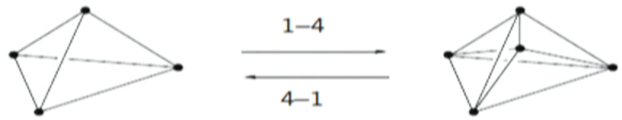

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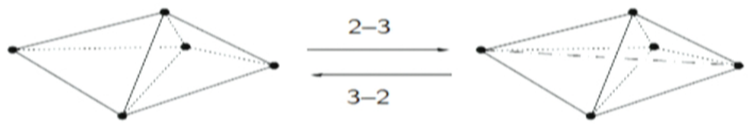
- 4–1 move: removes 4 old edges, $C^{n}_{old} = p^{n}_{old} + \frac{\partial S_\sigma}{\partial x^{old}_{n}} = 0$ (pre-constraints), $C^{n}_{old} = 0$ equiv. to eom for x^{old}_{n}



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- 3–2 move: removes 1 old edge, $C^{n}_{old} = p^{n}_{old} + \frac{\partial S_\sigma}{\partial x^{old}_{n}} = 0$ (pre-constraint)

'Correct dynamics' in Regge Calculus (role (b) of constraints)

- pre– and post–constraints restrict time evolution (connectivity of triangulation)
- at n pre–constraints of new piece of triangulation in conflict with underlying data, then either
  1. accept that cannot glue piece of triangulation to Σ_n $\Rightarrow$ perform other evolution move (e.g. 1–4 and 2–3 Pachner moves are always possible)
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option 1: when consider initial value problem

option 2: when consider combined initial and final value problem

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Pre– and post–constraints

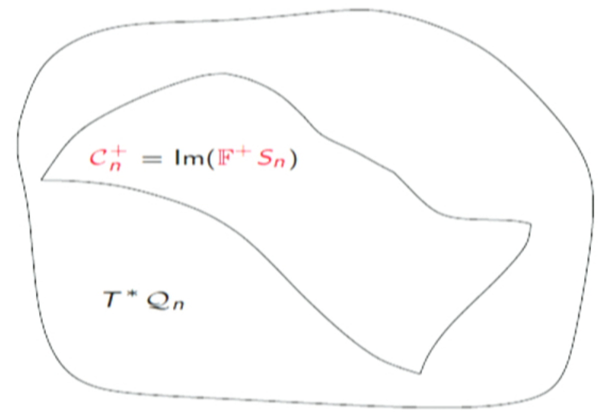
- need both pre– and post–constraints to classify DoFs
- pre– and post–constraints each form abelian sub-algebra

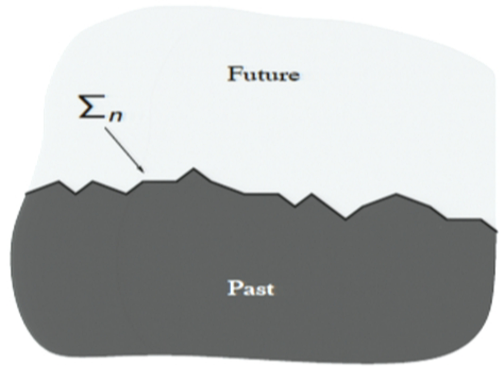
$$\{C_n^-, C_n'^-\} = 0 = \{C_n^+, C_n'^+\}$$

- however, generally

$$\{C_n^-, C_n^+\} \neq 0$$

Constraint (un-)matching





- *a posteriori*:
  if no complete *constraint matching*, i.e. $\mathcal{C}_n^+ \neq \mathcal{C}_n^-$, pre-constraints may fix *a priori* free parameters λ associated to post–constraints (in cont. $\dot{C}_{m'} = \{C_{m'}, H + \lambda^m C_m\} \overset{!}{=} 0$) [Dittrich, PH '11 and to appear]
- if some parameters remain free, obtain proper gauge transformations

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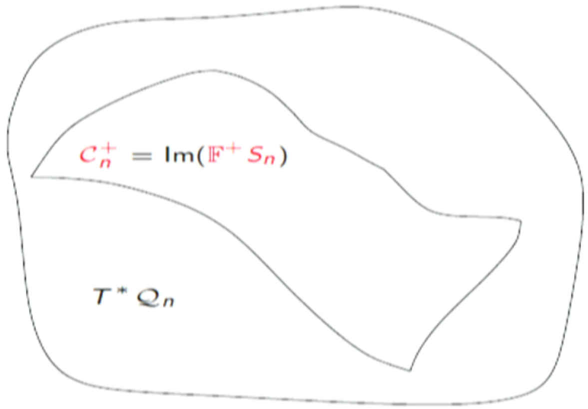

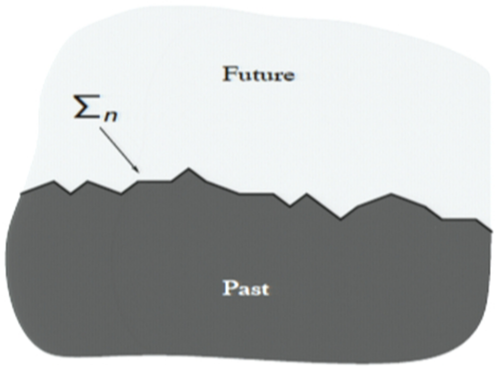

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Gauge symmetries (role (c) of constraints)

- pre–/post–constraints associated to left/right null–vectors of $\Omega^n = \frac{\partial^2 S_n}{\partial x_{n-1} \partial x_n}$

$$C_n^- = L_n({}^-p^n + \frac{\partial S_{n+1}}{\partial x_n}) \qquad L_n \Omega^{n+1} = 0$$

$$C_n^+ = R_n({}^+p^n - \frac{\partial S_n}{\partial x_n}) \qquad \Omega^n R_n = 0$$

- $L_n = R_n = Y_n$ null–vector matching, $Y_n \cdot H = 0$ (H Hessian of action)
  $\Rightarrow C_n^-, C_n^+$ 1st class and gauge generators [to appear]

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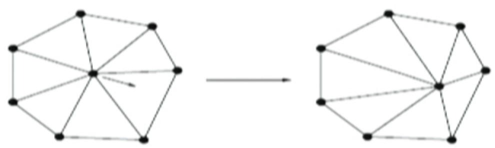
- $L_n = R_n = Y_n$ null–vector matching, $Y_n \cdot H = 0$ (H Hessian of action) $\Rightarrow$ C_n^-, C_n^+ 1st class and gauge generators [to appear]
- in Regge Calculus Y_n associated to vertices $\Rightarrow$ vertex displacement gauge symmetry



(in general, Regge gauge symmetry broken in presence of curvature [Rocek, Williams '84; Bahr, Dittrich '09; Dittrich, PH '09; etc.])

'Minimal slice' [to appear]

- turns out, no. of constraints depends on init. and final slice

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- *'minimal slice'*: smallest no. of variables, largest no. of constraints
  $\Rightarrow$ 'propagates' these constraints to all other slices
  $\Rightarrow$ fixes no. of constraints (on $\mathcal{P}_{tot}$)
- however "once gauge, always gauge"
- observable determination non–local (in discrete time), depends on 'minimal slice'
- tent moves: every slice a 'minimal slice'

Conclusions and challenges

- general canonical framework for discrete systems: can cope with varying phase space dim.
- equivalent to covariant formalism $\Rightarrow$ dynamics
- can apply to simplicial gravity: implement general discrete time evolution scheme <u>not</u> generated by constraints
- due to allowing for full freedom in evolution encounter new features in evolution of data (but get standard picture for restriction to constant phase space dimension)
- complete constraint classification to identify DoFs [to appear]

$\Rightarrow$ reduced phase space dimension preserved?

- understand 'covariant phase space' [to appear]

Quantization

- action as generating function $\Rightarrow$ direct connection between canonical framework and path integral
- heuristic idea: given wave function at n $\psi(x_n)$, obtain wave function at $(n+1)$ by

$$\text{``} \quad \psi(x_{n+1}) = \int dx_o Exp(iS_\sigma)\psi(x_n) \quad \text{''}$$

S_σ: action of glued simplex, x_o: edges going into the bulk, dx_o: some integration measure

- but now implement evolv. Hilbert spaces
- in Regge sector connect to recent developments on linking cov. and can. quantizations [Alesci,Bonzom,Freidel,Livine,Thiemann, Zipfel,...]