Title: Chern-Simons Theory, Vassiliev- Kontsevich Invariant, and Spinfoam Quantum Gravity

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Abstract: We construct the q-deformed spinfoam vertex amplitude using Chern-Simons theory on the boundary 3-sphere of the 4-simplex. The rigorous definition involves the construction of Vassiliev-Kontsevich invariant for trivalent knot graph. Under the semiclassical asymptotics, the q-deformed spinfoam amplitude reproduce Regge gravity with cosmological constant at nondegenerate critical configurations.

Outline

- Motivation
- Chern-Simons Theory
- Vassiliev-Kontsevich Invariant
- q-deformed Spinfoam Vertex Amplitude
- Relate to Discrete Gravity with Λ
- Summary

Motivation

- Looking for a mathematical framework for LQG dynamics

- Study the relations between $\left\{\begin{array}{l}\text{Spinfoam QG in 4d} \\ \text{Topological QFT in 3d} \\ \text{Conformal Field Theory in 2d}\end{array}\right.$

- Construct a finite q-deformed spinfoam model using topological invariant

- Reproduce discrete GR with cosmological constant in the asymptotics.

  (Reproduce Λ-Regge calculus using flat simplices)

Previous works are done by: Turaev, Viro, Mizoguchi, Tada, Crane, Yetter, Smolin, Major, Freidel, Krasnov, Noui, Roche, Fairbairn, Meusburger, M.H

The vertex amplitude (Euclidean)

$$A_v^q = \int \Psi_\gamma[A^\pm]\, e^{\frac{2\pi i}{h^+} S_{CS}[A^+] + \frac{2\pi i}{h^-} S_{CS}[A^-]}\, DA^+ DA^-$$

(Motivated by L. Smolin '95)

$\Psi_\gamma[A^\pm]$: projective spin-network function on graph γ embedded in S^3

(Dupuis & Livine '10)

A_v^q is rigorously defined by Vassiliev - Kontsevich Invariant

(Motivated by Freidel & Krasnov '99)

$$h^\pm = \pm \frac{8}{(1 \pm \beta)^2}\, \omega \qquad \omega = \Lambda \ell_p^2 = \ell_p^2 / \ell_c^2$$

Asymptotic regime : $\ell_p^2 \ll \text{Area}_f \ll \ell_c^2$

$$A^q_{4\text{-simplex}} \sim \left(\frac{2\pi}{\lambda}\right)^{12} e^{i\lambda \sum_f \gamma j_f \Theta_f}\, e^{i\omega V_\sigma} \left[1 + o\left(\tfrac{1}{\lambda}\right)\right]$$

The analysis can be generalized to simplicial manifold with many 4-simplices.

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The analysis can be generalized to simplicial manifold with many 4-simplices.

- The (q-deformed) spinfoam formalism of LQG defines a family of transition amplitudes, which are finite on all possible 2-complexes.

- The (q-deformed) spinfoam amplitude is a mathematical wall-defined object, without infinity, and relate GR in the semiclassical limit.

Chern - Simons Theory

* Chern - Simons action on 3-manifold M (TQFT)

$$S_{CS}[A] = \frac{k}{4\pi} \int_M \mathrm{Tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right)$$

A : G - connection , G : compact Lie group

Chern - Simons level/coupling : $k \in \mathbb{Z}$

* Chern - Simons expection value of Links

$$\langle \; {}^{R_1 \;\; R_2} \; \rangle_{CS} = \int {}^{R_1 \;\; R_2} \; e^{i S_{CS}[A]} \, \mathcal{D}A$$

framed Link L   Wilson loops

with regularization / normalization.

* Skein Relation for Jones Polynomial :

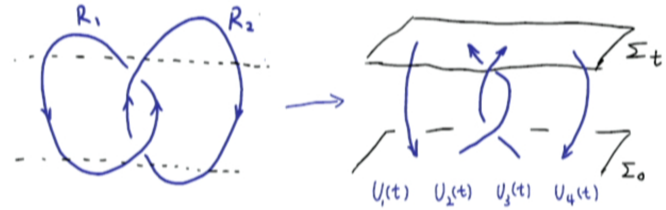
$$- q^{\frac{N}{2}} \langle \;\; \rangle_{CS} + \left(q^{\frac{1}{2}} - q^{-\frac{1}{2}} \right) \langle \;\; \rangle_{CS} + q^{\frac{N}{2}} \langle \;\; \rangle_{CS} = 0$$

trivial Loop : $\langle \bigcirc \rangle_{CS} = \dfrac{q^{\frac{N}{2}} - q^{\frac{N}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}$

$R = SU(N)$ fundamental rep. $q = e^{\frac{2\pi i}{N+k}}$

(E. Witten '89)

*



Wilson lines

EOM of Chern-Simons correlation functions

$$\frac{d}{dt}\left\langle \overset{R_1}{U_1}(t)\otimes\cdots\otimes\overset{R_n}{U_n}(t)\right\rangle_{CS}$$

$$=\frac{1}{k+N}\sum_{i<j=1}^{n}\frac{\dot z_i-\dot z_j}{z_i-z_j}\,\Omega_{ij}\left\langle \overset{R_1}{U_1}(t)\otimes\cdots\otimes\overset{R_n}{U_n}(t)\right\rangle_{CS}$$

Knizhnik - Zamolodchikov Equation

z : complex coordinate on complex plane Σ. $\dot z=\frac{d}{dt}z(t)$

* Infinitesimal Braiding

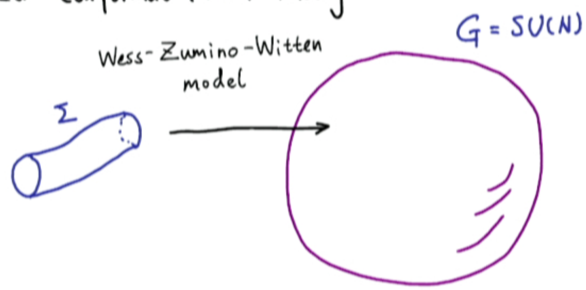
$$\Omega_{ij}=\sum_{k=1}^{\dim G} id\otimes\cdots\otimes id\otimes\overset{R_i}{T_k}\otimes\cdots\otimes\overset{R_j}{T_k}\otimes id\otimes\cdots\otimes id$$

$=$ 

chord diagram

* All the following constructions of invariants and spinfoam amplitude come from the solutions of Knizhnik - Zamolodchikov Equation.

* Knizhnik - Zamolodchikov Equation comes natually from 2d Conformal Field Theory



The operator algebra is a Kac-Moody algebra $\hat{g}_k$

$$[j_m^a, j_n^b] = i \sum_c f^{abc} j_{m+n}^c + k m \delta^{ab} \delta_{m+n,0}$$

$$\left([j_0^a \;\; j_0^b] = i \sum_c f^{abc} j_0^c \right)$$

chiral current : $j^a(z) = \sum_{n \in \mathbb{Z}} j_n^a z^{-n-1}$

central charge : $c = \frac{k \dim G}{k+N}$, k : Chern-Simons level

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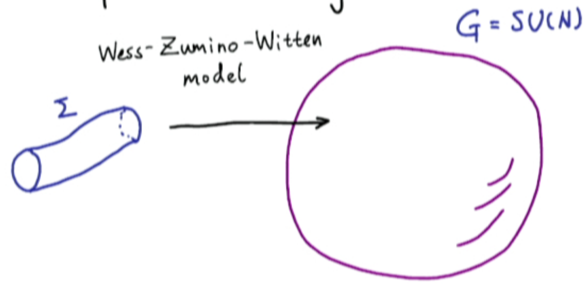

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Unitary rep. of the current algebra $\hat{g}_k$

finite # of $\hat{g}_k$-primary fields/states (ground states)

$\xrightarrow{j^a_n}$ descendants of each primary (finite # of $\hat{g}_k$-families)

($\hat{g}_k$-primary fields OPE : $j^a(z)\,\phi^r_R(w) = \frac{1}{z-w}\sum_s (\overset{R}{T}{}^a)^r{}_s\, \phi^s_R(w) + \text{regular}$)

$\widehat{su(2)}_k$: finite # of $\hat{g}_k$-families → finite # of spins

Constraint on the correlator of $\hat{g}_k$-primary fields

$$\left[\partial_{z_i} - \frac{1}{k+N}\sum_{j\neq i}\frac{\Omega_{ij}}{z_i - z_j}\right]\left\langle \phi_{R_1}(z_1)\cdots\phi_{R_n}(z_n)\right\rangle = 0$$

Knizhnik - Zamolodchikov Equation

(rep. indep.)

$*$ The KZ eqn. $\frac{d}{dt}\langle\ \rangle = \frac{1}{k+N}\sum_{i<j=1}^{n}\frac{\dot{z}_i - \dot{z}_j}{z_i - z_j}\ \overset{i}{|}\!\!-\!-\!-\!\!\overset{j}{|}\ \langle\ \rangle$

is a parallel transportation equation with a flat KZ connection on $\mathbb{C}^n$

$$\omega = \frac{1}{k+N}\sum_{i<j}^{n}\frac{\overset{i}{|}\!\!-\!-\!-\!\!\overset{j}{|}}{z_i - z_j}\left(dz_i - dz_j\right)$$

The solutions of KZ eqn are holonomies of ω.

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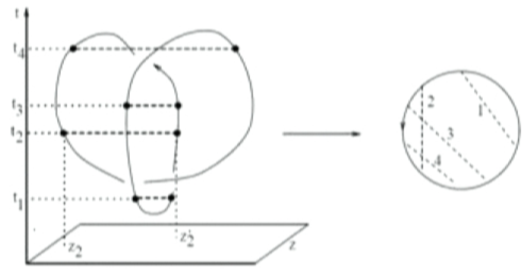
$*$ Kontsevich Integral (Kontsevich '93)

$$\langle \gamma \rangle = \sum_{n=0}^{\infty} \left(\frac{h}{2\pi i}\right)^n \int_{\substack{t_1 < \cdots < t_n \\ |z_i - z_i'| > \varepsilon}} \sum_{\substack{\text{Pairings} \\ P = (z_i z_i')}} (-1)^{\# P_\downarrow} D(\gamma, P) \bigwedge_{i=1}^{n} \frac{dz_i - dz_i'}{z_i - z_i'}$$

$\left(h = \frac{2\pi i}{k+N} \right)$

$D(\gamma, P)$ is a chord diagram based on γ



After regularization & normalization,

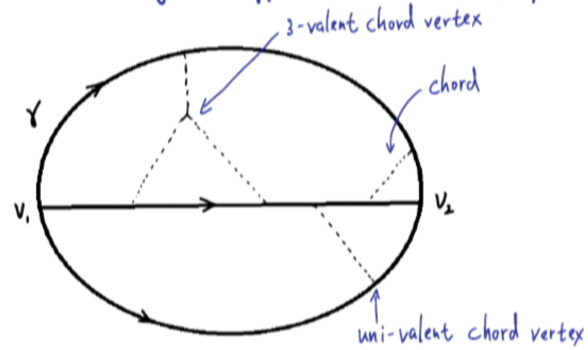
$\Rightarrow$ Perturbative Vassiliev - Kontsevich invariant

Vassiliev '90 Kontsevich '93 Bar-Natan '95

Vassiliev - Kontsevich Invariant (Combinatorial Definition)

Piunikhin '95, Le & Murakami '97, Murakami & Ohtsuki '97
Altschuler & Freidel '95, Dancso '08

Definition (Chord diagram supported on 3-valent graph)

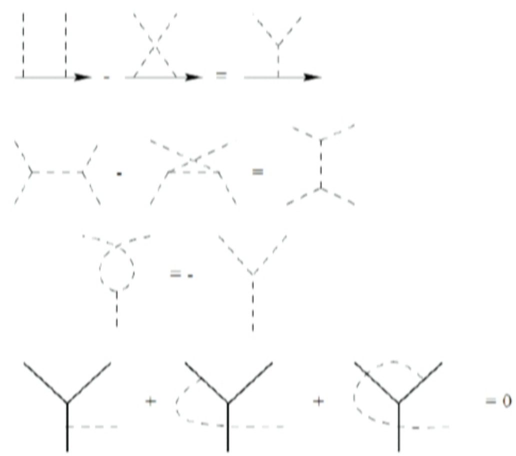


* There are 3-valent and uni-valent vertices for chords
* The uni-valent chord vertices are on γ

The chord diagrams are perturbative Feynman diagram of Chern-Simons theory.

Definition

Given a 3-valent graph γ embedded in 3-manifold, $\mathcal{A}(\gamma)$ is the $\mathbb{C}$-linear space spanned by the chord diagrams supported by γ, subject to the relations:



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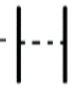
Three building blocks for Vassiliev-Kontsevich invariant

- Braiding $R = \exp \frac{ih}{2}$ 
- Unknot $\nu = | - \frac{h^2}{48}$  $+ \cdots$
- Associator $\Phi = ||| - \frac{h^2}{12}$  $+ \cdots$

$\Bigg\}$ Power series in h

Definition (Le & Murakami '97, Murakami & Ohtsuki '97, Dancso '08)

The Vassiliev - Kontsevich invariant

$$\hat{Z} : \gamma \mapsto \hat{Z}(\gamma) \in \mathcal{A}(\gamma) \leftarrow \text{linear space of chord diagram}$$

constructed by the following rules

- Crossings $\hat{Z}($  $) = R$ $\hat{Z}($  $) = R^{-1}$

- Maximum/Minimum $\hat{Z}($  $) = \nu^{-1}$ $\hat{Z}($  $) = 1$

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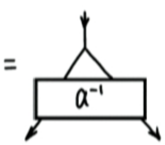
- Vertices $\hat{Z}($  $) =$ 

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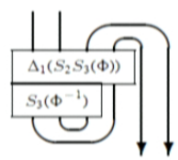
$a =$ 

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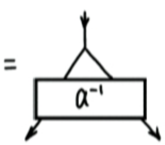
- Vertices $\hat{Z}($$) =$ 

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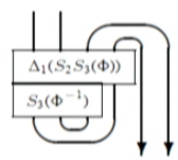
$a =$ 

Theorem (Murakami & Ohtsuki '97, Dancso '08)

$\hat{Z}(\gamma)$ is an isotopy invariant of the embedding $\gamma \hookrightarrow S^3$

$\downarrow$

Diff invariance

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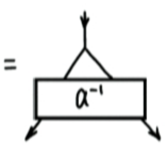
- Vertices $\hat{Z}($  $) =$ 

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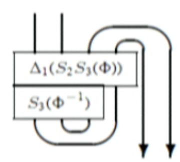
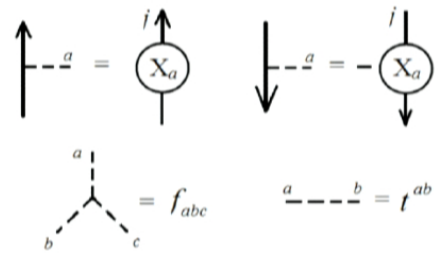
$a =$ 

Evaluate the chord diagram with Lie algebra $\mathfrak{g}$ & rep.



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Quadratic Casimir : $t^{ab} X_a X_b$

3-valent vertices in γ : intertwiners.

$\hat{Z}(\gamma) = \sum_{n=0}^{\infty} \hat{Z}_n(\gamma)\, h^n$ is a power series of chord diagrams

$$\hat{Z}(\gamma) \xrightarrow[\omega_{\mathfrak{g}}]{\text{Evaluation with } \mathfrak{g} \text{ order by order}} \omega_{\mathfrak{g}} \cdot \hat{Z}(\gamma)$$

Quantum Group invariant

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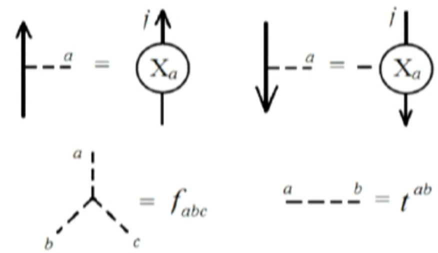

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Quantum Group invariant

q-deformed Spinfoam Amplitude

Heuristically, the vertex amplitude (Euclidean)

$$A_v^q = \int \Psi_\gamma[A^\pm]\, e^{\frac{2\pi i}{h^+} S_{CS}[A^+] + \frac{2\pi i}{h^-} S_{CS}[A^-]}\, DA^+ DA^-$$

(A^+, A^-) : $SU(2)\times SU(2)$ - connection

S_{CS} : Chern-Simons action on S^3

$$h^\pm = \pm \frac{8}{(1\pm\beta)^2}\,\omega \qquad \omega = \Lambda \ell_p^2 = \ell_p^2/\ell_c^2$$

$\Psi_\gamma[A^\pm]$: projective spin-network function on graph γ embedded in S^3

Given an $SU(2)$ spin-network function ψ_γ with $(j_e\ i_v)$

EPRL map : $\quad j_e \longmapsto j_e^\pm = \frac{|1\pm\beta|}{2}\, j_e \quad$ (spin(4) spins)

$$i_v \longmapsto I_v = i_v^{a_1\cdots a_n}\, C_{a_1}^{b_1^+ b_1^-} \cdots C_{a_n}^{b_n^+ b_n^-}\, P_{b_1^\pm \cdots b_n^\pm}^{c_1^\pm \cdots c_n^\pm}$$

(Spin(4) intertwiner)

$C_a^{b^+ b^-}(j, j^+, j^-)$ CG coefficients

Projector to the space of Spin(4) intertwiners

$\longrightarrow$ Construct an Spin(4) spin-network Ψ_γ from ψ_γ

Dupuis & Livine '10.

The (arbitrary valent) vertex amplitude $A_v^q = A_v^q[\psi_\gamma]$

is a functional on $CYL \subset \mathcal{H}_{kin}$ via Chern-Simons and EPRL map.

Motivated by L. Smolin '95

Rigorous definition: $A_v^q = \omega_{Spin(4)} \circ \hat{Z}(\gamma)_{h^+, h^-}$

$\uparrow$

Evaluation using spin(4) rep & intertwiner which are images of EPRL map.

Relate to Gambini, Griego, Pullin '98

Generalize to Lorentzian theory:

$$A_v^q = \omega_{SL(2,\mathbb{C})} \circ \hat{Z}(\gamma)_h$$

A finite spinfoam amplitude on a 2-complex

$$A^q(K) = \sum_{j_f\, i_e} \prod_f \mu(j_f) \prod_v A_v^q(j_f, i_e)$$

The (arbitrary valent) vertex amplitude $A_v^q = A_v^q[\psi_\gamma]$ is a functional on $CYL \subset \mathcal{H}_{kin}$ via Chern-Simons and EPRL map.

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Asymptotics (Euclidean, 4-simplex amplitude) M.H '11

Asymptotic regime: $\ell_p^2 \ll \mathrm{Area}_f \ll \ell_c^2$

We keep triangle area fixed, but scale

$$\ell_p^2 \longrightarrow \lambda^{-1}\ell_p^2 \qquad \lambda \to \infty$$
$$\ell_c^2 \longrightarrow \lambda\,\ell_c^2$$

From $\mathrm{Area} = j_f\,\ell_p^2$, $\omega = \ell_p^2/\ell_c^2$

$$j_f \to \lambda\, j_f \qquad \omega \to \lambda^{-2}\omega$$

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Consider the expansion and the scaling $j \to \lambda j$, $\omega \to \omega \lambda^{-2}$

- Braiding $R = \exp \frac{ih}{2} \, | \; |$ $\qquad \sim 1 + o(1)$
- Unknot $\nu = | - \frac{h^2}{48} \, | + \cdots$ $\qquad \sim 1 + o(\frac{1}{\lambda})$
- Associator $\Phi = |\,|\,| - \frac{h^2}{12} \, |\,|\,| + \cdots \sim 1 + o(\frac{1}{\lambda})$

with a certain nondeg. boundary condition of 4-simplex

$$A^{q}_{4\text{-simplex}} \sim \left(\frac{2\pi}{\lambda}\right)^{12} e^{i\lambda \sum_f \gamma \tilde{j}_f \Theta_f} \underbrace{e^{i\omega V_\sigma}}_{\text{contribution from } R.} \Big[1 + \underbrace{o(\tfrac{1}{\lambda})}_{\nu,\ \Phi}\Big]$$

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contribution from R. $\qquad \nu, \Phi$

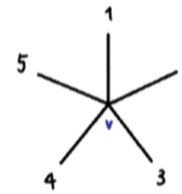
Some details:

$$A^{q}_{4\text{-simplex}} = \int \prod_{a=1}^{5} dg_a^+ dg_a^- \, K^{\omega} \underbrace{\prod_{a<b} \langle j_{ab}^{\pm}, n_{ab} | (g_a^{\pm})^{-1} g_b^{\pm} | j_{ab}^{\pm}, n_{ba} \rangle}_{\text{usual Euclidean spinfoam integrand}}$$

expansion of chord diagrams (Feynman diagrams)

From K^{ω}

n univalent vertices



$$\frac{\langle j_{ab} n_{ab} | g_a^{-1} X_{k_1} \cdots X_{k_n} g_b | j_{ab} n_{ba} \rangle}{\langle j_{ab} n_{ab} | g_a^{-1} g_b | j_{ab} n_{ba} \rangle}$$

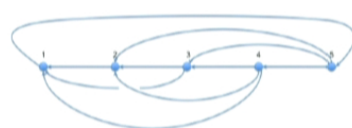
$j_{ab} \to \lambda j_{ab}$

$$\overset{\lambda\to\infty}{=} \lambda j_{ab} (g_b n_{ba})^{k_1} \cdots \lambda j_{ab} (g_b n_{ba})^{k_n} \left[1 + o\left(\tfrac{1}{\lambda}\right) \right] \sim \lambda^{n}$$

A chord diagram with n uni-valent & m tri-valent vertices of order $h^{\frac{m+n}{2}}$ $(h \propto \omega)$, contributing $\lambda^{-(m+n)}$ from coupling constant

→ a diagram with m  contribut λ^{-m}

→ leading contribution: $m=0$ only comes from R-matrix



$$A^{q}_{4\text{-simplex}} \sim \left(\frac{2\pi}{\lambda}\right)^{12} \underbrace{e^{i\lambda \sum_f \gamma \hat{j}_f \Theta_f} \; e^{i\omega V_\sigma}}_{\Lambda\text{-Regge action with flat simplex}} \left[1 + o\left(\tfrac{1}{\lambda}\right)\right]$$

Λ-Regge action with flat simplex

consistent with $\ell_p^2 \ll \ell_\sigma^2 \ll \ell_c^2$

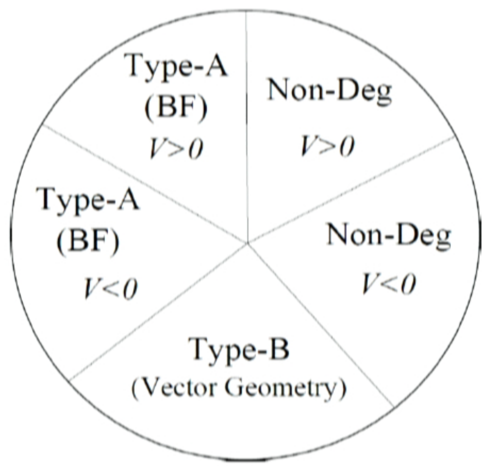
scale of 4-simplex

On a simplicial complex K

$$A^{g}(K) = \sum_{j_f, i_e} \prod_f \mu(j_f) \prod_v A_v^{g}(j_f, i_e)$$

Asymptotics of spinfoam amplitude on a simplicial complex

classify the critical configurations into 5 types



M.H & M.Zhang '11.

Recall the asymptotic formula:

$$f(\lambda) = \int dx\ a(x)\ e^{\lambda S(x)} \qquad \text{with } \operatorname{Re} S \leq 0$$

As $\lambda \to \infty$

$$f(\lambda) = \sum_{x_c} a(x_c) \left(\frac{2\pi}{\lambda}\right)^{\frac{r(x_c)}{2}} \frac{e^{i\,\mathrm{Ind}\,H'(x_c)}}{\sqrt{|\det_r H'(x_c)|}}\ e^{\lambda S(x_c)} \left[1 + o\left(\tfrac{1}{\lambda}\right)\right]$$

action at critical point

on different type of geometries:

Nondeg, $V_4 > 0$ or $V_4 < 0$

$$e^{\lambda S(x_c)} = e^{-i\lambda\left[\varepsilon\,\mathrm{sgn} V_4 \sum_f \gamma j_f \Theta_f + \pi \sum_e n_e \sum_{f \subset t_e} j_f\right]}\ e^{i\sum_\sigma \omega V_\sigma}$$

Deg-A, $V_4 > 0$ or $V_4 < 0$ (asymptotics of BF type)

$$e^{\lambda S(x_c)} = e^{-i\lambda\left[\varepsilon\,\mathrm{sgn} V_4 \sum_f j_f \Theta_f + \pi \sum_e n_e \sum_{f \subset t_e} j_f\right]}$$

Deg-B

$$e^{\lambda S(x_c)} = e^{-i\lambda \sum_f j_f \Phi_f}$$

← an angle from vector geometry.

Summary

- We formulate the spinfoam vertex amplitude in terms of Chern-Simons theory and Vassiliev-Kontsevich invariant.

- The constuction is essentially the solution of Knizhnik-Zamolodchikov Equation, with regularization & normalization, combined with EPRL map.

- The resulting spinfoam model is finite, and relate discrete GR with Λ in the regime $\ell_p^2 \ll \ell_\sigma^2 \ll \ell_c^2$.

Some details:

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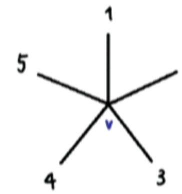

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$\lambda \to \infty$

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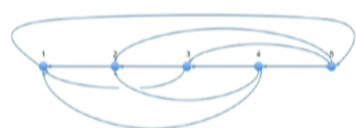

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