Title: Constraining Conformal Field Theories with a Higher Spin Symmetry

Date: Dec 06, 2011 02:00 PM

URL: http://pirsa.org/11120025

Abstract: We study the constraints imposed by the existence of a single higher spin conserved current on a three dimensional conformal field theory. A single higher spin conserved current implies the existence of an infinite number of higher spin conserved currents. The correlation functions of the stress tensor and the conserved currents are then shown to be equal to those of a free field theory. Namely a theory of N free bosons or free fermions. This is an extension of the Coleman-Mandula theorem to CFT's, which do not have a conventional S matrix. We also briefly discuss the case where the higher spin symmetries are ``slightly'' broken.


(j1=0, j2) Δ ≥ j2 + 1
is a conserved current

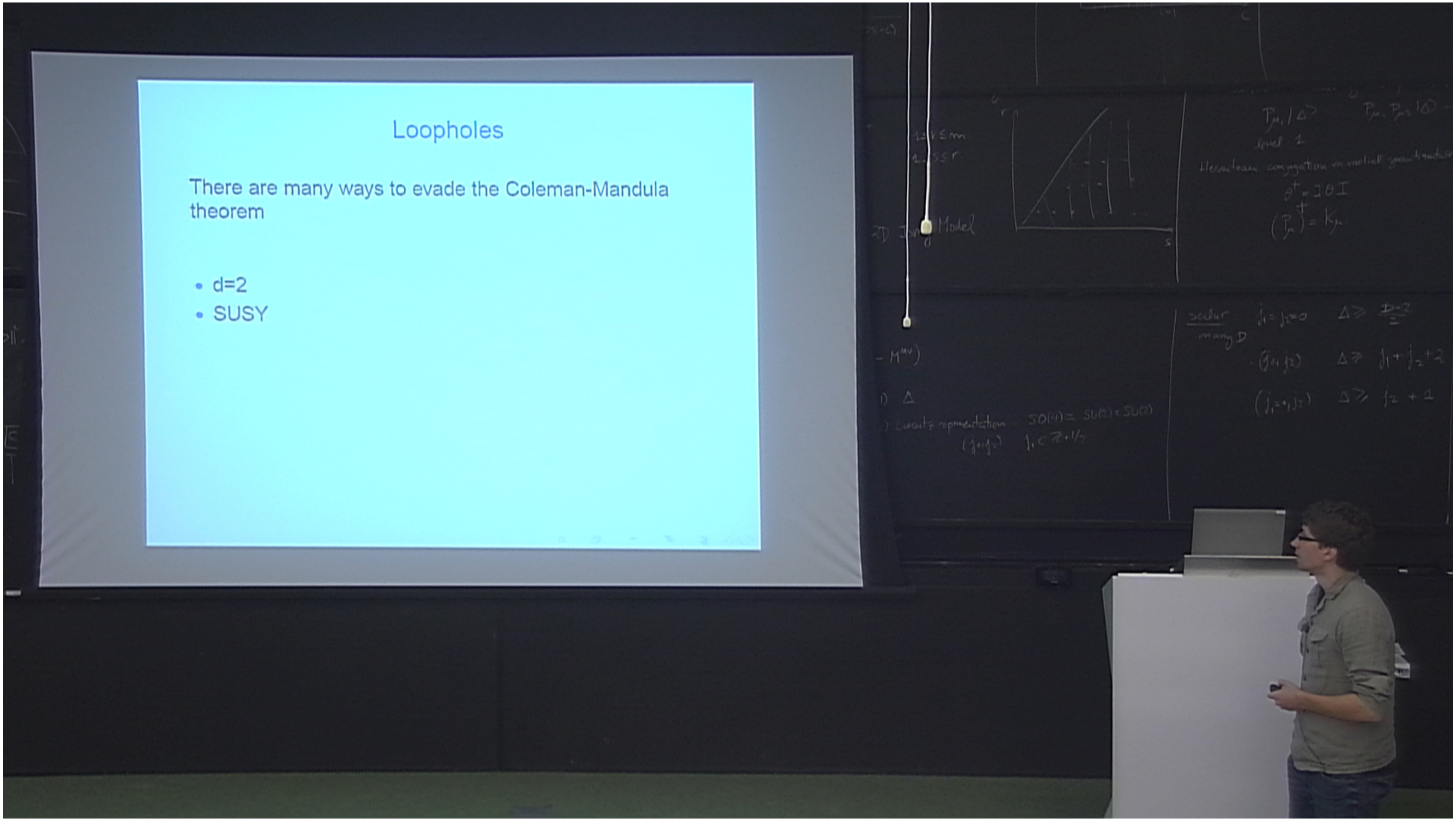
The Coleman-Mandula theorem (mod subtleties)

Assumptions:

- the S matrix exists: a theory has a mass gap (theory is IR free);
- the S matrix is nontrivial: everything scatters into something;
- the Poincare group is part of the symmetry group;

Conclusion:

- The symmetry group is the direct product of an int
  symmetry group and the Poincare group.


Loopholes
There are many ways to evade the Coleman-Mandula theorem
d=2
SUSY

Loopholes

There are many ways to evade the Coleman-Mandula theorem

- d=2
- SUSY
- CFT
- AdS
- d>2
- superalgebra (HLS 1975)
- the S matrix does not exist
- Vasiliev theory

Is there a Coleman-Mandula theorem for AdS physics? Or, in other words...

Known symmetries of nontrivial CFTs

Before trying to prove that something is impossible let's summarize what we know is possible:

- symmetry can be infinite dimensional in $d = 2 \rightarrow d > 2$;
- SUSY $\rightarrow$ generators or currents of half-integer spin;
- internal symmetries are definitely allowed;
- no examples of non-trivial CFTs with conserved currents of spin higher than two.

Could we have more symmetries while having non-trivial correlation functions?

Known symmetries of nontrivial CFTs

Before trying to prove that something is impossible let's summarize what we know is possible:

- symmetry can be infinite dimensional in $d = 2 \to d > 2$;
- SUSY $\to$ generators or currents of half-integer spin;
- internal symmetries are definitely allowed;
- no examples of non-trivial CFTs with conserved currents of spin higher than two.

Could we have more symmetries while having non-trivial correlation functions?

The answer

We arrive at the conclusion that the answer to this question is

No

Let’s proceed to the assumptions and conclusions...

Assumptions

Let's consider a set of QFTs for which the following is true:

- CFT: $\mathcal{H} = \bigoplus[\mathcal{O}_{\Delta,s}]$, OPE, j_2, cluster decomposition;
- the theory is unitary;
- the theory contains conserved current j_s of spin higher than two $s > 2$;
- the two-point function of stress tensors is finite.

Additional assumptions

- $d = 3$ (for $d > 3$ the same set of ideas is applicable);
- the stress tensor (or conserved current of spin two) is unique.

Assumptions

Let's consider a set of QFTs for which the following is true:

- CFT: $\mathcal{H} = \bigoplus[\mathcal{O}_{\Delta,s}]$, OPE, j_2, cluster decomposition;
- the theory is unitary;
- the theory contains conserved current j_s of spin higher than two $s > 2$;
- the two-point function of stress tensors is finite.

Additional assumptions

- $d = 3$ (for $d > 3$ the same set of ideas is applicable);
- the stress tensor (or conserved current of spin two) is unique.

Conclusion

The theory contains an infinite number of currents j_s that appears in the OPE of $j_2 j_2$.
Their correlation functions

$$\langle j_{s_1}(x_1) \ldots j_{s_n}(x_n) \rangle$$

are fixed to be the free boson or the free fermion ones up to a free integer number.
This number is the coefficient in the two-point function of stress tensors.
Start with N bosons or fermions. Compute the correlat
functions of $O(N)$ singlet bilinear currents.

Outline

- General idea
- Exploring the twist gap
- Conserved currents sector
- Bilocal operators
- Theories with higher spin symmetries broken at the $\frac{1}{N}$ level

General idea

We consider CFT on the plane.
We start from the extra conserved current j_s, build the extra symmetry charges

$$
\begin{aligned}
Q_s^\zeta &= \int_{\Sigma_{d-1}} *j_s^\zeta \\
j_s^\zeta &= j_{\mu\mu_1\ldots\mu_{s-1}}\zeta^{\mu_1\ldots\mu_{s-1}} \\
[Q_s^\zeta, \mathcal{O}(x)] &= \int_{S_{x+\epsilon}} *j_s^\zeta(x+\epsilon)\mathcal{O}(x)
\end{aligned}
$$

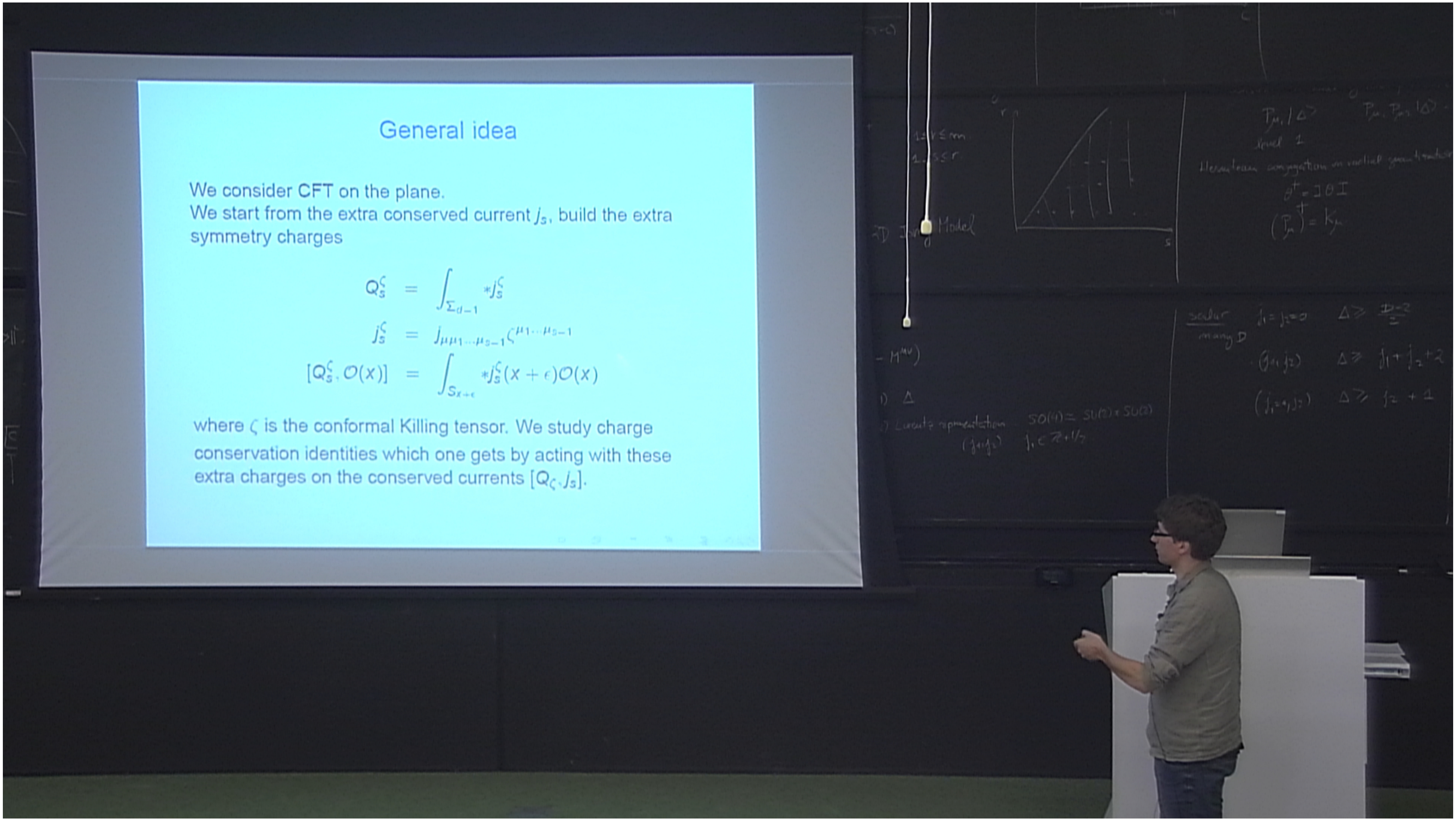
where ζ is the conformal Killing tensor. We study charge conservation identities which one gets by acting with these extra charges on the conserved currents $[Q_\zeta, j_s]$.

General idea

We consider CFT on the plane.
We start from the extra conserved current j_s, build the extra symmetry charges

$$
\begin{aligned}
Q_s^\zeta &= \int_{\Sigma_{d-1}} *j_s^\zeta \\
j_s^\zeta &= j_{\mu\mu_1\ldots\mu_{s-1}}\zeta^{\mu_1\ldots\mu_{s-1}} \\
[Q_s^\zeta, \mathcal{O}(x)] &= \int_{S_{x+\epsilon}} *j_s^\zeta(x+\epsilon)\mathcal{O}(x)
\end{aligned}
$$

where ζ is the conformal Killing tensor. We study charge conservation identities which one gets by acting with these extra charges on the conserved currents $[Q_\zeta, j_s]$.


General idea
We consider CFT on the plane.
We start from the extra conserved current j_s, build the extra symmetry charges
$Q_s^\zeta = \int_{\Sigma_{d-1}} *j_s^\zeta$
$j_s^\zeta = j_{\mu\mu_1\dots\mu_{s-1}}\zeta^{\mu_1\dots\mu_{s-1}}$
$[Q_s^\zeta, \mathcal{O}(x)] = \int_{S_{x+\epsilon}} *j_s^\zeta(x+\epsilon)\mathcal{O}(x)$
where ζ is the conformal Killing tensor. We study charge conservation identities which one gets by acting with these extra charges on the conserved currents $[Q_\zeta, j_s]$.

Twist gap

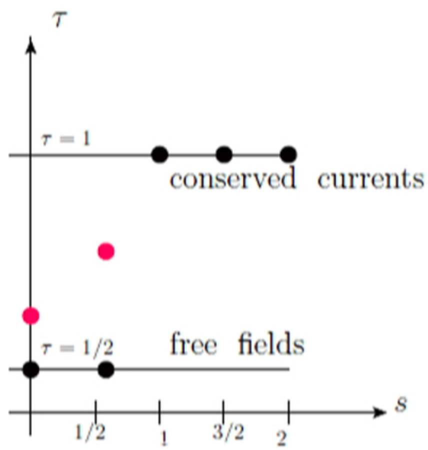


Figure: Spectrum of the unitary CFT in $d = 3$

The unitary constrains the possible dimensions of operators as follows

$$\Delta \geq s + 1/2, \quad s = 0, 1/2$$
$$\Delta \geq s + 1, \quad s \geq 1$$

Thus, if we introduce the twist $\tau = \Delta - s$ then the operators with the twist

$$1/2 \leq \tau < 1$$

could have only 0 or 1/2 spin.

Twist gap II

Let's imagine that we have some CFT such that there is a scalar operator ϕ_Δ with the twist lying inside the twist gap

$$\Delta = \frac{1}{2} + \gamma, \quad \gamma < 1/2.$$

Suppose also that it is charged under the $j_4 \to Q_4$.
Let's build the charge Q_{---} using CKT along the minus direction.

Twist gap III

Thus, from the higher spin WI we conclude that

$$\langle \phi_\Delta(x_1)\phi_\Delta(x_2)\phi_\Delta(x_3)\phi_\Delta(x_4)\rangle = \langle \phi_\Delta(x_1)\phi_\Delta(x_2)\rangle \ \langle \phi_\Delta(x_3)\phi_\Delta(x_4)\rangle + \ldots$$

Due to the fact that all operators couple to stress tensor $\langle \phi_\Delta \phi_\Delta T\rangle \neq 0$ and the fact that $\langle TT\rangle$ is finite, the stress tensor should be present in the OPE. El-Showk, Papadodimas

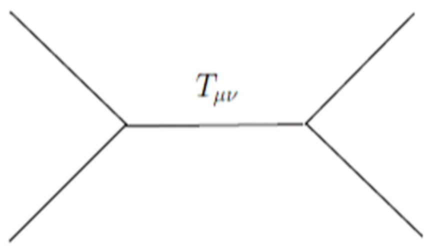


Figure: Stress tensor should be present in the OPE

Free fields

Let's consider the free scalar field $\phi(x)$ and let's consider the charge built using constant CKT ζ. Then the action of this charge on the free field is

$$[Q_s^\zeta, \phi(x)] = \zeta^{\mu_1 \dots \mu_{s-1}} \partial_{\mu_1} \dots \partial_{\mu_{s-1}} \phi(x)$$

Consider now the charge conservation identity that we get by acting on the correlation function

$$\langle [Q_s^\zeta, \phi(x_1)\phi(x_2)\dots\phi(x_n)] \rangle = 0$$

in momentum space it takes the form

$$(\sum_{i=1}^{n} k_i^{s-1}) \langle \phi(k_1)\phi(k_2)\dots\phi(k_n) \rangle = 0$$

so that the correlation functions factorize!

Conserved currents: basic properties

To attack the sector of conserved currents let's recall the basic properties of three point functions of conserved currents (Giombi-Prakash-Yin, Costa-Penedones-Poland-Rychkov).

$$\langle j_{s_1} j_{s_2} j_{s_3} \rangle = \langle \text{boson} \rangle + \langle \text{fermion} \rangle + \langle \text{odd} \rangle$$

where

$$\mathcal{F}_{even} = e^{\frac{1}{2}(Q_1+Q_2+Q_3)} e^{P_1+P_2} (b \cosh P_3 + f \sinh P_3)$$

and the odd piece is given by

$$\langle j_{s_1}(\vec{x}_1, \lambda_1) j_{s_2}(\vec{x}_2, \lambda_2) j_{s_3}(\vec{x}_3, \lambda_3) \rangle_{odd} \sim \int dt d^3\vec{x}_0 t^{s_1+s_2+s_3-1} \frac{(\lambda_1 \not{x}_{10} \not{x}_{02} \lambda_2)^{(s_1+s_2-s_3)} (\lambda_1 \not{x}_{10} \not{x}_{03} \lambda_3)^{(s_1+s_3-s_2)} (\lambda_2 \not{x}_{20} \not{x}_{03} \lambda_3)^{(s_2+s_3-s_1)}}{(x_{10}^2)^{2s_1+1} (x_{20}^2)^{2s_2+1} (x_{30}^2)^{2s_3+1}}$$

Conserved currents: basic properties

To attack the sector of conserved currents let's recall the basic properties of three point functions of conserved currents (Giombi-Prakash-Yin, Costa-Penedones-Poland-Rychkov).

$$\langle j_{s_1} j_{s_2} j_{s_3} \rangle = \langle \text{boson} \rangle + \langle \text{fermion} \rangle + \langle \text{odd} \rangle$$

where

$$\mathcal{F}_{even} = e^{\frac{1}{2}(Q_1+Q_2+Q_3)} e^{P_1+P_2} (b \cosh P_3 + f \sinh P_3)$$

and the odd piece is given by

$$\langle j_{s_1}(\vec{x}_1, \lambda_1) j_{s_2}(\vec{x}_2, \lambda_2) j_{s_3}(\vec{x}_3, \lambda_3) \rangle_{odd} \sim \int dt d^3 \vec{x}_0 t^{s_1+s_2+s_3-1} \frac{(\lambda_1 \not{x}_{10} \not{x}_{02} \lambda_2)^{(s_1+s_2-s_3)} (\lambda_1 \not{x}_{10} \not{x}_{03} \lambda_3)^{(s_1+s_3-s_2)} (\lambda_2 \not{x}_{20} \not{x}_{03} \lambda_3)^{(s_2+s_3-s_1)}}{(x_{10}^2)^{2s_1+1} (x_{20}^2)^{2s_2+1} (x_{30}^2)^{2s_3+1}}$$

Conserved currents: basic properties

To attack the sector of conserved currents let's recall the basic properties of three point functions of conserved currents (Giombi-Prakash-Yin, Costa-Penedones-Poland-Rychkov).

$$\langle j_{s_1} j_{s_2} j_{s_3} \rangle = \langle \text{boson} \rangle + \langle \text{fermion} \rangle + \langle \text{odd} \rangle$$

where

$$\mathcal{F}_{even} = e^{\frac{1}{2}(Q_1+Q_2+Q_3)} e^{P_1+P_2} (b \cosh P_3 + f \sinh P_3)$$

and the odd piece is given by

$$\langle j_{s_1}(\vec{x}_1, \lambda_1) j_{s_2}(\vec{x}_2, \lambda_2) j_{s_3}(\vec{x}_3, \lambda_3) \rangle_{odd} \sim \int dt d^3 \vec{x}_0 t^{s_1+s_2+s_3-1} \frac{(\lambda_1 \not{x}_{10} \not{x}_{02} \lambda_2)^{(s_1+s_2-s_3)} (\lambda_1 \not{x}_{10} \not{x}_{03} \lambda_3)^{(s_1+s_3-s_2)} (\lambda_2 \not{x}_{20} \not{x}_{03} \lambda_3)^{(s_2+s_3-s_1)}}{(x_{10}^2)^{2s_1+1} (x_{20}^2)^{2s_2+1} (x_{30}^2)^{2s_3+1}}$$

Conserved currents: basic properties

To attack the sector of conserved currents let's recall the basic properties of three point functions of conserved currents (Giombi-Prakash-Yin, Costa-Penedones-Poland-Rychkov).

$$\langle j_{s_1} j_{s_2} j_{s_3} \rangle = \langle \text{boson} \rangle + \langle \text{fermion} \rangle + \langle \text{odd} \rangle$$

where

$$\mathcal{F}_{even} = e^{\frac{1}{2}(Q_1+Q_2+Q_3)} e^{P_1+P_2} (b \cosh P_3 + f \sinh P_3)$$

and the odd piece is given by

$$\langle j_{s_1}(\vec{x}_1, \lambda_1) j_{s_2}(\vec{x}_2, \lambda_2) j_{s_3}(\vec{x}_3, \lambda_3) \rangle_{odd} \sim \int dt d^3\vec{x}_0 t^{s_1+s_2+s_3-1}$$
$$\frac{(\lambda_1 \not{x}_{10} \not{x}_{02} \lambda_2)^{(s_1+s_2-s_3)} (\lambda_1 \not{x}_{10} \not{x}_{03} \lambda_3)^{(s_1+s_3-s_2)} (\lambda_2 \not{x}_{20} \not{x}_{03} \lambda_3)^{(s_2+s_3-s_1)}}{(x_{10}^2)^{2s_1+1} (x_{20}^2)^{2s_2+1} (x_{30}^2)^{2s_3+1}}$$

Conserved currents: basic properties

Other useful properties to remember are

$$\langle \mathcal{O}\mathcal{O}T \rangle \neq 0$$

and also

$$\langle j_s j_s j_{s'} \rangle = 0$$

when s' is odd.

We will be again interested in all-minus charges $Q_s = Q_{-\ldots-}$.
So let's introduce the following notations

$$\langle j_{s_1 - \ldots -}(x_1) j_{s_2 - \ldots -}(x_2) j_{s_3 - \ldots -}(x_3) \rangle = \langle s_1 s_2 s_3 \rangle$$

Rule of thumb: whenever you think that there should be a tensor index — it is the minus index.

Conserved currents: basic properties

Other useful properties to remember are

$$\langle \mathcal{O}\mathcal{O}T \rangle \neq 0$$

and also

$$\langle j_s j_s j_{s'} \rangle = 0$$

when s' is odd.
We will be again interested in all-minus charges $Q_s = Q_{-\dots-}$.
So let's introduce the following notations

$$\langle j_{s_1 - \dots -}(x_1) j_{s_2 - \dots -}(x_2) j_{s_3 - \dots -}(x_3) \rangle = \langle s_1 s_2 s_3 \rangle$$

Rule of thumb: whenever you think that there should be a tensor index — it is the minus index.

Conserved currents: action of the higher spin charges

As the next step consider the action of minus-charge on the all-minus component of conserved currents. Again, unitarity fixes it up to several constants

$$[Q_s, j_k] = \sum_{i=-s}^{s} c_i \partial^{(s-i)} j_{k+i}.$$

We can do a little bit better

$$[Q_s, j_2] \sim \partial j_{s-1}.$$

This term must be there! And this opens the flow...

$$[Q_s, X] \sim Y$$

$$[Q_s, Y] \sim X$$

from $\langle [Q, XY] \rangle = 0$.

Conserved currents: action of the higher spin charges

As the next step consider the action of minus-charge on the all-minus component of conserved currents. Again, unitarity fixes it up to several constants

$$[Q_s, j_k] = \sum_{i=-s}^{s} c_i \partial^{(s-i)} j_{k+i}.$$

We can do a little bit better

$$[Q_s, j_2] \sim \partial j_{s-1}.$$

This term must be there! And this opens the flow...

$$[Q_s, X] \sim Y$$

$$[Q_s, Y] \sim X$$

from $\langle [Q, XY] \rangle = 0$.

Conserved currents: action of the higher spin charges

As the next step consider the action of minus-charge on the all-minus component of conserved currents. Again, unitarity fixes it up to several constants

$$[Q_s, j_k] = \sum_{i=-s}^{s} c_i \partial^{(s-i)} j_{k+i}.$$

We can do a little bit better

$$[Q_s, j_2] \sim \partial j_{s-1}.$$

This term must be there! And this opens the flow...

$$[Q_s, X] \sim Y$$

$$[Q_s, Y] \sim X$$

from $\langle [Q, XY] \rangle = 0$.

Conserved currents: example

Consider the theory that contains spin four current j_4. From it we build minus charge Q_4. When acting on the stress tensor

$$[Q_4, 2] \sim \partial 4$$

Let's consider the $\langle [Q_4, 224] \rangle = 0$ CCI.
On general grounds there will be

$$\partial_{x_1} \langle 4(x_1) 2(x_2) 4(x_3) \rangle \neq 0$$

thus, we get an algebraic equation

$$c_{422} \partial_{x_1} \langle 4(x_1) 2(x_2) 4(x_3) \rangle + c_{222} \partial^3_{x_1} \langle 2(x_1) 2(x_2) 4(x_3) \rangle + c_{022} \partial^5_{x_1} \langle 0(x_1) 2(x_2) 4(x_3) \rangle + \ldots = 0$$

on the coefficients...

Conserved currents: example

Consider the theory that contains spin four current j_4. From it we build minus charge Q_4. When acting on the stress tensor

$$[Q_4, 2] \sim \partial 4$$

Let's consider the $\langle [Q_4, 224] \rangle = 0$ CCI.
On general grounds there will be

$$\partial_{x_1} \langle 4(x_1) 2(x_2) 4(x_3) \rangle \neq 0$$

thus, we get an algebraic equation

$$c_{422} \partial_{x_1} \langle 4(x_1) 2(x_2) 4(x_3) \rangle + c_{222} \partial^3_{x_1} \langle 2(x_1) 2(x_2) 4(x_3) \rangle + c_{022} \partial^5_{x_1} \langle 0(x_1) 2(x_2) 4(x_3) \rangle + \ldots = 0$$

on the coefficients...

Conserved currents: results

From this simple exercise we can learn that

- there are three families of solutions (boson, fermion, odd); Is there interacting HS CFT with odd parts?
- if 4 is present, 6 is necessary present;
- for boson and odd the scalar 0 is necessary present.

Repeating a similar exercise for the scalar $\langle 022\rangle$ one can show that uniqueness of the stress tensor restricts

$$\begin{aligned} \langle 222\rangle &= \langle \mathrm{boson}\rangle + \langle \mathrm{odd}\rangle \\ \langle 222\rangle &= \langle \mathrm{fermion}\rangle \end{aligned}$$

This is one of the many examples when boson and fermion solutions are separated.

Energy one point function

Knowing something about three point functions of stress tensors allows one to compute the so-called one point energy correlator .

$$\langle \mathcal{O}[\Psi] | \hat{\mathcal{E}}(\vec{n}) | \mathcal{O}[\Psi] \rangle$$

$$\hat{\mathcal{E}}(\vec{n}) = \lim_{r \to \infty} r^2 \int_{-\infty}^{\infty} dt n^i T_i^0(t, r\vec{n})$$

(Hofman-Maldacena)

Some intuition about this object:

- measure the energy flow at infinity;
- small coupling - jets and showering;
- strong coupling - uniform flow.

Energy one point function: no showering, no odd piece

Consider the case when three point function of stress tensors is boson+odd. Then one can show that

$$\langle T_{11} - T_{22} | \mathcal{E}(\vec{n}) | T_{11} - T_{22} \rangle = \frac{q^0}{2\pi}(1 + \cos 4\theta + d_{\mathrm{odd}} \sin 4\theta)$$

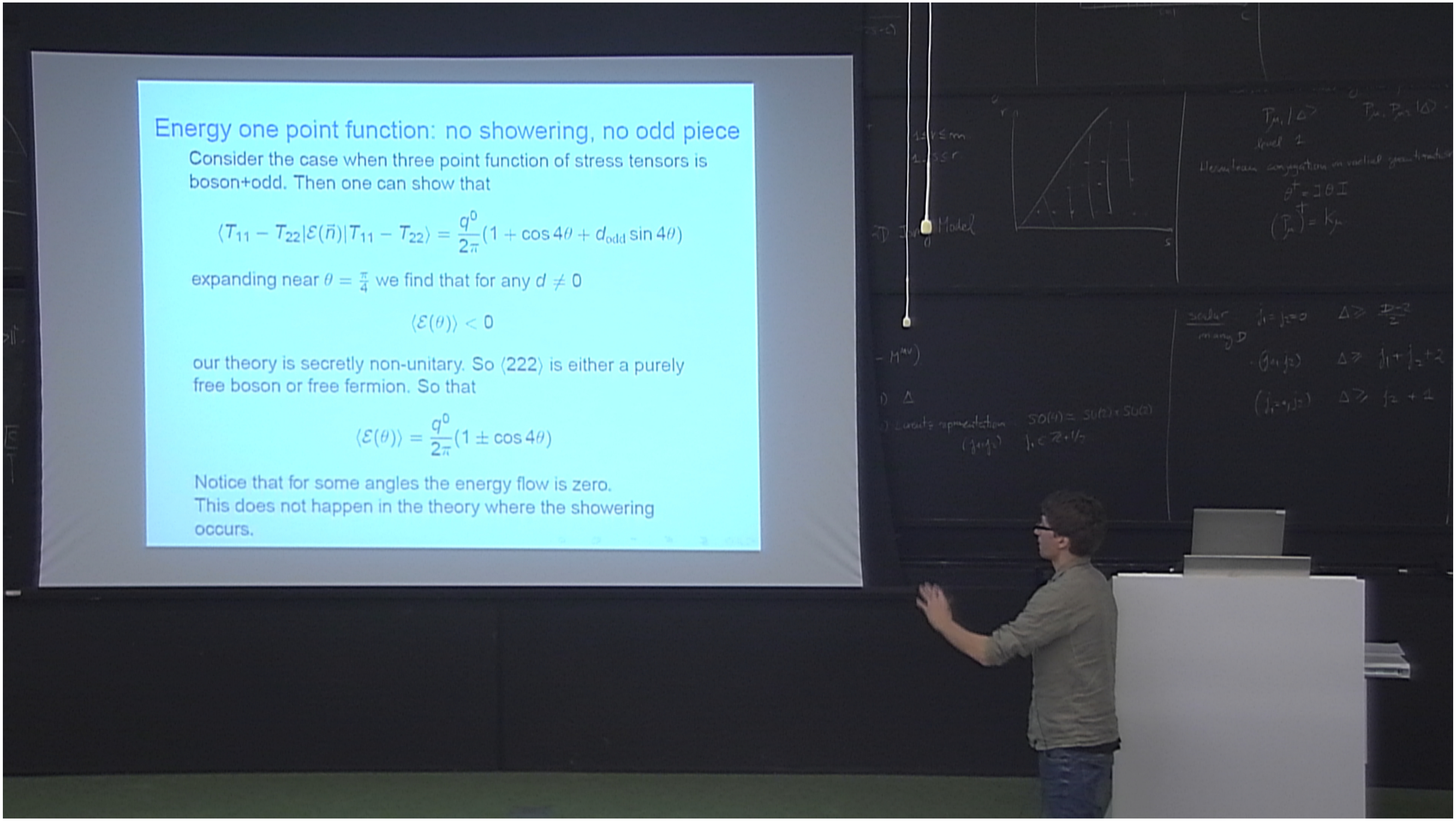
expanding near $\theta = \frac{\pi}{4}$ we find that for any $d \neq 0$

$$\langle \mathcal{E}(\theta) \rangle < 0$$

our theory is secretly non-unitary. So $\langle 222 \rangle$ is either a purely free boson or free fermion. So that

$$\langle \mathcal{E}(\theta) \rangle = \frac{q^0}{2\pi}(1 \pm \cos 4\theta)$$

Notice that for some angles the energy flow is zero.
This does not happen in the theory where the showering occurs.


Energy one point function: no showering, no odd piece
Consider the case when three point function of stress tensors is boson+odd. Then one can show that
$\langle T_{11} - T_{22} | \mathcal{E}(\vec{n}) | T_{11} - T_{22} \rangle = \frac{q^0}{2\pi}(1 + \cos 4\theta + d_{\rm odd} \sin 4\theta)$
expanding near $\theta = \frac{\pi}{4}$ we find that for any $d \neq 0$
$\langle \mathcal{E}(\theta) \rangle < 0$
our theory is secretly non-unitary. So $\langle 222 \rangle$ is either a purely free boson or free fermion. So that
$\langle \mathcal{E}(\theta) \rangle = \frac{q^0}{2\pi}(1 \pm \cos 4\theta)$
Notice that for some angles the energy flow is zero.
This does not happen in the theory where the showering occurs.

Four point functions

One can also consider the four point functions of four scalars 0.
After showing that

$$[Q_4, 0] = \partial^3 0 + \partial 2$$

we get the differential equation

$$\partial^3 \langle 0000 \rangle + \partial \langle 2000 \rangle + ... = 0.$$

This time we need to solve genuine differential equations for the functions of cross ratios...
The result is that the solution is fixed up to one constant

$$\langle 0000 \rangle = \langle \text{disconnected} \rangle + \frac{1}{c} \langle \text{connected} \rangle$$

Four point functions: conclusion

- using only one additional charge we can fix two correlation functions $\langle 0000 \rangle$ and $\langle 2000 \rangle$;
- $\langle 2000 \rangle_{odd}$ must be set to zero to obey CCI, by the OPE it sets to zero odd piece of three point functions;
- the free fermion story is very similar.

We got this using only one additional charge in a very explicit way...
But we have an infinite number of them. So there should be a shorter way to the answer that uses all the symmetries!

Four point functions: conclusion

- using only one additional charge we can fix two correlation functions $\langle 0000 \rangle$ and $\langle 2000 \rangle$;
- $\langle 2000 \rangle_{odd}$ must be set to zero to obey CCI, by the OPE it sets to zero odd piece of three point functions;
- the free fermion story is very similar.

We got this using only one additional charge in a very explicit way...
But we have an infinite number of them. So there should be a shorter way to the answer that uses all the symmetries!

Taking the light cone limit

Let's consider the light cone limit of two conserved currents $j_s(x)j_{s'}(0)$. There are three types of limits that project to three different parts of three point functions (boson, fermion, odd). For simplicity we present here the bosonic limit

$$\underline{j_s(x)j_{s'}(0)}_b = \left(\lim_{y_{12}\to 0^+} + \lim_{y_{12}\to 0^-}\right)|y_{12}| \lim_{x_{12}^+\to 0} j_s(x_1)j_{s'}(x_2)$$

where we implicitly eliminated all operators of twist less than one.

$$ds^2 = dx^+dx^- + dy^2$$

Taking the light cone limit

Let's consider the light cone limit of two conserved currents $j_s(x)j_{s'}(0)$. There are three types of limits that project to three different parts of three point functions (boson, fermion, odd). For simplicity we present here the bosonic limit

$$\underline{j_s(x)j_{s'}(0)}_b = \left(\lim_{y_{12}\to 0^+} + \lim_{y_{12}\to 0^-}\right)|y_{12}| \lim_{x_{12}^+\to 0} j_s(x_1)j_{s'}(x_2)$$

where we implicitly eliminated all operators of twist less than one.

$$ds^2 = dx^+dx^- + dy^2$$

Simplification of the three point functions

The crucial simplification that occurs in the limit is the following

$$\langle j_s j_{s'} j_{s''}(x_3)\rangle = \partial_1^s \partial_s^{s'} \langle \phi(x_1)\phi^*(x_2) j_{s''}(x_3)\rangle_{free}$$

where

$$\langle \phi(x_1)\phi^*(x_2) j_{s''}(x_3)\rangle = \frac{1}{\sqrt{\hat{x}_{13}\hat{x}_{23}}}\left(\frac{\hat{x}_{12}}{\hat{x}_{13}\hat{x}_{23}}\right)^{s''}$$

$$\hat{x}_1 = x_1^-, \quad \hat{x}_2 = x_2^-, \quad \hat{x}_3 = x_3^- - \frac{y_{13}^2}{x_{13}^+}$$

and all indices are minuses as usual.
This allows us to analyze infinitely many CCIs in a very simple manner.

Simplification of the three point functions

The crucial simplification that occurs in the limit is the following

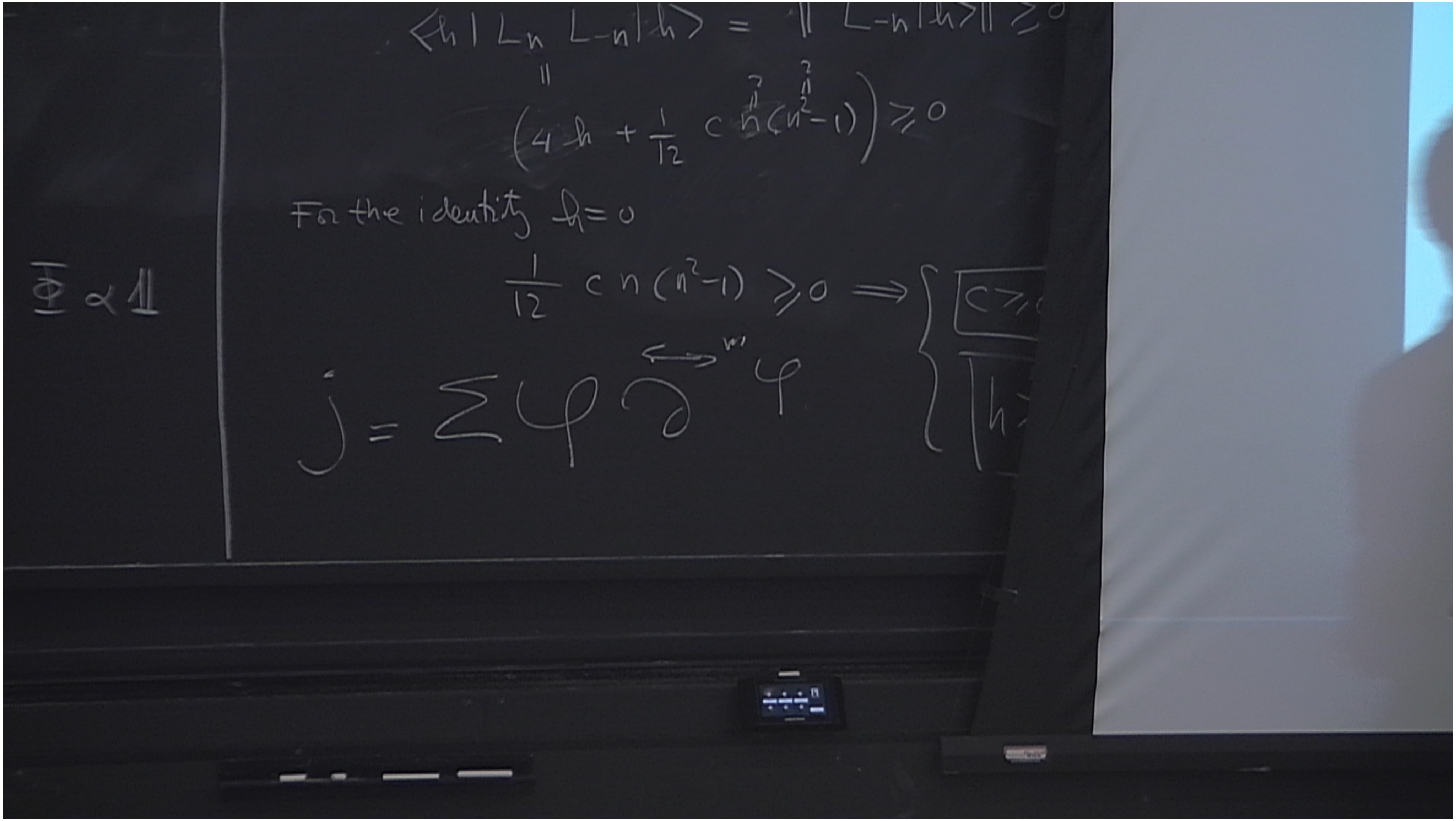
$$\langle j_s j_{s'} {}_b j_{s''}(x_3)\rangle = \partial_1^s \partial_s^{s'} \langle \phi(x_1)\phi^*(x_2) j_{s''}(x_3)\rangle_{free}$$

where

$$\langle \phi(x_1)\phi^*(x_2) j_{s''}(x_3)\rangle = \frac{1}{\sqrt{\hat{x}_{13}\hat{x}_{23}}}\left(\frac{\hat{x}_{12}}{\hat{x}_{13}\hat{x}_{23}}\right)^{s''}$$

$$\hat{x}_1 = x_1^-, \quad \hat{x}_2 = x_2^-, \quad \hat{x}_3 = x_3^- - \frac{y_{13}^2}{x_{13}^+}$$

and all indices are minuses as usual.
This allows us to analyze infinitely many CCIs in a very simple manner.


⟨h| L_n L_-n |h⟩ = || L_-n |h⟩ || ≥ 0
(4h + 1/12 c n(n²−1)) ≥ 0
For the identity h = 0
1/12 c n(n²−1) ≥ 0 ⟹
Φ ∝ 𝟙
j = Σ φ ∂ φ

$\Phi)|0\rangle||^2 = 0$

$\Phi = 0$

$\Phi = 0 \iff \Phi \propto \mathbb{1}$

$\partial_1 \partial_2 \frac{1}{X}$

$(4h + \frac{1}{12} c\, n(\ldots$

For the identity $h = 0$

$\frac{1}{12} c\, n(n^2-1)$

$j = \sum \varphi \overleftrightarrow{\partial}^{m} \varphi$

$= 0$

$\Longleftrightarrow \quad \Phi \propto \mathbb{1}$

$\frac{1}{|x|}$

$$\left(4h + \frac{1}{12} c\, n(n^2-1)\right) \geq$$

For the identity $h = 0$

$$\frac{1}{12} c\, n(n^2-1) \geq 0 \Longrightarrow$$

$$j = \sum \varphi\, \partial^n \varphi$$

Simplification of the three point functions

The crucial simplification that occurs in the limit is the following

$$\langle \underline{j_s j_{s'}}_b j_{s''}(x_3)\rangle = \partial_1^s \partial_s^{s'} \langle \underline{\phi(x_1)\phi^*(x_2)} j_{s''}(x_3)\rangle_{free}$$

where

$$\langle \underline{\phi(x_1)\phi^*(x_2)} j_{s''}(x_3)\rangle = \frac{1}{\sqrt{\hat{x}_{13}\hat{x}_{23}}}\left(\frac{\hat{x}_{12}}{\hat{x}_{13}\hat{x}_{23}}\right)^{s''}$$

$$\hat{x}_1 = x_1^-, \quad \hat{x}_2 = x_2^-, \quad \hat{x}_3 = x_3^- - \frac{y_{13}^2}{x_{13}^+}$$

and all indices are minuses as usual.
This allows us to analyze infinitely many CCIs in a very simple manner.

Simplification of the three point functions

The crucial simplification that occurs in the limit is the following

$$\langle j_s j_{s'} j_{s''}(x_3)\rangle = \partial_1^s \partial_s^{s'} \langle \phi(x_1)\phi^*(x_2) j_{s''}(x_3)\rangle_{free}$$

where

$$\langle \phi(x_1)\phi^*(x_2) j_{s''}(x_3)\rangle = \frac{1}{\sqrt{\hat{x}_{13}\hat{x}_{23}}}\left(\frac{\hat{x}_{12}}{\hat{x}_{13}\hat{x}_{23}}\right)^{s''}$$

$$\hat{x}_1 = x_1^-, \quad \hat{x}_2 = x_2^-, \quad \hat{x}_3 = x_3^- - \frac{y_{13}^2}{x_{13}^+}$$

and all indices are minuses as usual.
This allows us to analyze infinitely many CCIs in a very simple manner.

Simplification of the three point functions

The crucial simplification that occurs in the limit is the following

$$\langle \underline{j_s j_{s'}}_b j_{s''}(x_3)\rangle = \partial_1^s \partial_s^{s'} \langle \underline{\phi(x_1)\phi^*(x_2)} j_{s''}(x_3)\rangle_{free}$$

where

$$\langle \underline{\phi(x_1)\phi^*(x_2)} j_{s''}(x_3)\rangle = \frac{1}{\sqrt{\hat{x}_{13}\hat{x}_{23}}}\left(\frac{\hat{x}_{12}}{\hat{x}_{13}\hat{x}_{23}}\right)^{s''}$$

$$\hat{x}_1 = x_1^-, \quad \hat{x}_2 = x_2^-, \quad \hat{x}_3 = x_3^- - \frac{y_{13}^2}{x_{13}^+}$$

and all indices are minuses as usual.
This allows us to analyze infinitely many CCIs in a very simple manner.

Getting an infinite number of currents

Imagine we have a current of spin s. Then we know that

$$[Q_s, j_2] = \partial j_s + \cdots$$

and from $\langle [Q_s, j_2 j_s] \rangle = 0$

$$[Q_s, j_s] = \partial^{2s-3} j_2 + \cdots$$

Let's assume

$$\langle j_2 j_2 j_2 \rangle|_b \neq 0$$

We now consider the charge conservation identity $\langle [Q_s, \underline{j_2 j_2}_b j_s] \rangle = 0$ we get

$$0 = \langle \underline{[Q_s, j_2]} j_2 j_s \rangle + \langle j_2 \underline{[Q_s, j_2]} j_s \rangle + \langle \underline{j_2 j_2} [Q_s, j_s] \rangle$$

Getting an infinite number of currents

Imagine we have a current of spin s. Then we know that

$$[Q_s, j_2] = \partial j_s + \cdots$$

and from $\langle [Q_s, j_2 j_s] \rangle = 0$

$$[Q_s, j_s] = \partial^{2s-3} j_2 + \cdots$$

Let's assume

$$\langle j_2 j_2 j_2 \rangle |_b \neq 0$$

We now consider the charge conservation identity $\langle [Q_s, \underline{j_2 j_2}_b j_s] \rangle = 0$ we get

$$0 = \langle \underline{[Q_s, j_2]} j_2 j_s \rangle + \langle j_2 \underline{[Q_s, j_2]} j_s \rangle + \langle \underline{j_2 j_2} [Q_s, j_s] \rangle$$

Getting an infinite number of currents II

This equation takes the form

$$0 = \partial_1^2 \partial_2^2 \Lambda$$

where Λ is given by

$$\left[\gamma(\partial_1^{s-1} + (-1)^s \partial_2^{s-1})\langle \underline{\phi\phi^*} j_s\rangle_{free} + \sum_{k=1}^{2s-1} \tilde{\alpha}_k \partial_3^{2s-1-k} \langle \underline{\phi\phi^*} j_k\rangle_{free}\right]$$

the only solution of this equation is such that $\tilde{\alpha}_k \neq 0$ for $k = 2, 4, ..., 2s-2$

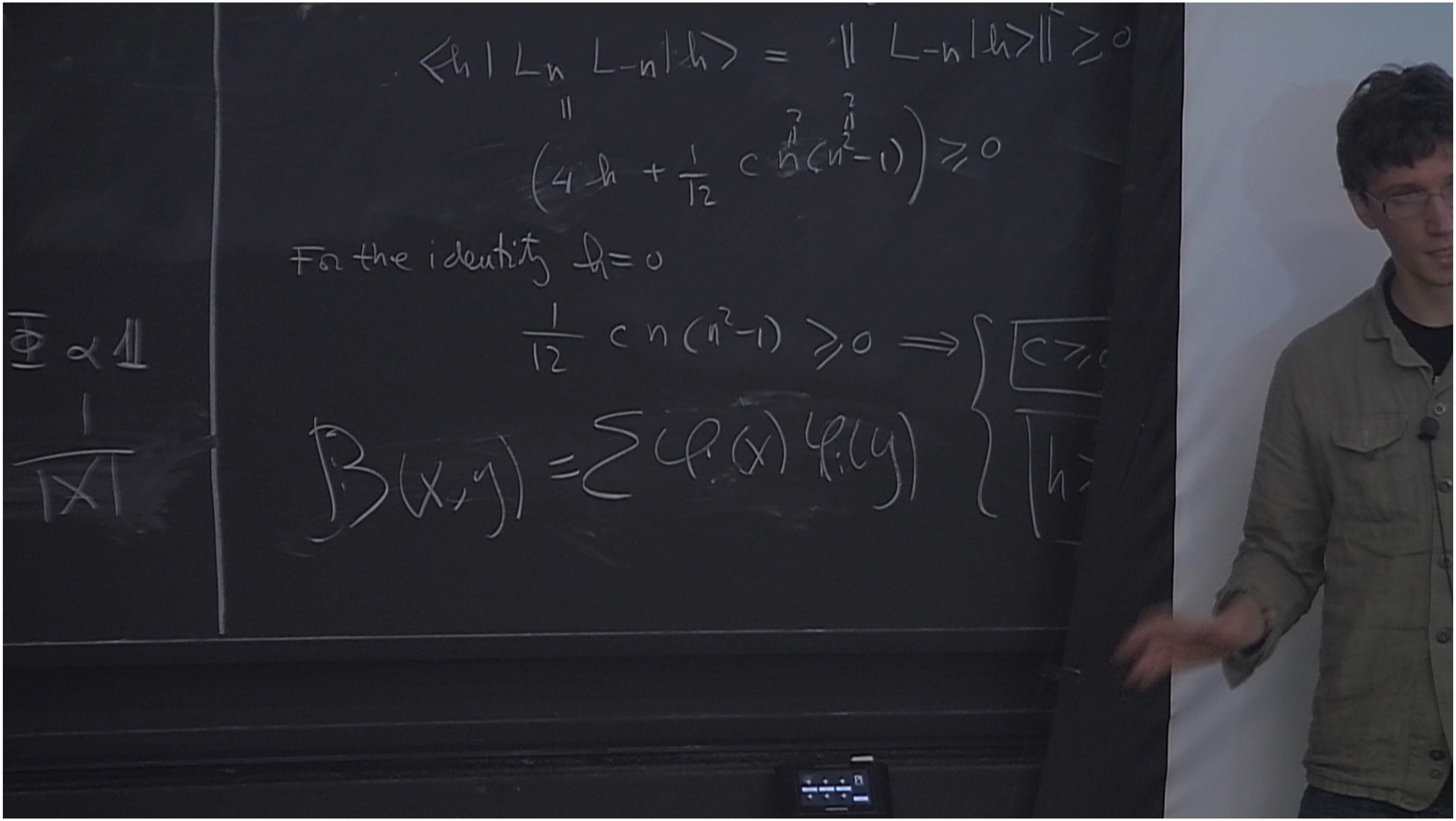
Looking for the bilocal operator

To find the bilocal that consists of two free fields we take the light-cone limit of two stress tensors

$$\underline{j_2(x)j_2(y)}_b = \partial_1^2\partial_2^2 B(\underline{x, y})$$

if $\langle 222\rangle_b \neq 0$ then $B(x, y)$, at least, contains j_2 in it.

The fact that $\langle B(\underline{x_1, x_2})j_s(x_3)\rangle \propto \langle \underline{\phi(x_1)\phi^*(x_2)}j_s(x_3)\rangle_{free}$ implies that B transforms as two weight $1/2$ fields under conformal transformations.



⟨h| L_n L_-n |h⟩ = || L_-n |h⟩ ||² ≥ 0
(4h + 1/12 c n(n² − 1)) ≥ 0
For the identity h = 0
1/12 c n(n² − 1) ≥ 0 ⟹
B(x,y) = Σ φ_i(x) φ_i(y)
Φ ∝ 1
1/|x|

Looking for the bilocal operator

To find the bilocal that consists of two free fields we take the light-cone limit of two stress tensors

$$\underline{j_2(x)j_2(y)}_b = \partial_1^2 \partial_2^2 B(\underline{x, y})$$

if $\langle 222 \rangle_b \neq 0$ then $B(x, y)$, at least, contains j_2 in it.

The fact that $\langle B(\underline{x_1, x_2}) j_s(x_3) \rangle \propto \langle \underline{\phi(x_1)\phi^*(x_2)} j_s(x_3) \rangle_{free}$ implies that B transforms as two weight $1/2$ fields under conformal transformations.

Looking for the bilocal operator II

In general these would be quasi-bilocals...

However, as we would like to argue in the theories with higher spin symmetries $B(\underline{x, y})$ behaves as the true bilocal operator, namely the normal ordered product of two free fields.

This will be done first by showing that

$$[Q_s, B(\underline{x_1, x_2})] = (\partial_1^{s-1} + \partial_2^{s-1}) B(\underline{x_1, x_2})$$

where Q_s is built from the current j_s that appears in the OPE of $j_2 j_2$.

And then showing that this implies that correlators have the free field form...

Proof of the simple transformation law

We would like to compute $[Q_s, B(\underline{x_1, x_2})]$.
We can compute

$$[Q_s, \underline{j_2 j_2}_b] = \underline{[Q, j_2] j_2}_b + \underline{j_2 [Q, j_2]}_b .$$

The action of Q_s commutes with the limit and we can write $[Q_s, j_2]$ in terms of currents and derivatives (with indices and derivatives all along the minus directions).

Thus, in the end we can write

$$[Q_s, B(\underline{x_1, x_2})] = (\partial_1^{s-1} + \partial_2^{s-1})\tilde{B}(\underline{x_1, x_2}) + (\partial_1^{s-1} - \partial_2^{s-1})B'(\underline{x_1, x_2})$$

where $\tilde{B}$ contains all even spin currents and B' all odd spin ones.

Proof that $B' = 0$

Let's assume that B' contains some odd spin s' current. Then

- consider $\langle [Q_{s'}, B' \;\; j_2] \rangle = 0$. This shows that there is j_1 such that $\langle B' \;\; j_1 \rangle \neq 0$;
- consider $\langle [Q_s, B \;\; j_1] \rangle = 0$. This shows that $\langle j_s j_s j_1 \rangle \neq 0$.

Thus, we reach a contradiction and $B' = 0$.

Proof of the simple transformation law

We would like to compute $[Q_s, B(\underline{x_1, x_2})]$.
We can compute

$$[Q_s, \underline{j_2 j_2}_b] = \underline{[Q, j_2] j_2}_b + \underline{j_2 [Q, j_2]}_b .$$

The action of Q_s commutes with the limit and we can write $[Q_s, j_2]$ in terms of currents and derivatives (with indices and derivatives all along the minus directions).

Thus, in the end we can write

$$[Q_s, B(\underline{x_1, x_2})] = (\partial_1^{s-1} + \partial_2^{s-1})\tilde{B}(\underline{x_1, x_2}) + (\partial_1^{s-1} - \partial_2^{s-1})B'(\underline{x_1, x_2})$$

where $\tilde{B}$ contains all even spin currents and B' all odd spin ones.

Proof that $B' = 0$

Let's assume that B' contains some odd spin s' current. Then

- consider $\langle [Q_{s'}, B' \;\; j_2] \rangle = 0$. This shows that there is j_1 such that $\langle B' \;\; j_1 \rangle \neq 0$;
- consider $\langle [Q_s, B \;\; j_1] \rangle = 0$. This shows that $\langle j_s j_s j_1 \rangle \neq 0$.

Thus, we reach a contradiction and $B' = 0$.

Proof that $\tilde{B} = B$

- Consider $\langle [Q_s, B \;\; j_2] \rangle = 0$. One sees then that $\langle \tilde{B} \, j_2 \rangle \neq 0$;
- consider $B - \tilde{B}$ where we normalize $\tilde{B}$ in such a way that the difference does not contain j_2;
- imagine that $B - \tilde{B}$ contains some current $j_{s'}$ to show that all the other currents will be also absent consider

$$\langle [Q_{s'}, (B - \tilde{B}) \;\; j_2] \rangle = 0$$

Again the chain nature of charge conservation identities and the structure of correlation functions are extremely restrictive.

Solving for correlation functions of B's

Now, once we argued that

$$[Q_s, B(\underline{x_1, x_2})] = (\partial_1^{s-1} + \partial_2^{s-1}) B(\underline{x_1, x_2})$$

is true, then we can consider any n point function of bilinears $\langle B(\underline{x_1, x_2}) \cdots B(\underline{x_{2n-1}, x_{2n}}) \rangle$. We have an infinite number of constraints from all the conserved charges. These constraints take the form

$$\sum_{i=1}^{2n} \partial_i^{s-1} \langle B(\underline{x_1, x_2}) \cdots B(\underline{x_{2n-1}, x_{2n}}) \rangle = 0 \,, \qquad s = 2, 4, 6, \cdots$$

Solving for correlation functions of B's

One can show that this implies factorization in x^- and then using conformal, rotational and permutation symmetry we get that all correlation functions are fixed in terms of one constant which appears in two point function of stress tensors

$$\langle B(\underline{x_1, x_2}) B(\underline{x_3, x_4}) \rangle = \tilde{N} \left(\frac{1}{d_{13} d_{24}} + \frac{1}{d_{14} d_{23}} \right)$$

Thus, to get n-point correlation function we can use

$$B(\underline{x_1, x_2}) = \sum_{i=1}^{N} \underline{: \phi_i(x_1) \phi_i(x_2) :}$$

and then analytically continue in $N \to \tilde{N}$.

Solving for correlation functions of B's

One can show that this implies factorization in x^- and then using conformal, rotational and permutation symmetry we get that all correlation functions are fixed in terms of one constant which appears in two point function of stress tensors

$$\langle B(\underline{x_1, x_2})B(\underline{x_3, x_4})\rangle = \tilde{N}\left(\frac{1}{d_{13}d_{24}} + \frac{1}{d_{14}d_{23}}\right)$$

Thus, to get n-point correlation function we can use

$$B(\underline{x_1, x_2}) = \sum_{i=1}^{N} \underline{:\phi_i(x_1)\phi_i(x_2):}$$

and then analytically continue in $N \to \tilde{N}$.

Quantization of $\tilde{N}$

However, we would like to argue now that unitarity restricts $\tilde{N}$ to be integer.

The basic idea for showing that $\tilde{N}$ is quantized uses the fact that the spectrum of the theory depends on N.
Consider the following operator in the theory of N bosons

$$\mathcal{O}_q = \delta^{[i_1,\cdots,i_q]}_{[j_1,\cdots,j_q]}(\phi^{i_1}\partial\phi^{i_2}\partial^2\phi^{i_3}\ldots\partial^{q-1}\phi^{i_q})(\phi^{j_1}\partial\phi^{j_2}\partial^2\phi^{j_3}\ldots\partial^{q-1}\phi^{j_q})$$

The two point function of this operator takes the form

$$\langle\mathcal{O}_q\mathcal{O}_q\rangle \propto \tilde{N}(\tilde{N}-1)(\tilde{N}-2)\cdots(\tilde{N}-(q-1))$$

Now, imagine that $\tilde{N}$ was not integer. Then we could consider this operator for $q=[\tilde{N}]+2$, where $[\tilde{N}]$ is an integer part of $\tilde{N}$. Then we find that two point function becomes

$$\langle\mathcal{O}_{[\tilde{N}]+2}\mathcal{O}_{[\tilde{N}]+2}\rangle = (\text{positive})(\tilde{N}-[\tilde{N}]-1)$$

Higher spin symmetries broken at $\frac{1}{N}$ order

It is not complicated to generalize our consideration to the cases when the higher spin symmetries are broken at the $\frac{1}{N}$ order in large N limit.

In this case the dimensions of conserved currents will

$$\Delta_s = s + 1 + O(\frac{1}{N})$$

and the divergence of the currents will take the form

$$\partial_\mu j^\mu = \frac{1}{\sqrt{N}} \sum \mathcal{O}_i$$

where we assumed vector-like large N expansion and also the fact that we know the spectrum of operators at $N = \infty$.

The operators in the RHS should have the right quantum numbers.

In a completely analogous way to the exact symmetries we can analyze all possible structures in the RHS.

Now what will happen with the CCIs?

Shadow Charge Conservation Identities

In the case of fermions we would get

$$\partial_\mu j^\mu{}_{---} = \frac{1}{\sqrt{N}}[a_1 \partial_- \tilde{j}_0 j_{--} + a_2\ \tilde{j}_0 \partial_- j_{--}].$$

If we consider $\langle j_2 j_2 j_2 \rangle$ CCI we would get terms like

$$\int_V (\partial \langle j_2(x) j_2 \rangle) \langle \tilde{j}_0(x) j_2 j_2 \rangle$$

using the fact that all indices are minus this can be rewritten as

$$\partial^5 \int_V \frac{1}{|x - x_i|} \langle \tilde{j}_0(x) j_2 j_2 \rangle$$

Now notice that the integral has the all properties of $\langle j_0 j_2 j_2 \rangle$.

This is the mechanism of the appearance of the twist one scalar in the CCI story. The operator $\tilde{j}_0$ of dimension 2 is substituted by the scalar of dimension $d - 2$ which is what sometimes is called "shadow" field.

Conclusions

- we analyzed the problem of possible symmetries of CFTs in $d > 2$;
- in $d = 3$ using unitarity, conformal symmetry and uniqueness of stress tensor we showed that addition of conserved currents of spin $s > 2$ makes the theory trivial (a-la Coleman-Mandula);
- our analysis heavily relied on the structure of three point functions of conserved currents in $d = 3$. The analysis for $d > 3$ is completely analogous and should be easy especially using the simplicity of the light-cone limit;
- using the same approach we can analyze the cases when higher spin symmetries are broken at $\frac{1}{N}$ order;

Conclusions II

- any higher spin symmetric theory in AdS that preserves symmetry at the quantum level in the bulk with higher spin preserving boundary condition is described by free fields at the boundary;
- there is more freedom to have a theory which has higher spin symmetry at the classical level only;
- we illustrated an example of bootstrap in higher dimensional field theories by fixing correlation functions without ever talking about the Lagrangian.

Open problems

- $d > 3$;
- possible spectrum of operators away from conserved currents (”minimal models“);
- multiple (more than two) stress tensors;
- any weakly coupled theory (pure YM, $\mathcal{N} = 4$...);
- to find correlation functions of theories with higher spin symmetry broken at $\frac{1}{N}$ at leading order in N.

Thank you!