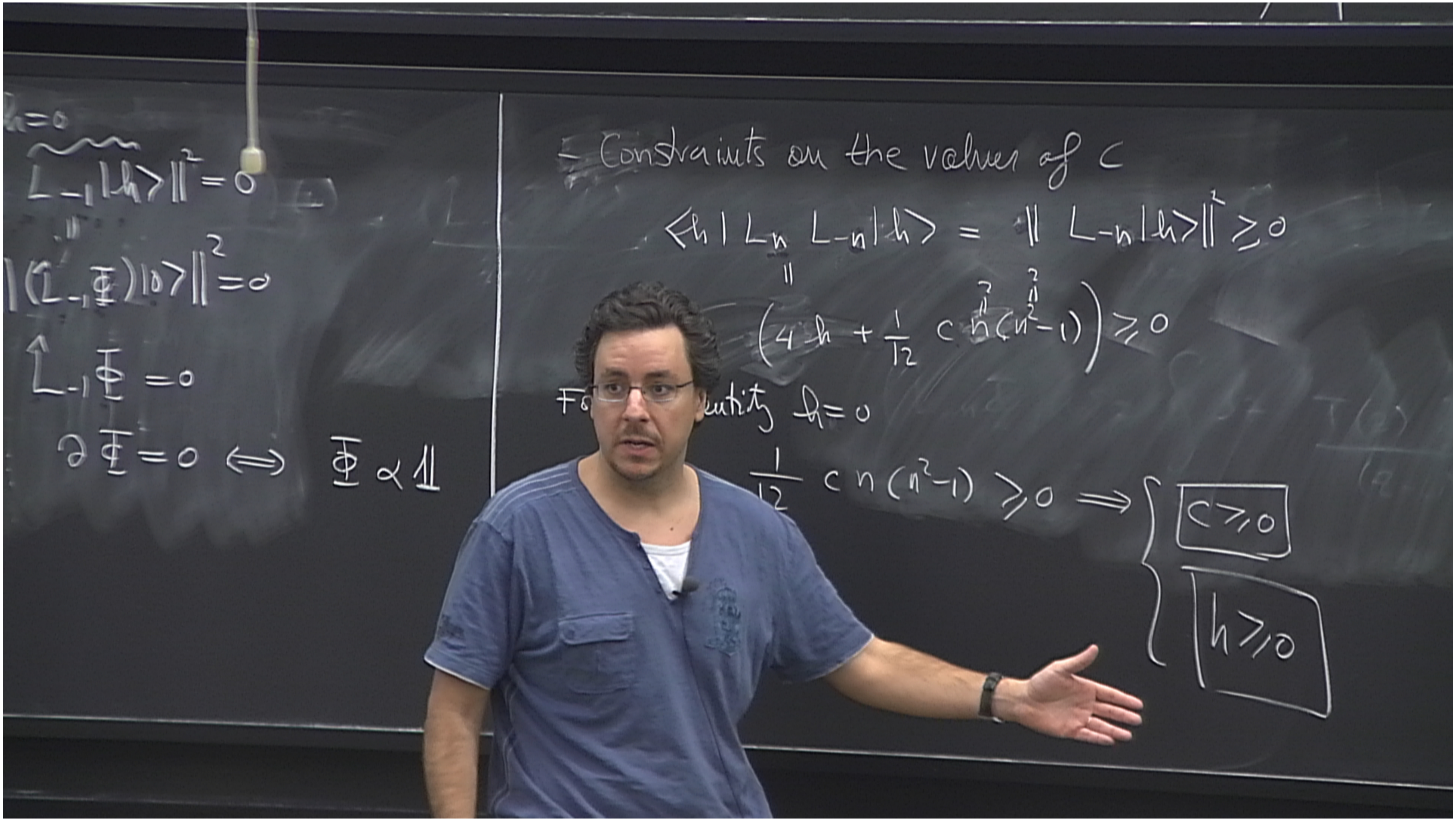
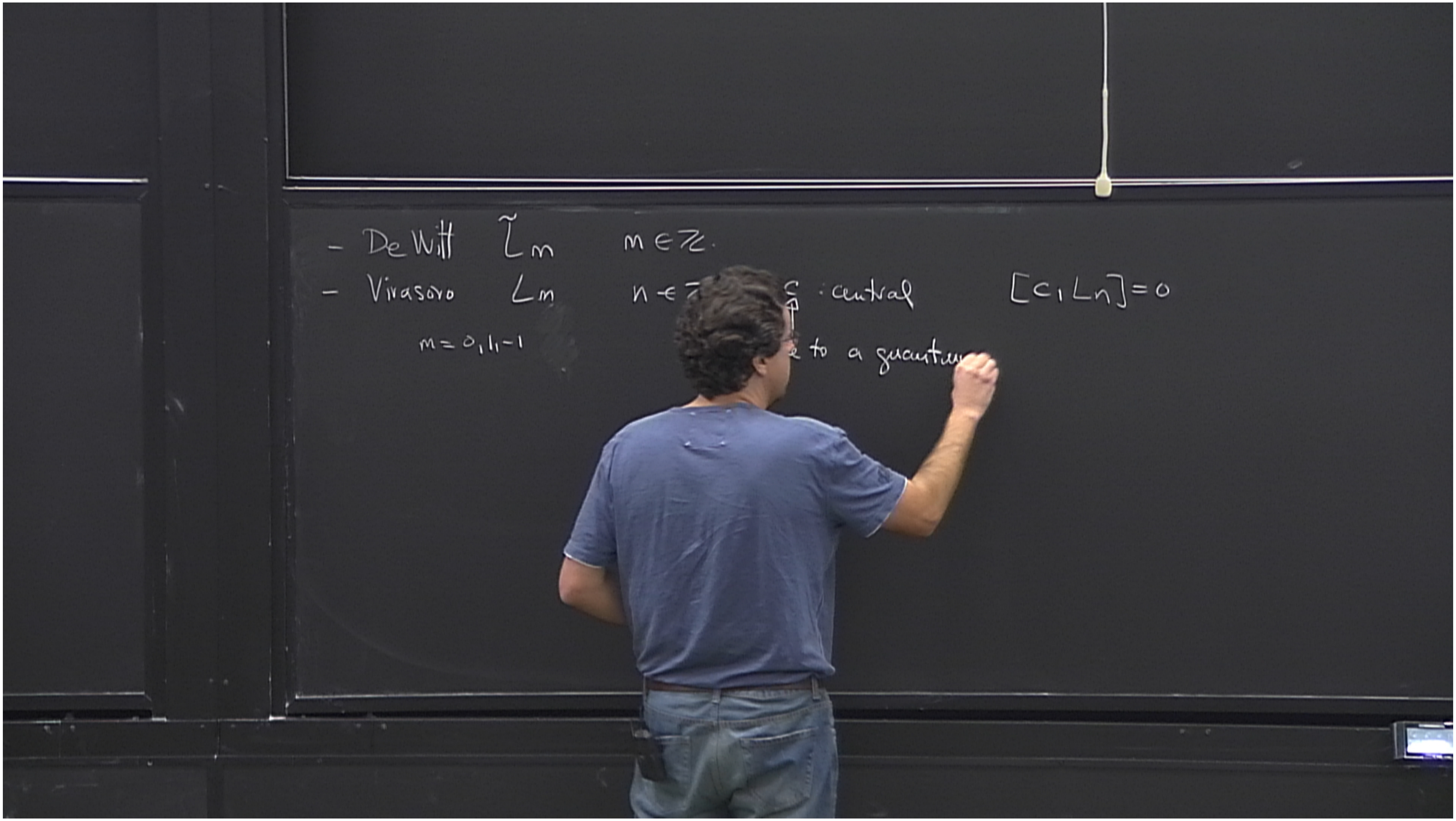
Title: Conformal Field Theory - Lecture 12

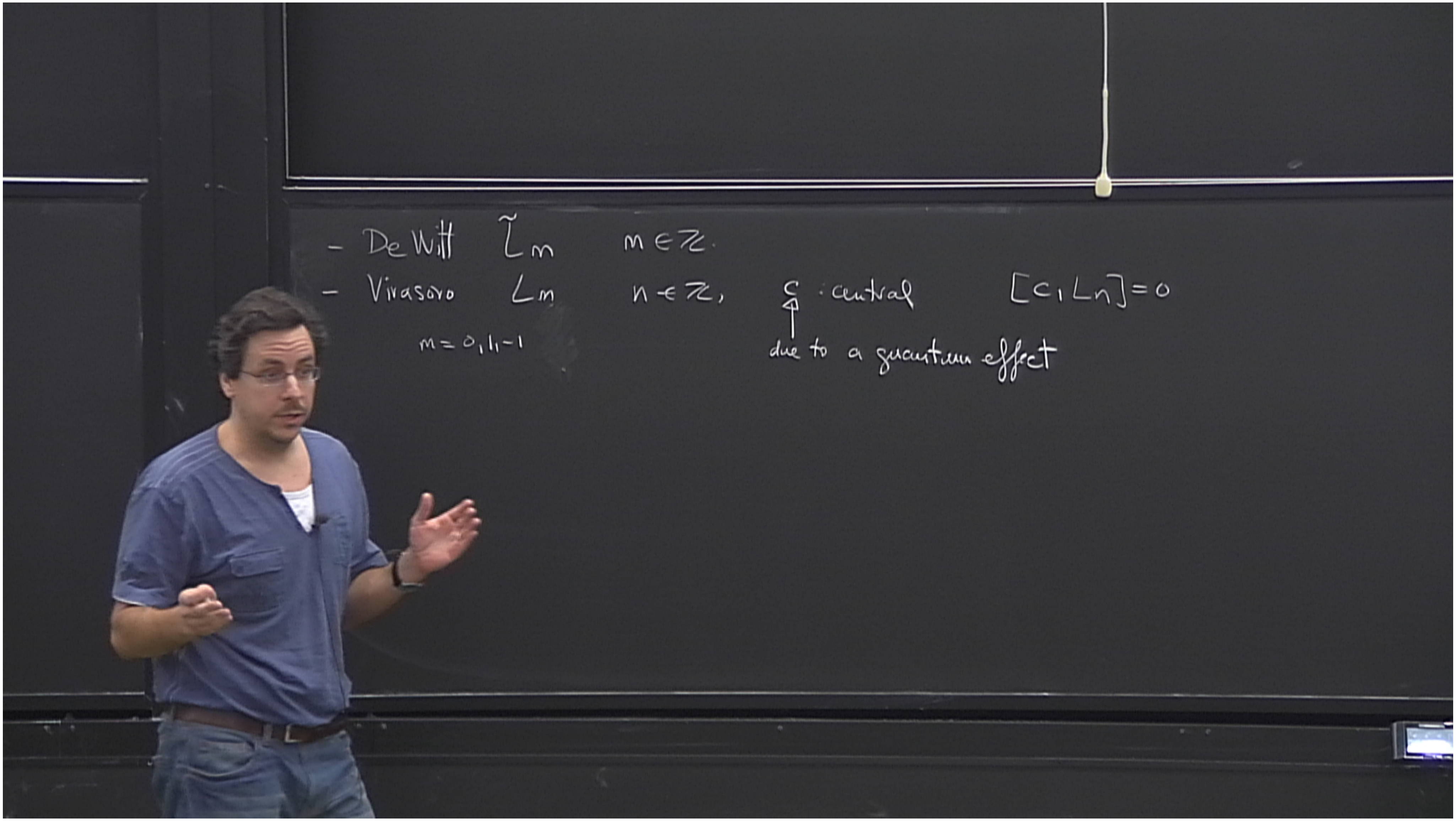
Date: Dec 06, 2011 10:30 AM

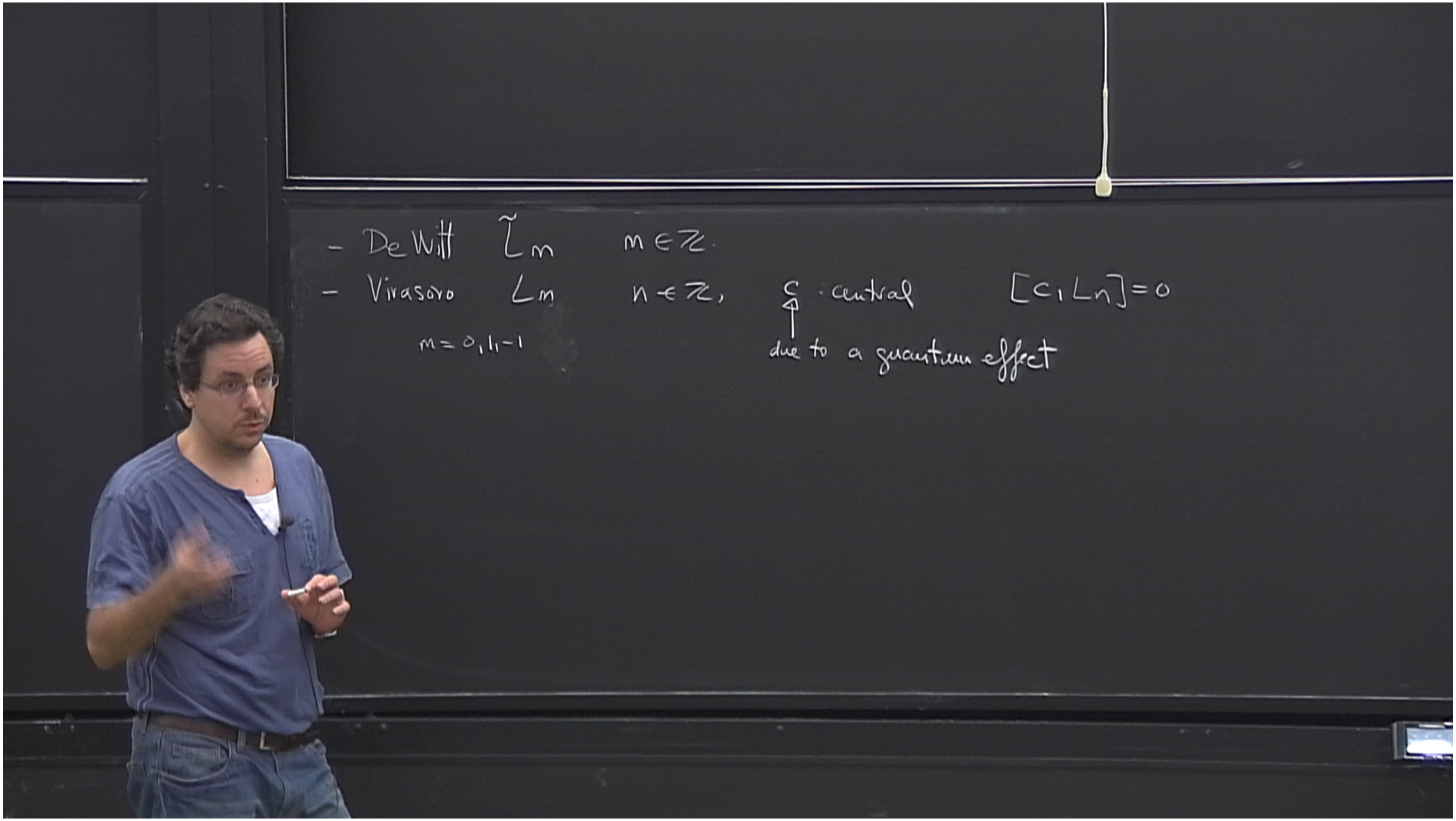
URL: http://pirsa.org/11120015

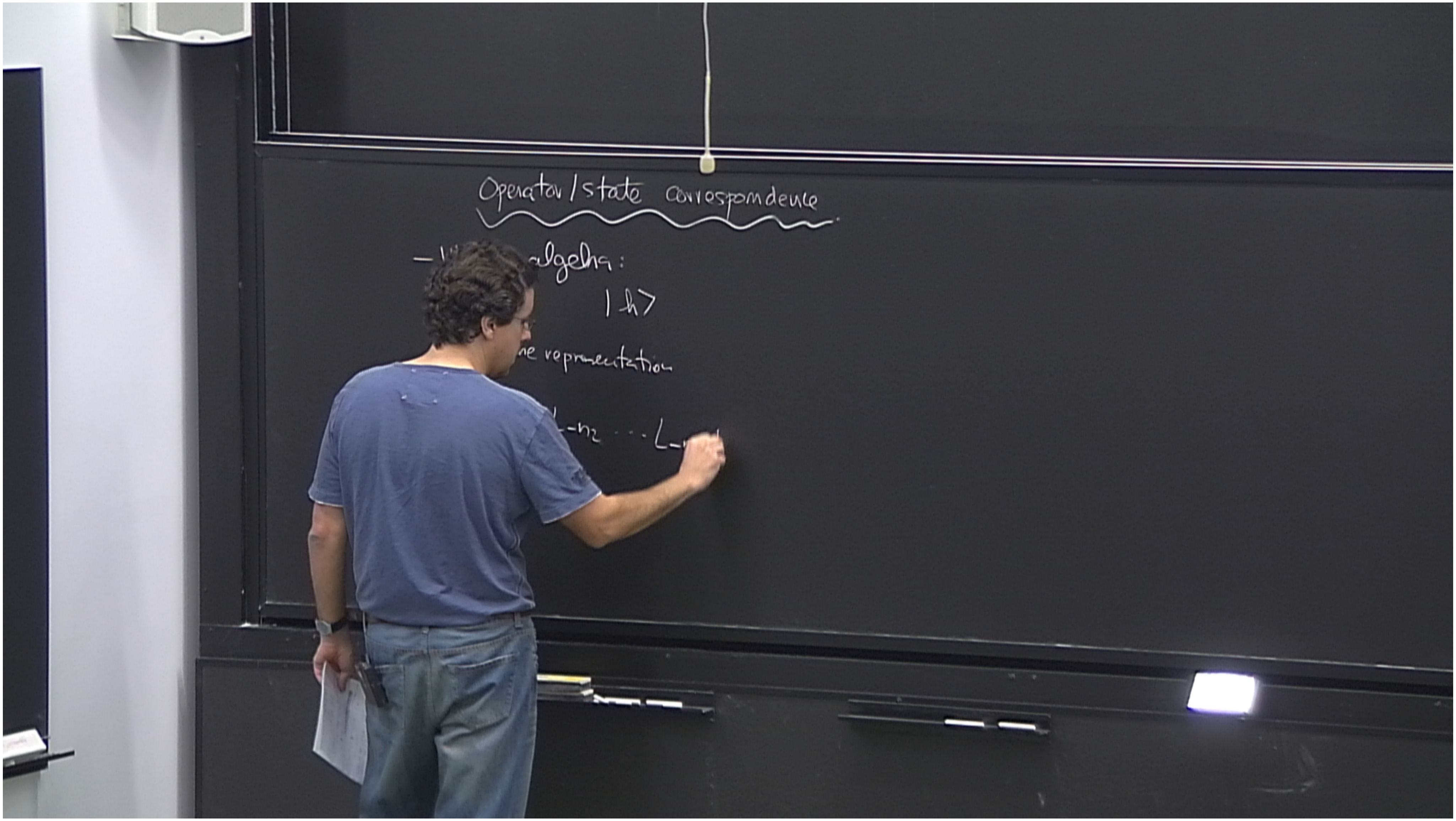
Abstract:

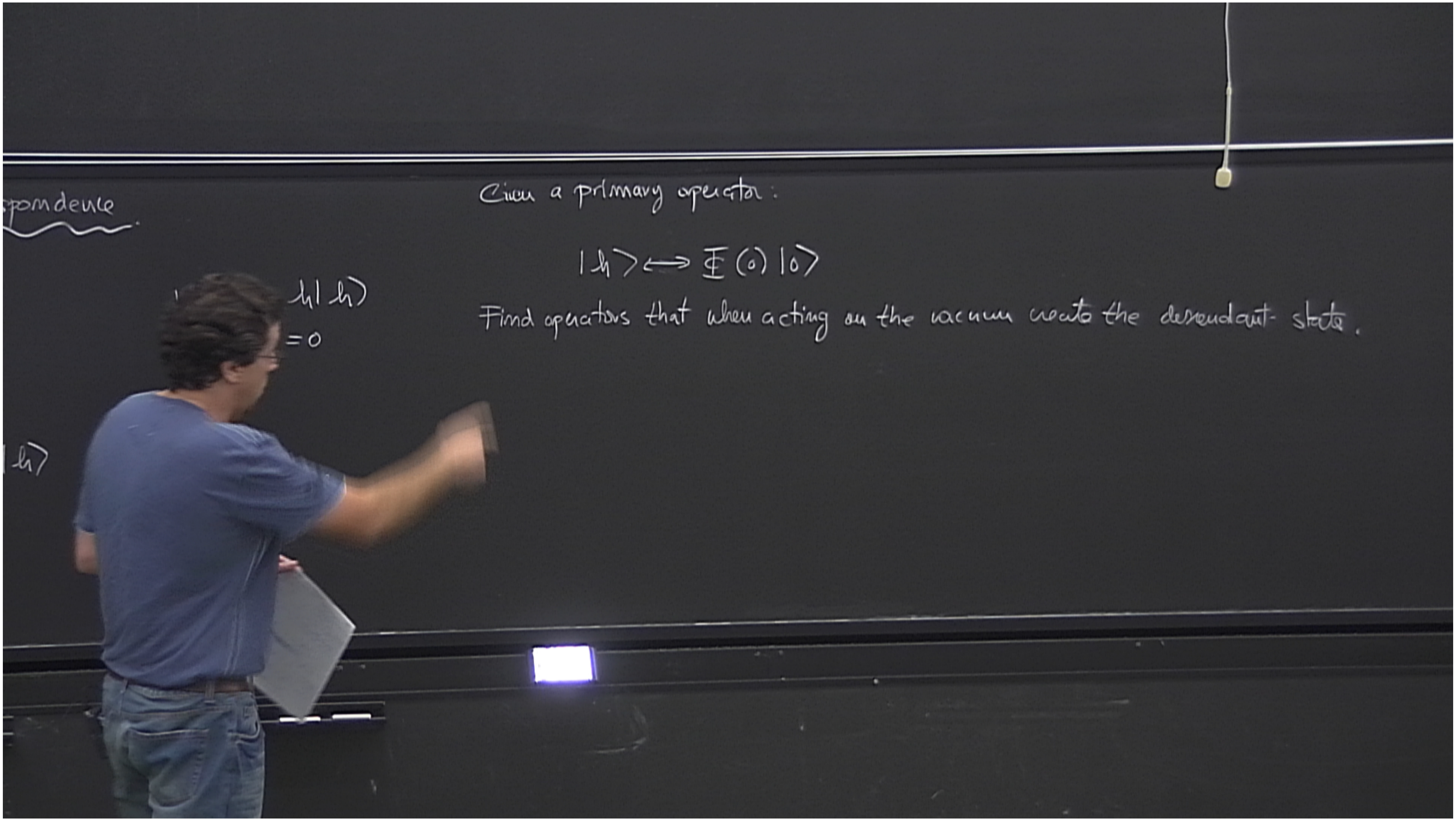

Constraints on the values of c
⟨h| L_n L_-n |h⟩ = ‖ L_-n |h⟩‖² ≥ 0
(4h + 1/12 c n(n²-1)) ≥ 0
h=0
‖L_-1 |h⟩‖² = 0
‖(L_-1 Φ)|0⟩‖² = 0
L_-1 Φ = 0
∂Φ = 0 ⟺ Φ ∝ 1
1/12 c n(n²-1) ≥ 0 ⟹
c ≥ 0
h ≥ 0


- De Witt L̃m m ∈ Z.
- Virasoro Lm n ∈ Z
c · central
[c, Ln] = 0
m = 0, 1, -1
to a


- De Witt $\tilde{L}_m$ $m \in \mathbb{Z}$.
- Virasoro L_m $n \in \mathbb{Z}$, c : central $[c, L_n] = 0$
$m = 0, 1, -1$
due to a quantum effect


- De Witt L̃m m ∈ ℤ.
- Virasoro Lm n ∈ ℤ, c · central [c, Ln] = 0
m = 0, 1, -1
due to a quantum effect


Operator/state correspondence
algebra:
$|h\rangle$
representation
$L_{-n_2} \cdots L_{-}$


spondence
Given a primary operator:
|h⟩ ⟷ Φ(0)|0⟩
h| h⟩
Find operators that when acting on the vacuum create the descendant state.
= 0
|h⟩

spondence

$L_0|h\rangle = h|h\rangle$
$L_m|h\rangle = 0$

Given a primary operator:

$$|h\rangle \longleftrightarrow \Phi(0)|0\rangle$$

Find operators that when acting on the vacuum create the descendant state.

$$|h\rangle \quad \underset{1 \le n_1 \le n_2 \le \cdots \le n_\ell}{\longleftrightarrow} \quad (\hat{L}_{-n_1} \cdots \hat{L}_{-n_\ell}\Phi)(0)|0\rangle$$

endant Operator from OPE

$$(z)\,\Phi(w) = \frac{(\hat{L}_0 \Phi)(w)}{(z-w)^2} + \frac{(\hat{L}_{-1}\Phi)(w)}{z-w} + (\hat{L}_{-2}\Phi)(w) + \cdots = \sum_{n=0}^{\infty} (\hat{L}_{-n}\Phi)(w)\,(z-w)^{n-2}.$$

$$\ldots)(w) + (\hat{L}_{-2}\Phi)(w) + \ldots = \sum_{n=0}^{\infty} (\hat{L}_{-n}\Phi)(w)\,(z-w)^{n-2}$$

$$(\hat{L}_{-n}\Phi)(w) = \oint_{C_w} dz \, \frac{T(z)\,\Phi(w)}{(z-w)^{n-1}}$$

1) $\Phi = \mathbb{1}$

$T(z)\cdot\mathbb{1} =$

$$\ldots(w) + (\hat{L}_{-2}\Phi)(w) + \ldots = \sum_{n=0}^{\infty} (\hat{L}_{-n}\Phi)(w)\,(z-w)^{n-2}.$$

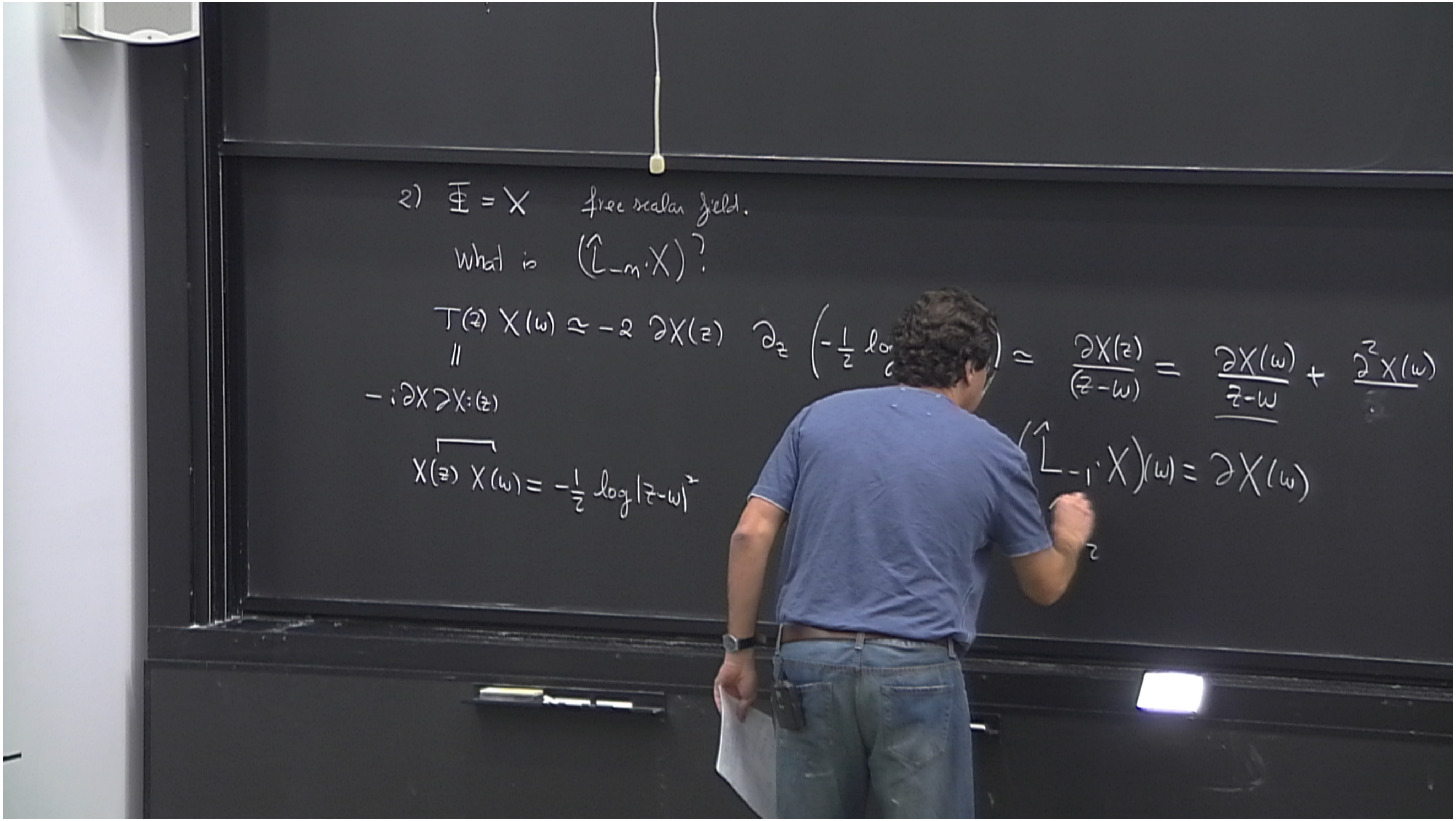
$$(\hat{L}_{-n}\Phi)(w) = \oint_{C_w} dz\, \frac{T(z)\,\Phi(w)}{(z-w)^{n-1}}$$

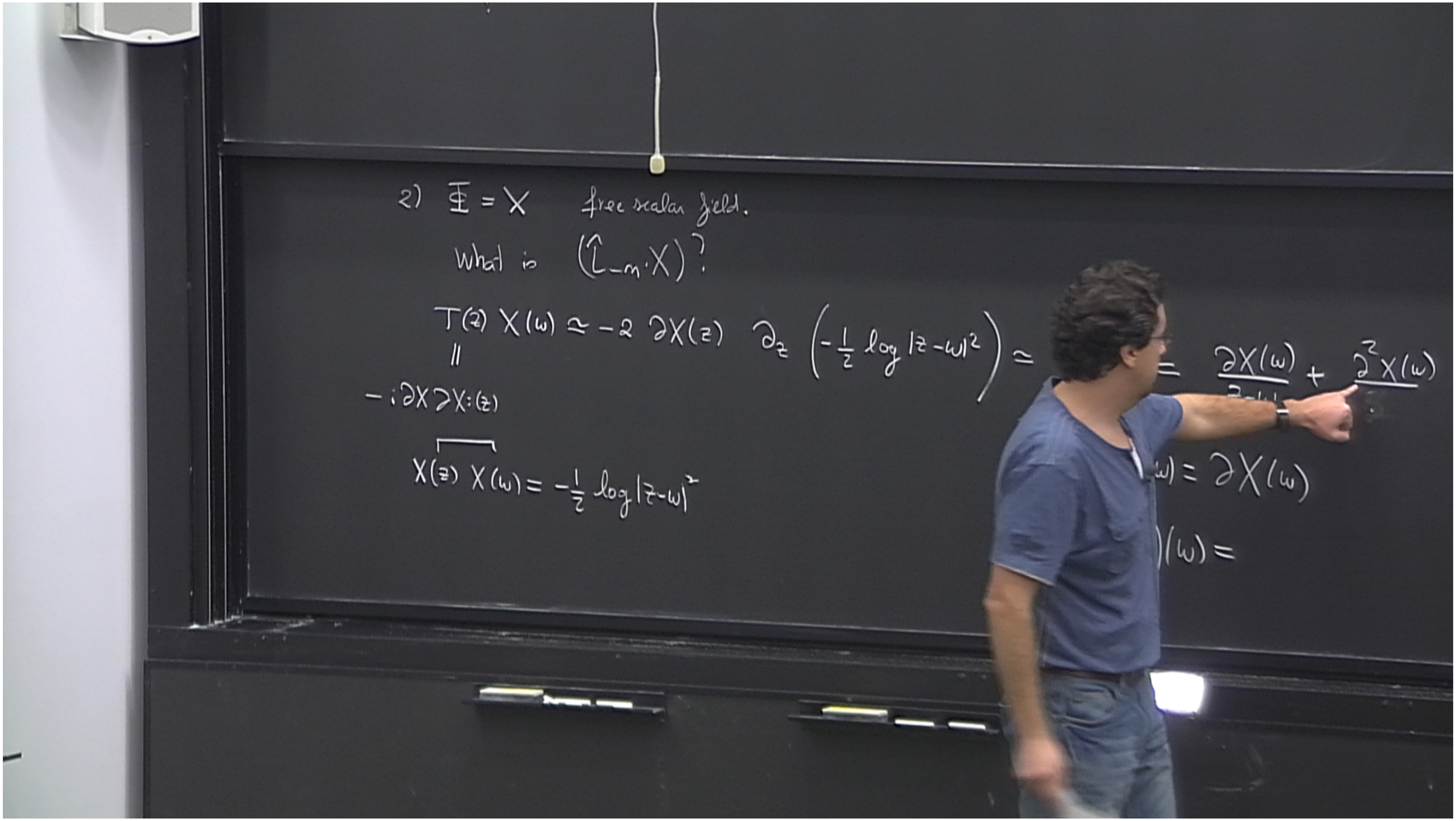
1) $\Phi = \mathbb{1}$

$$T(z)\cdot\mathbb{1} = \frac{0}{z^2} + \frac{\overset{0}{\overset{\shortparallel}{\partial\mathbb{1}}}}{z} + T(z)$$

$$T(z) = (\hat{L}_{-2}\cdot\mathbb{1})(z)$$

T(z) is a descendant of the identity operator.


2) $\Phi = X$ free scalar field.
What is $(\hat{L}_{-n} \cdot X)$?
$T(z)\, X(w) \simeq -2\, \partial X(z)\, \partial_z \left(-\frac{1}{2} \log \right.$
$\simeq \frac{\partial X(z)}{(z-w)} = \frac{\partial X(w)}{z-w} + \partial^2 X(w)$
$-:\partial X \partial X:(z)$
$X(z)\, X(w) = -\frac{1}{2} \log |z-w|^2$
$(\hat{L}_{-1} \cdot X)(w) = \partial X(w)$


2) $\Phi = X$ free scalar field.
What is $(\hat{L}_{-n} \cdot X)$?
$T(z)\, X(w) \simeq -2\, \partial X(z)\; \partial_z \left(-\frac{1}{2} \log |z-w|^2 \right) \simeq$
$\partial X(w) + \partial^2 X(w)$
$\| $
$-:\partial X \partial X:(z)$
$X(z)\, X(w) = -\frac{1}{2} \log |z-w|^2$
$w) = \partial X(w)$
$)(w) =$

$$\left(-\frac{1}{2}\log|z-w|^2\right) \simeq \frac{\partial X(z)}{(z-w)} = \frac{\partial X(w)}{z-w} + \partial^2 X(w) + \partial^3 X(w)\,(z-w) + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}\,\partial^{n+1} X(w)\cdot(z-w)^{n-2}.$$

$$(\hat{L}_{-1}\cdot X)(w) = \partial X(w)$$

$$(\hat{L}_{-2}\cdot X)(w) = \partial^2 X(w)$$

$$\boxed{(\hat{L}_{-n} X)(w) = \frac{1}{(n-1)!}\,\partial^n X(w)}$$

The state

$$L_{-2}|h\rangle \longleftrightarrow \partial^2 X(0)|h\rangle$$

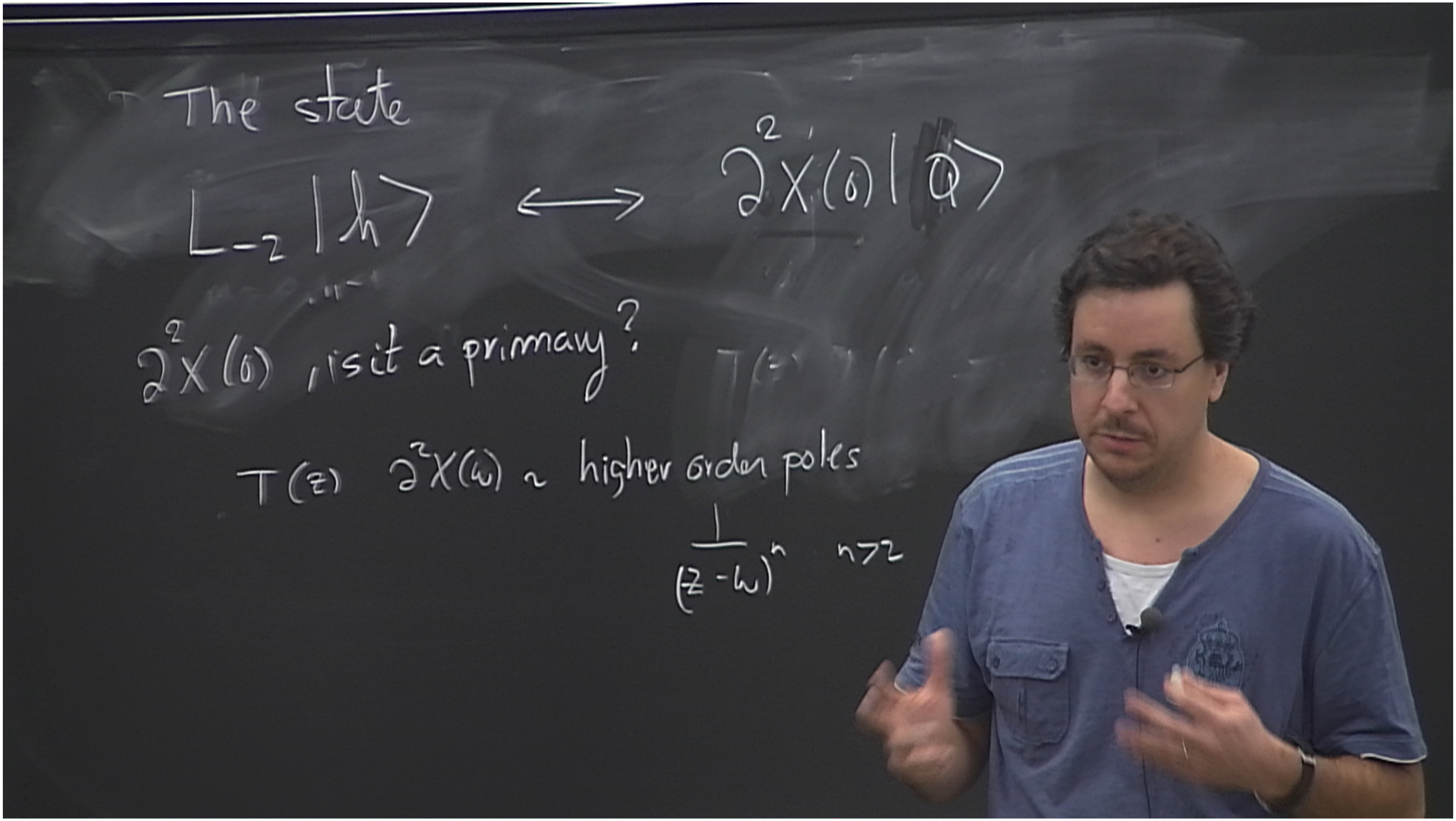
The state

$$L_{-2}|h\rangle \longleftrightarrow \partial^2 X(0)|0\rangle$$

$\partial^2 X(0)$, is it a primary?

$$T(z)\ \partial^2 X(w) \sim \text{higher order poles}$$

$$\frac{1}{(z-w)^n} \quad n>2$$


The state
$L_{-2}|h\rangle \longleftrightarrow \partial^2 X(0)|0\rangle$
$\partial^2 X(0)$, is it a primary?
$T(z)\ \partial^2 X(w) \sim$ higher order poles
$\frac{1}{(z-w)^n}\quad n>2$

$= (\hat{L}_{-2}\mathbb{1})(z)$

descendant of the identity operator.

$\sum_{n=0}^{+\infty} \frac{1}{n!} \partial^{n+1} X(w) \cdot (z-w)^{n-2}.$

$\boxed{X(w)}$

The state

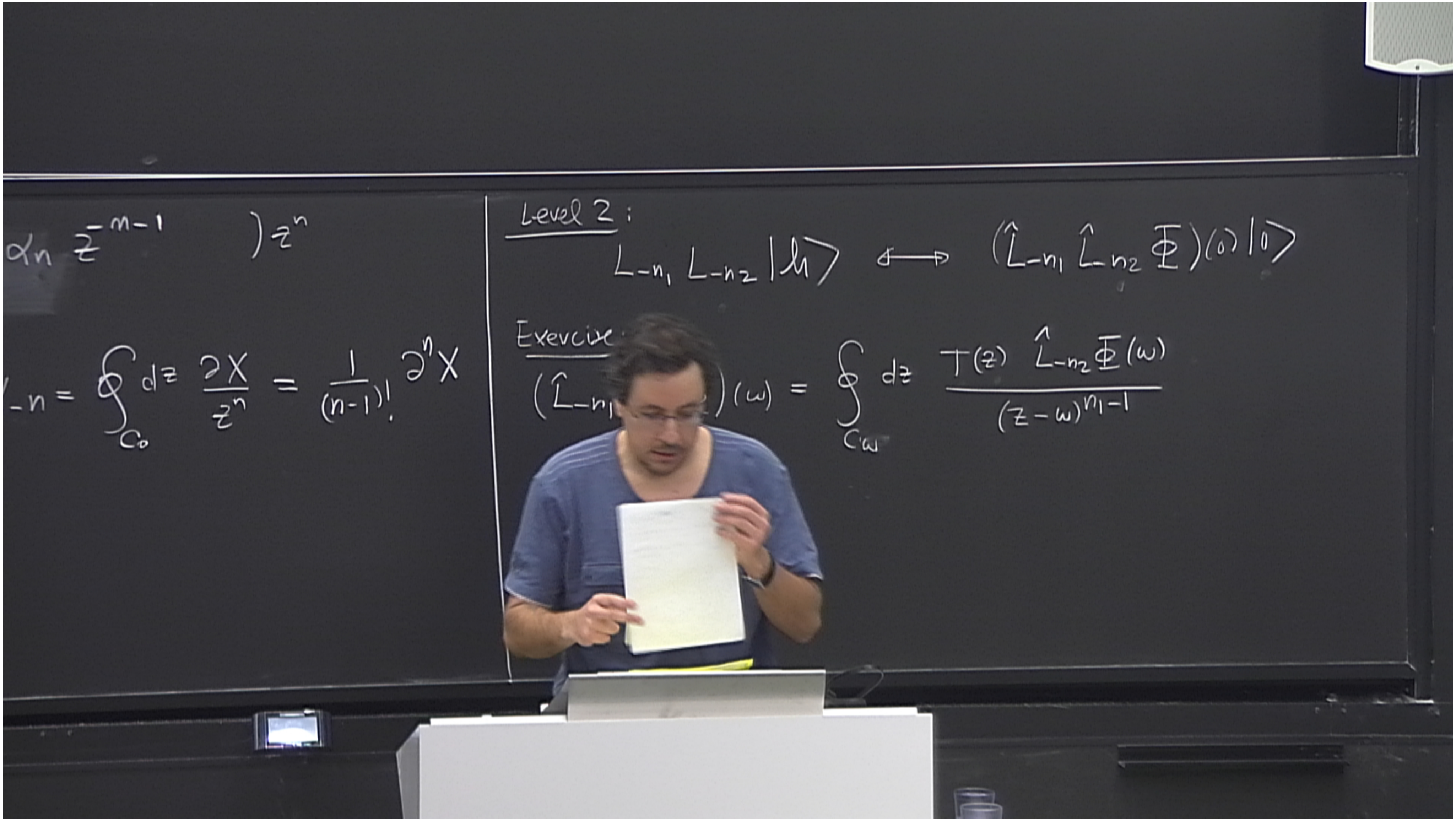
$L_{-2}|h\rangle \longleftrightarrow \partial^2 X(0)|0\rangle$

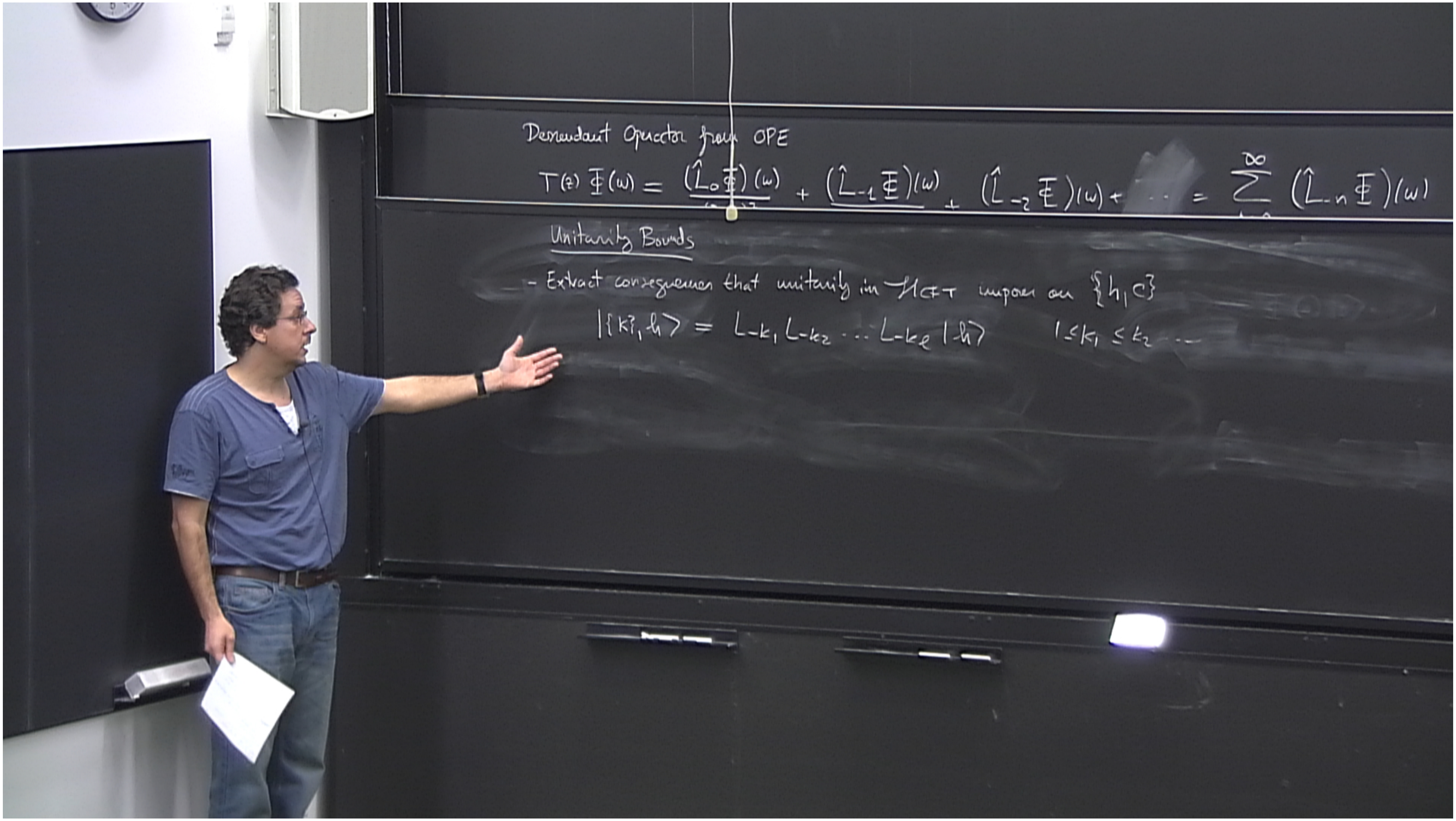
$\partial^2 X(0)$, is it a primary?

$T(z)\ \partial^2 X(w) \sim$ higher order poles

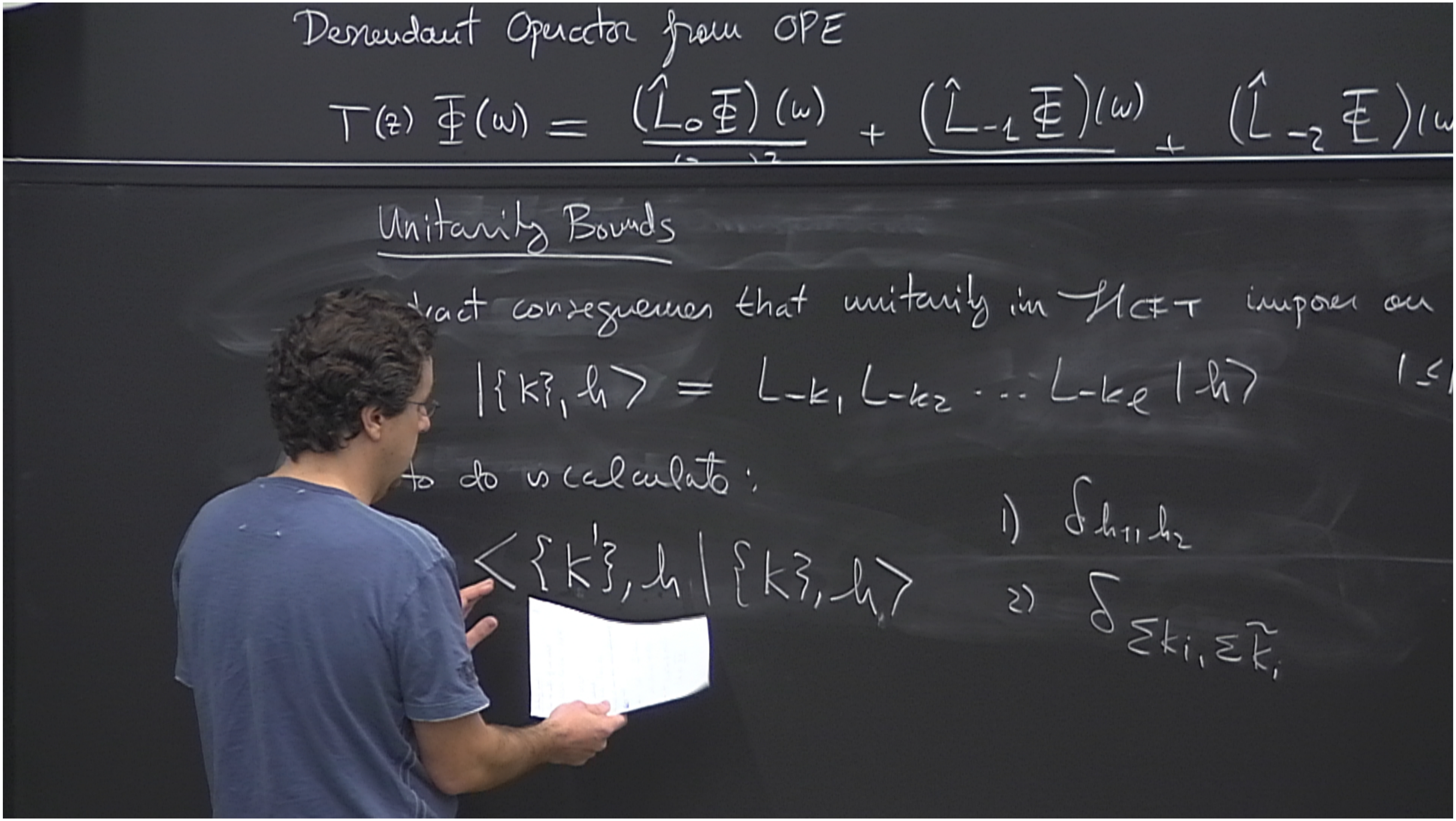
$\frac{1}{(z-w)^n}\ n>2$

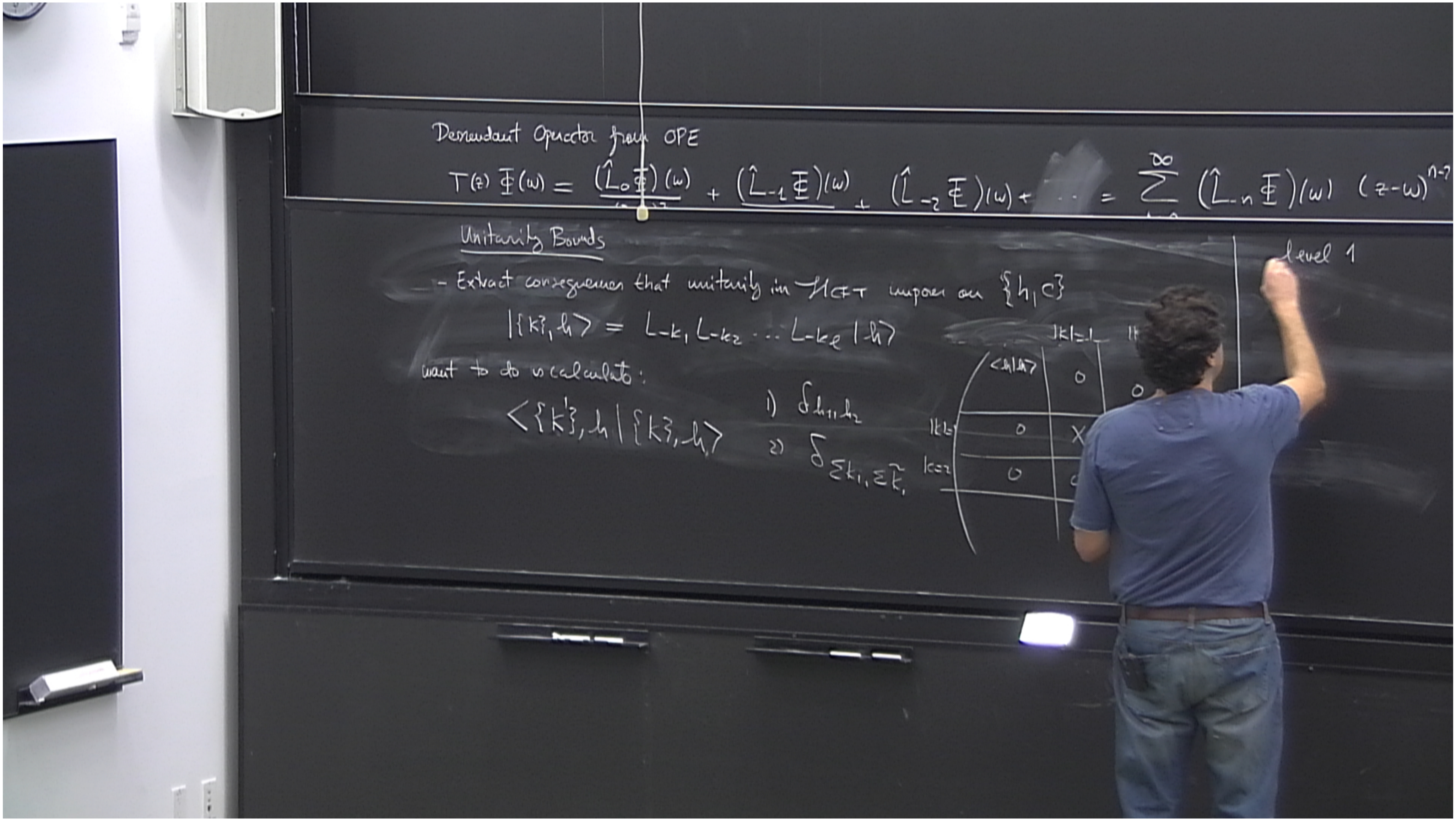
$\partial X = \sum_n \alpha_n z^{-n-1}$


$\alpha_n z^{-n-1}$ $) z^n$
$_{-n} = \oint_{C_0} dz \frac{\partial X}{z^n} = \frac{1}{(n-1)!} \partial^n X$
Level 2 :
$L_{-n_1} L_{-n_2} |h\rangle \longleftrightarrow (\hat{L}_{-n_1} \hat{L}_{-n_2} \Phi)(0)|0\rangle$
Exercise :
$(\hat{L}_{-n_1}$ $)(\omega) = \oint_{C_\omega} dz \frac{T(z) \hat{L}_{-n_2} \Phi(\omega)}{(z-\omega)^{n_1-1}}$



Descendant Operator from OPE
T(z) Φ(w) = (L̂₀ Φ)(w) + (L̂₋₁ Φ)(w) + (L̂₋₂ Φ)(w) + ... = Σ^∞ (L̂₋ₙ Φ)(w)
Unitarity Bounds
- Extract consequences that unitarity in H_CFT imposes on {h, c}
|{k}, h⟩ = L₋k₁ L₋k₂ ... L₋kℓ |h⟩   1 ≤ k₁ ≤ k₂ ...


Descendant Operator from OPE
Unitarity Bounds
consequences that unitarity in 2d CFT impose on
to do is calculate:
1) δ h1, h2
2) δ Σki, Σk̃i


Descendant Operator from OPE
$T(z)\,\Phi(w) = (\hat{L}_0 \Phi)(w) + (\hat{L}_{-1}\Phi)(w) + (\hat{L}_{-2}\Phi)(w) + \dots = \sum^{\infty} (\hat{L}_{-n}\Phi)(w)\,(z-w)^{n-2}$
Unitarity Bounds
- Extract consequences that unitarity in $\mathcal{H}_{CFT}$ impose on $\{h, c\}$
$|\{k\}, h\rangle = L_{-k_1} L_{-k_2} \dots L_{-k_\ell} |h\rangle$
want to do a calculate:
$\langle \{k'\}, h | \{k\}, h\rangle$
1) δ_{h_1, h_2}
2) $\delta_{\Sigma k_i, \Sigma \tilde{k}_i}$
$\langle h|h\rangle$
level 1

Descendant Operator from OPE

$$T(z)\,\Phi(w) = \frac{(\hat{L}_0 \Phi)(w)}{(z-w)^2} + \frac{(\hat{L}_{-1}\Phi)(w)}{(z-w)} + (\hat{L}_{-2}\Phi)(w) + \dots = \sum_{n=0}^{\infty} (\hat{L}_{-n}\Phi)(w)\,(z-w)^{n-2}$$

1) $\Phi = \mathbb{1}$

$$T(z)\cdot \mathbb{1} = \frac{0}{z^2} + \frac{\partial \mathbb{1}}{z} + T(z) \dots$$

Unitarity Bounds

- Extract consequences that unitarity in $\mathcal{H}_{CFT}$ imposes on $\{h, c\}$

$$|\{k\}, h\rangle = L_{-k_1} L_{-k_2} \dots L_{-k_\ell} |h\rangle$$

want to do is calculate:

$$\langle \{k'\}, h \,|\, \{k\}, h\rangle$$

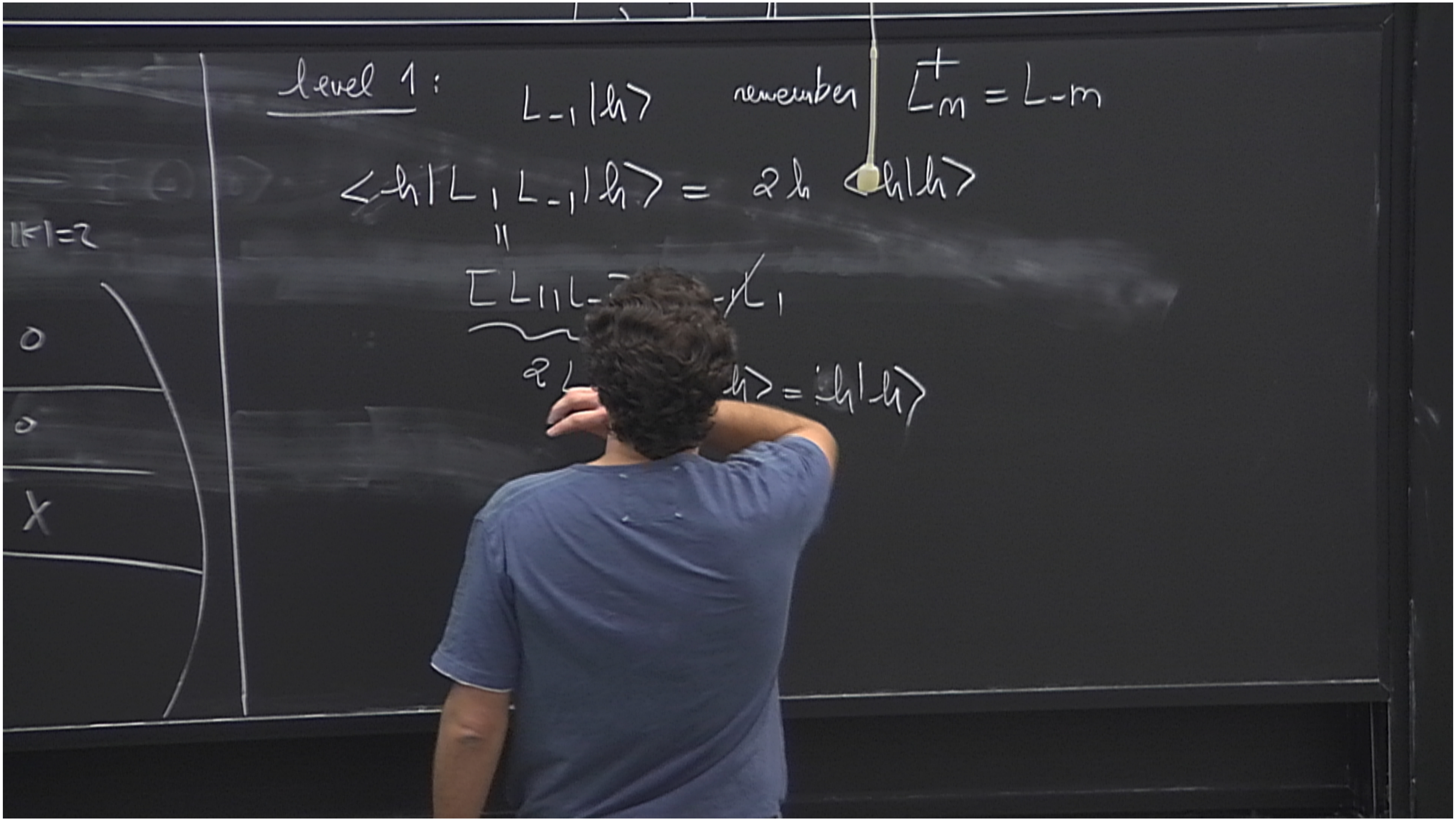
1) δ_{h_1, h_2}

2) $\delta_{\Sigma k_i, \Sigma k'_i}$

$$|k| = \sum k_i \qquad [L_0, L_{-k}] = k L_{-k}$$

	$\|k\|=1$	$\|k\|=2$
$\langle h \| h\rangle$	0	0
$\|k\|=1$: 0	X	0
$\|k\|=2$: 0	0	X
	0	

level 1:


level 1 :
L_{-1}|h>
remember L_m^† = L_{-m}
<h|L_1 L_{-1}|h> = 2h <h|h>

$$\sum_{n=0}^{\infty} (\hat{L}_{-n}\Phi)(w)\,(z-w)^{n-2}$$

$$\oint_w dz\,\frac{T(z)\,\Phi(w)}{(z-w)^{n-1}}$$

1) $\Phi = \mathbb{1}$

$$T(z)\cdot\mathbb{1} = \frac{0}{z^2} + \frac{\overset{0}{\overset{\shortparallel}{\partial\mathbb{1}}}}{z} + T(z)$$

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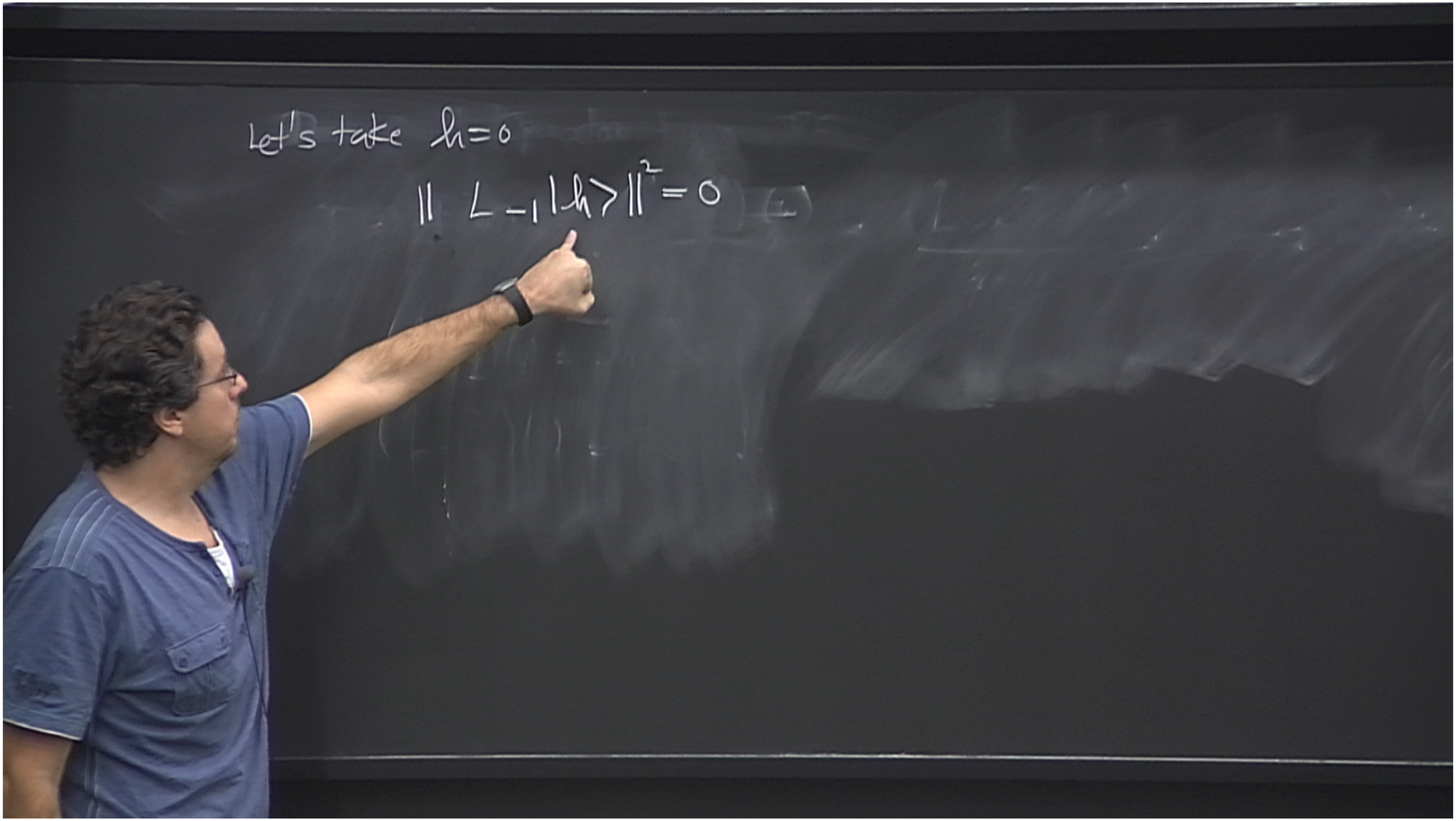
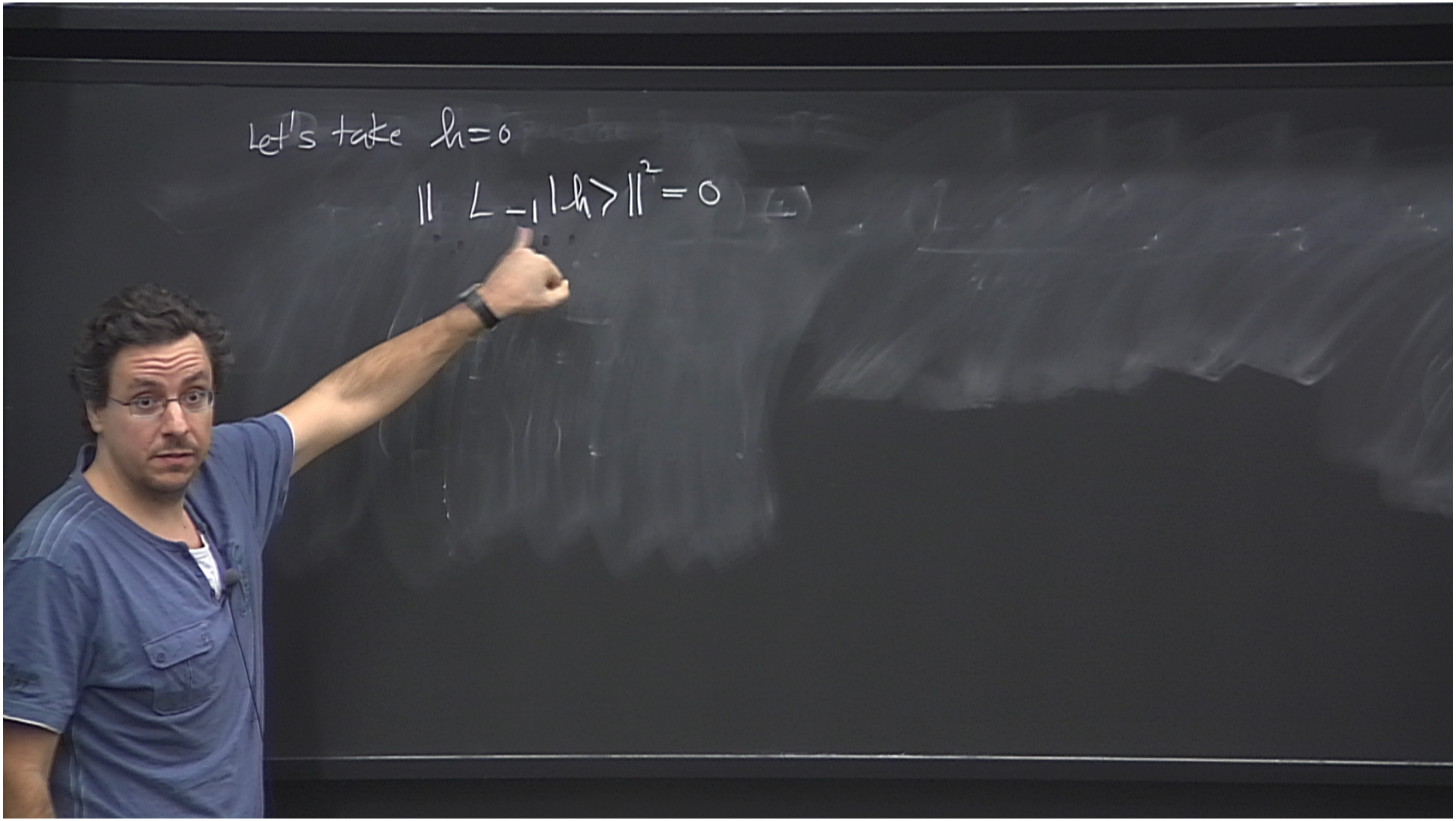
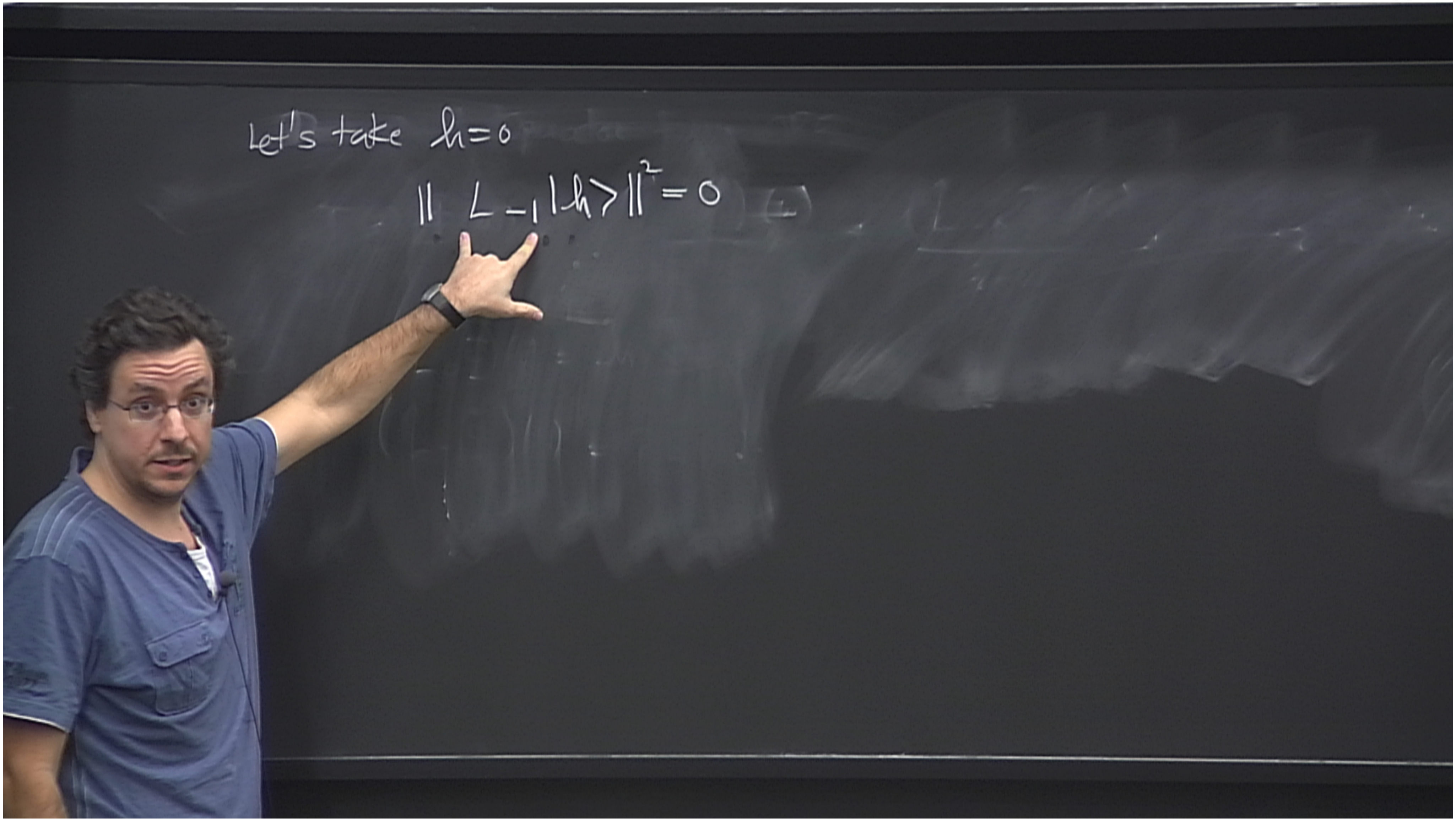
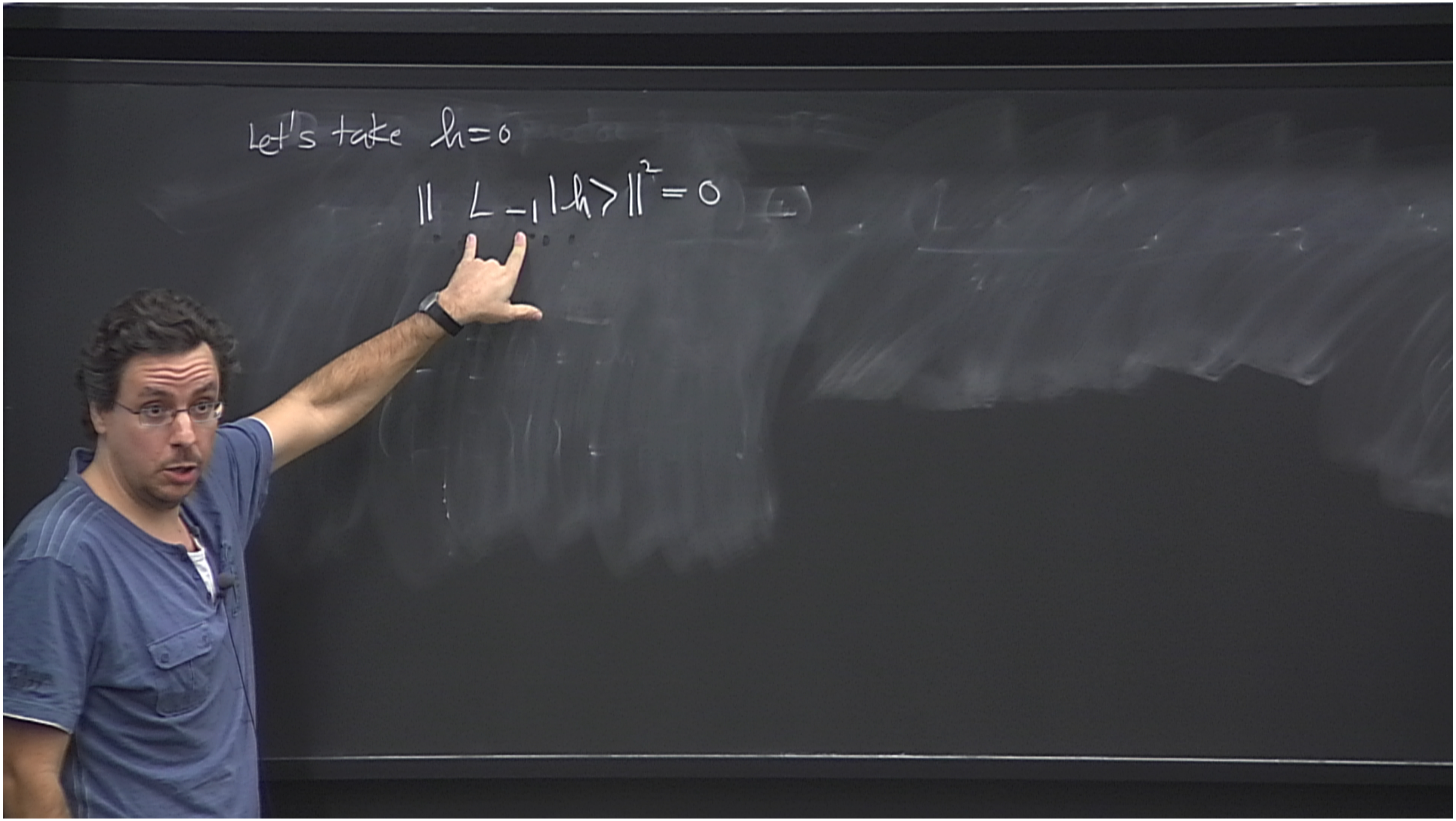

Let's take $h=0$
$\| L_{-1} |h\rangle \|^2 = 0$


Let's take h=0
|| L_-1 |h> ||² = 0


Let's take h=0
|| L_{-1} |h> ||^2 = 0


Let's take $h=0$
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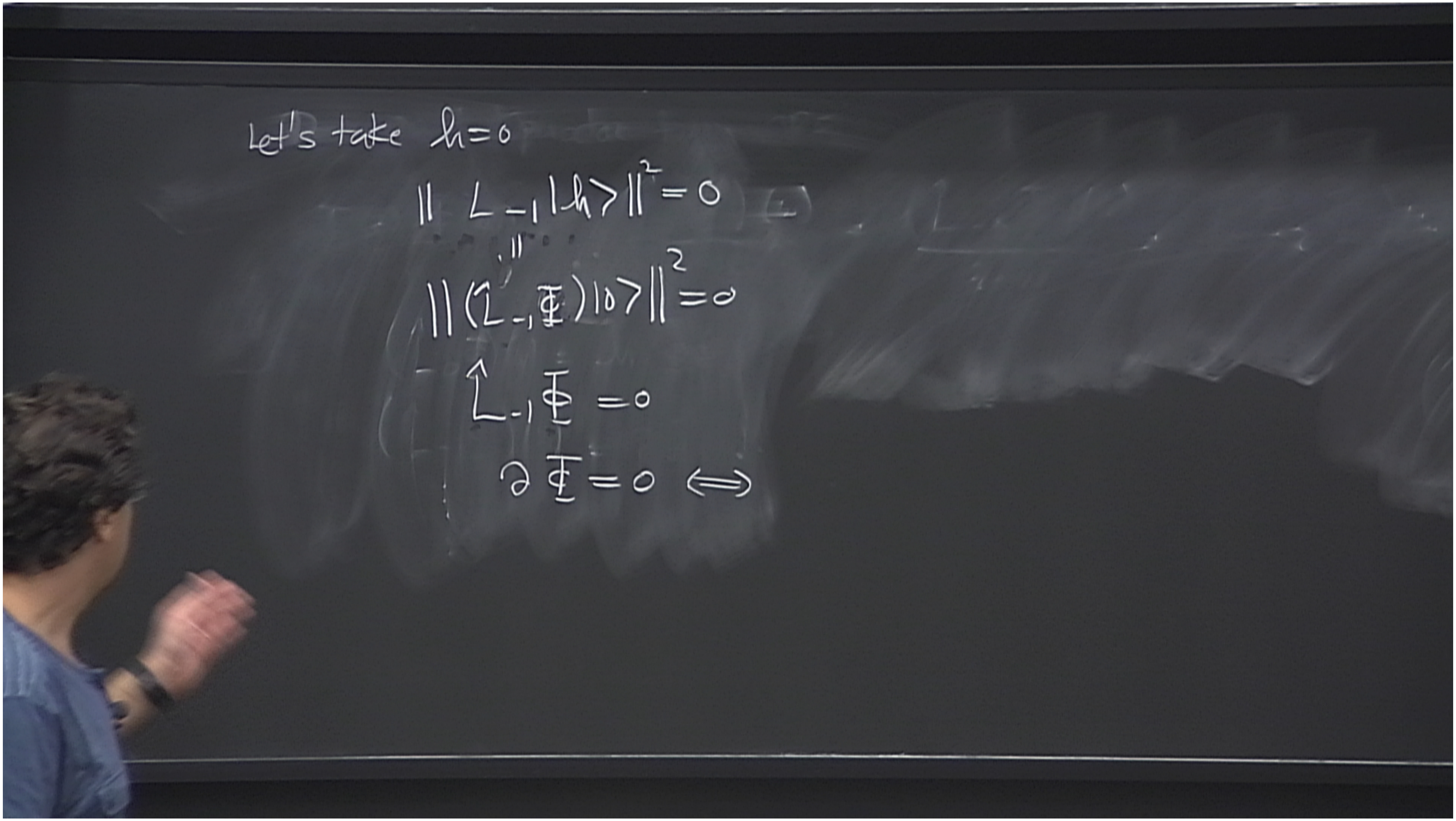

Let's take $h=0$
$\| L_{-1}|h\rangle \|^2 = 0$
$\| (L_{-1}\Phi)|0\rangle \|^2 = 0$
$L_{-1}\Phi = 0$
$\partial \Phi = 0 \Longleftrightarrow$

$h=0$

$\|L_{-1}|h\rangle\|^2=0$

$\|(L_{-1}\Phi)|0\rangle\|^2=0$

$L_{-1}\Phi=0$

$\partial\Phi=0 \iff \Phi \propto \mathbb{1}$

- Constraints on the values of c

$h=0$

$\| L_{-1}|h\rangle \|^2 = 0$

$\| (L_{-1}\Phi)|0\rangle \|^2 = 0$

$L_{-1}\Phi = 0$

$\partial\Phi = 0 \iff \Phi \propto \mathbb{1}$

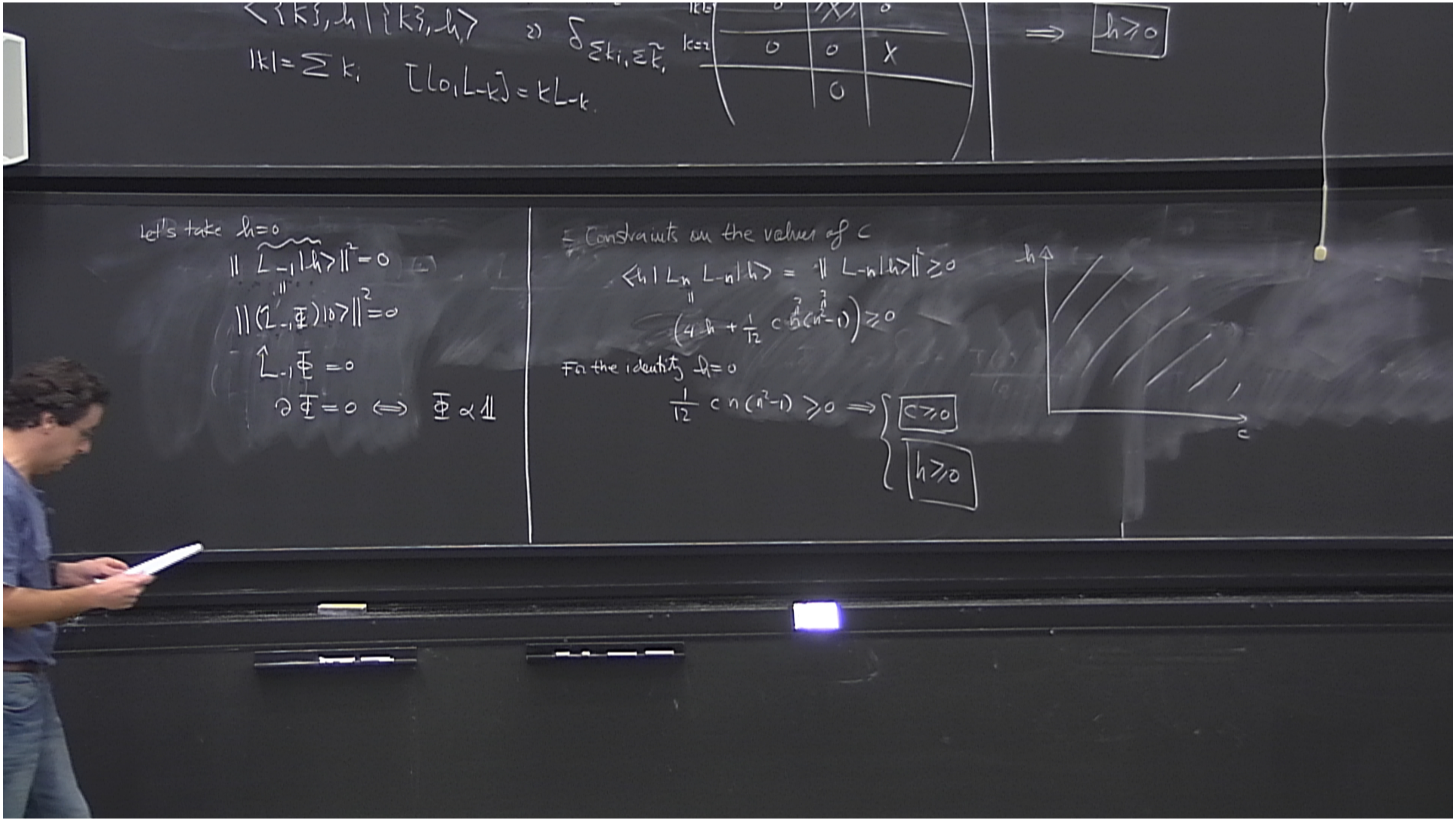
Constraints on the values of c

$$\langle h| L_n L_{-n} |h\rangle = \| L_{-n}|h\rangle \|^2 \geq 0$$

$$4h + \frac{1}{12} c\, n(n^2-1) \geq 0$$

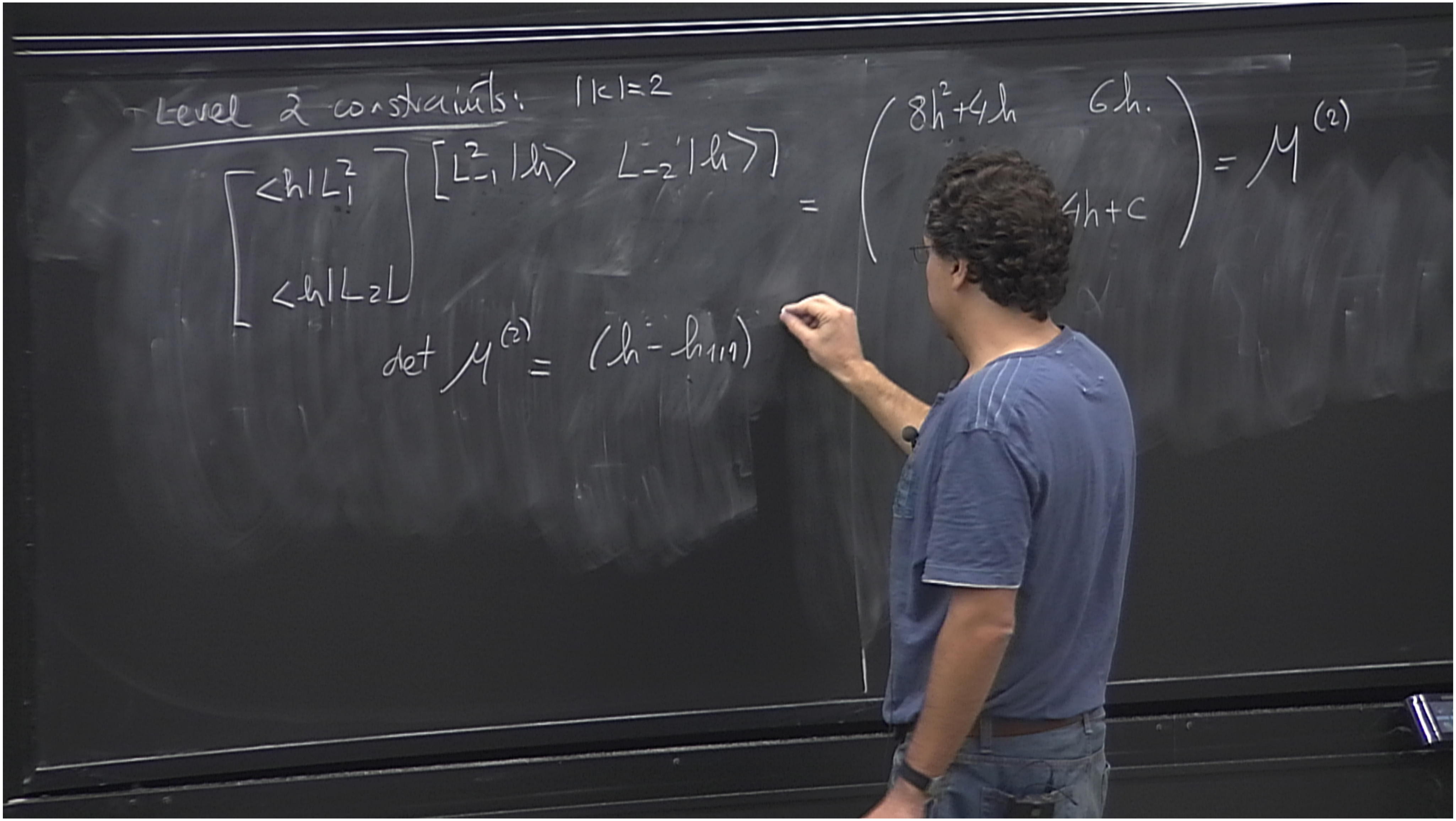
For the identity $h=0$

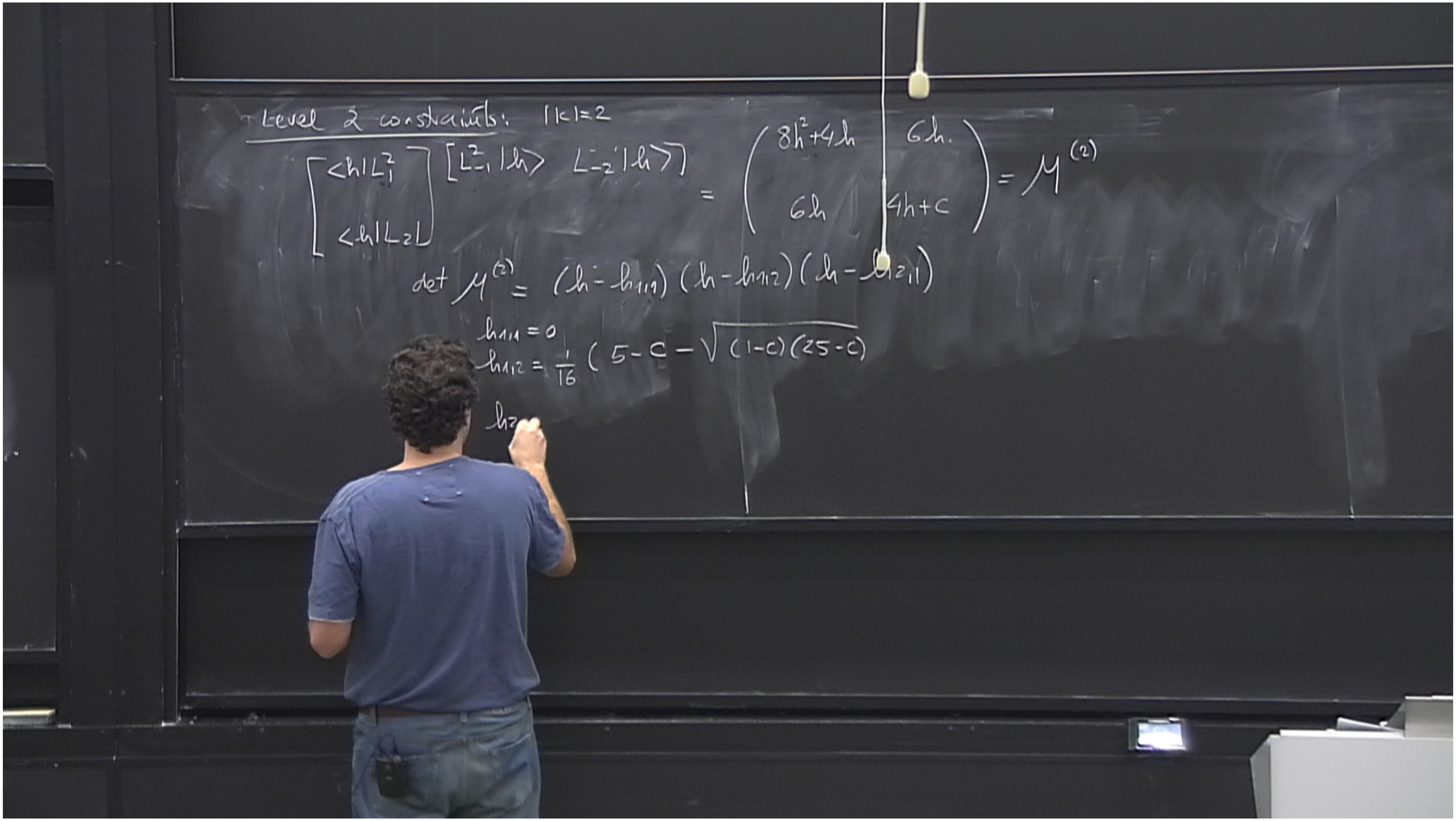
$$\frac{1}{12} c\, n$$

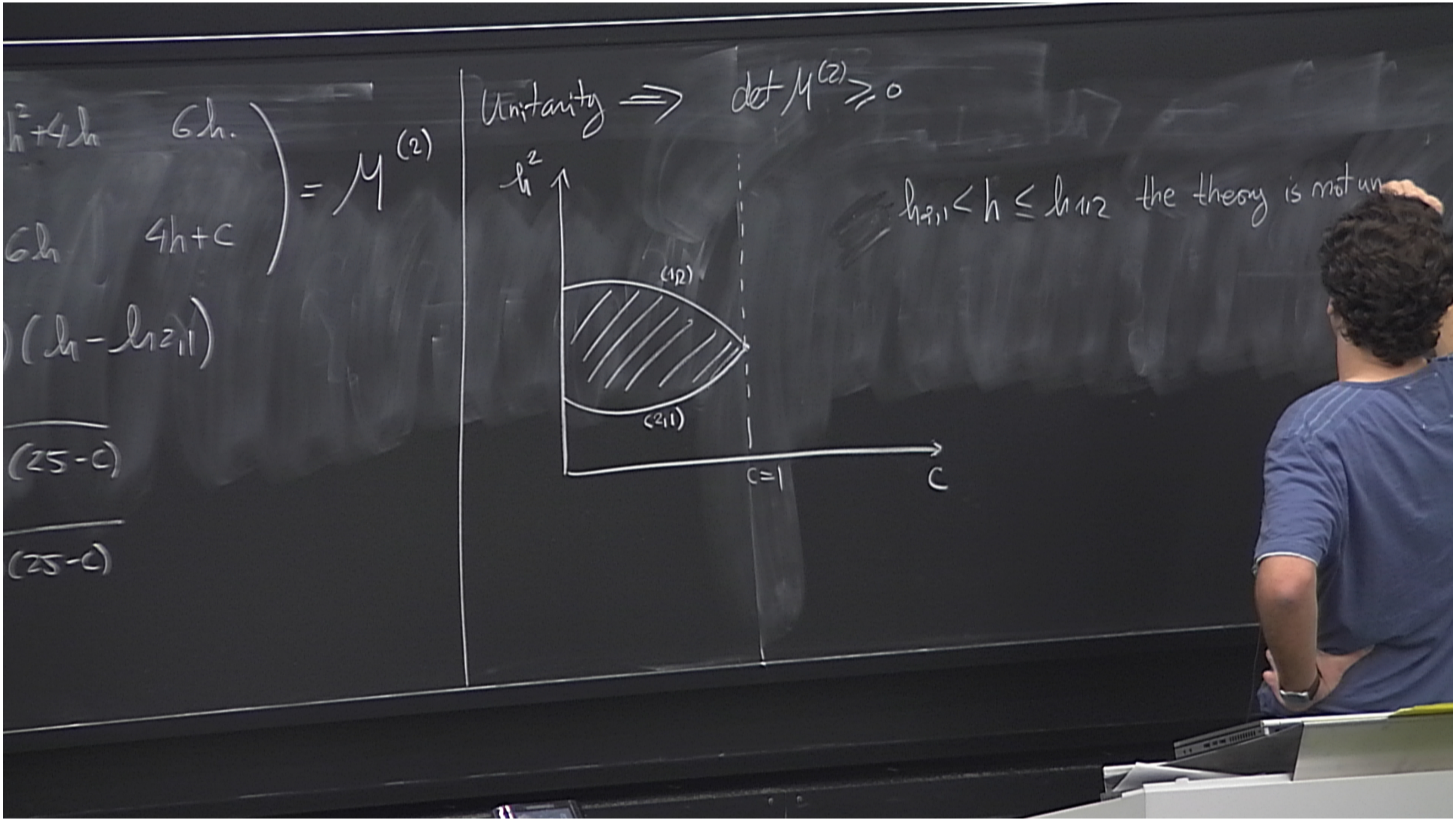

$|k| = \sum k_i$
$[L_0, L_{-k}] = k L_{-k}$
2) $\delta_{\sum k_i, \sum \tilde{k}_i}$
$\Rightarrow \boxed{h \geq 0}$
Let's take $h=0$
$\| L_{-1} |h\rangle \|^2 = 0$
$\| (L_{-1} \Phi) |0\rangle \|^2 = 0$
$L_{-1} \Phi = 0$
$\partial \Phi = 0 \iff \Phi \propto \mathbb{1}$
Constraints on the values of c
$\langle h | L_n L_{-n} | h \rangle = \| L_{-n} |h\rangle \|^2 \geq 0$
$\left(4h + \frac{1}{12} c\, n(n^2-1)\right) \geq 0$
For the identity $h=0$
$\frac{1}{12} c\, n(n^2-1) \geq 0 \Rightarrow \boxed{c \geq 0}$
$\boxed{h \geq 0}$
h
c

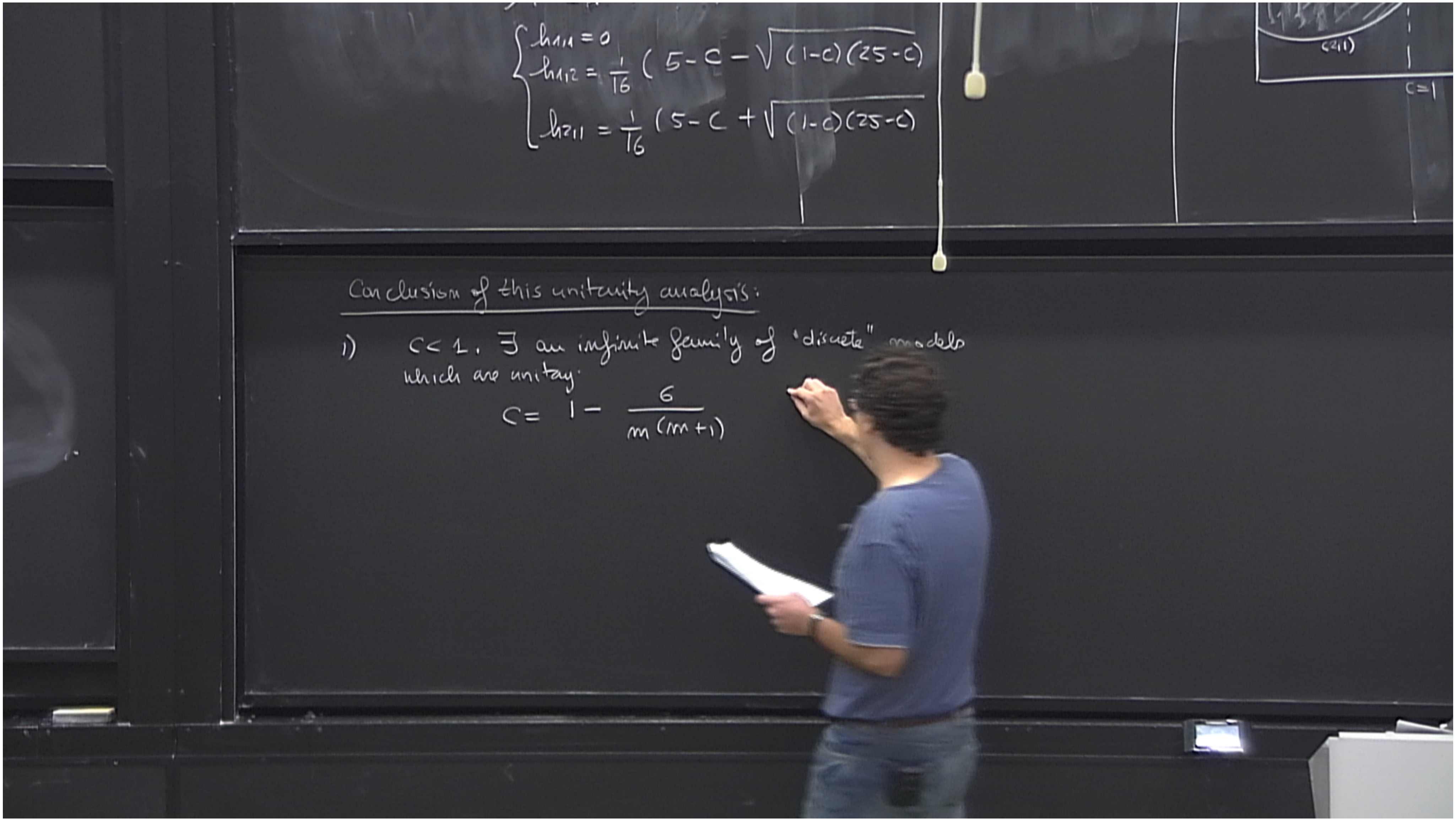
Level 2 constraints: $|k| = 2$

$$\begin{bmatrix} \langle h| L_1^2 \\ \langle h| L_2 \end{bmatrix} \begin{bmatrix} L_{-1}^2 |h\rangle & L_{-2} |h\rangle \end{bmatrix} = \begin{pmatrix} 8h^2+4h & 6h \\ 6h & 4h+c \end{pmatrix}$$


Level 2 constraints: |k|=2
[<h|L_1^2 ; <h|L_2] [L_{-1}^2|h> L_{-2}|h>] = (8h^2+4h 6h. ; 4h+c) = M^(2)
det M^(2) = (h - h_{1,1})


Level 2 constraints: $|k|=2$
$\begin{bmatrix} \langle h|L_1^2 \\ \langle h|L_2 \end{bmatrix} \begin{bmatrix} L_{-1}^2|h\rangle & L_{-2}|h\rangle \end{bmatrix} = \begin{pmatrix} 8h^2+4h & 6h \\ 6h & 4h+c \end{pmatrix} = M^{(2)}$
$\det M^{(2)} = (h-h_{1,1})(h-h_{1,2})(h-h_{2,1})$
$h_{1,1}=0$
$h_{1,2} = \frac{1}{16}(5-c-\sqrt{(1-c)(25-c)}$
h_2


Unitarity ⟹ det M^(2) ≥ 0
h²+4h 6h.
6h 4h+c
= M^(2)
(h − h2,1)
(25 − c)
(25 − c)
h²
(1,2)
(2,1)
c=1
c
h2,1 < h ≤ h1,2 the theory is not un


$h_{1,1} = 0$
$h_{1,2} = \frac{1}{16} (5 - c - \sqrt{(1-c)(25-c)}$
$h_{2,1} = \frac{1}{16} (5 - c + \sqrt{(1-c)(25-c)}$
(2,1)
c=1
Conclusion of this unitarity analysis:
1) c< 1. ∃ an infinite family of "discrete" models which are unitary
$c = 1 - \frac{6}{m(m+1)}$

$$\begin{cases} h_{1,1} = 0 \\ h_{1,2} = \frac{1}{16}\left(5 - c - \sqrt{(1-c)(25-c)}\right) \\ h_{2,1} = \frac{1}{16}\left(5 - c + \sqrt{(1-c)(25-c)}\right) \end{cases}$$

$c(2,1)$

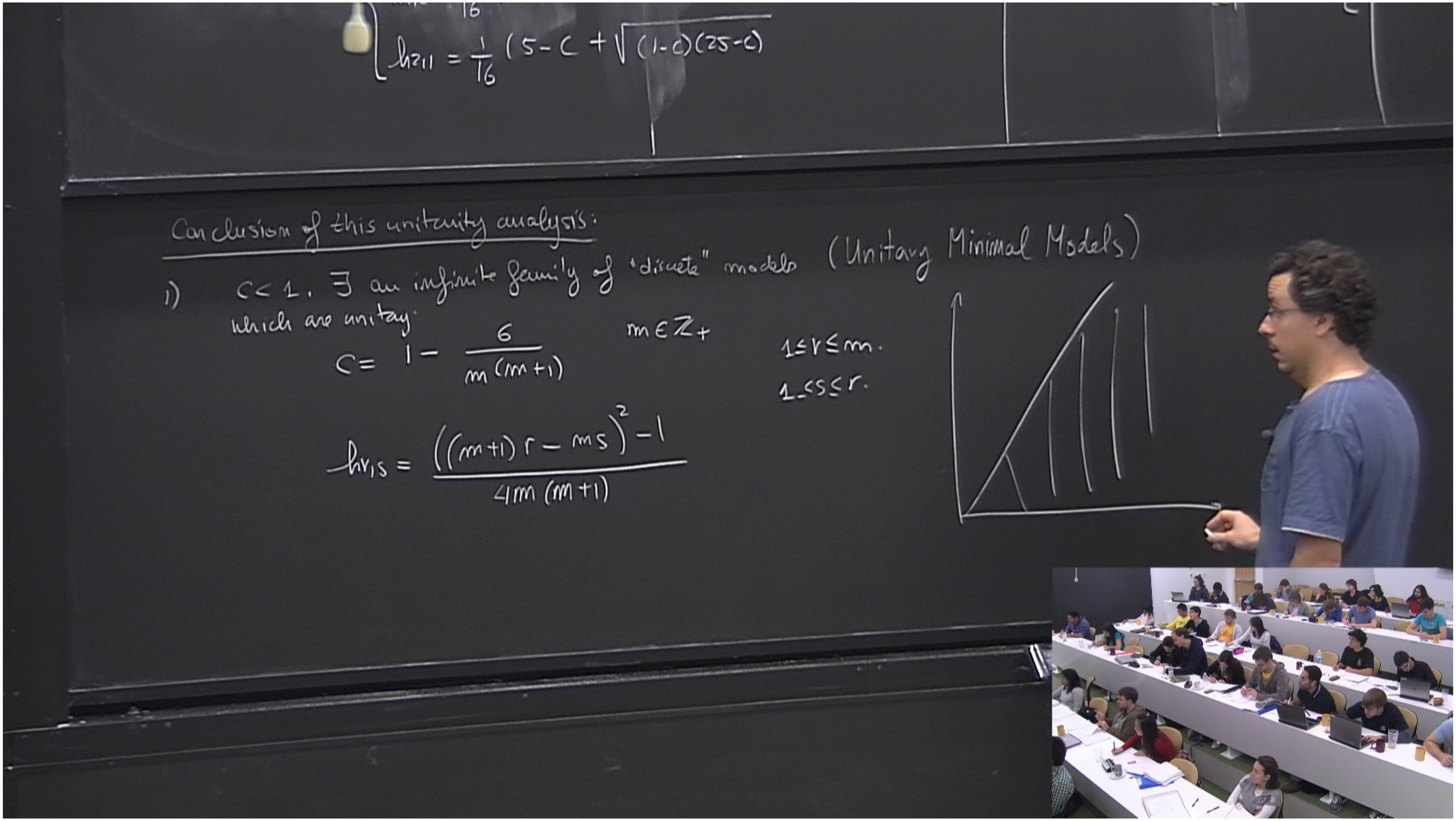
$c=1$

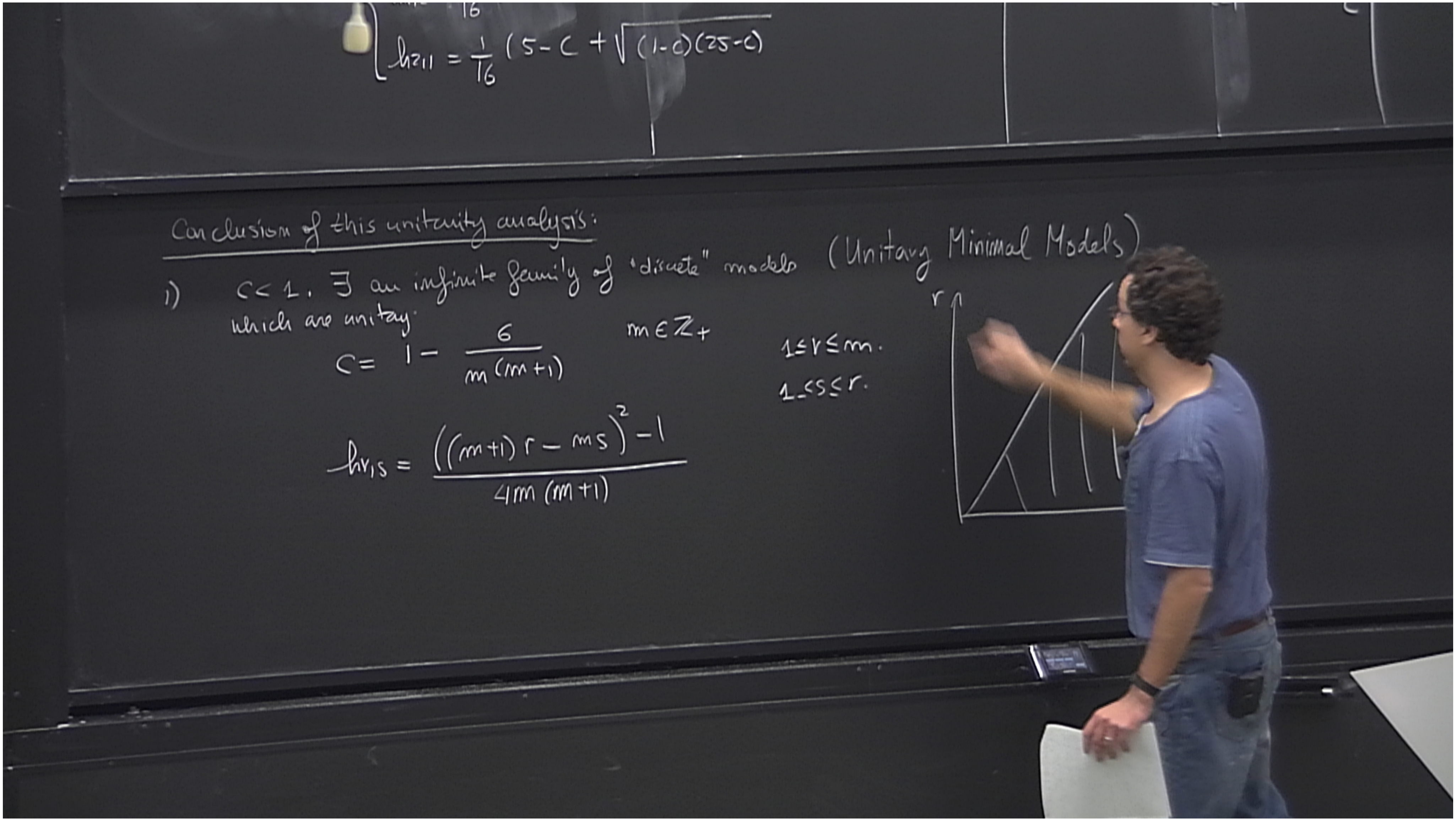
Conclusion of this unitarity analysis:

1) $c < 1$. $\exists$ an infinite family of "discrete" models which are unitary.

$$c = 1 - \frac{6}{m(m+1)} \qquad m \in \mathbb{Z}_+$$

$$h_{r,s} = \frac{\left((m+1)r - ms\right)^2 - 1}{4m(m+1)}$$


$h_{2,1} = \frac{1}{16}\left(5 - c + \sqrt{(1-c)(25-c)}\right.$
Conclusion of this unitarity analysis:
1) $c < 1$. $\exists$ an infinite family of "discrete" models (Unitary Minimal Models)
which are unitary.
$c = 1 - \frac{6}{m(m+1)}$ $m \in \mathbb{Z}_+$
$1 \leq r \leq m.$
$1 \leq s \leq r.$
$h_{r,s} = \frac{((m+1)r - ms)^2 - 1}{4m(m+1)}$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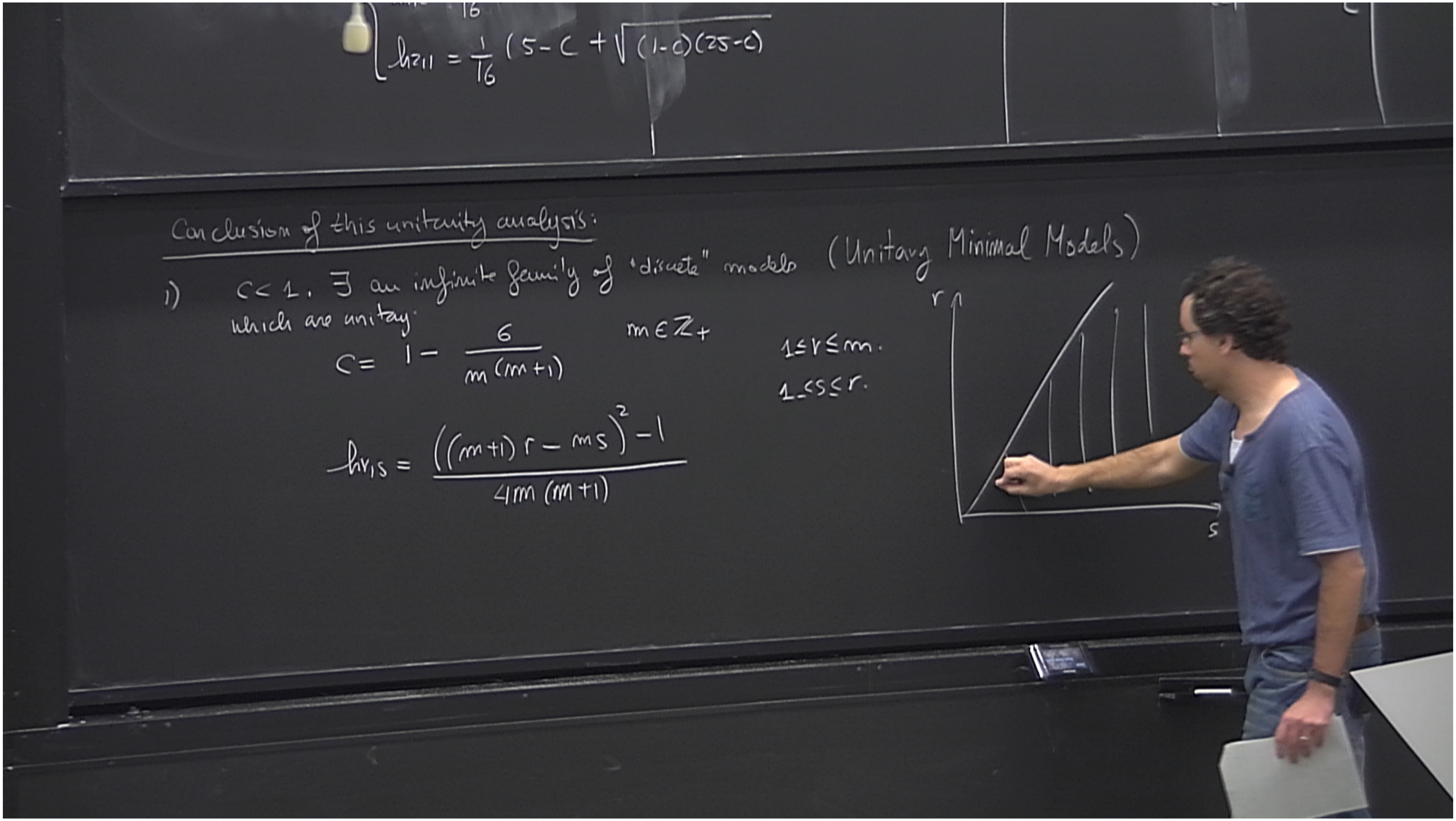
r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$1 \le r \le m$.
$1 \le s \le r$.
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r
s

$$h_{2,1} = \frac{1}{16}\left(5 - c + \sqrt{(1-c)(25-c)}\right.$$

Conclusion of this unitarity analysis:

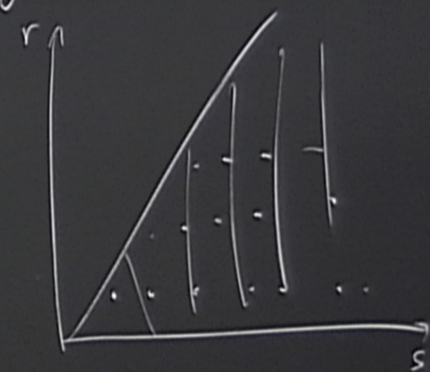
1) $c < 1$. $\exists$ an infinite family of "discrete" models (Unitary Minimal Models) which are unitary.

$$c = 1 - \frac{6}{m(m+1)} \qquad m \in \mathbb{Z}_+$$

$1 \le r \le m$.

$1 \le s \le r$.

$$h_{r,s} = \frac{\left((m+1)r - ms\right)^2 - 1}{4m(m+1)}$$



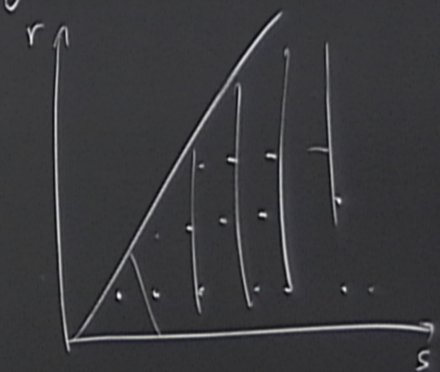
$\Rightarrow$ $m = 3$ $\left(c = \frac{1}{2}\right)$ Universality class of the 2D Ising Model

$$h_{2,1} = \frac{1}{16}\left(5 - c + \sqrt{(1-c)(25-c)}\right.$$

Conclusion of this unitarity analysis:

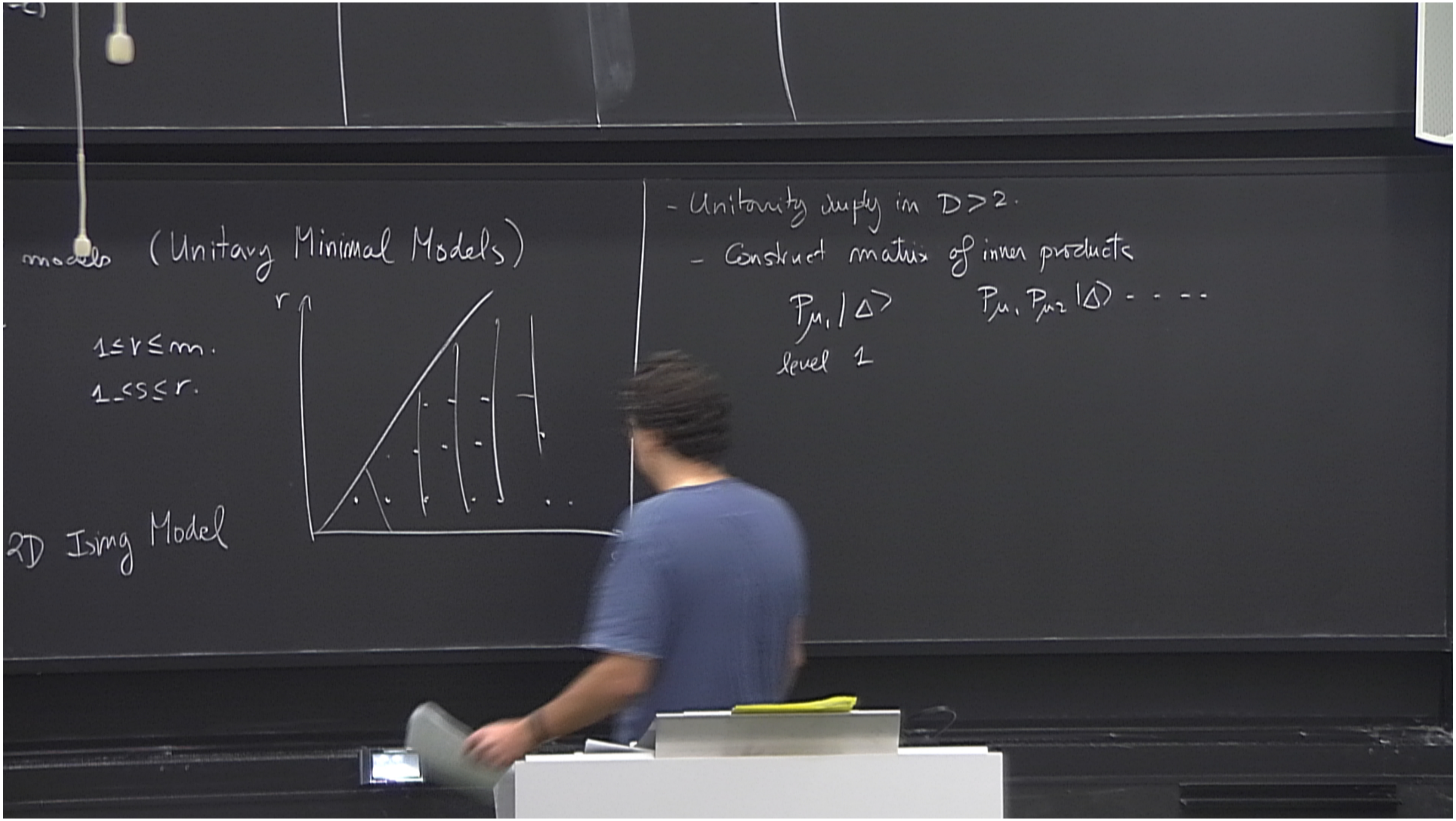
1) $c < 1$. $\exists$ an infinite family of "discrete" models (Unitary Minimal Models) which are unitary.

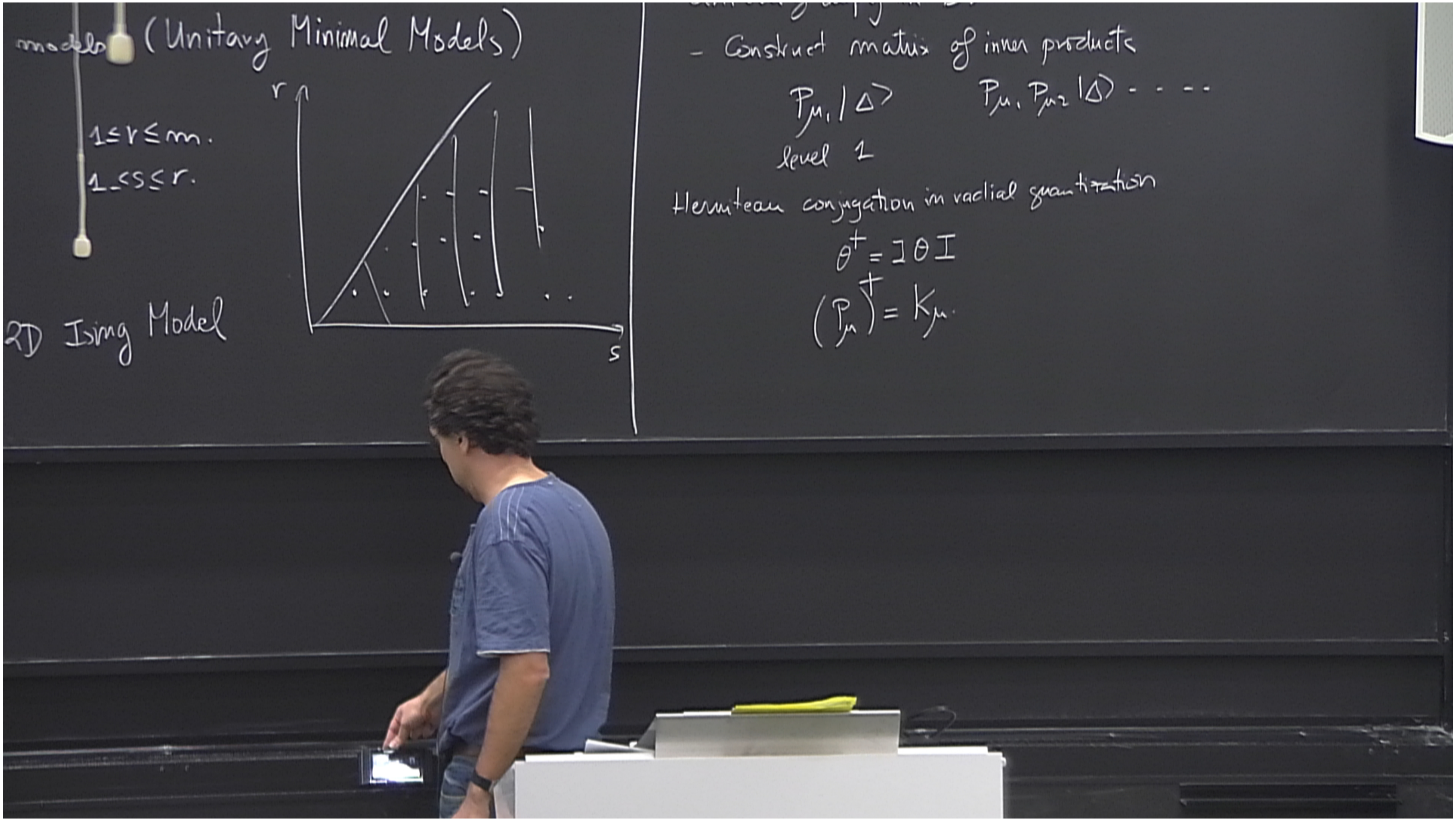
$$c = 1 - \frac{6}{m(m+1)} \qquad m \in \mathbb{Z}_+ \qquad 1 \le r \le m. \quad 1 \le s \le r.$$

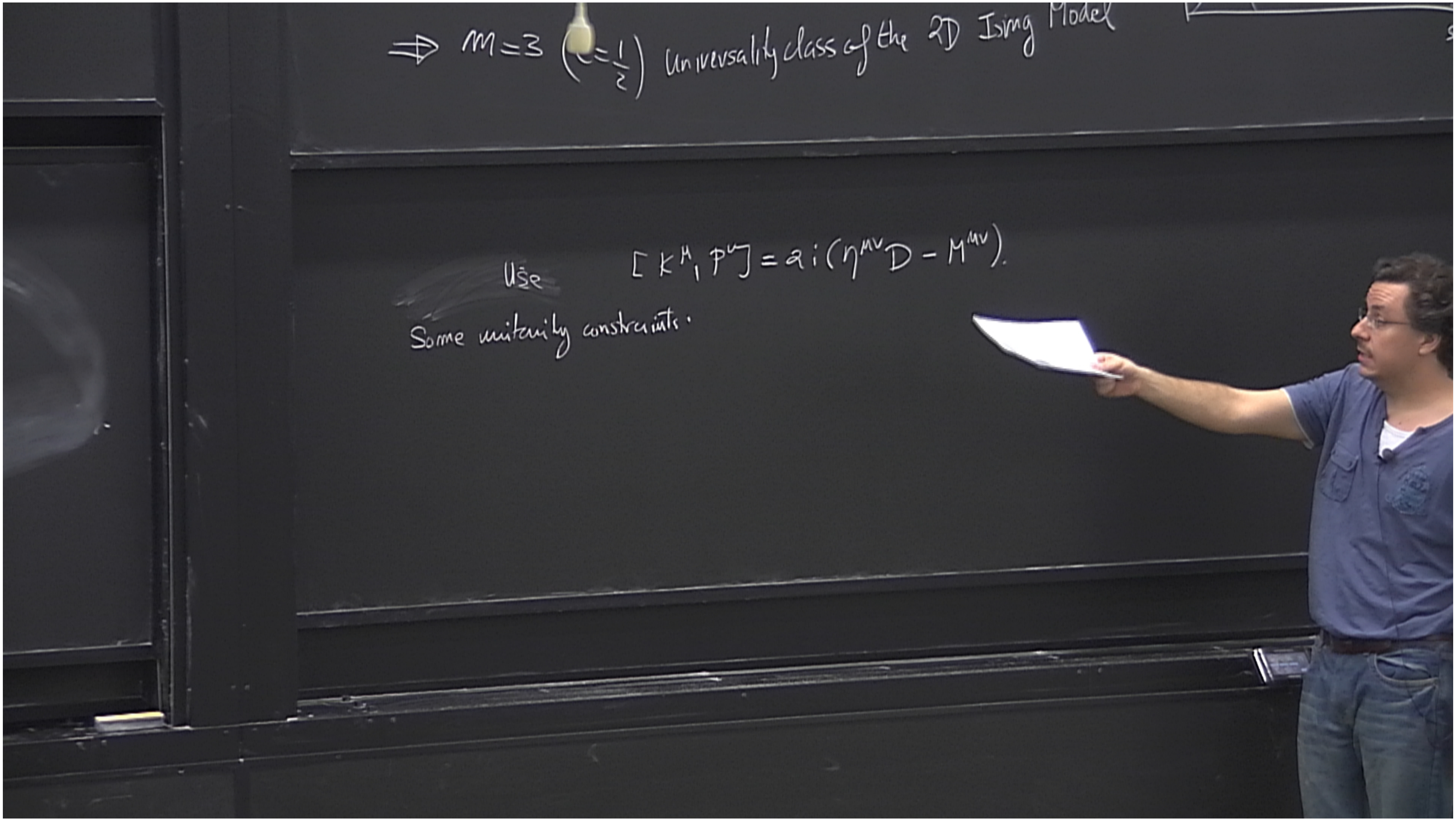


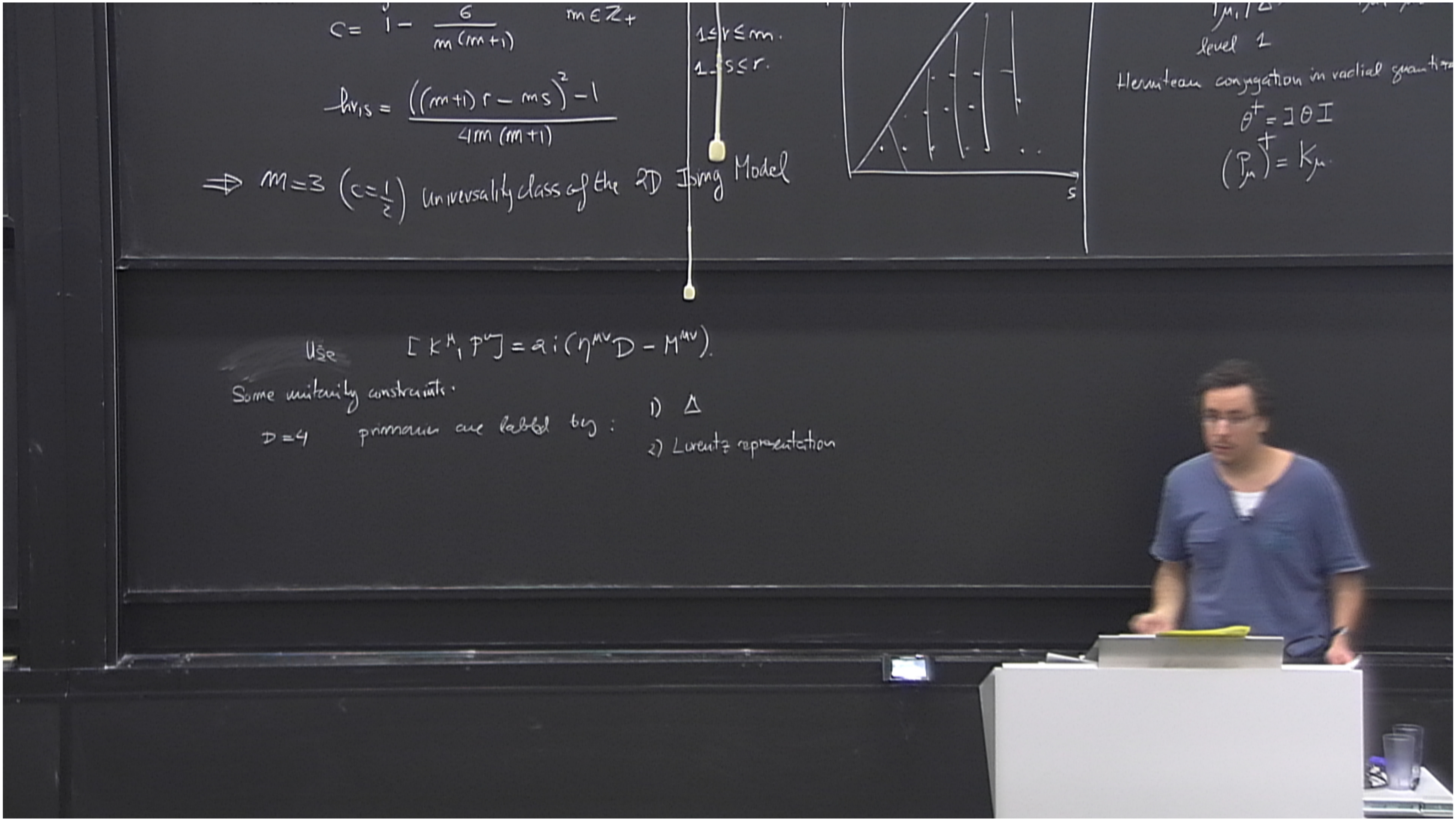
$$h_{r,s} = \frac{\left((m+1)r - ms\right)^2 - 1}{4m(m+1)}$$

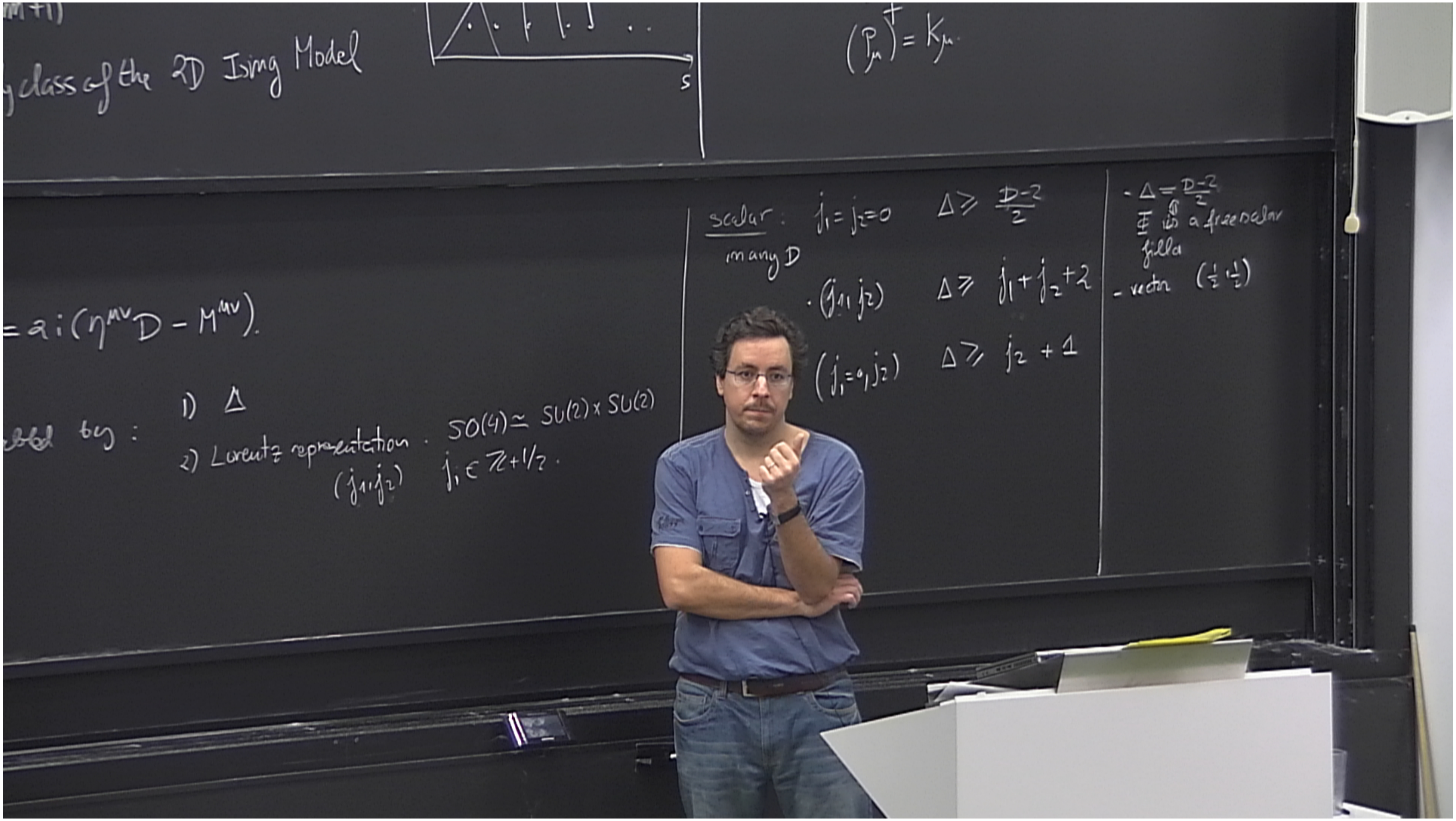
$\Rightarrow$ $m = 3$ $\left(c = \frac{1}{2}\right)$ Universality class of the 2D Ising Model


models (Unitary Minimal Models)
r
1≤r≤m.
1≤s≤r.
2D Ising Model
- Unitarity imply in D>2.
- Construct matrix of inner products
P_μ1 |Δ>
P_μ1 P_μ2 |Δ> - - - -
level 1


models (Unitary Minimal Models)
1≤r≤m.
1≤s≤r.
r
s
2D Ising Model
- Construct matrix of inner products
$P_{\mu_1}|\Delta\rangle$ $P_{\mu_1}P_{\mu_2}|\Delta\rangle$ - - - -
level 1
Hermitean conjugation in radial quantization
$\theta^\dagger = I\theta I$
$(P_\mu)^\dagger = K_\mu$


⇒ $m=3$ ($c=\frac{1}{2}$) Universality class of the 2D Ising Model
Use $[K^\mu, P^\nu] = 2i(\eta^{\mu\nu} D - M^{\mu\nu})$.
Some unitarity constraints.


c = 1 − 6 / m(m+1)   m ∈ Z+
1 ≤ r ≤ m.
1 ≤ s ≤ r.
h_r,s = ((m+1) r − m s)^2 − 1 / 4m (m+1)
⇒ m = 3 (c = 1/2) Universality class of the 2D Ising Model
s
level 1
Hermitean conjugation in radial quantization
θ† = I θ I
(P_μ)† = K_μ.
Use [K^μ, P^ν] = 2i (η^μν D − M^μν).
Some unitarity constraints.
D = 4   primaries are labeled by :
1) Δ
2) Lorentz representation


class of the 2D Ising Model
s
$(P_\mu)^\dagger = K_\mu$
$= 2i(\eta^{\mu\nu} D - M^{\mu\nu})$.
by :
1) Δ
2) Lorentz representation . $SO(4) \simeq SU(2) \times SU(2)$
(j_1, j_2) $j_i \in \mathbb{Z} + 1/2$.
scalar : $j_1 = j_2 = 0$ $\Delta \geq \frac{D-2}{2}$
many D
$\cdot (j_1, j_2)$ $\Delta \geq j_1 + j_2 + 2$
$(j_1 = 0, j_2)$ $\Delta \geq j_2 + 1$
$\cdot \Delta = \frac{D-2}{2}$
Φ is a free scalar field
- vector $(\frac{1}{2}, \frac{1}{2})$