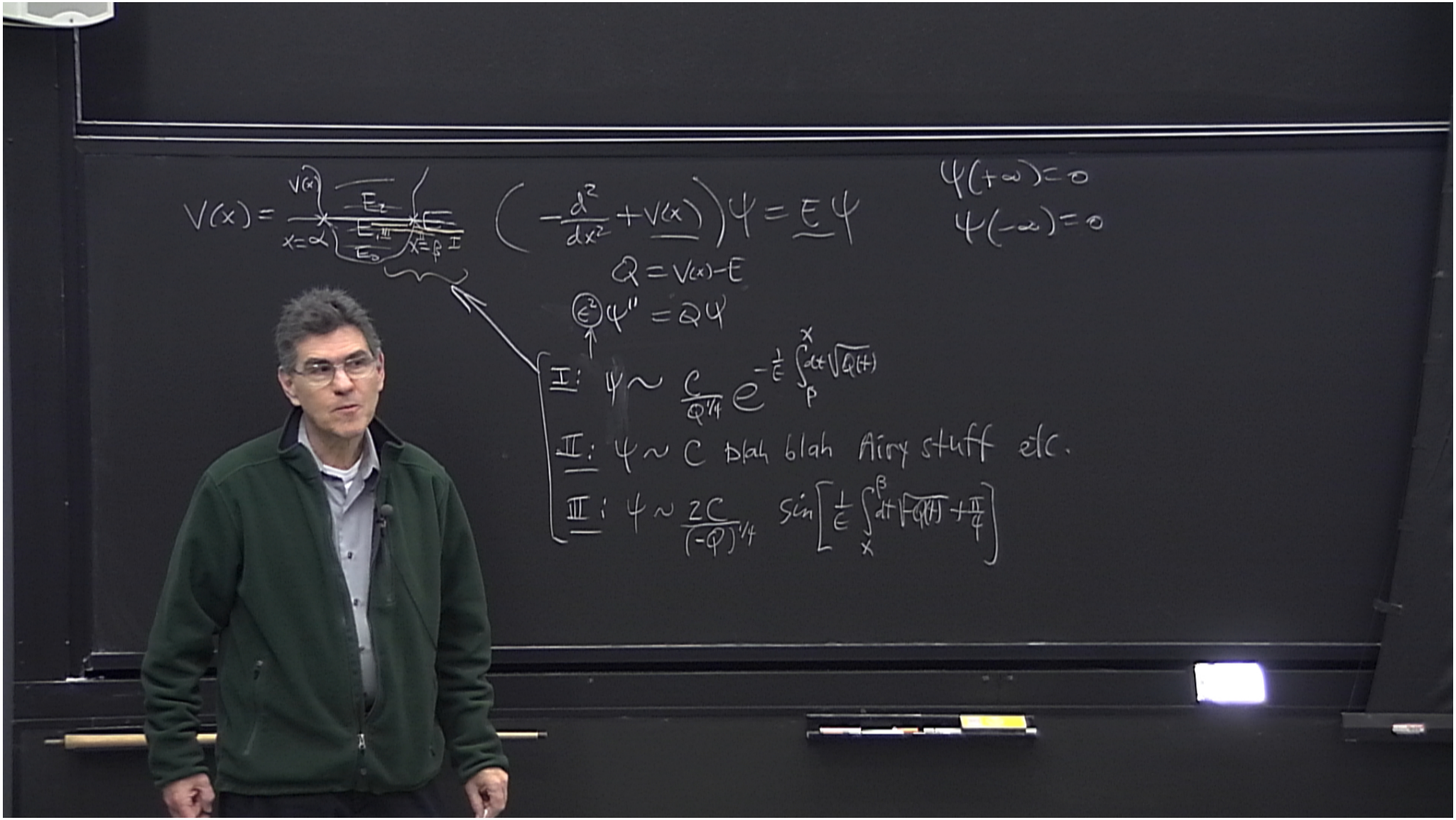
Title: Mathematical Physics - Lecture 15

Date: Dec 09, 2011 09:00 AM

URL: http://pirsa.org/11120009

Abstract:


V(x) =
x=α
x=β
$\left(-\frac{d^2}{dx^2}+V(x)\right)\psi = E\psi$
$\psi(+\infty)=0$
$\psi(-\infty)=0$
$Q = V(x)-E$
$\epsilon^2\psi'' = Q\psi$
I: $\psi \sim \frac{C}{Q^{1/4}} e^{-\frac{1}{\epsilon}\int_p^x dt\sqrt{Q(t)}}$
II: $\psi \sim$ C blah blah Airy stuff etc.
III: $\psi \sim \frac{2C}{(-Q)^{1/4}} \sin\left[\frac{1}{\epsilon}\int_x^\beta dt\sqrt{Q(t)} + \frac{\pi}{4}\right]$

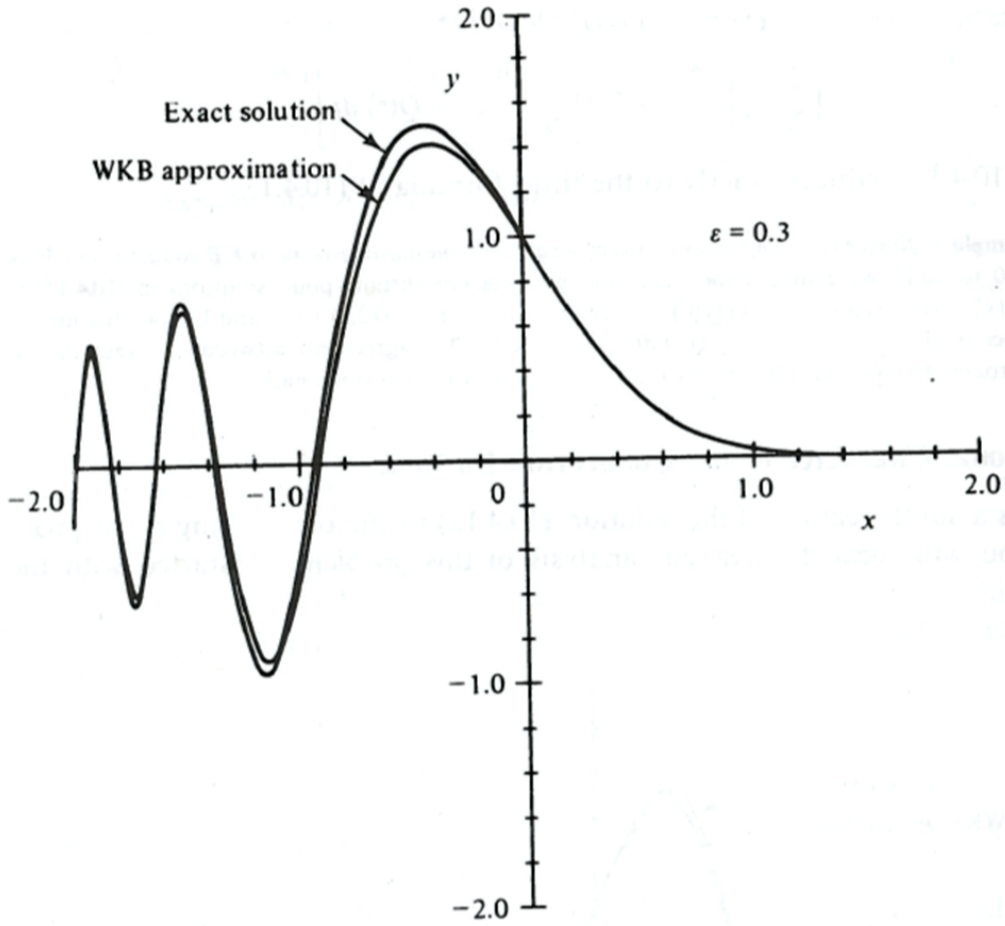
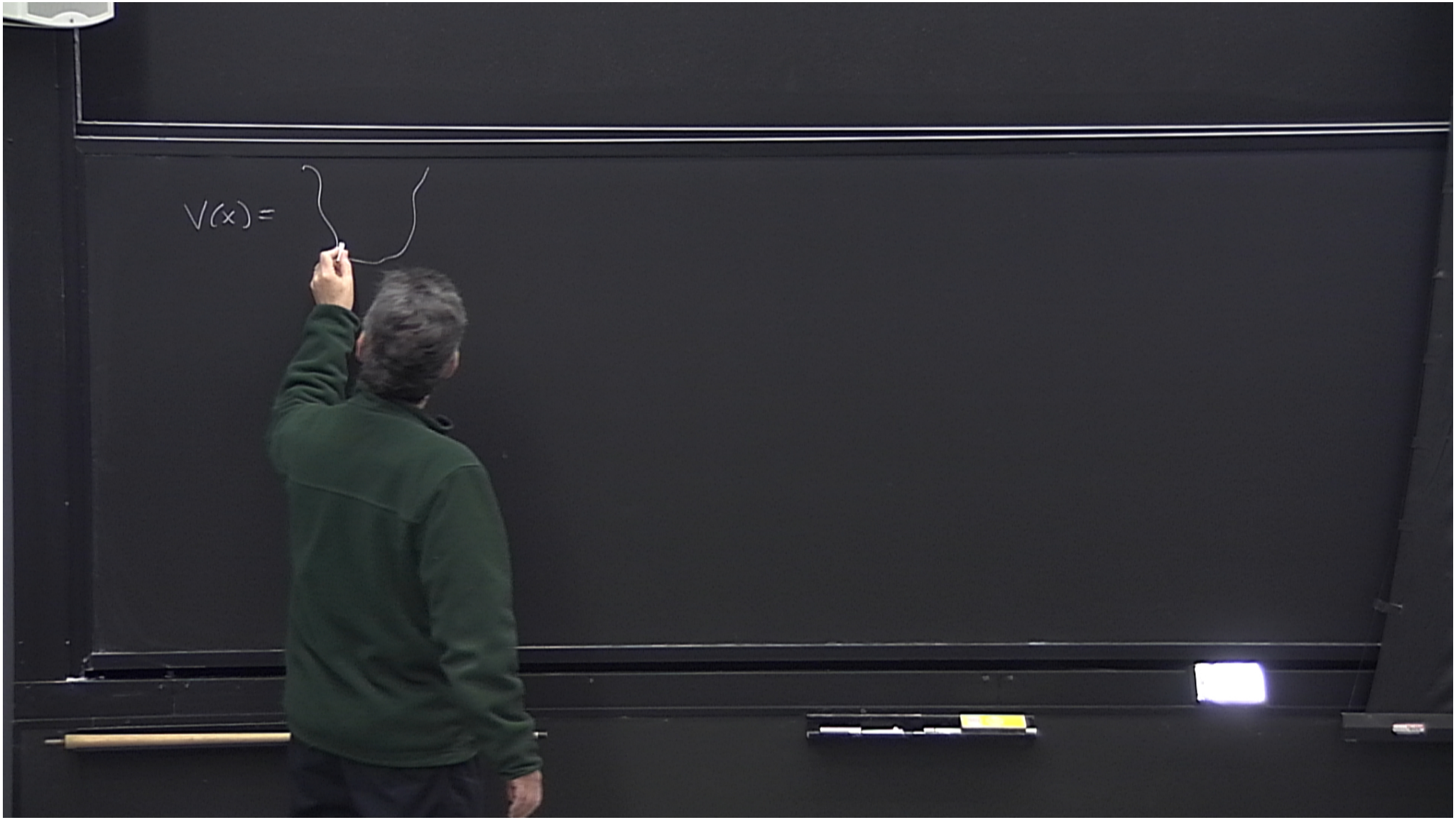
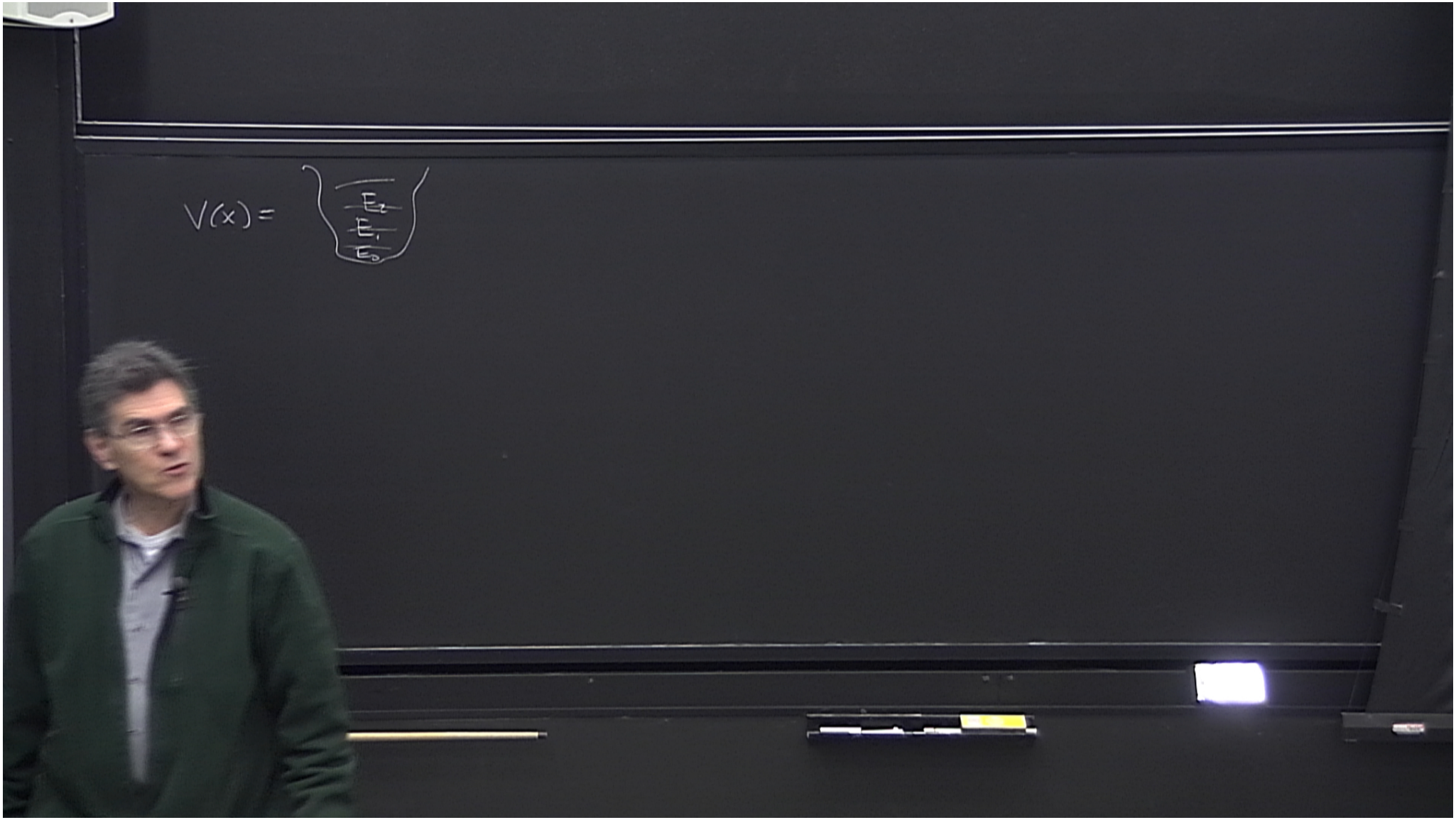
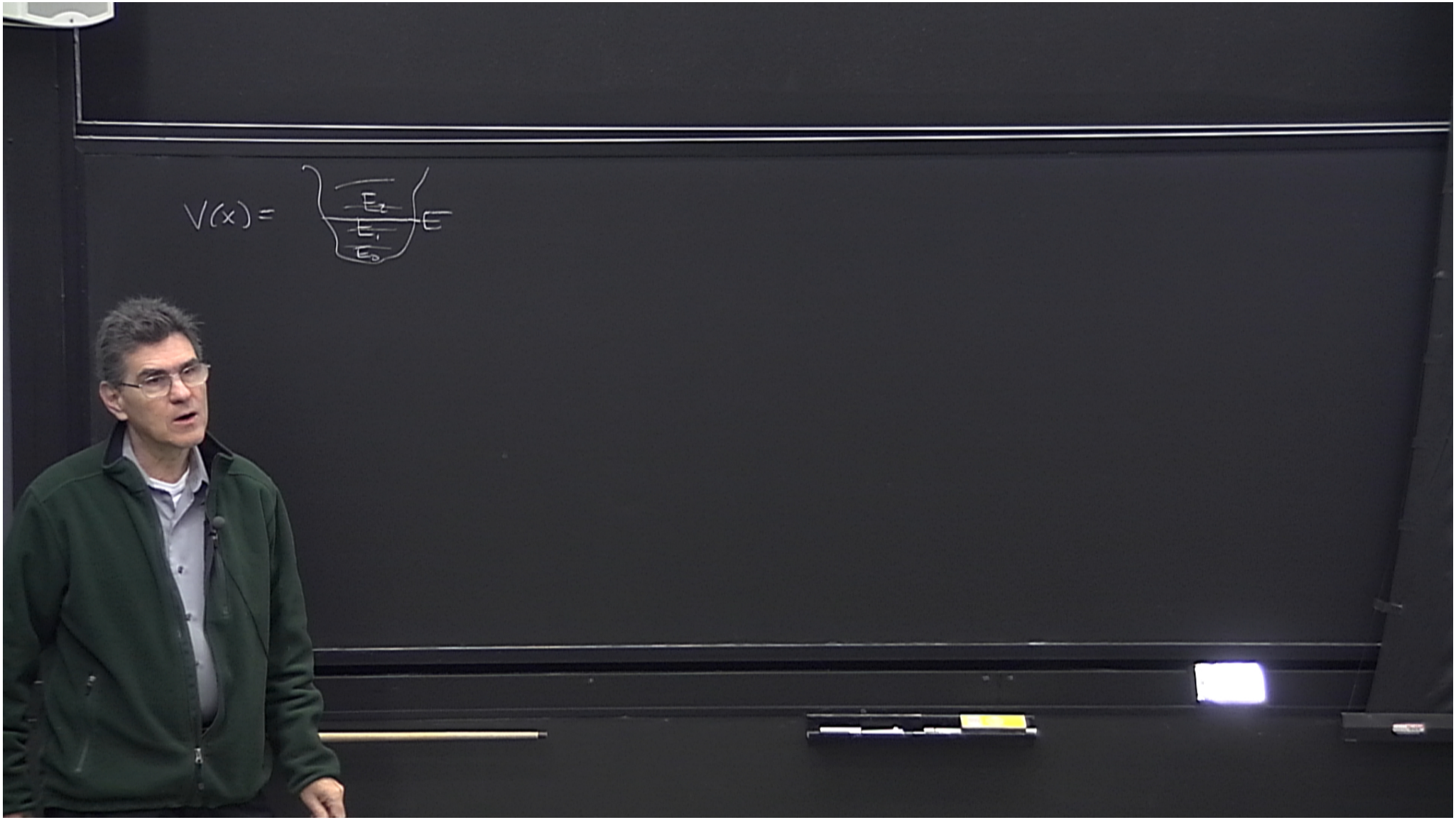
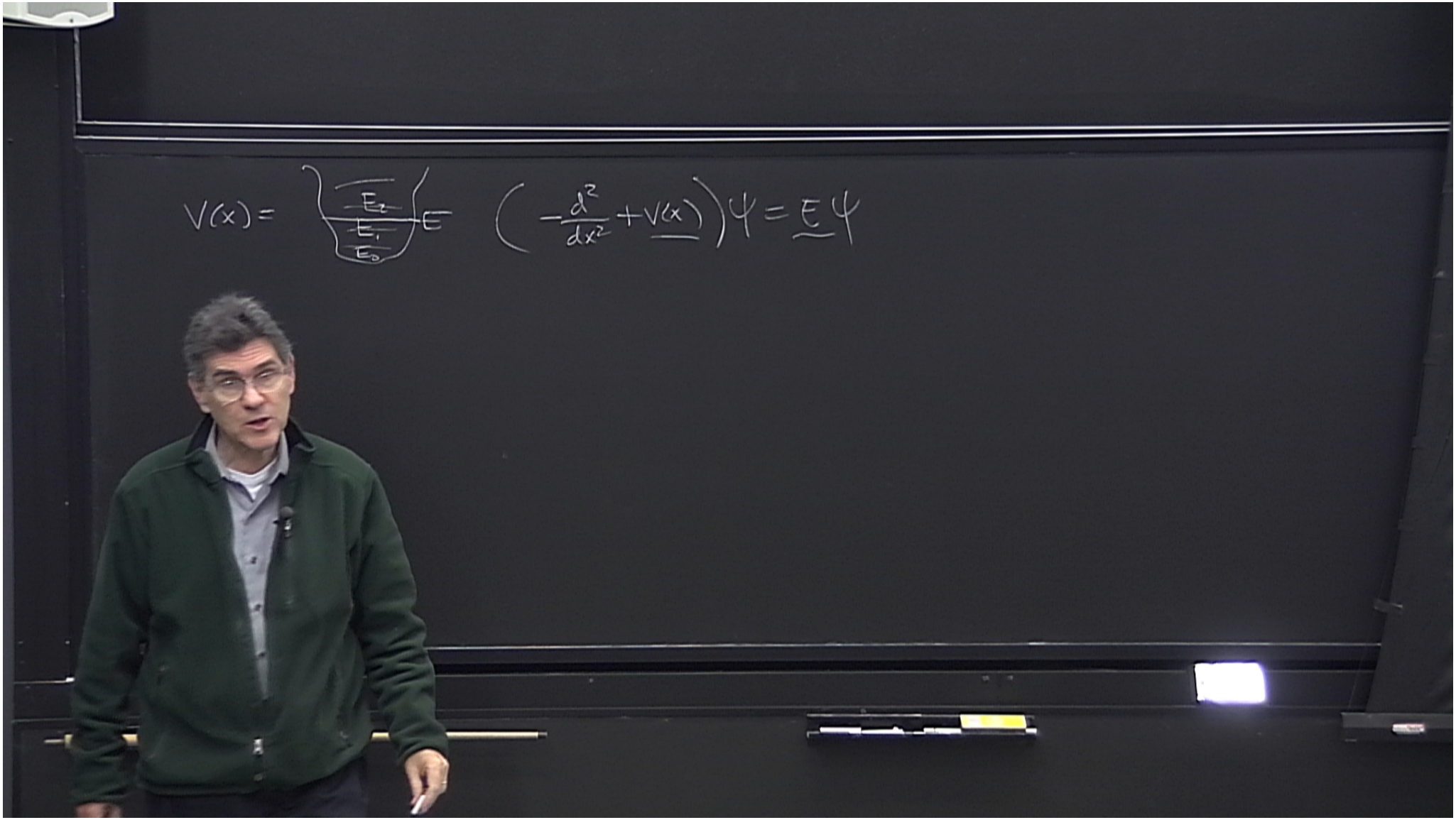


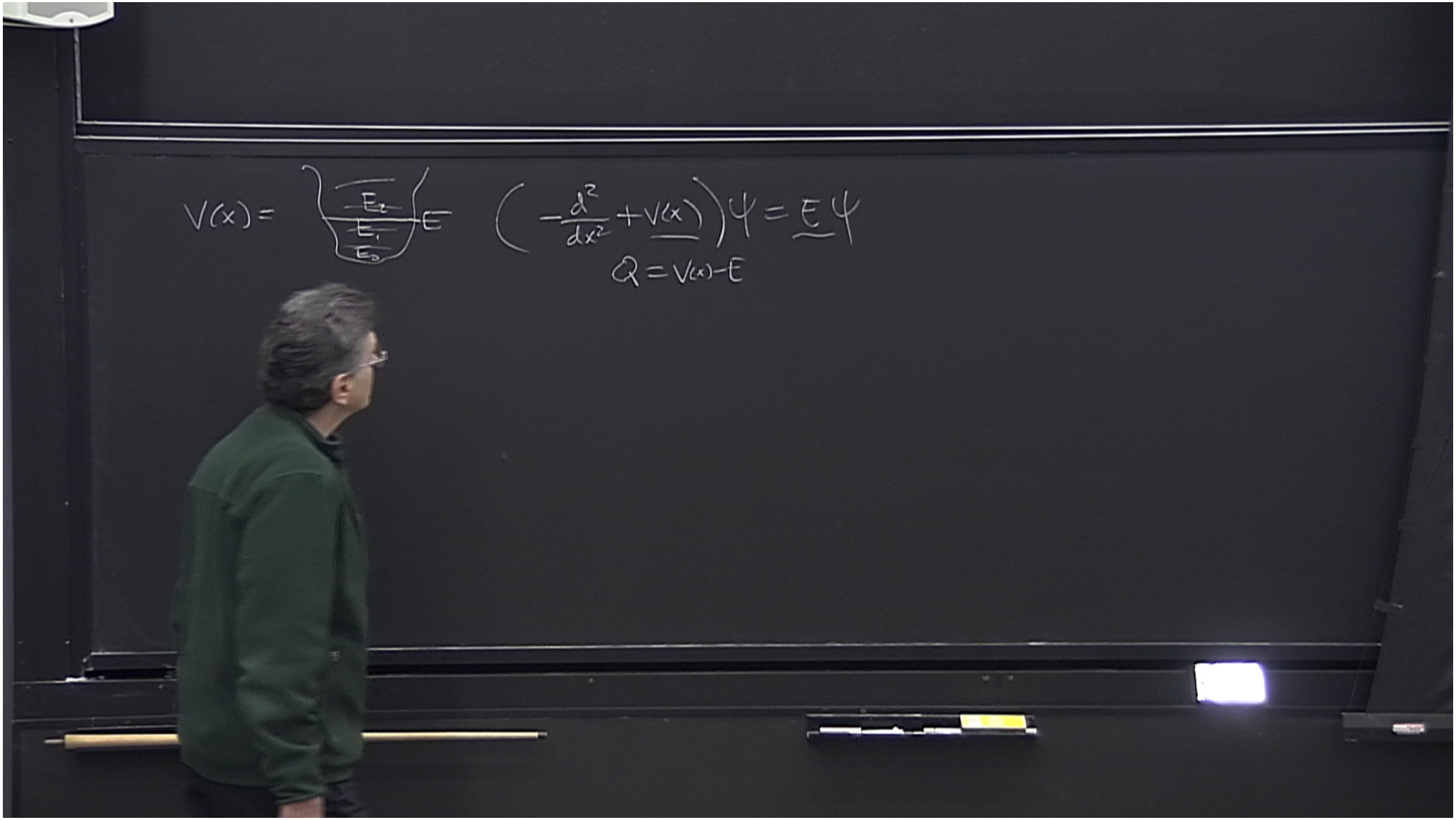
Figure 10.11 Same as in Fig. 10.10 except that $\varepsilon = 0.3$.

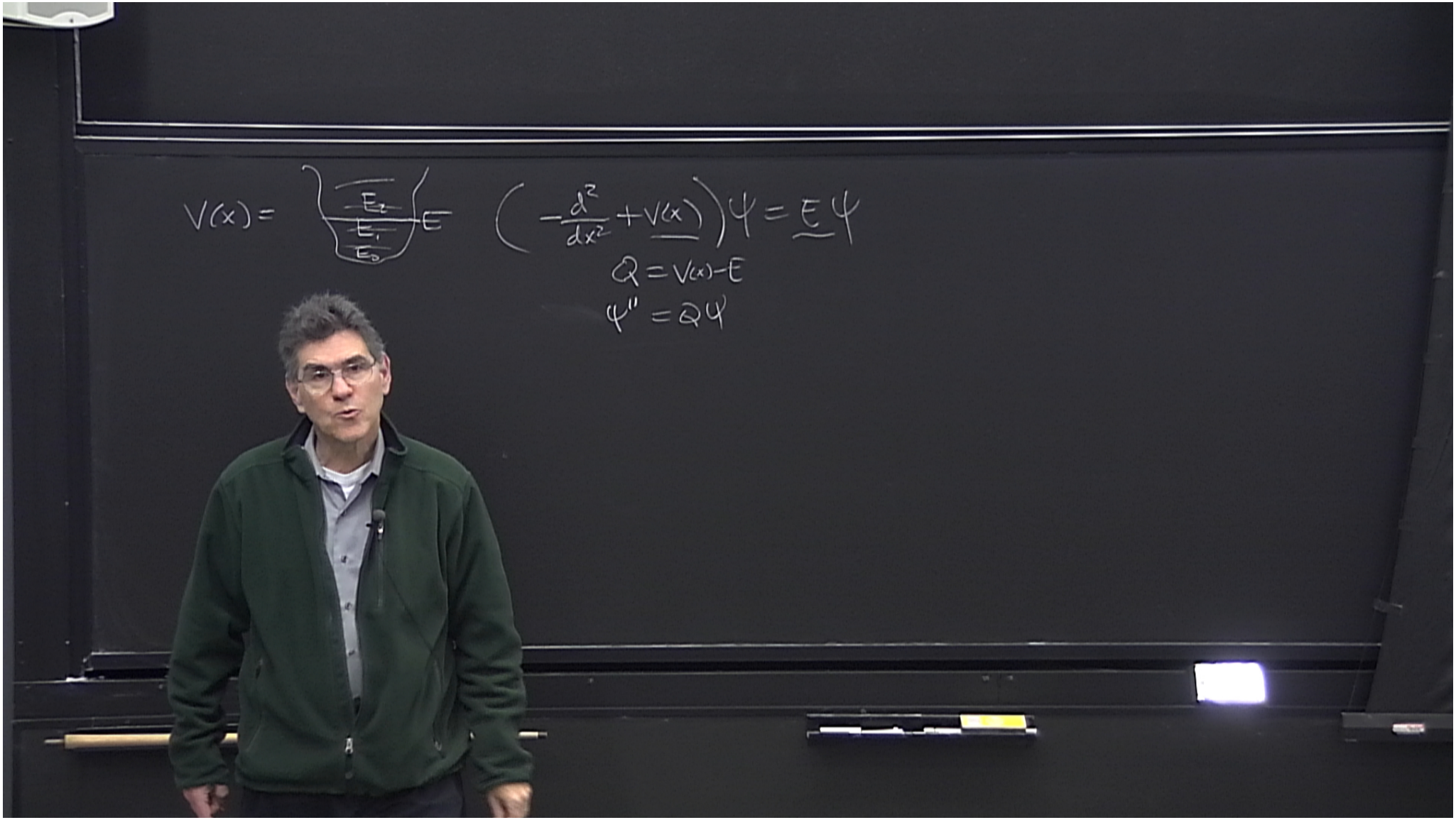

V(x) =

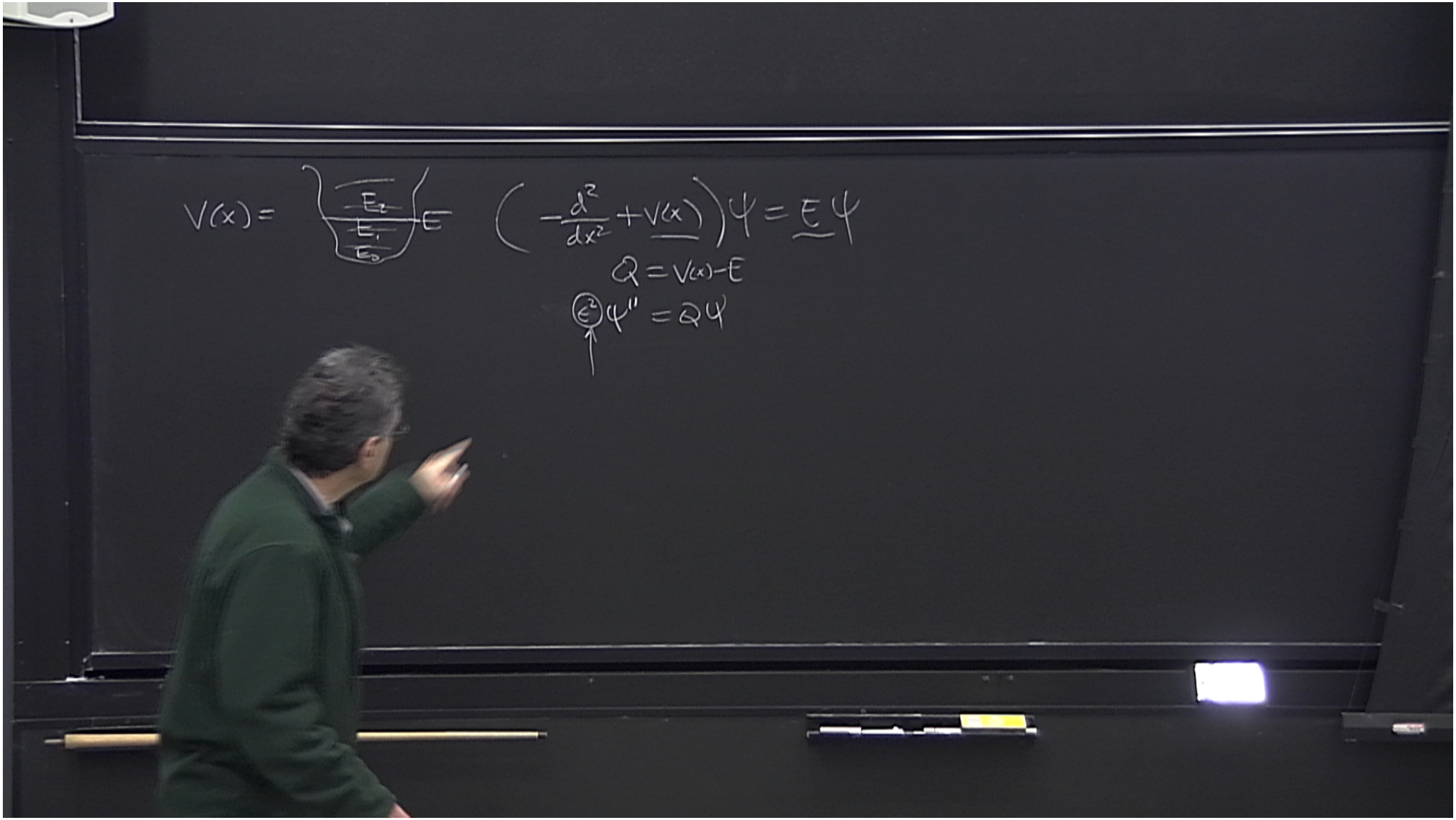

V(x) =
E2
E1
E0

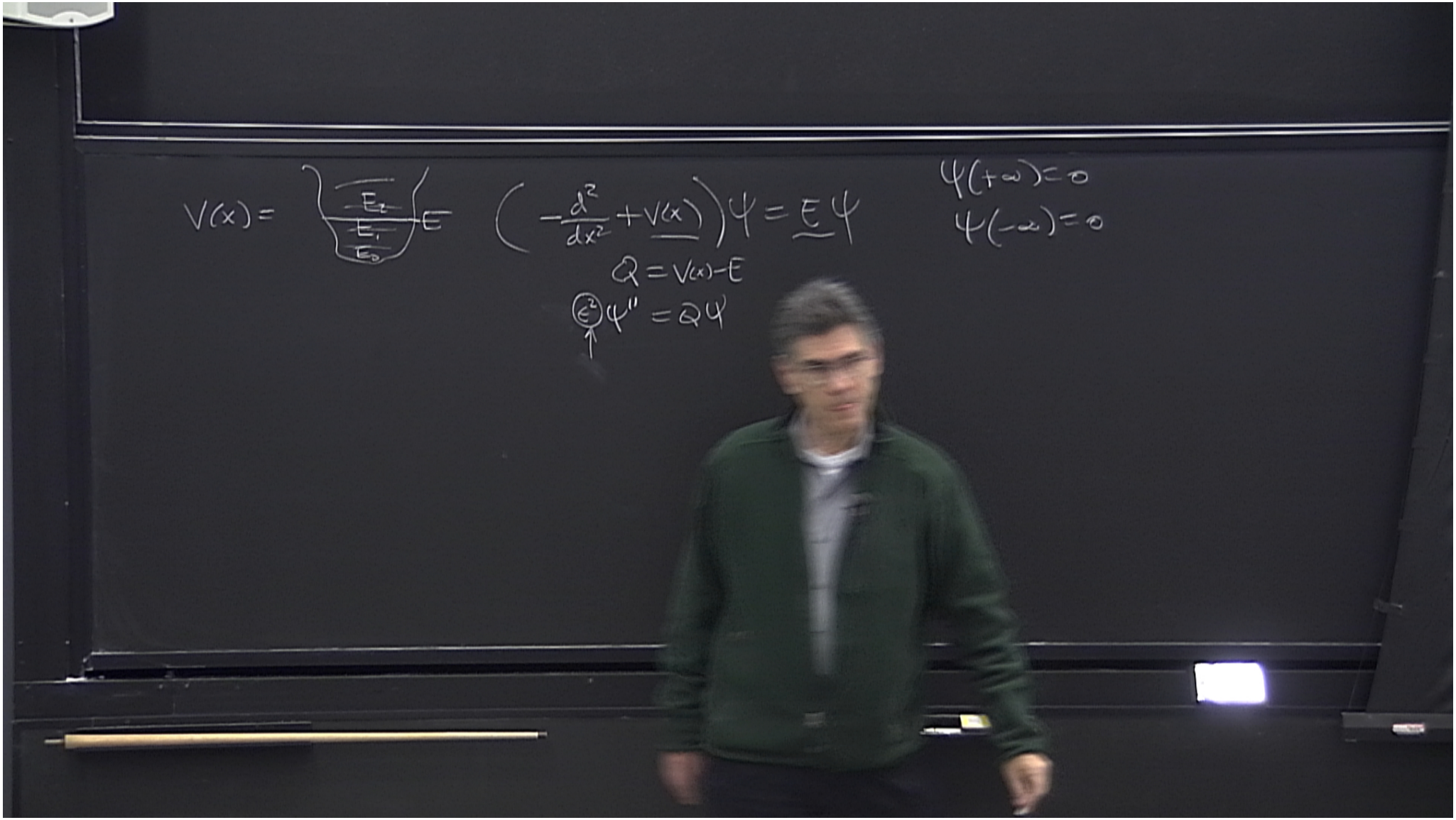

V(x) =
E₂
E
E₁
E₀

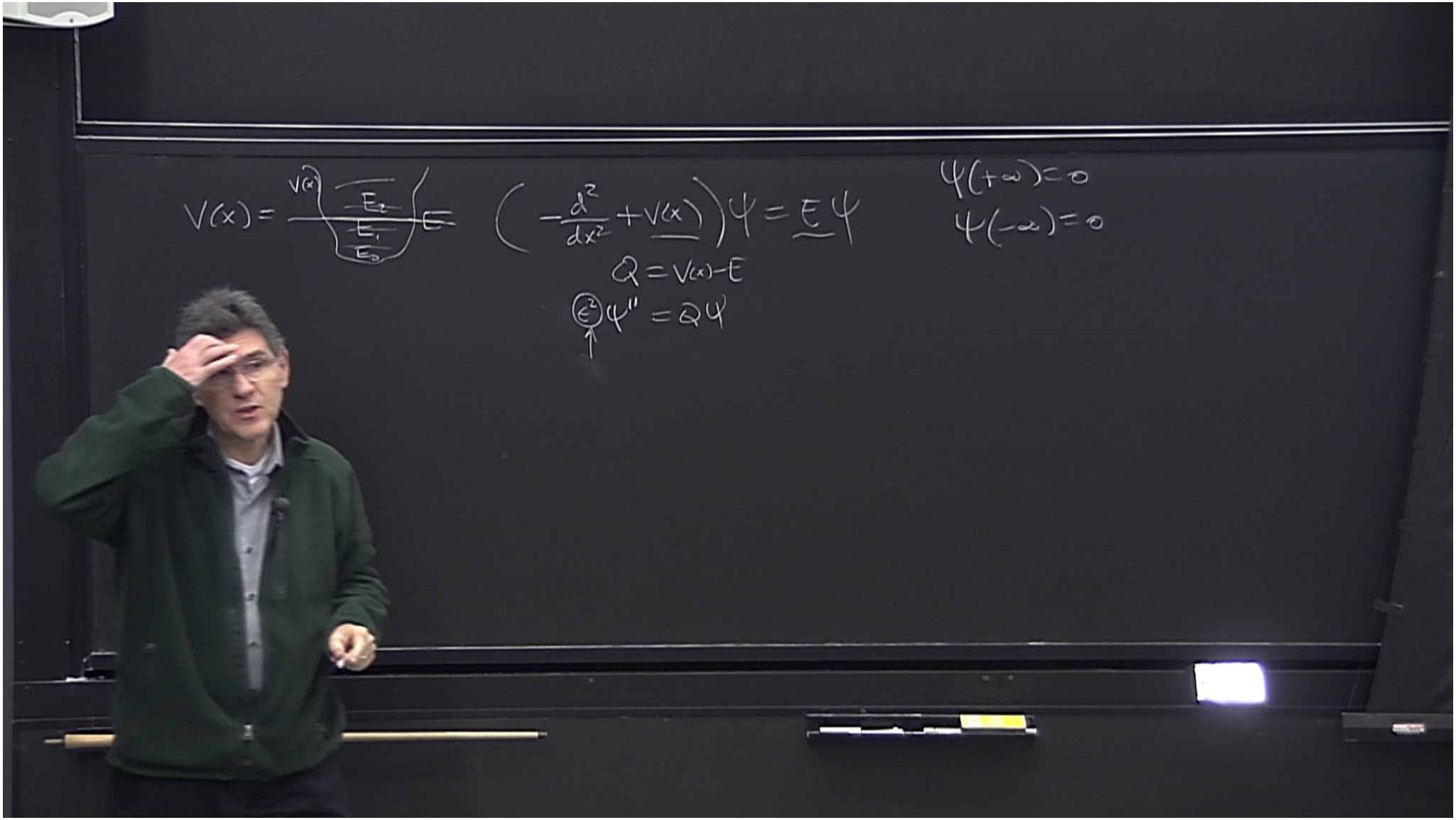

V(x) =
E₂
E
E₁
E₀
$\left(-\frac{d^2}{dx^2}+V(x)\right)\psi = E\psi$

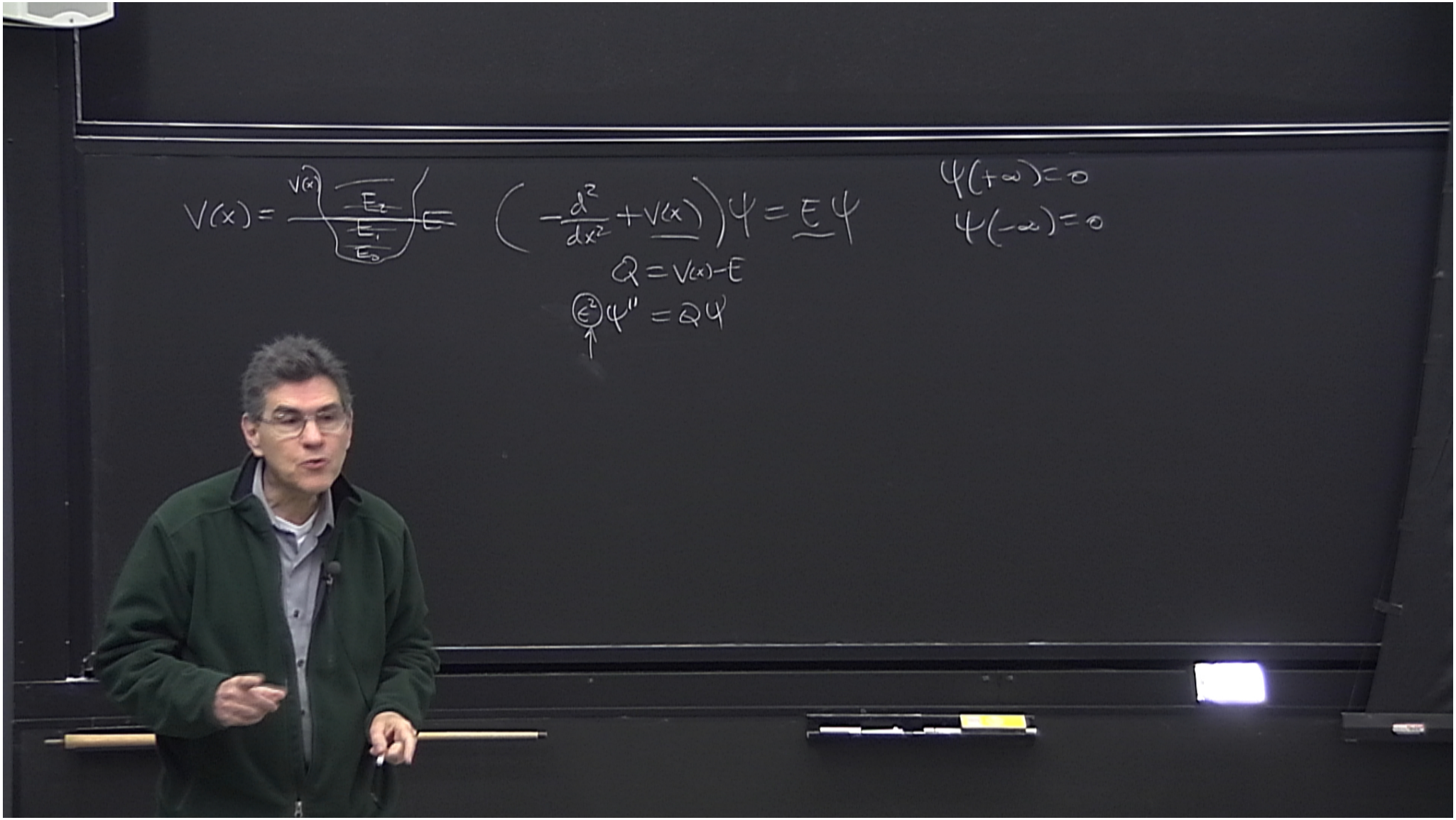

V(x) =
E₂
E
E₁
E₀
$\left(-\frac{d^2}{dx^2}+V(x)\right)\psi = E\psi$
Q = V(x) − E

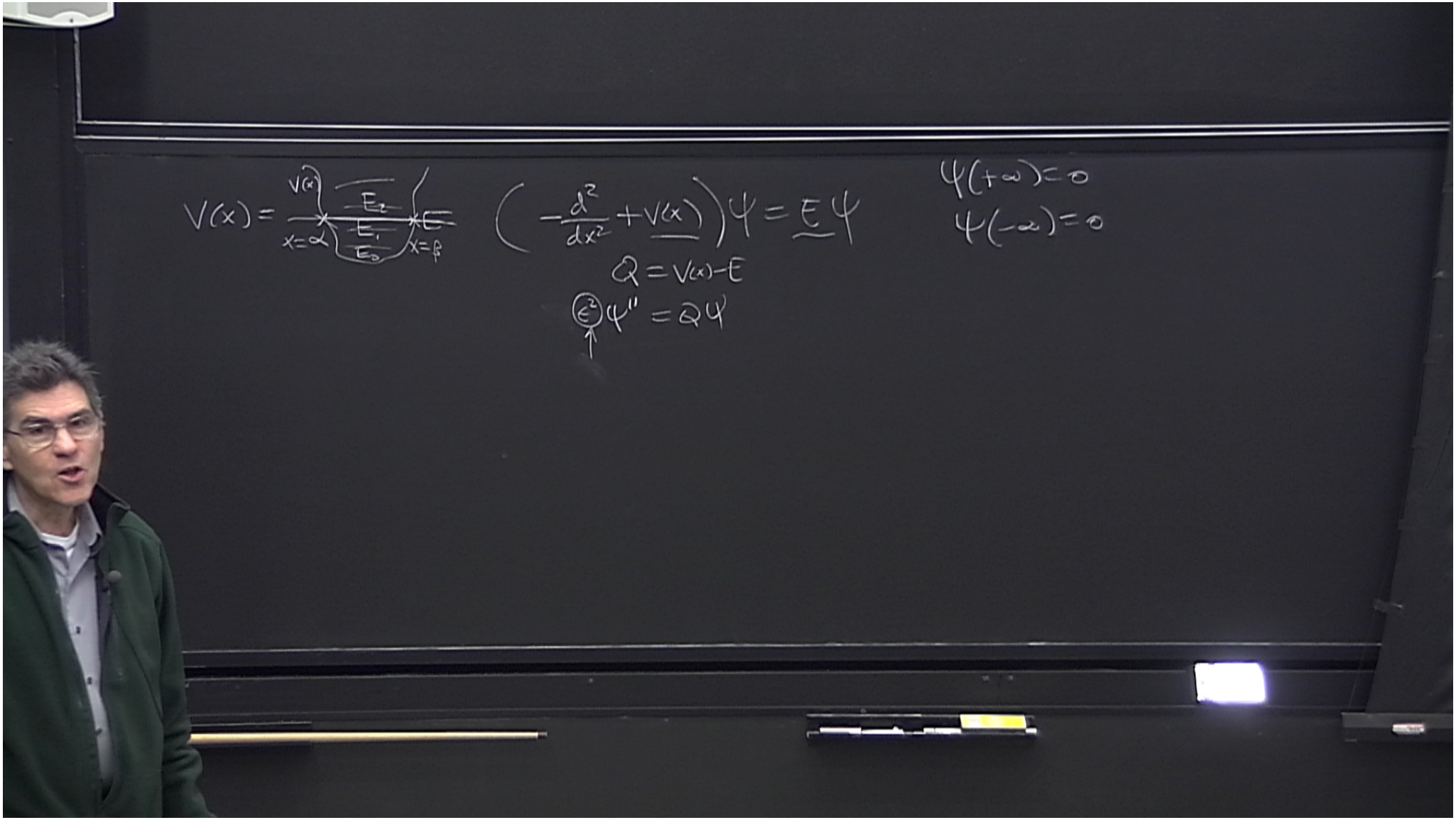

V(x) =
E₂
E
E₁
E₀
$\left(-\frac{d^2}{dx^2}+V(x)\right)\psi = E\psi$
$Q = V(x) - E$
$\psi'' = Q\psi$

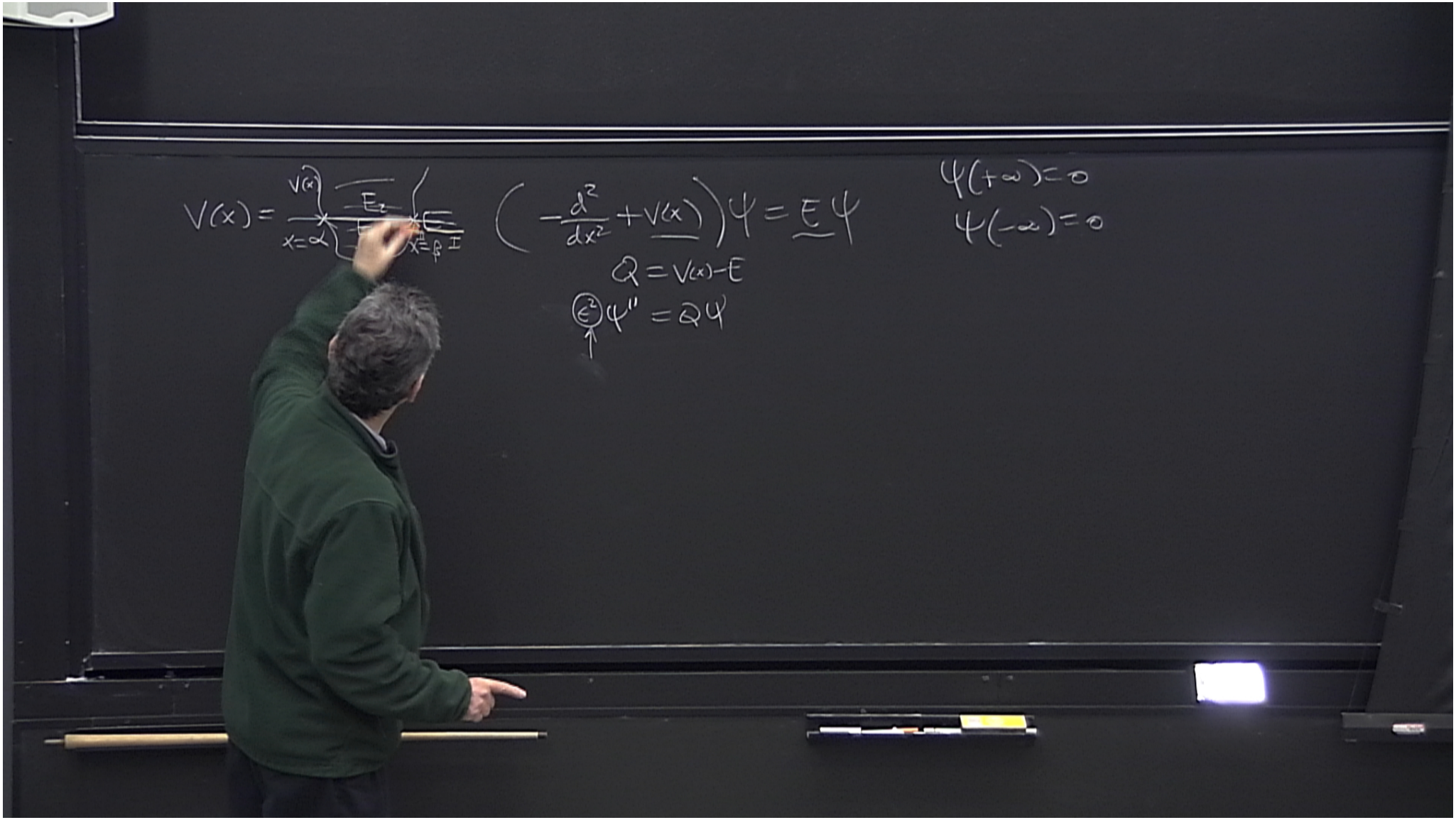

V(x) =
E₂
E
E₁
E₀
$\left(-\frac{d^2}{dx^2}+V(x)\right)\psi = E\psi$
$Q = V(x) - E$
$\epsilon^2 \psi'' = Q\psi$

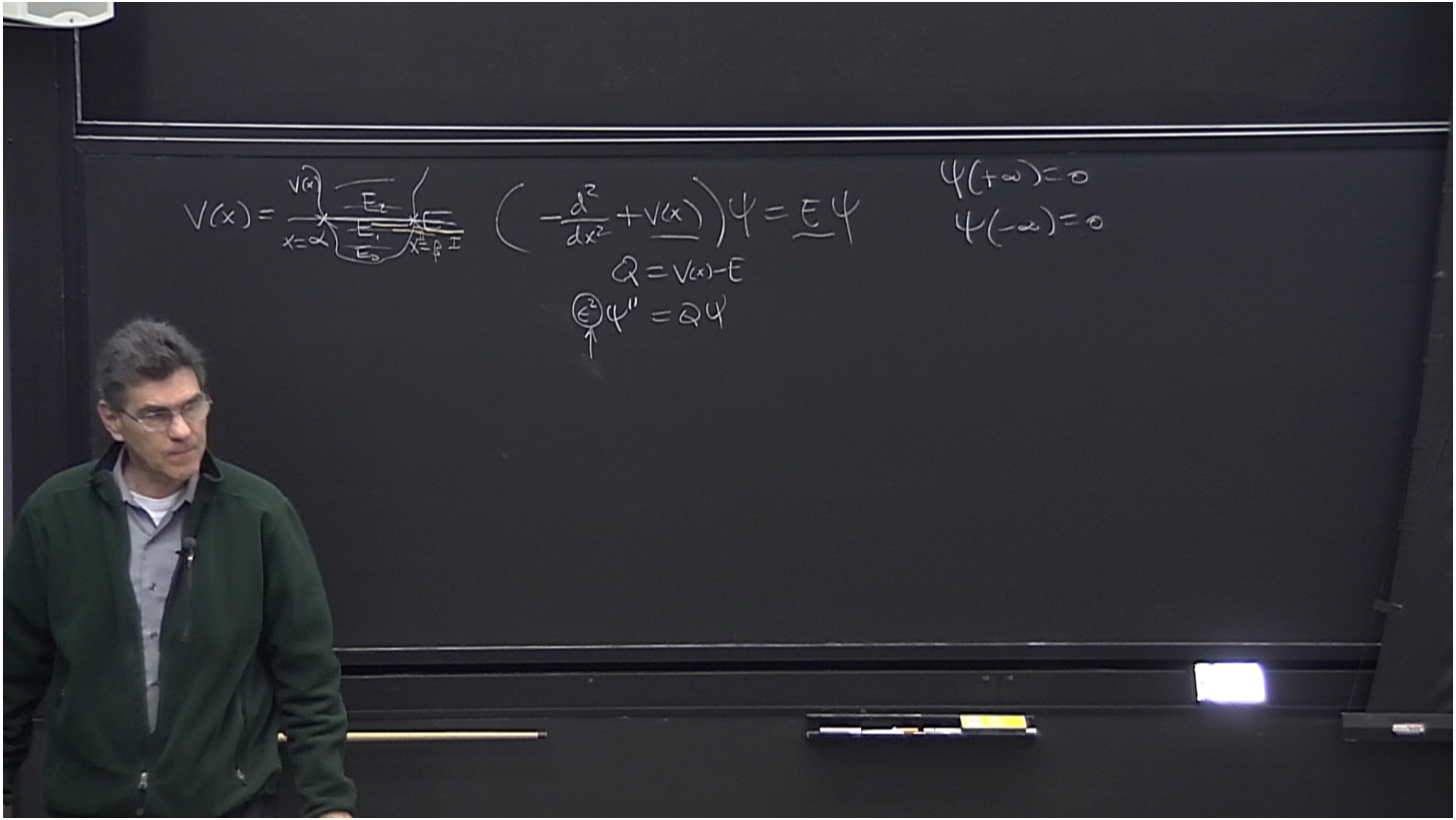

$V(x) =$
E_2
E_1
E_0
E
$\left(-\frac{d^2}{dx^2} + V(x)\right)\psi = E\psi$
$Q = V(x) - E$
$\epsilon^2 \psi'' = Q\psi$
$\psi(+\infty) = 0$
$\psi(-\infty) = 0$

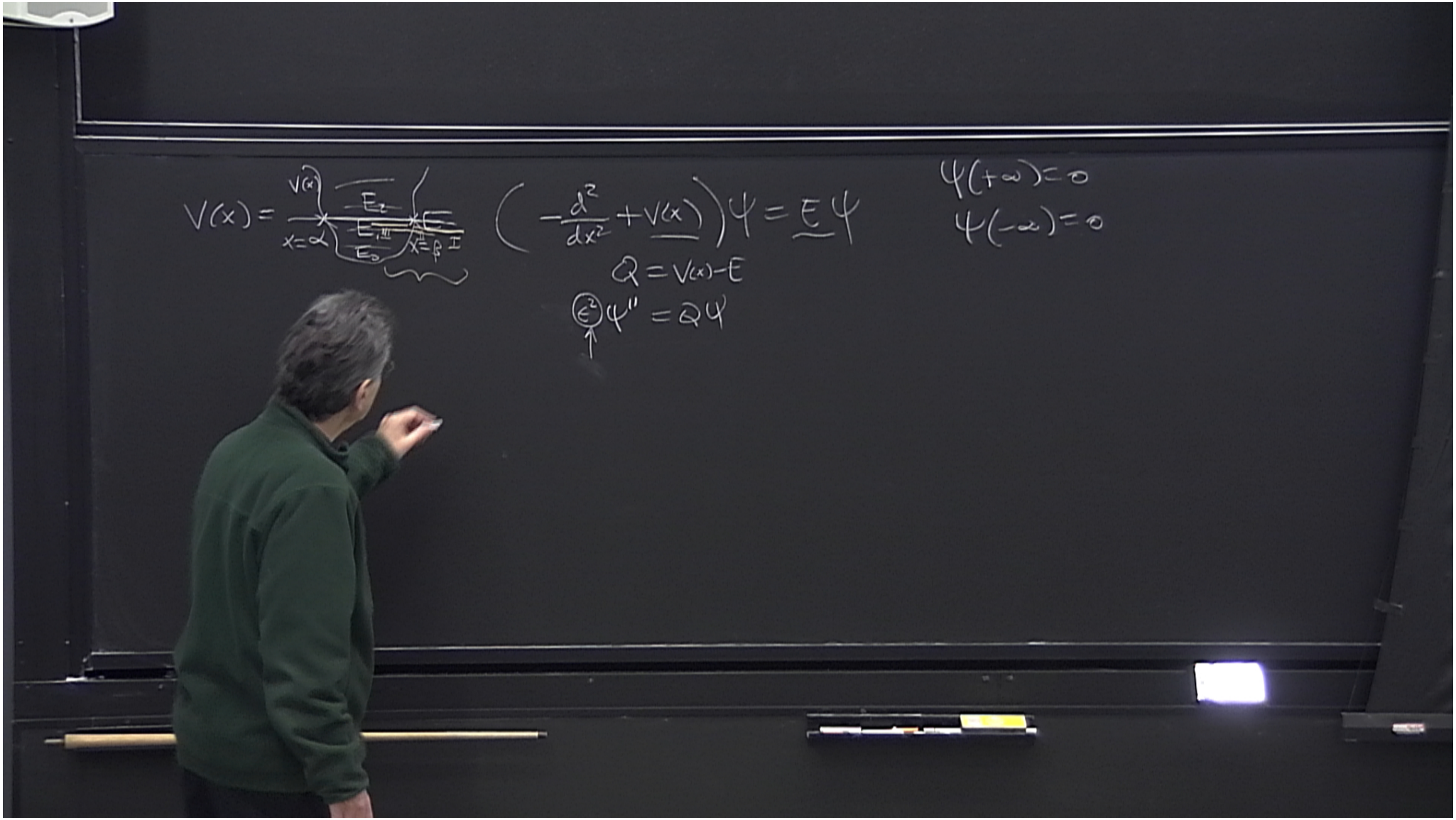

V(x) = V(x) E₂ E E₁ E₀
$\left(-\frac{d^2}{dx^2}+V(x)\right)\psi = E\psi$
$\psi(+\infty)=0$
$\psi(-\infty)=0$
$Q = V(x)-E$
$\epsilon^2\psi'' = Q\psi$

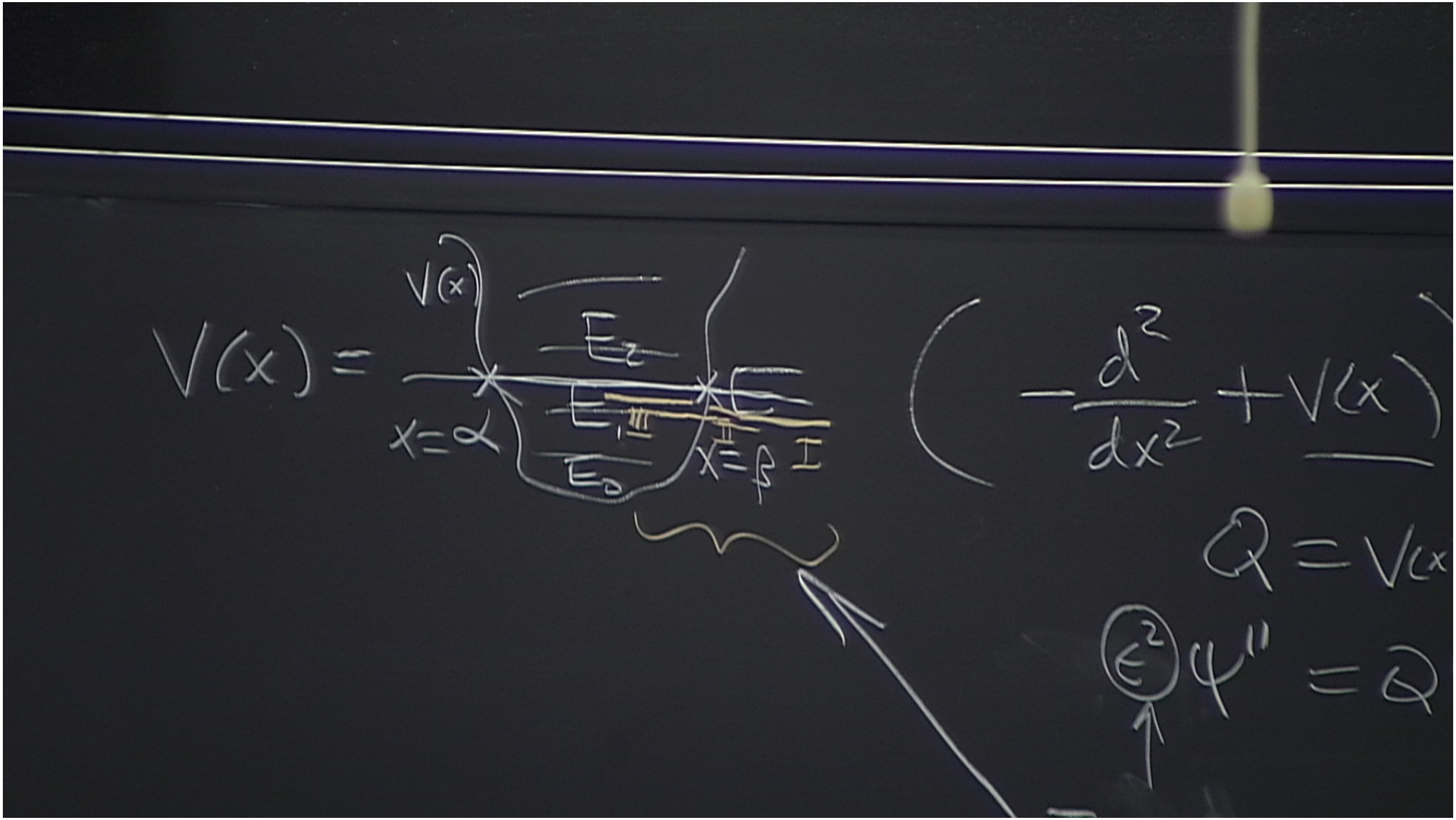

V(x) =
V(x)
E₂
E
E₁
E₀
$\left(-\frac{d^2}{dx^2}+V(x)\right)\psi = E\psi$
$\psi(+\infty)=0$
$\psi(-\infty)=0$
$Q = V(x)-E$
$\epsilon^2\psi'' = Q\psi$

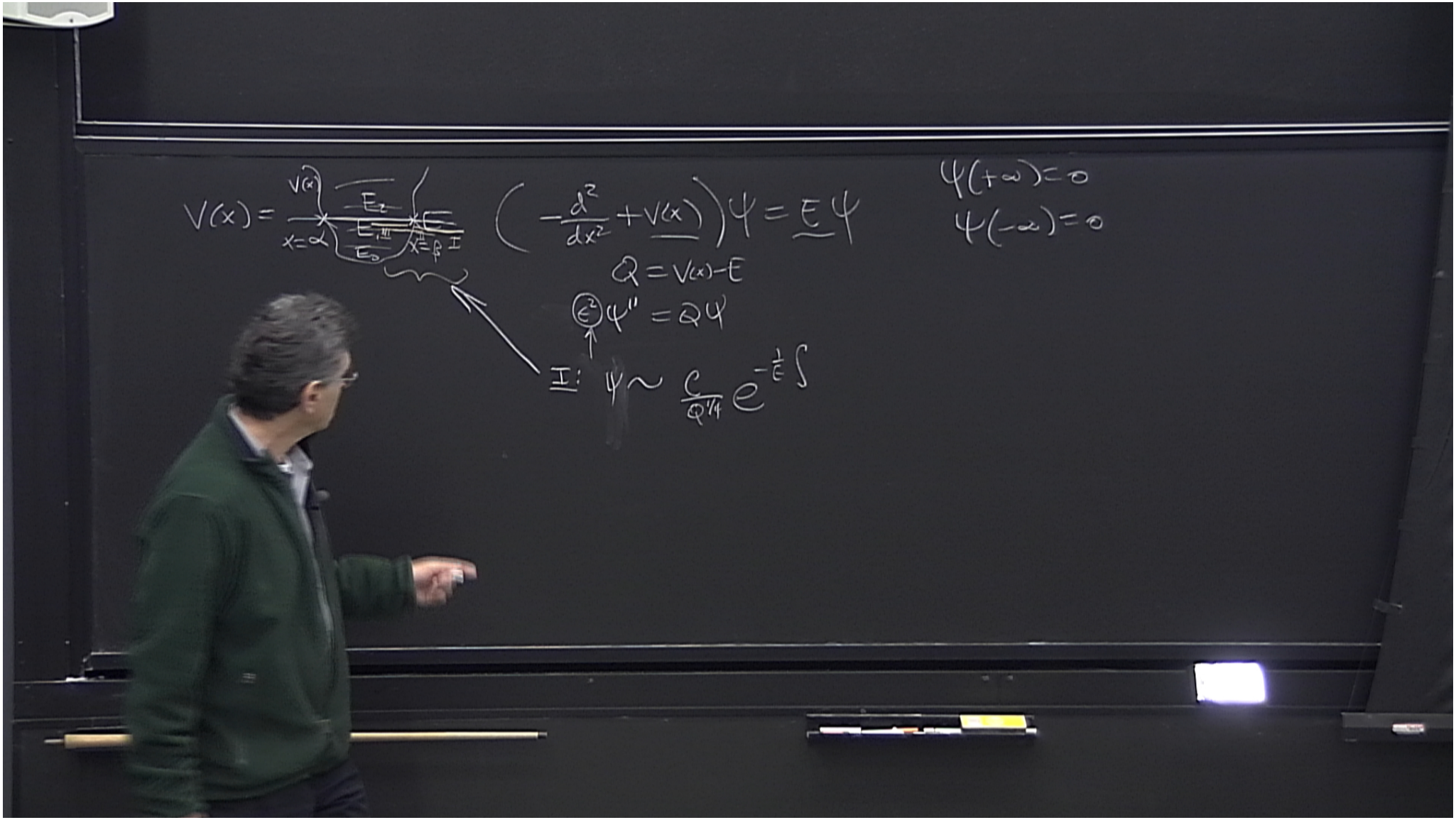

$V(x) = \begin{cases} E_2 & x<\alpha \\ E_1 & \alpha<x<\beta \\ E_0 & x>\beta \end{cases}$
$\left(-\frac{d^2}{dx^2} + V(x)\right)\psi = E\psi$
$Q = V(x) - E$
$\epsilon^2 \psi'' = Q\psi$
$\psi(+\infty) = 0$
$\psi(-\infty) = 0$

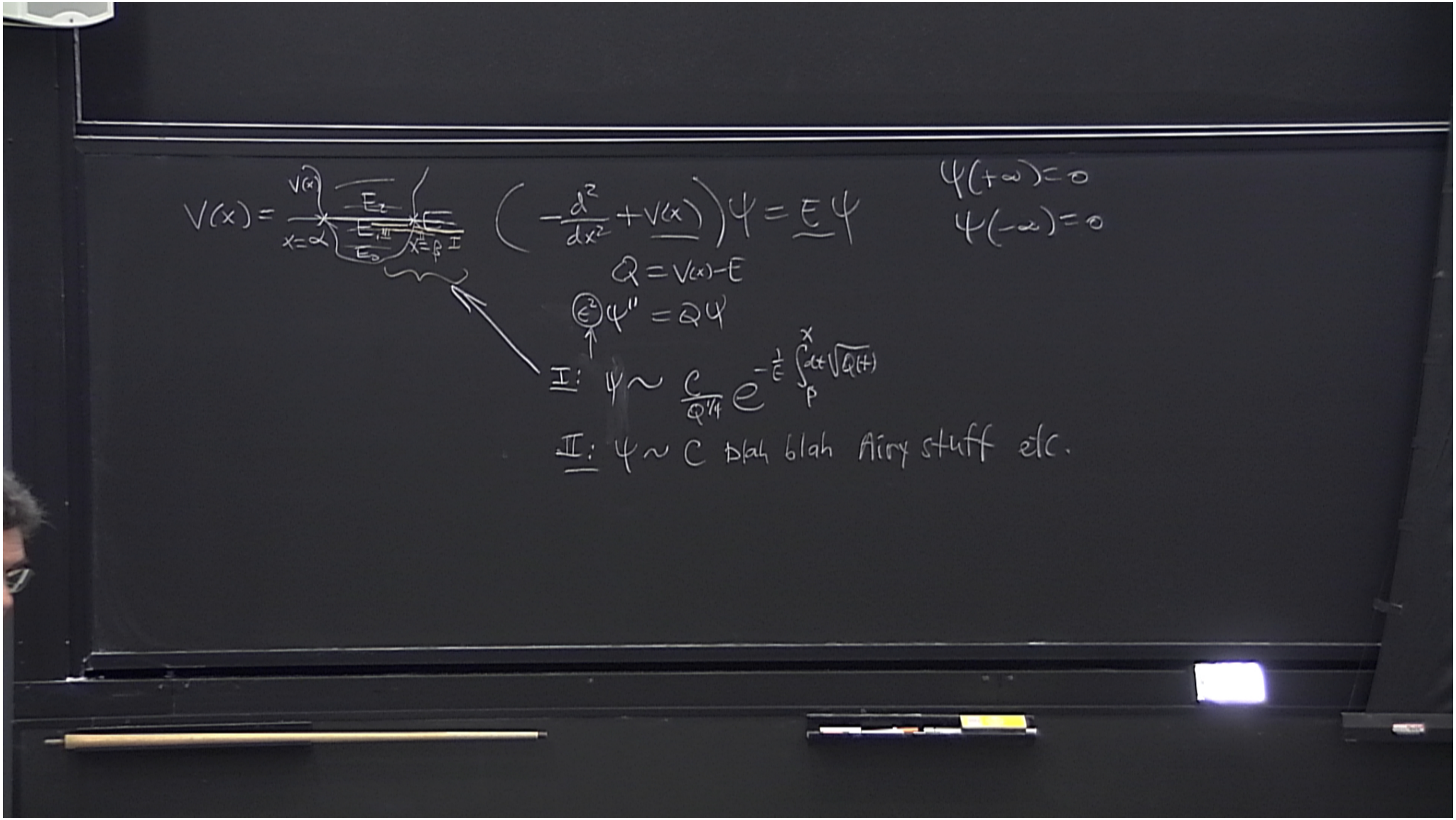

V(x) =
V(x)
E
x=α
x=β
I
$\left(-\frac{d^2}{dx^2}+V(x)\right)\psi = E\psi$
$\psi(+\infty)=0$
$\psi(-\infty)=0$
$Q = V(x)-E$
$\epsilon^2\psi'' = Q\psi$

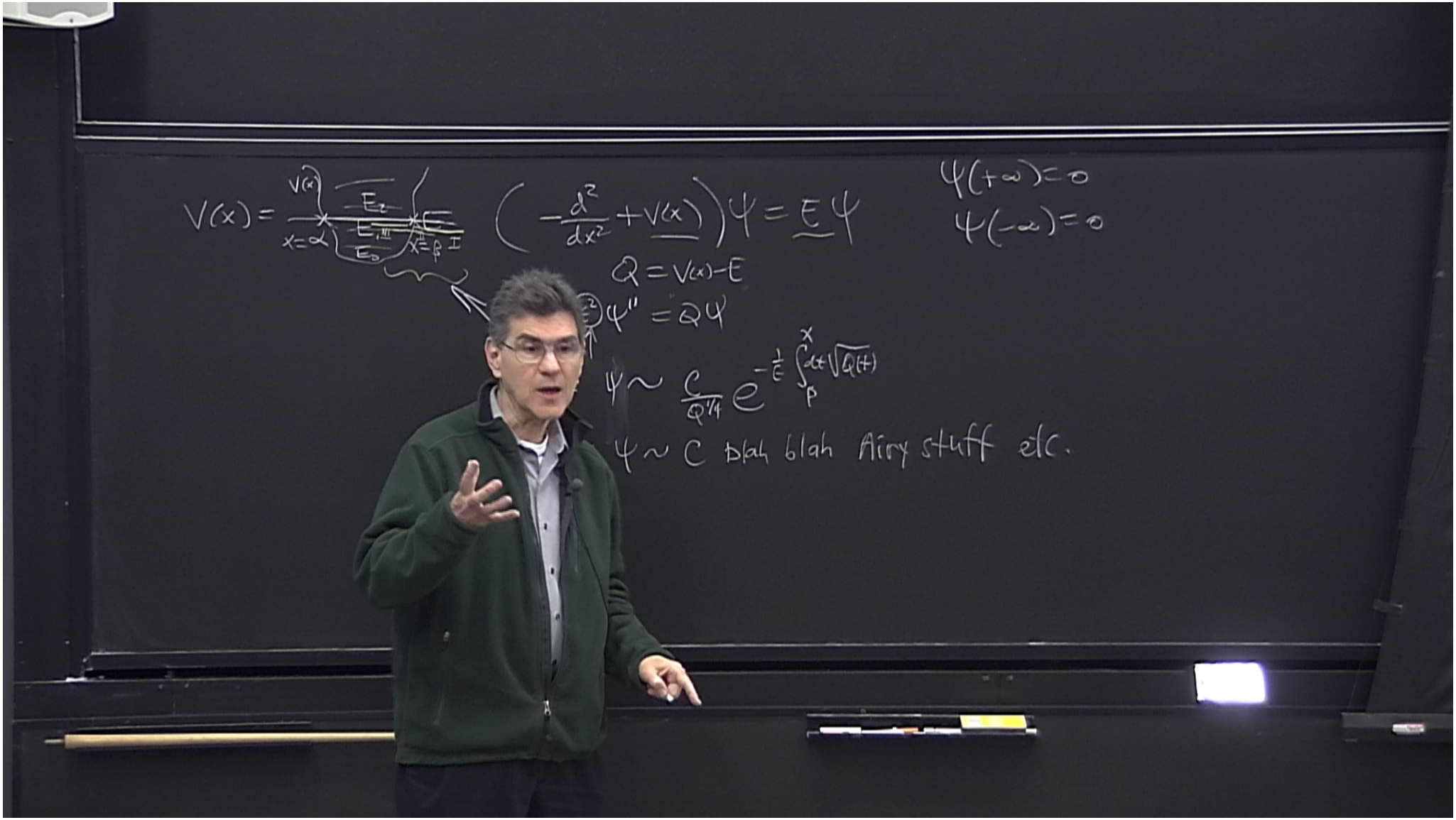

V(x) = V(x) E₂ E₁ E₀ x=α x=β I
$\left(-\frac{d^2}{dx^2}+V(x)\right)\psi = E\psi$
$\psi(+\infty)=0$
$\psi(-\infty)=0$
$Q = V(x)-E$
$\epsilon^2\psi'' = Q\psi$

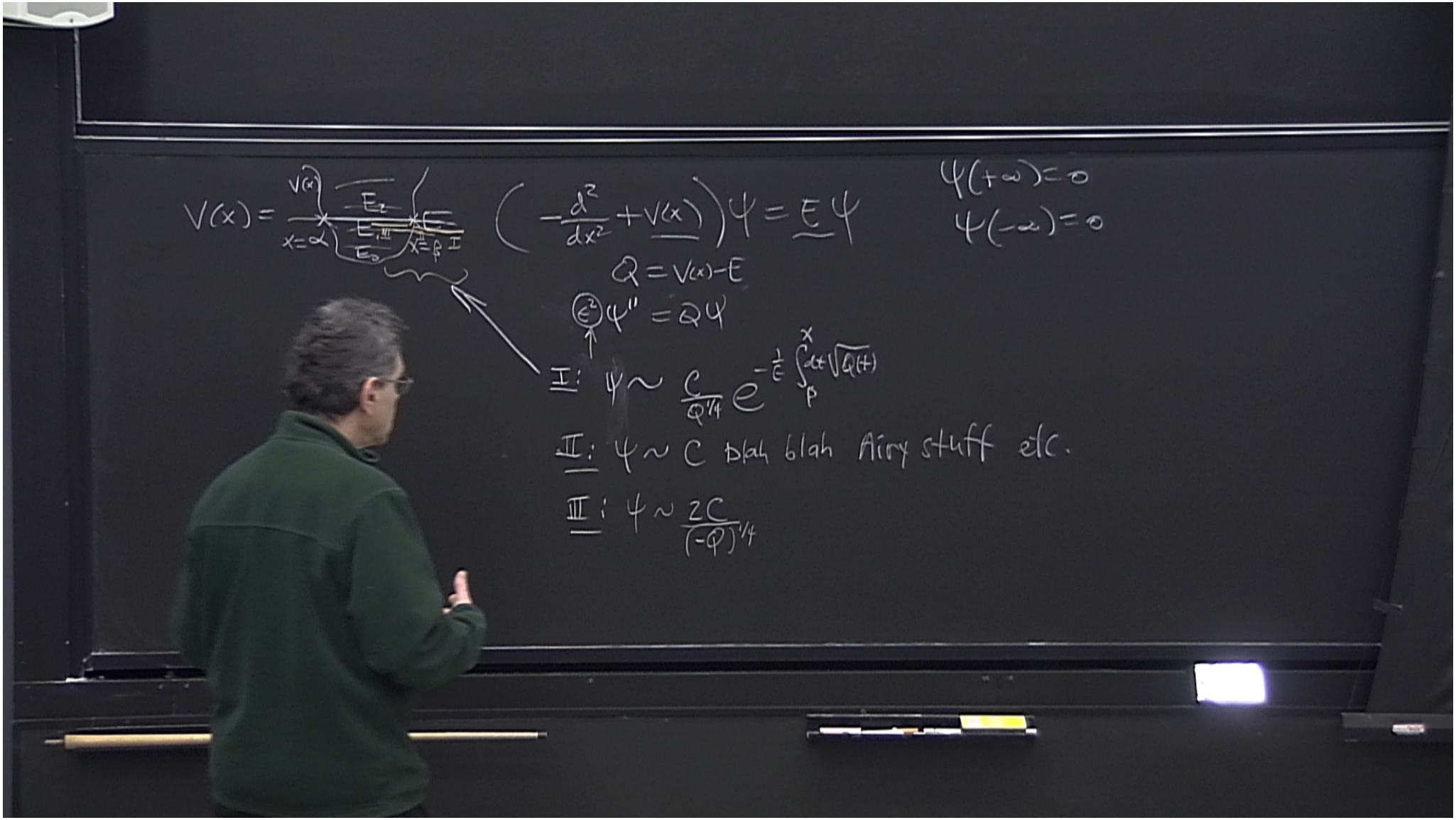

V(x) =
V(x)
E₂
E₁, III
E₀
x=α
x=β
II
I
$\left(-\frac{d^2}{dx^2}+V(x)\right)\psi = E\psi$
$\psi(+\infty)=0$
$\psi(-\infty)=0$
$Q = V(x)-E$
$\epsilon^2\psi'' = Q\psi$

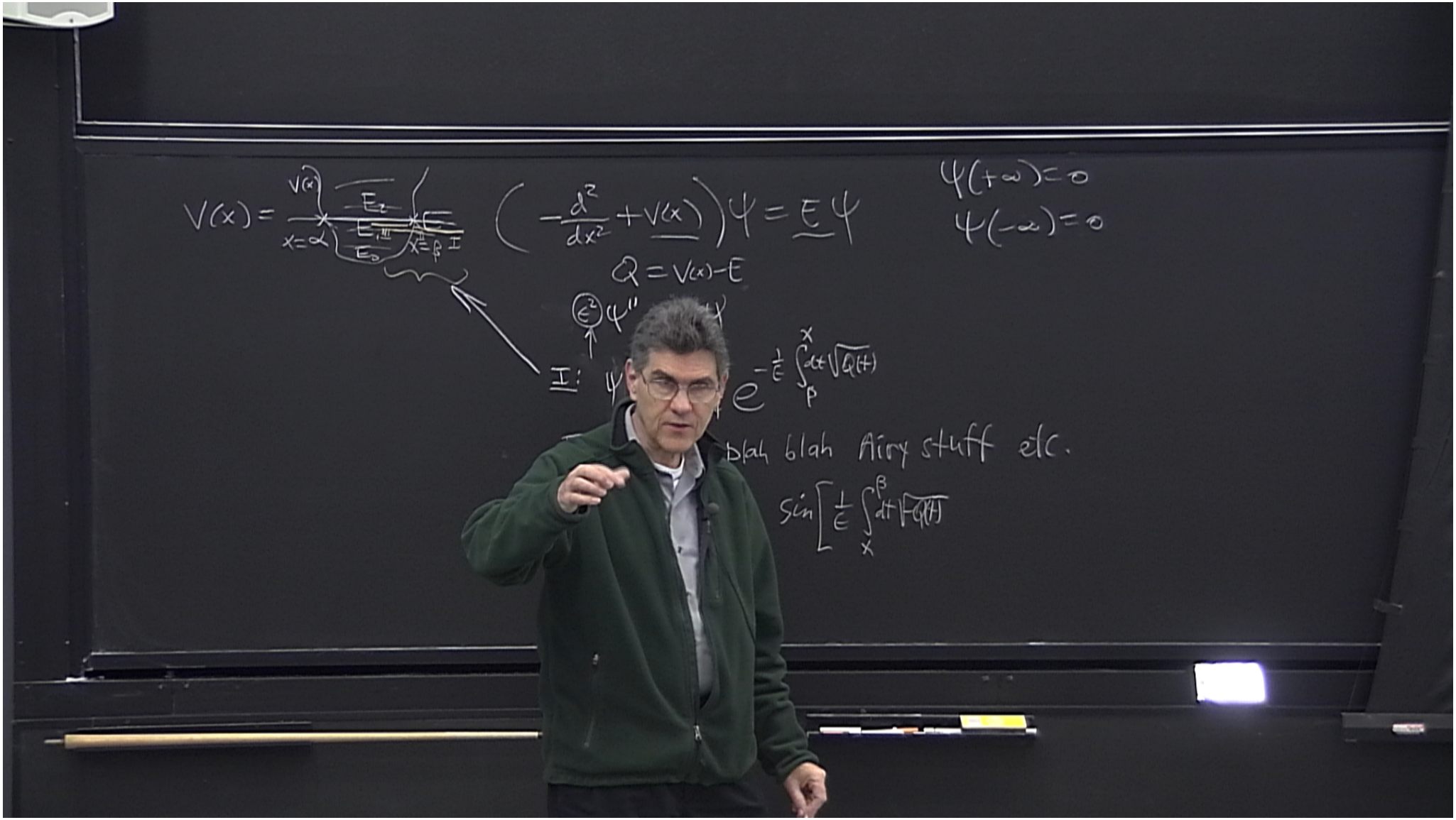

V(x) =
V(x)
x=α
x=β
I
II
III
$-\frac{d^2}{dx^2} + V(x)$
Q = V(x
$\epsilon^2 \psi'' = Q$

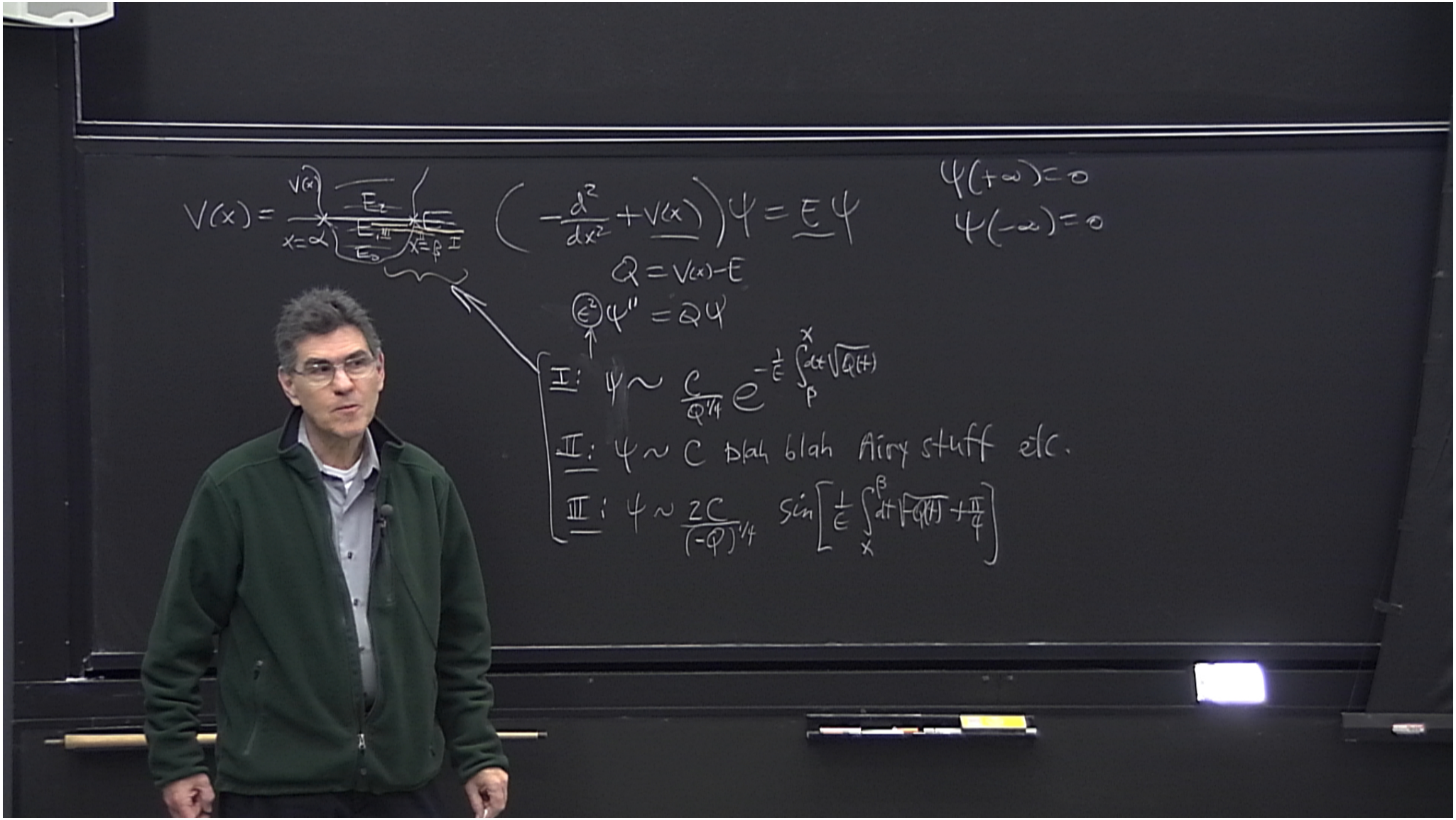

V(x) =
V(x)
E₂
E₁, III
E₀
x=α
x=β
II
I
$\left(-\frac{d^2}{dx^2}+V(x)\right)\psi = E\psi$
$\psi(+\infty)=0$
$\psi(-\infty)=0$
$Q = V(x)-E$
$\epsilon^2\psi'' = Q\psi$
I: $\psi \sim \frac{c}{Q^{1/4}} e^{-\frac{1}{\epsilon}\int}$


V(x) =
V(x)
E₂
E₁,III
E₀
x=α
x=β
I
II
$\left(-\frac{d^2}{dx^2}+V(x)\right)\psi = E\psi$
$\psi(+\infty)=0$
$\psi(-\infty)=0$
$Q = V(x)-E$
$\epsilon^2\psi'' = Q\psi$
I: $\psi \sim \frac{C}{Q^{1/4}} e^{-\frac{1}{\epsilon}\int_\beta^x dt\sqrt{Q(t)}}$
II: $\psi \sim C$ blah blah Airy stuff etc.


$V(x) =$
$x=\alpha$
$x=\beta$
I
II
III
$\left(-\frac{d^2}{dx^2} + V(x)\right)\psi = E\psi$
$\psi(+\infty) = 0$
$\psi(-\infty) = 0$
$Q = V(x) - E$
$\psi'' = Q\psi$
$\psi \sim \frac{C}{Q^{1/4}} e^{-\frac{1}{\epsilon}\int_\beta^x dt \sqrt{Q(t)}}$
$\psi \sim C$ blah blah Airy stuff etc.


V(x) =
V(x)
E2
E1
E0
x=α
x=β
I
$\left(-\frac{d^2}{dx^2}+V(x)\right)\psi = E\psi$
$\psi(+\infty)=0$
$\psi(-\infty)=0$
$Q = V(x)-E$
$\epsilon^2\psi'' = Q\psi$
I: $\psi \sim \frac{C}{Q^{1/4}} e^{-\frac{1}{\epsilon}\int_\beta^x dt\sqrt{Q(t)}}$
II: $\psi \sim C$ blah blah Airy stuff etc.
III: $\psi \sim \frac{2C}{(-Q)^{1/4}}$


V(x) =
x=α
x=β
I
II
III
$\left(-\frac{d^2}{dx^2}+V(x)\right)\psi = E\psi$
$\psi(+\infty)=0$
$\psi(-\infty)=0$
$Q = V(x)-E$
$\epsilon^2\psi''$
I: ψ
$e^{-\frac{1}{\epsilon}\int_\beta^x dt\sqrt{Q(t)}}$
blah blah Airy stuff etc.
$\sin\left[\frac{1}{\epsilon}\int_x^\beta dt\sqrt{Q(t)}\right.$


V(x) =
x=α
x=β
I
$\left(-\frac{d^2}{dx^2}+V(x)\right)\psi = E\psi$
$\psi(+\infty)=0$
$\psi(-\infty)=0$
$Q = V(x)-E$
$\epsilon^2\psi'' = Q\psi$
I: $\psi \sim \frac{C}{Q^{1/4}} e^{-\frac{1}{\epsilon}\int_\beta^x dt\sqrt{Q(t)}}$
II: $\psi \sim C$ blah blah Airy stuff etc.
III: $\psi \sim \frac{2C}{(-Q)^{1/4}} \sin\left[\frac{1}{\epsilon}\int_x^\beta dt\sqrt{-Q(t)}+\frac{\pi}{4}\right]$

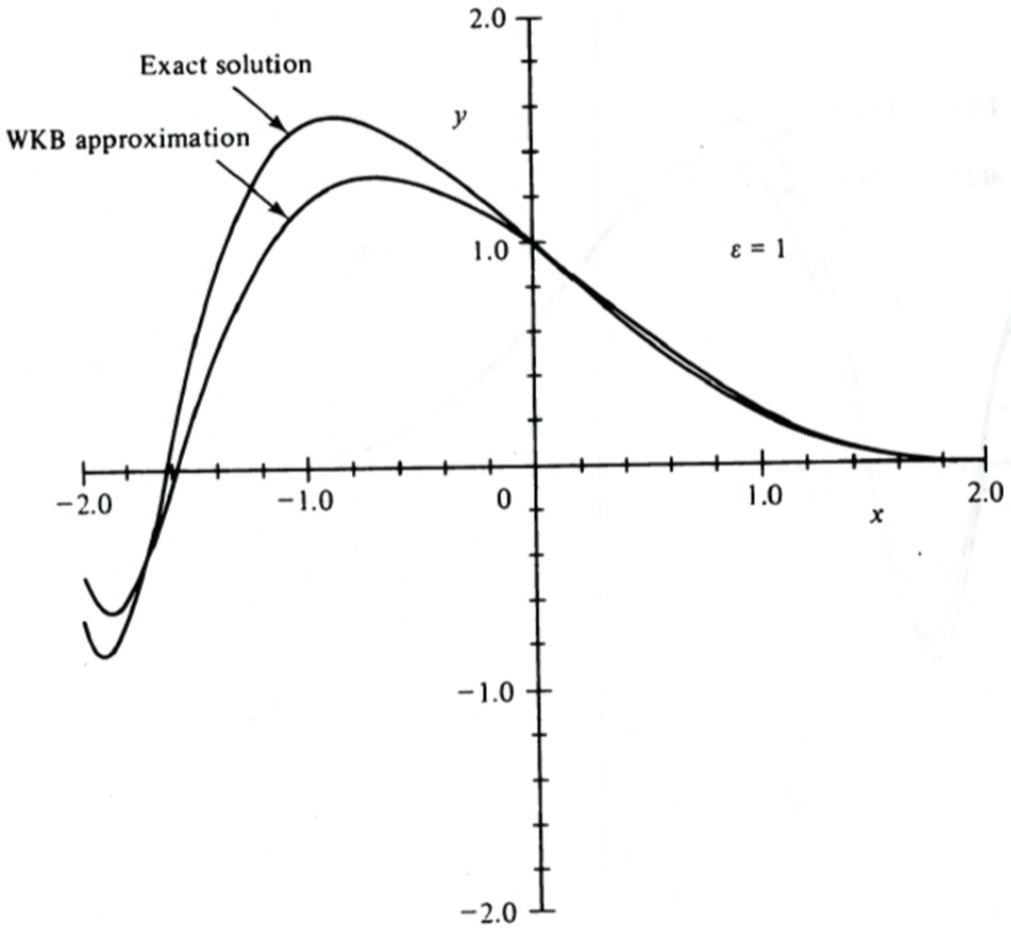
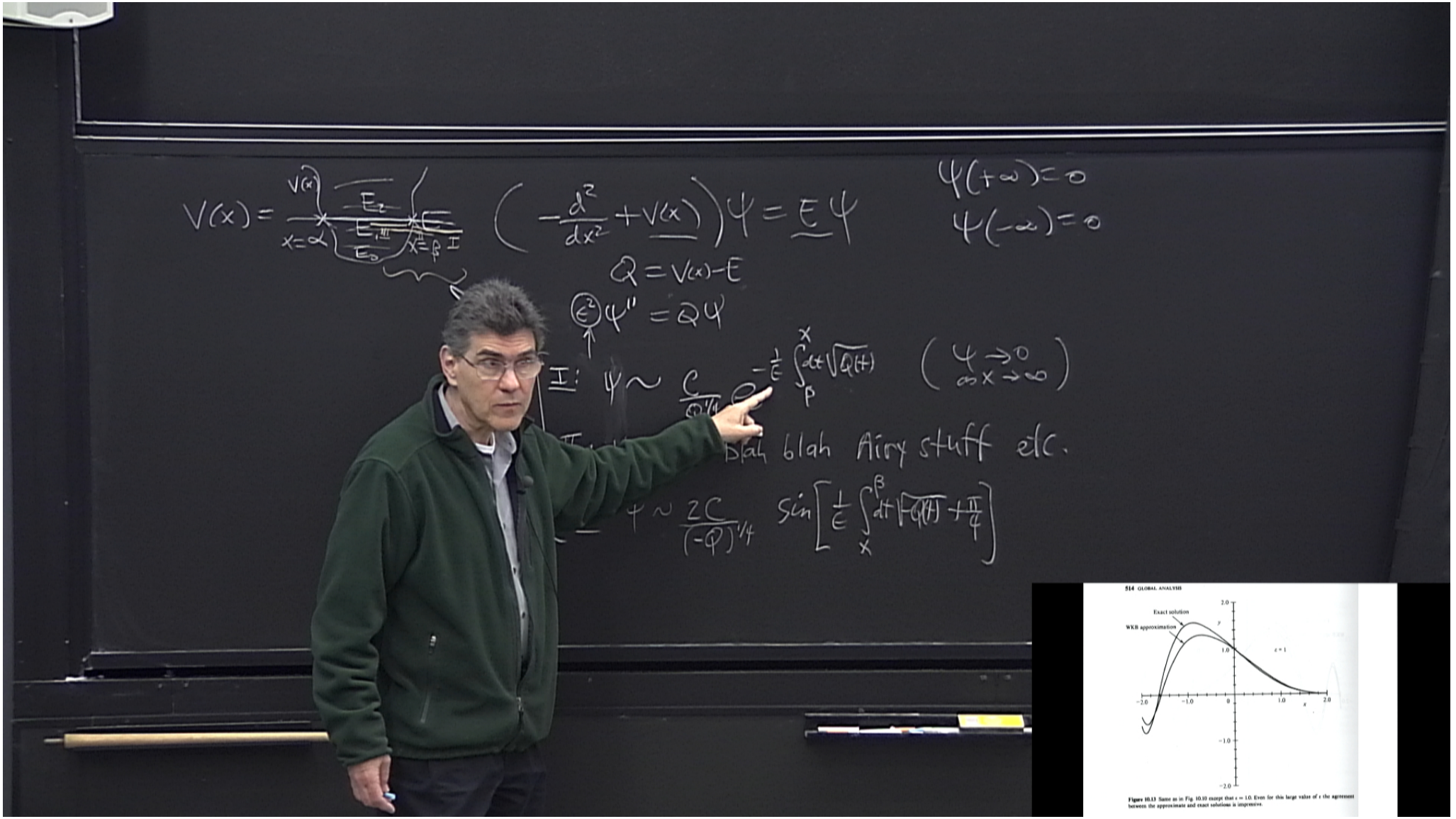
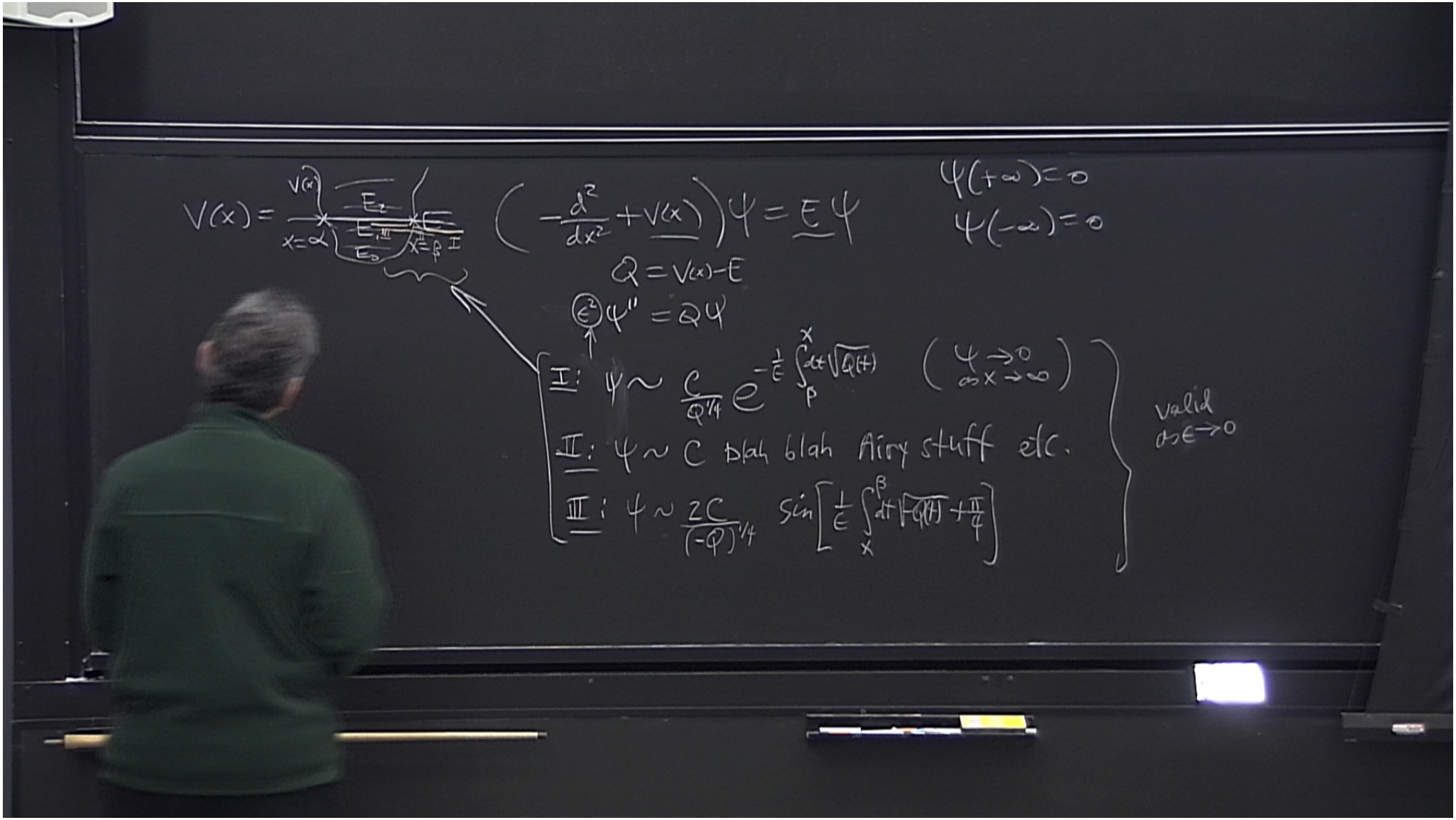
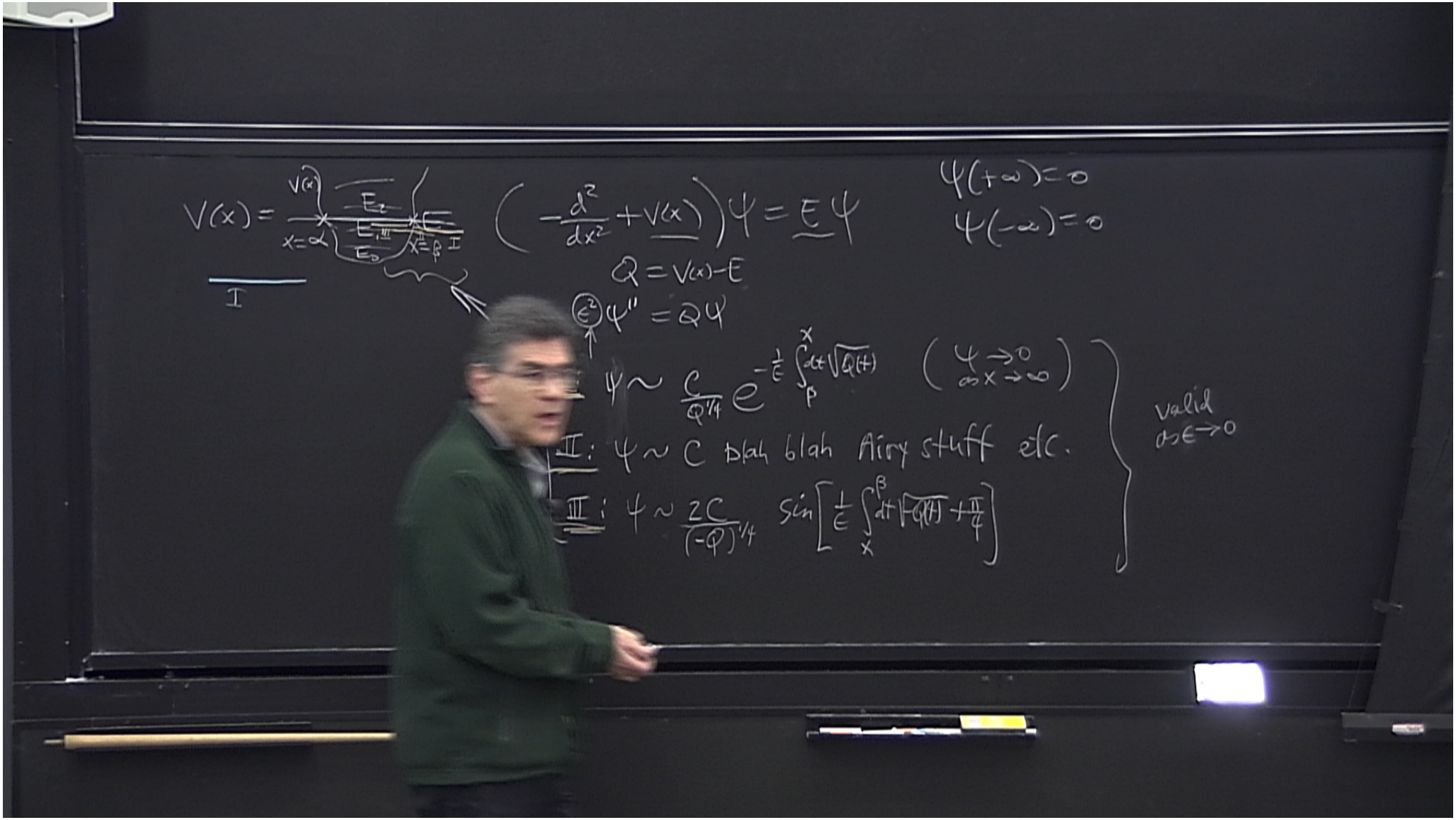


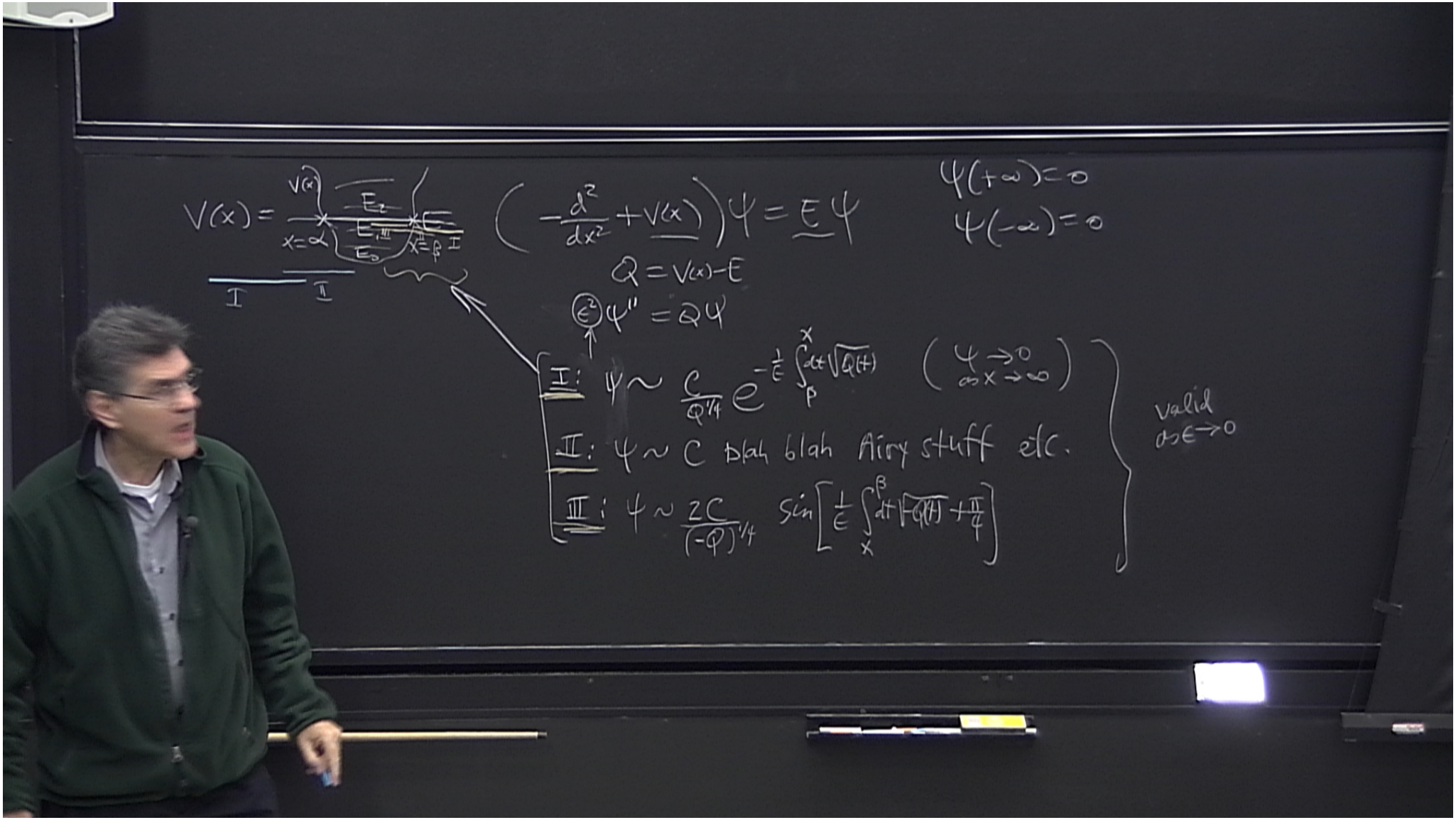
Figure 10.13 Same as in Fig. 10.10 except that $\varepsilon = 1.0$. Even for this large value of ε the agreement between the approximate and exact solutions is impressive.

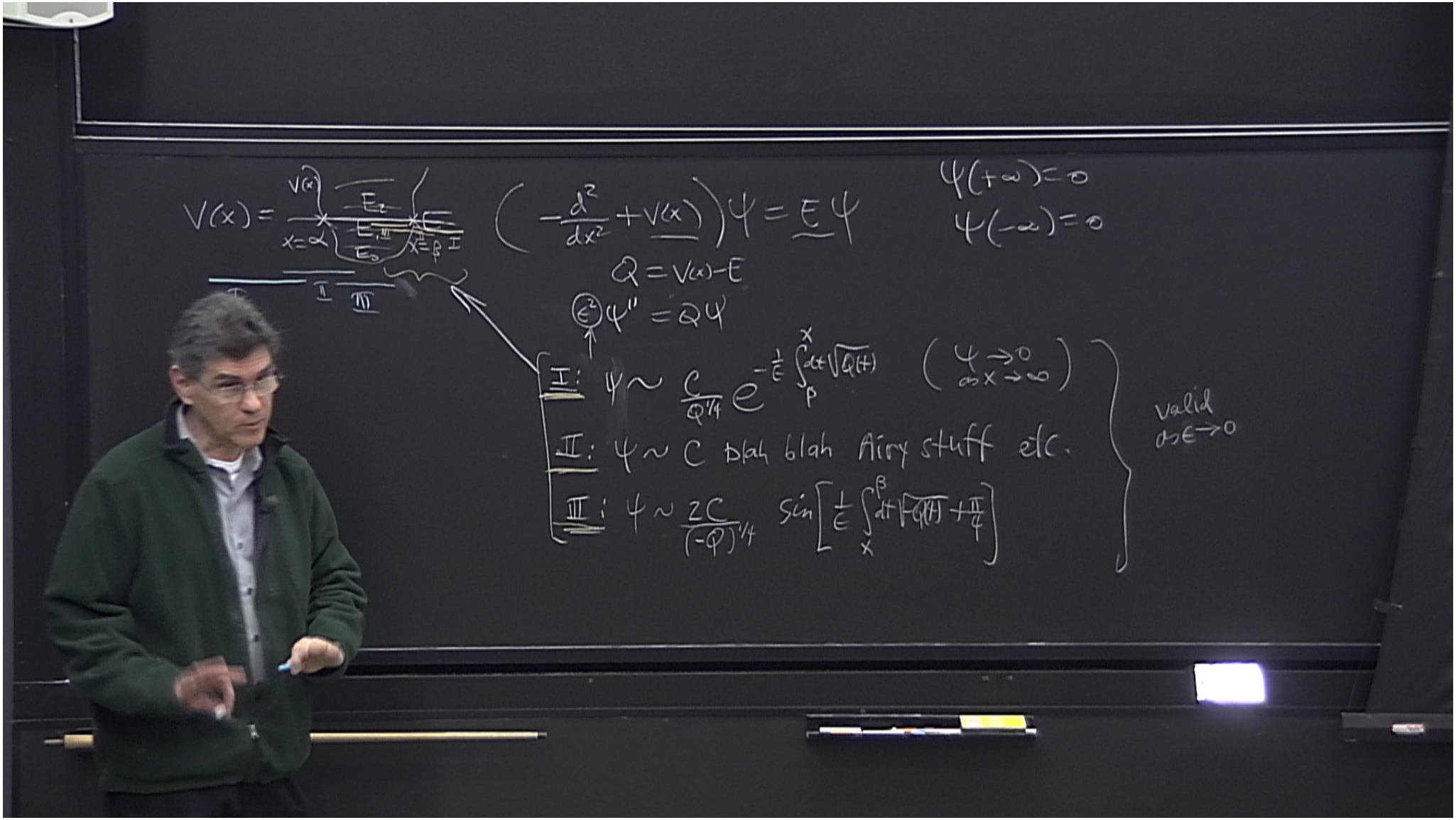


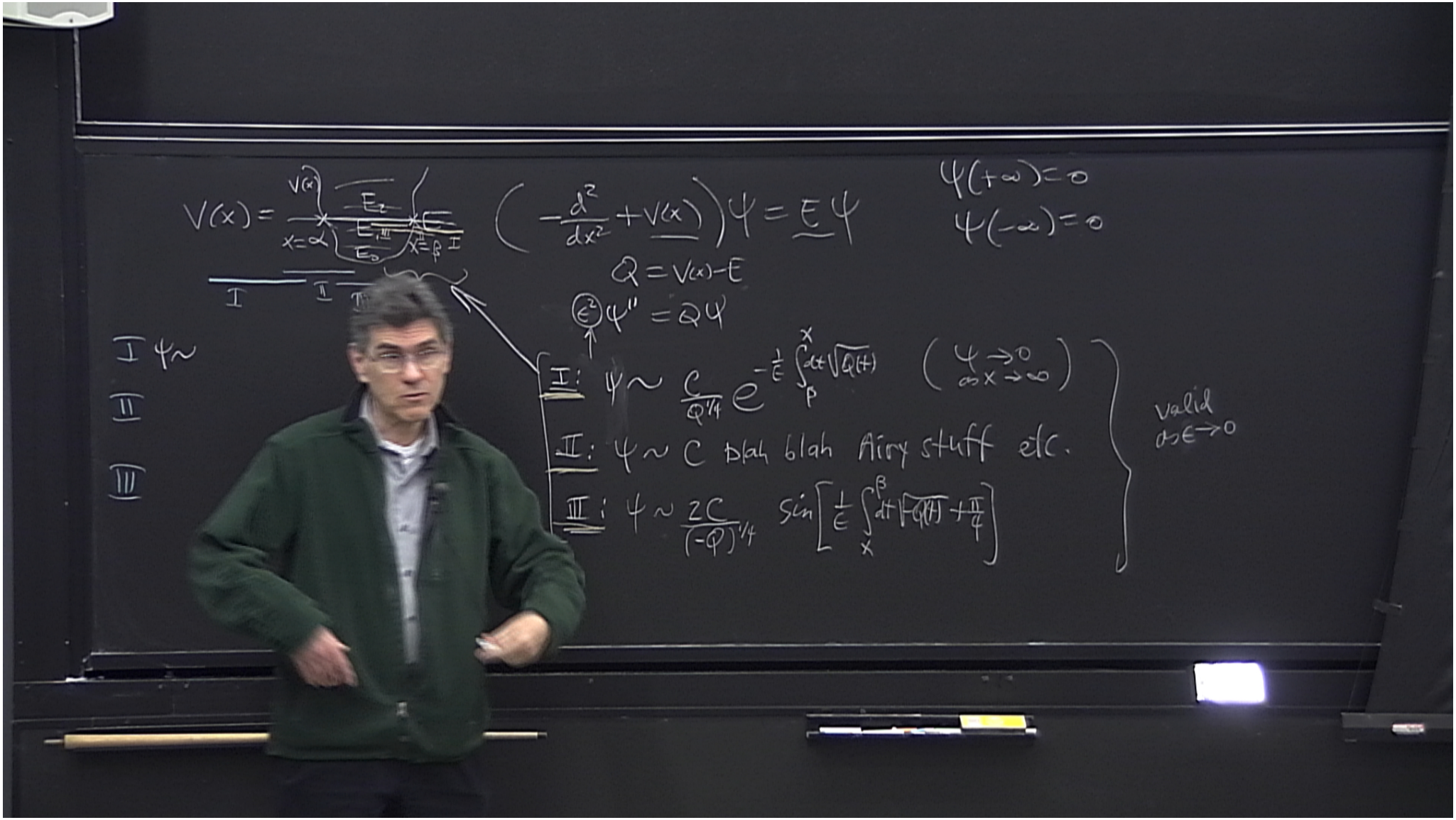
V(x) =
x=α
x=β
E₂
E₁
E₀
I
$\left(-\frac{d^2}{dx^2}+V(x)\right)\psi = E\psi$
$\psi(+\infty)=0$
$\psi(-\infty)=0$
$Q = V(x)-E$
$\epsilon^2\psi'' = Q\psi$
I: $\psi \sim \frac{C}{Q^{1/4}} e^{-\frac{1}{\epsilon}\int_\beta^x dt\sqrt{Q(t)}}$
$\left(\psi \to 0 \text{ as } x\to\infty\right)$
blah blah Airy stuff etc.
$\psi \sim \frac{2C}{(-Q)^{1/4}} \sin\left[\frac{1}{\epsilon}\int_x^\beta dt\sqrt{-Q(t)}+\frac{\pi}{4}\right]$
514 GLOBAL ANALYSIS
Exact solution
WKB approximation
y
x
ε = 1
2.0
1.0
−1.0
−2.0
0
Figure 10.13 Same as in Fig. 10.10 except that ε = 1.0. Even for this large value of ε the agreement between the approximate and exact solutions is impressive.

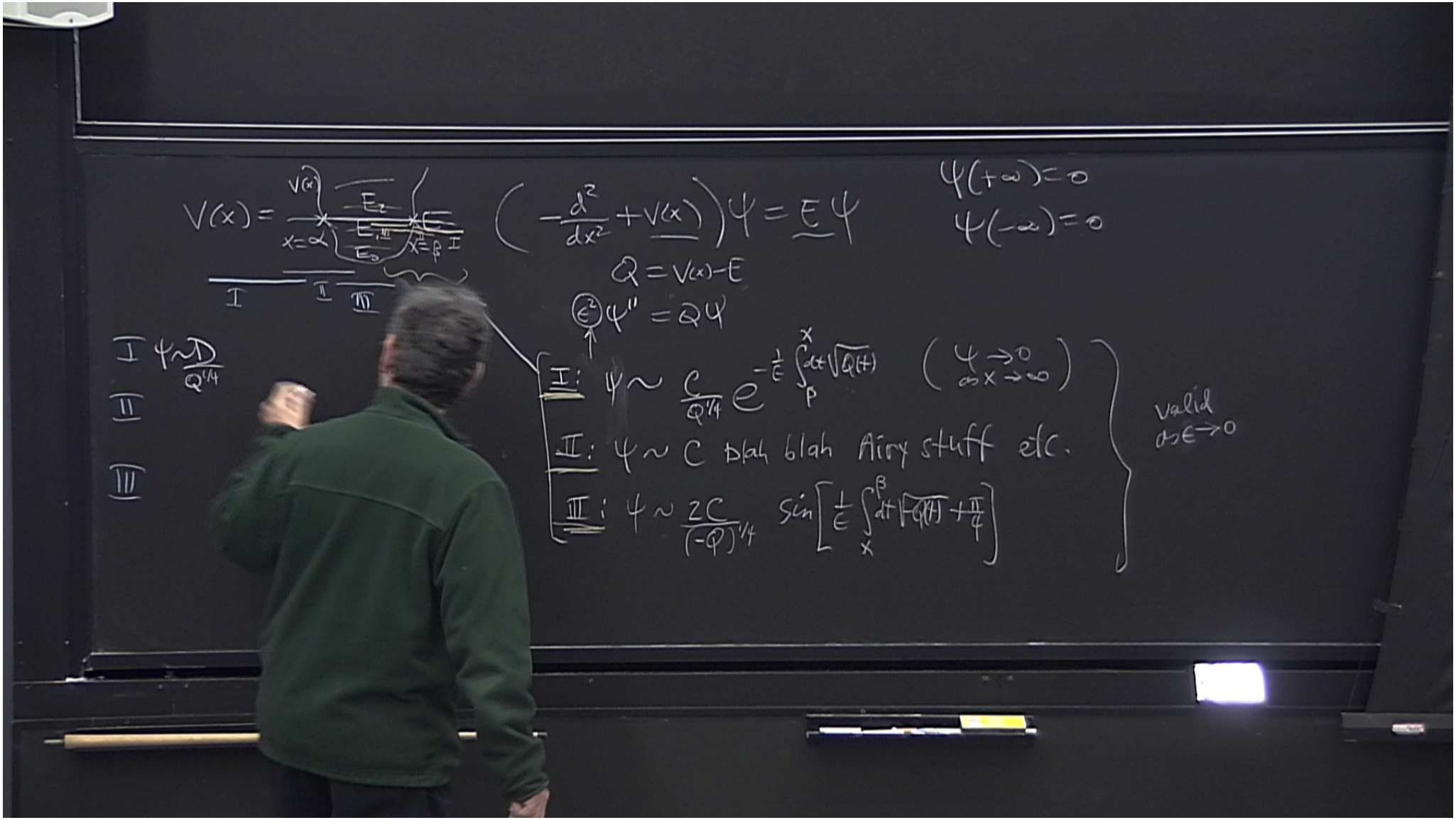

V(x) =
x=α
x=β
E₂
E₁
E₀
I
II
III
(−d²/dx² + V(x))ψ = Eψ
ψ(+∞)=0
ψ(−∞)=0
Q = V(x)−E
ε²ψ'' = Qψ
I: ψ ~ C/Q^{1/4} e^{−(1/ε)∫_β^x dt √Q(t)}
(ψ→0 as x→∞)
II: ψ ~ C blah blah Airy stuff etc.
III: ψ ~ 2C/(−Q)^{1/4} sin[(1/ε)∫_x^β dt √−Q(t) + π/4]
valid as ε→0


V(x) =
x=α
x=β
I
II
III
$\left(-\frac{d^2}{dx^2}+V(x)\right)\psi = E\psi$
$\psi(+\infty)=0$
$\psi(-\infty)=0$
$Q = V(x)-E$
$\epsilon^2\psi'' = Q\psi$
$\psi \sim \frac{C}{Q^{1/4}} e^{-\frac{1}{\epsilon}\int_\beta^x dt\sqrt{Q(t)}}$ $\left(\begin{matrix}\psi\to 0\\ x\to\infty\end{matrix}\right)$
II: $\psi \sim$ C blah blah Airy stuff etc.
III: $\psi \sim \frac{2C}{(-Q)^{1/4}} \sin\left[\frac{1}{\epsilon}\int_x^\beta dt\sqrt{-Q(t)}+\frac{\pi}{4}\right]$
Valid as $\epsilon\to 0$


$V(x) =$
$V(x)$
E_2
E_1
E_3
$x=\alpha$
$x=\beta$
I
II
III
$\left(-\frac{d^2}{dx^2} + V(x)\right)\psi = E\psi$
$\psi(+\infty) = 0$
$\psi(-\infty) = 0$
$Q = V(x) - E$
$\epsilon^2 \psi'' = Q\psi$
I: $\psi \sim \frac{C}{Q^{1/4}} e^{-\frac{1}{\epsilon}\int_\beta^x dt \sqrt{Q(t)}}$ $\left(\begin{matrix}\psi \to 0 \\ \text{as } x \to \infty\end{matrix}\right)$
II: $\psi \sim C$ blah blah Airy stuff etc.
III: $\psi \sim \frac{2C}{(-Q)^{1/4}} \sin\left[\frac{1}{\epsilon}\int_\alpha^\beta dt \sqrt{Q(t)} + \frac{\pi}{4}\right]$
Valid as $\epsilon \to 0$


$V(x) =$
$x=\alpha$
$x=\beta$
E_2
E_1
E_0
I
II
III
$\left(-\frac{d^2}{dx^2} + V(x)\right)\psi = E\psi$
$\psi(+\infty) = 0$
$\psi(-\infty) = 0$
$Q = V(x) - E$
$\epsilon^2 \psi'' = Q\psi$
I: $\psi \sim \frac{C}{Q^{1/4}} e^{-\frac{1}{\epsilon}\int_p^x dt \sqrt{Q(t)}}$ $\left(\begin{matrix}\psi \to 0 \\ \infty x \to \infty\end{matrix}\right)$
II: $\psi \sim C$ blah blah Airy stuff etc.
III: $\psi \sim \frac{2C}{(-Q)^{1/4}} \sin\left[\frac{1}{\epsilon}\int_x^\beta dt \sqrt{Q(t)} + \frac{\pi}{4}\right]$
Valid as $\epsilon \to 0$


V(x) =
x=α
x=β
$\left(-\frac{d^2}{dx^2}+V(x)\right)\psi = E\psi$
$\psi(+\infty)=0$
$\psi(-\infty)=0$
$Q = V(x)-E$
$\epsilon^2\psi'' = Q\psi$
I $\psi\sim$
II
III
I: $\psi \sim \frac{C}{Q^{1/4}} e^{-\frac{1}{\epsilon}\int_p^x dt\sqrt{Q(t)}}$ $\left(\begin{matrix}\psi\to 0\\ \infty x\to\infty\end{matrix}\right)$
II: $\psi\sim C$ blah blah Airy stuff etc.
III: $\psi \sim \frac{2C}{(-Q)^{1/4}} \sin\left[\frac{1}{\epsilon}\int_x^\beta dt\sqrt{Q(t)}+\frac{\pi}{4}\right]$
Valid as $\epsilon\to 0$


$V(x) =$
$x=\alpha$
$x=\beta$
E_2
E_1
E_0
I
II
III
$\left(-\frac{d^2}{dx^2} + V(x)\right)\psi = E\psi$
$\psi(+\infty) = 0$
$\psi(-\infty) = 0$
$Q = V(x) - E$
$\epsilon^2 \psi'' = Q\psi$
I $\psi \sim \frac{D}{Q^{1/4}}$
II
III
I: $\psi \sim \frac{C}{Q^{1/4}} e^{-\frac{1}{\epsilon}\int_p^x dt \sqrt{Q(t)}}$ $\left(\begin{matrix}\psi \to 0 \\ as\ x \to \infty\end{matrix}\right)$
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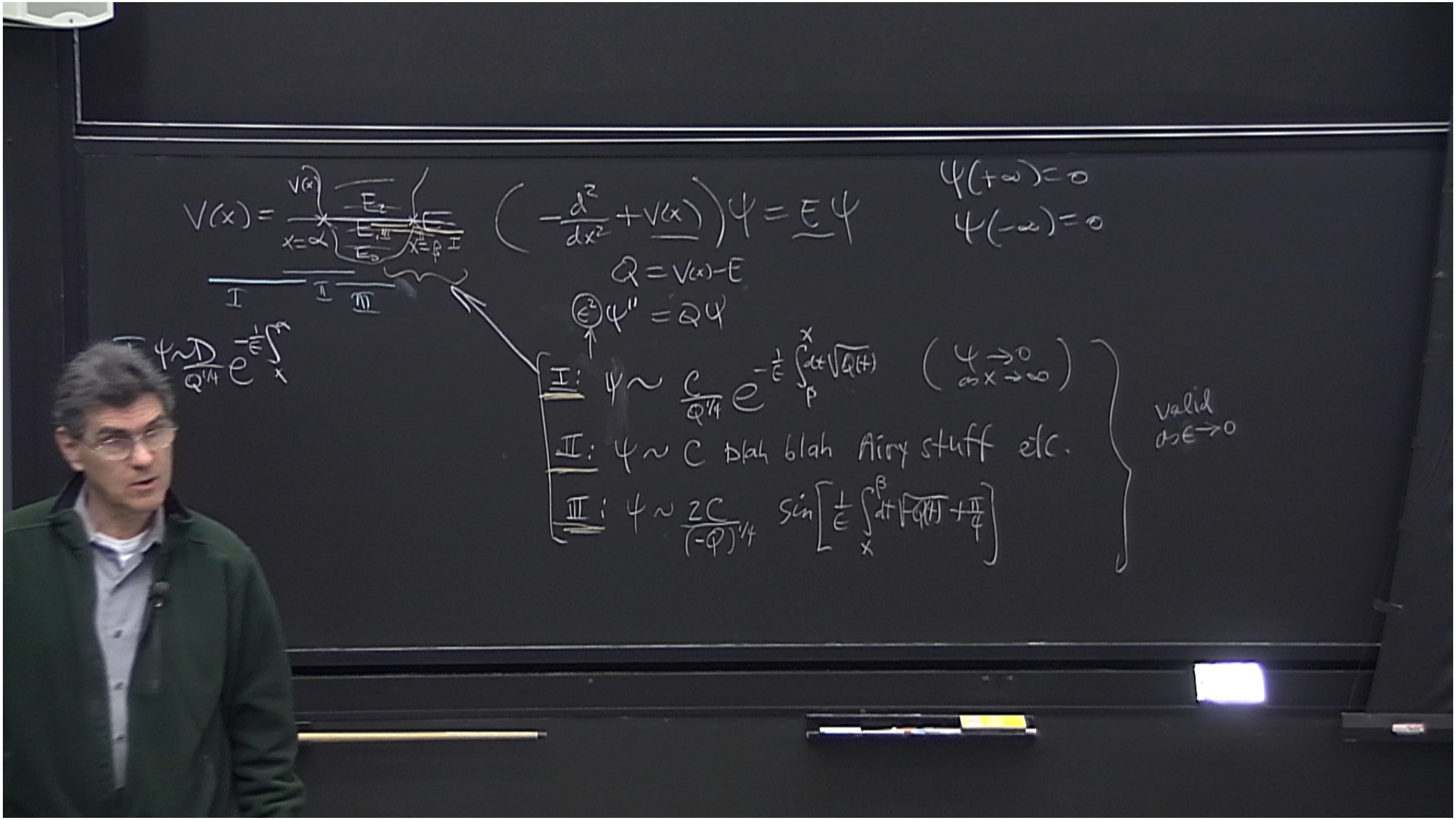
V(x) =
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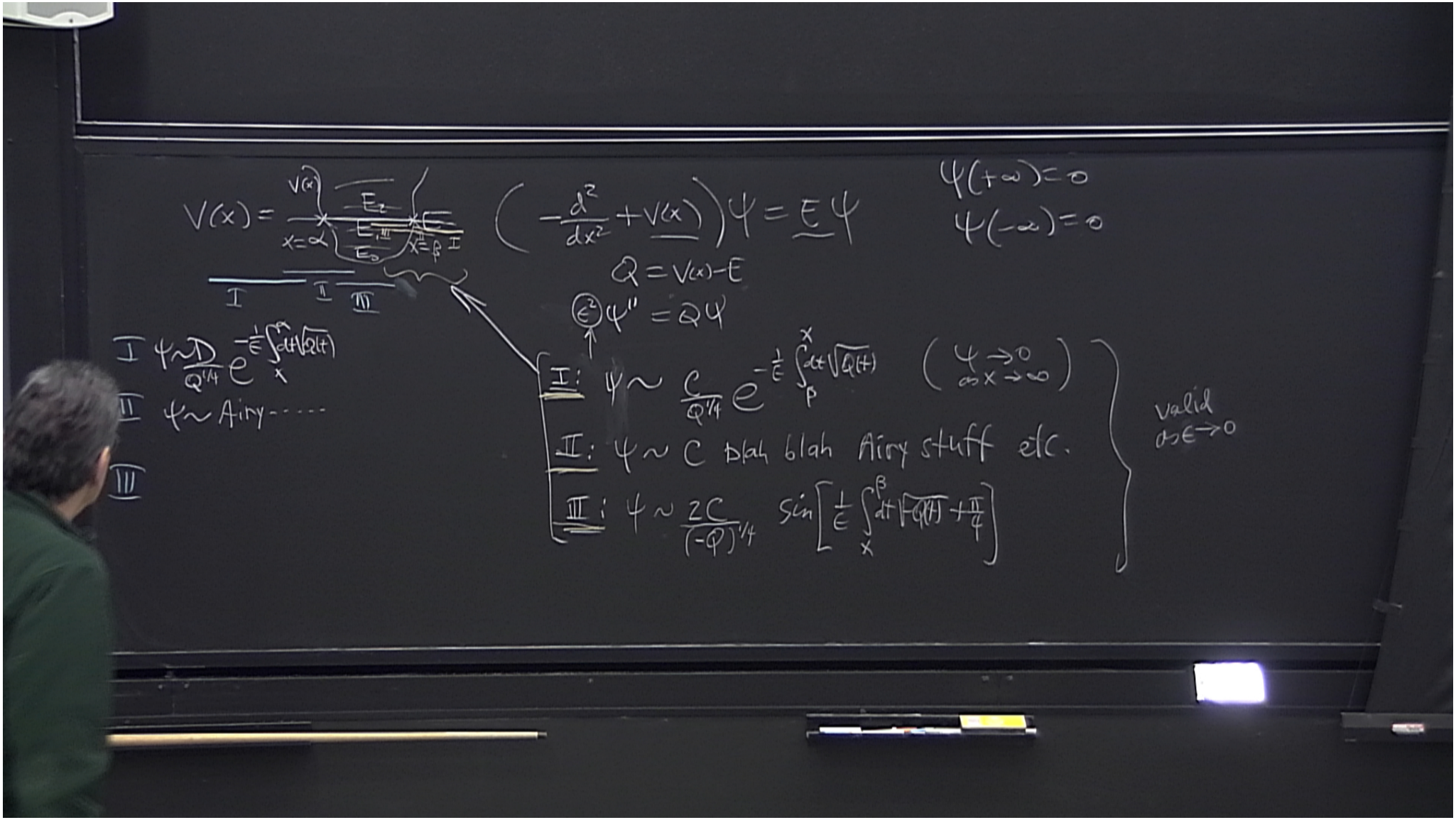
valid as $\epsilon \to 0$
$\psi \sim \frac{D}{Q^{1/4}} e^{-\frac{1}{\epsilon}\int_x^\infty}$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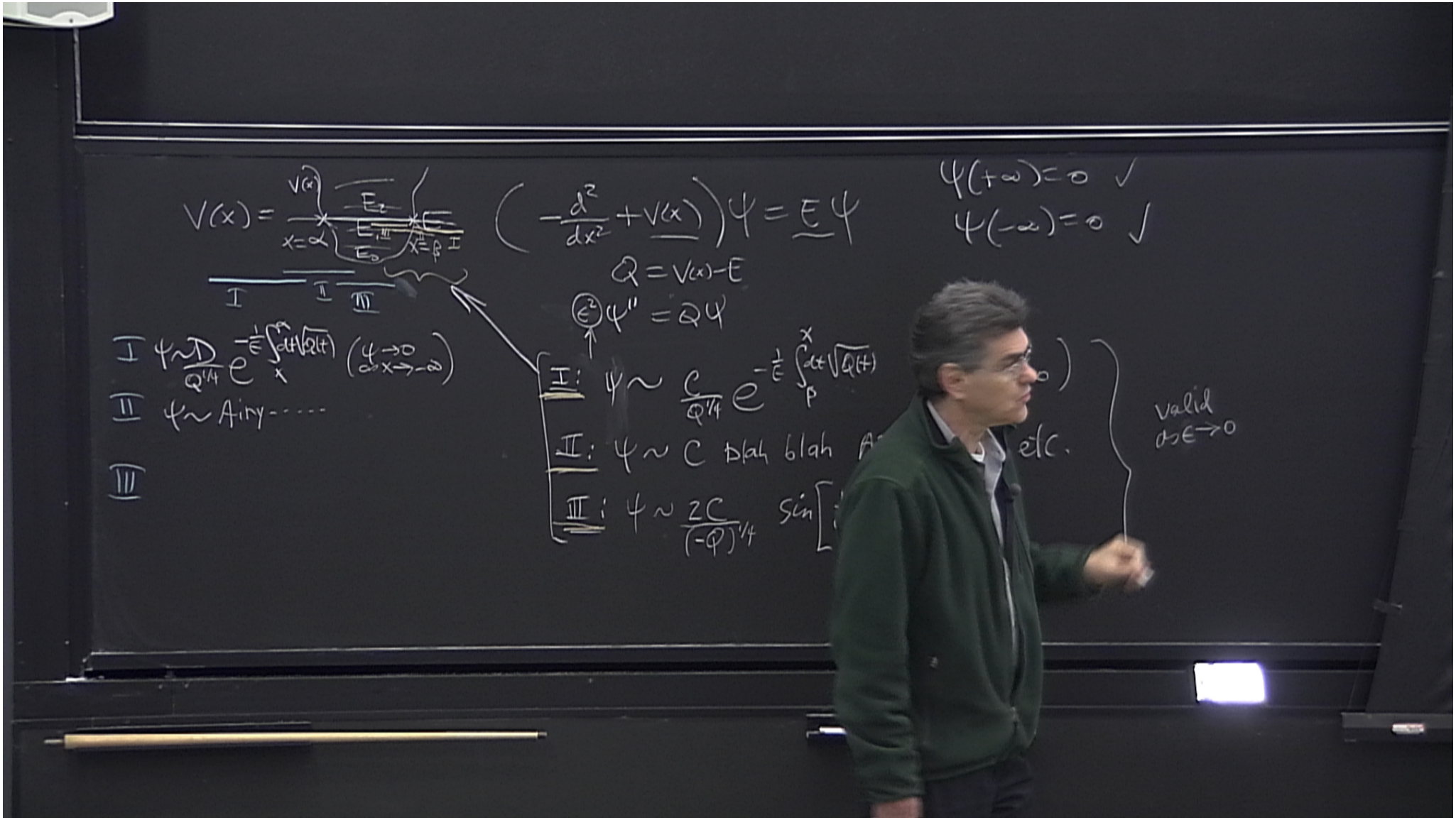

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II: $\psi \sim C$ blah blah
etc.
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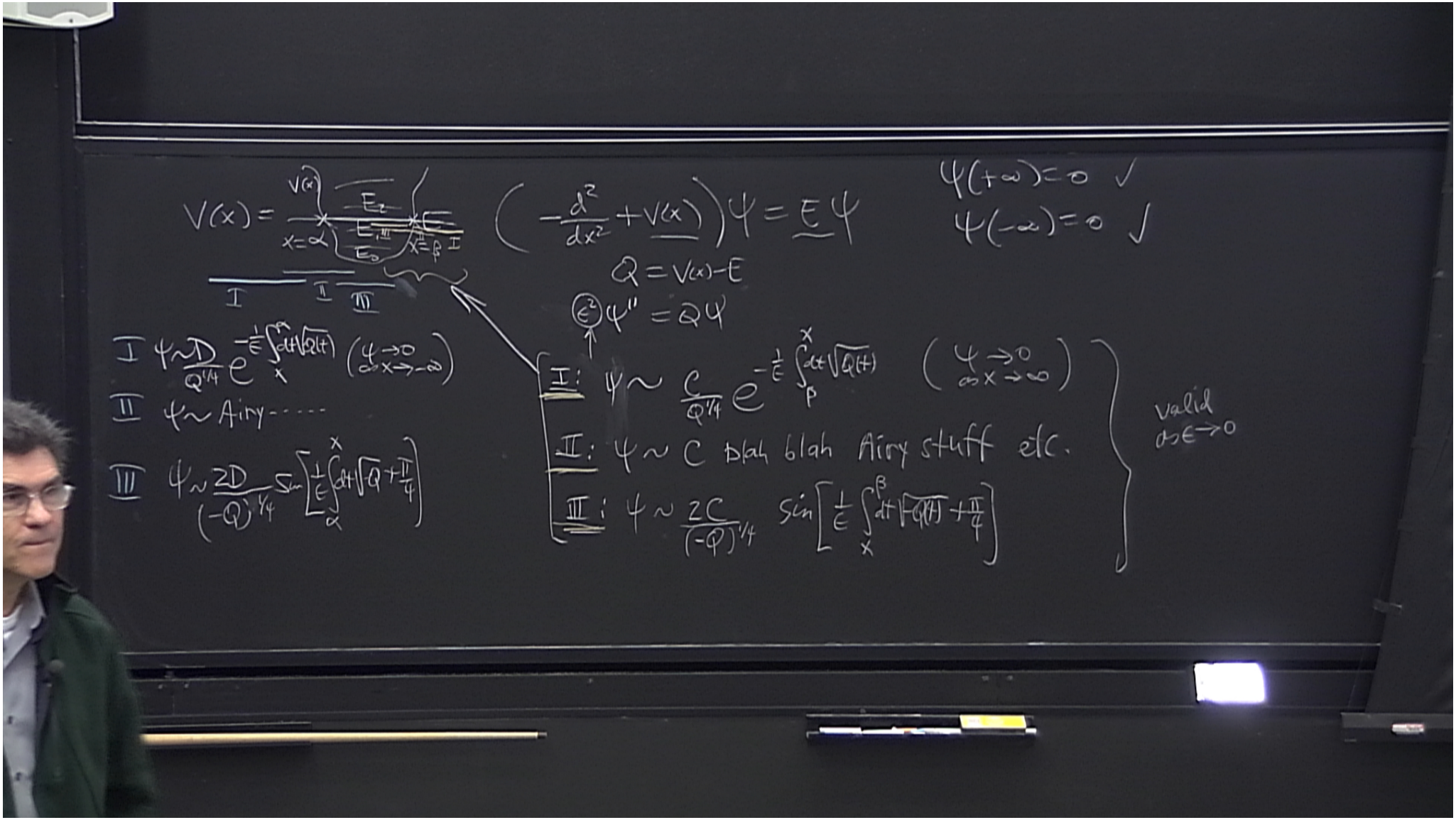
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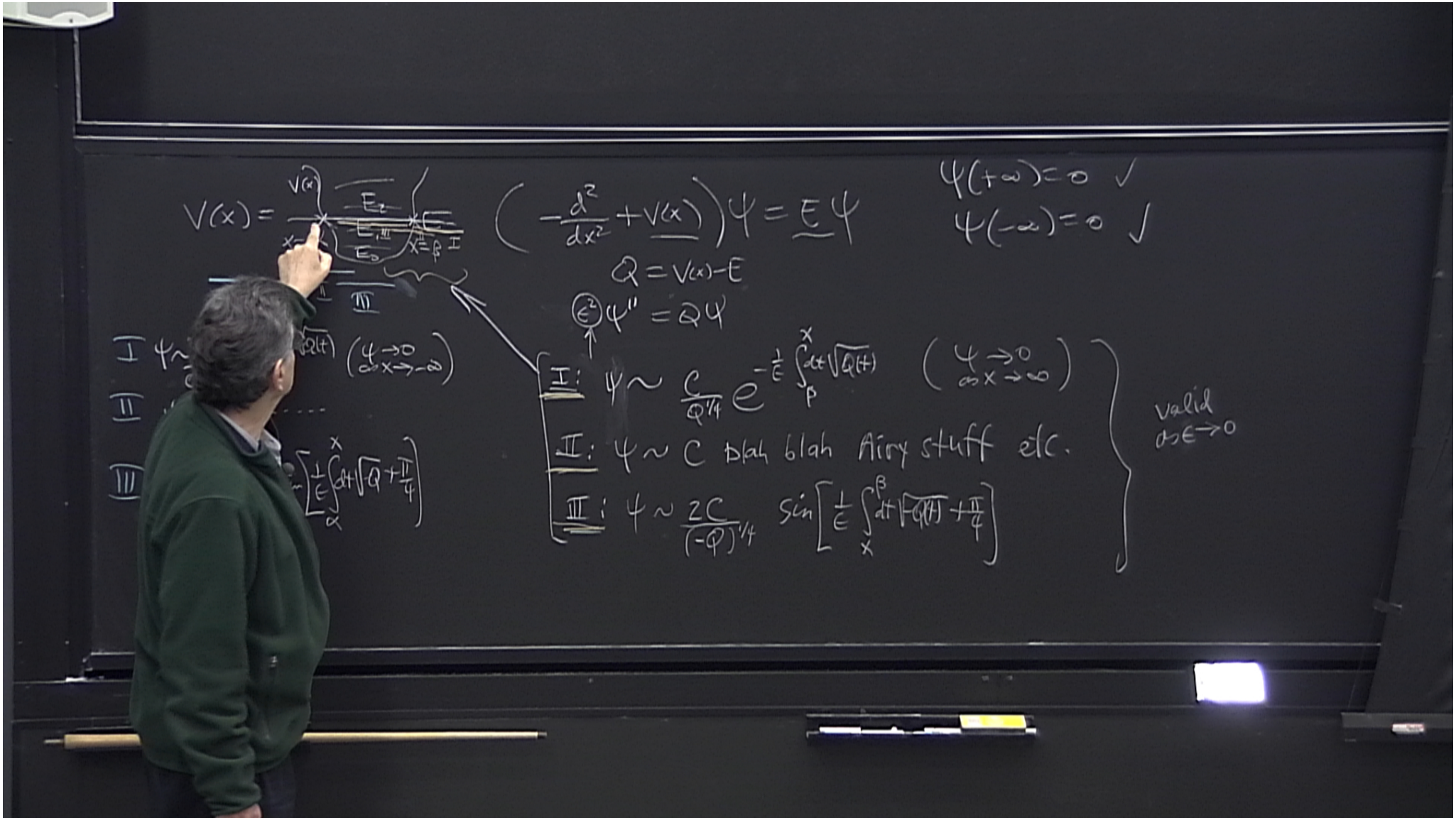
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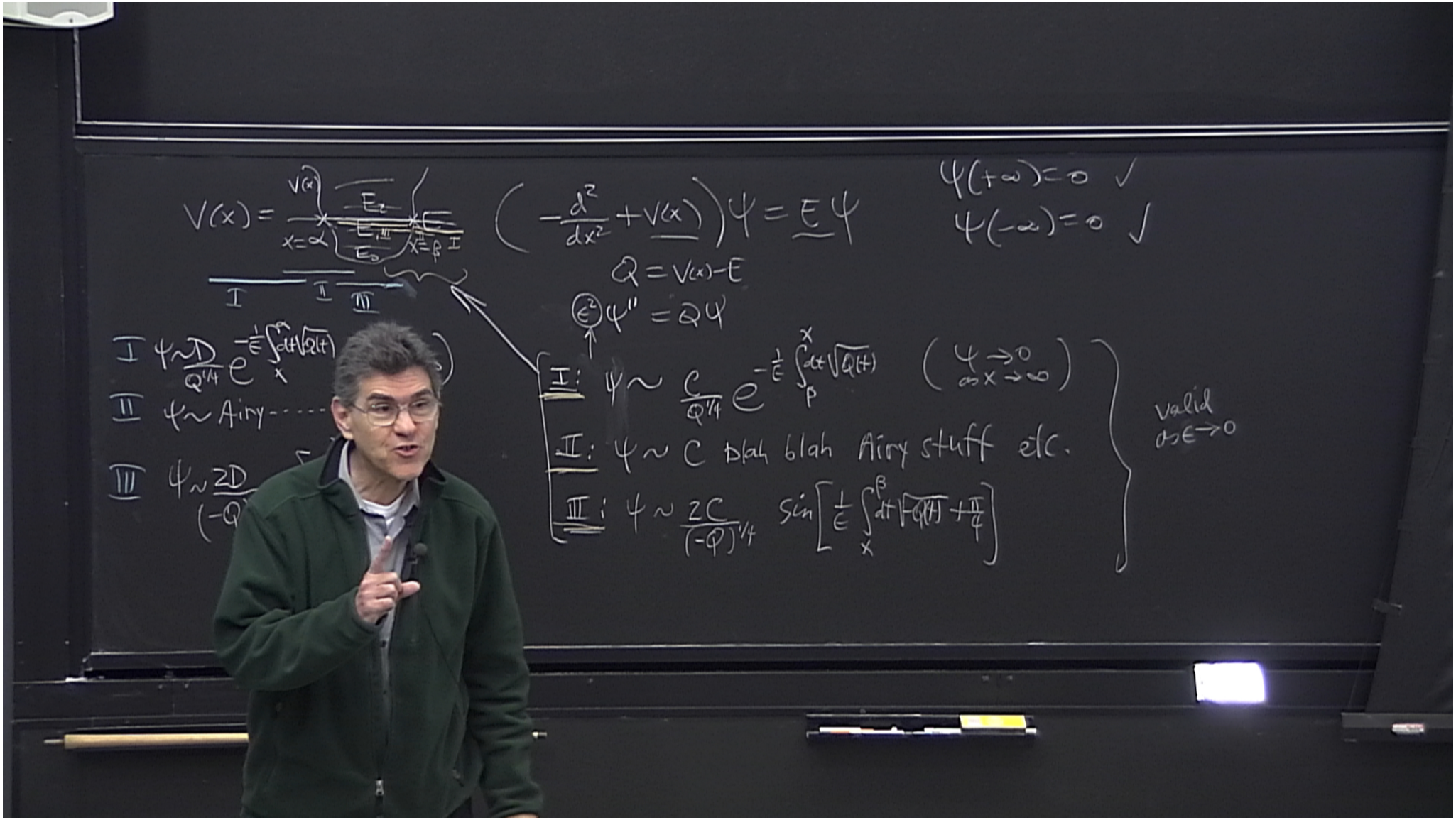
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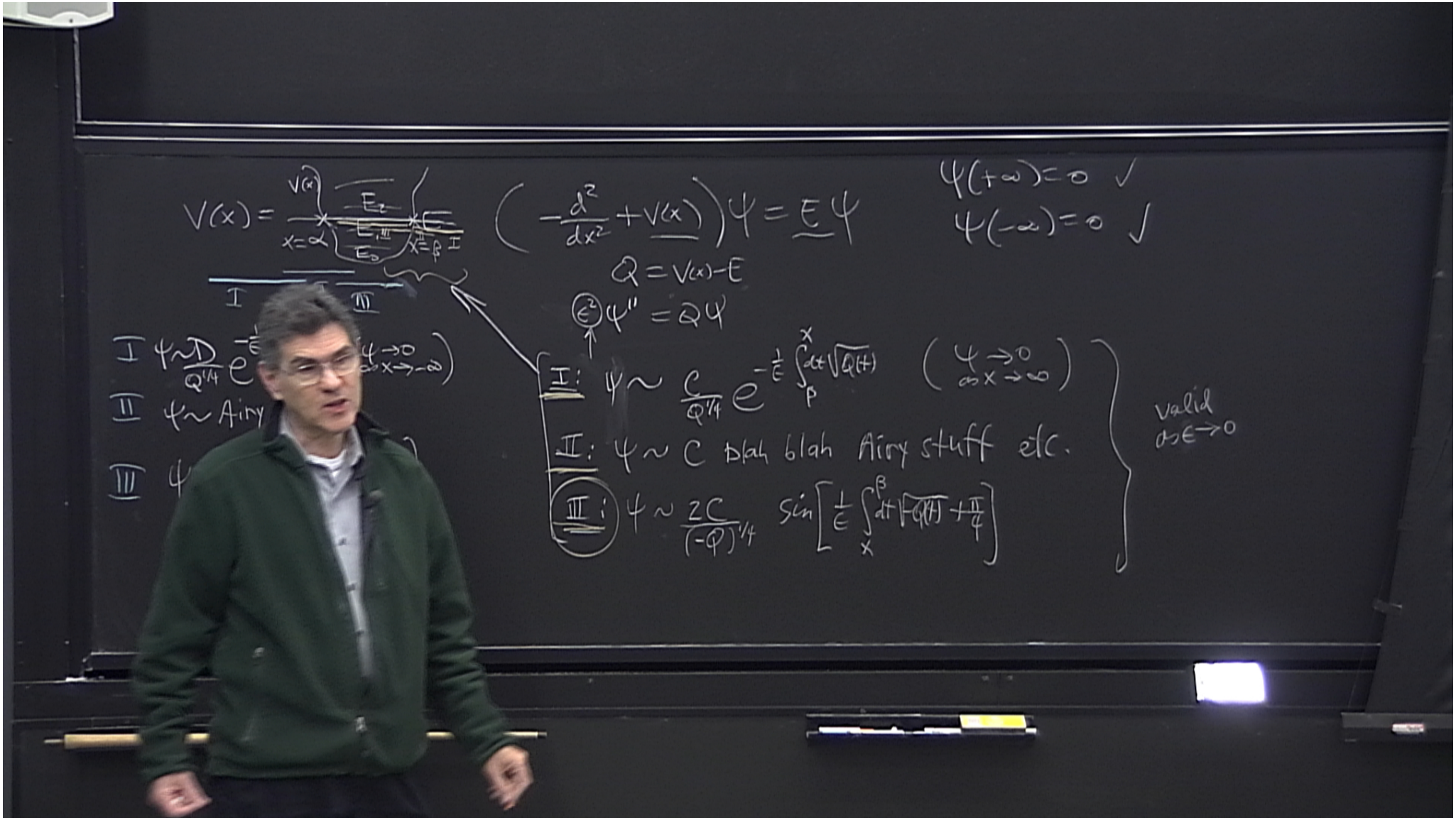
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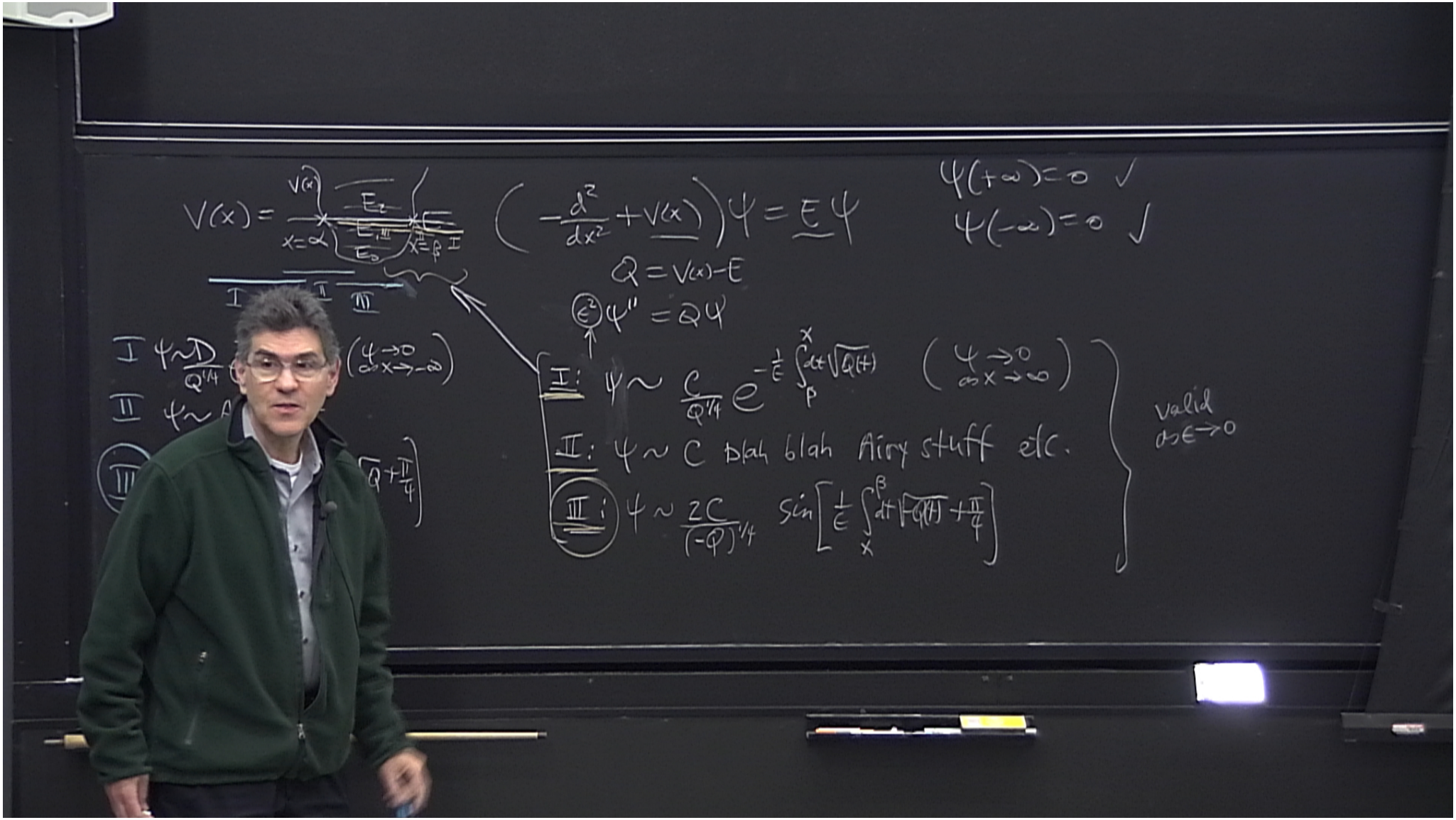
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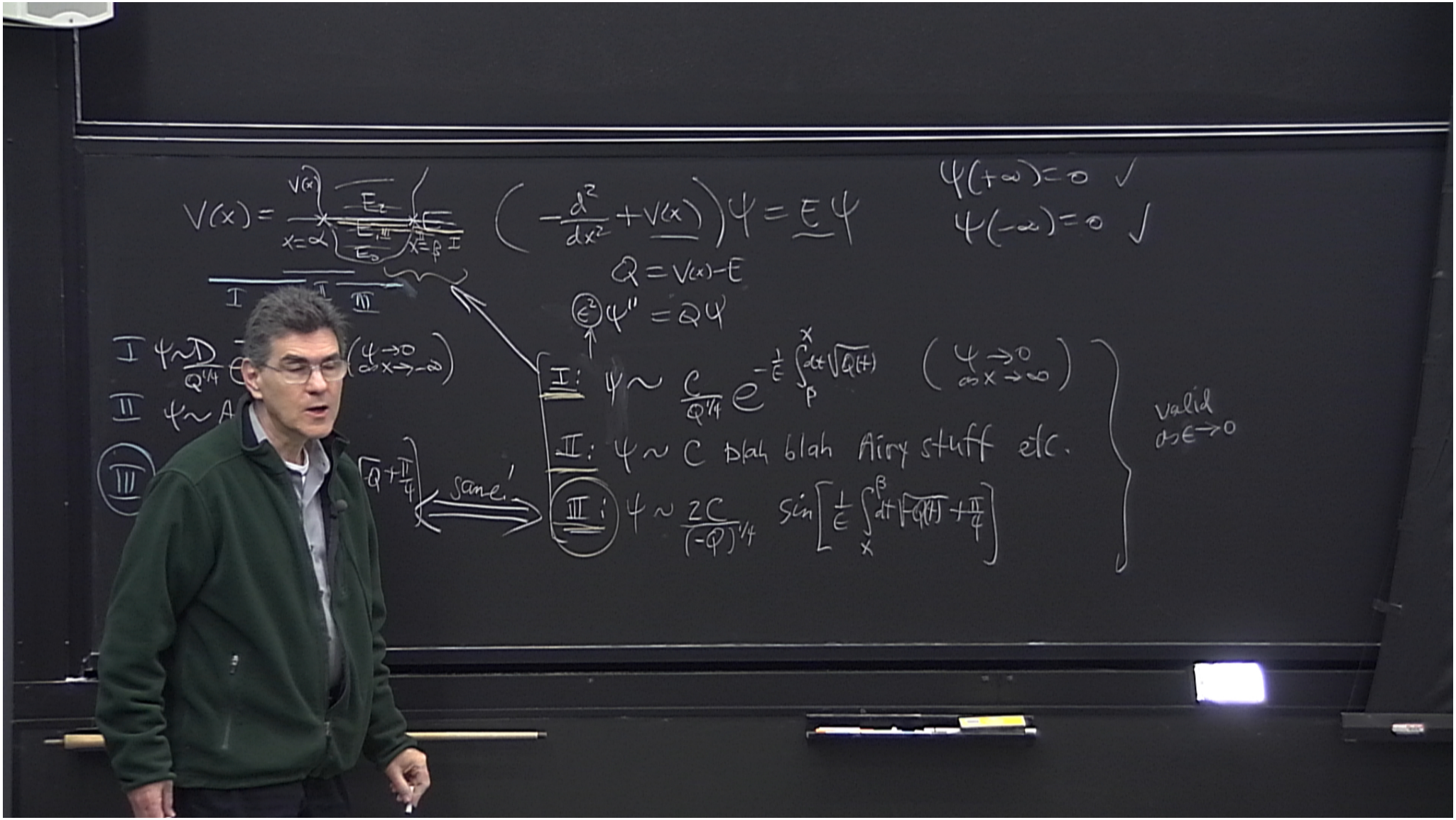

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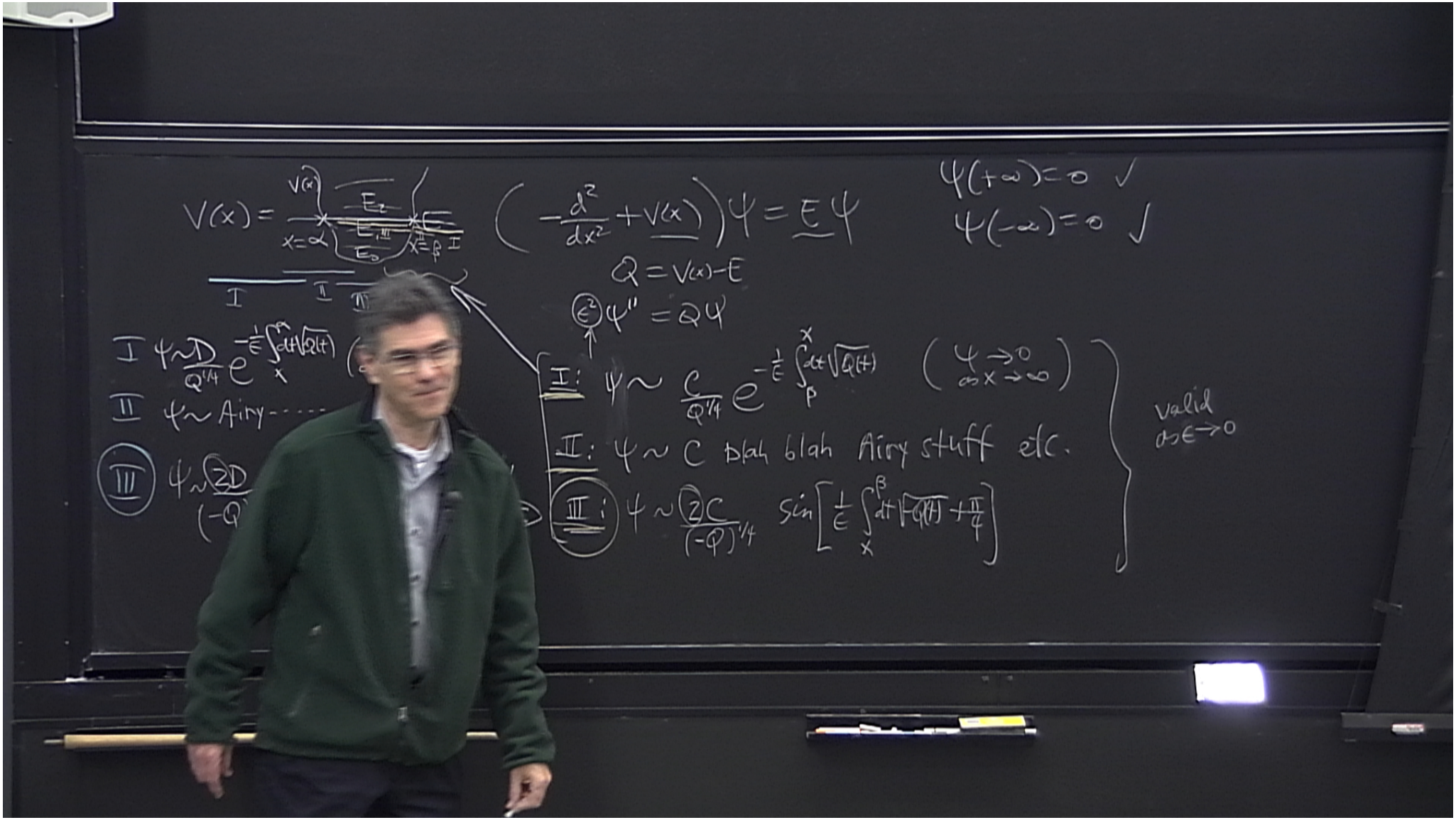
same!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I: $\psi \sim \frac{C}{Q^{1/4}} e^{-\frac{1}{\epsilon}\int_{\beta}^{x} dt\sqrt{Q(t)}}$ $\left(\begin{matrix}\psi\to 0\\ \infty\, x\to\infty\end{matrix}\right)$
II: $\psi \sim C$ blah blah Airy stuff etc.
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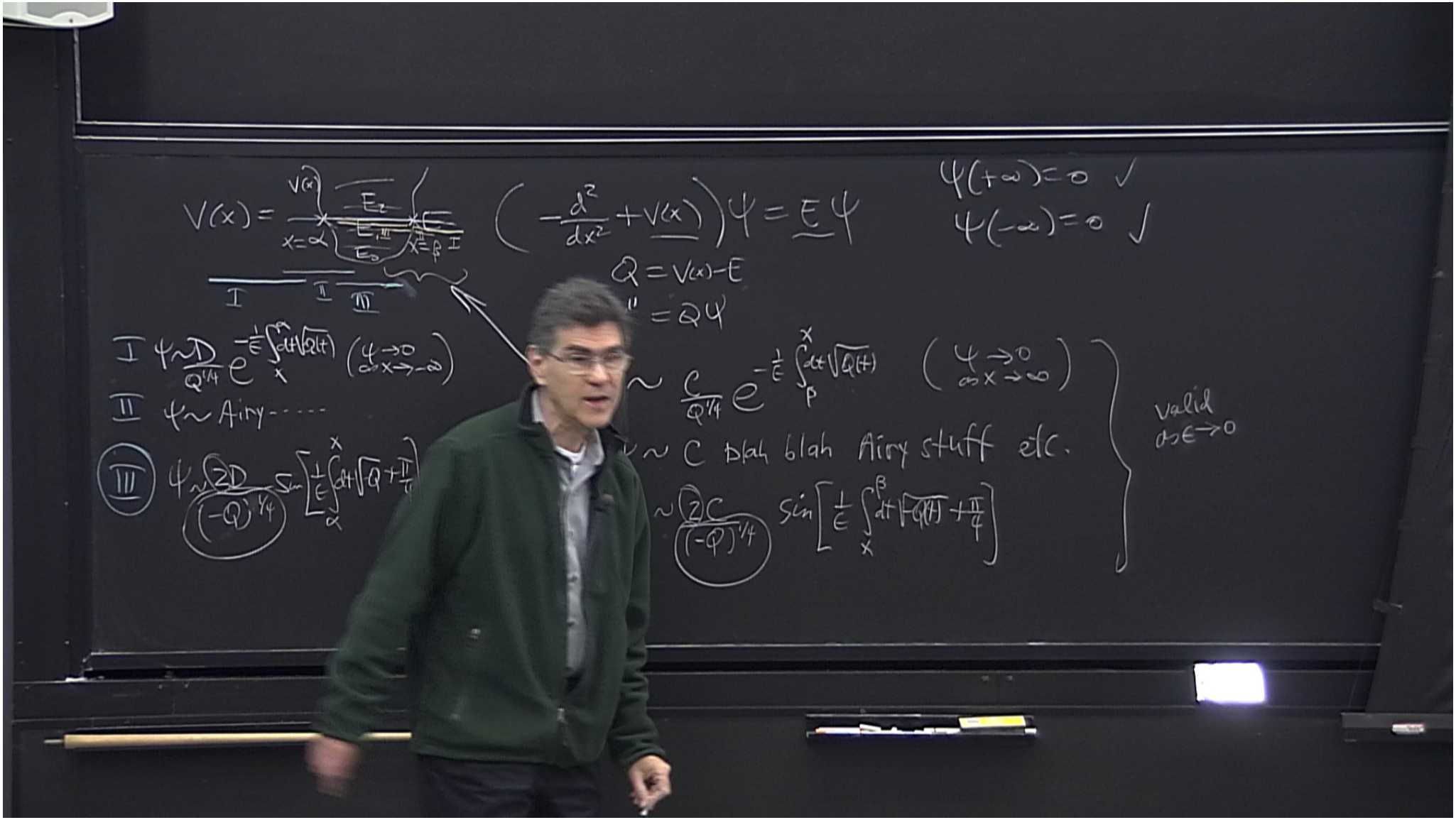
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III $\psi \sim \frac{2D}{(-Q)^{1/4}} \sin\left[\frac{1}{\epsilon}\int_\alpha^x dt\sqrt{Q}+\frac{\pi}{4}\right]$
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$\sim$ C blah blah Airy stuff etc.
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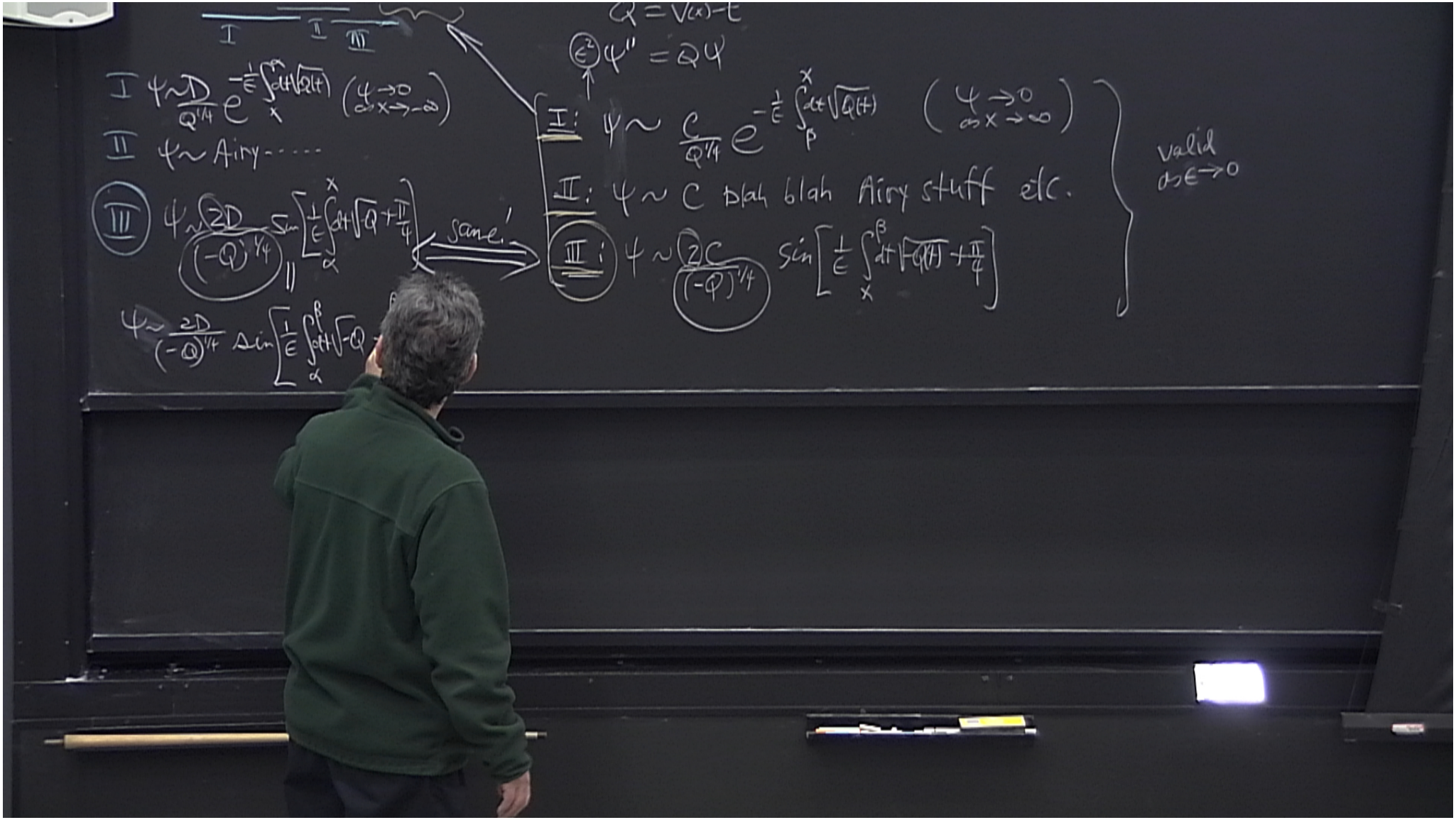
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I: $\psi \sim \frac{C}{Q^{1/4}} e^{-\frac{1}{\epsilon}\int_\beta^x dt \sqrt{Q(t)}}$ $(\psi \to 0 \text{ as } x \to \infty)$
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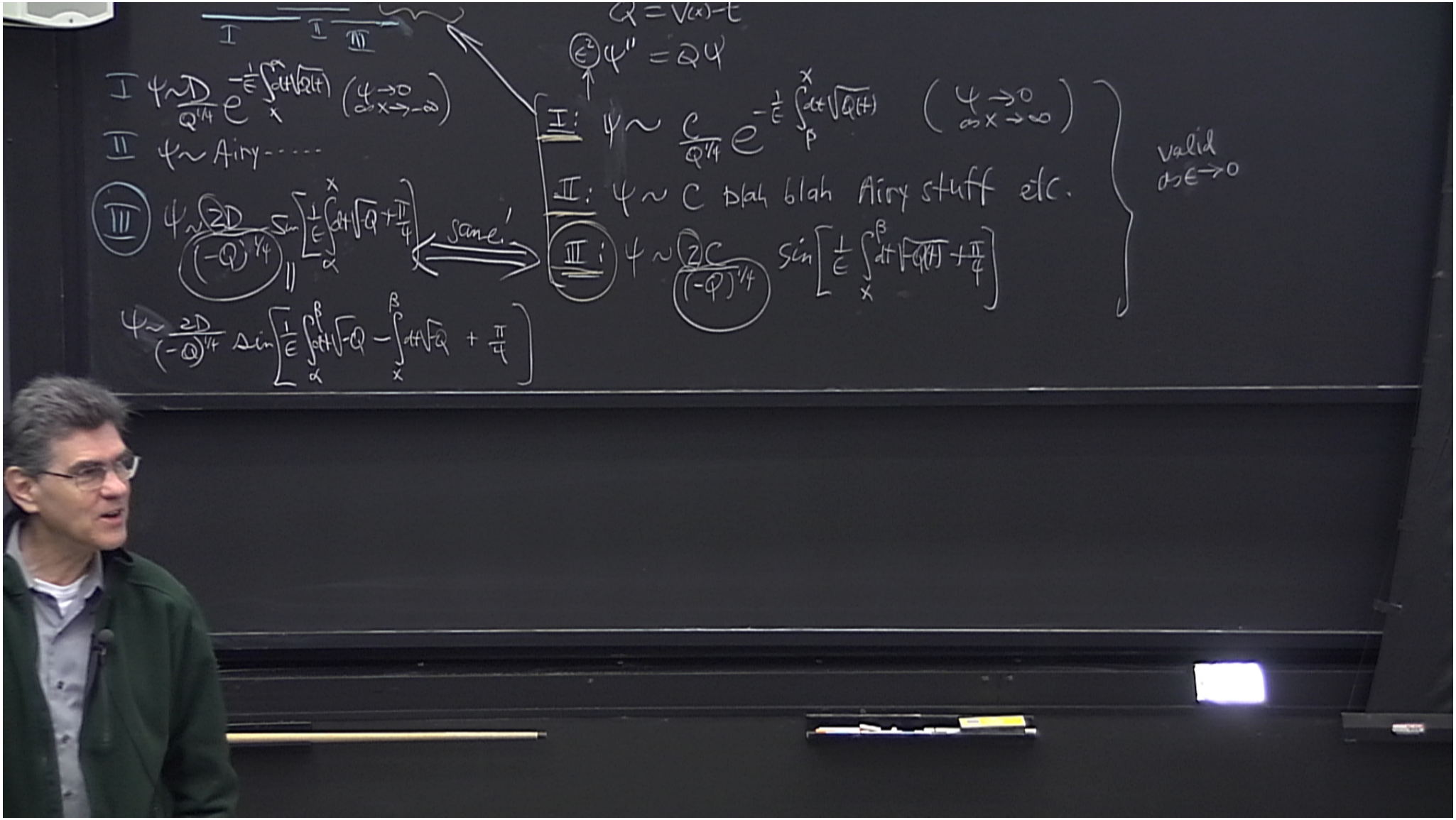
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$\psi \sim \frac{2D}{(-Q)^{1/4}} \sin\left[\frac{1}{\epsilon}\int_\alpha^\beta dt \sqrt{-Q} - \int_x^\beta dt \sqrt{Q} + \frac{\pi}{4}\right]$
same!
I: $\psi \sim \frac{C}{Q^{1/4}} e^{-\frac{1}{\epsilon}\int_\beta^x dt \sqrt{Q(t)}}$ $(\psi \to 0 \text{ as } x \to \infty)$
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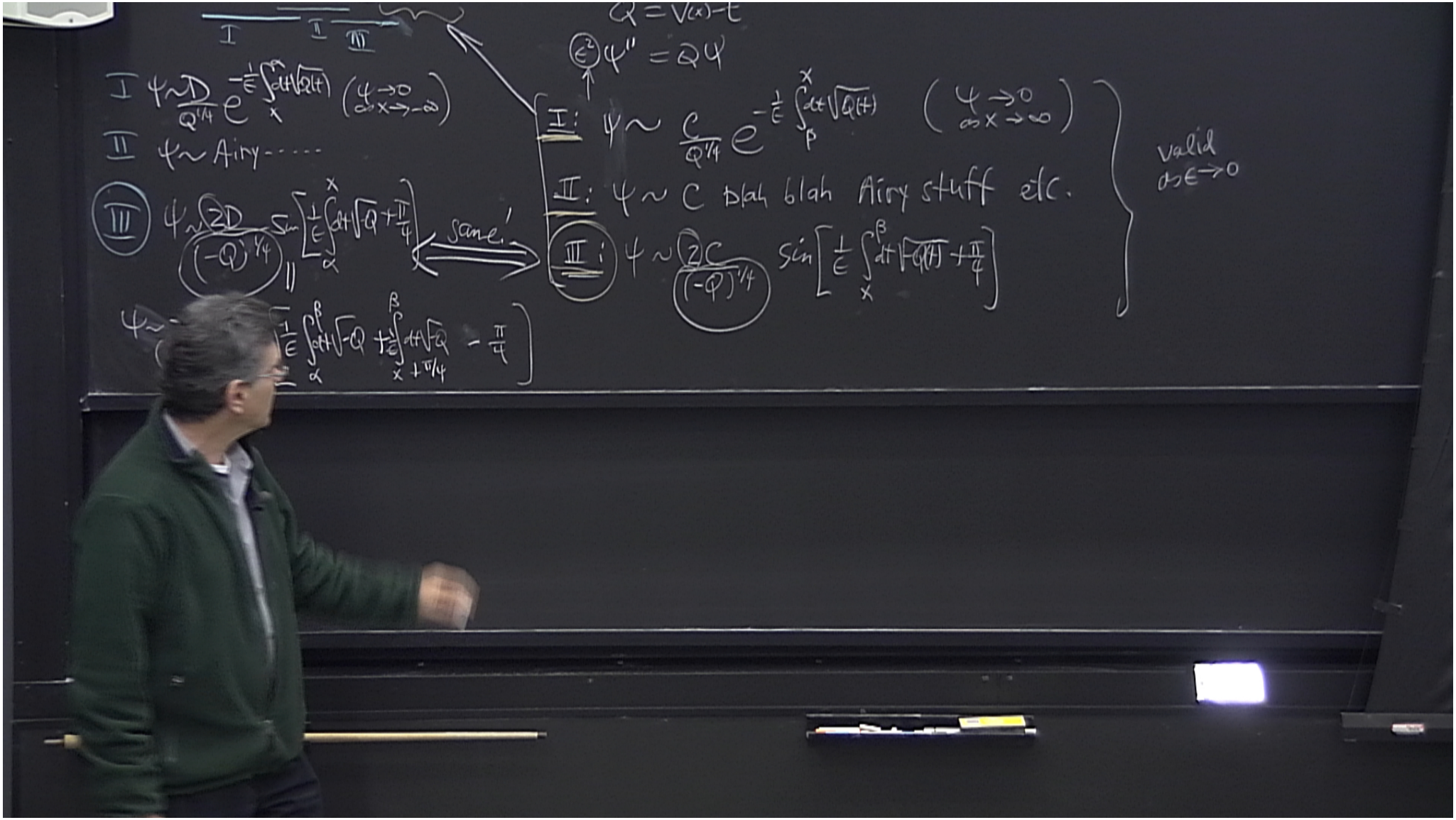
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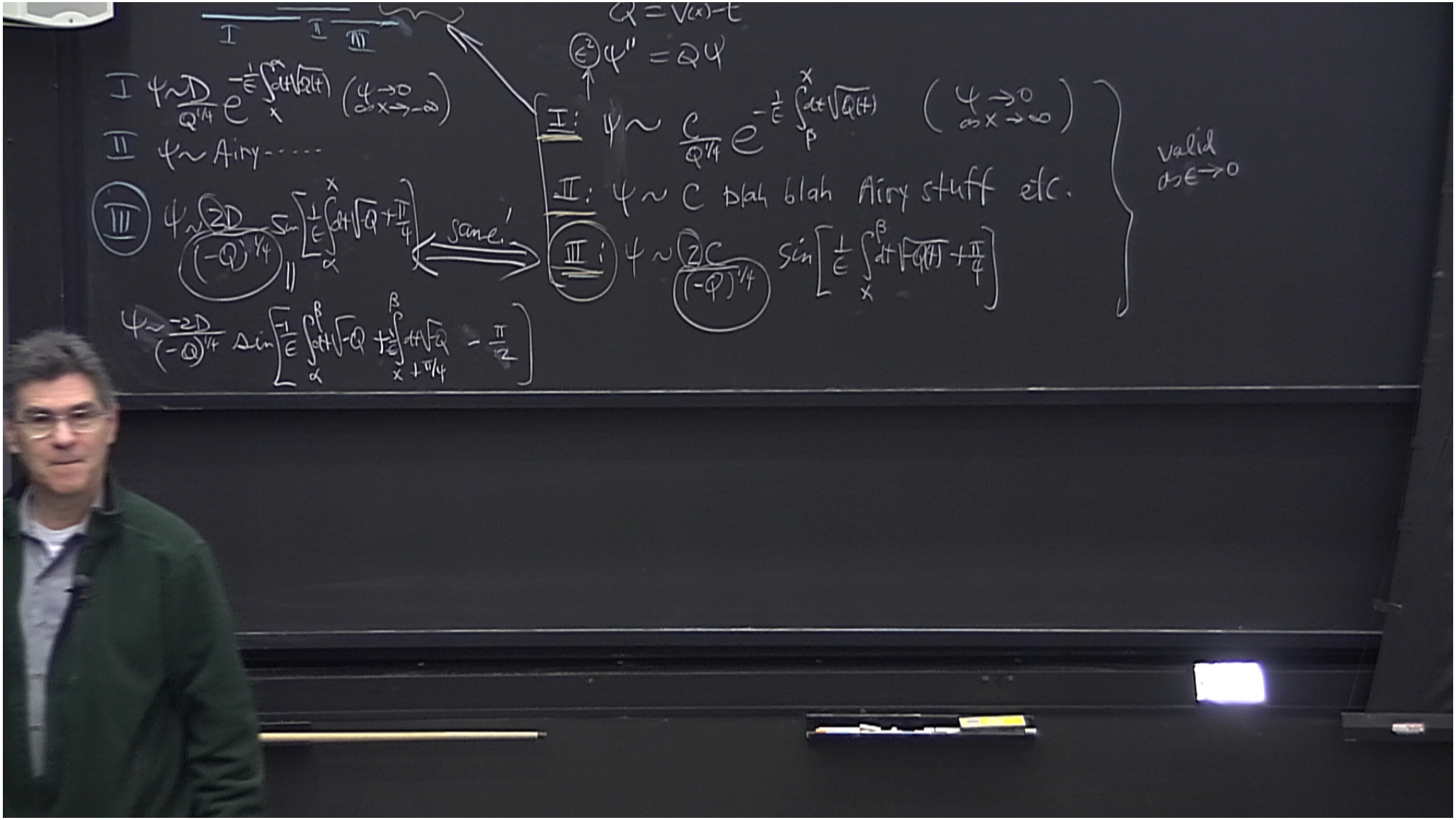
II: $\psi \sim C$ blah blah Airy stuff etc.
III: $\psi \sim \frac{2C}{(-Q)^{1/4}} \sin\left[\frac{1}{\epsilon}\int_x^\beta dt \sqrt{-Q(t)} + \frac{\pi}{4}\right]$
same!
valid as $\epsilon \to 0$
$\psi \sim$ ($\frac{1}{\epsilon}\int_\alpha^\beta dt \sqrt{-Q} + \frac{1}{\epsilon}\int_x^\beta dt \sqrt{Q} - \frac{\pi}{4}$)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same!
I: $\psi \sim \frac{C}{Q^{1/4}} e^{-\frac{1}{\epsilon}\int_\beta^x dt \sqrt{Q(t)}}$ $(\psi \to 0 \text{ as } x \to \infty)$
II: $\psi \sim C$ blah blah Airy stuff etc.
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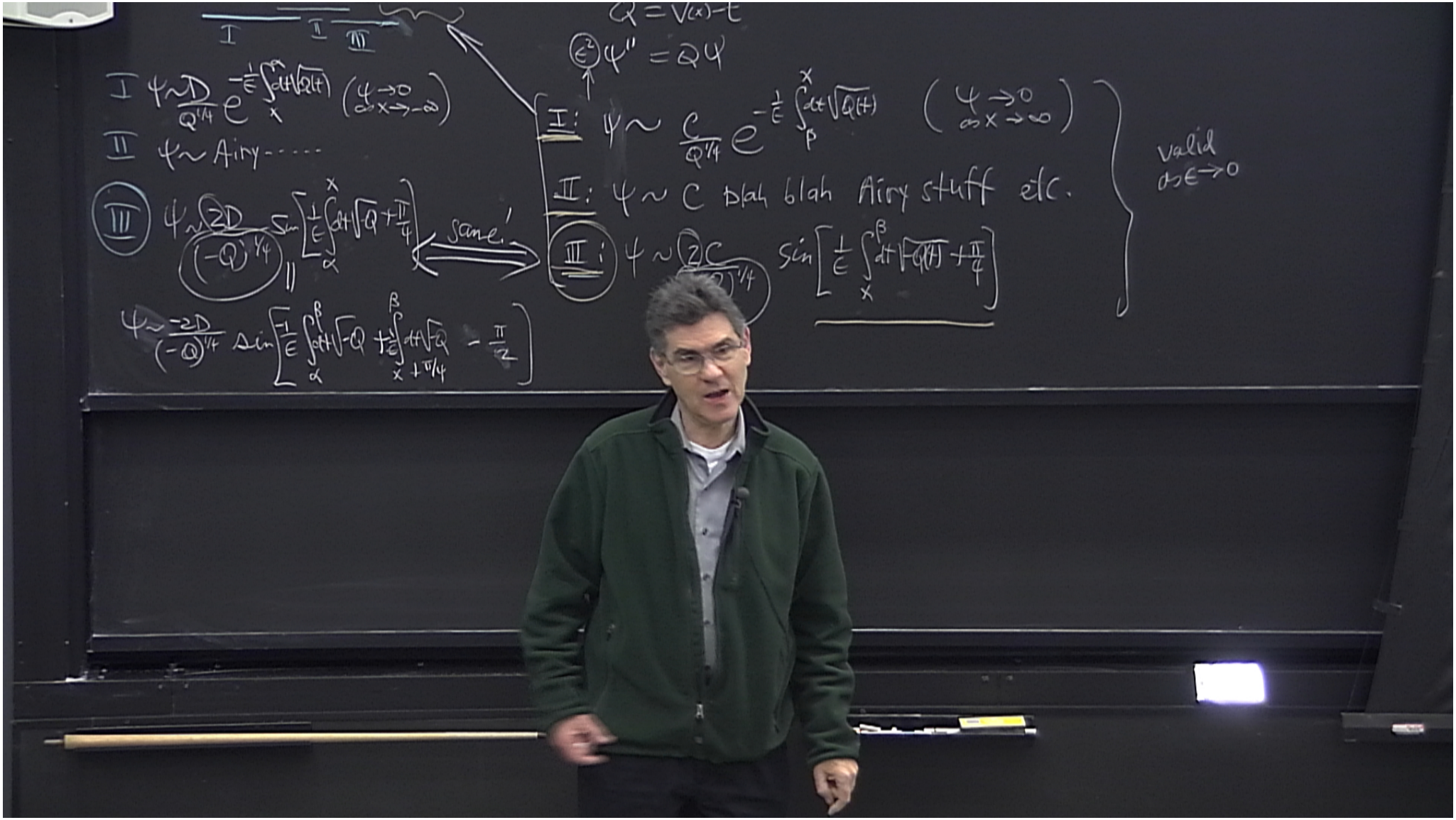
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$(-Q)^{1/4}$
same!
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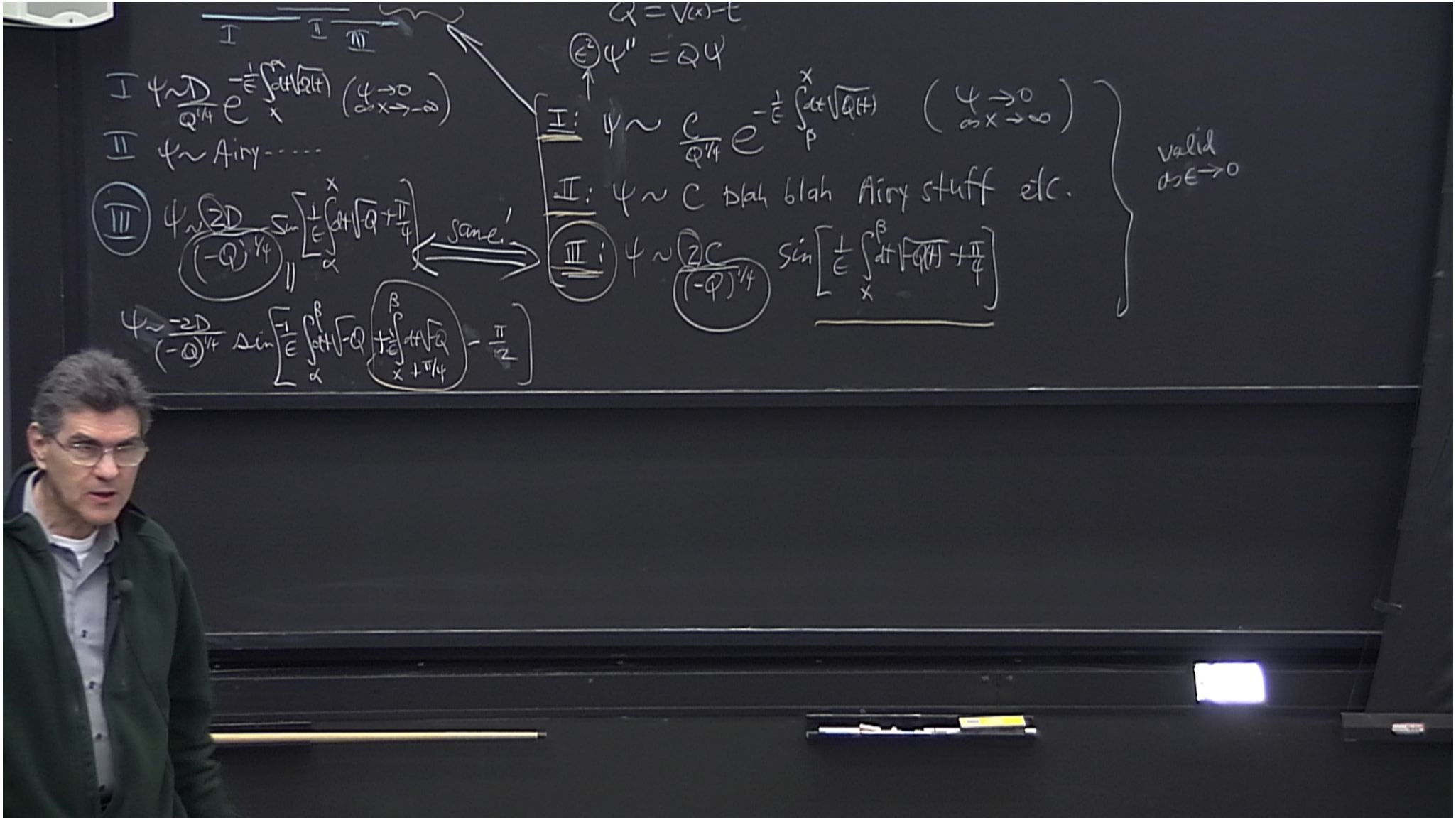
valid as $\epsilon \to 0$
$\psi \sim \frac{-2D}{(-Q)^{1/4}} \sin\left[\frac{-1}{\epsilon}\int_\alpha^\beta dt \sqrt{-Q} + \frac{1}{\epsilon}\int_x^\beta dt \sqrt{Q} - \frac{\pi}{4}\right]$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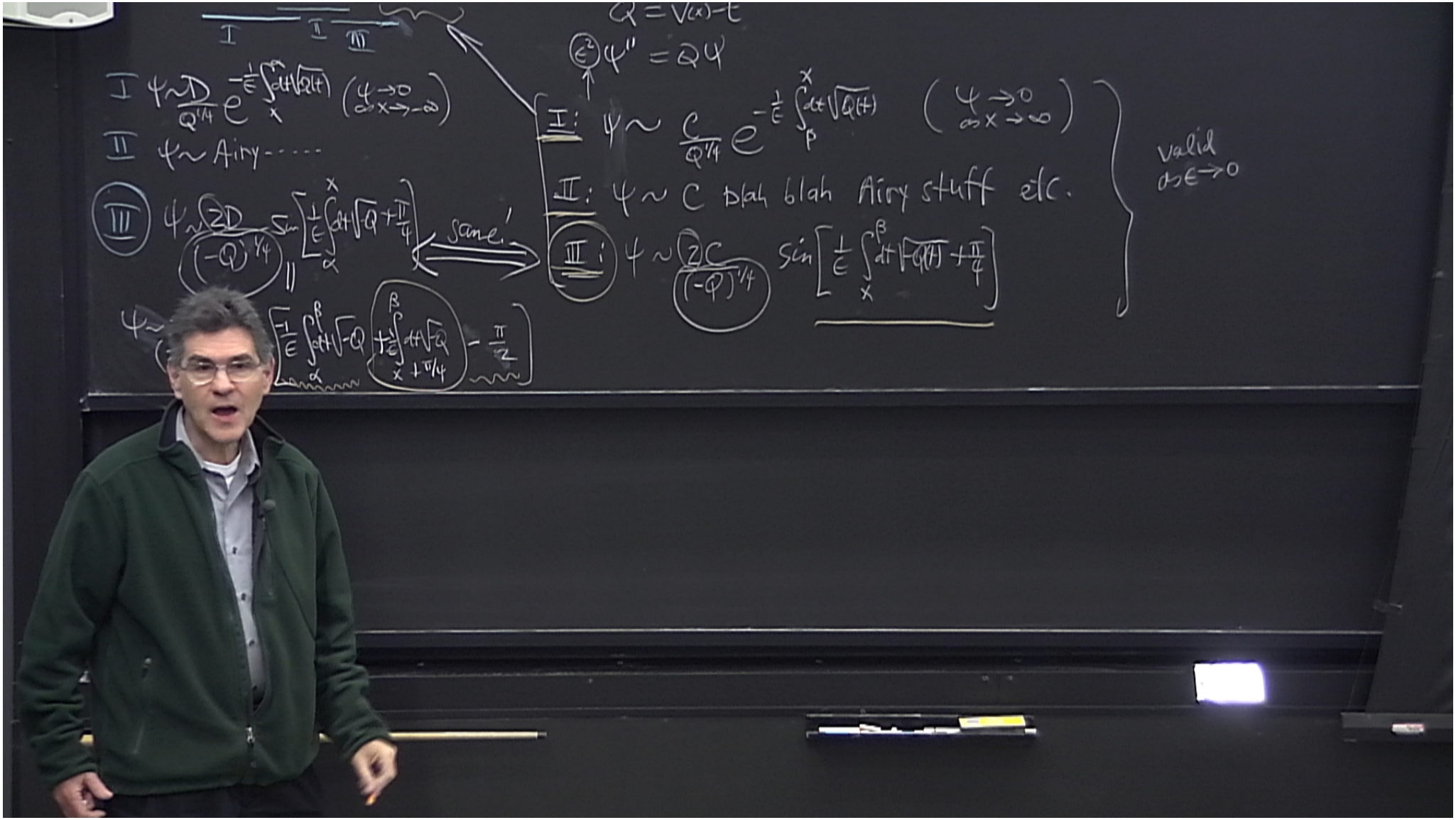
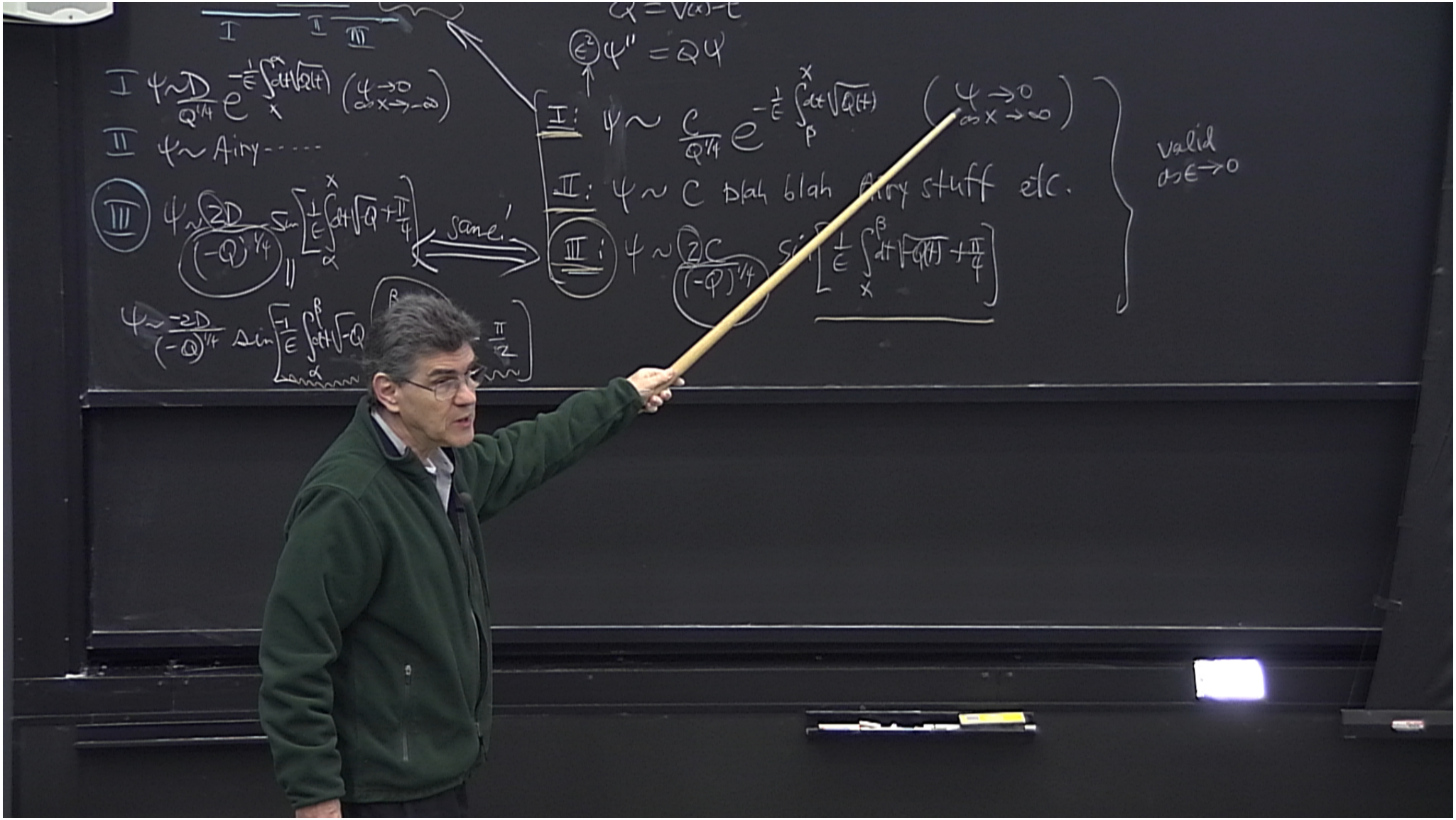
II $\psi \sim$ Airy.....
III $\psi \sim \frac{2D}{(-Q)^{1/4}} \sin\left[\frac{1}{\epsilon}\int_\alpha^x dt \sqrt{Q} + \frac{\pi}{4}\right]$
$\psi \sim \frac{-2D}{(-Q)^{1/4}} \sin\left[\frac{-1}{\epsilon}\int_\alpha^\beta dt \sqrt{-Q} + \frac{1}{\epsilon}\int_x^\beta dt \sqrt{Q} + \frac{\pi}{4}\right] - \frac{\pi}{2}$
same!
I: $\psi \sim \frac{C}{Q^{1/4}} e^{-\frac{1}{\epsilon}\int_\beta^x dt \sqrt{Q(t)}}$ $(\psi \to 0 \text{ as } x \to \infty)$
II: $\psi \sim C$ blah blah Airy stuff etc.
III: $\psi \sim \frac{2C}{(-Q)^{1/4}} \sin\left[\frac{1}{\epsilon}\int_x^\beta dt \sqrt{-Q(t)} + \frac{\pi}{4}\right]$
valid as $\epsilon \to 0$


I
II
III
$Q = V(x) - E$
$\epsilon^2 \psi'' = Q\psi$
I $\psi \sim \frac{D}{Q^{1/4}} e^{-\frac{1}{\epsilon}\int_x^\infty dt \sqrt{Q(t)}}$ $(\psi \to 0 \text{ as } x \to -\infty)$
II $\psi \sim$ Airy .....
III $\psi \sim \frac{2D}{(-Q)^{1/4}} \sin\left[\frac{1}{\epsilon}\int_\alpha^x dt \sqrt{Q} + \frac{\pi}{4}\right]$
$= \sin\left[\frac{1}{\epsilon}\int_\alpha^\beta dt \sqrt{-Q} + \frac{1}{\epsilon}\int_x^\beta dt \sqrt{Q} + \frac{\pi}{4}\right] - \frac{\pi}{2}$
same!
I: $\psi \sim \frac{C}{Q^{1/4}} e^{-\frac{1}{\epsilon}\int_\beta^x dt \sqrt{Q(t)}}$ $(\psi \to 0 \text{ as } x \to \infty)$
II: $\psi \sim C$ blah blah Airy stuff etc.
III: $\psi \sim \frac{2C}{(-Q)^{1/4}} \sin\left[\frac{1}{\epsilon}\int_x^\beta dt \sqrt{-Q(t)} + \frac{\pi}{4}\right]$
valid as $\epsilon \to 0$


I
II
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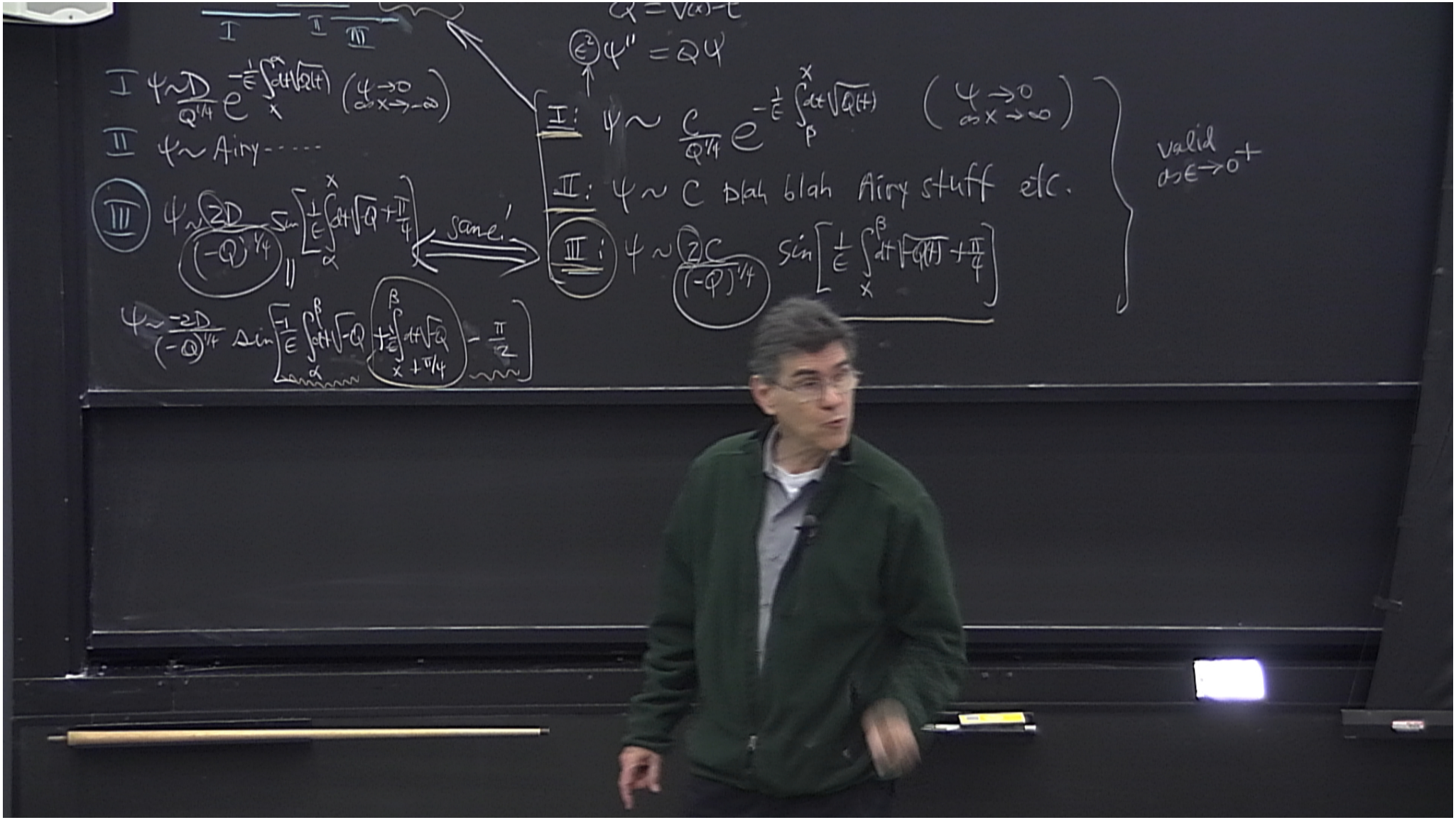
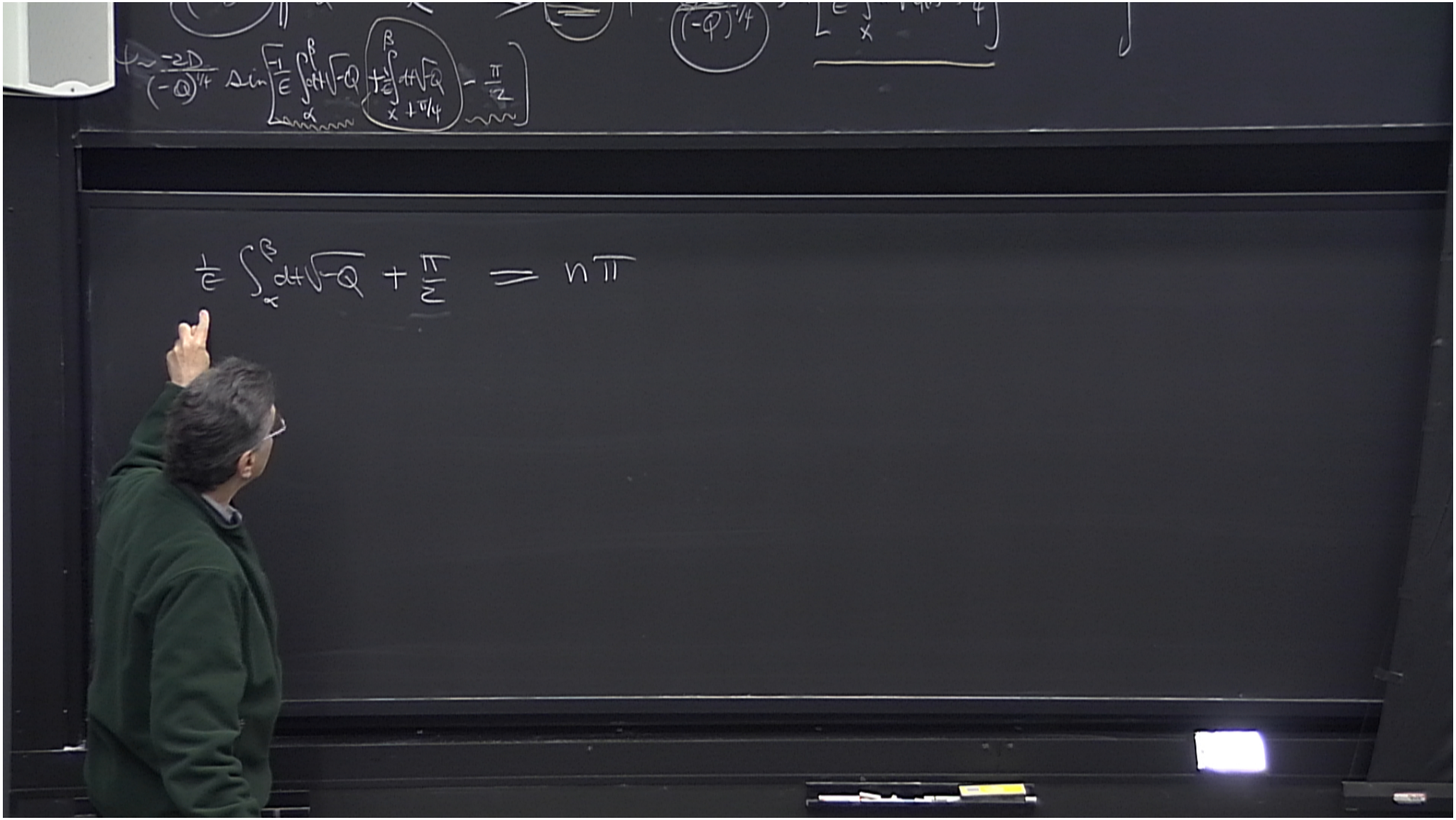
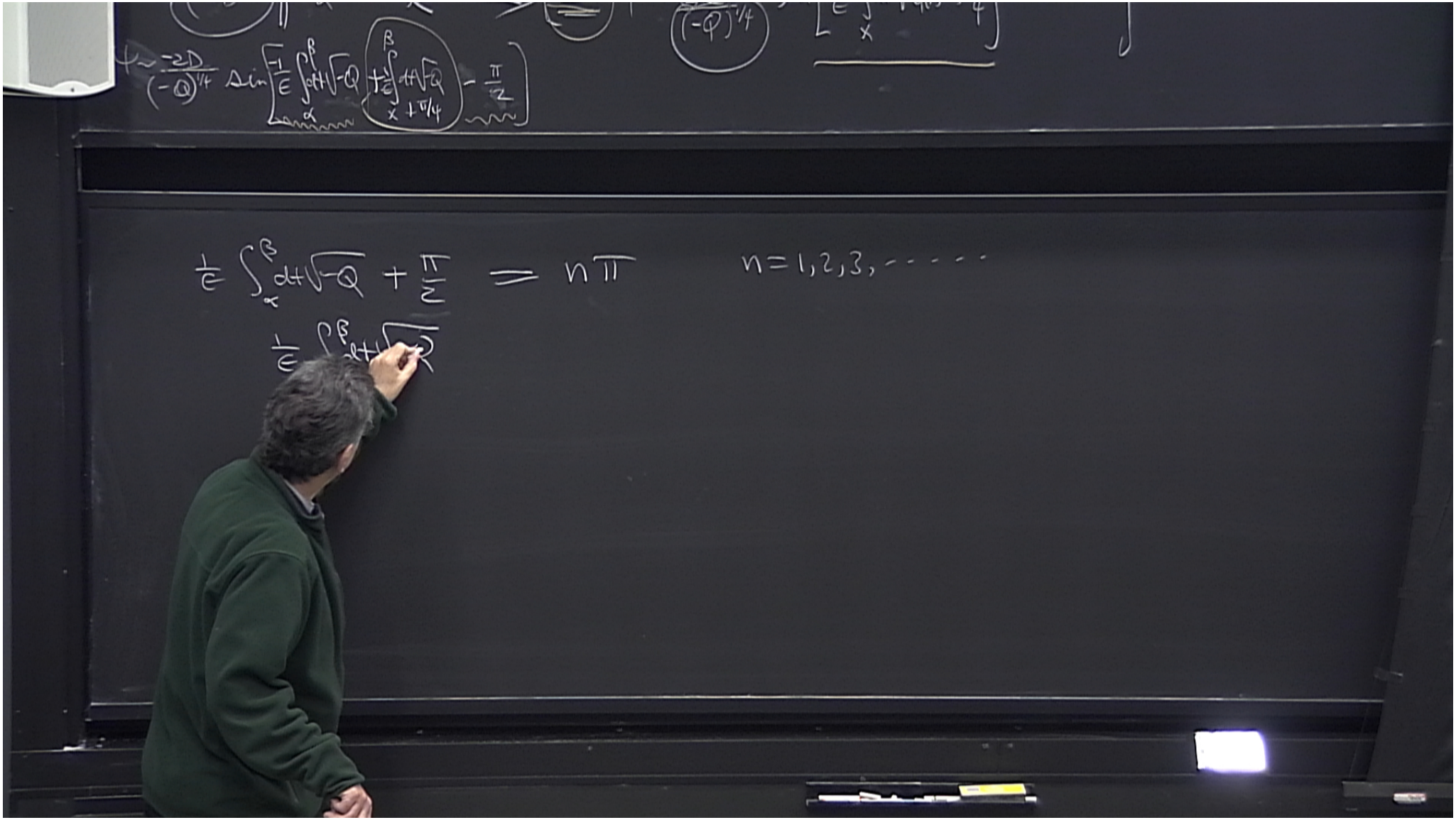
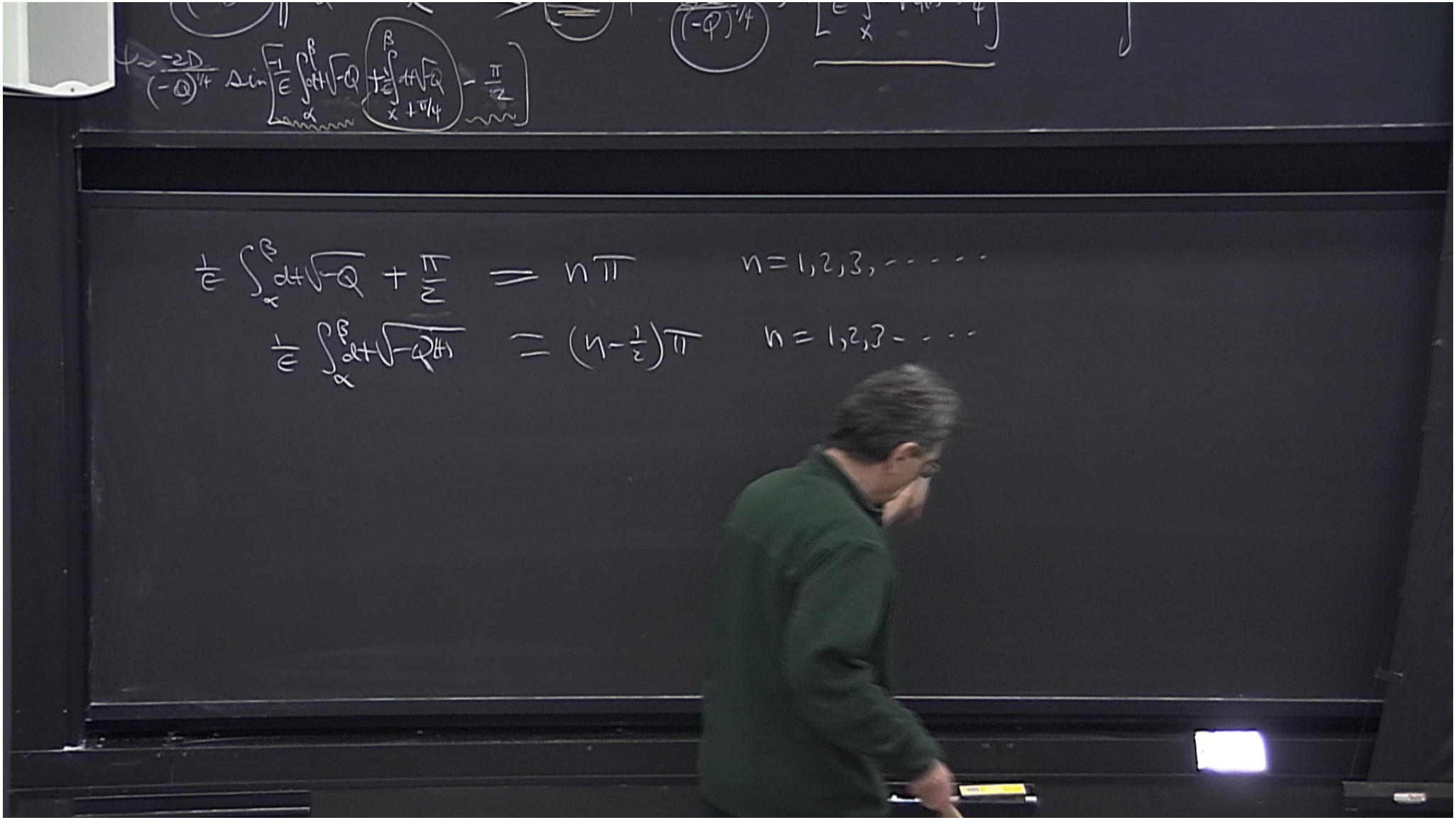
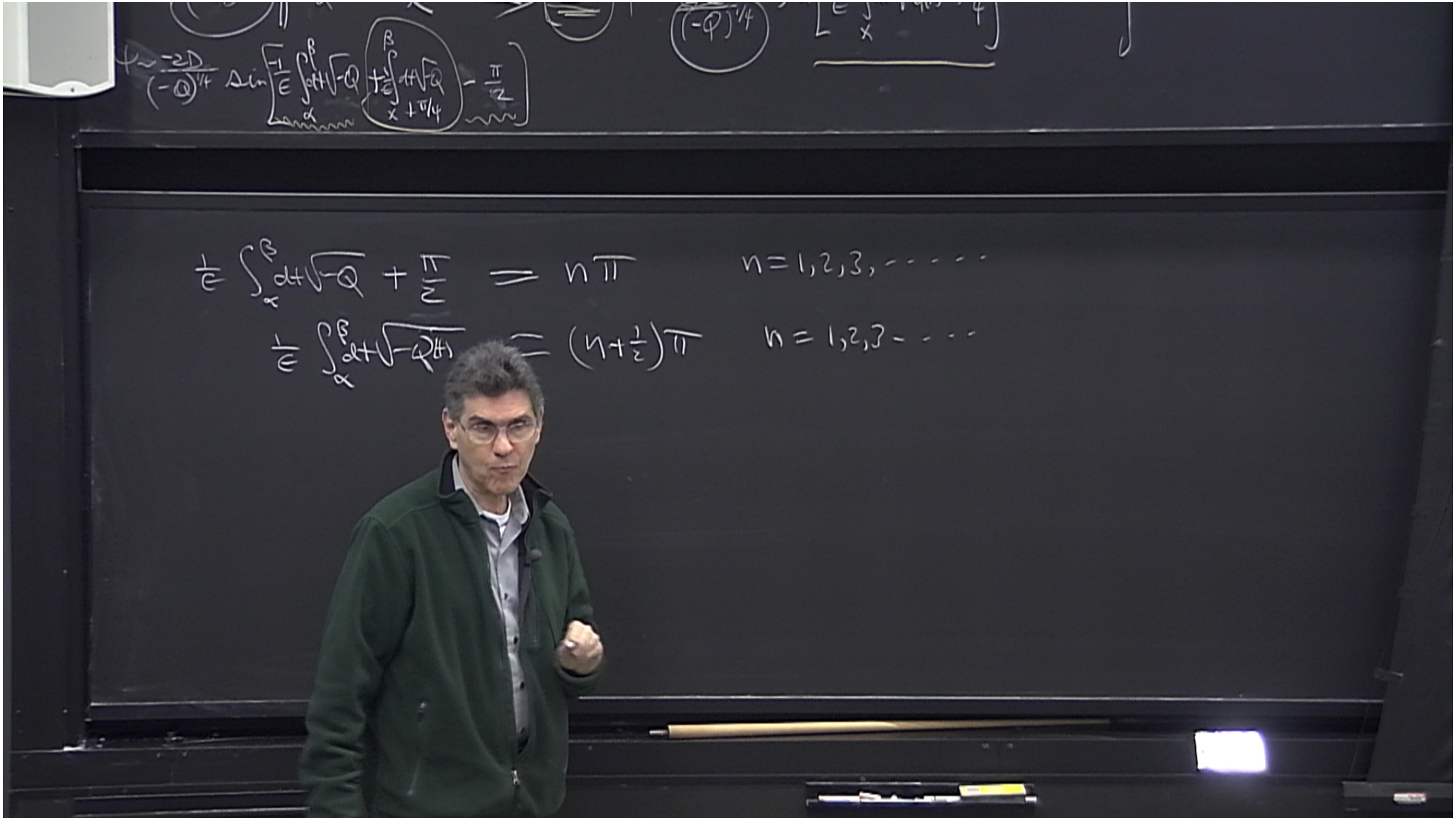
$\epsilon^2\psi'' = Q\psi$
I: $\psi \sim \frac{C}{Q^{1/4}} e^{-\frac{1}{\epsilon}\int_\beta^x dt\sqrt{Q(t)}}$ $\left(\psi \to 0 \text{ as } x \to \infty\right)$
II: $\psi \sim C$ blah blah Airy stuff etc.
III: $\psi \sim \frac{2C}{(-Q)^{1/4}} \sin\left[\frac{1}{\epsilon}\int_x^\beta dt\sqrt{-Q(t)} + \frac{\pi}{4}\right]$
Valid as $\epsilon \to 0$
same!

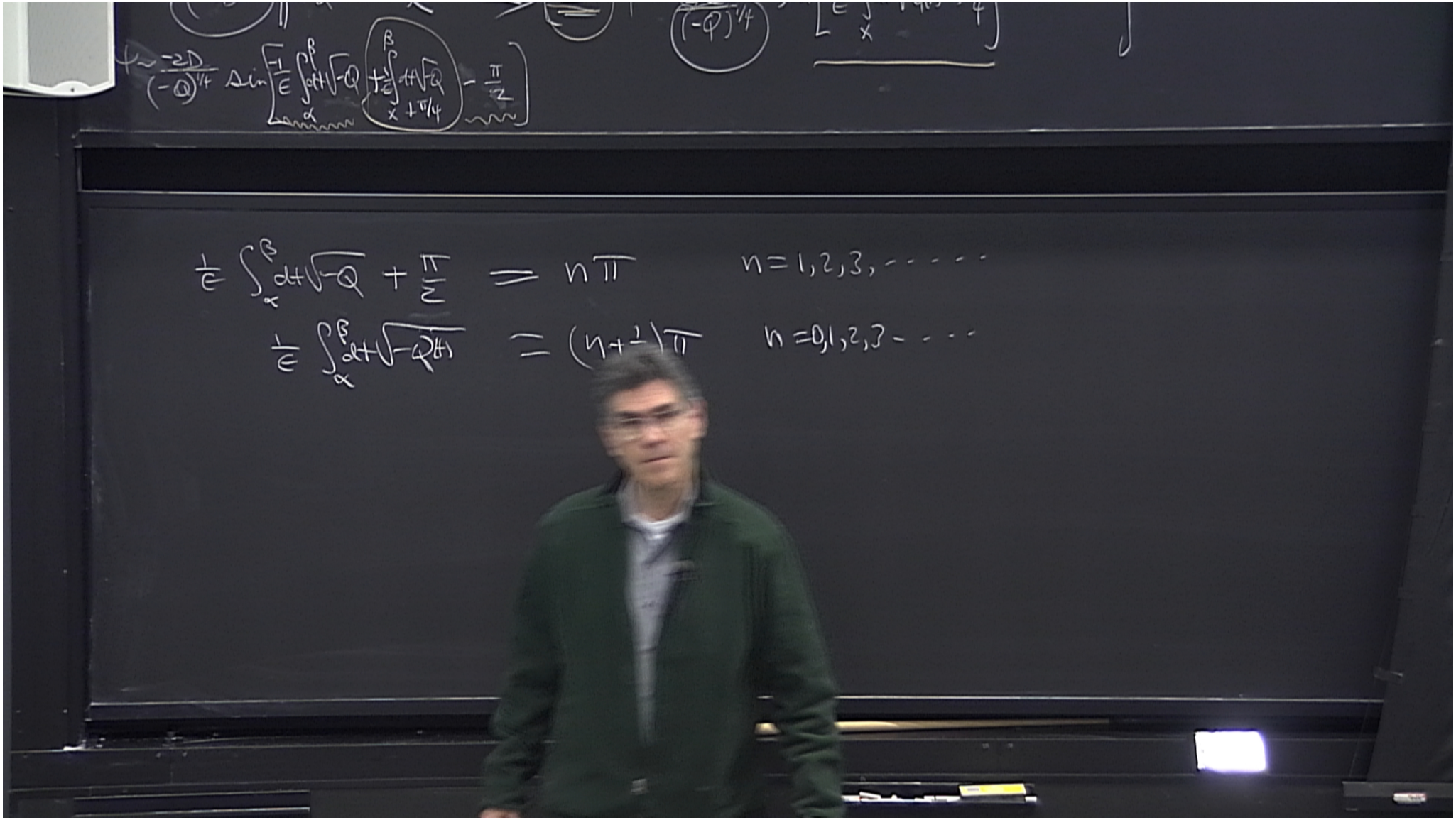

I
II
III
$Q = V(x) - E$
$\epsilon^2 \psi'' = Q\psi$
I $\psi \sim \frac{D}{Q^{1/4}} e^{-\frac{1}{\epsilon}\int_x^{\alpha} dt \sqrt{Q(t)}}$ $(\psi \to 0 \text{ as } x \to -\infty)$
II $\psi \sim$ Airy · · · · ·
III $\psi \sim \frac{2D}{(-Q)^{1/4}} \sin\left[\frac{1}{\epsilon}\int_{\alpha}^{x} dt \sqrt{Q} + \frac{\pi}{4}\right]$
$\psi \sim \frac{-2D}{(-Q)^{1/4}} \sin\left[\frac{1}{\epsilon}\int_{\alpha}^{\beta} dt \sqrt{-Q} + \frac{1}{\epsilon}\int_{x}^{\beta} dt \sqrt{Q} + \pi/4 - \frac{\pi}{2}\right]$
same!
I: $\psi \sim \frac{C}{Q^{1/4}} e^{-\frac{1}{\epsilon}\int_{\beta}^{x} dt \sqrt{Q(t)}}$ $(\psi \to 0 \text{ as } x \to \infty)$
II: $\psi \sim C$ blah blah Airy stuff etc.
III: $\psi \sim \frac{2C}{(-Q)^{1/4}} \sin\left[\frac{1}{\epsilon}\int_{x}^{\beta} dt \sqrt{Q(t)} + \frac{\pi}{4}\right]$
valid as $\epsilon \to 0^+$

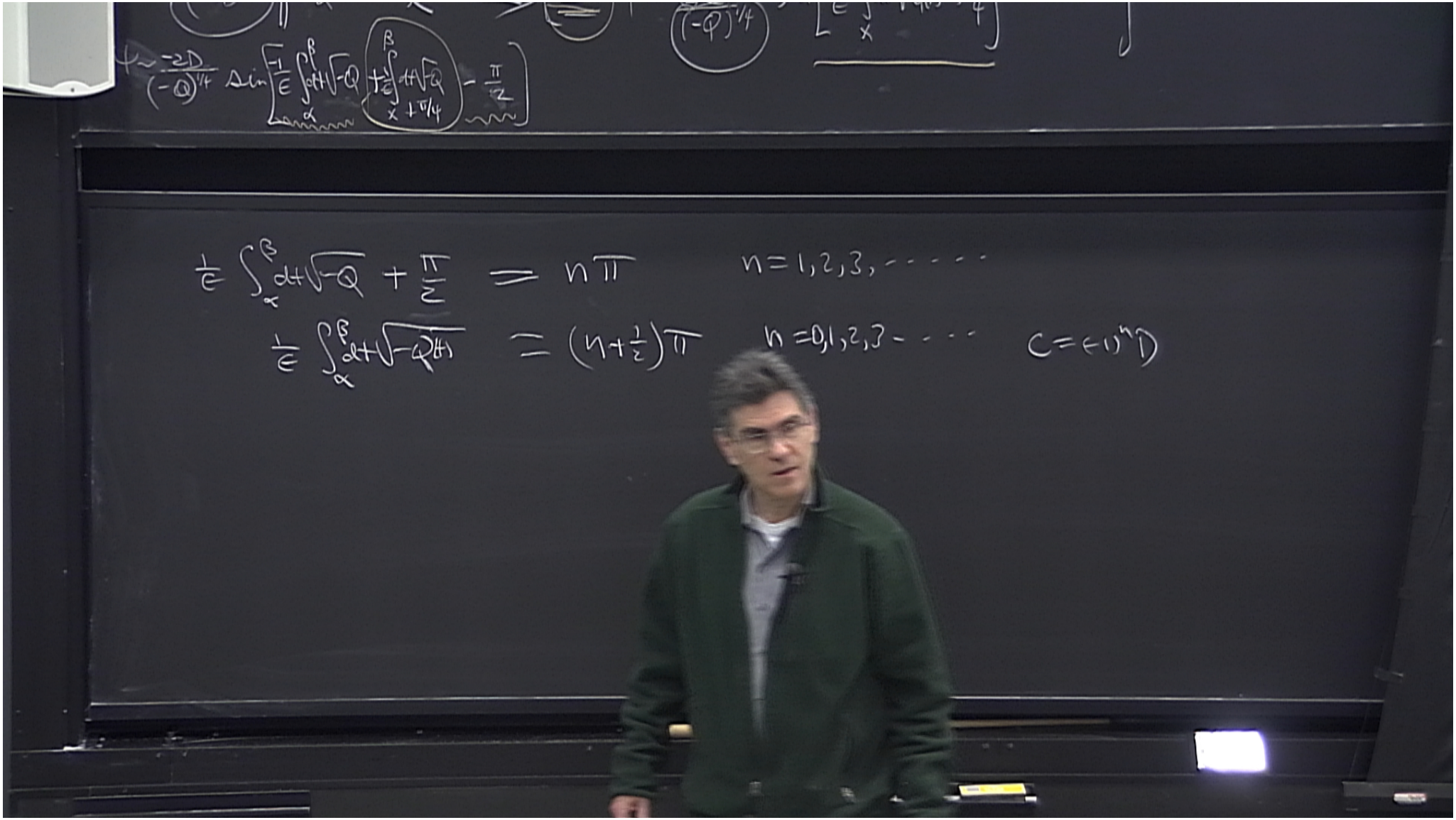

$\frac{1}{\epsilon}\int_{\alpha}^{\beta} dt\sqrt{-Q} + \frac{\pi}{2} = n\pi$

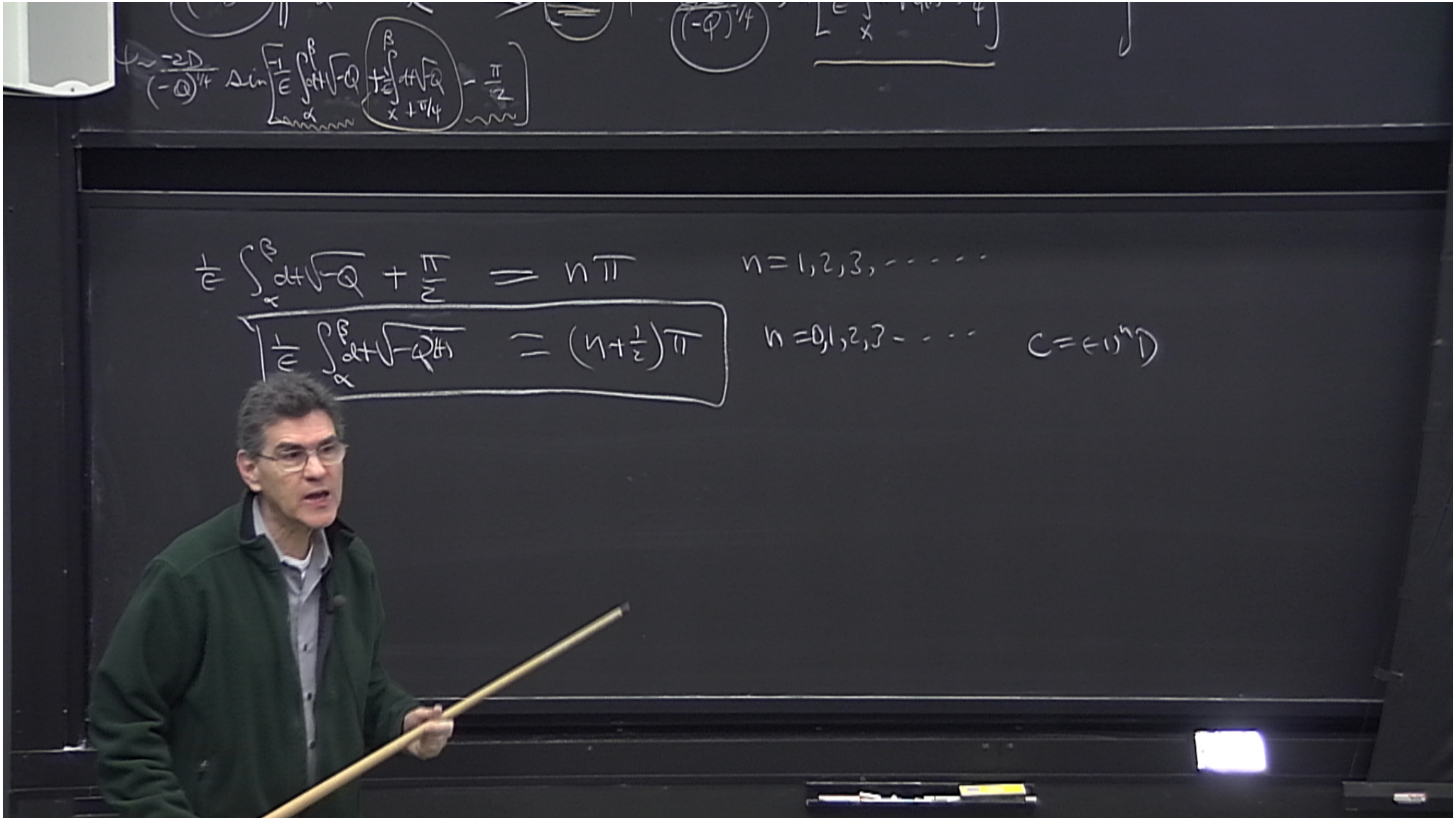

$\frac{1}{\epsilon}\int_{\alpha}^{\beta} dt\sqrt{-Q} + \frac{\pi}{2} = n\pi \qquad n = 1, 2, 3, \ldots$

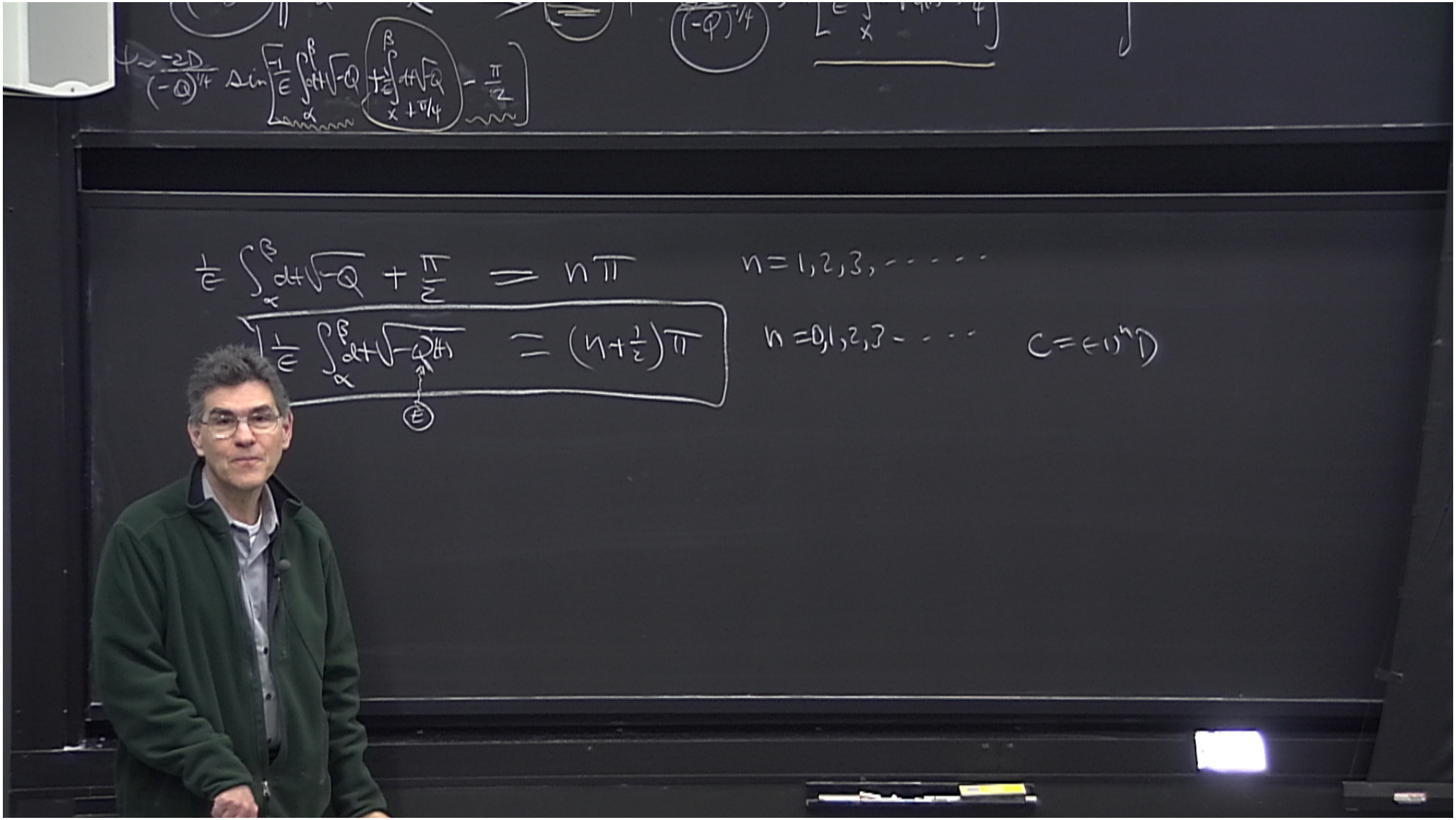

$\frac{1}{\epsilon}\int_{\alpha}^{\beta} dt\sqrt{-Q} + \frac{\pi}{2} = n\pi \qquad n = 1,2,3,\ldots$
$\frac{1}{\epsilon}\int_{\alpha}^{\beta} dt\sqrt{-Q(t)} = (n-\frac{1}{2})\pi \qquad n = 1,2,3\ldots$

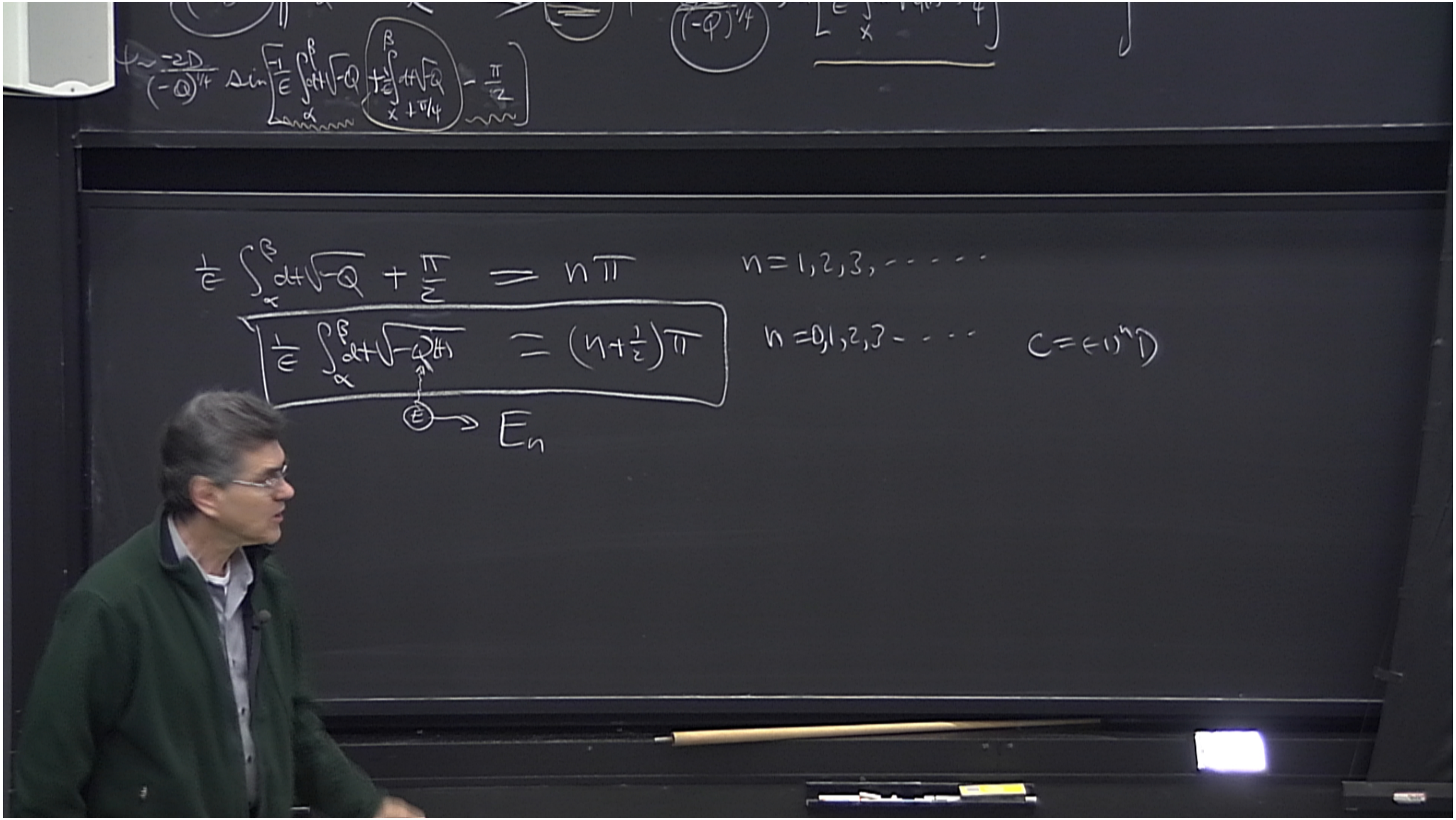

$\frac{1}{\epsilon}\int_\alpha^\beta dt\sqrt{-Q} + \frac{\pi}{2} = n\pi \qquad n = 1,2,3,\ldots$
$\frac{1}{\epsilon}\int_\alpha^\beta dt\sqrt{-Q(t)} = (n+\frac{1}{2})\pi \qquad n = 1,2,3\ldots$

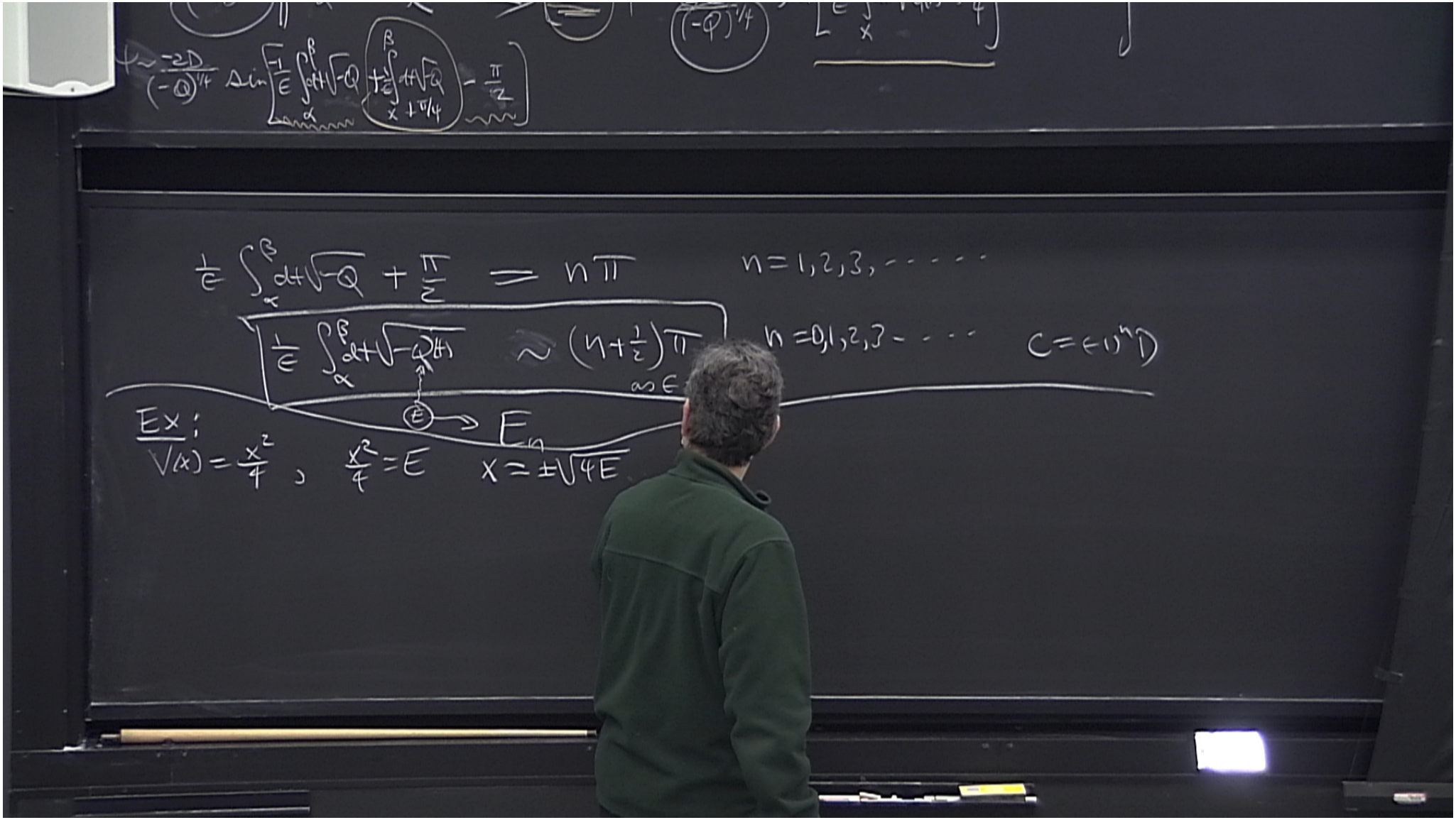

$\frac{1}{\epsilon}\int_{\alpha}^{\beta} dt\sqrt{-Q} + \frac{\pi}{2} = n\pi \qquad n = 1,2,3,\ldots$
$\frac{1}{\epsilon}\int_{\alpha}^{\beta} dt\sqrt{-Q(t)} = (n+\frac{1}{2})\pi \qquad n = 0,1,2,3\ldots$

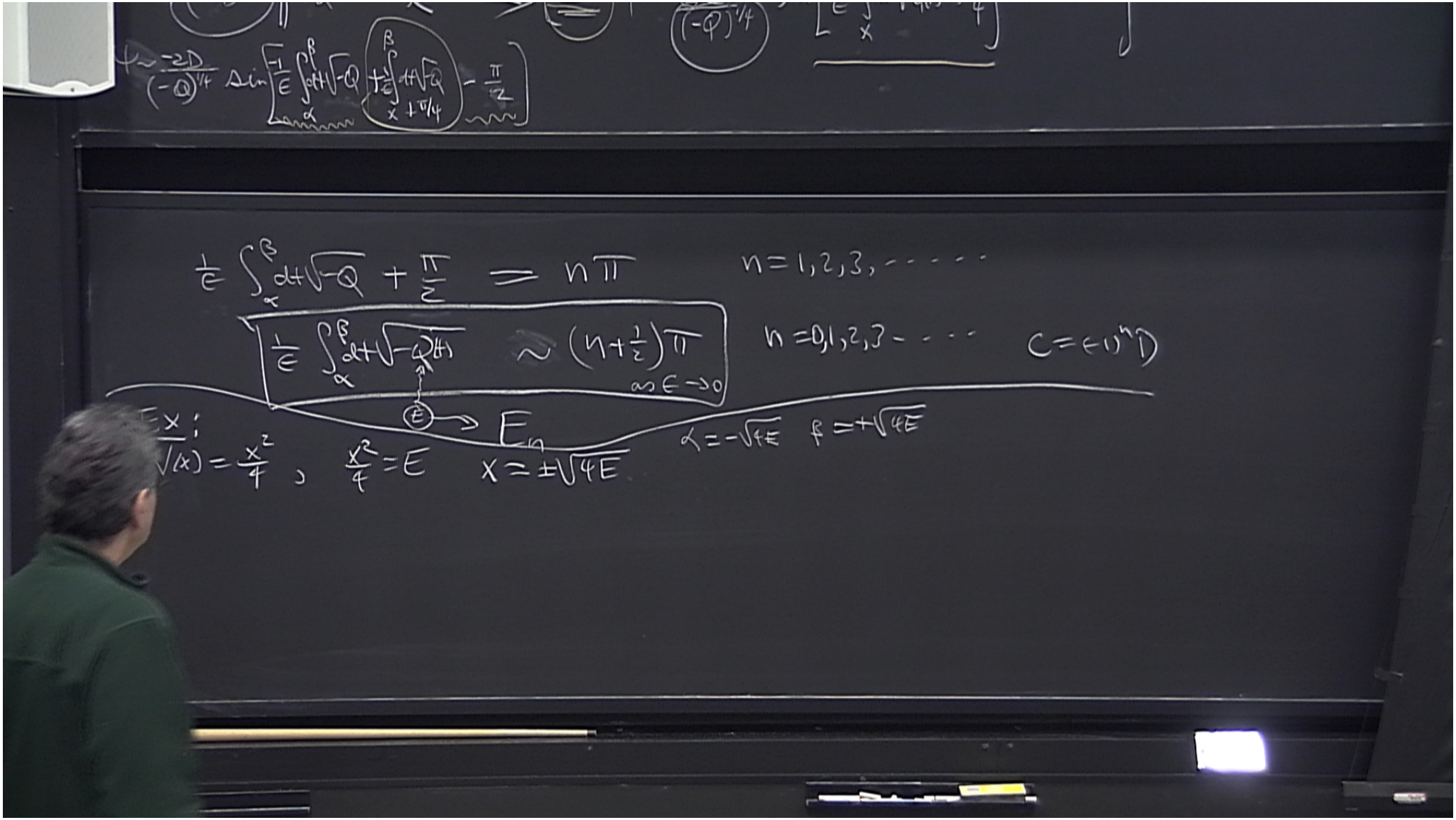

$\frac{1}{\epsilon}\int_{\alpha}^{\beta} dt\sqrt{-Q} + \frac{\pi}{2} = n\pi \qquad n = 1,2,3,\cdots$
$\frac{1}{\epsilon}\int_{\alpha}^{\beta} dt\sqrt{-Q(t)} = (n+\frac{1}{2})\pi \qquad n = 0,1,2,3\cdots \qquad C = \epsilon(n+\frac{1}{2})$

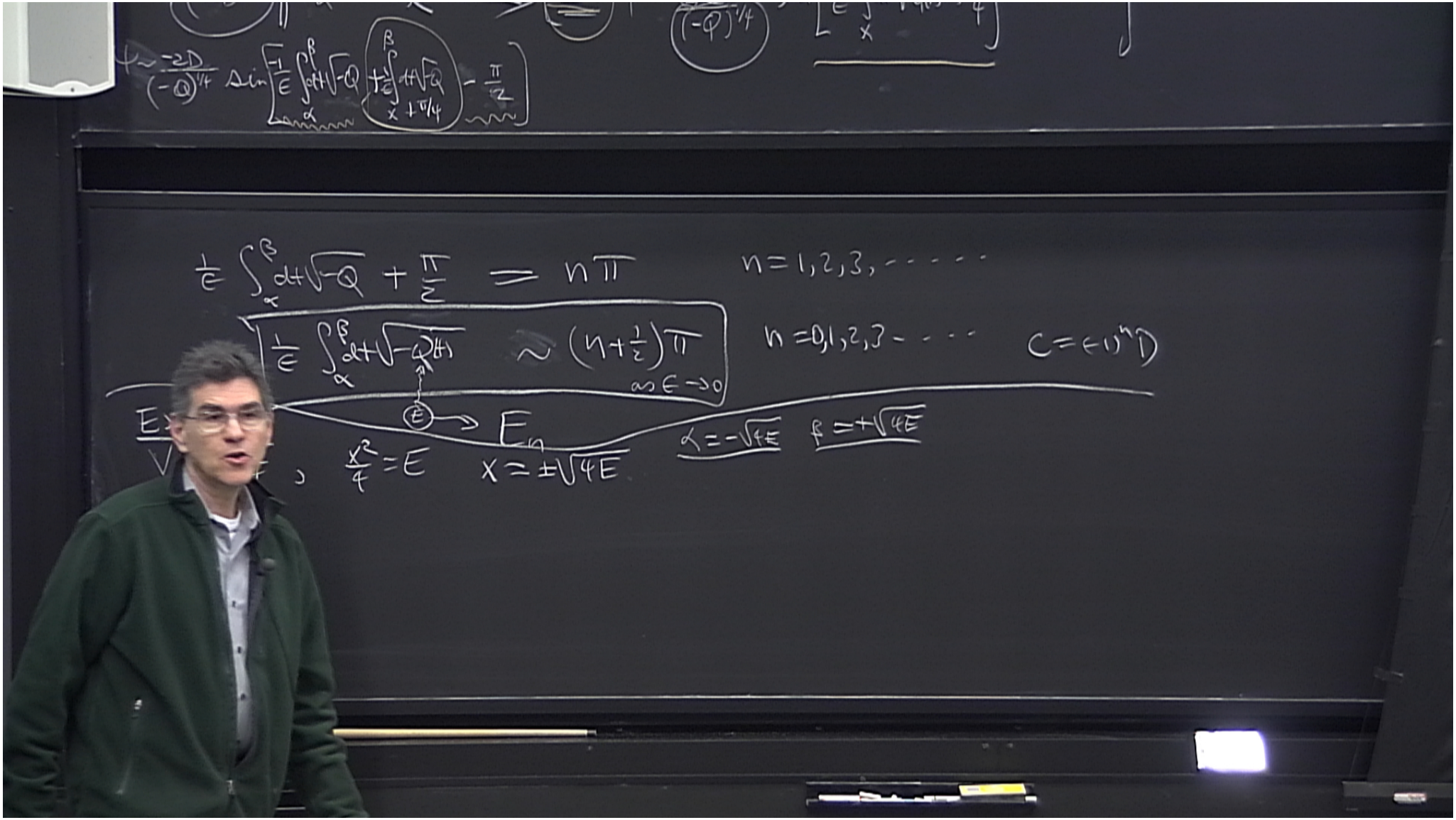

$\frac{1}{\epsilon}\int_\alpha^\beta dt\sqrt{-Q} + \frac{\pi}{2} = n\pi$
$n = 1, 2, 3, \ldots$
$\frac{1}{\epsilon}\int_\alpha^\beta dt\sqrt{-Q(t)} = (n+\frac{1}{2})\pi$
$n = 0, 1, 2, 3 \ldots$
$C = (-1)^n D$

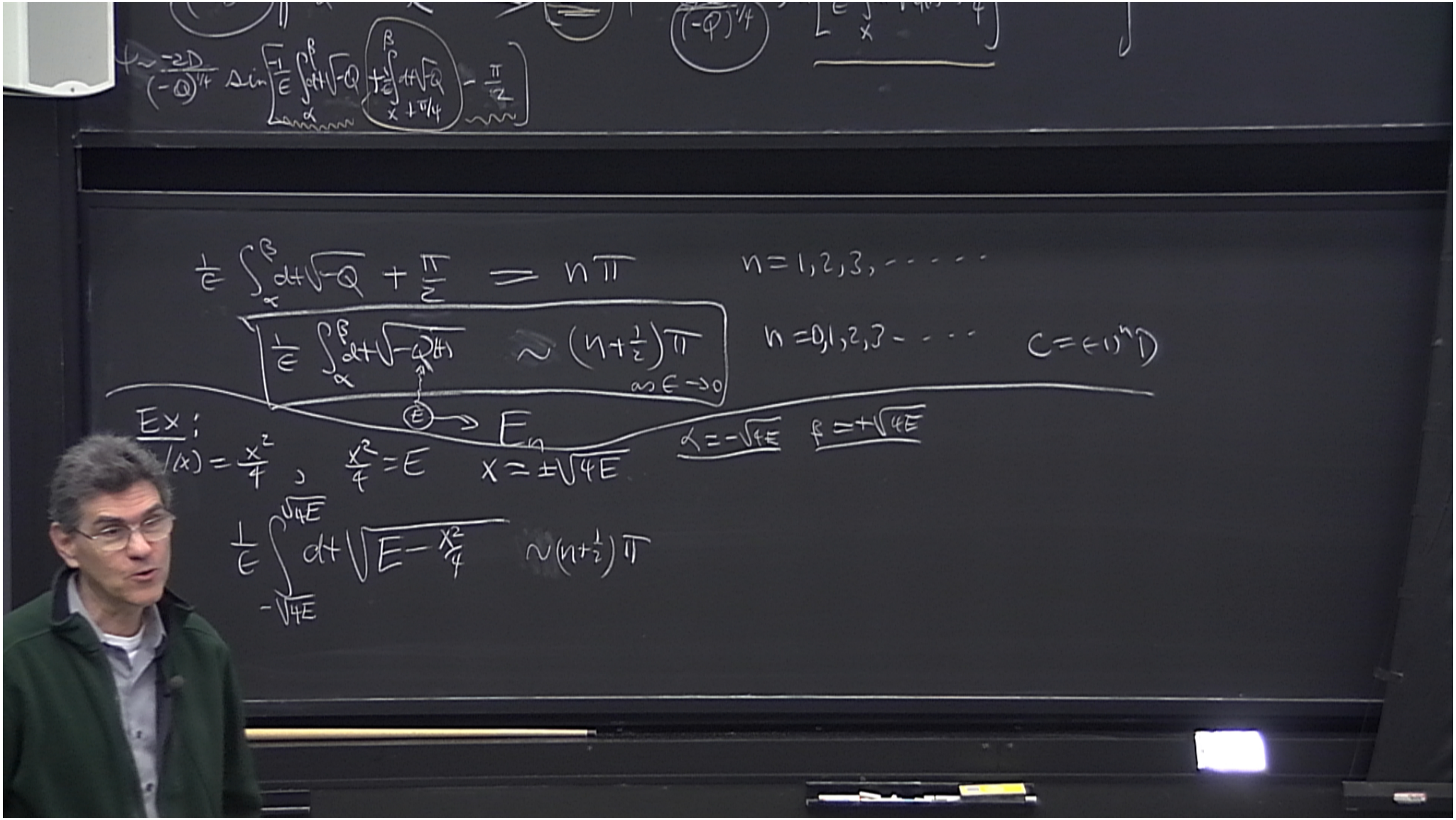

$\frac{1}{\epsilon}\int_{\alpha}^{\beta} dt\sqrt{-Q} + \frac{\pi}{2} = n\pi$
$n = 1, 2, 3, \cdots$
$\frac{1}{\epsilon}\int_{\alpha}^{\beta} dt\sqrt{-Q(t)} = (n+\frac{1}{2})\pi$
$n = 0, 1, 2, 3 \cdots$
$C = \epsilon(n)$
ϵ


$\frac{1}{\epsilon}\int_\alpha^\beta dt\sqrt{-Q} + \frac{\pi}{2} = n\pi$
$n = 1, 2, 3, \ldots$
$\frac{1}{\epsilon}\int_\alpha^\beta dt\sqrt{-Q(t)} = (n+\frac{1}{2})\pi$
$n = 0, 1, 2, 3 \ldots$
$C = \epsilon(n)$
E_n


$\frac{1}{\epsilon}\int_\alpha^\beta dt\sqrt{-Q} + \frac{\pi}{2} = n\pi$
$n = 1, 2, 3, \ldots$
$\frac{1}{\epsilon}\int_\alpha^\beta dt\sqrt{-Q(t)} \sim (n+\frac{1}{2})\pi$
$n = 0, 1, 2, 3 \ldots$
$C = \epsilon(n)$
E_n
Ex:
$V(x) = \frac{x^2}{4}$, $\frac{x^2}{4} = E$ $x = \pm\sqrt{4E}$


$\frac{1}{\epsilon}\int_\alpha^\beta dt\sqrt{-Q} + \frac{\pi}{2} = n\pi$
$n = 1,2,3,\ldots$
$\frac{1}{\epsilon}\int_\alpha^\beta dt\sqrt{-Q(t)} \sim (n+\frac{1}{2})\pi$
as $\epsilon \to 0$
$n = 0,1,2,3\ldots$
$C = \epsilon(n)$
$\epsilon \to E_n$
$\alpha = -\sqrt{4E}$
$\beta = +\sqrt{4E}$
$\frac{x^2}{4} = E$
$x = \pm\sqrt{4E}$


$\frac{1}{\epsilon}\int_\alpha^\beta dt\sqrt{-Q} + \frac{\pi}{2} = n\pi$
$n = 1,2,3,\ldots$
$\frac{1}{\epsilon}\int_\alpha^\beta dt\sqrt{-Q(t)} \sim (n+\frac{1}{2})\pi$
as $\epsilon \to 0$
$n = 0,1,2,3\ldots$
$C = \epsilon(n)$
E_n
$\frac{x^2}{4} = E$
$x = \pm\sqrt{4E}$
$\alpha = -\sqrt{4E}$
$\beta = +\sqrt{4E}$


$\frac{1}{\epsilon}\int_\alpha^\beta dt\sqrt{-Q} + \frac{\pi}{2} = n\pi$
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$\frac{1}{\epsilon}\int_\alpha^\beta dt\sqrt{-Q(t)} \sim (n+\frac{1}{2})\pi$
as $\epsilon \to 0$
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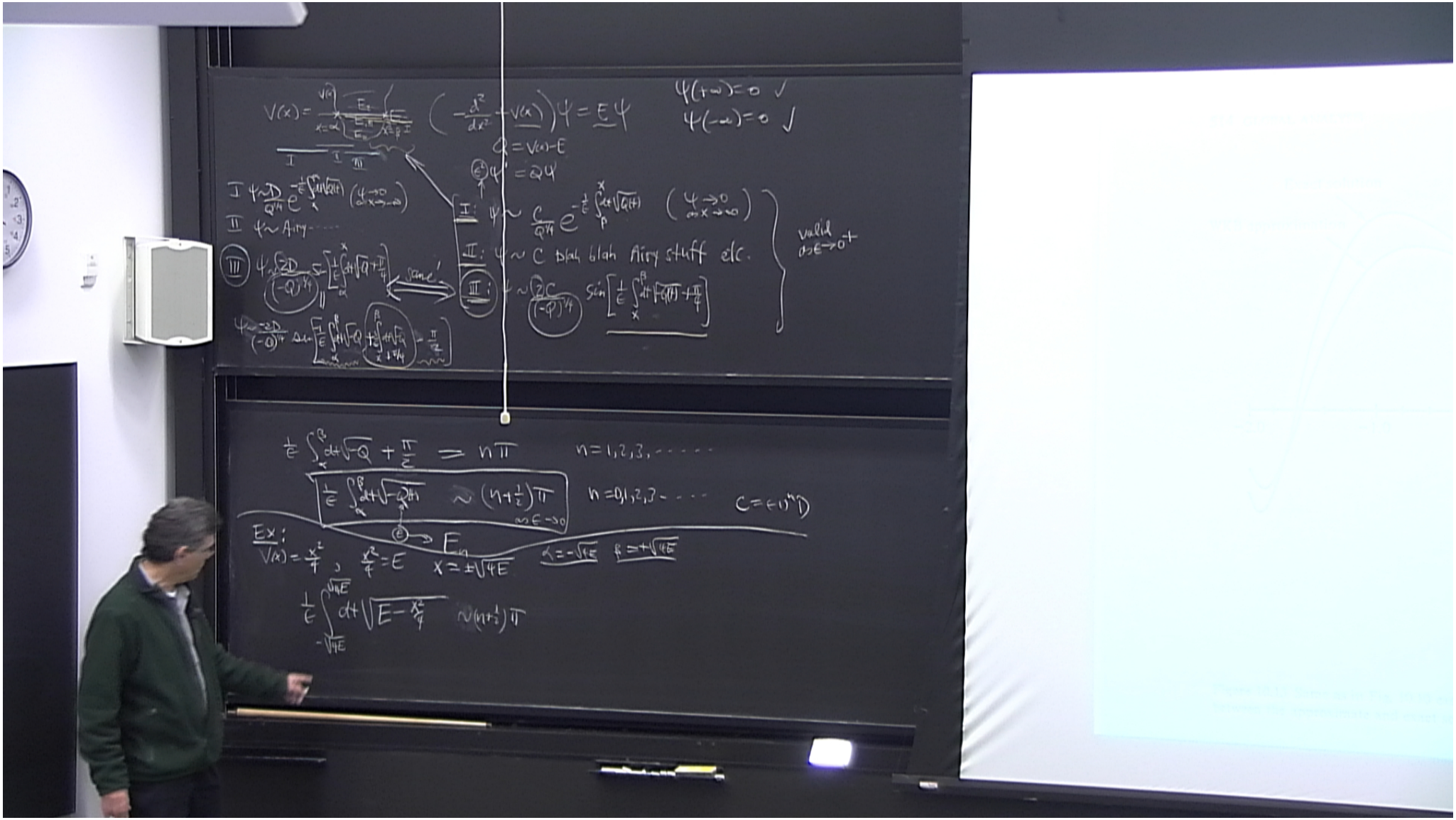
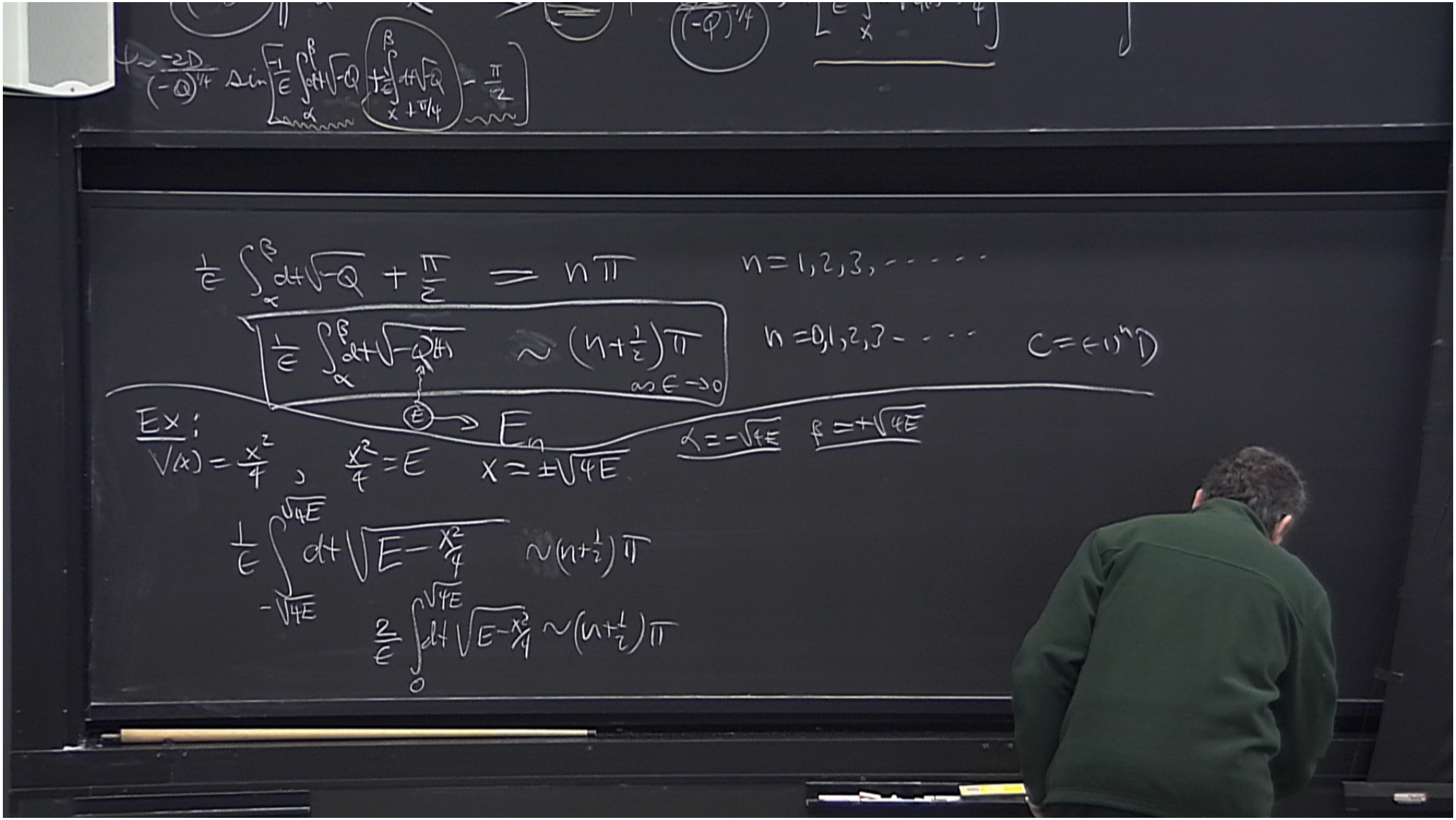
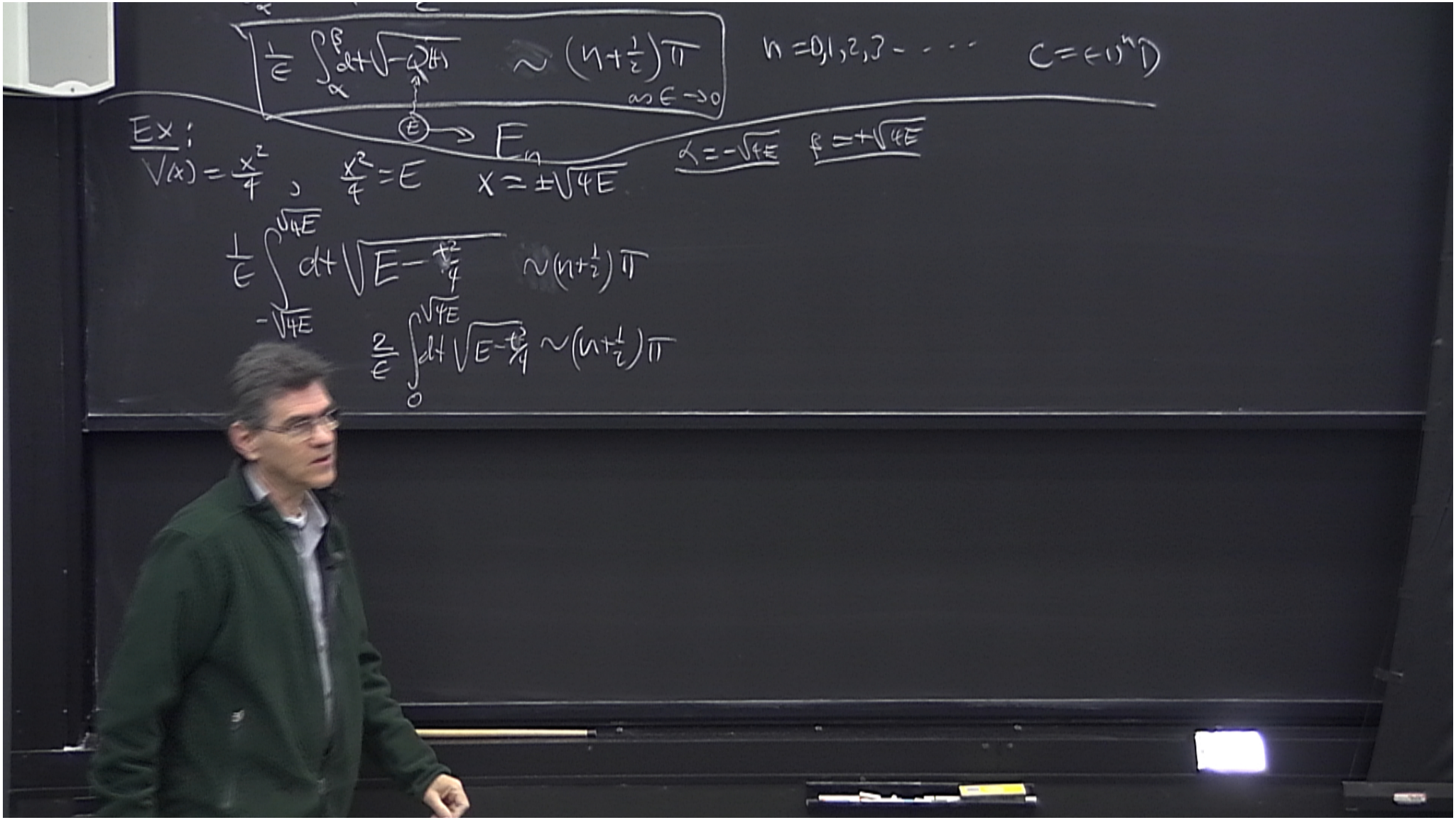
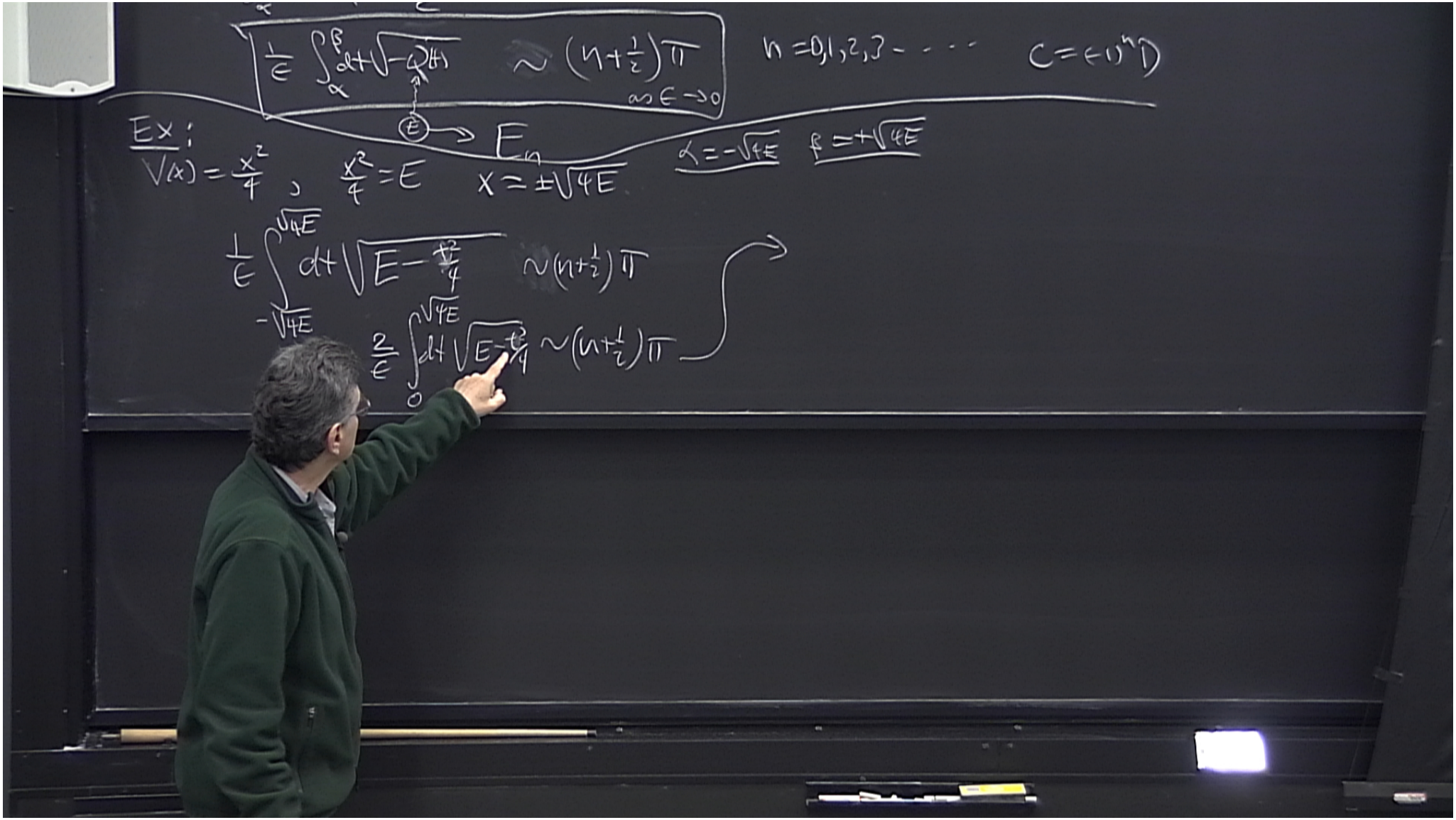
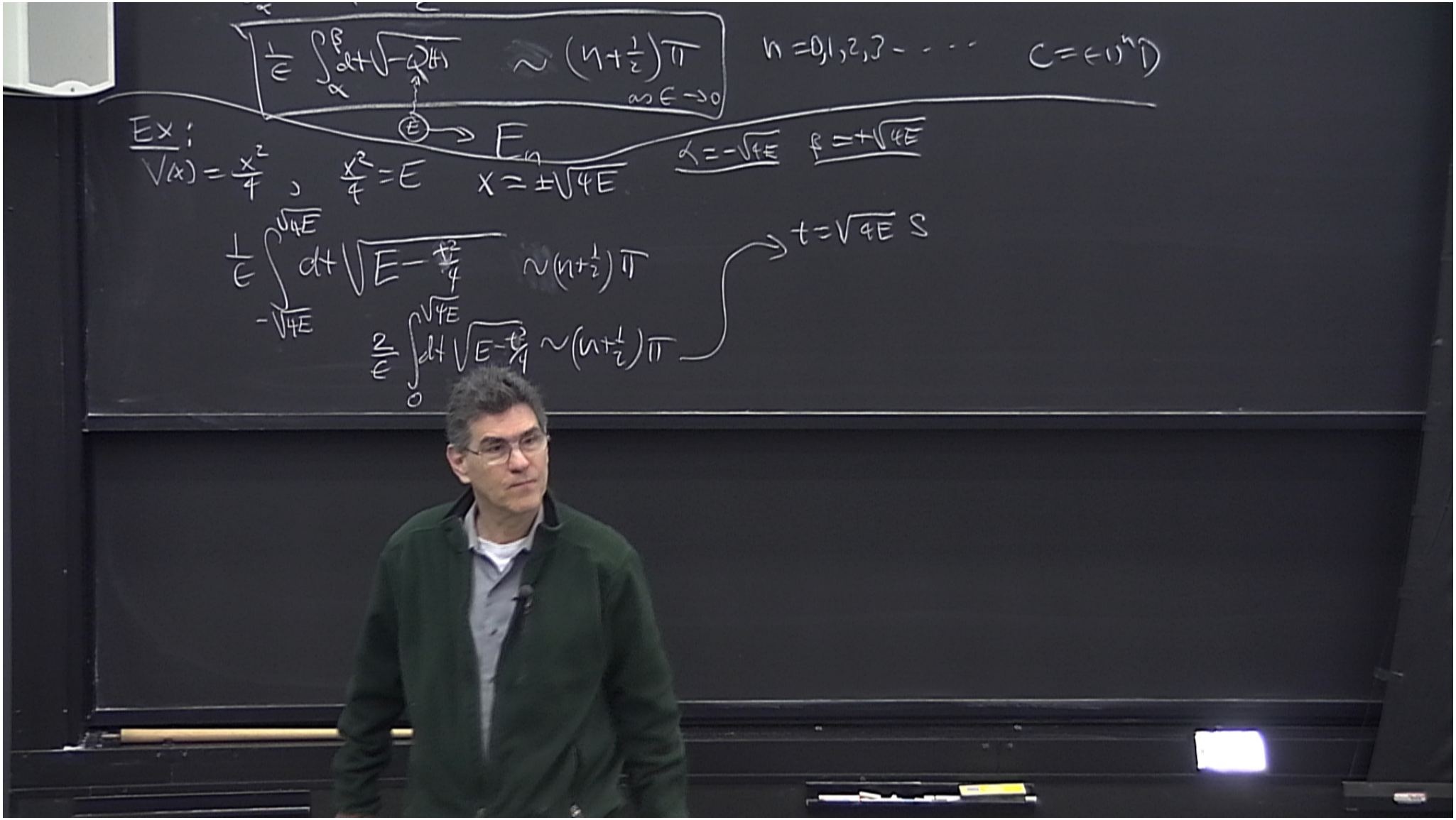
$\epsilon \to E_n$
Ex:
$V(x) = \frac{x^2}{4}$, $\frac{x^2}{4} = E$ $x = \pm\sqrt{4E}$
$\alpha = -\sqrt{4E}$ $\beta = +\sqrt{4E}$
$\frac{1}{\epsilon}\int_{-\sqrt{4E}}^{\sqrt{4E}} dt\sqrt{E - \frac{x^2}{4}} \sim (n+\frac{1}{2})\pi$

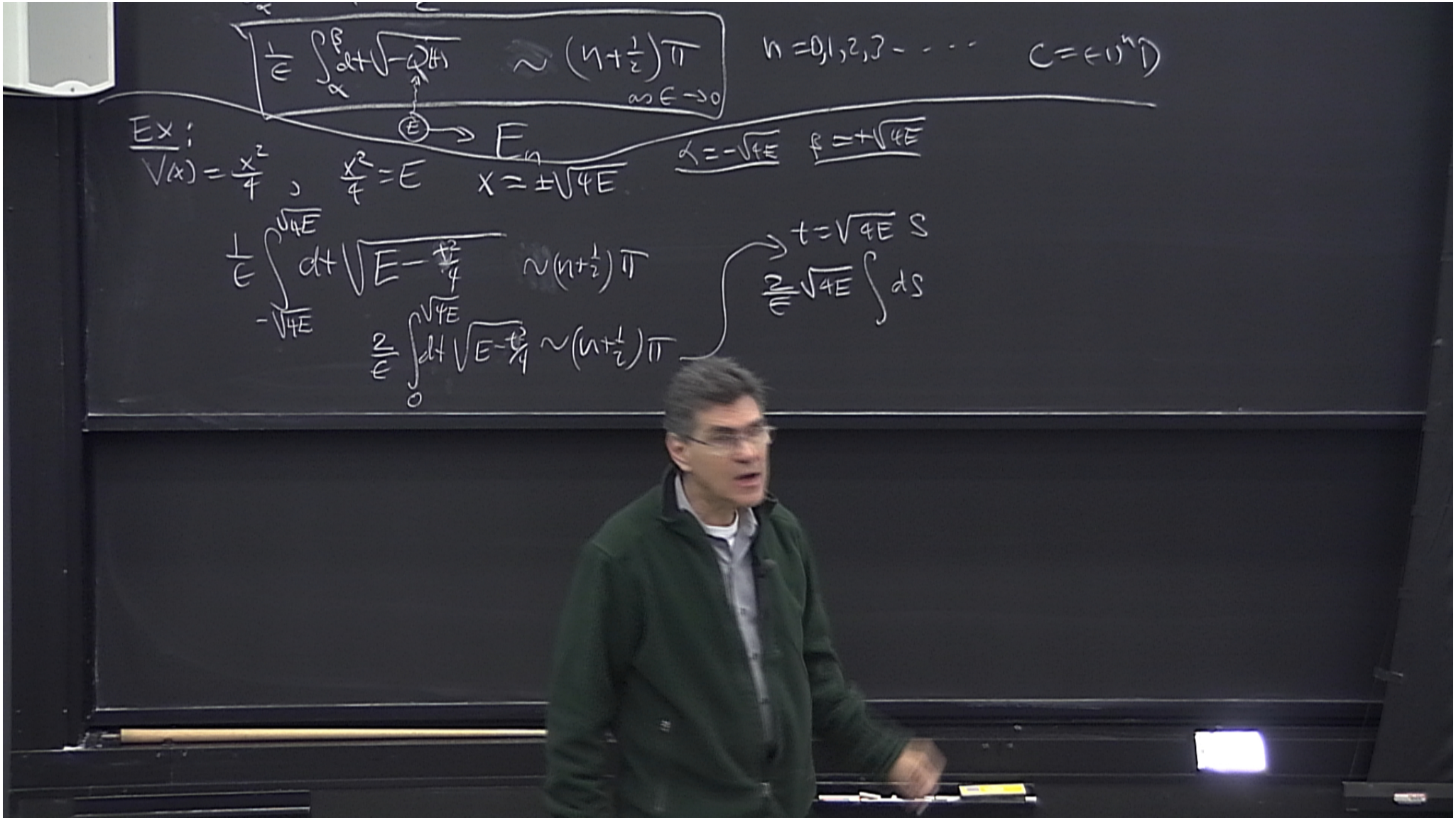

$V(x)$
$\left(-\frac{d^2}{dx^2}+V(x)\right)\psi=E\psi$
$\psi(+\infty)=0$ ✓
$\psi(-\infty)=0$ ✓
$Q=V(x)-E$
$\psi''=Q\psi$
I: $\psi\sim\frac{C}{Q^{1/4}}e^{-\frac{1}{\epsilon}\int_p^x\sqrt{Q(t)}}$ $\left(\psi\to 0,\ x\to-\infty\right)$
II: $\psi\sim$ C blah blah Airy stuff etc.
III: $\psi\sim\frac{2C}{(-Q)^{1/4}}\sin\left[\frac{1}{\epsilon}\int_x^\beta\sqrt{Q(t)}+\frac{\pi}{4}\right]$
Valid $\epsilon\to 0^+$
same!
$\frac{1}{\epsilon}\int_\alpha^\beta dt\sqrt{-Q}+\frac{\pi}{2}=n\pi$ $n=1,2,3,\ldots$
$\frac{1}{\epsilon}\int_\alpha^\beta dt\sqrt{-Q}\sim\left(n+\frac{1}{2}\right)\pi$ $n=0,1,2,3\ldots$ $C=(-1)^n D$
$\epsilon\to 0$
E_n
Ex: $V(x)=\frac{x^2}{4}$, $\frac{x^2}{4}=E$ $x=\pm\sqrt{4E}$ $\alpha=-\sqrt{4E}$ $\beta=+\sqrt{4E}$
$\frac{1}{\epsilon}\int_{-\sqrt{4E}}^{\sqrt{4E}}dt\sqrt{E-\frac{x^2}{4}}\sim\left(n+\frac{1}{2}\right)\pi$

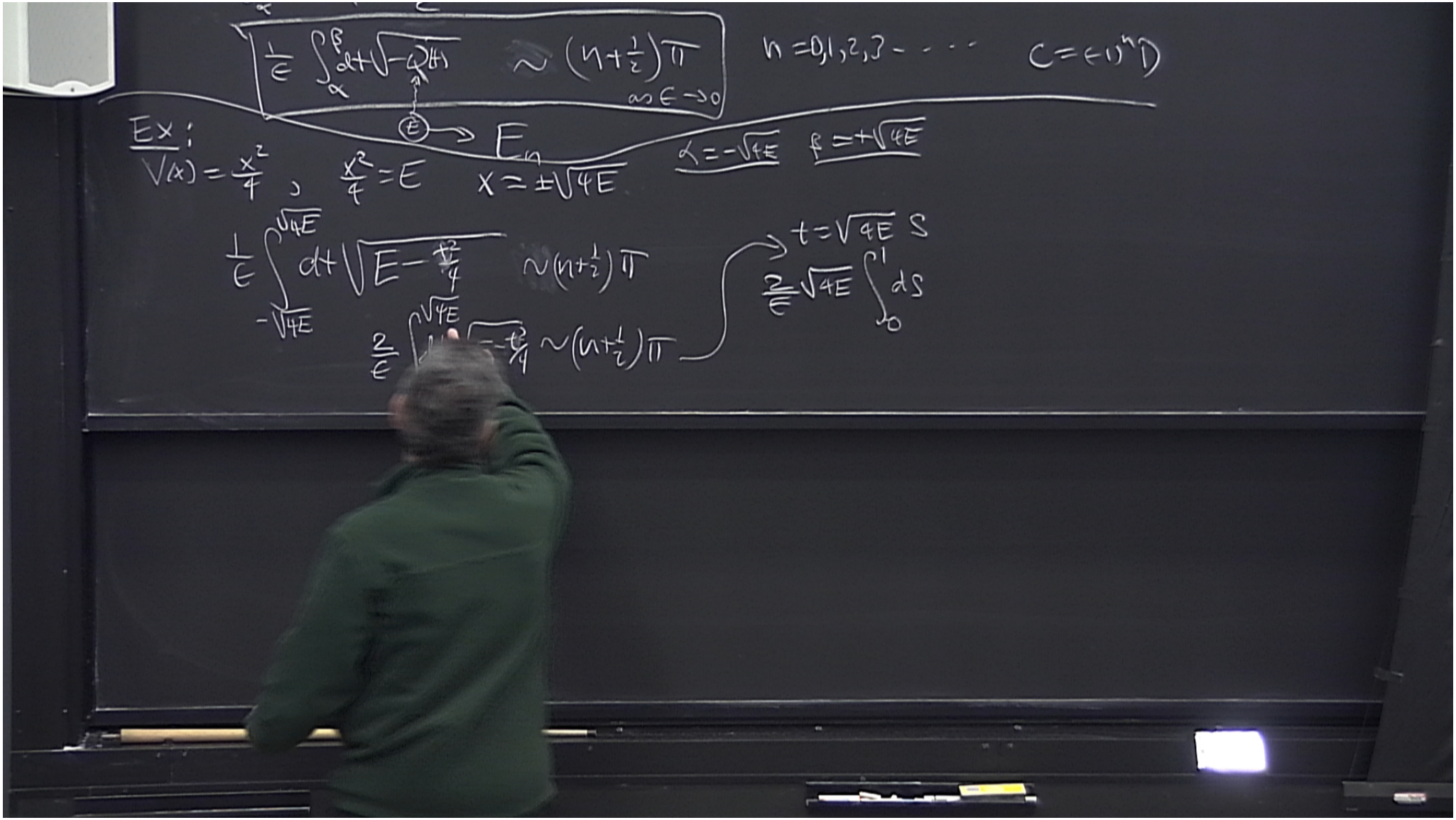

$\frac{1}{\epsilon}\int_\alpha^\beta dt\sqrt{-Q} + \frac{\pi}{2} = n\pi$
$n = 1,2,3,\ldots$
$\frac{1}{\epsilon}\int_\alpha^\beta dt\sqrt{-Q(t)} \sim (n+\frac{1}{2})\pi$
as $\epsilon \to 0$
$n = 0,1,2,3\ldots$
$C = \epsilon(\eta)$
$\epsilon \to E_n$
Ex:
$V(x) = \frac{x^2}{4}$, $\frac{x^2}{4} = E$ $x = \pm\sqrt{4E}$
$\alpha = -\sqrt{4E}$ $\beta = +\sqrt{4E}$
$\frac{1}{\epsilon}\int_{-\sqrt{4E}}^{\sqrt{4E}} dt\sqrt{E - \frac{x^2}{4}} \sim (n+\frac{1}{2})\pi$
$\frac{2}{\epsilon}\int_0^{\sqrt{4E}} dt\sqrt{E - \frac{x^2}{4}} \sim (n+\frac{1}{2})\pi$


$\frac{1}{\epsilon}\int_\alpha^\beta dt\sqrt{-q(t)}\sim(n+\frac{1}{2})\pi$
as $\epsilon\to 0$
$n=0,1,2,3\cdots$
$C=\epsilon U^n D$
E
E_n
Ex:
$V(x)=\frac{x^2}{4}$, $\frac{x^2}{4}=E$ $x=\pm\sqrt{4E}$
$\alpha=-\sqrt{4E}$ $\beta=+\sqrt{4E}$
$\frac{1}{\epsilon}\int_{-\sqrt{4E}}^{\sqrt{4E}} dt\sqrt{E-\frac{t^2}{4}}\sim(n+\frac{1}{2})\pi$
$\frac{2}{\epsilon}\int_0^{\sqrt{4E}} dt\sqrt{E-\frac{t^2}{4}}\sim(n+\frac{1}{2})\pi$


$\frac{1}{\epsilon}\int_\alpha^\beta dt\sqrt{-Q(t)} \sim (n+\frac{1}{2})\pi$
as $\epsilon \to 0$
$n = 0, 1, 2, 3 - - - -$
$C = (-1)^n D$
$E \to E_n$
Ex:
$V(x) = \frac{x^2}{4}$, $\frac{x^2}{4} = E$ $x = \pm\sqrt{4E}$
$\alpha = -\sqrt{4E}$ $\beta = +\sqrt{4E}$
$\frac{1}{\epsilon}\int_{-\sqrt{4E}}^{\sqrt{4E}} dt\sqrt{E - \frac{t^2}{4}} \sim (n+\frac{1}{2})\pi$
$\frac{2}{\epsilon}\int_0^{\sqrt{4E}} dt\sqrt{E - \frac{t^2}{4}} \sim (n+\frac{1}{2})\pi$


$\frac{1}{\epsilon}\int_\alpha^\beta dt\sqrt{-q(t)}\sim(n+\frac{1}{2})\pi$
as $\epsilon\to 0$
$n=0,1,2,3\cdots$
$C=\epsilon U^n D$
E
E_n
Ex:
$V(x)=\frac{x^2}{4}$, $\frac{x^2}{4}=E$ $x=\pm\sqrt{4E}$
$\alpha=-\sqrt{4E}$ $\beta=+\sqrt{4E}$
$\frac{1}{\epsilon}\int_{-\sqrt{4E}}^{\sqrt{4E}} dt\sqrt{E-\frac{t^2}{4}}\sim(n+\frac{1}{2})\pi$
$\frac{2}{\epsilon}\int_0^{\sqrt{4E}} dt\sqrt{E-\frac{t^2}{4}}\sim(n+\frac{1}{2})\pi$
$t=\sqrt{4E}\,s$


Ex:
$V(x) = \frac{x^2}{4}$
$\alpha = -\sqrt{4E}$
$\beta = +\sqrt{4E}$
$t = \sqrt{4E}\, s$


$\frac{1}{\epsilon}\int_\alpha^\beta dt\sqrt{-q(t)}\sim\left(n+\frac{1}{2}\right)\pi$
as $\epsilon\to 0$
$n=0,1,2,3\cdots$
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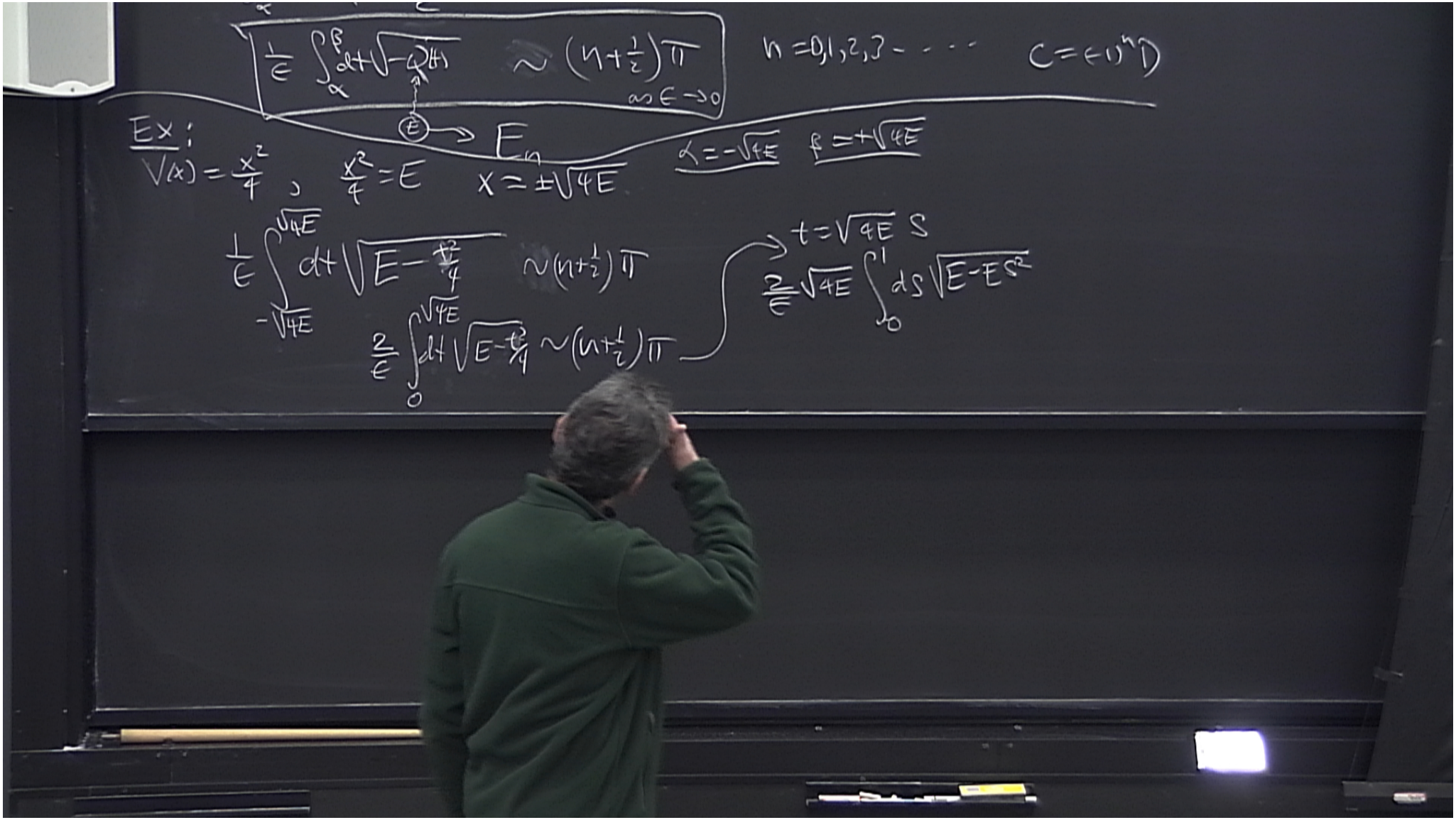
Ex:
$V(x)=\frac{x^2}{4}$, $\frac{x^2}{4}=E$ $x=\pm\sqrt{4E}$
$\alpha=-\sqrt{4E}$ $\beta=+\sqrt{4E}$
$\frac{1}{\epsilon}\int_{-\sqrt{4E}}^{\sqrt{4E}} dt\sqrt{E-\frac{t^2}{4}}\sim\left(n+\frac{1}{2}\right)\pi$
$\frac{2}{\epsilon}\int_{-\sqrt{4E}}^{\sqrt{4E}}\sqrt{-\frac{t^2}{4}}\sim\left(n+\frac{1}{2}\right)\pi$
$t=\sqrt{4E}\,s$
$\frac{2}{\epsilon}\sqrt{4E}\int_0^1 ds$


$\frac{1}{\epsilon}\int_\alpha^\beta dt\sqrt{-q(t)} \sim (n+\frac{1}{2})\pi$
as $\epsilon \to 0$
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Ex:
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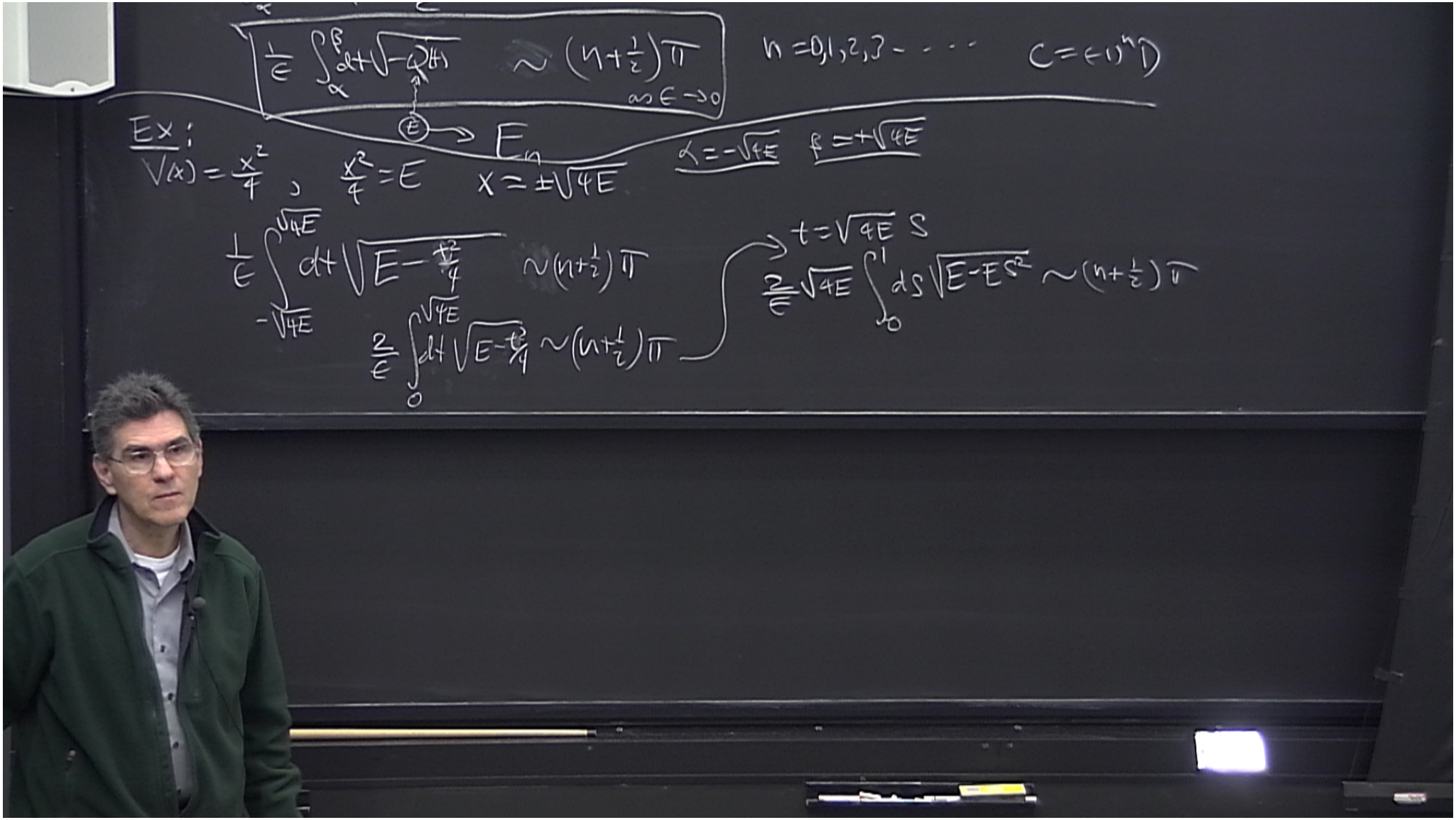
$\alpha = -\sqrt{4E}$
$\beta = +\sqrt{4E}$
$\frac{1}{\epsilon}\int_{-\sqrt{4E}}^{\sqrt{4E}} dt\sqrt{E-\frac{t^2}{4}} \sim (n+\frac{1}{2})\pi$
$\frac{2}{\epsilon}\int_0^{\sqrt{4E}} dt\sqrt{E-\frac{t^2}{4}} \sim (n+\frac{1}{2})\pi$
$t = \sqrt{4E}\,s$
$\frac{2}{\epsilon}\sqrt{4E}\int_0^1 ds\sqrt{E-Es^2}$


$\frac{1}{\epsilon}\int_\alpha^\beta dt\sqrt{-Q(t)} \sim (n+\frac{1}{2})\pi$
as $\epsilon \to 0$
$n = 0, 1, 2, 3 \ldots$
$C = (-1)^n D$
$E \to E_n$
Ex:
$V(x) = \frac{x^2}{4}$, $\frac{x^2}{4} = E$ $x = \pm\sqrt{4E}$
$\alpha = -\sqrt{4E}$ $\beta = +\sqrt{4E}$
$\frac{1}{\epsilon}\int_{-\sqrt{4E}}^{\sqrt{4E}} dt\sqrt{E - \frac{t^2}{4}} \sim (n+\frac{1}{2})\pi$
$\frac{2}{\epsilon}\int_0^{\sqrt{4E}} dt\sqrt{E - \frac{t^2}{4}} \sim (n+\frac{1}{2})\pi$
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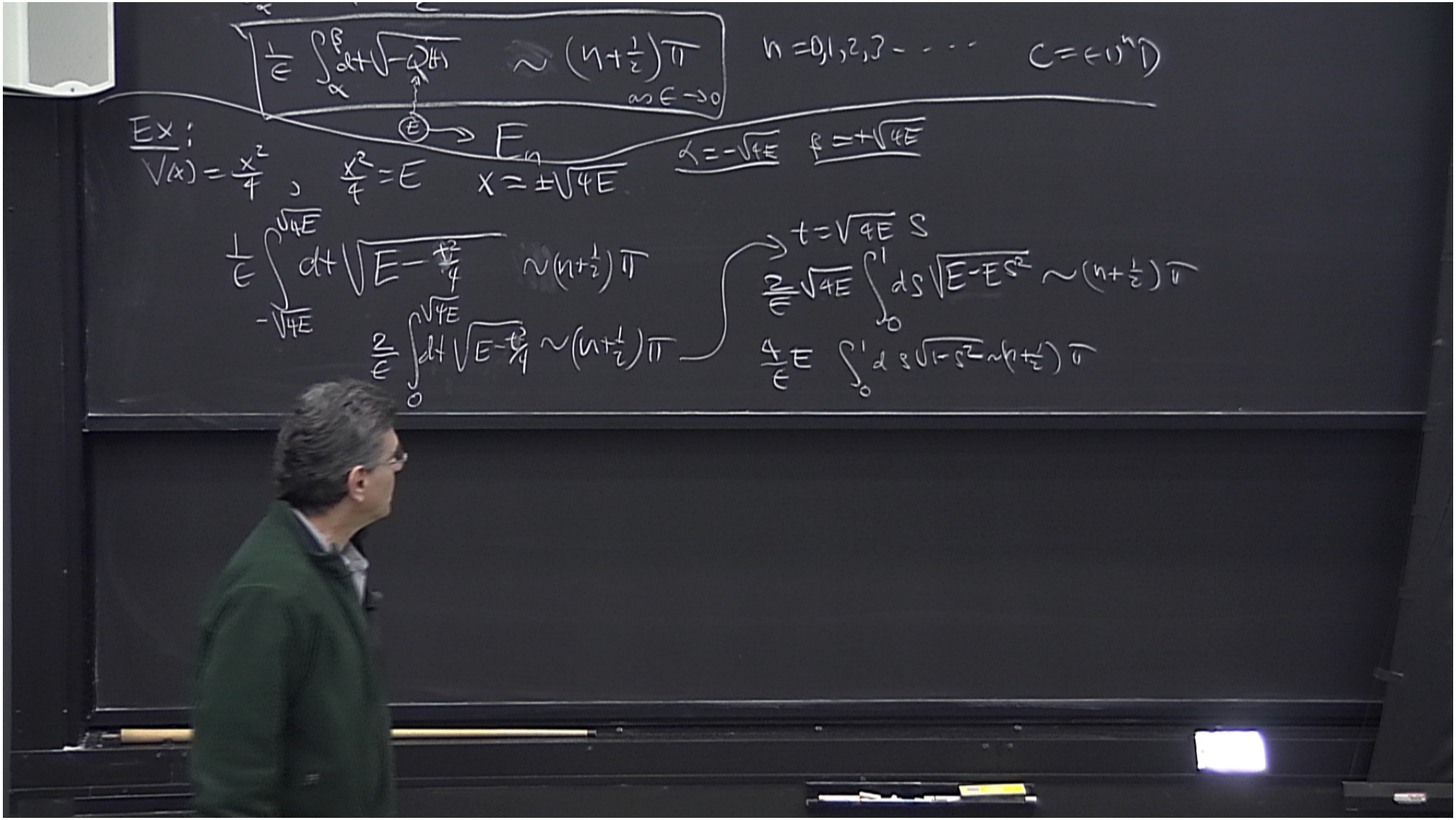
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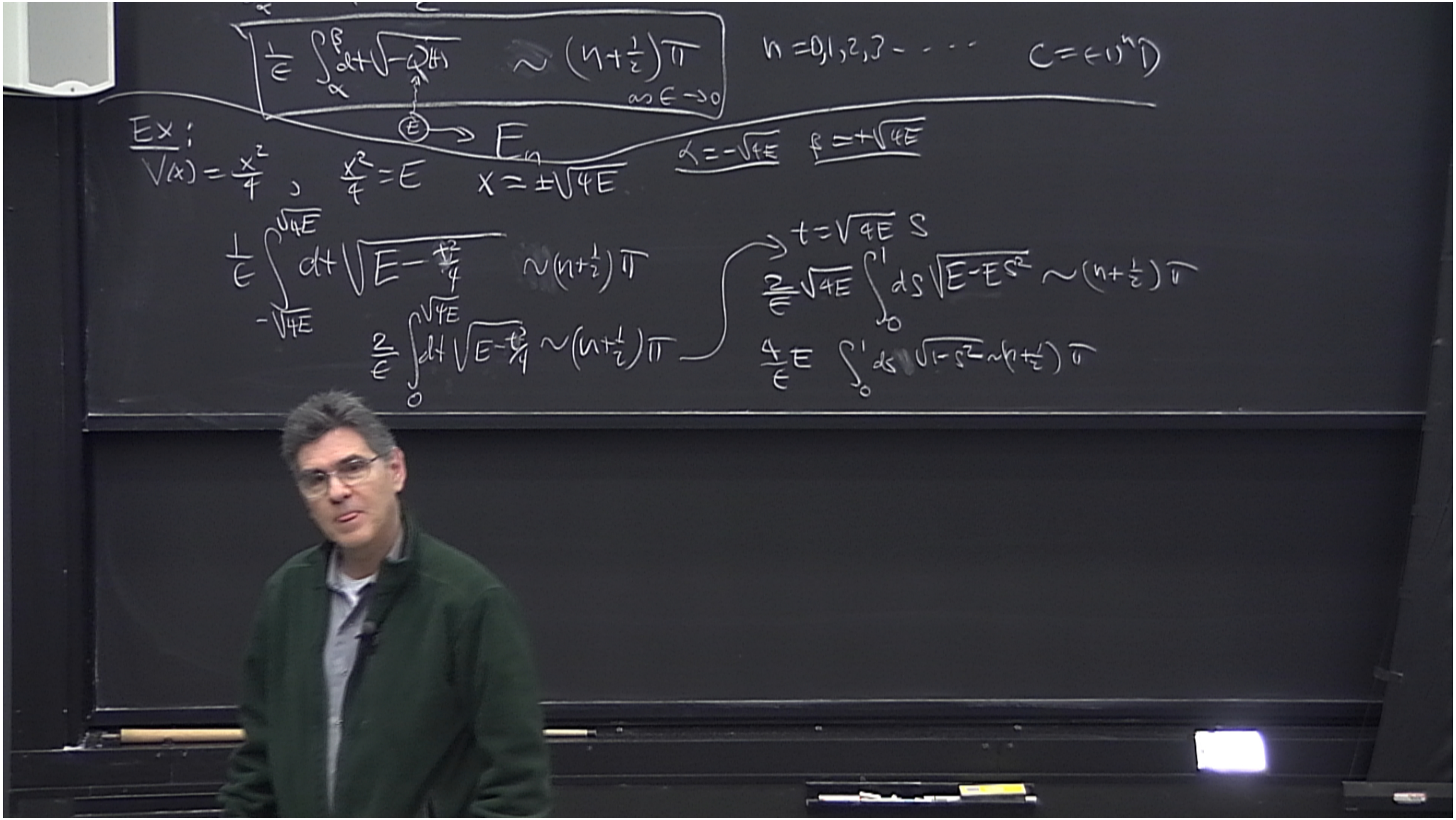
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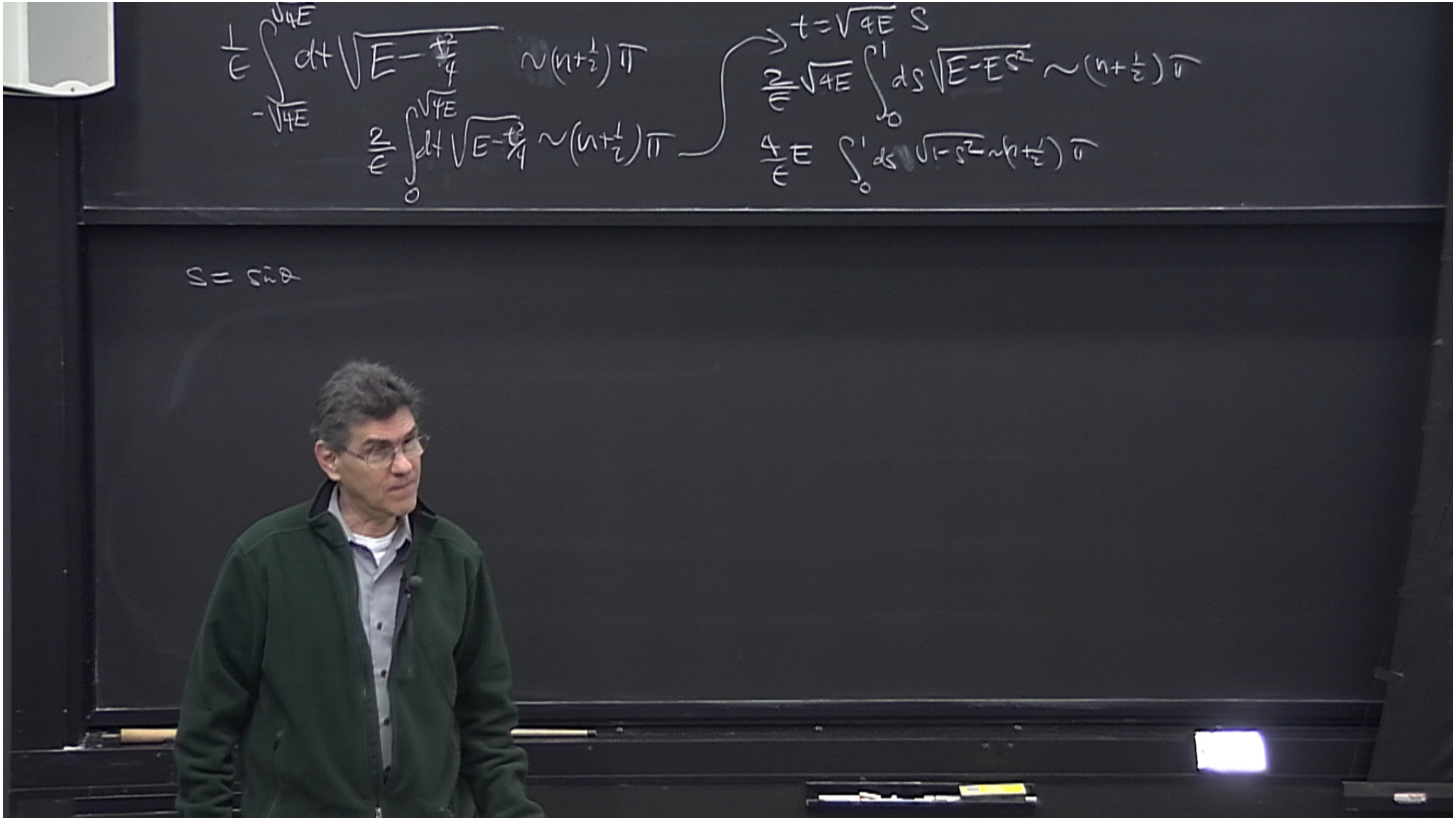
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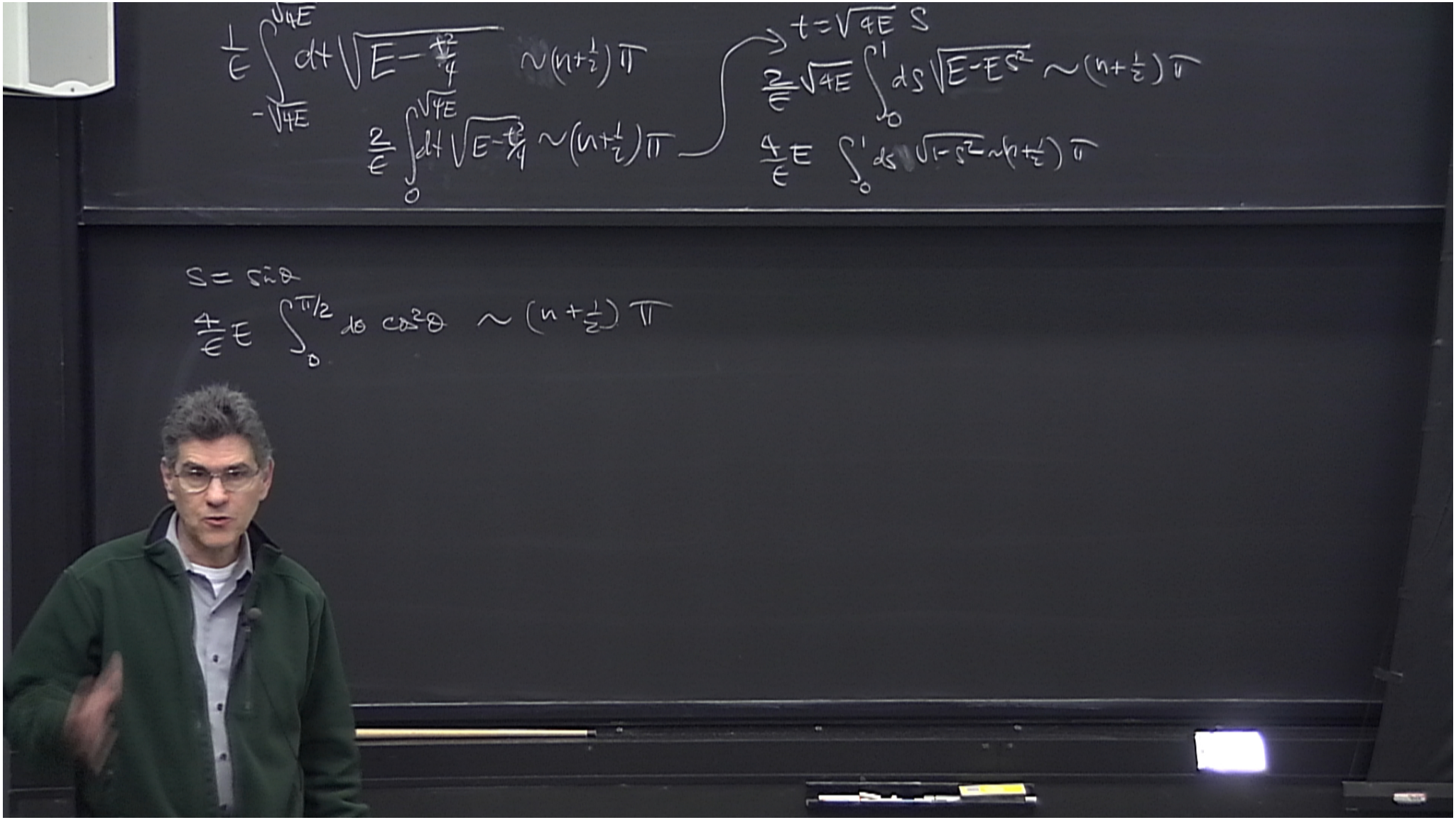
$s=\sin\theta$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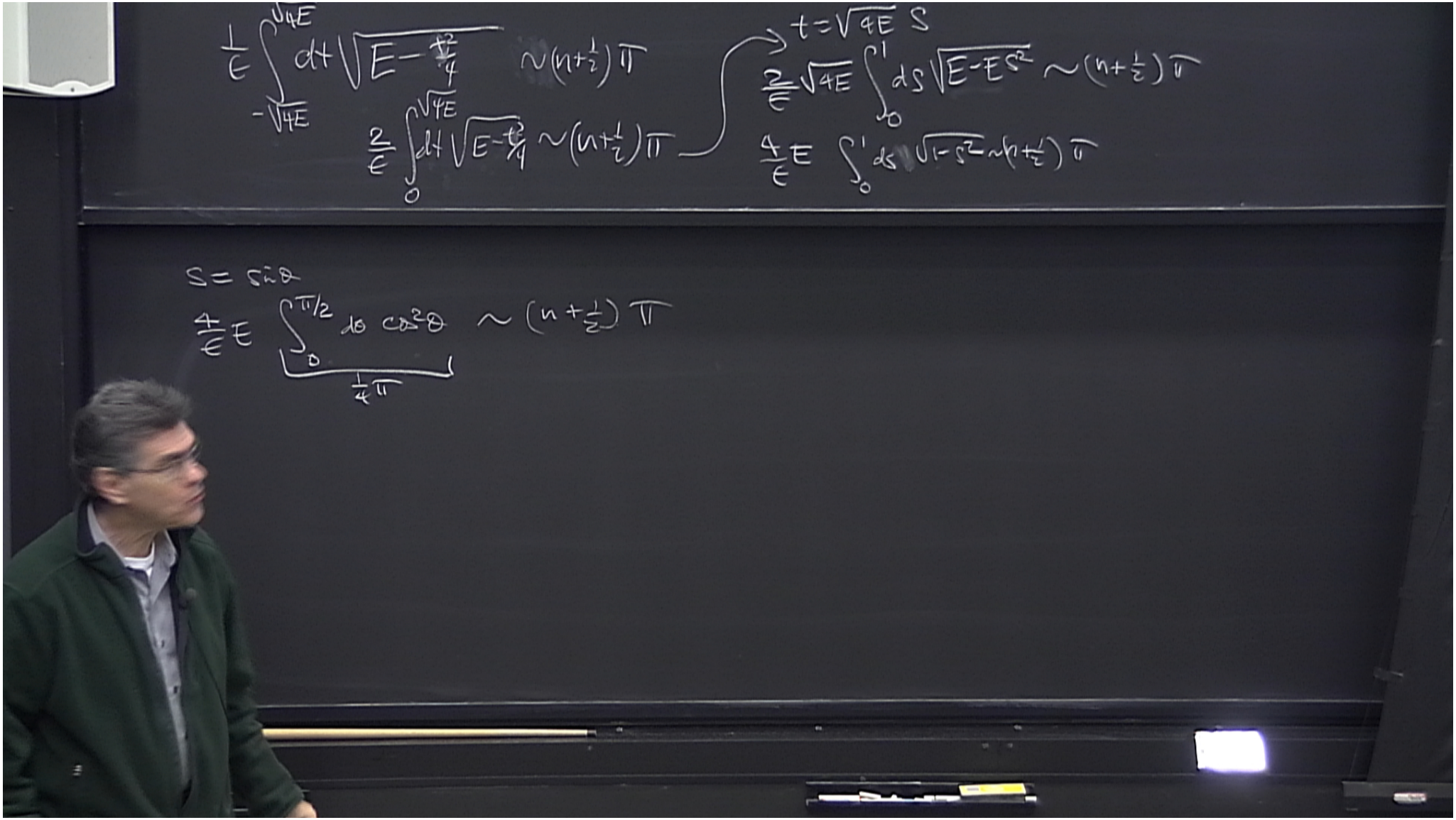
$\frac{4}{\epsilon}E\int_0^{\pi/2} d\theta \cos^2\theta \sim (n+\frac{1}{2})\pi$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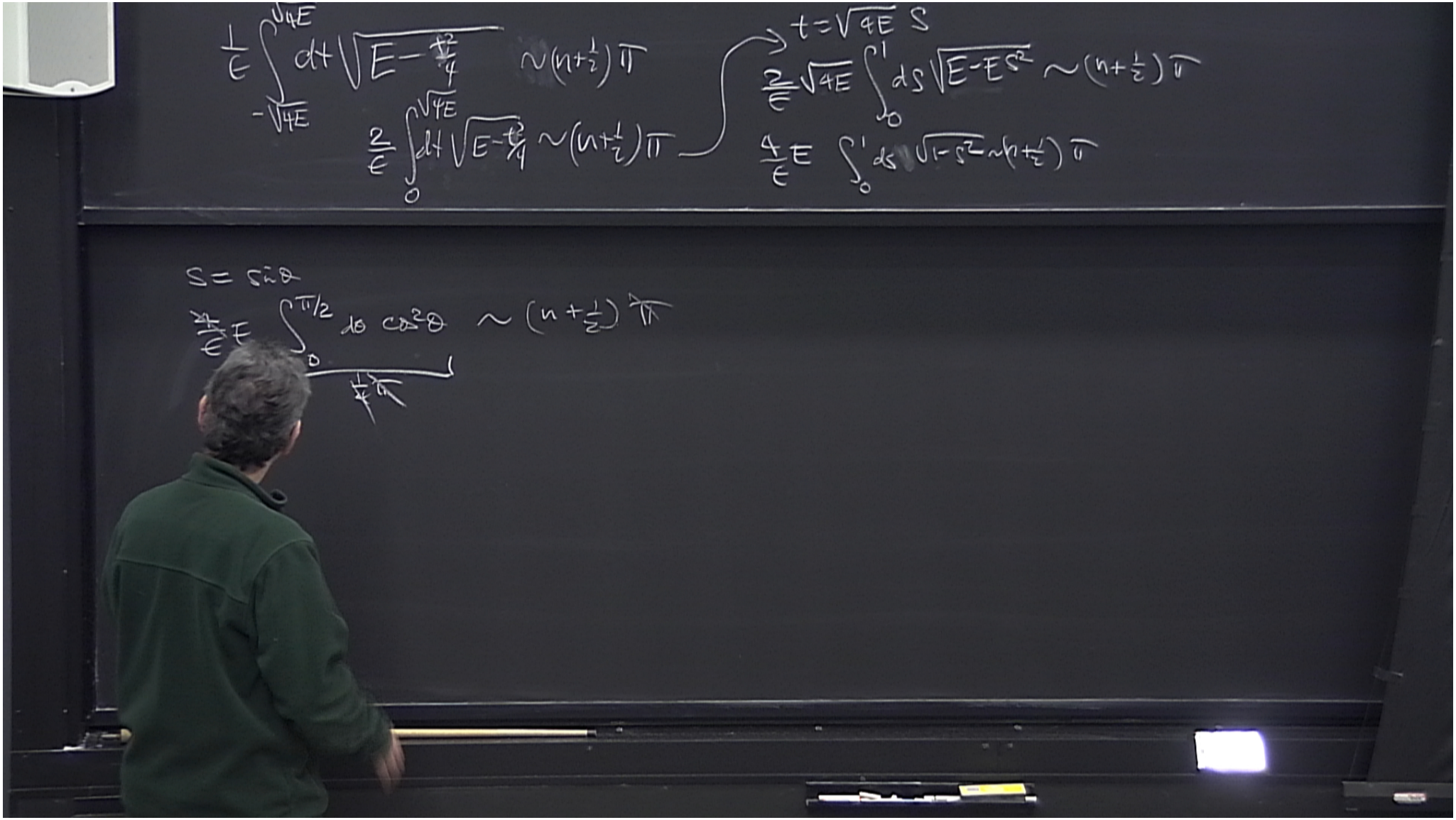
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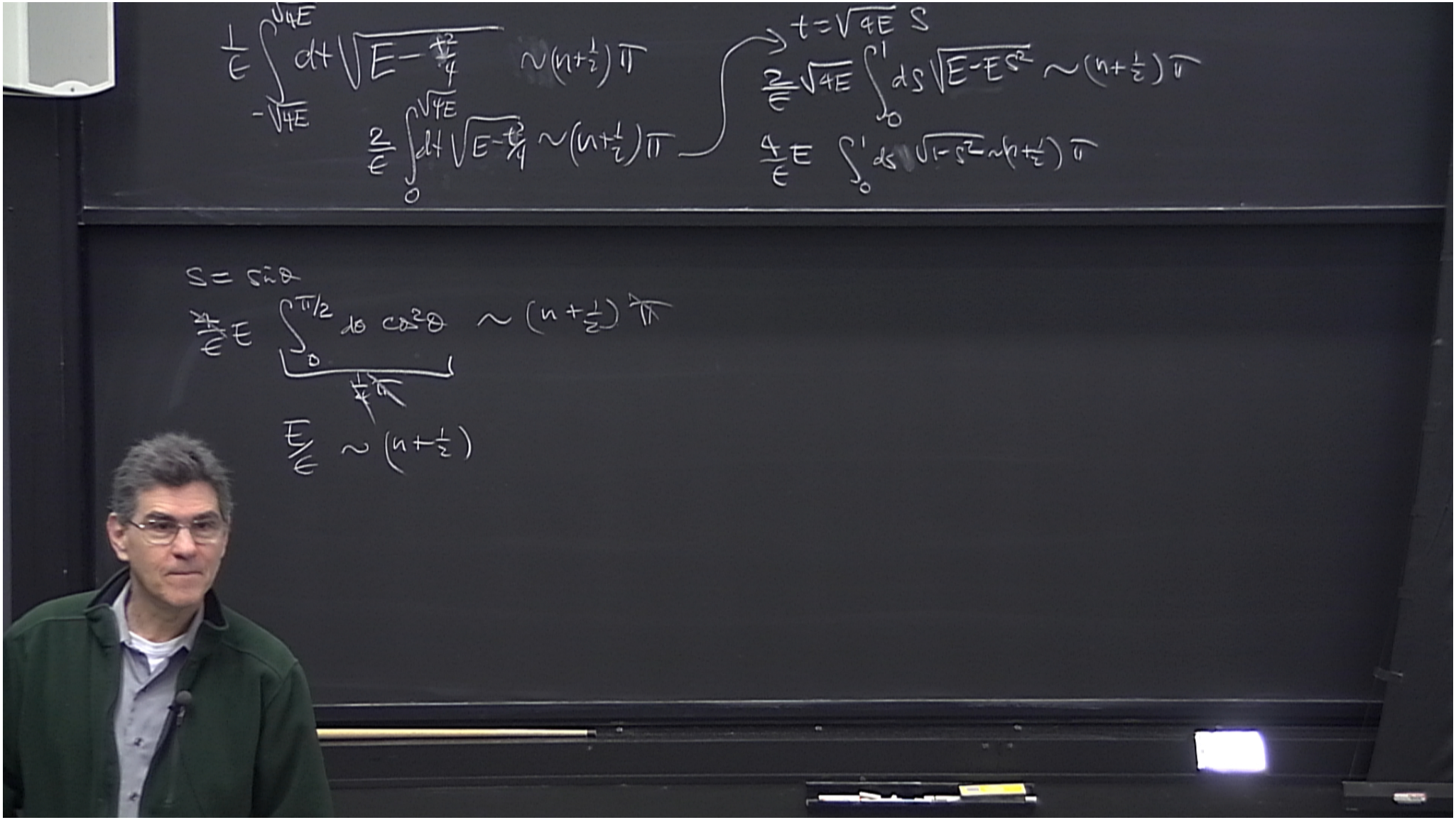
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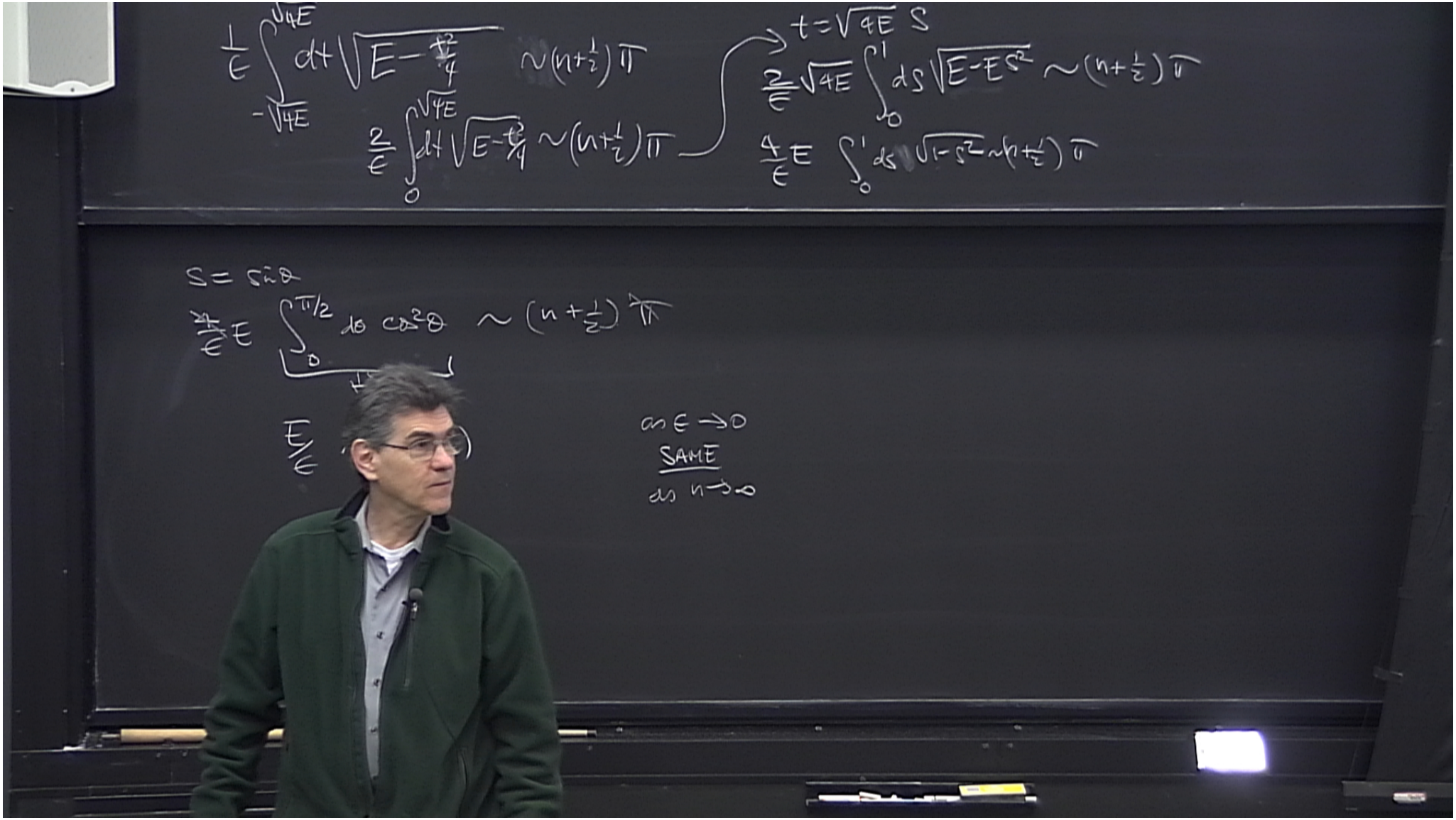
$\frac{1}{4}\pi$
$\frac{E}{\epsilon} \sim (n+\frac{1}{2})$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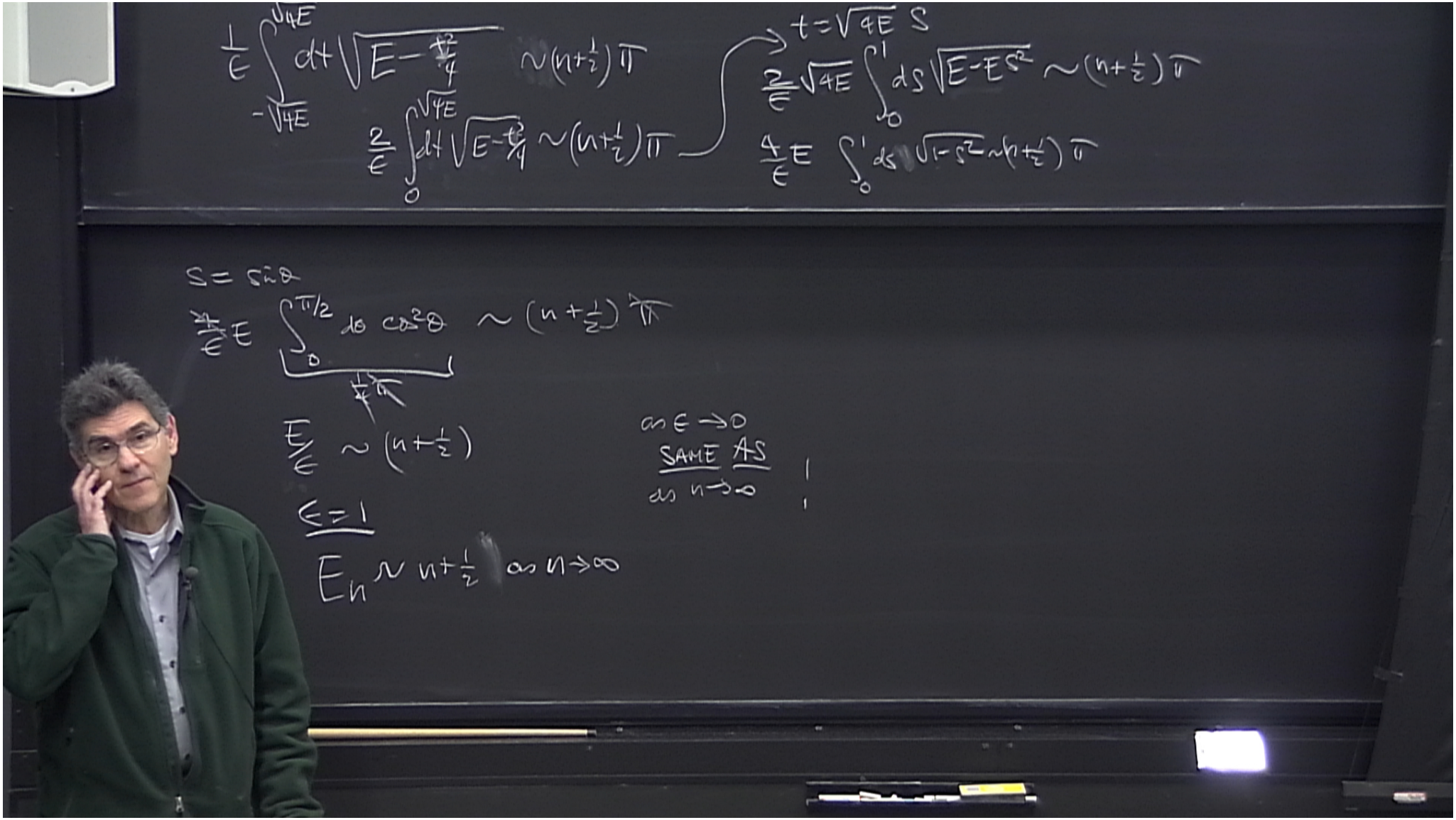
$s=\sin\theta$
$\frac{4}{\epsilon}E\int_0^{\pi/2} d\theta\cos^2\theta \sim (n+\frac{1}{2})\pi$
as $\epsilon\to 0$
SAME
as $n\to\infty$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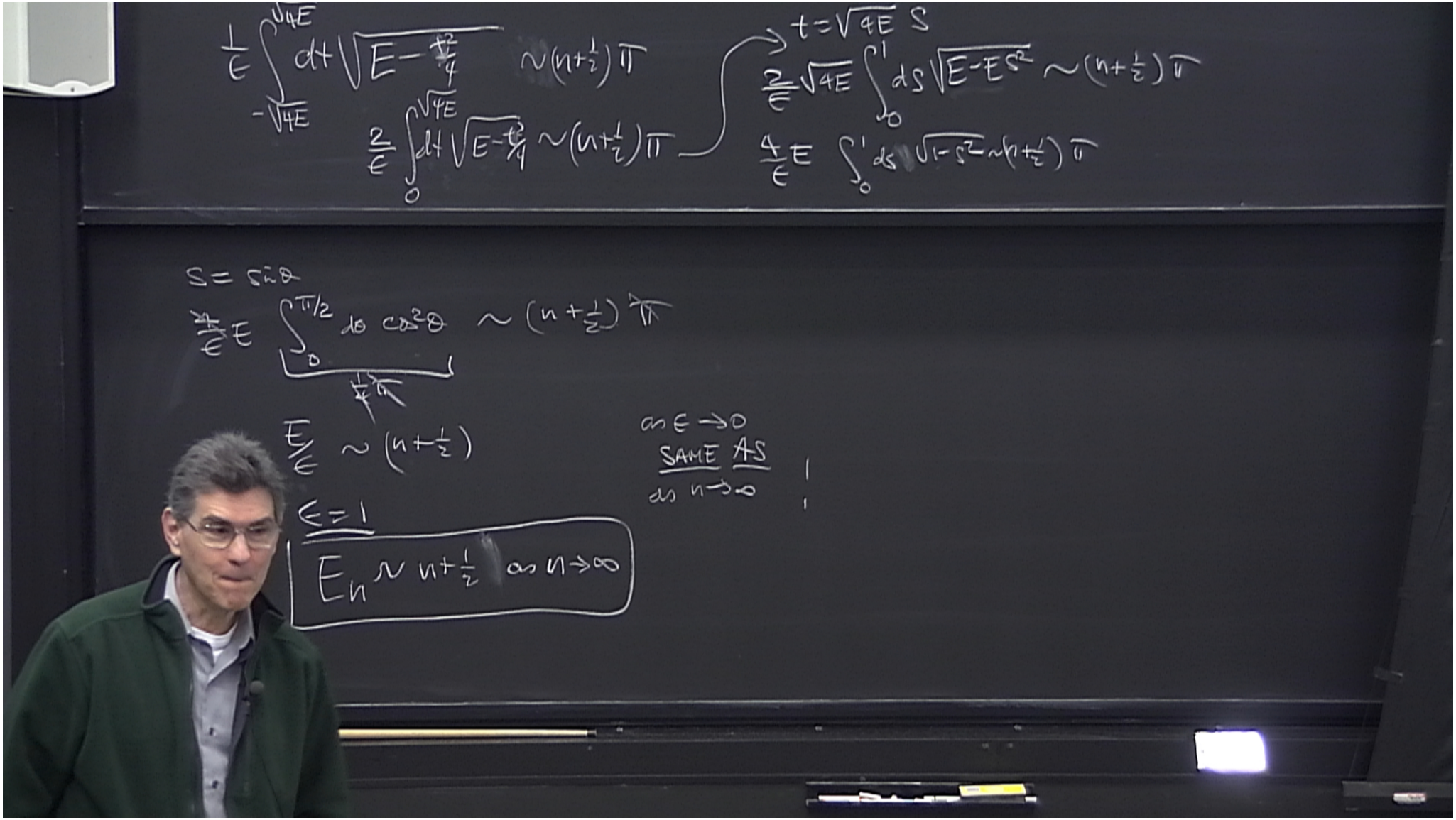
$\frac{\pi}{4}$
$\frac{E}{\epsilon} \sim (n+\frac{1}{2})$
as $\epsilon\to 0$
SAME AS
as $n\to\infty$
!
$\epsilon=1$
$E_n \sim n+\frac{1}{2}$ as $n\to\infty$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as $\epsilon \to 0$
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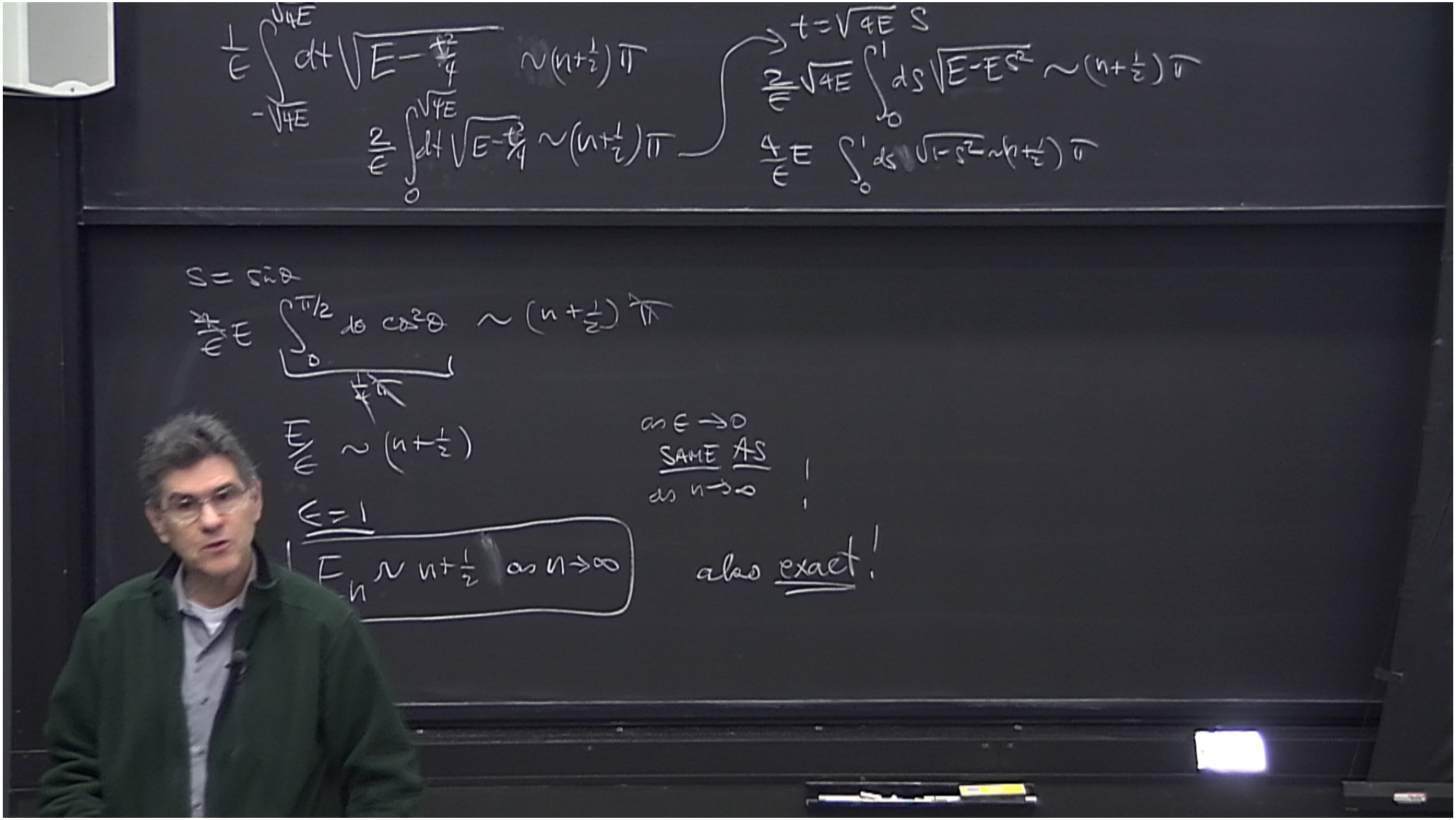
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SAME AS
as $n\to\infty$
!
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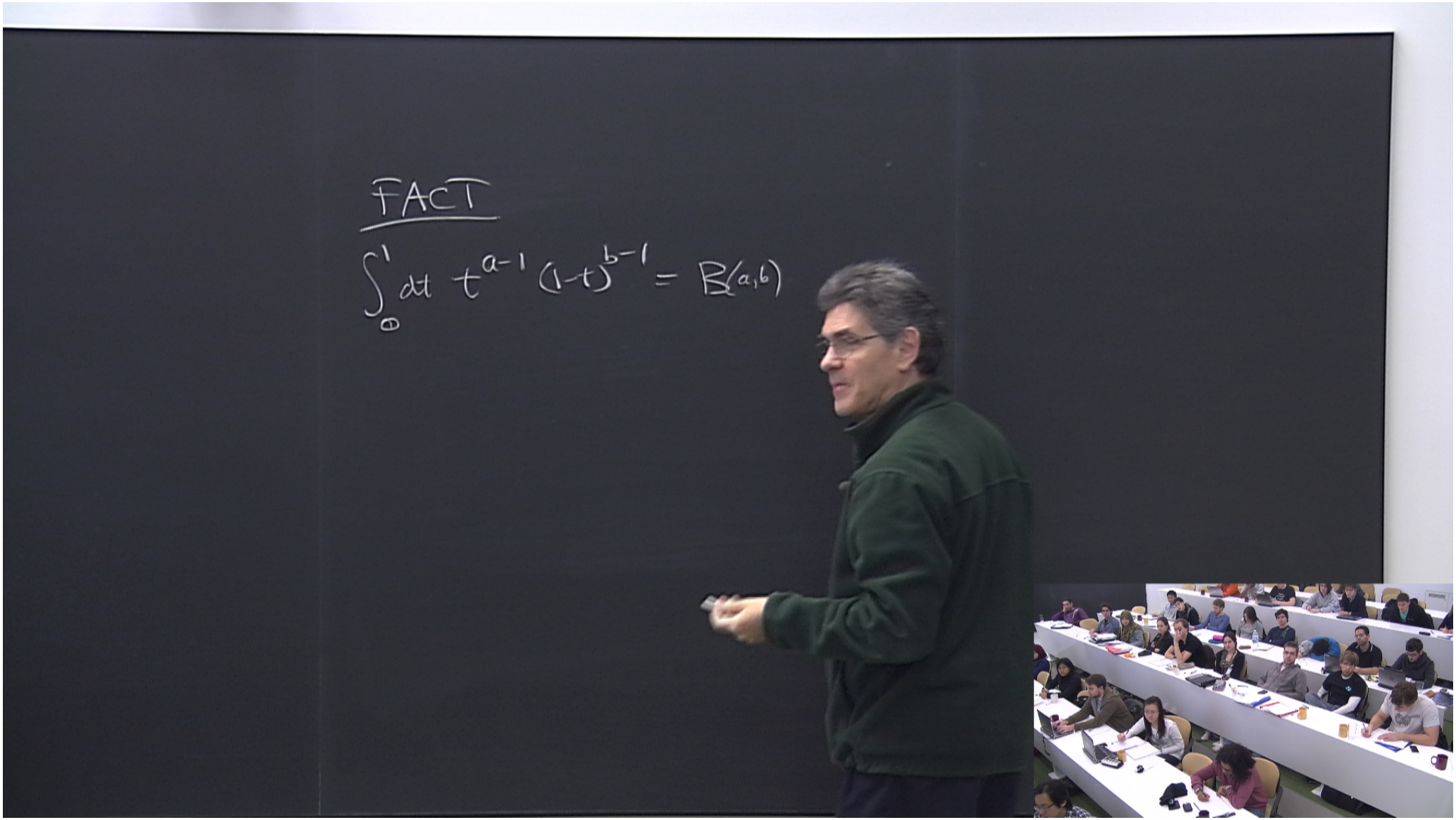
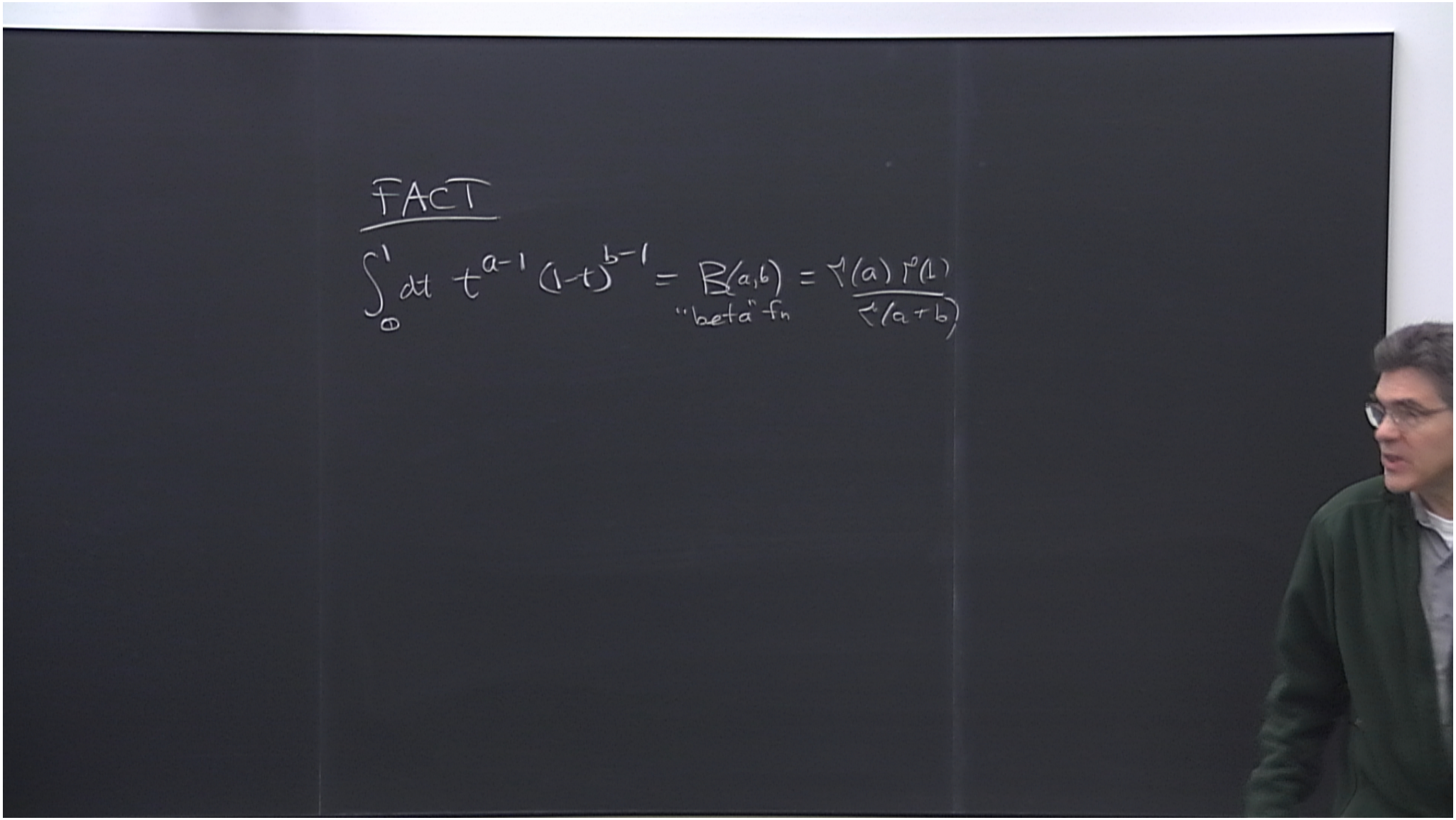
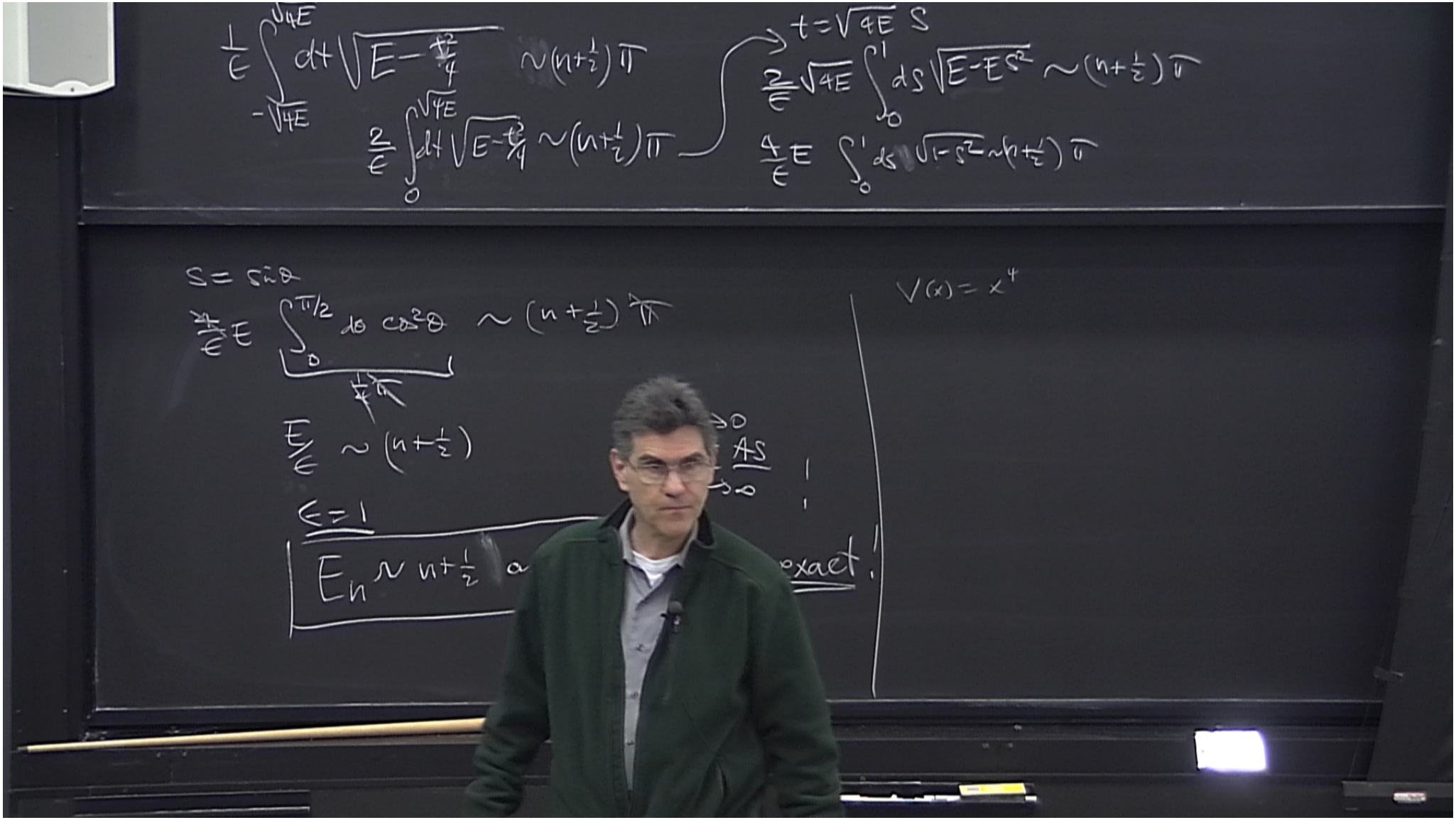
$E_n \sim n+\frac{1}{2}$ as $n\to\infty$
also exact!


FACT
$\int_0^1 dt\ t^{a-1}(1-t)^{b-1} = B(a,b)$


FACT
$\int_0^1 dt\, t^{a-1}(1-t)^{b-1} = B(a,b) = \frac{\Gamma(a)\Gamma(1)}{\Gamma(a+b)}$
"beta" fn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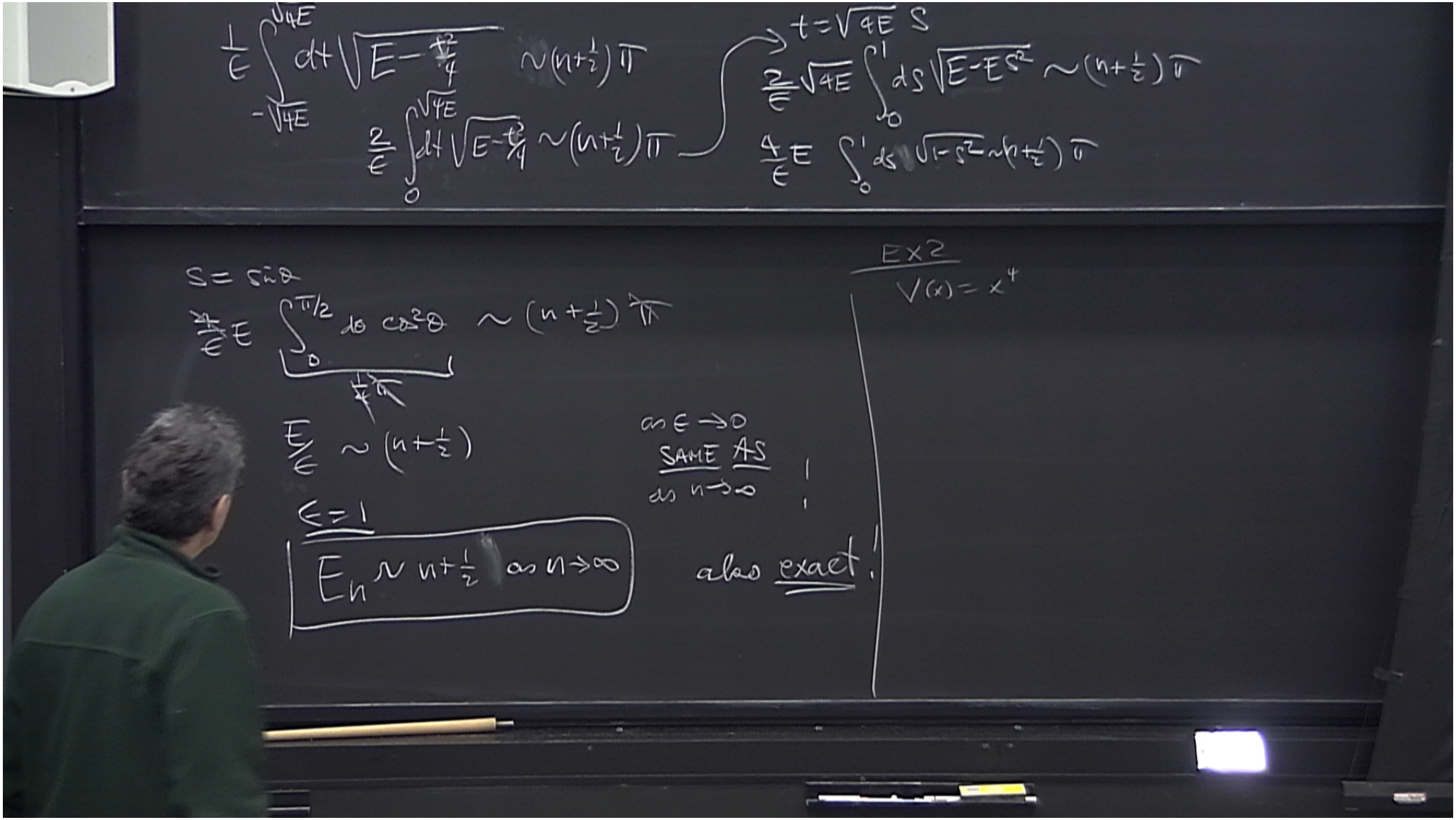
$E_n \sim n+\frac{1}{2}$ a
xact!
$V(x)=x^4$
AS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$\frac{\pi}{4}$
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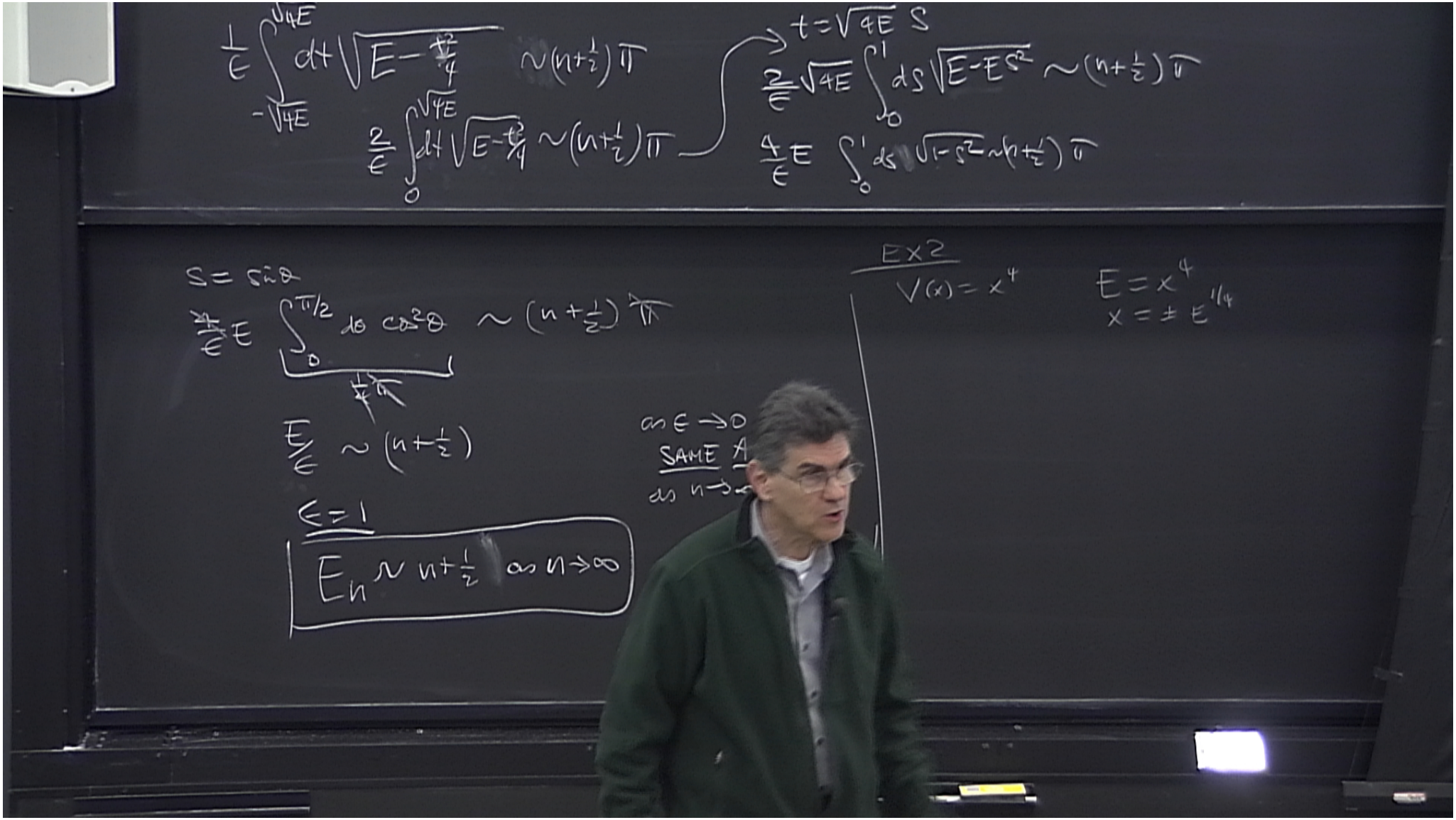
$\epsilon=1$
$E_n \sim n+\frac{1}{2}$ as $n\to\infty$
as $\epsilon\to 0$
SAME AS
as $n\to\infty$
!
also exact!
EX2
$V(x)=x^4$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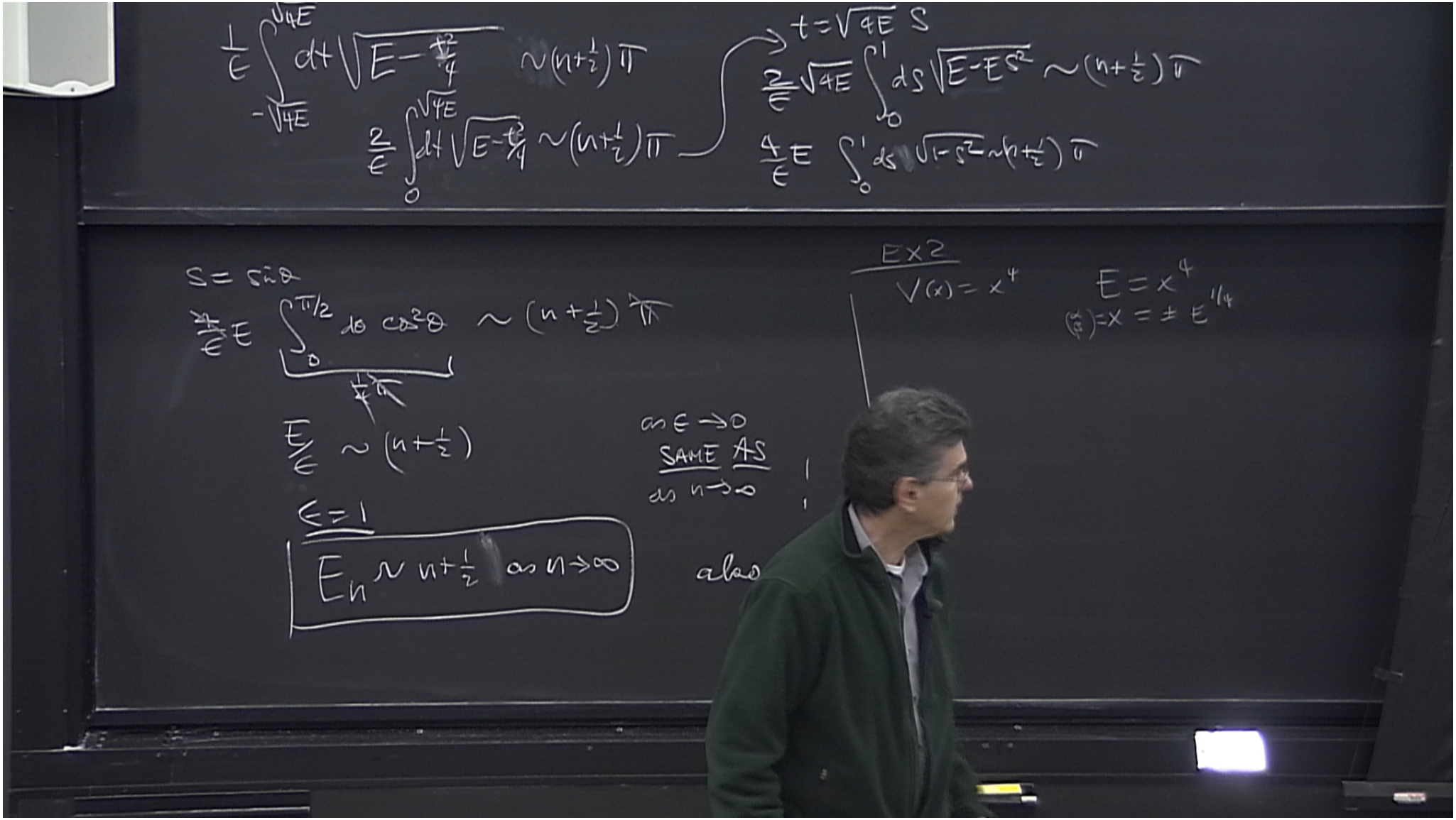
$\epsilon=1$
$E_n \sim n+\frac{1}{2}$ as $n\to\infty$
as $\epsilon\to 0$
SAME
as $n\to$
EX2
$V(x)=x^4$
$E=x^4$
$x=\pm E^{1/4}$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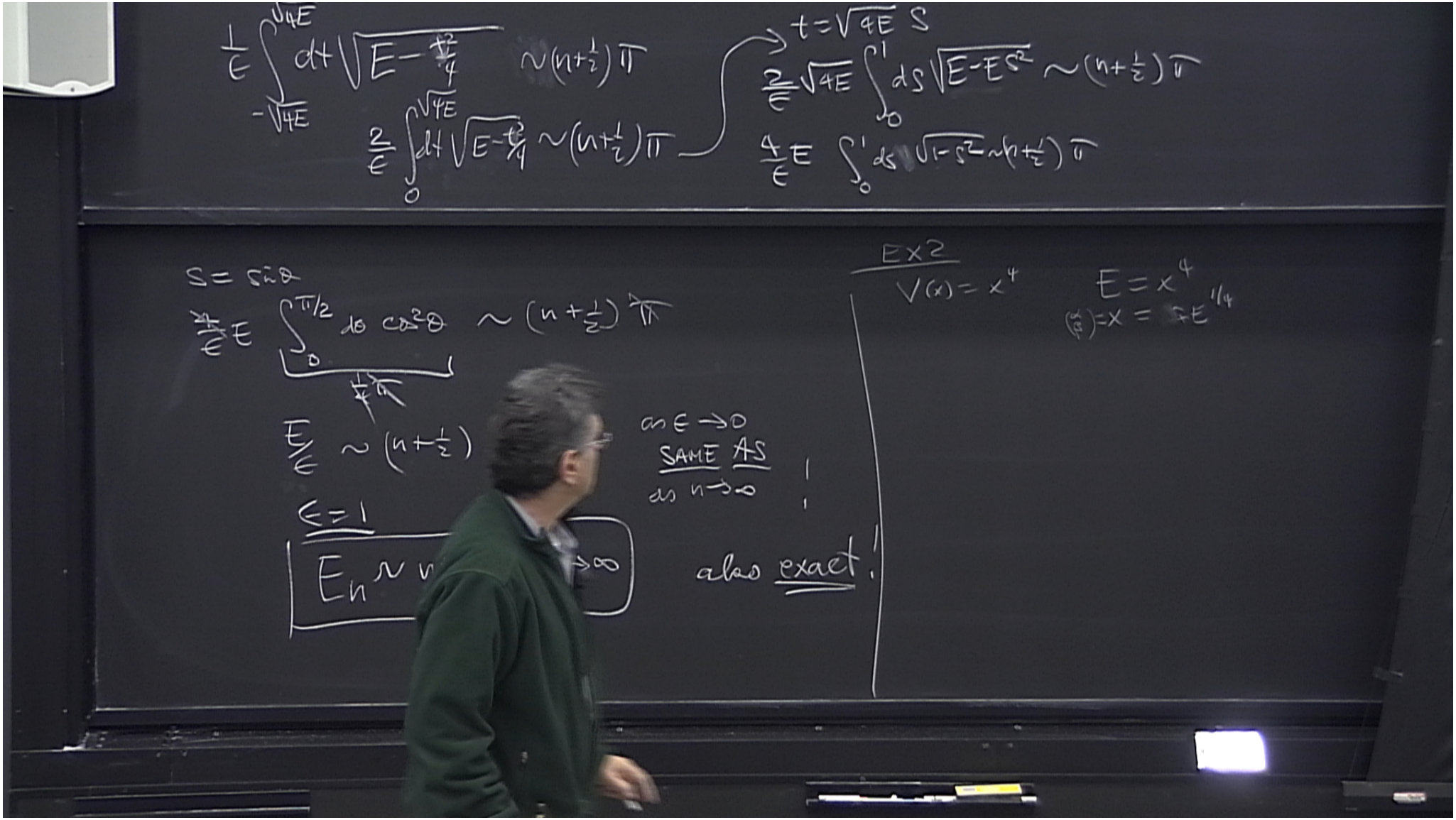
$\frac{\pi}{4}$
$\frac{E}{\epsilon} \sim (n+\frac{1}{2})$
$\epsilon=1$
$E_n \sim n+\frac{1}{2}$ as $n\to\infty$
as $\epsilon\to 0$
SAME AS
as $n\to\infty$
!
!
also
EX2
$V(x)=x^4$
$E=x^4$
$(\sqrt{\ })\ x=\pm E^{1/4}$


EX2
$V(x) = x^4$
$E = x^4$
$x = E^{1/4}$
$s = \sin\theta$
as $\epsilon \to 0$
SAME AS
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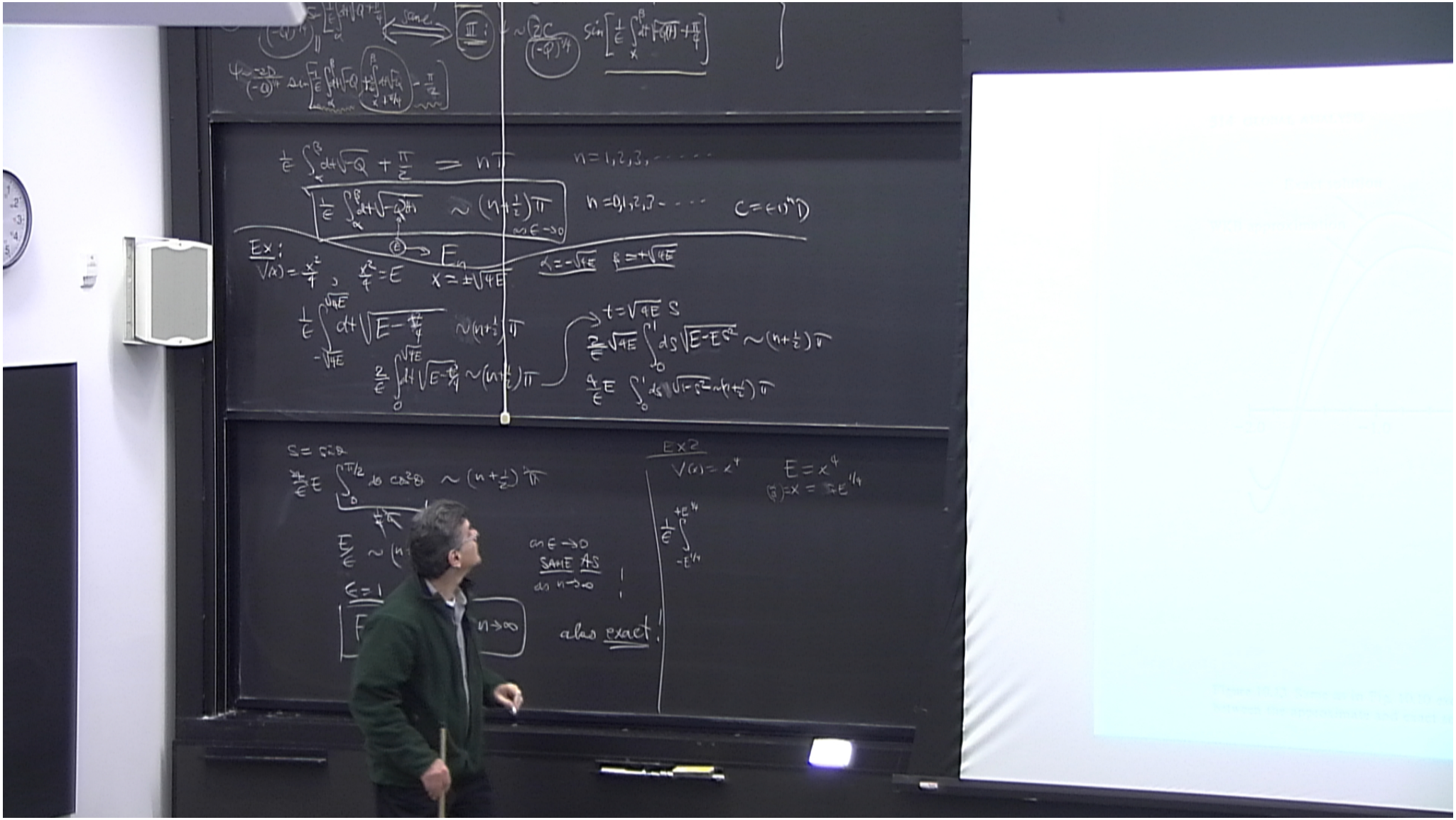
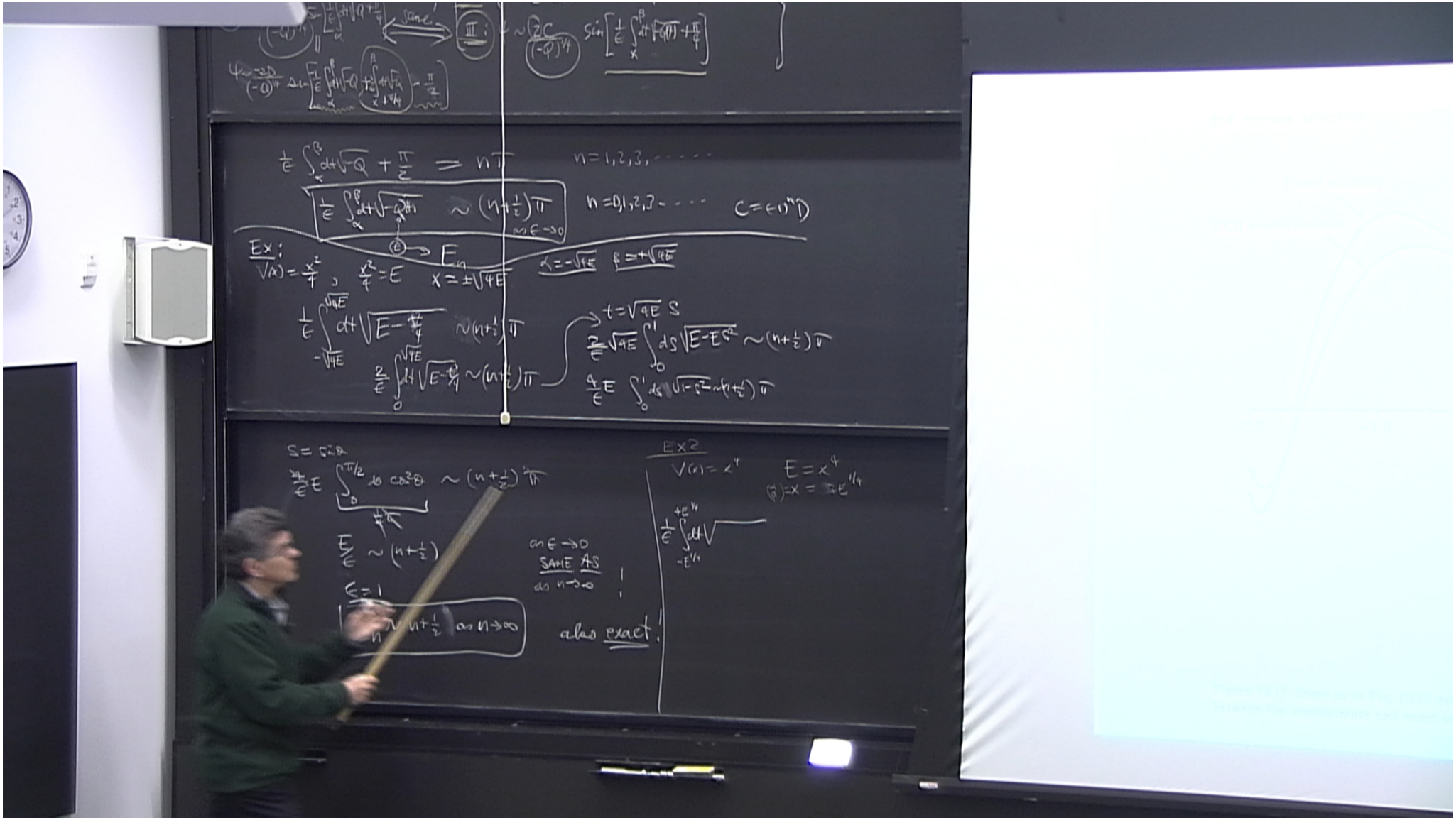
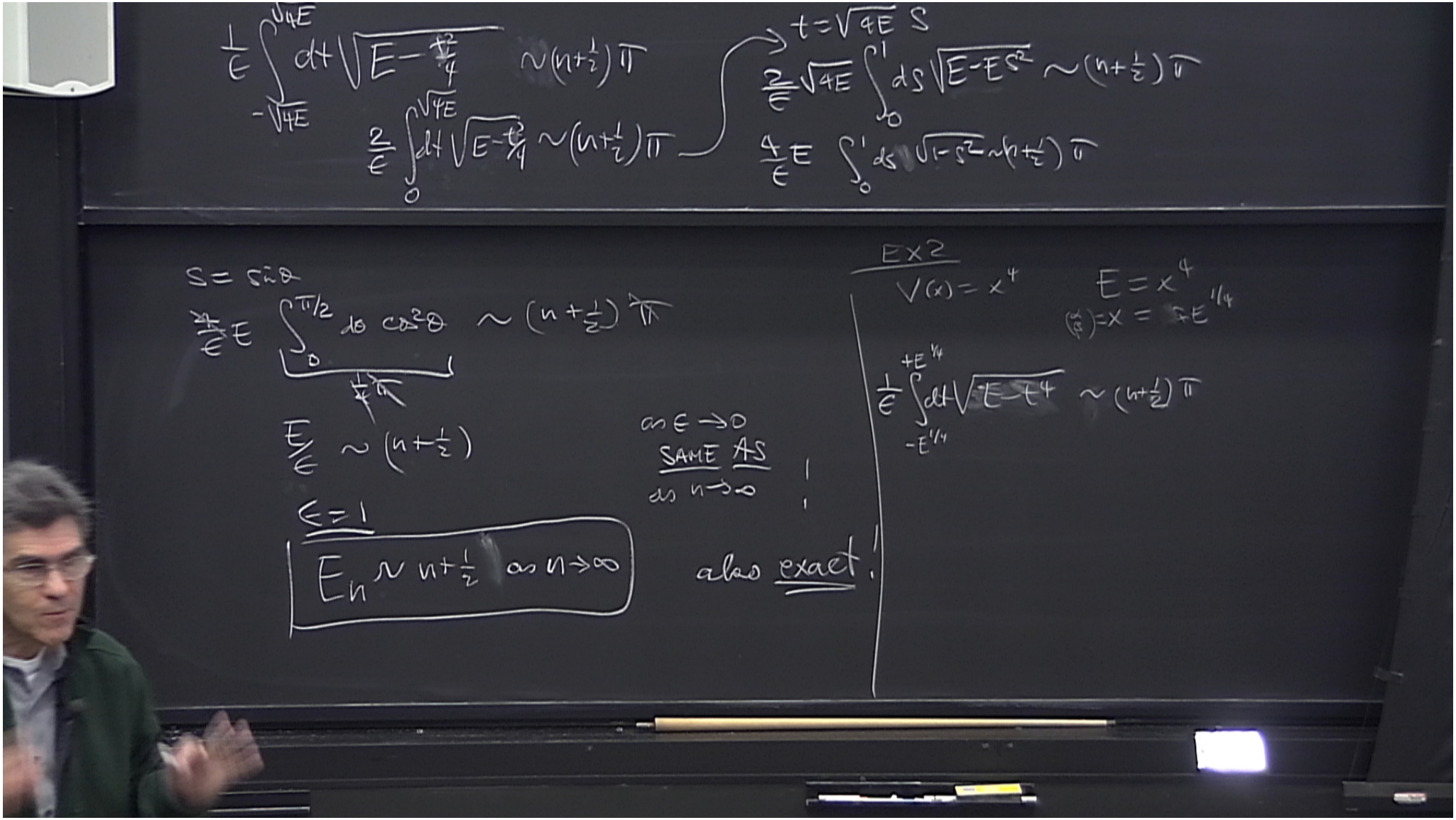
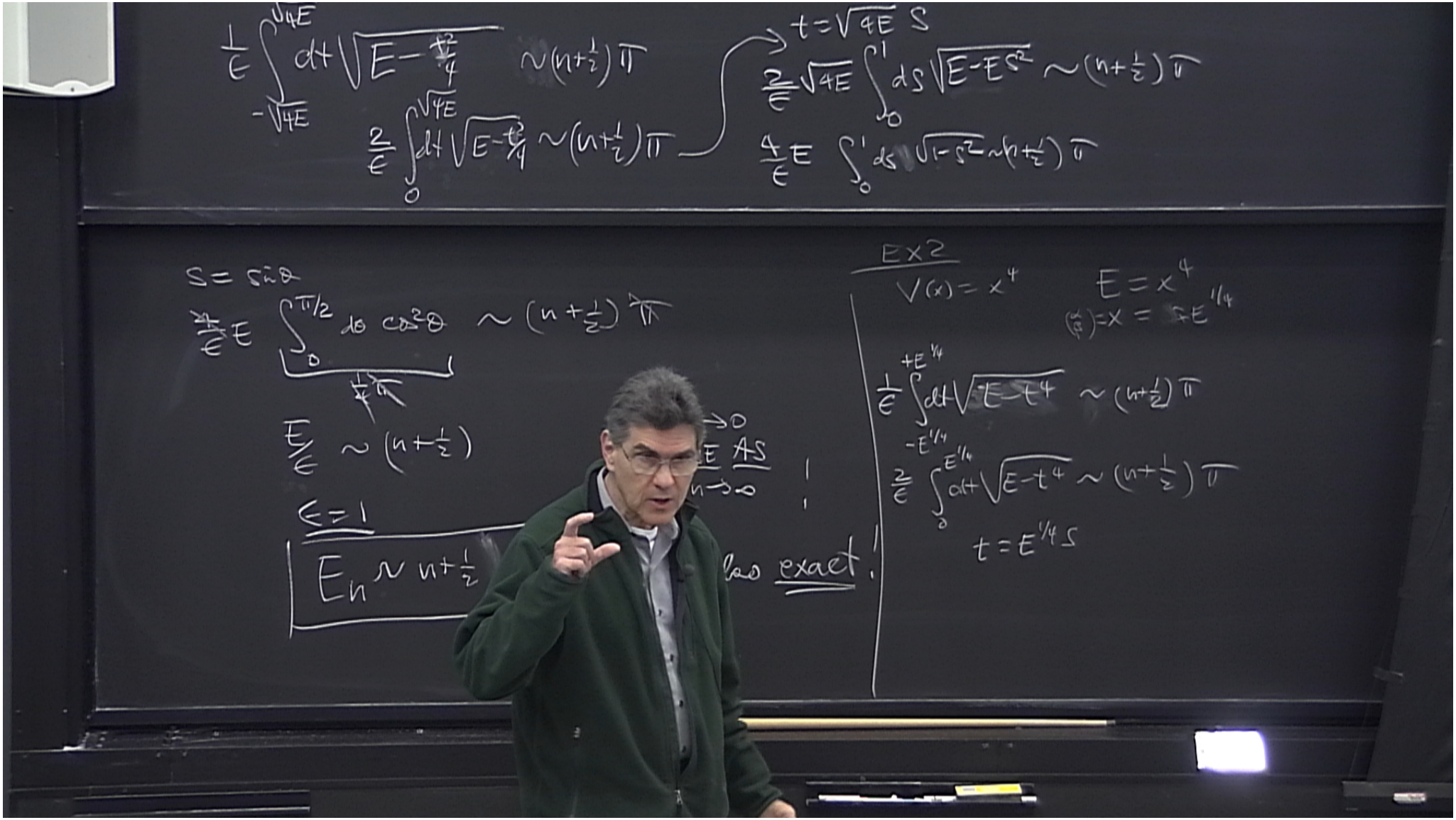
$\epsilon = 1$
also exact!


Ex:
Ex2
SAME AS
also exact!


Ex:
Ex2
same
same
same


$\frac{1}{\epsilon}\int_{-\sqrt{4E}}^{\sqrt{4E}} dt \sqrt{E-\frac{t^2}{4}} \sim (n+\frac{1}{2})\pi$
$\frac{2}{\epsilon}\int_0^{\sqrt{4E}} dt \sqrt{E-\frac{t^2}{4}} \sim (n+\frac{1}{2})\pi$
$t=\sqrt{4E}\,s$
$\frac{2}{\epsilon}\sqrt{4E}\int_0^1 ds \sqrt{E-Es^2} \sim (n+\frac{1}{2})\pi$
$\frac{4}{\epsilon}E\int_0^1 ds \sqrt{1-s^2} \sim (n+\frac{1}{2})\pi$
$s=\sin\theta$
$\frac{4}{\epsilon}E\int_0^{\pi/2} d\theta \cos^2\theta \sim (n+\frac{1}{2})\pi$
$\frac{\pi}{4}$
$\frac{E}{\epsilon} \sim (n+\frac{1}{2})$
$\epsilon=1$
$E_n \sim n+\frac{1}{2}$ as $n\to\infty$
as $\epsilon\to 0$
SAME AS
as $n\to\infty$
!
!
also exact!
EX2
$V(x)=x^4$
$E=x^4$
$(\sqrt{\ })\ x=\pm E^{1/4}$
$\frac{1}{\epsilon}\int_{-E^{1/4}}^{+E^{1/4}} dt \sqrt{E-t^4} \sim (n+\frac{1}{2})\pi$



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$E=x^4$
$\binom{\alpha}{\beta}=x=sE^{1/4}$
$\frac{1}{\epsilon}\int_{-E^{1/4}}^{+E^{1/4}} dt \sqrt{E-t^4} \sim (n+\frac{1}{2})\pi$
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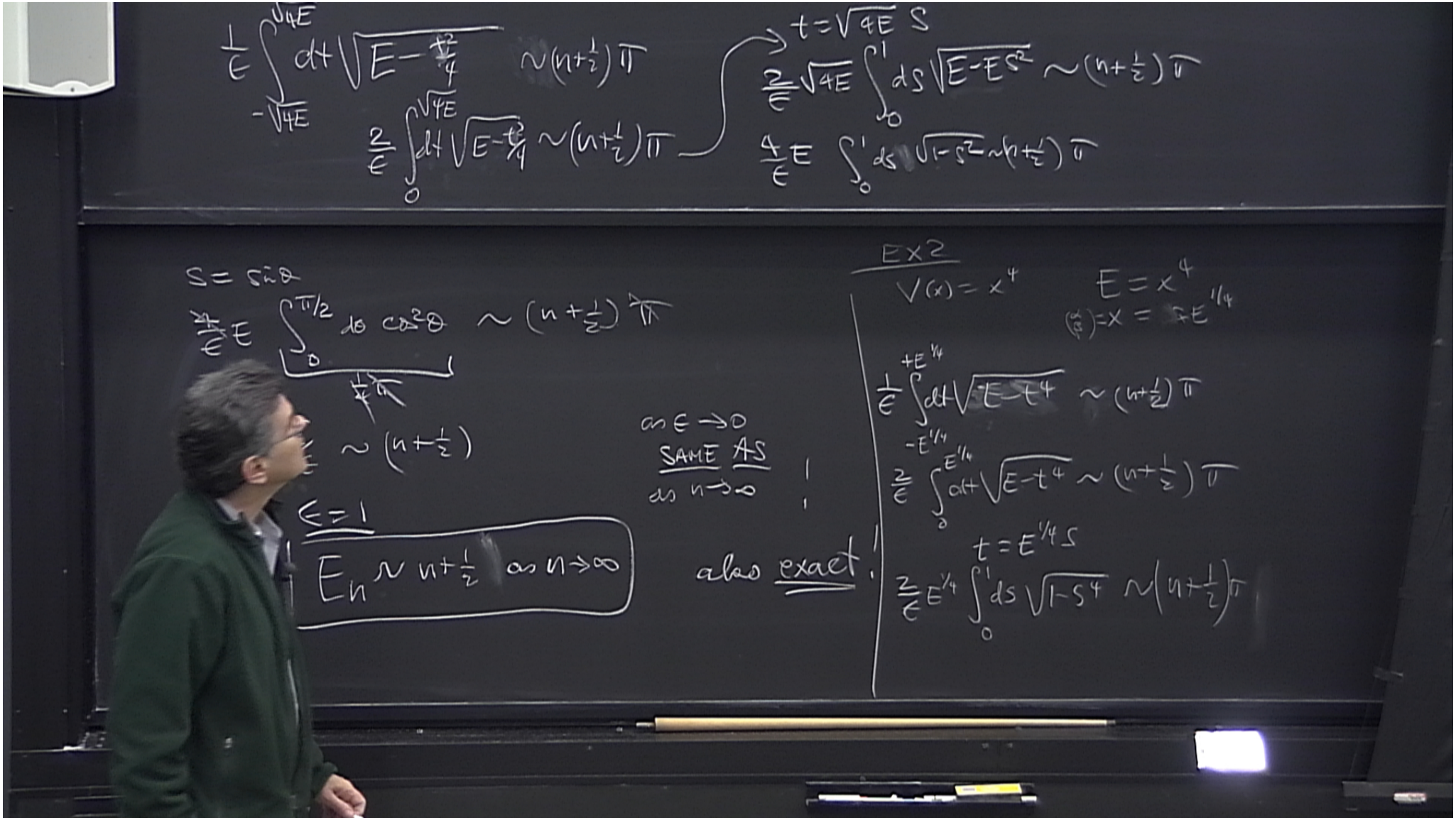
$t=E^{1/4}s$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$V(x)=x^4$
$E=x^4$
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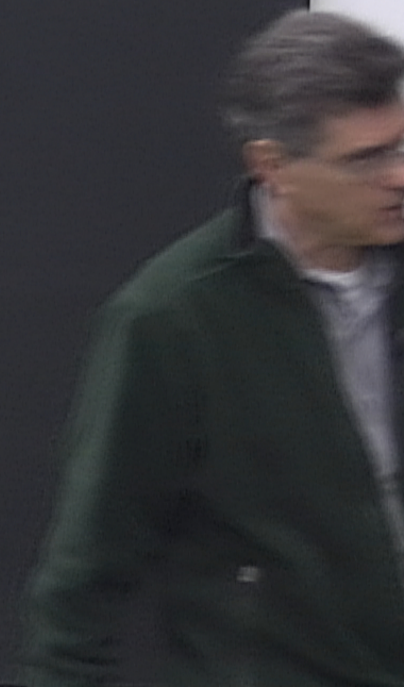
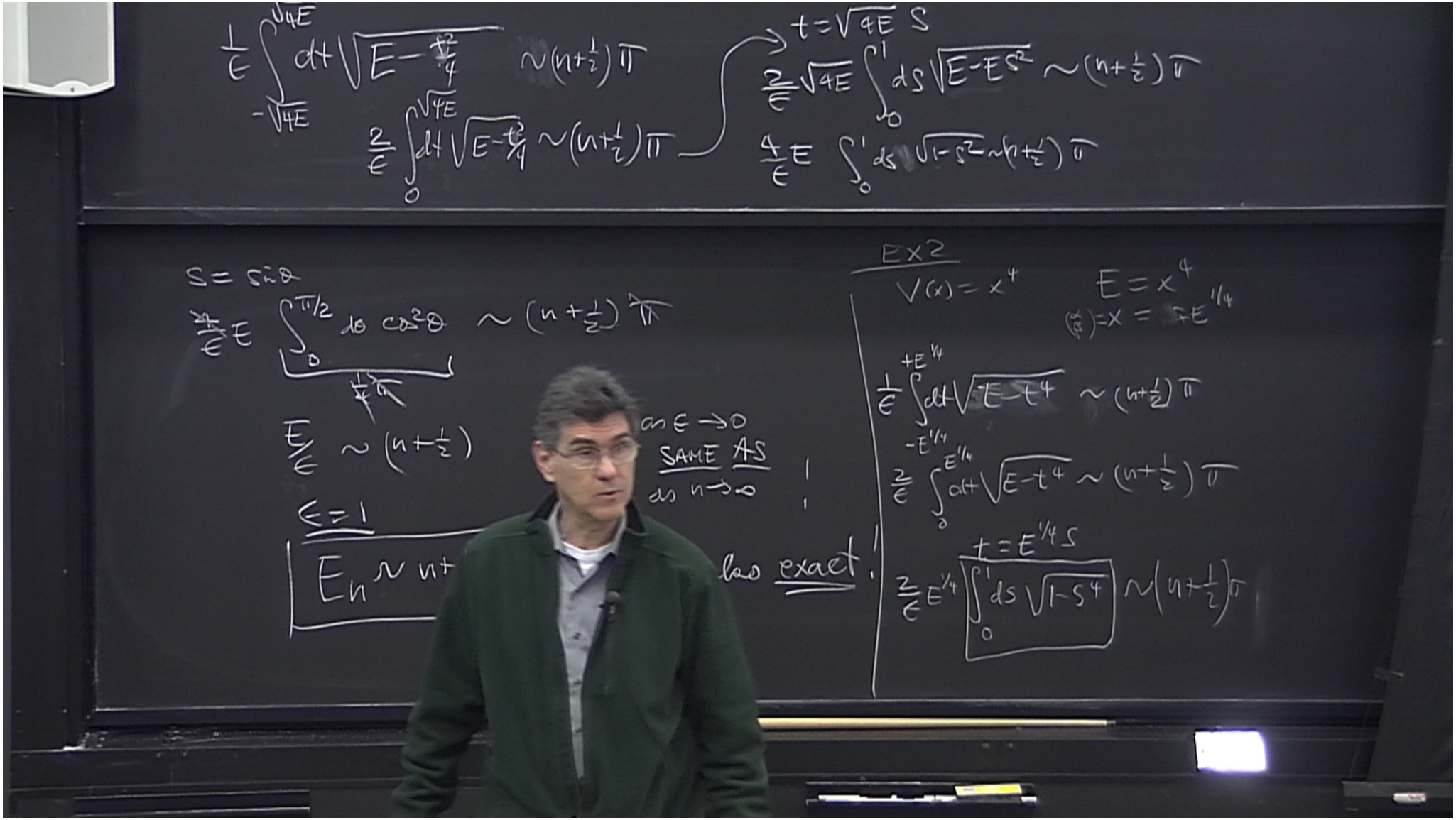
$t=E^{1/4}s$
$\frac{2}{\epsilon}E^{1/4}\int_0^1 ds \sqrt{1-s^4} \sim (n+\frac{1}{2})\pi$

FACT

$$\int_0^1 dt\, t^{a-1}(1-t)^{b-1} = B(a,b) = \frac{\Gamma(a)\Gamma(1)}{\Gamma(a+b)}$$

"beta" fn

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$V(x) = x^4$
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$t = E^{1/4} s$
$s = \sin\theta$
$E_n \sim n +$
as $\epsilon \to 0$
SAME AS
as $n \to \infty$
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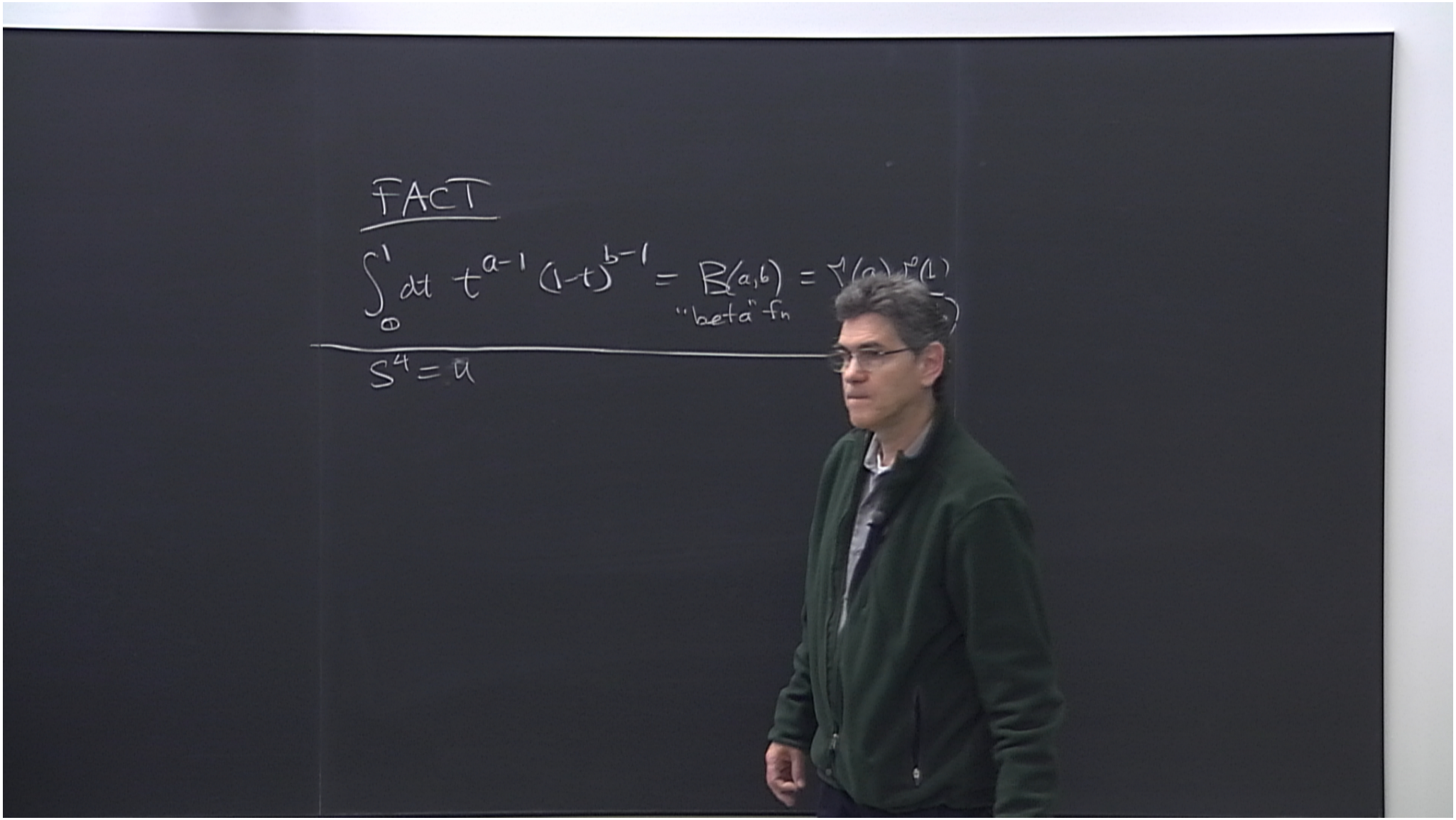
!
too exact!

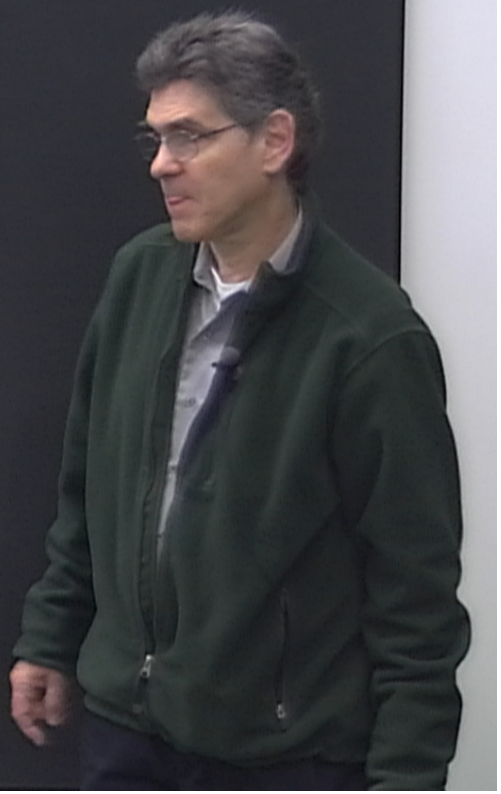

FACT
$\int_0^1 dt\, t^{a-1}(1-t)^{b-1} = B(a,b) = \Gamma(a)\Gamma(b)$
"beta" fn
$S^4 = 4$

FACT

$$\int_0^1 dt\, t^{a-1}(1-t)^{b-1} = B(a,b) = \frac{\Gamma(a)\Gamma(1)}{\Gamma(a+b)}$$

"beta" fn

$s^4 = u \qquad s = u^{1/4} \qquad ds = \frac{1}{4} u^{\frac{1}{4}-1} du$



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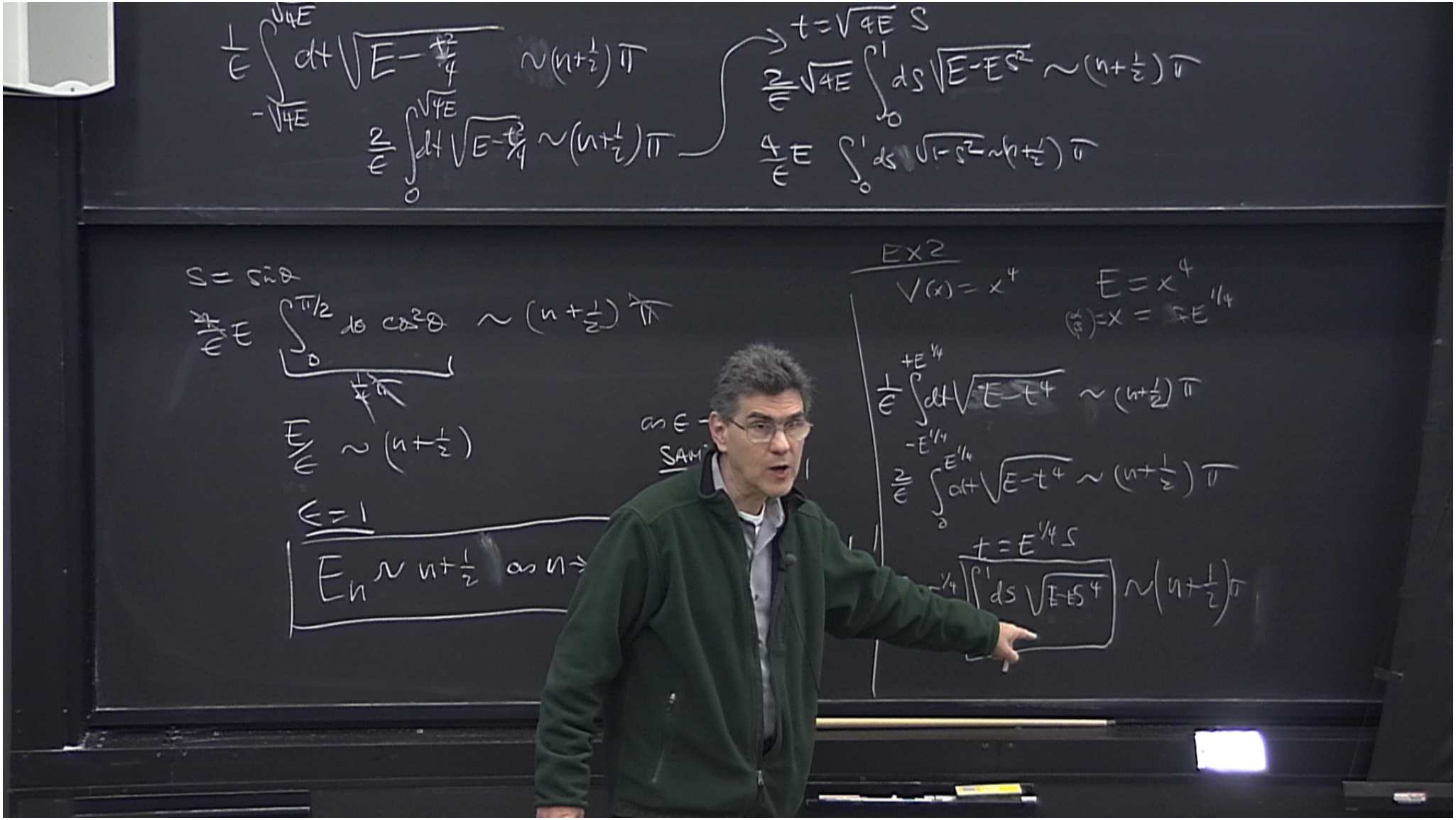
$\frac{2}{\epsilon} E^{1/4}$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$V(x) = x^4$
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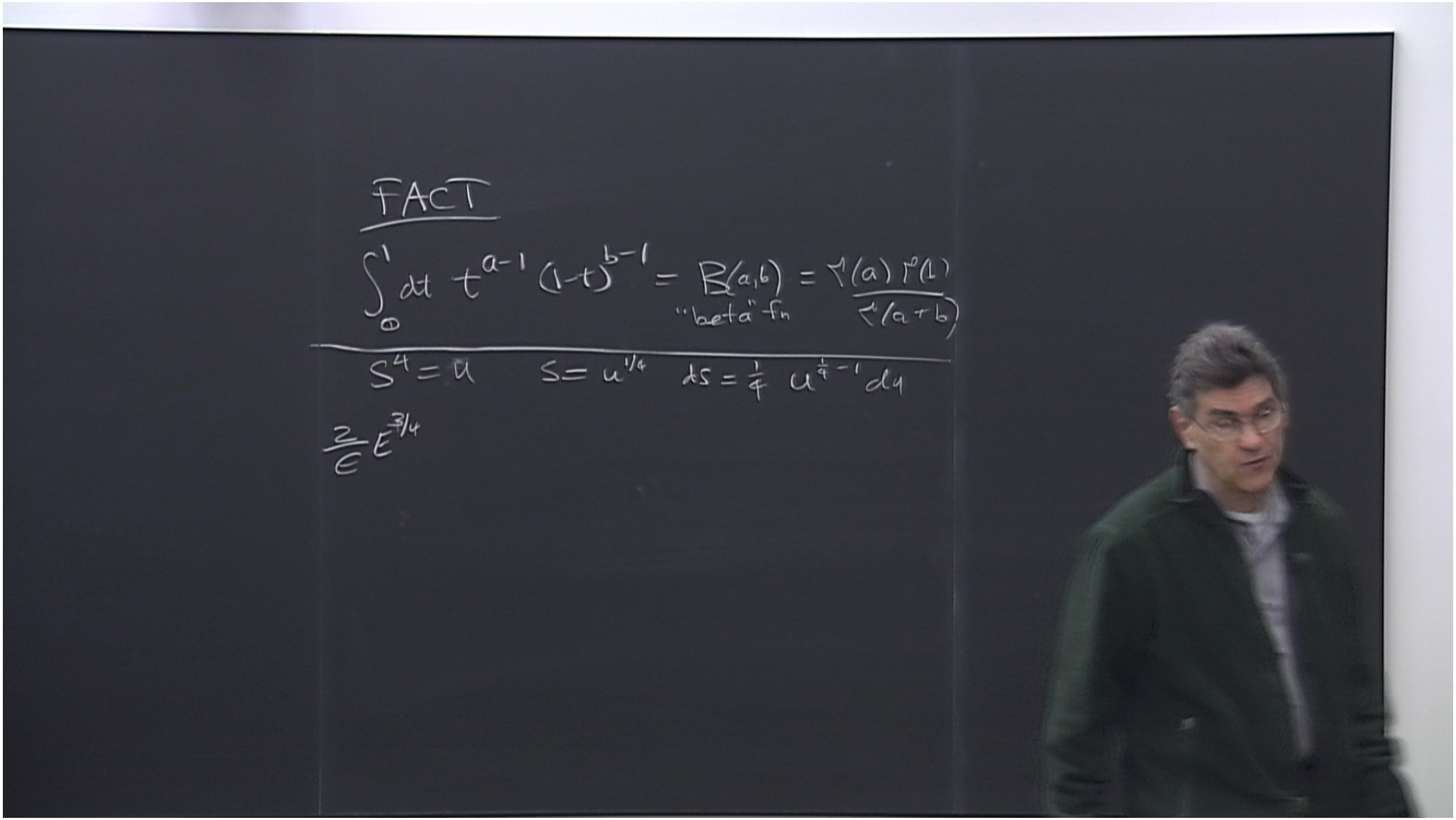
$E_n \sim n + \frac{1}{2}$ as $n \to$


FACT
$\int_0^1 dt\, t^{a-1}(1-t)^{b-1} = B(a,b) = \frac{\Gamma(a)\Gamma(1)}{\Gamma(a+b)}$
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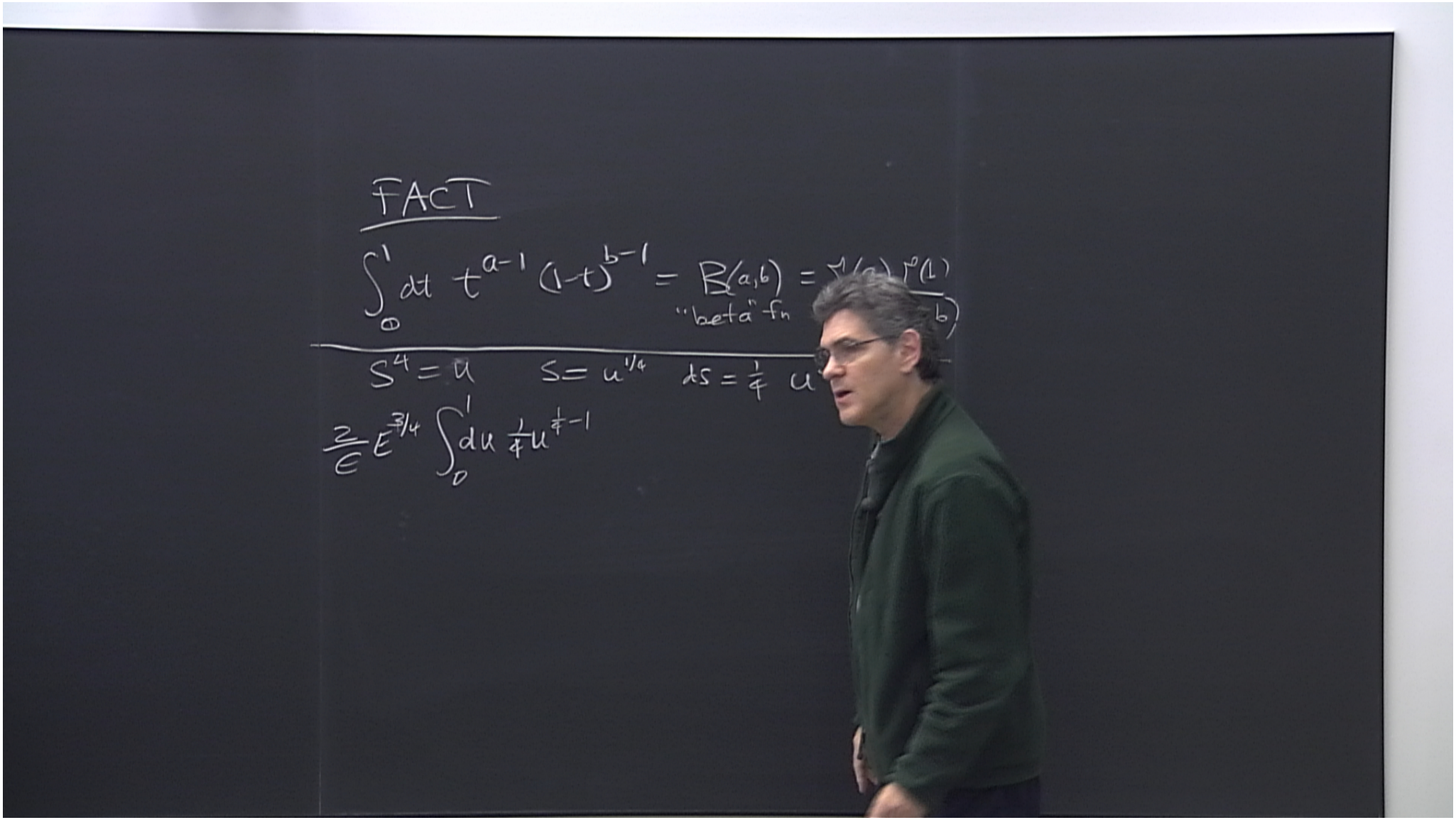
$s^4 = u \quad s = u^{1/4} \quad ds = \frac{1}{4} u^{\frac{1}{4}-1} du$
$\frac{2}{\epsilon} E^{3/4}$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$\int_0^1 dt\, t^{a-1}(1-t)^{b-1} = B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$
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$s^4 = u \quad s = u^{1/4} \quad ds = \frac{1}{4} u^{-3/4} du$
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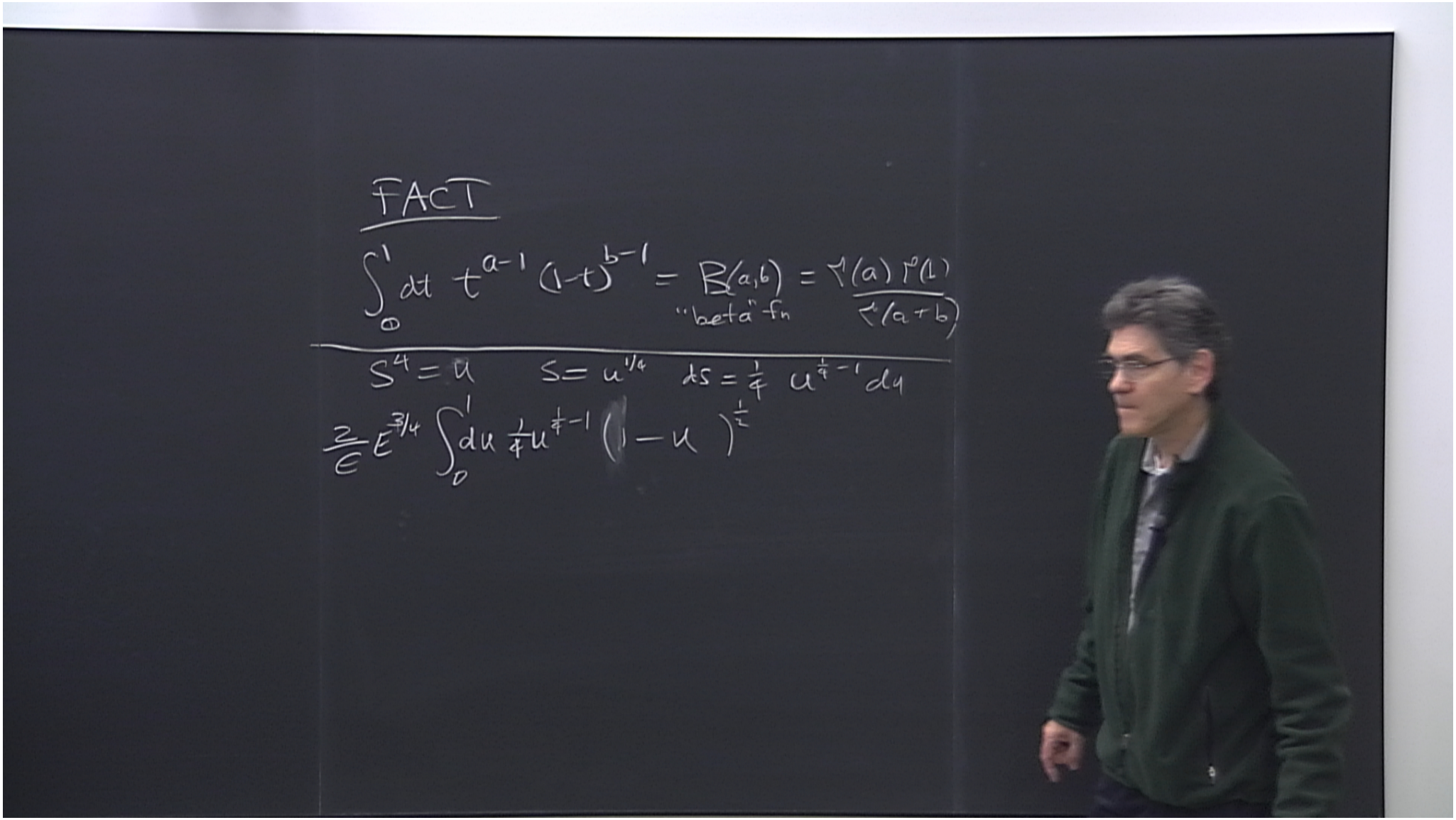
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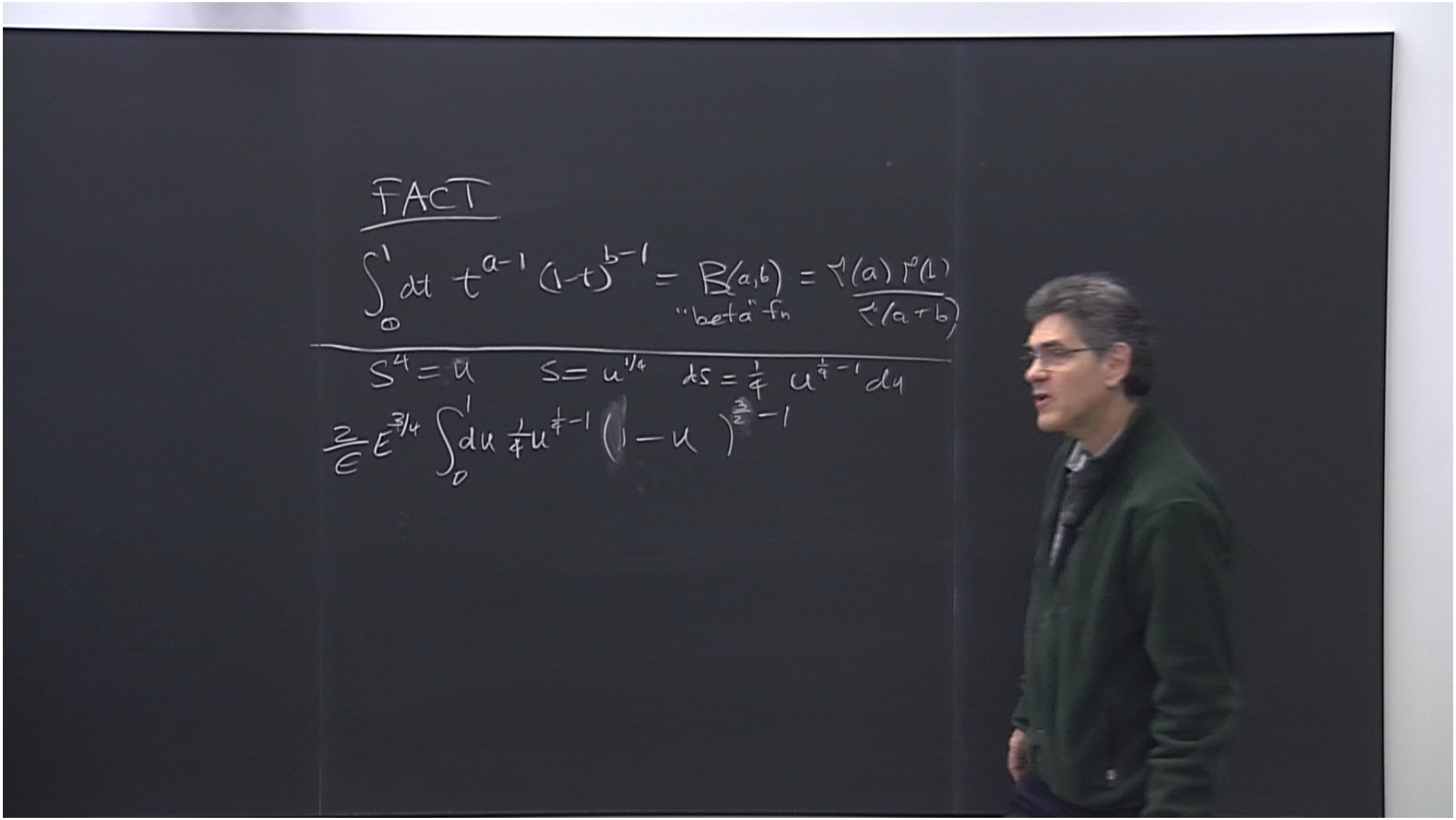
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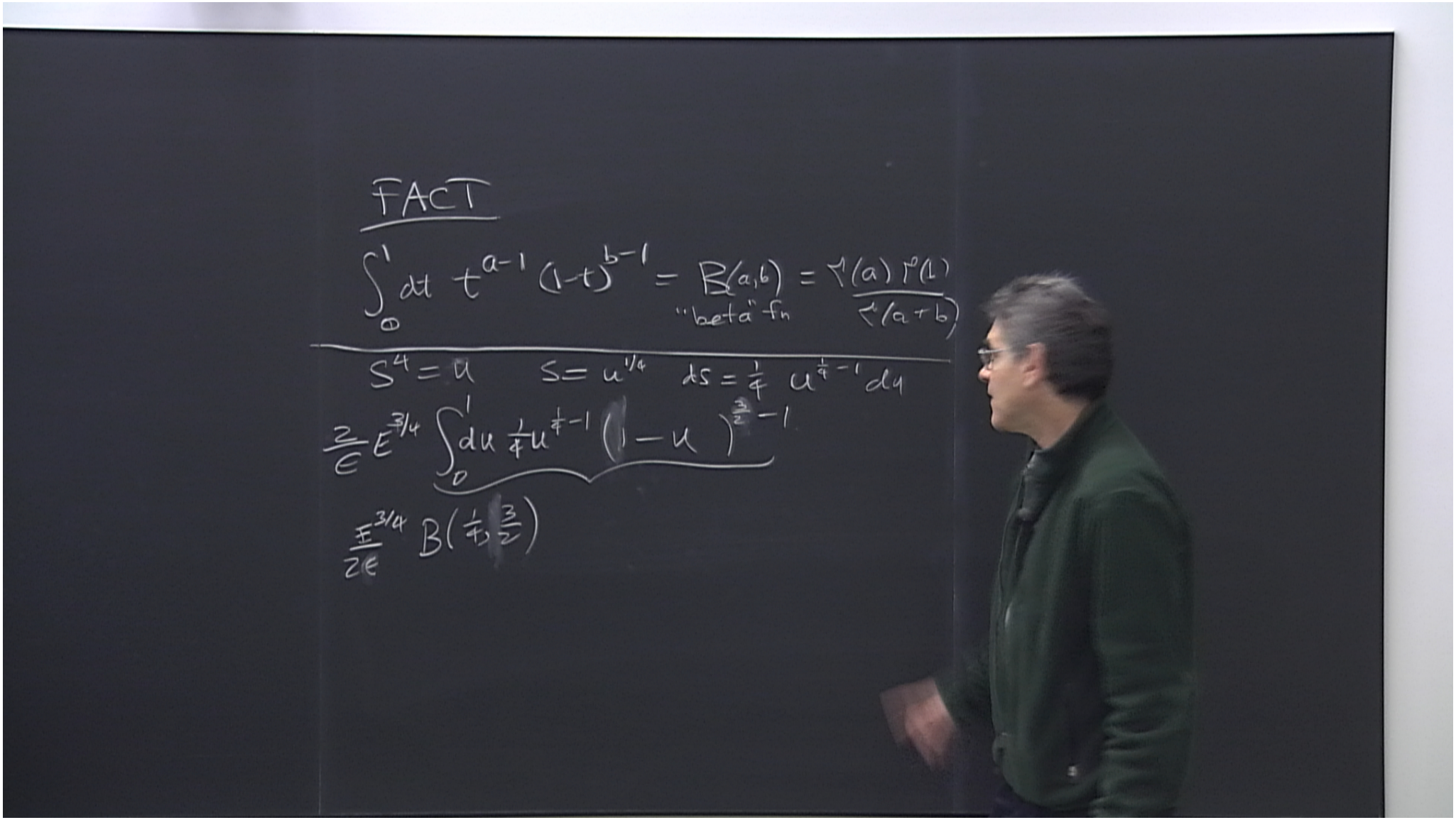
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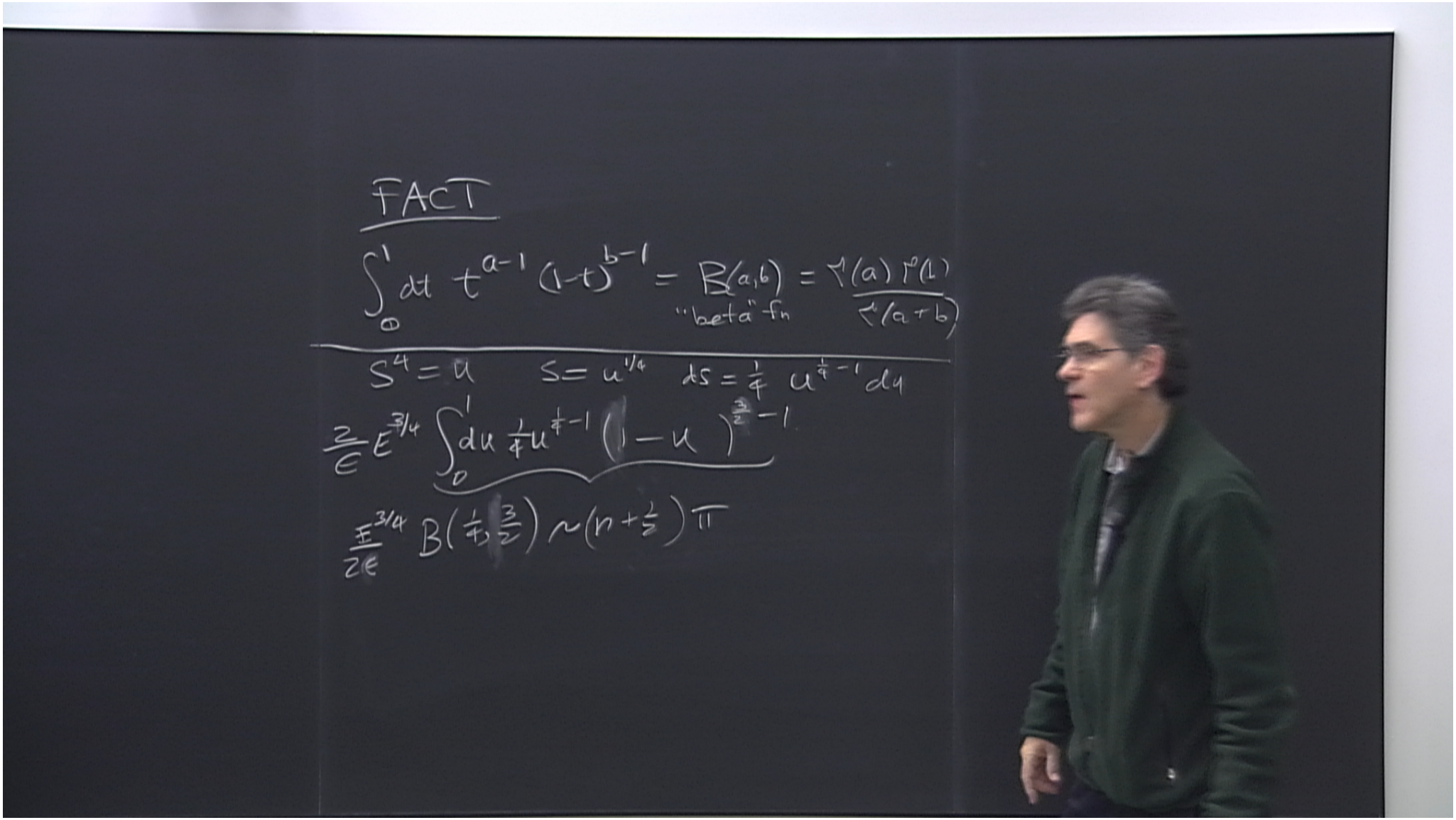
$\frac{E^{3/4}}{2\epsilon} B(\frac{1}{4}, \frac{3}{2})$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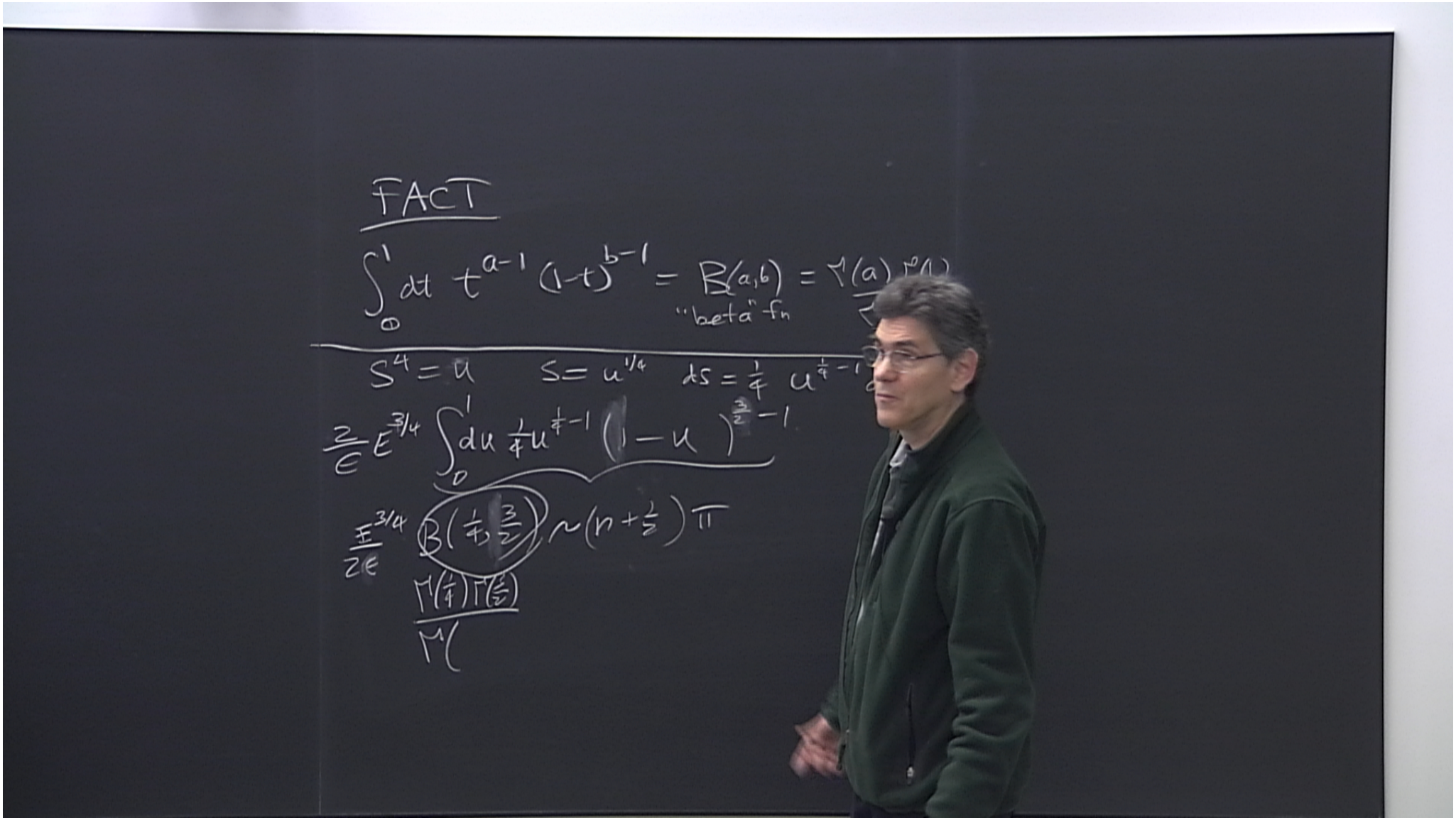
$\frac{E^{3/4}}{2\epsilon} B(\frac{1}{4}, \frac{3}{2}) \sim (n+\frac{1}{2})\pi$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$\frac{E^{3/4}}{2\epsilon} B\left(\frac{1}{4}, \frac{3}{2}\right) \sim \left(n+\frac{1}{2}\right)\pi$
$\frac{\Gamma(\frac{1}{4})\Gamma(\frac{3}{2})}{\Gamma($

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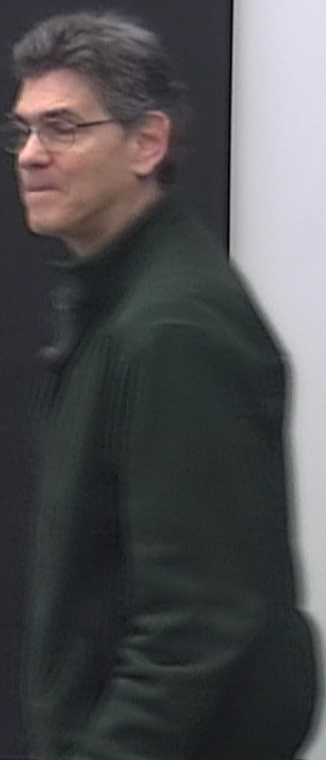
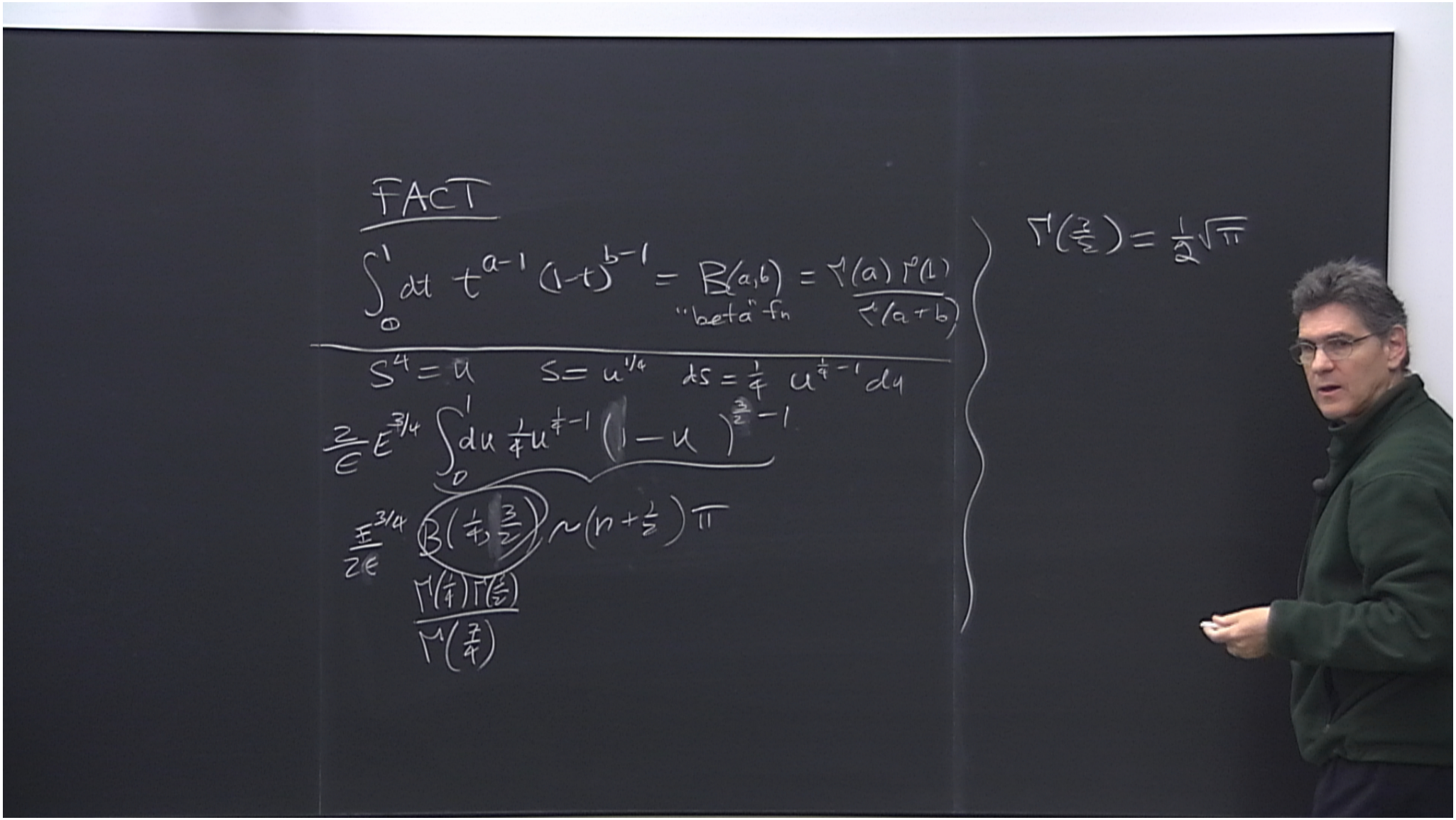
$$\frac{\Gamma(\frac{1}{4})\Gamma(\frac{3}{2})}{\Gamma(\frac{7}{4})}$$

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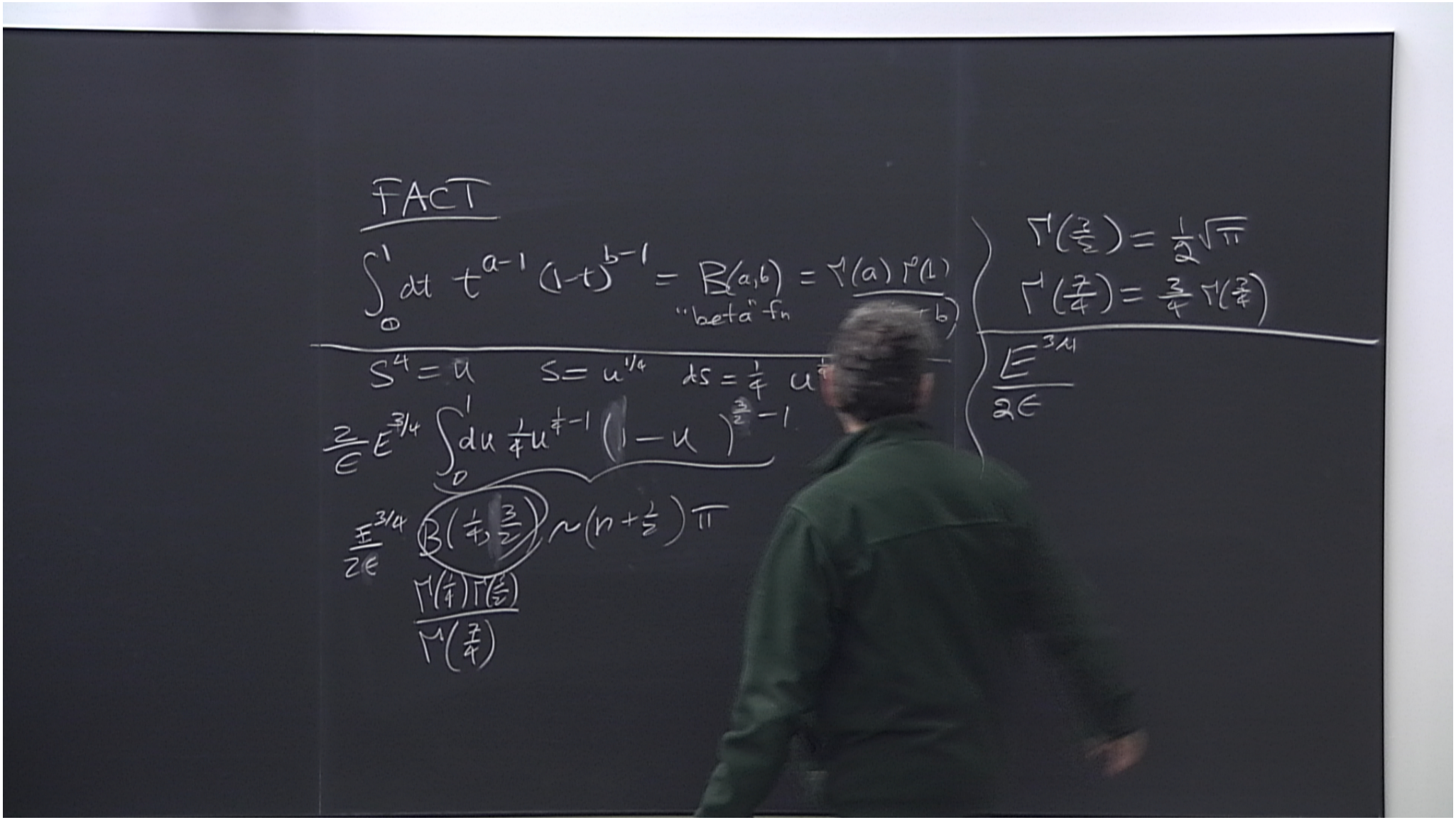
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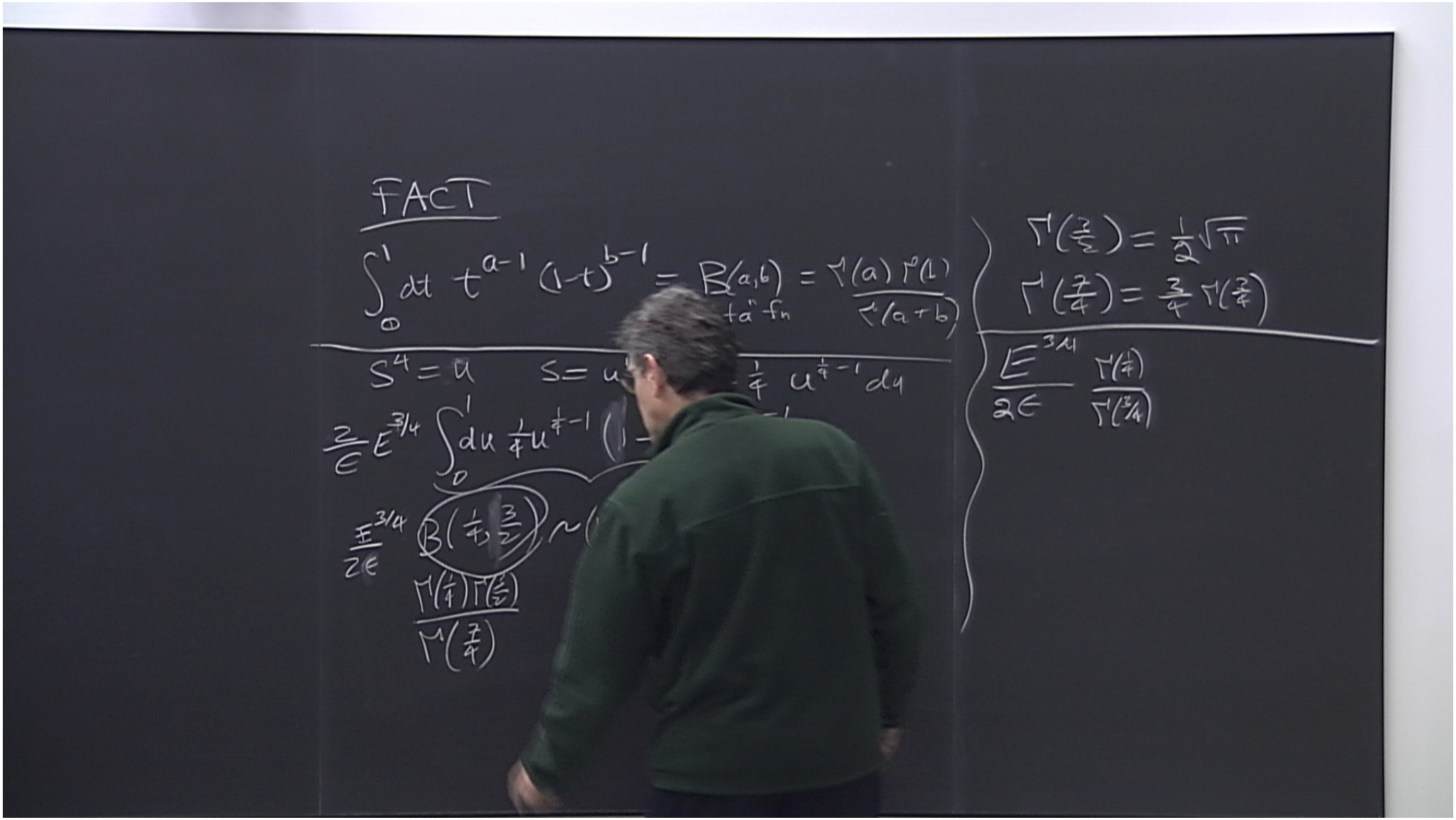
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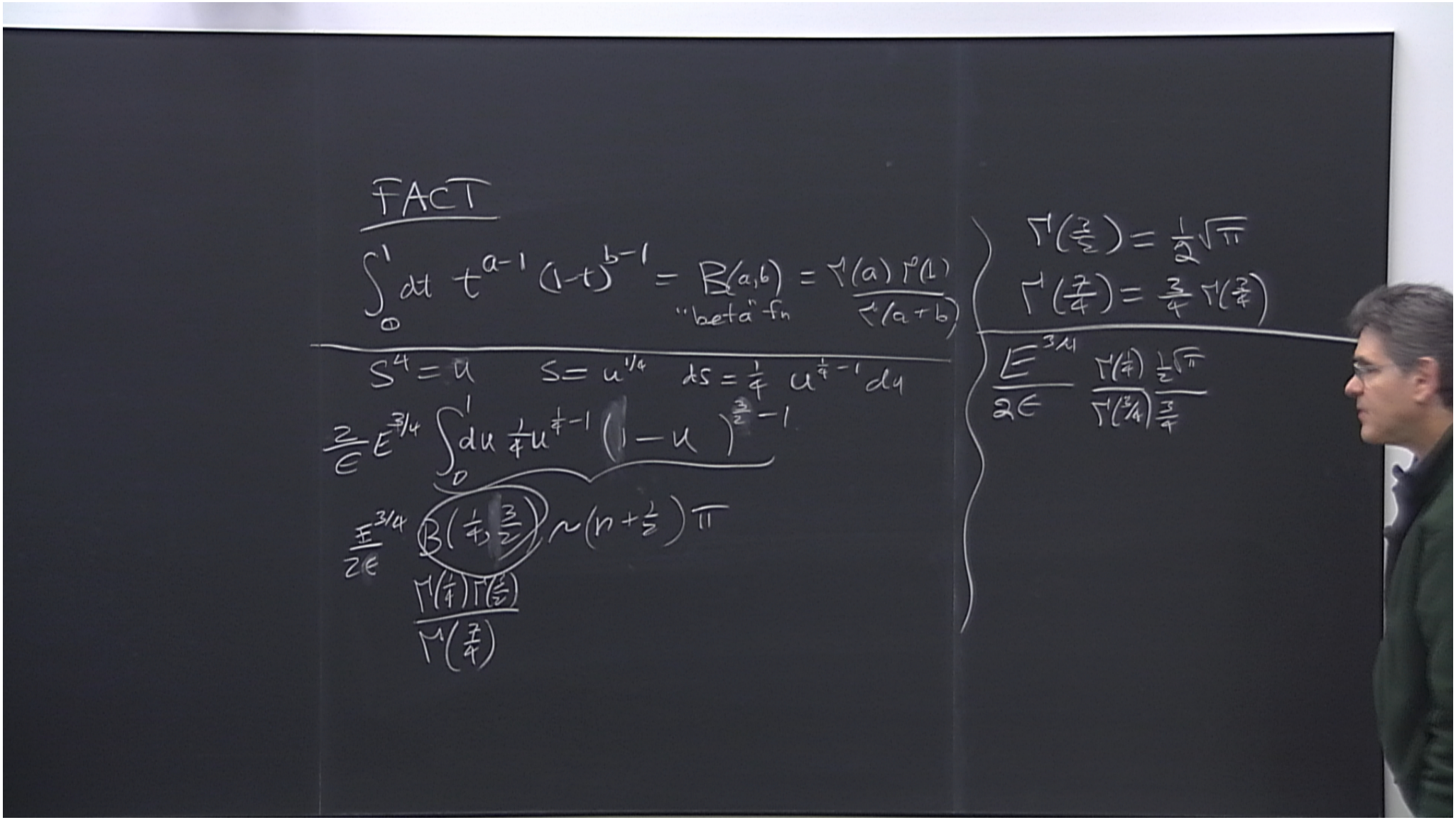
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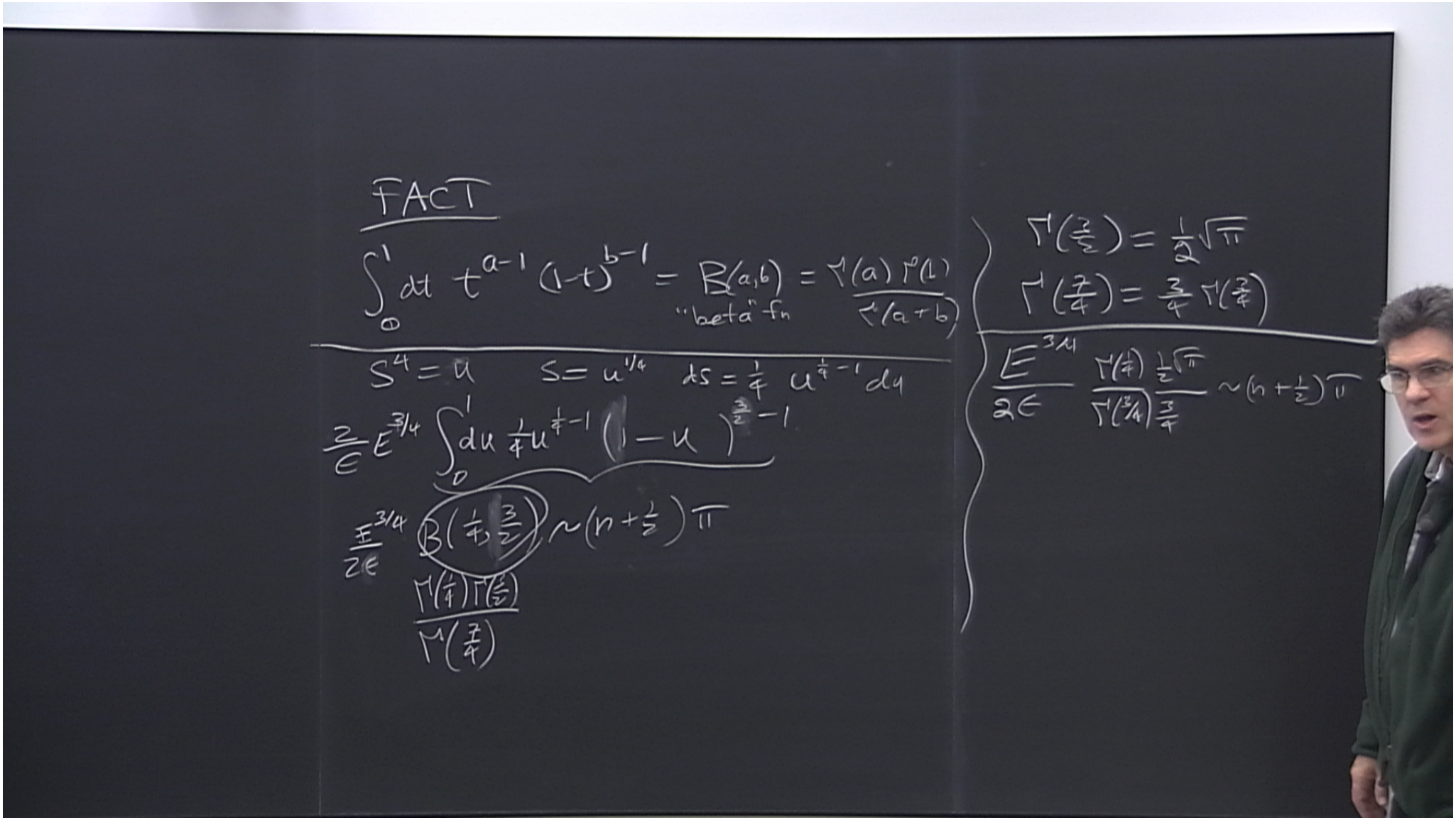
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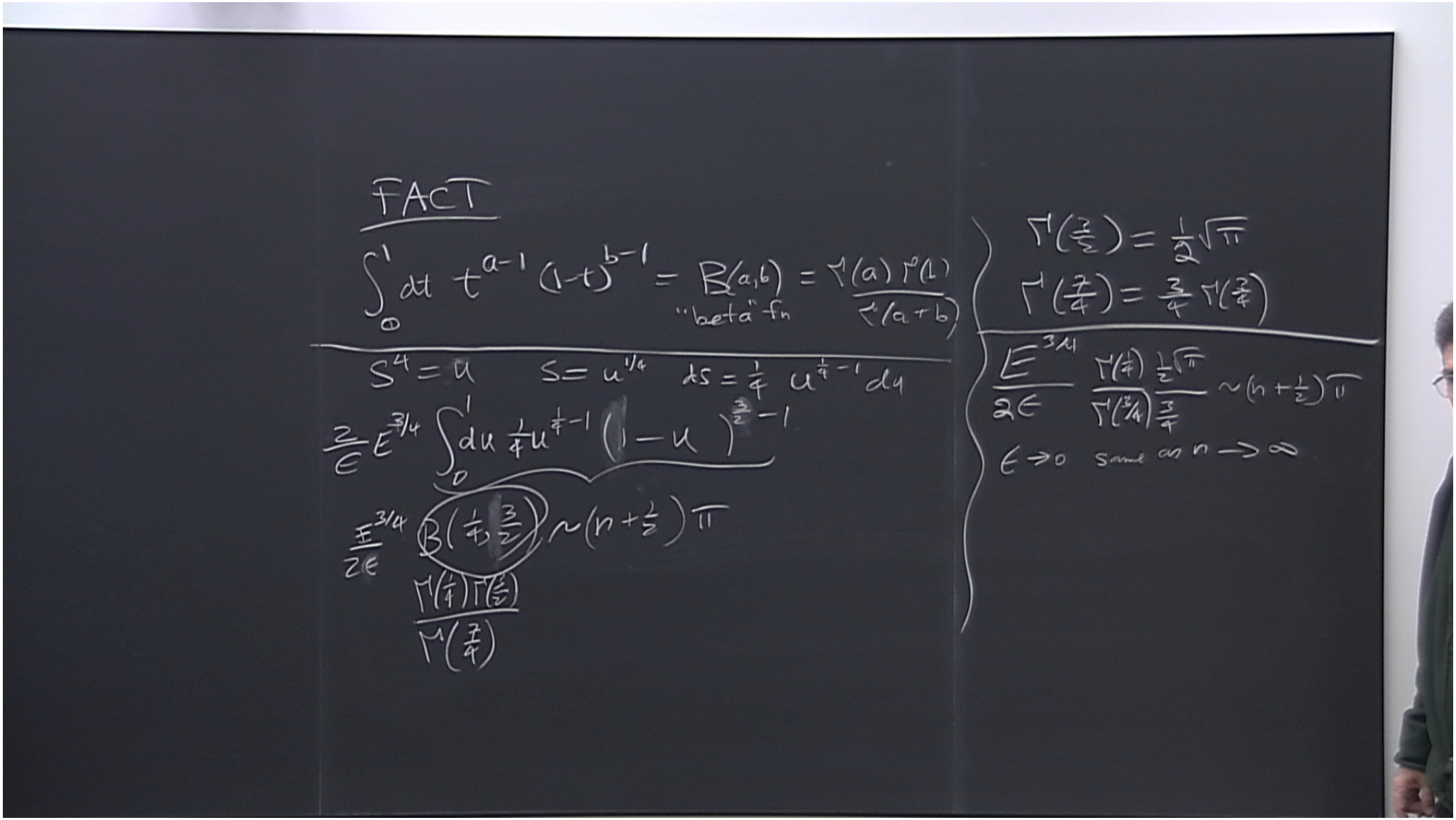
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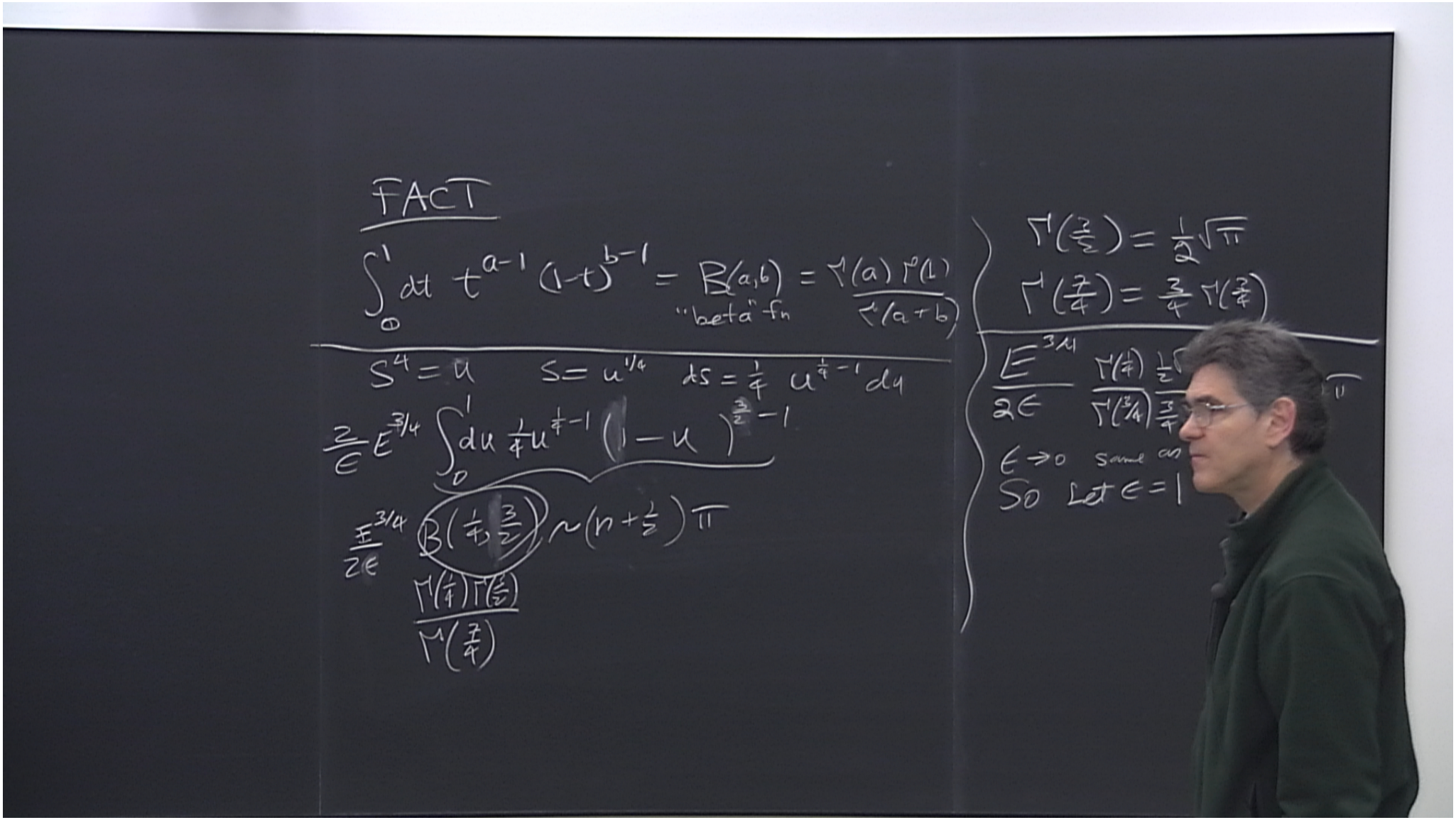
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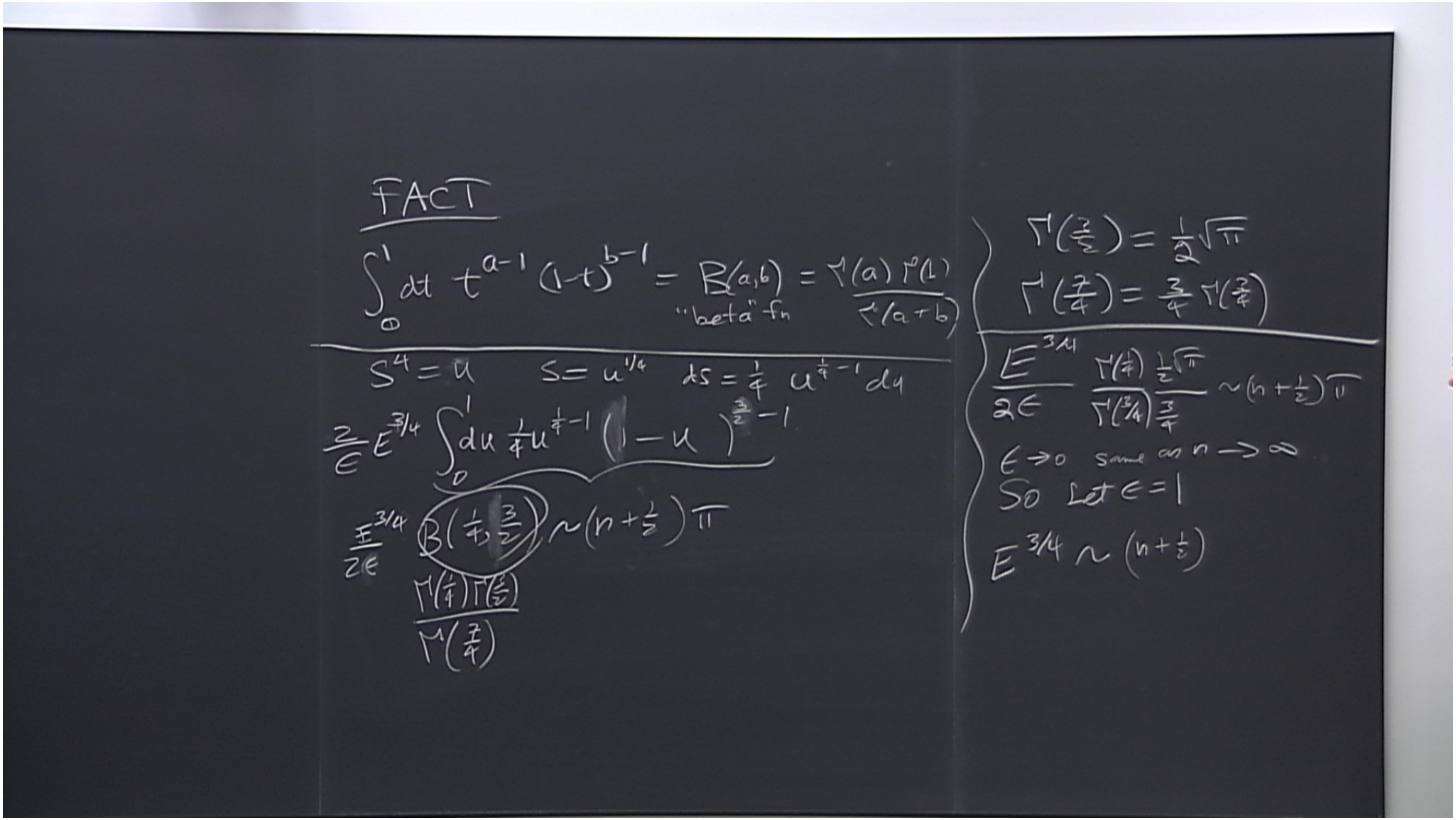
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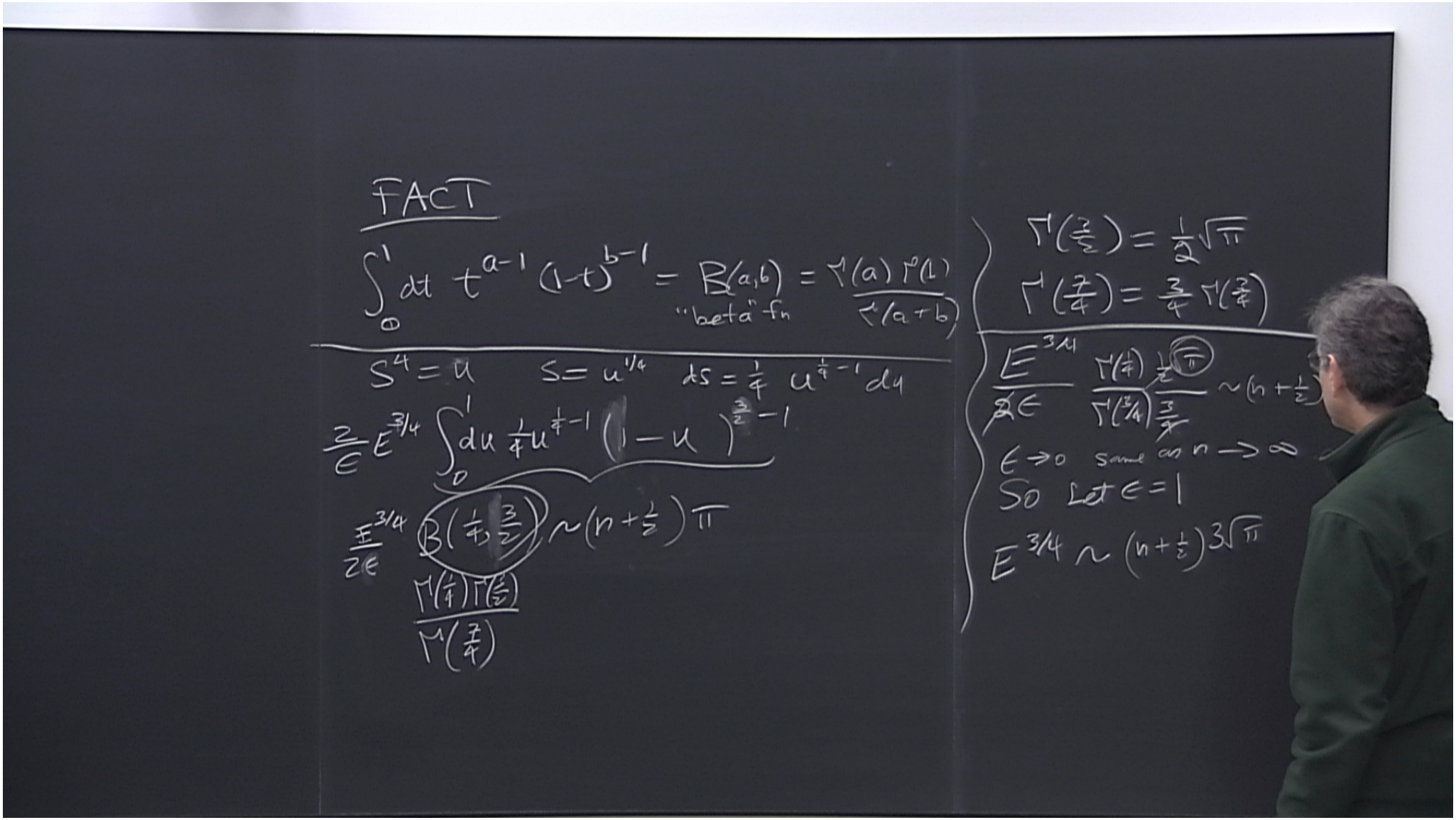
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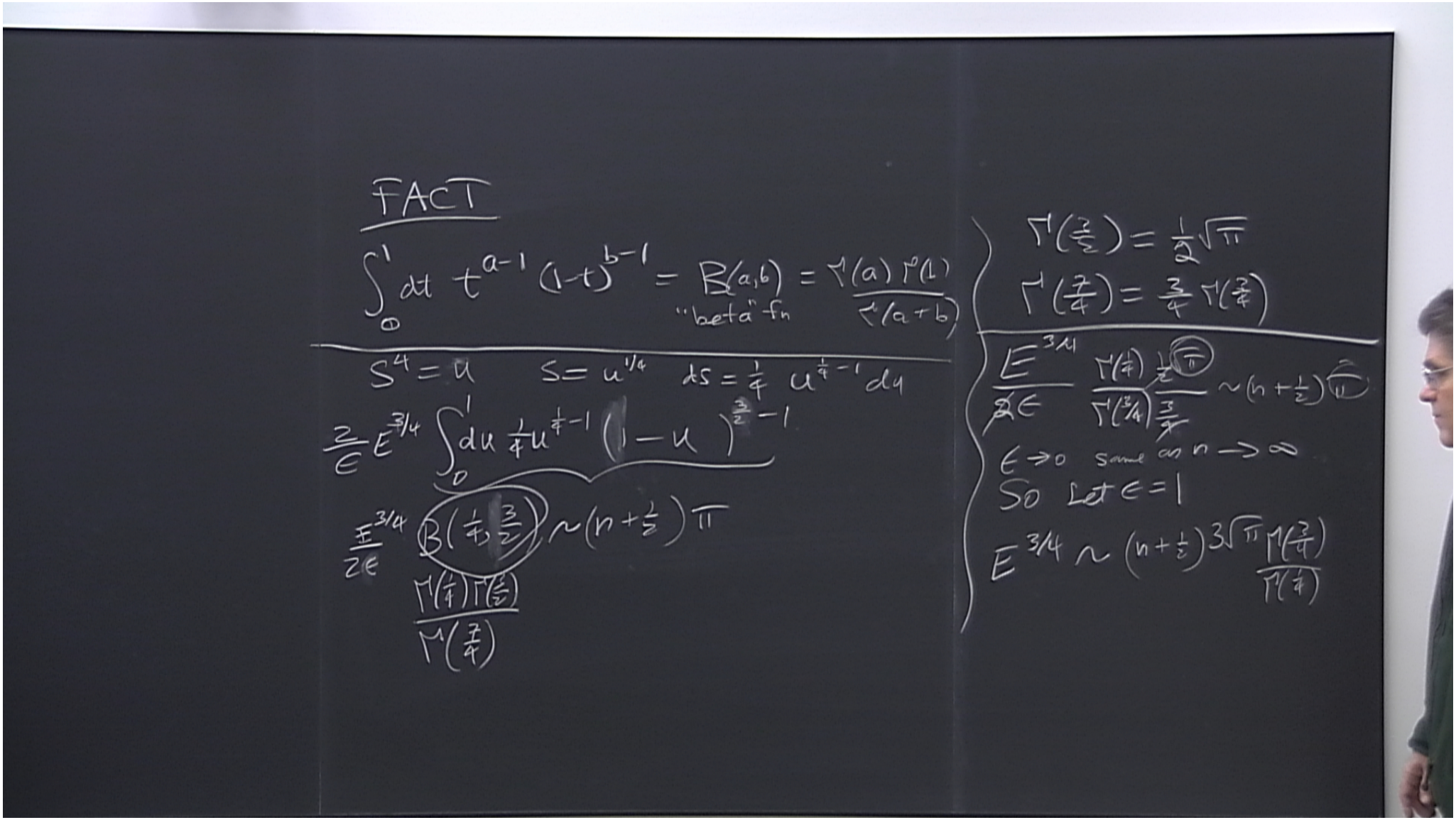
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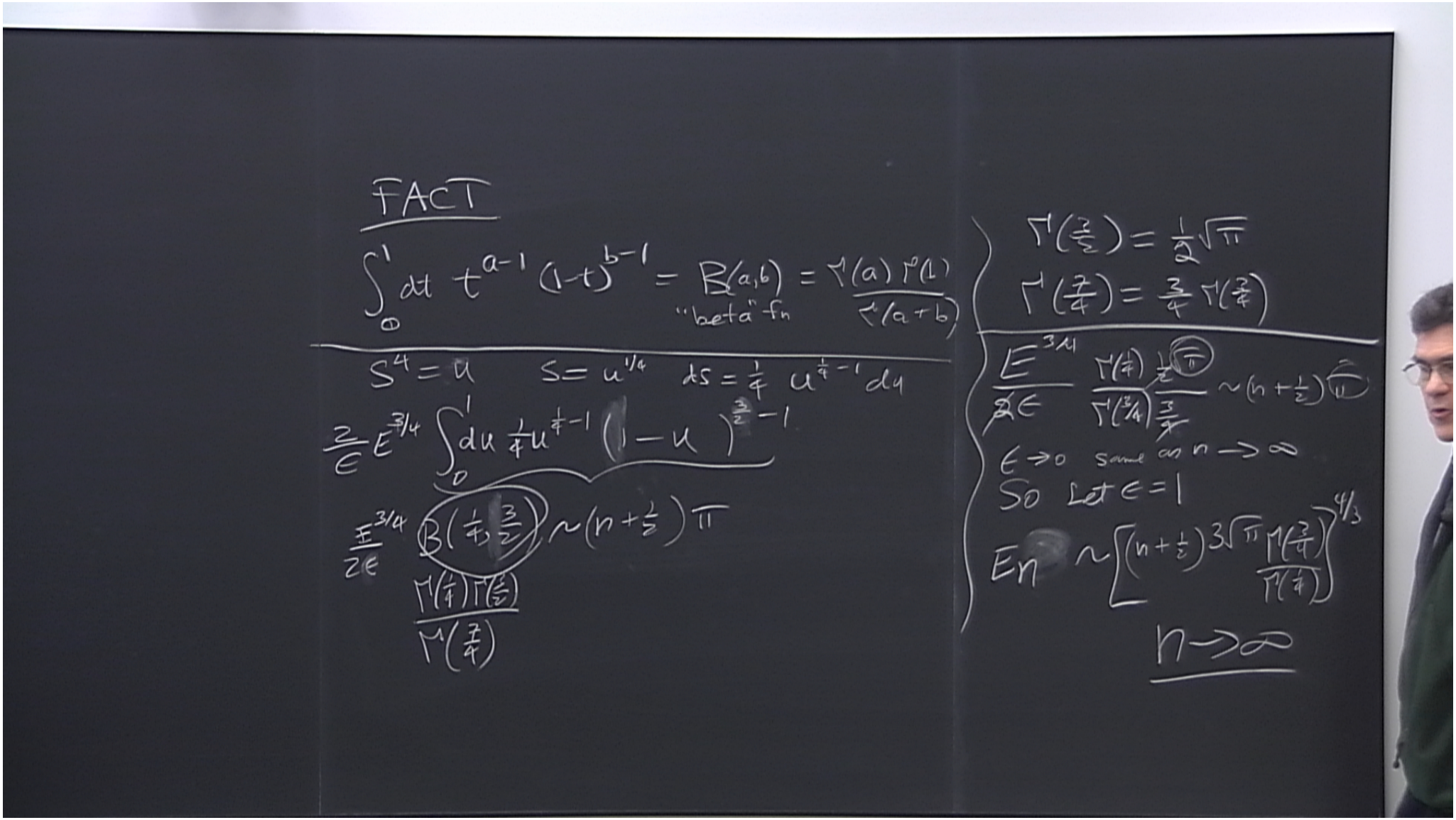
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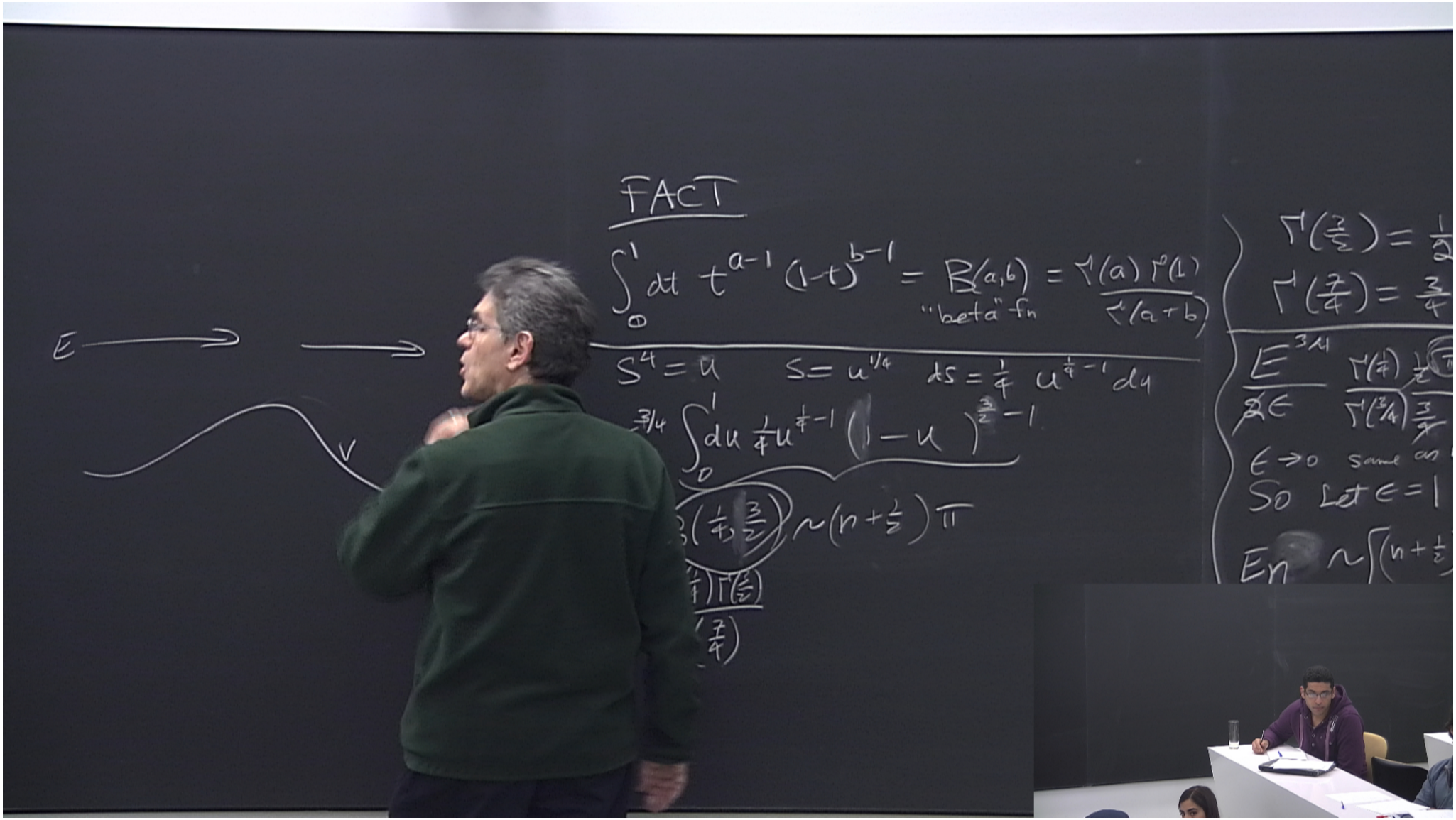
$E_n \sim \left[(n+\frac{1}{2})3\sqrt{\pi}\frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})}\right]^{4/3}$
$n \to \infty$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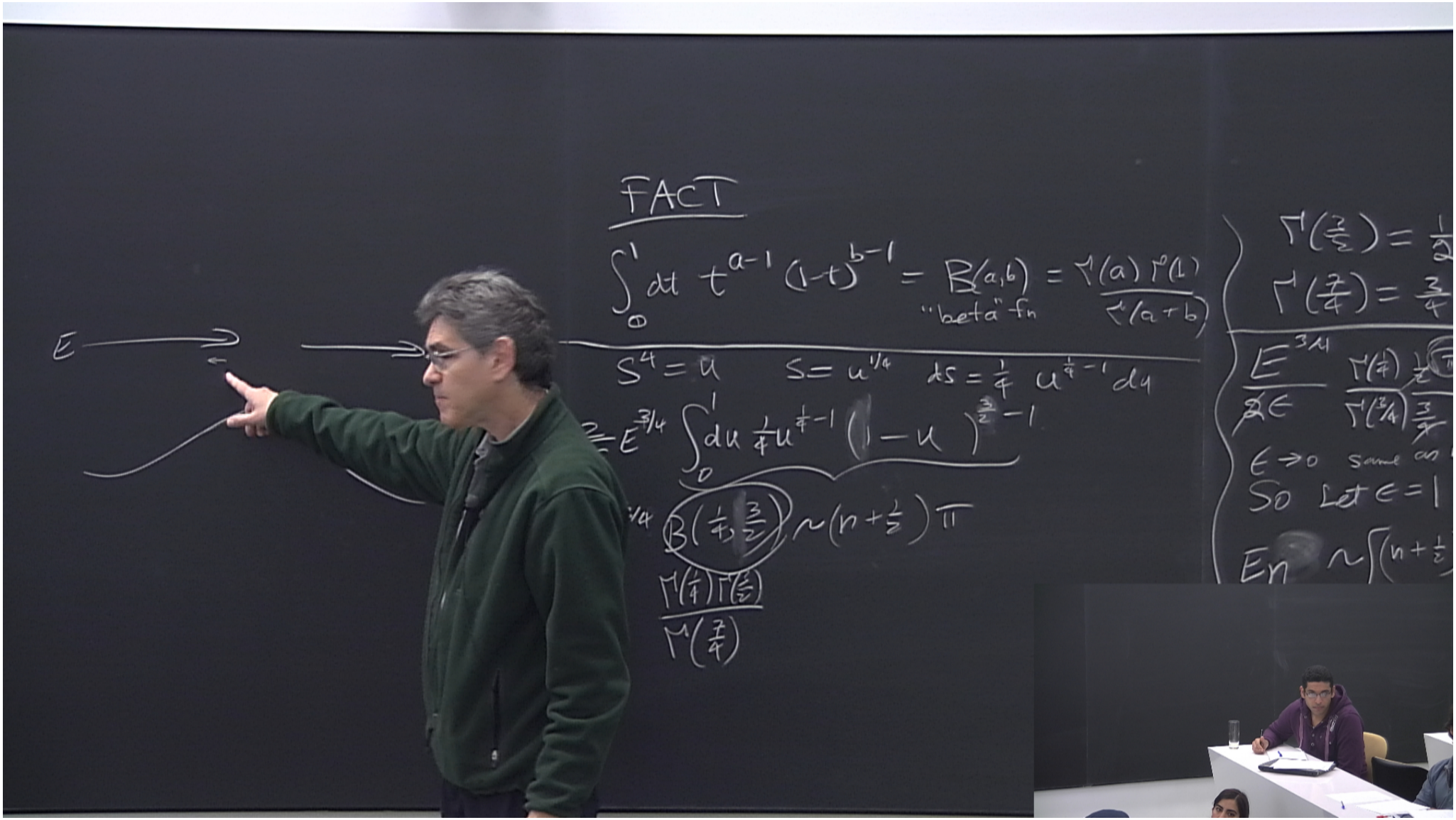
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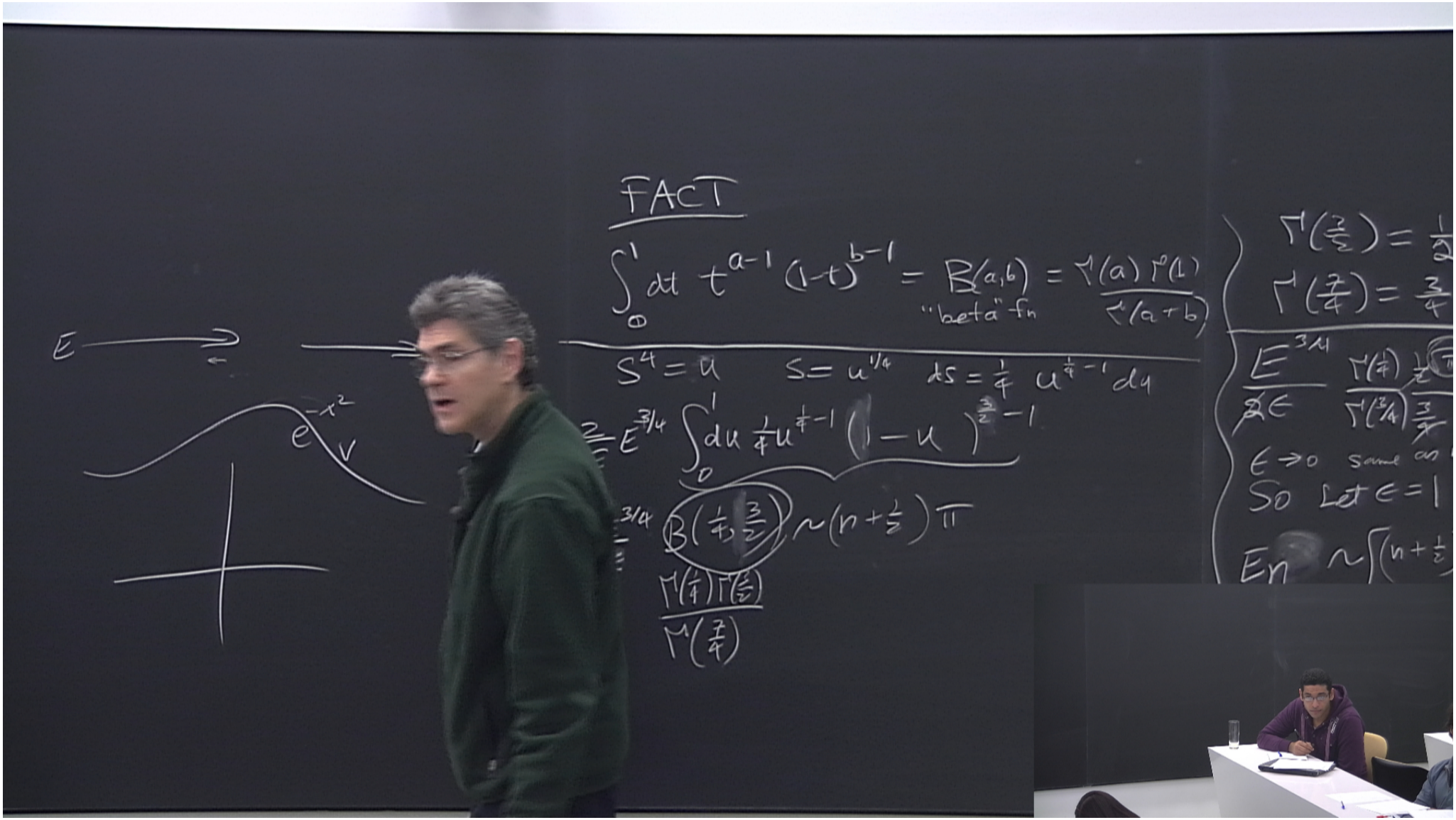
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E
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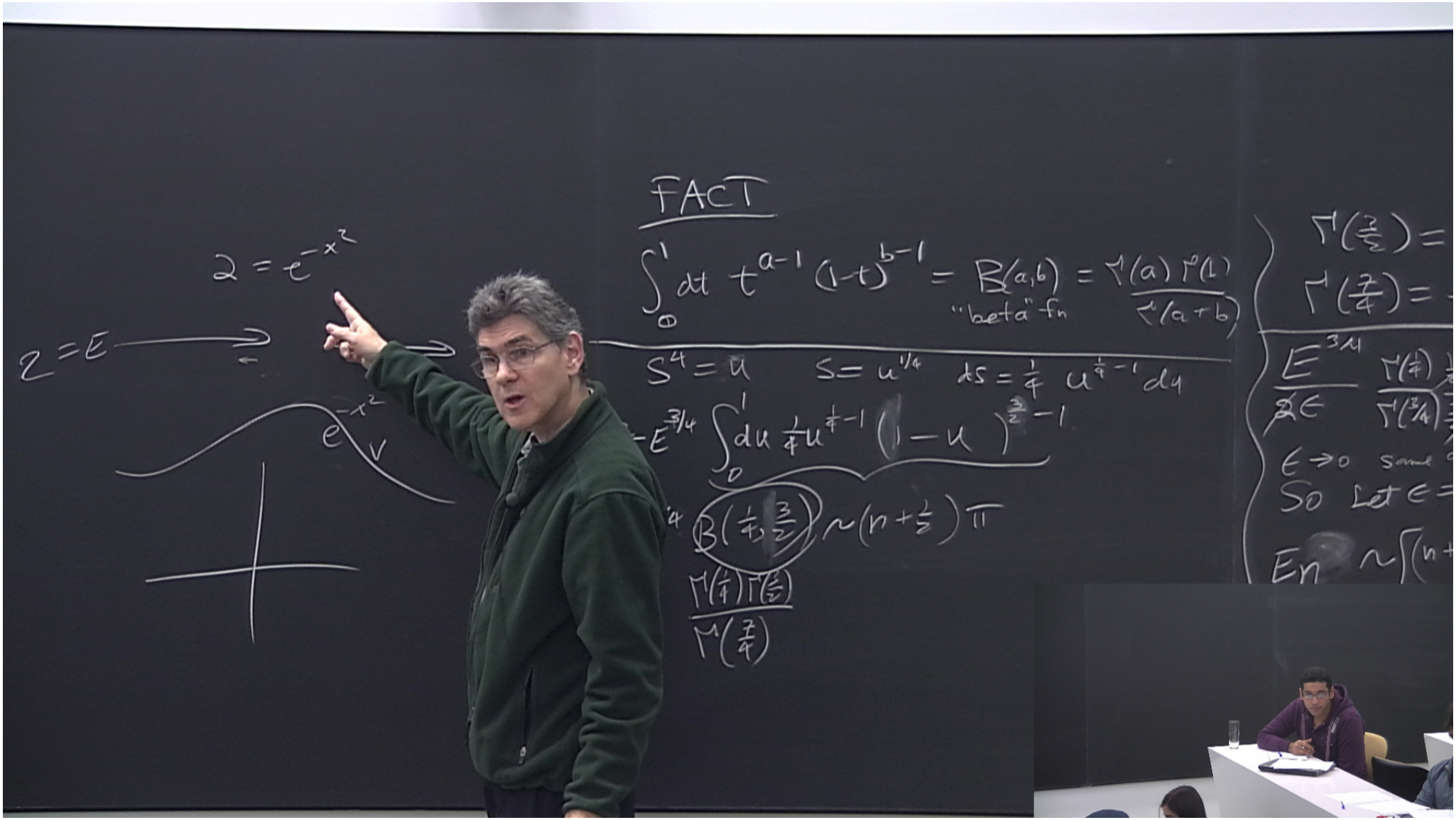
e^{-x^2}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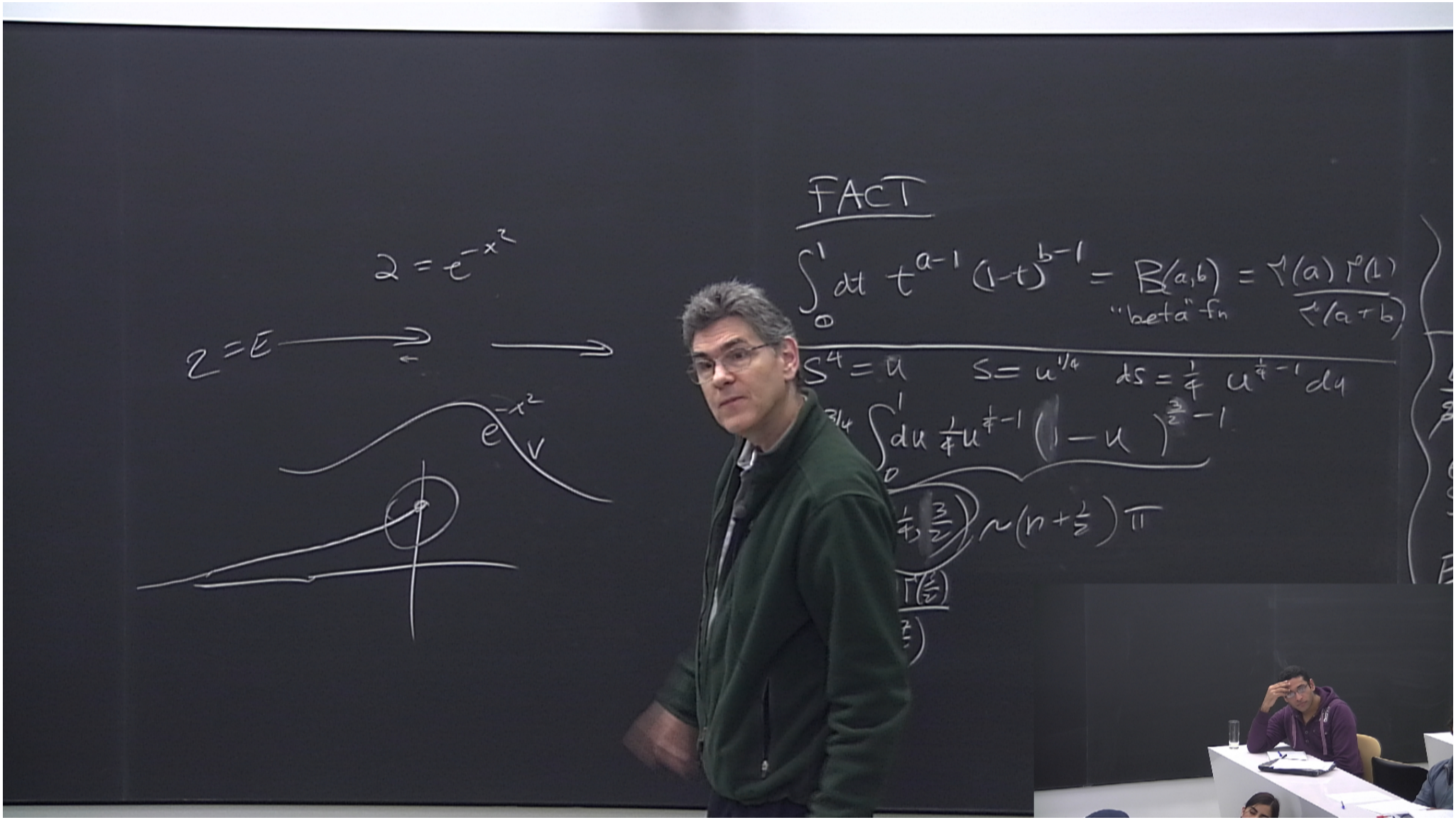
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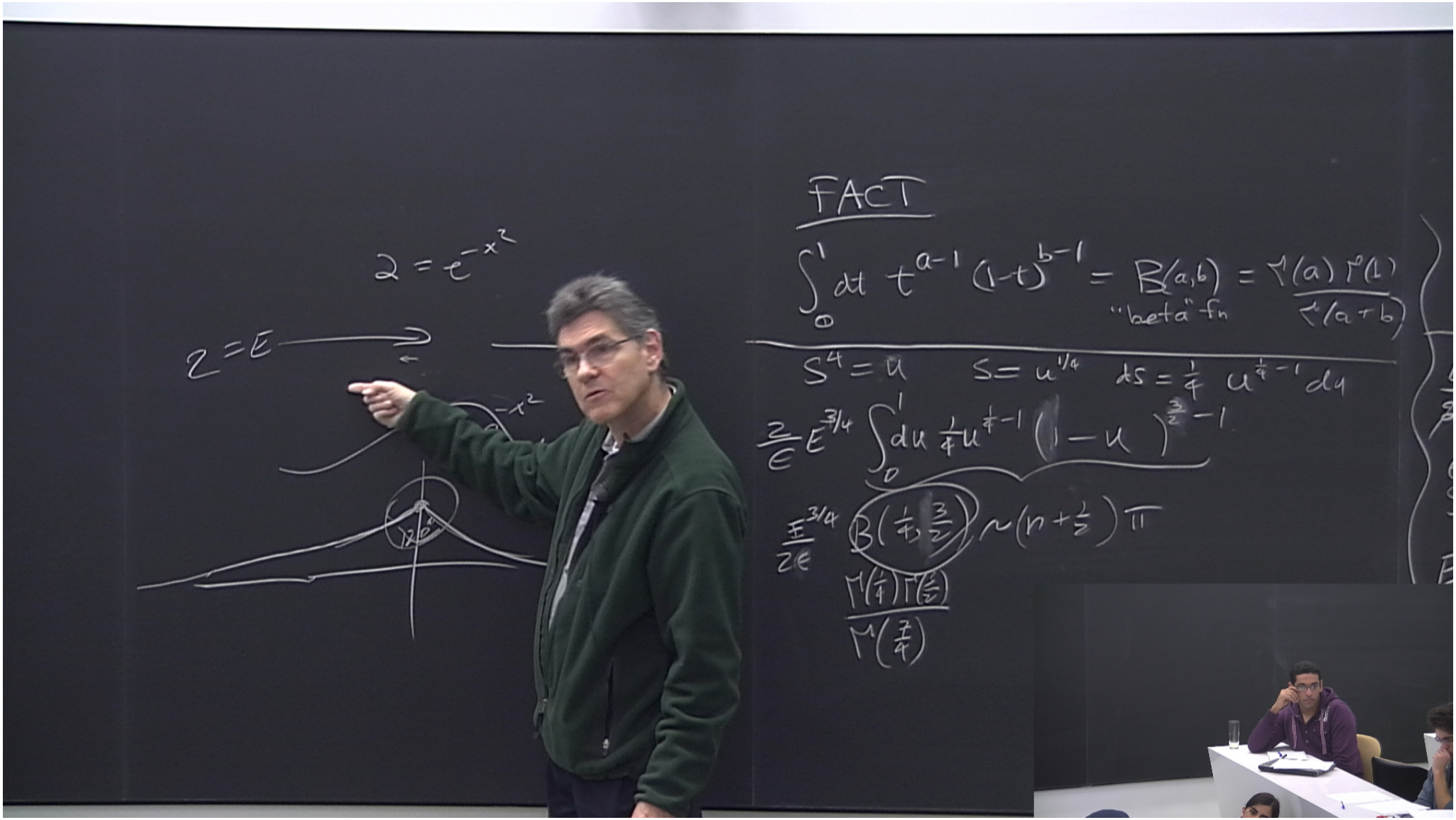
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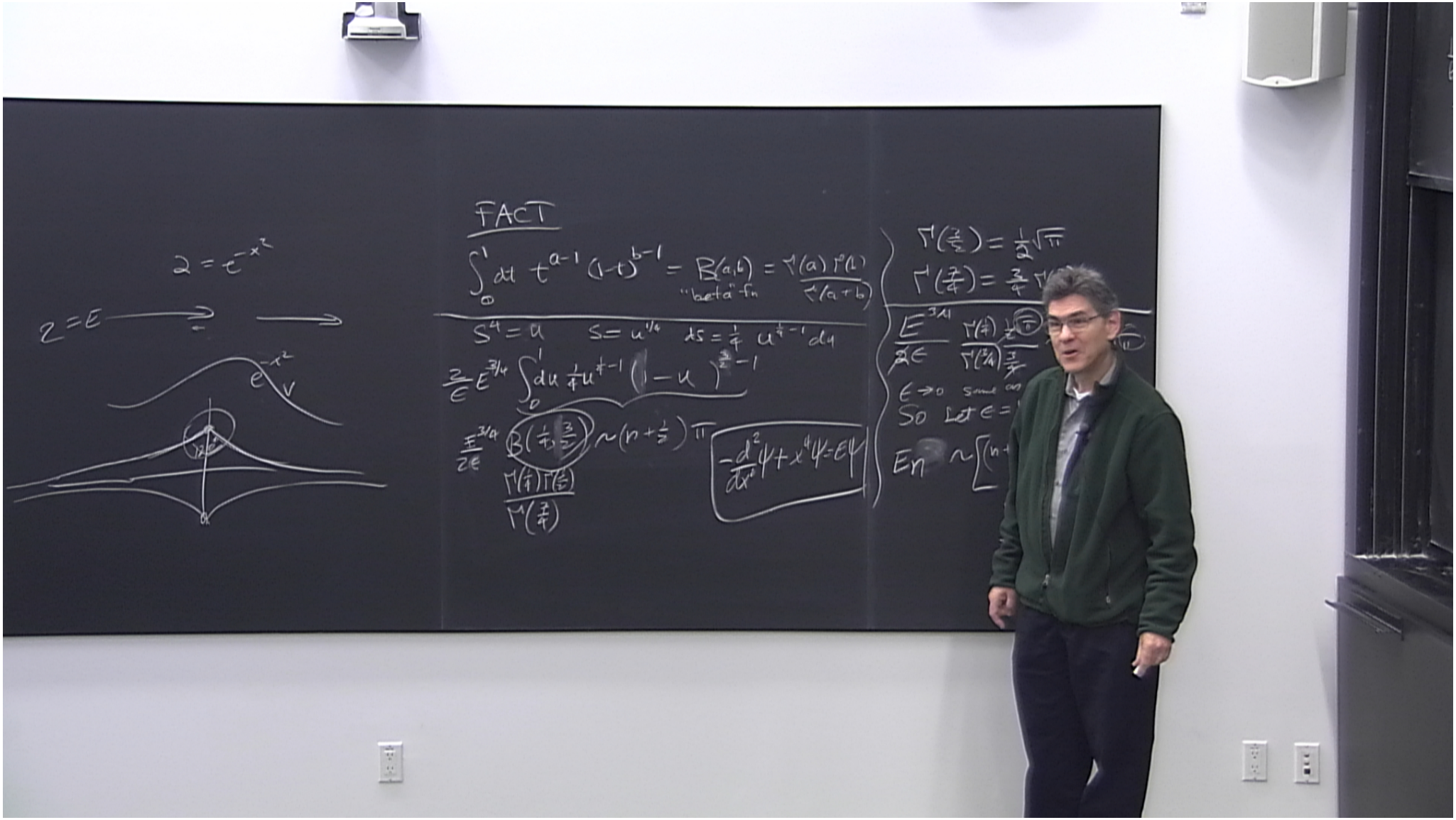
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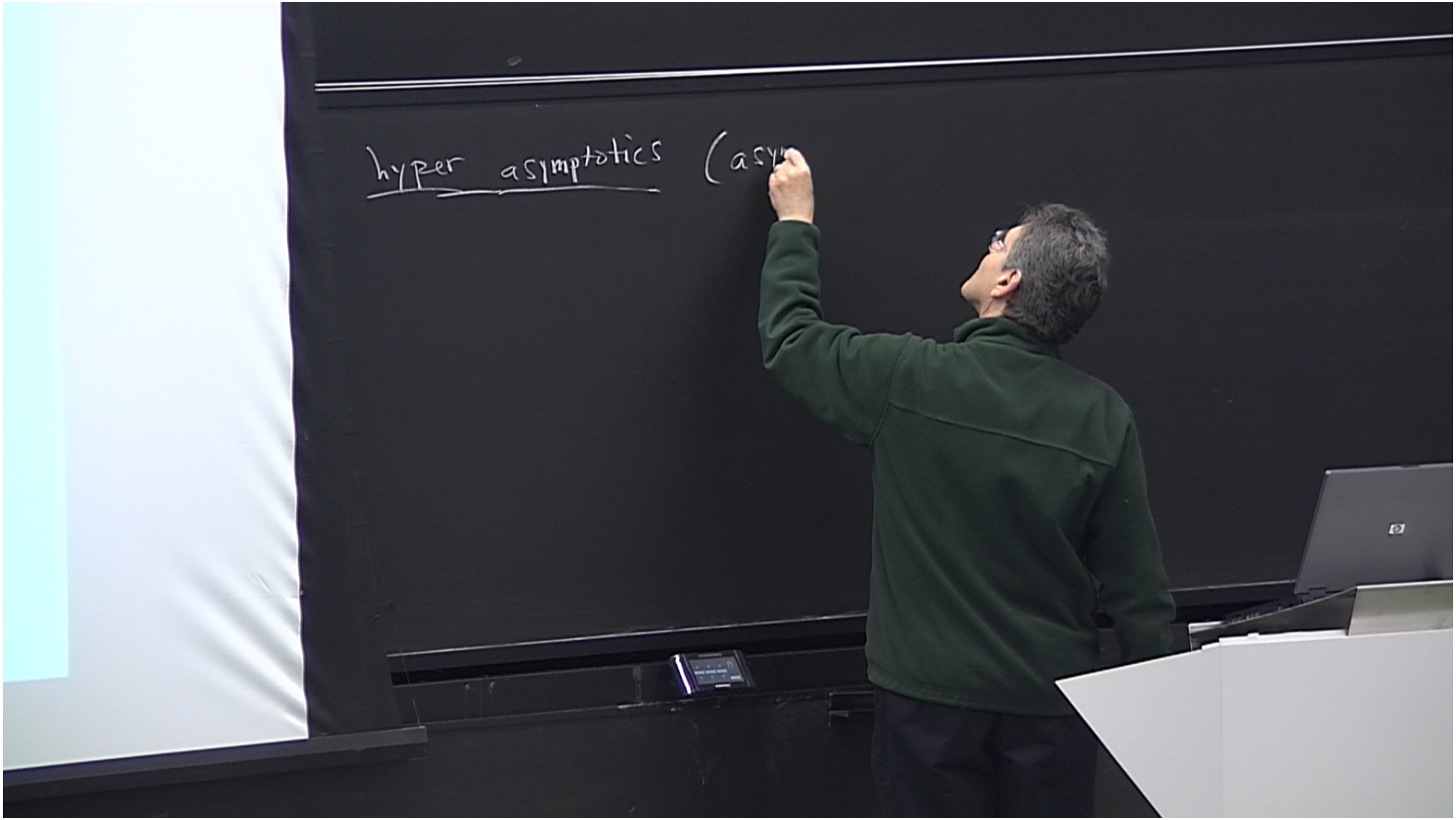
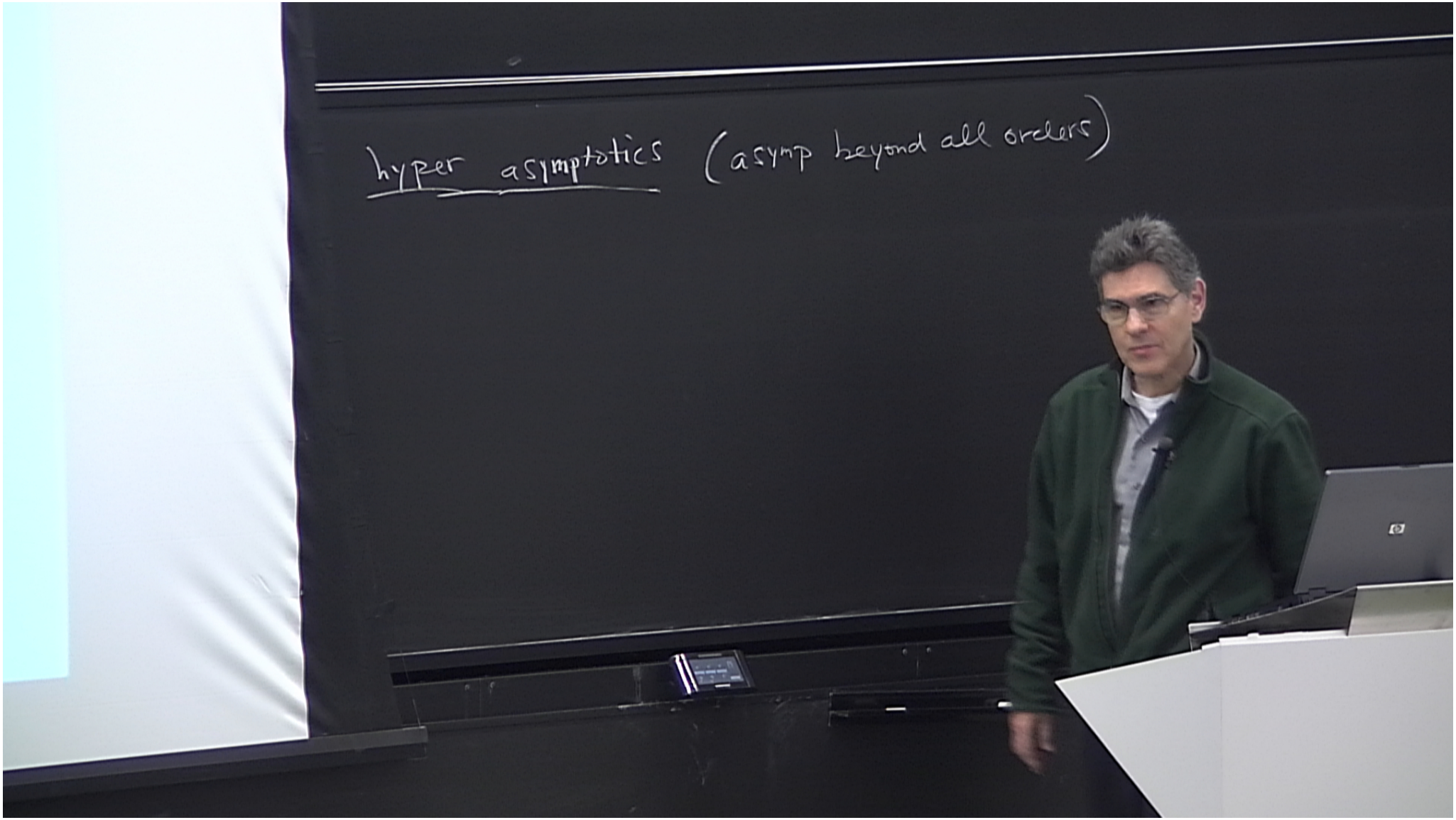
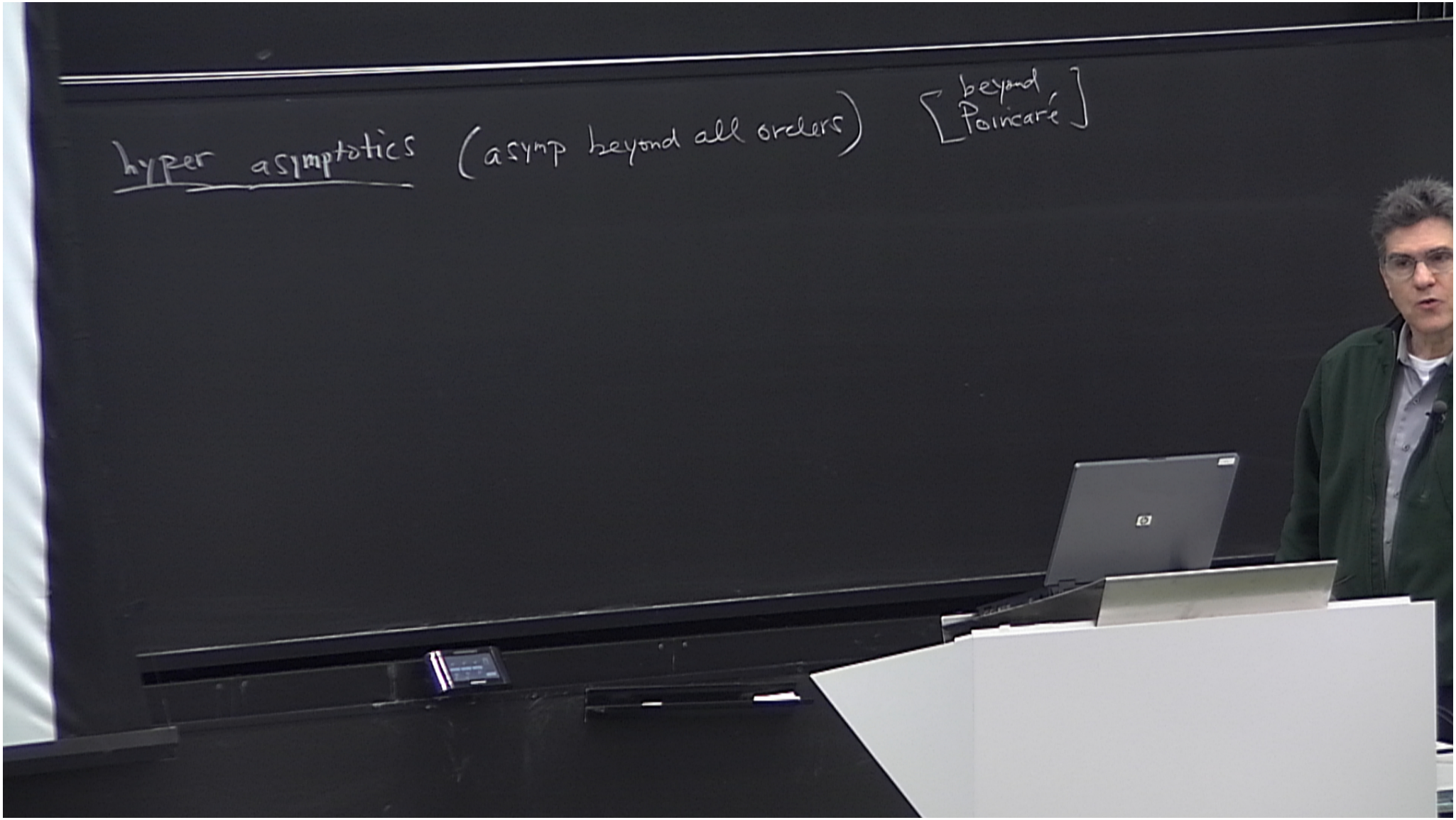
Table 10.4 Comparison of exact eigenvalues E_n for $y'' = (x^4 - E)y$ $[y(\pm\infty) = 0]$ and the WKB prediction for E_n in (10.5.15): $E_n \sim [3\Gamma(\frac{3}{4})(n + \frac{1}{2})\sqrt{\pi}/\Gamma(\frac{1}{4})]^{4/3}$ $(n \to \infty)$

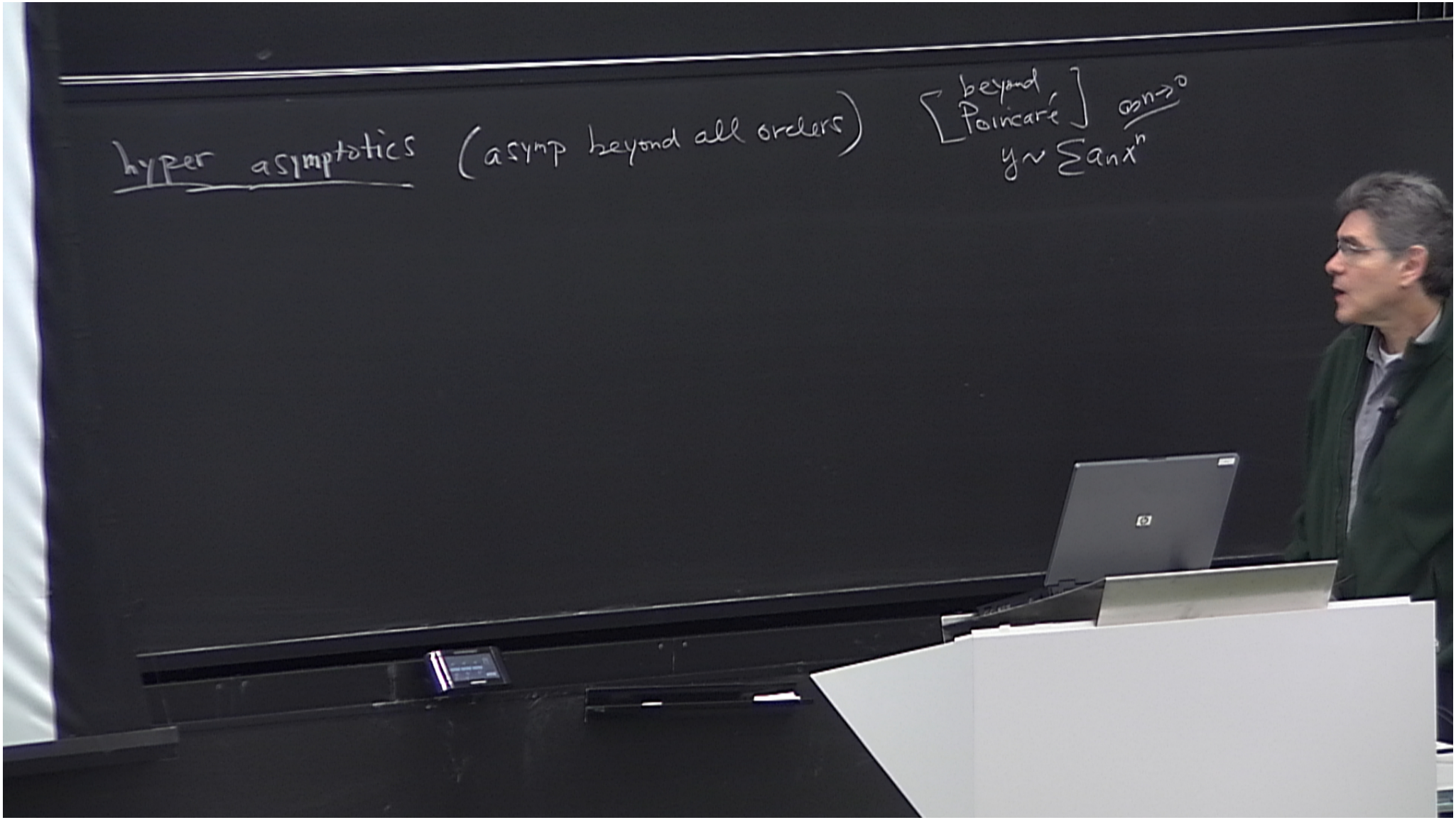
Observe that the relative error [% relative error = 100 (WKB E_n − exact E_n)/(exact E_n)] decreases rapidly as n increases. [Note that $\Gamma(\frac{3}{4})/\Gamma(\frac{1}{4}) \doteq 0.337\,99$.]

n	Exact E_n	WKB E_n	Relative error, %
0	1.060	0.867	−18.00
2	7.456	7.414	−0.56
4	16.262	16.234	−0.17
6	26.528	26.506	−0.08
8	37.923	37.904	−0.05
10	50.256	50.240	−0.03


hyper asymptotics (asy


hyper asymptotics (asymp beyond all orders)


hyper asymptotics (asymp beyond all orders) [beyond Poincaré]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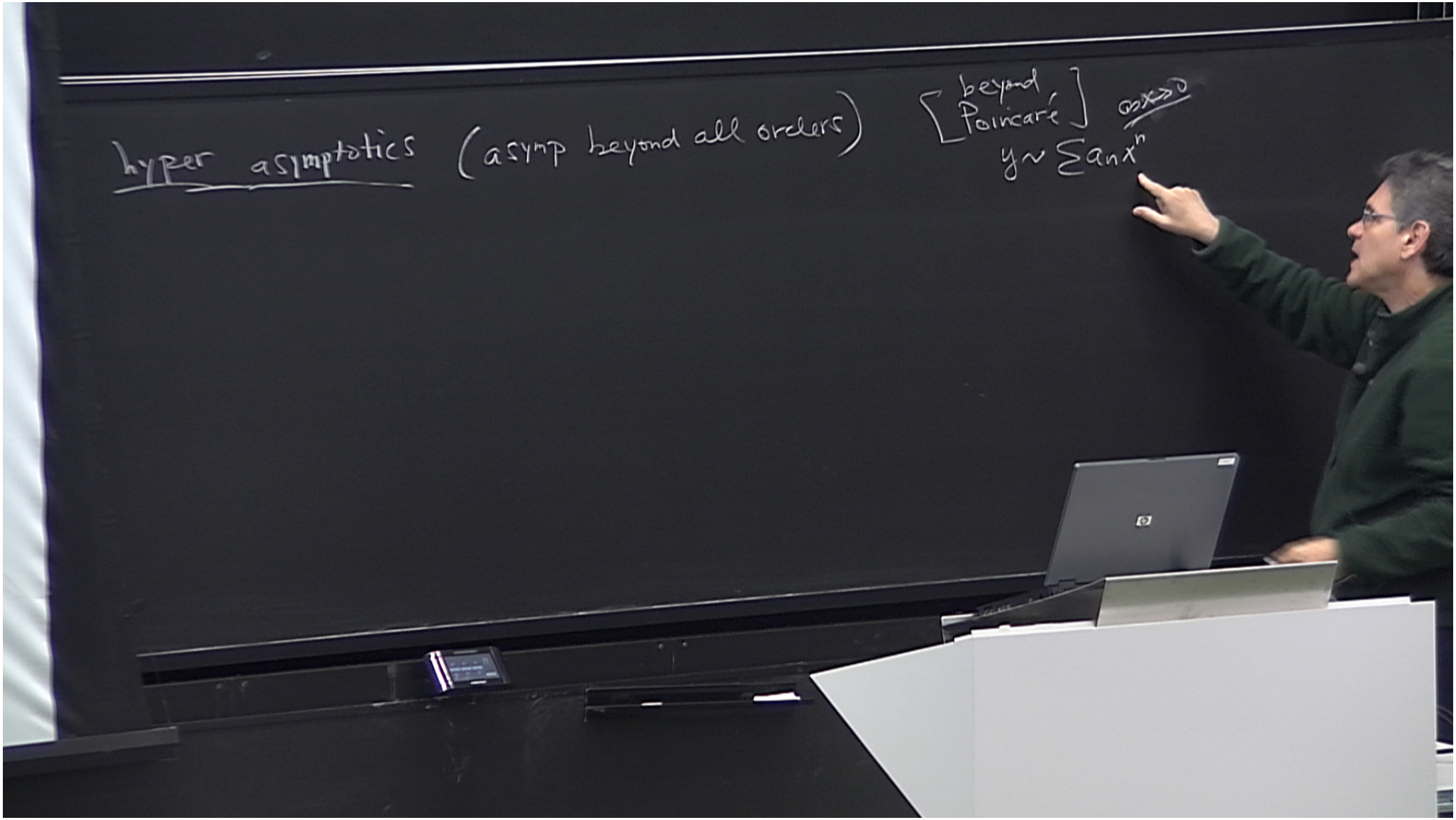
[beyond Poincaré]
$a_{3n} \to 0$
$y \sim \sum a_n x^n$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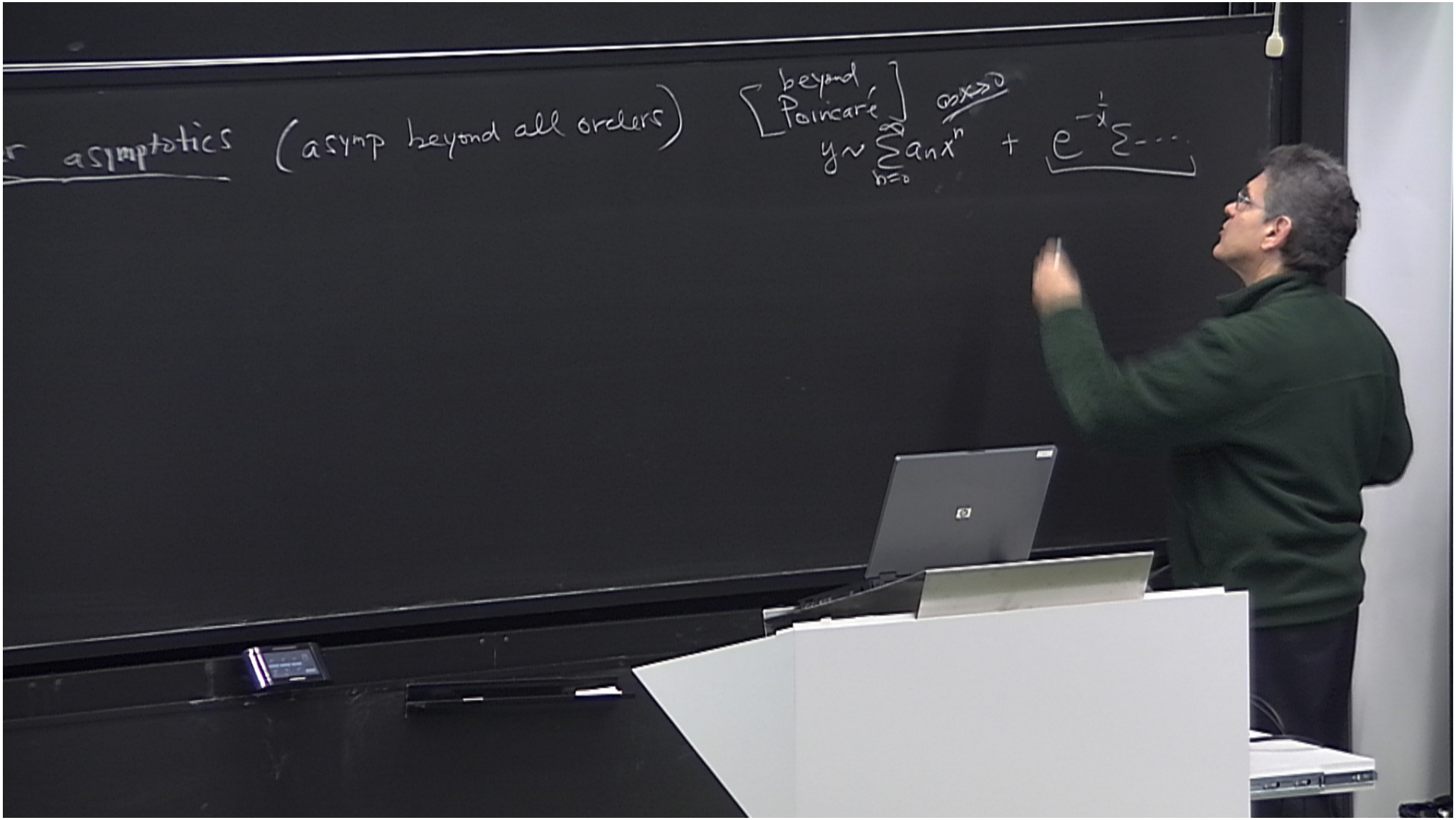
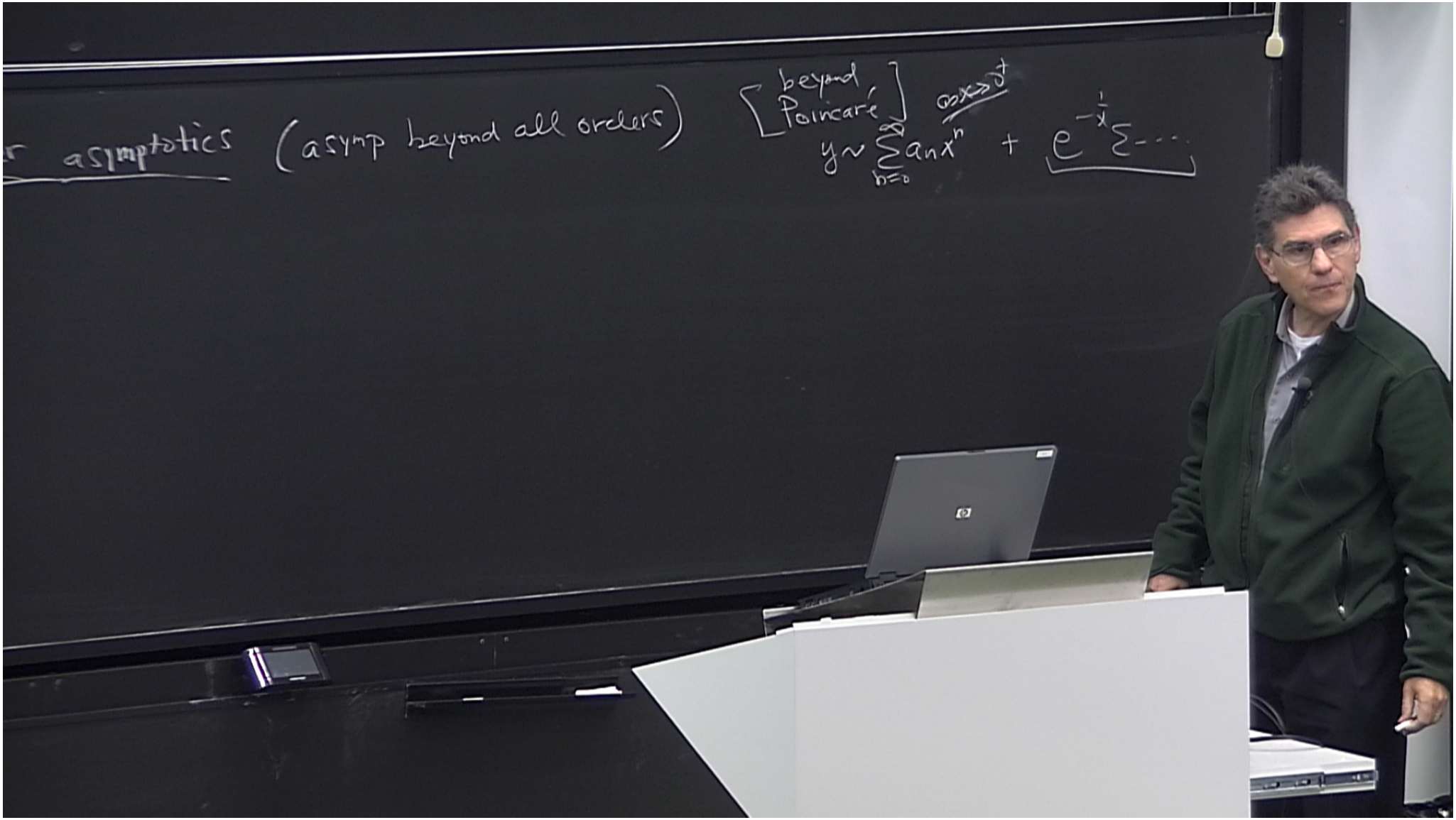

r asymptotics (asymp beyond all orders)
[beyond Poincaré]
$y \sim \sum_{n=0}^{\infty} a_n x^n + e^{-\frac{1}{x}} \Sigma \cdots$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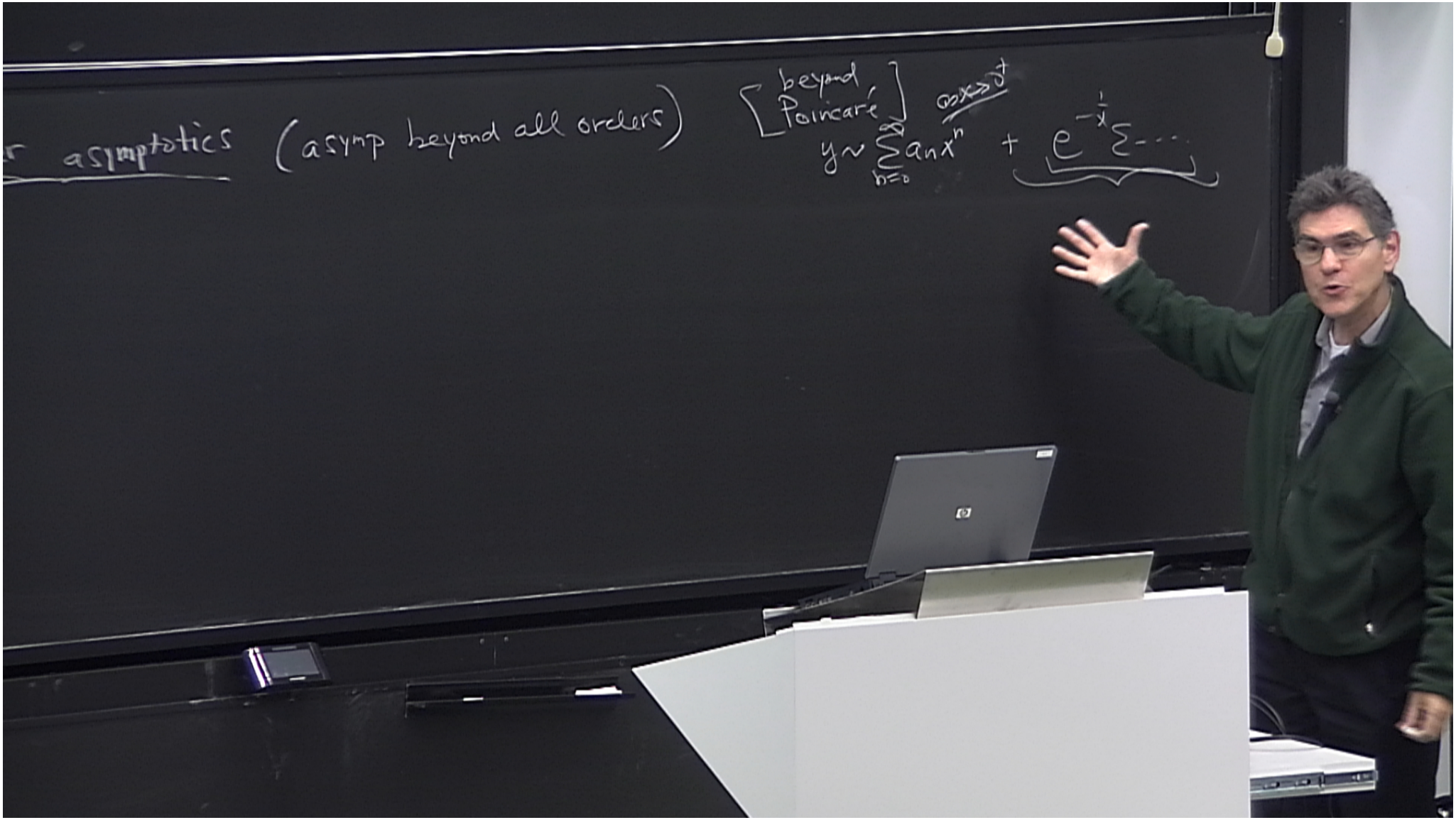
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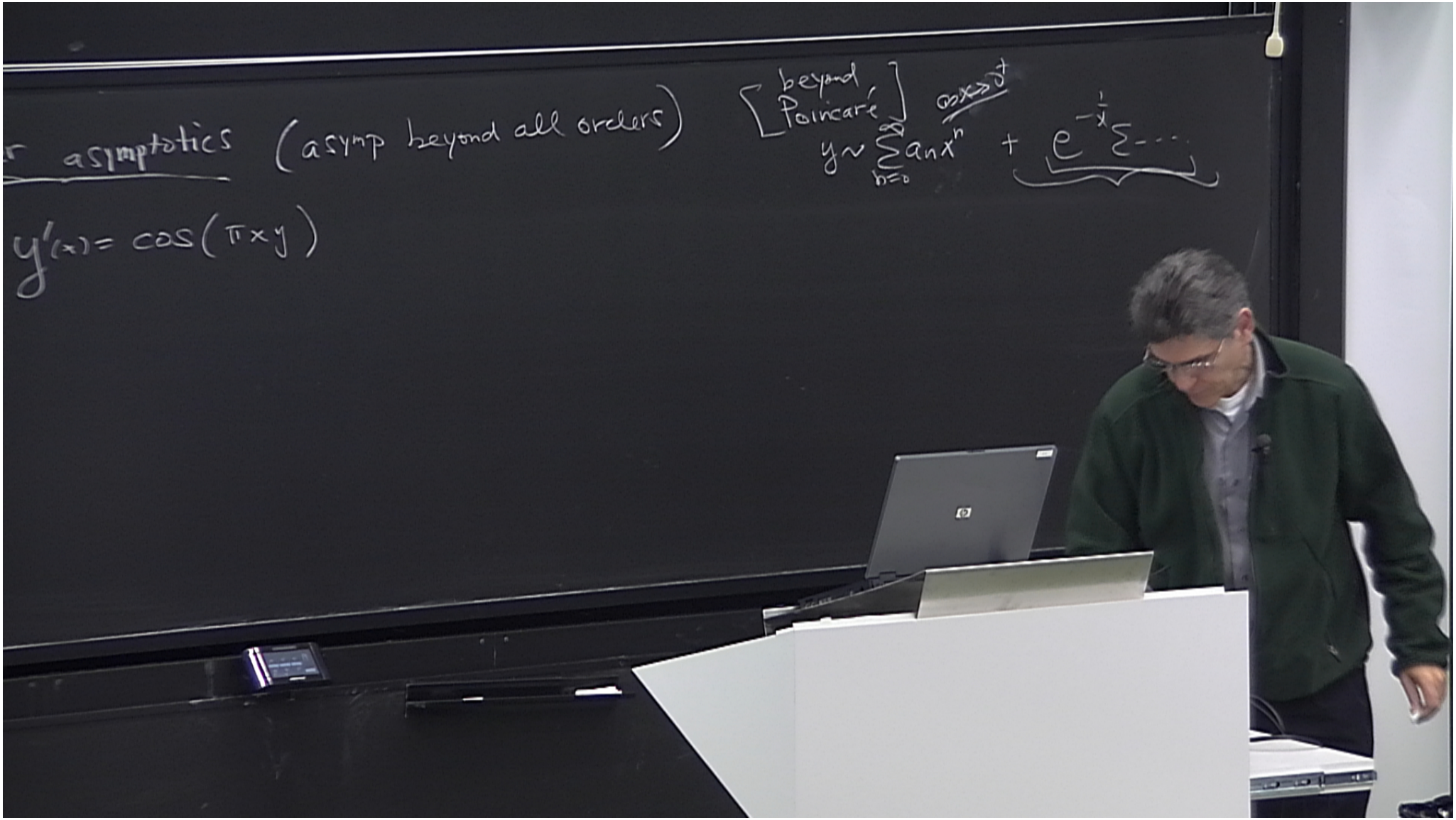
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$y' = \cos(\pi x y)$
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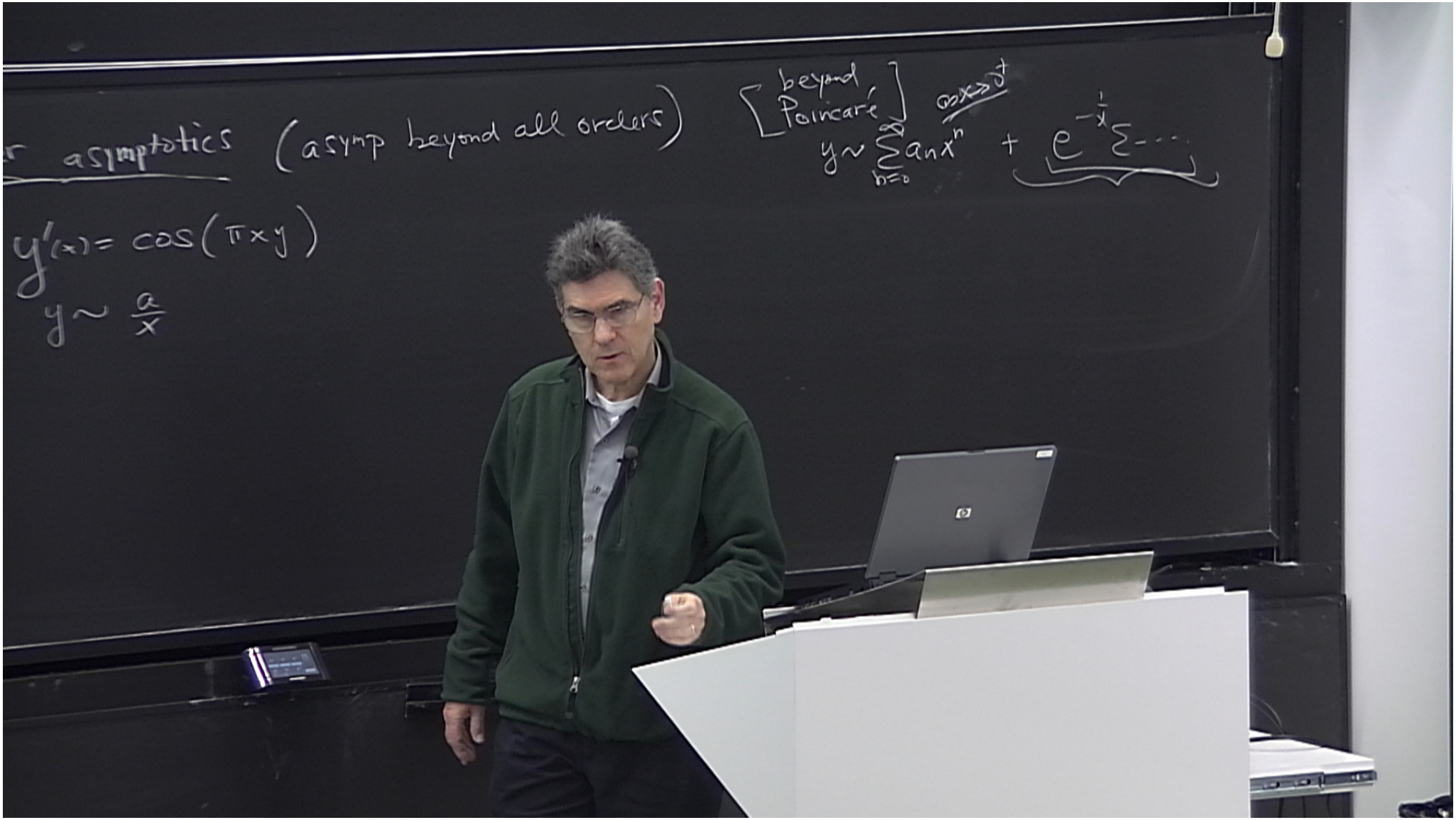
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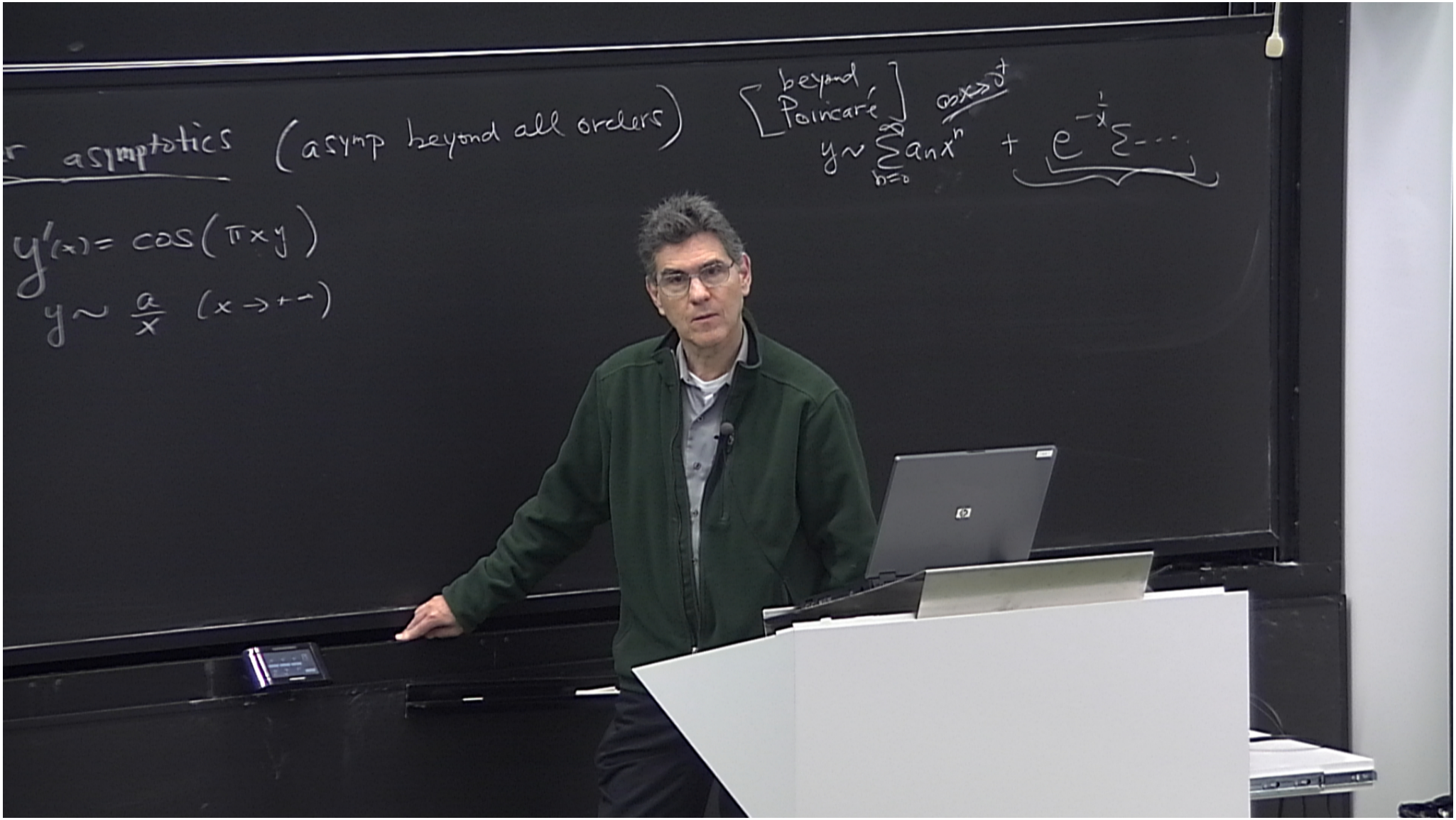
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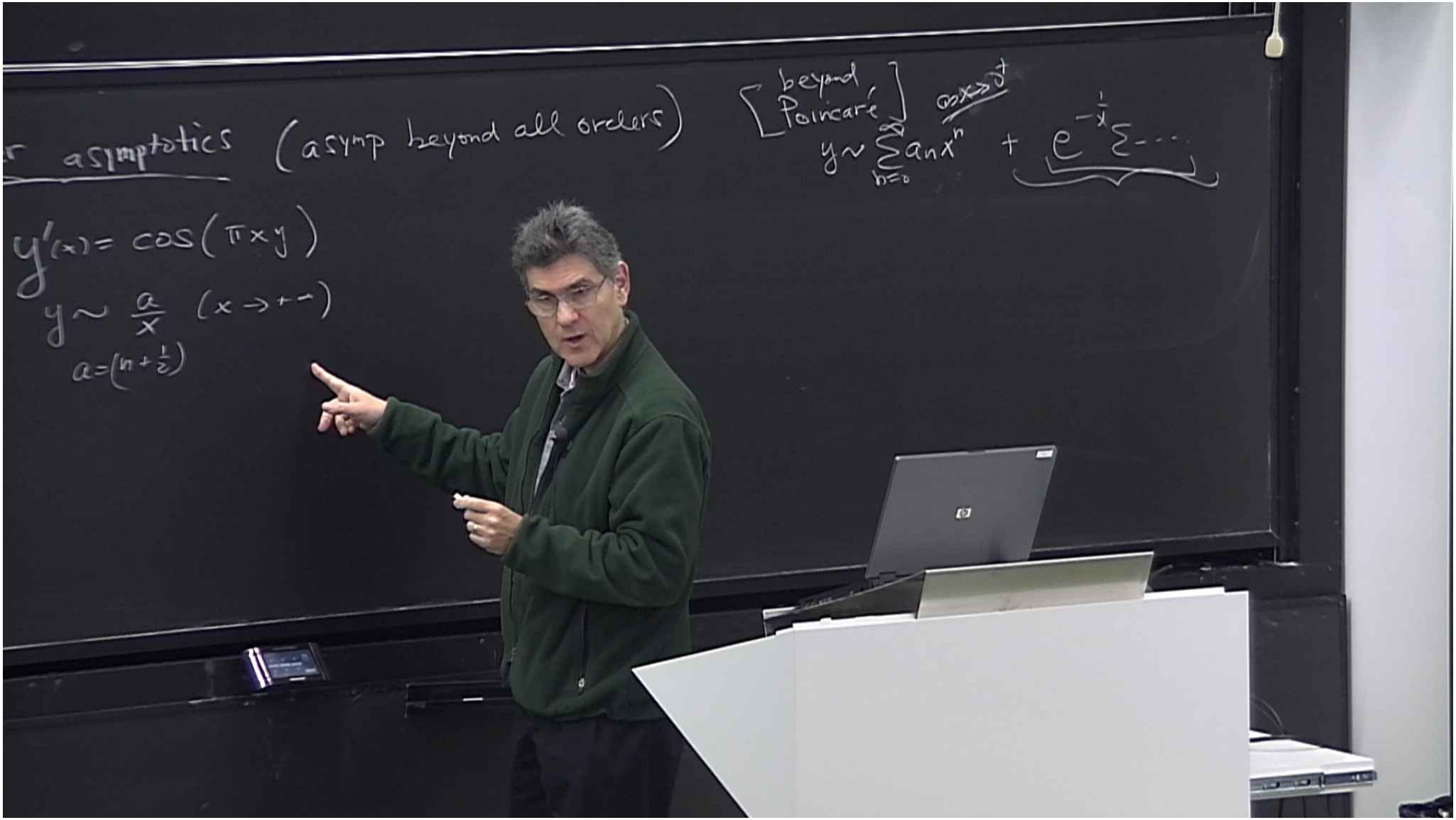
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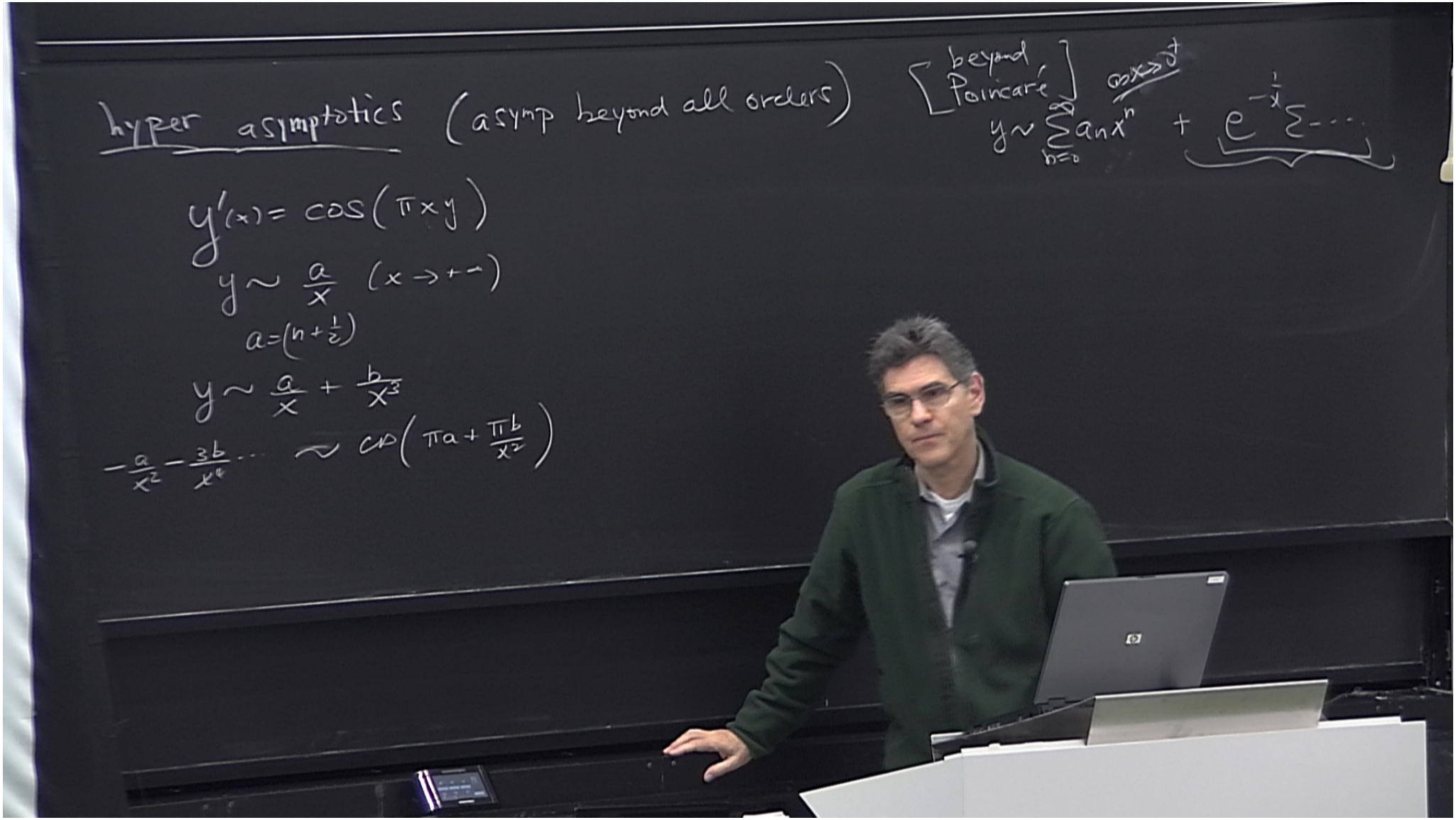
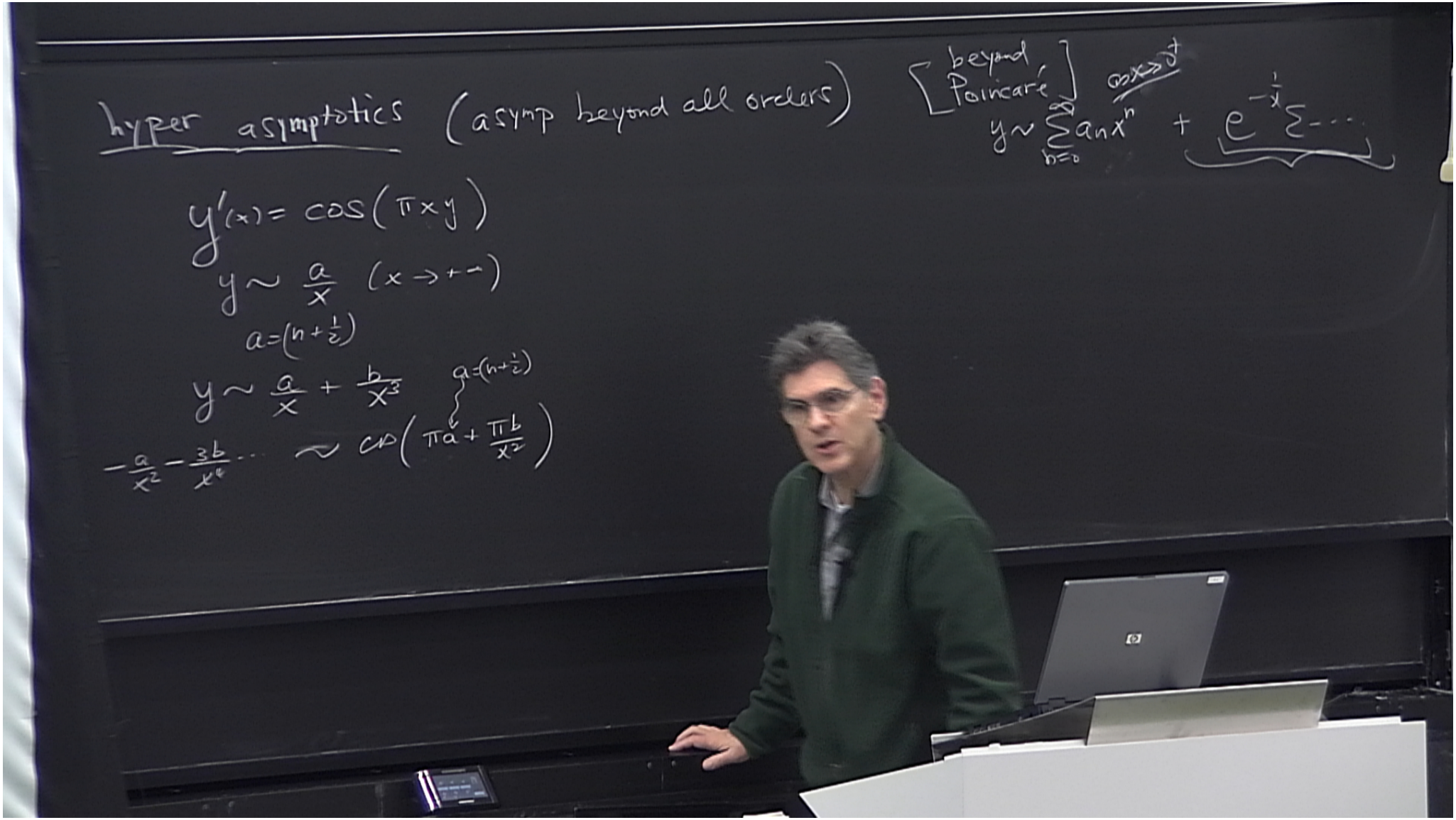
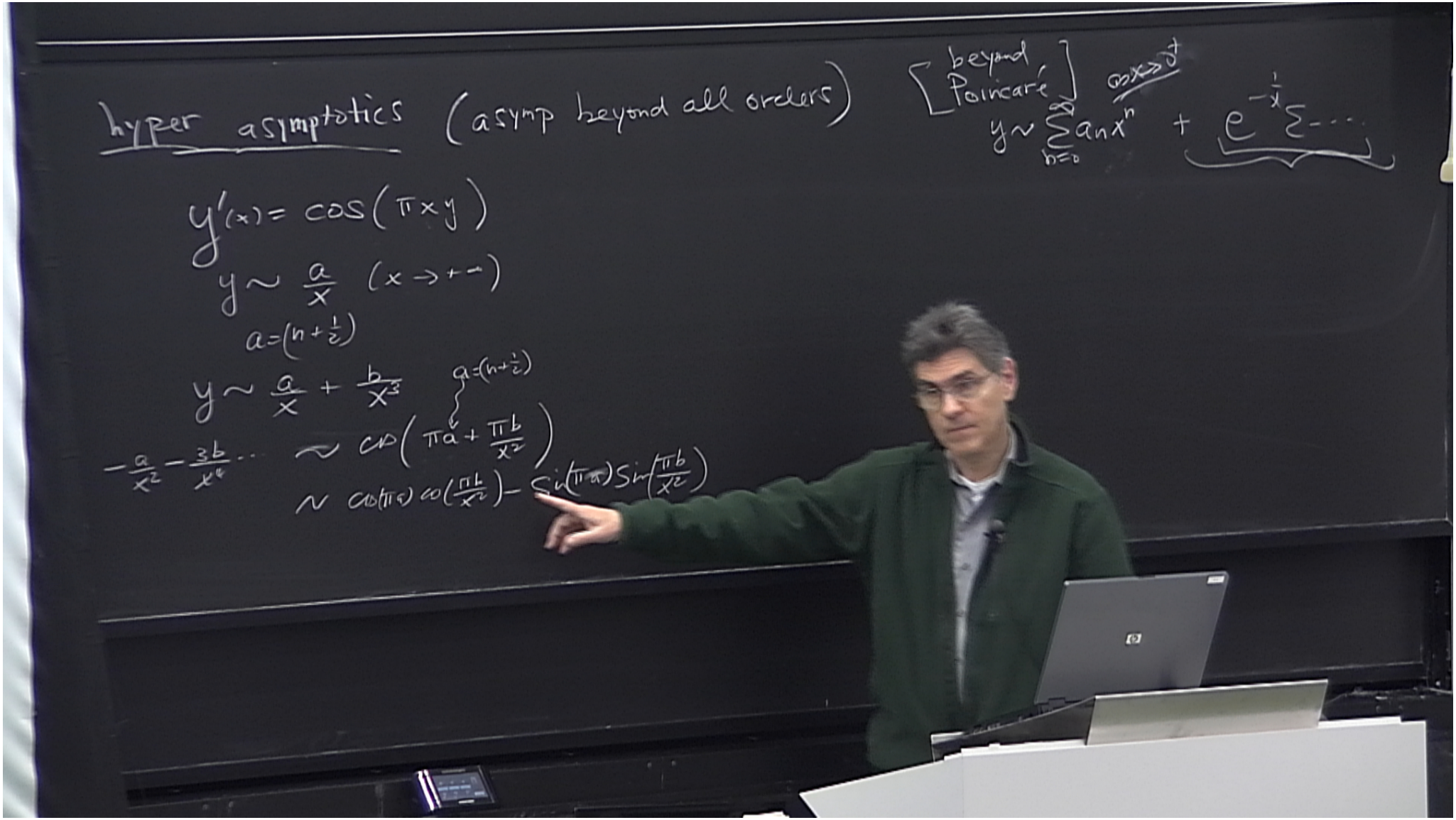
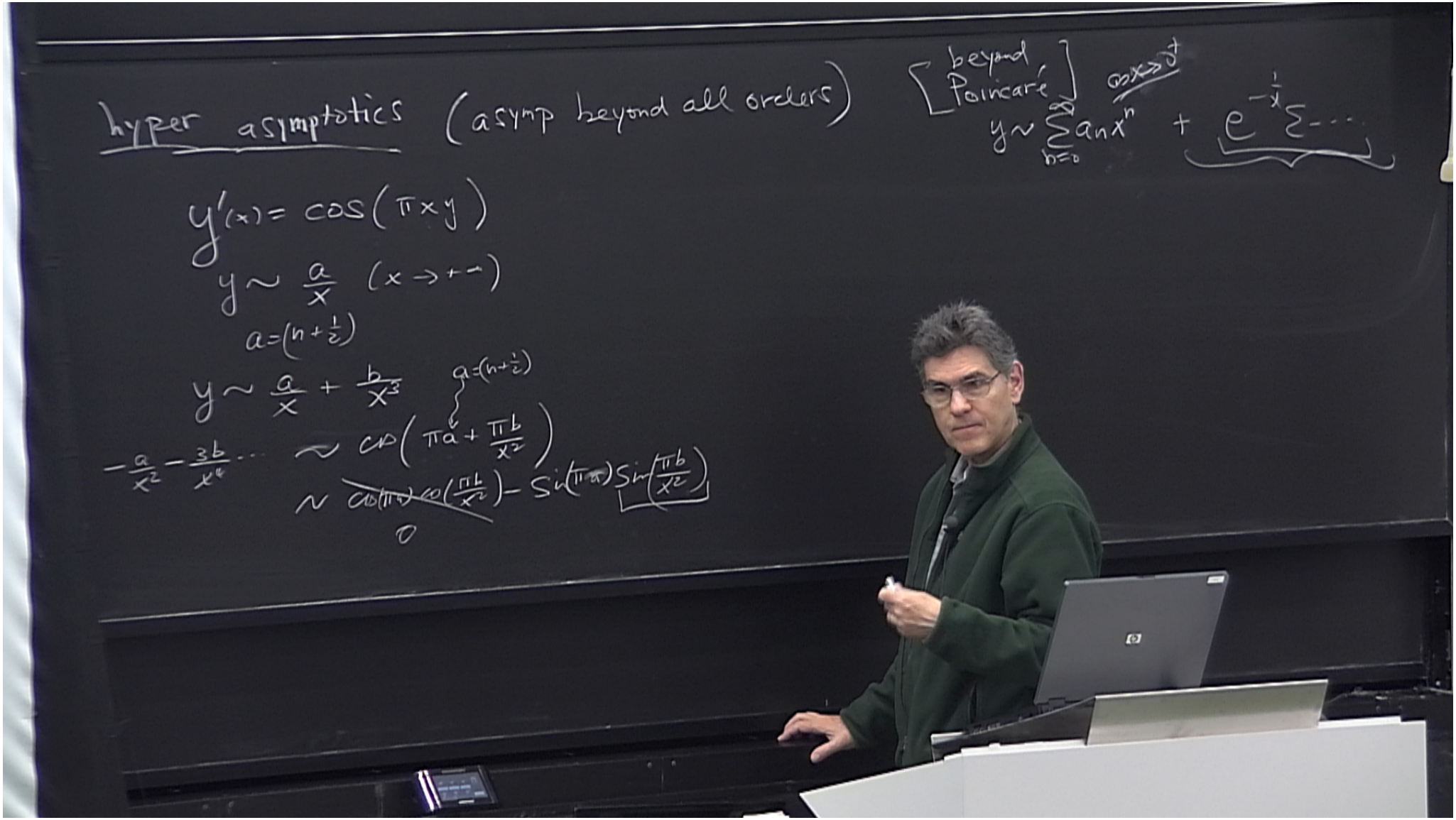
$a = (n + \frac{1}{2})$
$y \sim \sum_{n=0}^{\infty} a_n x^n + e^{-\frac{1}{x}} \Sigma \cdots$

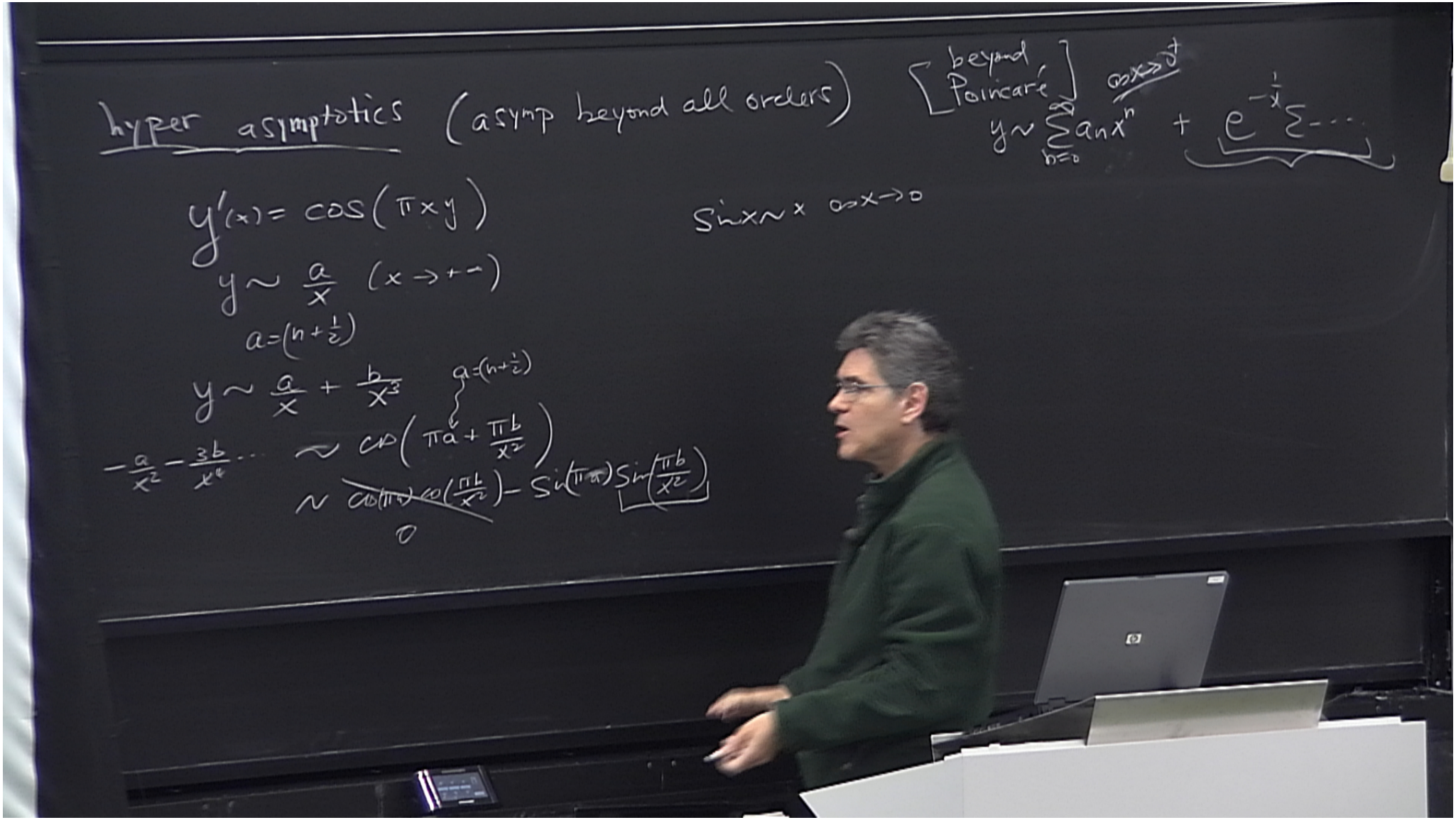
$y'(x) = \cos(\pi x y)$
$y \sim \frac{a}{x} \quad (x \to +\infty)$
$a = (n + \frac{1}{2})$
$y \sim \frac{a}{x} + \frac{b}{x^3}$

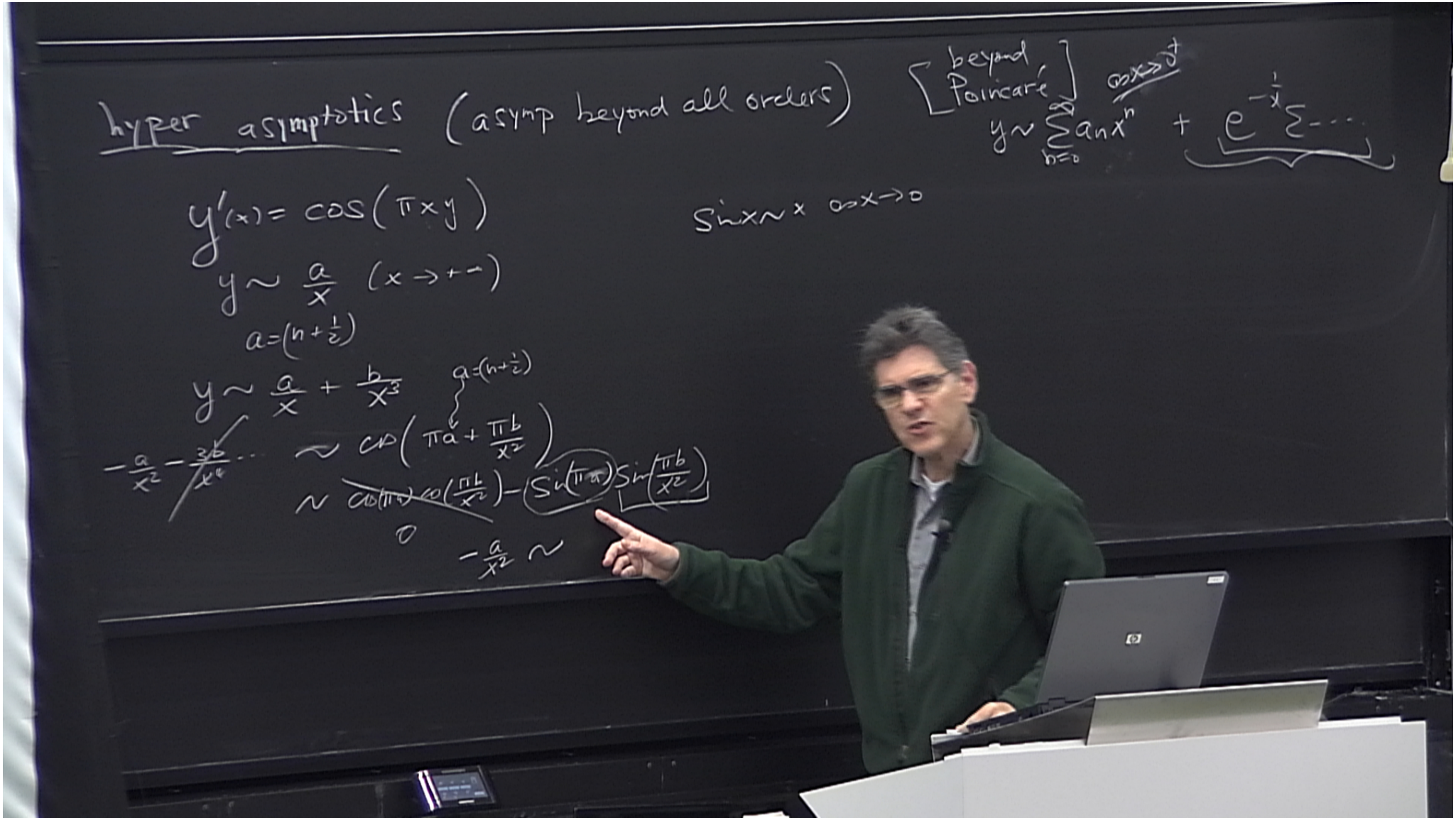

hyper asymptotics (asymp beyond all orders)
[beyond Poincaré]
$y' = \cos(\pi x y)$
$y \sim \frac{a}{x}$ $(x \to +\infty)$
$a = (n + \frac{1}{2})$
$y \sim \frac{a}{x} + \frac{b}{x^3}$
$-\frac{a}{x^2} - \frac{3b}{x^4} \cdots \sim \cos\left(\pi a + \frac{\pi b}{x^2}\right)$
$\sum_{n=0}^{\infty} a_n x^n$ + $e^{-\frac{1}{x}} \Sigma \cdots$

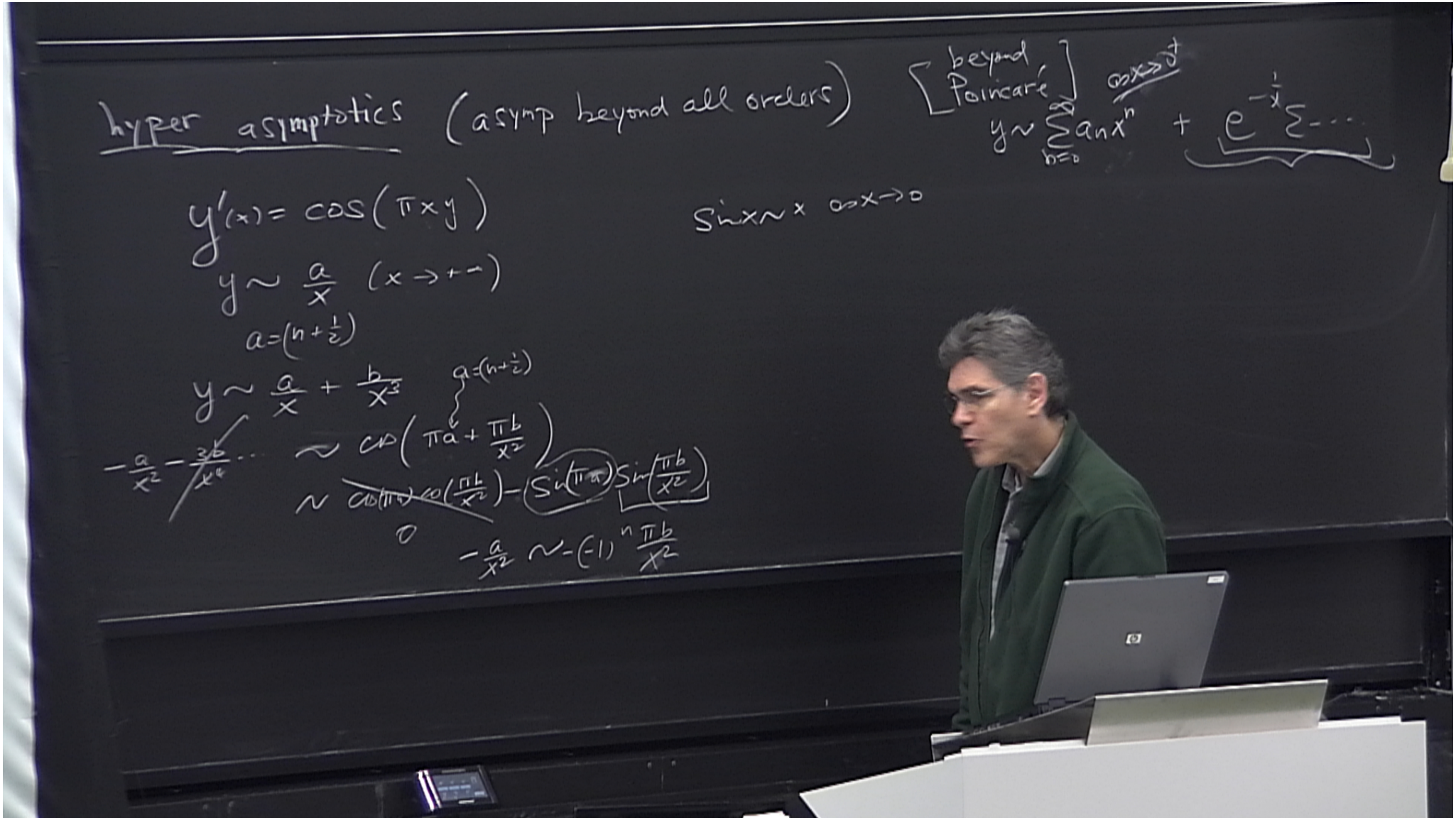

hyper asymptotics (asymp beyond all orders)
[beyond Poincaré]
$y \sim \sum_{n=0}^{\infty} a_n x^n$ as $x \to 0^+$
$+ e^{-\frac{1}{x}} \Sigma \cdots$
$y'(x) = \cos(\pi x y)$
$y \sim \frac{a}{x}$ $(x \to +\infty)$
$a = (n + \frac{1}{2})$
$y \sim \frac{a}{x} + \frac{b}{x^3}$
$a = (n+\frac{1}{2})$
$-\frac{a}{x^2} - \frac{3b}{x^4} \cdots \sim \cos\left(\pi a + \frac{\pi b}{x^2}\right)$

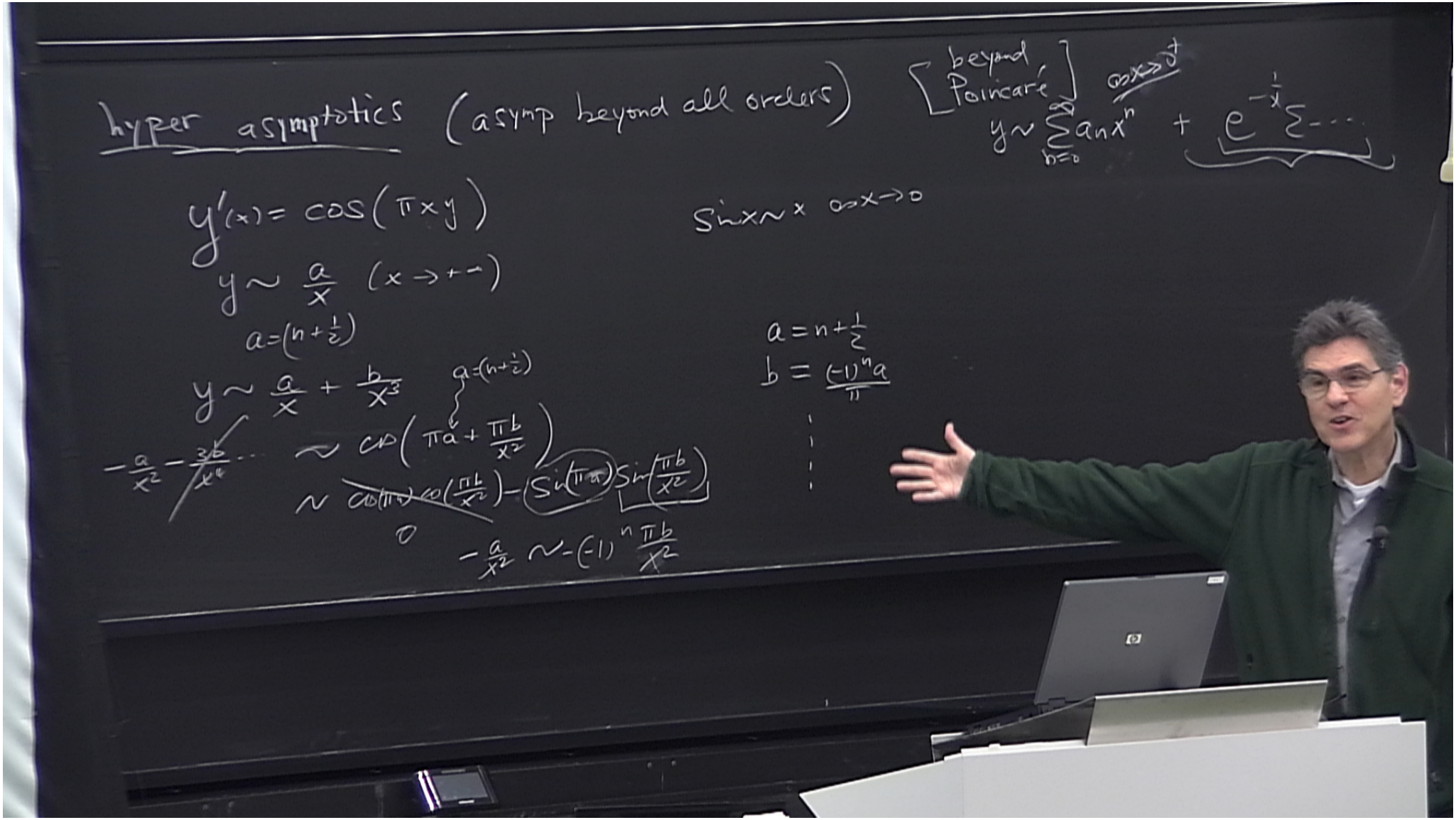

hyper asymptotics (asymp beyond all orders)
[beyond Poincaré]
$y \sim \sum_{n=0}^{\infty} a_n x^n + e^{-\frac{1}{x}} \Sigma \cdots$
$x \to 0$
$y'(x) = \cos(\pi x y)$
$y \sim \frac{a}{x} \quad (x \to +\infty)$
$a = (n + \frac{1}{2})$
$y \sim \frac{a}{x} + \frac{b}{x^3}$
$a = (n+\frac{1}{2})$
$-\frac{a}{x^2} - \frac{3b}{x^4} \cdots \sim \cos\left(\pi a + \frac{\pi b}{x^2}\right)$
$\sim \cos(\pi a)\cos\left(\frac{\pi b}{x^2}\right) - \sin(\pi a)\sin\left(\frac{\pi b}{x^2}\right)$

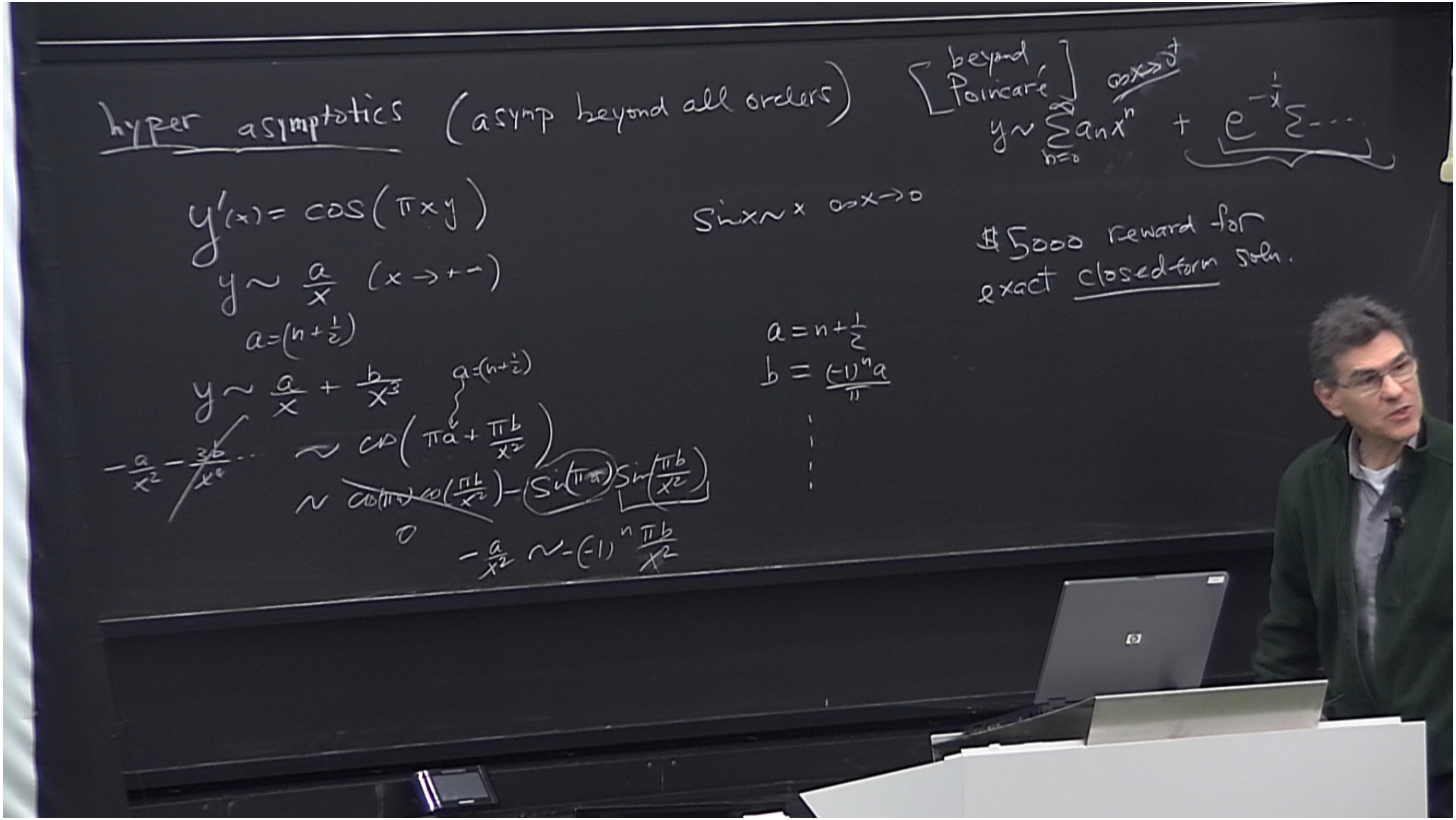

hyper asymptotics (asymp beyond all orders)
[beyond Poincaré]
$y' = \cos(\pi x y)$
$y \sim \frac{a}{x}$ $(x \to +-)$
$a = (n + \frac{1}{2})$
$y \sim \frac{a}{x} + \frac{b}{x^3}$
$a = (n+\frac{1}{2})$
$-\frac{a}{x^2} - \frac{3b}{x^4} \cdots \sim \cos\left(\pi a + \frac{\pi b}{x^2}\right)$
$\sim \cos(\pi a)\cos\left(\frac{\pi b}{x^2}\right) - \sin(\pi a)\sin\left(\frac{\pi b}{x^2}\right)$
0

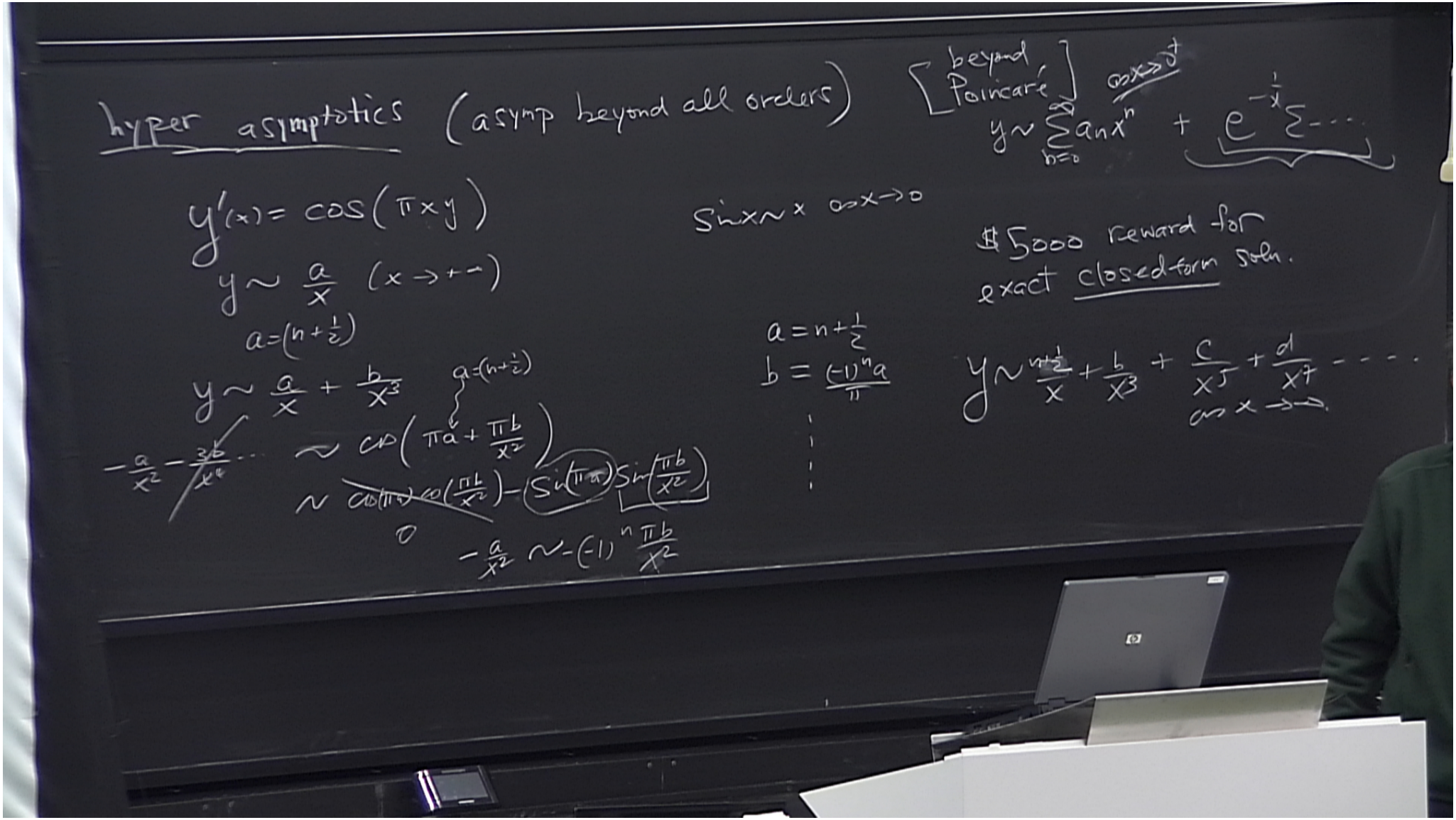

hyper asymptotics (asymp beyond all orders)
[beyond Poincaré]
$y' = \cos(\pi x y)$
$y \sim \frac{a}{x}$ $(x \to +-)$
$a = (n + \frac{1}{2})$
$y \sim \frac{a}{x} + \frac{b}{x^3}$
$a = (n+\frac{1}{2})$
$\sim \cos(\pi a + \frac{\pi b}{x^2})$
$\sim \cos(\pi n)\cos(\frac{\pi b}{x^2}) - \sin(\pi n)\sin(\frac{\pi b}{x^2})$
$\sin x \sim x$ $0 \to x \to 0$

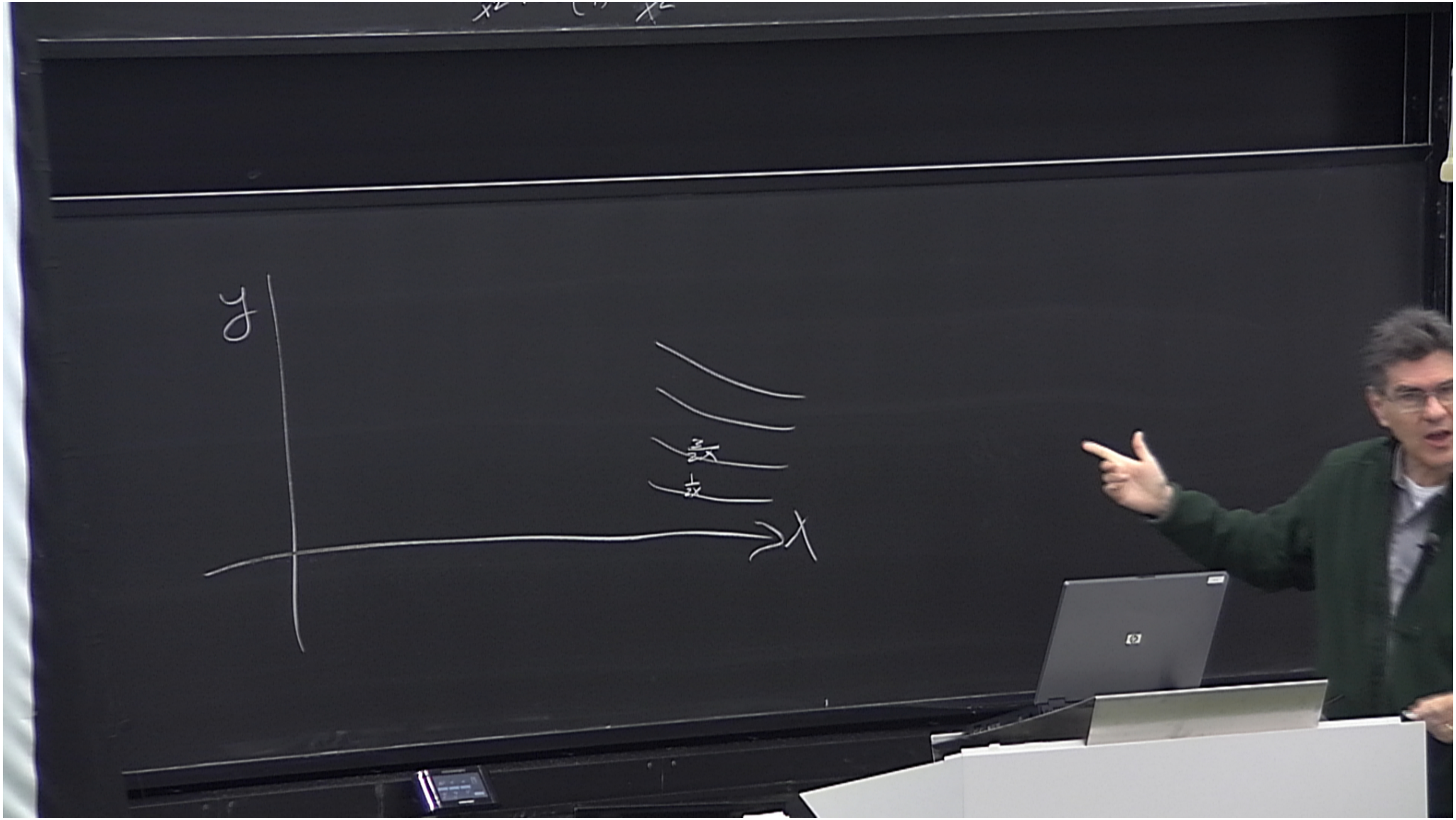

hyper asymptotics (asymp beyond all orders)
[beyond Poincaré]
$y' = \cos(\pi x y)$
$y \sim \frac{a}{x}$ $(x \to +-)$
$a = (n + \frac{1}{2})$
$y \sim \frac{a}{x} + \frac{b}{x^3}$
$\sin x \sim x$ $x \to 0$
$-\frac{a}{x^2} - \frac{3b}{x^4} \cdots$
$\sim \cos(\pi a + \frac{\pi b}{x^2})$
$\sim \cos(\pi n)\cos(\frac{\pi b}{x^2}) - \sin(\pi n)\sin(\frac{\pi b}{x^2})$
0
$-\frac{a}{x^2} \sim$

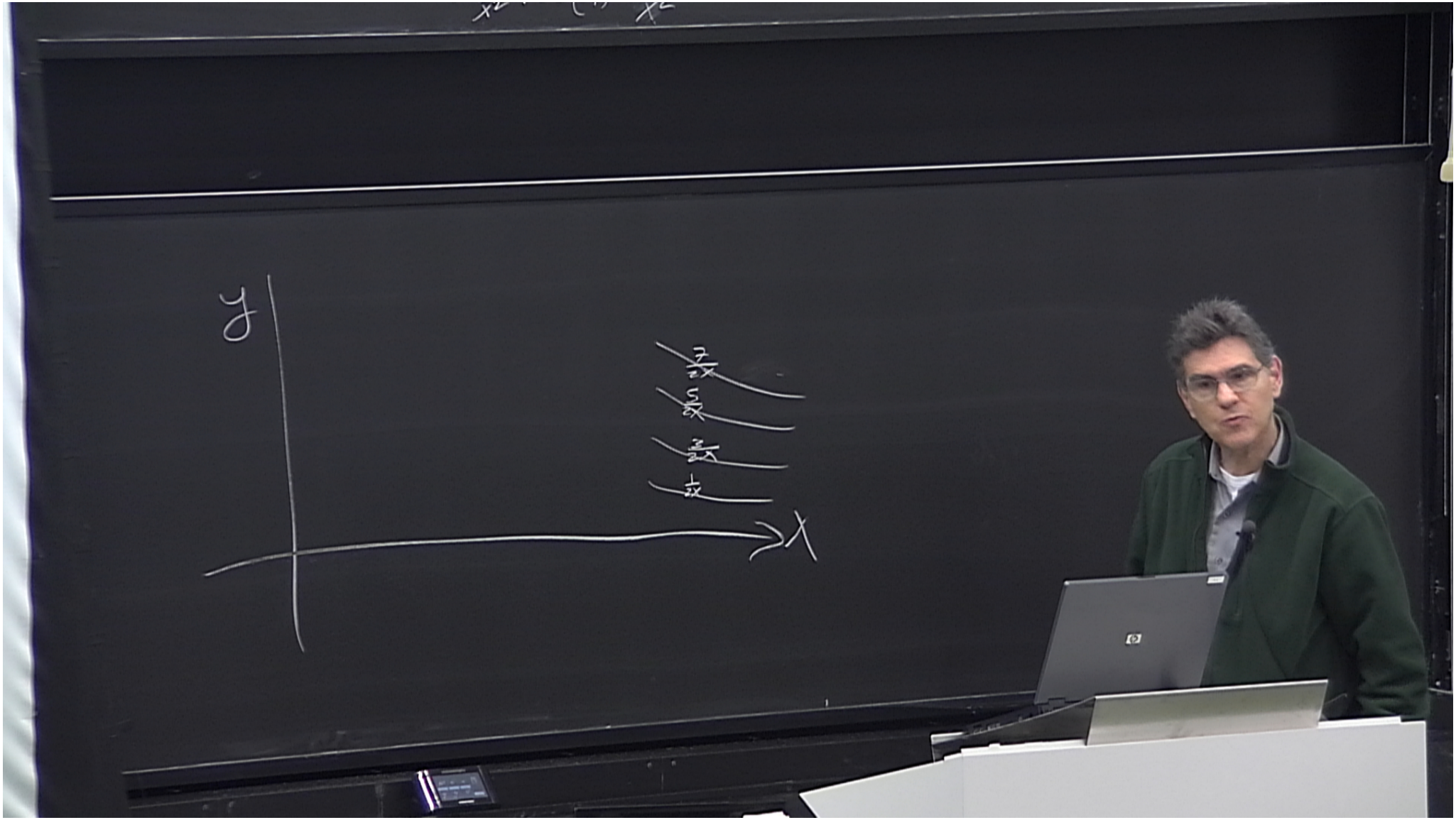

hyper asymptotics (asymp beyond all orders)
[beyond Poincaré]
$y' = \cos(\pi x y)$
$y \sim \frac{a}{x}$ $(x \to +-)$
$a = (n + \frac{1}{2})$
$y \sim \frac{a}{x} + \frac{b}{x^3}$
$a = (n+\frac{1}{2})$
$\sim \cos(\pi a + \frac{\pi b}{x^2})$
$\sim \cos(\pi n)\cos(\frac{\pi b}{x^2}) - \sin(\pi n)\sin(\frac{\pi b}{x^2})$
0
$-\frac{a}{x^2} \sim -(-1)^n \frac{\pi b}{x^2}$
$\sin x \sim x$ $\quad x \to 0$
$y \sim \sum_{n=0}^{\infty} a_n x^n + e^{-\frac{1}{x}} \Sigma \cdots$

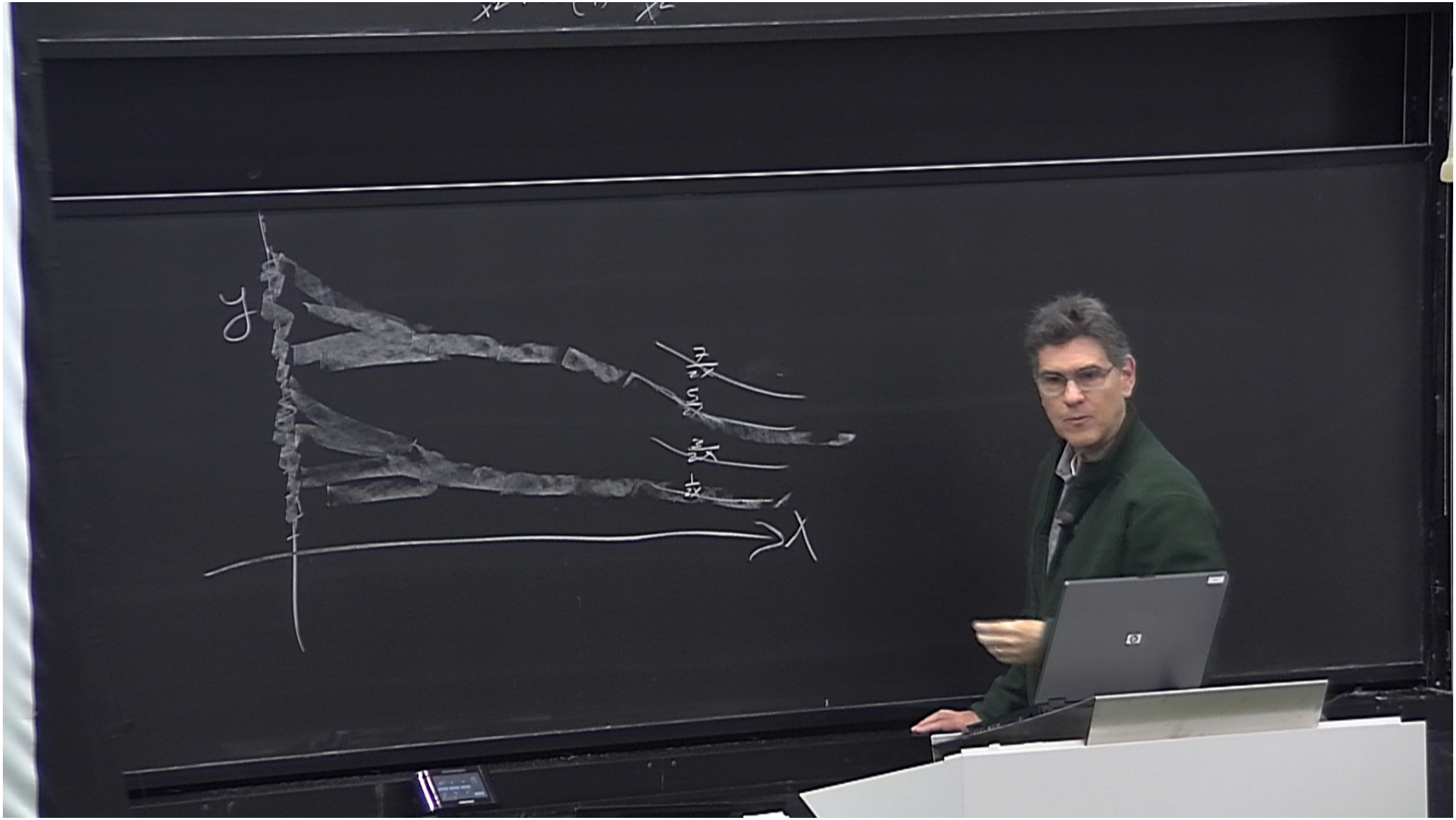

hyper asymptotics (asymp beyond all orders)
[beyond Poincaré]
$y' = \cos(\pi x y)$
$y \sim \frac{a}{x}$ $(x \to +-)$
$a = (n+\frac{1}{2})$
$y \sim \frac{a}{x} + \frac{b}{x^3}$
$a = (n+\frac{1}{2})$
$\sin x \sim x$ $x \to 0$
$a = n + \frac{1}{2}$
$b = \frac{(-1)^n a}{\pi}$
$-\frac{a}{x^2} = (-1)^n \frac{\pi b}{x^2}$


hyper asymptotics (asymp beyond all orders)
[beyond Poincaré]
$y \sim \sum_{n=0}^{\infty} a_n x^n + e^{-\frac{1}{x}} \Sigma \cdots$ as $x \to 0^+$
$y'(x) = \cos(\pi x y)$
$y \sim \frac{a}{x}$ $(x \to +-)$
$a = (n+\frac{1}{2})$
$y \sim \frac{a}{x} + \frac{b}{x^3}$
$a=(n+\frac{1}{2})$
$\sim \cos(\pi a + \frac{\pi b}{x^2})$
$\sim \cos(\pi n)\cos(\frac{\pi b}{x^2}) - \sin(\pi n)\sin(\frac{\pi b}{x^2})$
0
$-\frac{a}{x^2} \sim -(-1)^n \frac{\pi b}{x^2}$
$\sin x \sim x$ as $x \to 0$
$a = n+\frac{1}{2}$
$b = \frac{(-1)^n a}{\pi}$
$5000 reward for exact closed-form soln.


hyper asymptotics (asymp beyond all orders)
[beyond Poincaré]
$y' = \cos(\pi x y)$
$y \sim \frac{a}{x}$ $(x \to +-)$
$a = (n+\frac{1}{2})$
$y \sim \frac{a}{x} + \frac{b}{x^3}$
$a = (n+\frac{1}{2})$
$\sim \cos(\pi a + \frac{\pi b}{x^2})$
$\sim \cos(\pi)\cos(\frac{\pi b}{x^2}) - \sin(\pi)\sin(\frac{\pi b}{x^2})$
0
$-\frac{a}{x^2} \sim -(-1)^n \frac{\pi b}{x^2}$
$\sin x \sim x$ as $x \to 0$
$a = n+\frac{1}{2}$
$b = \frac{(-1)^n a}{\pi}$
$y \sim \sum_{n=0}^{\infty} a_n x^n + e^{-\frac{1}{x}} \Sigma \cdots$ as $x \to 0^+$
$5000 reward for exact closed-form soln.
$y \sim \frac{n+\frac{1}{2}}{x} + \frac{b}{x^3} + \frac{c}{x^5} + \frac{d}{x^7} \cdots$
as $x \to \infty$.


y
x


y
x


y
x

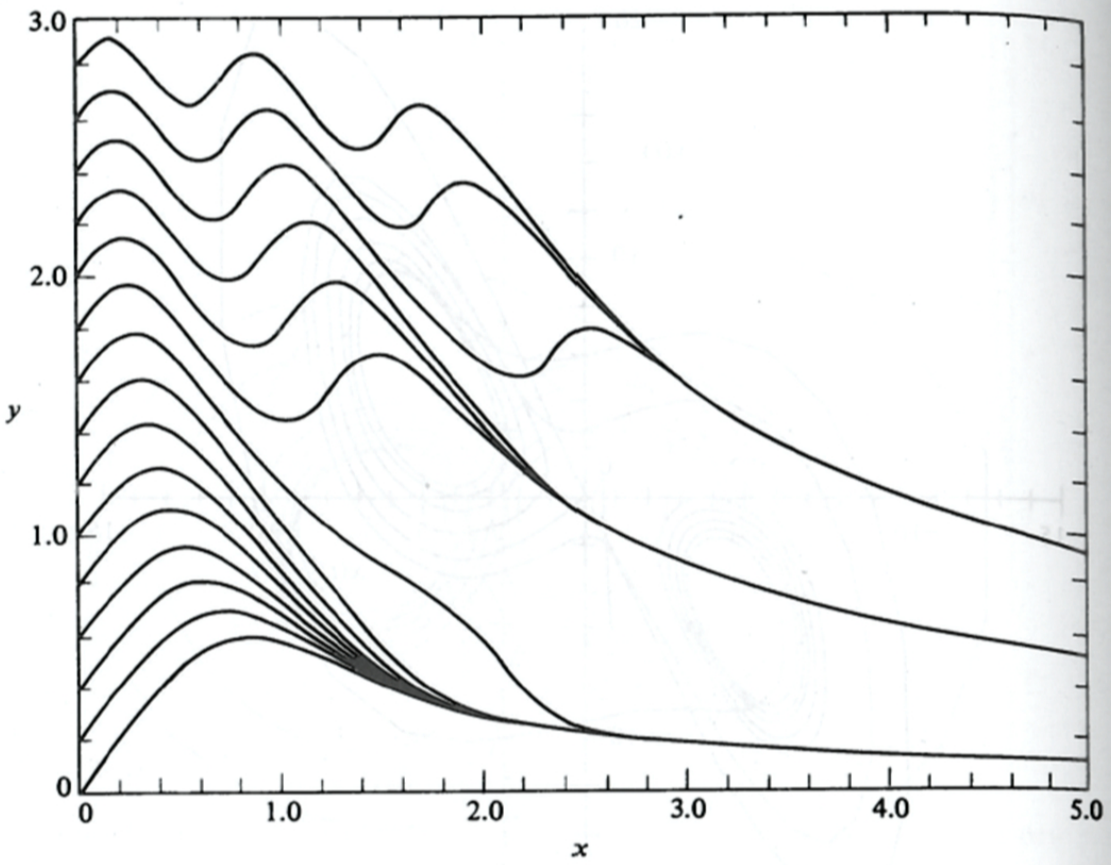
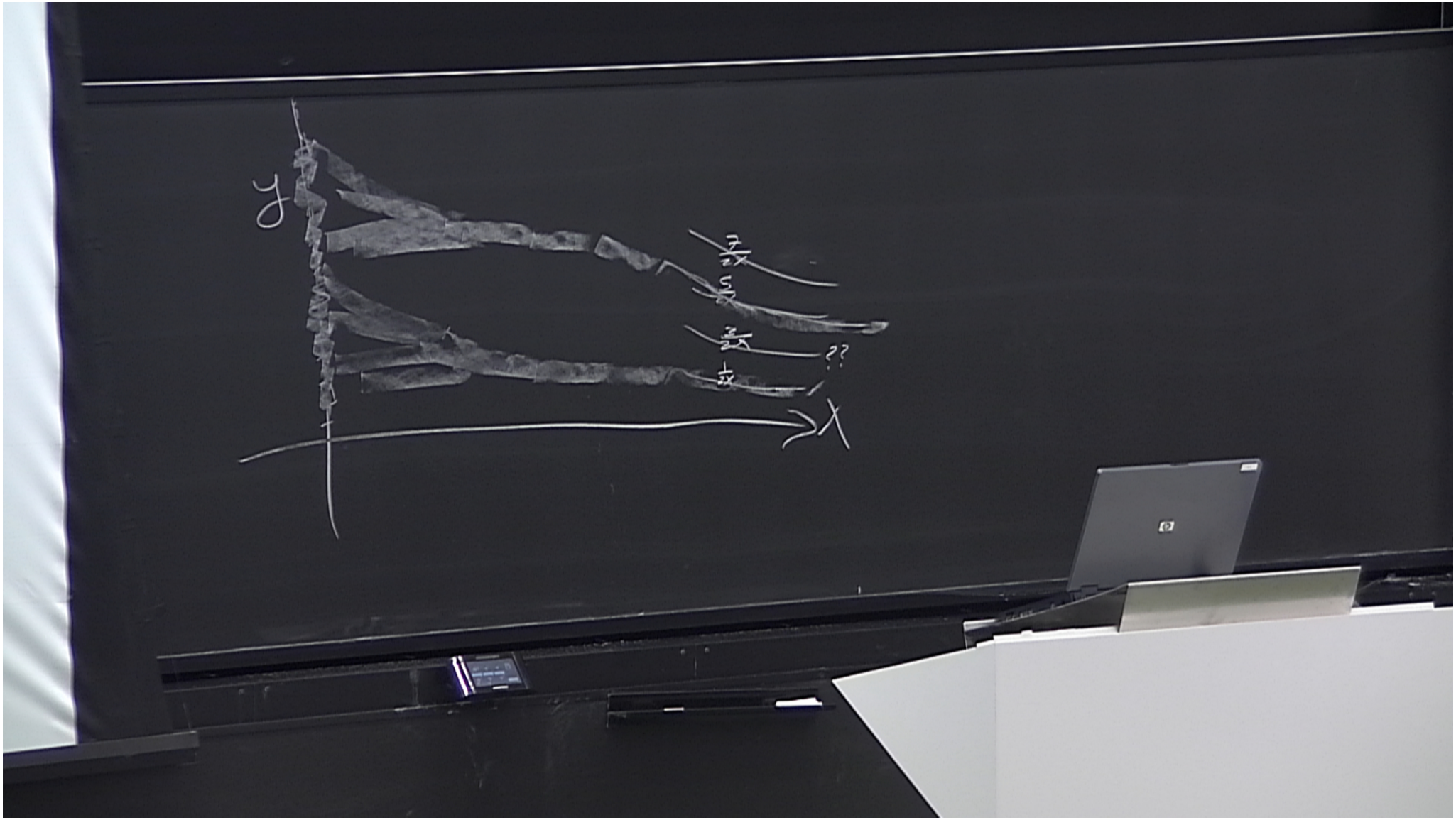
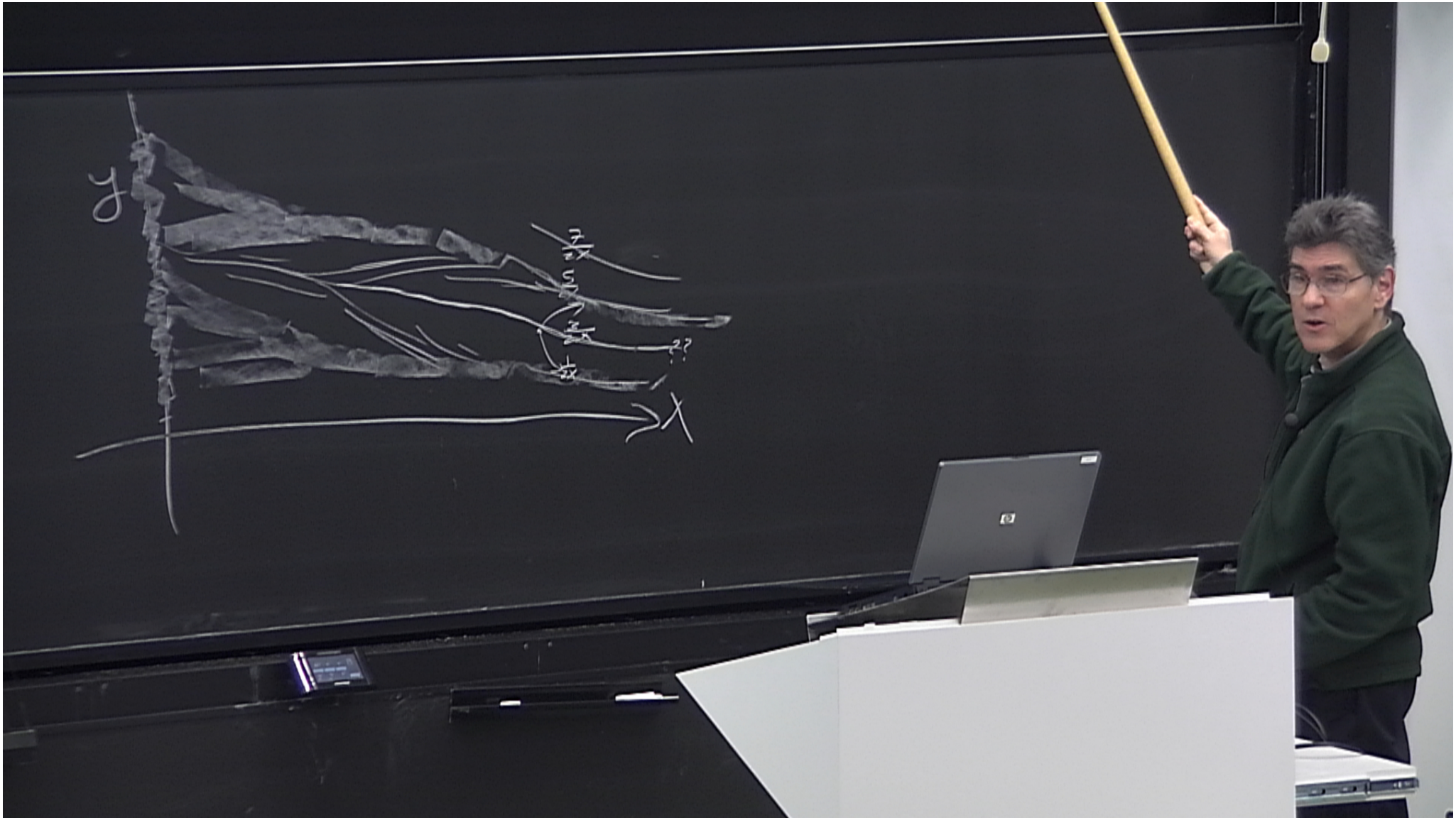
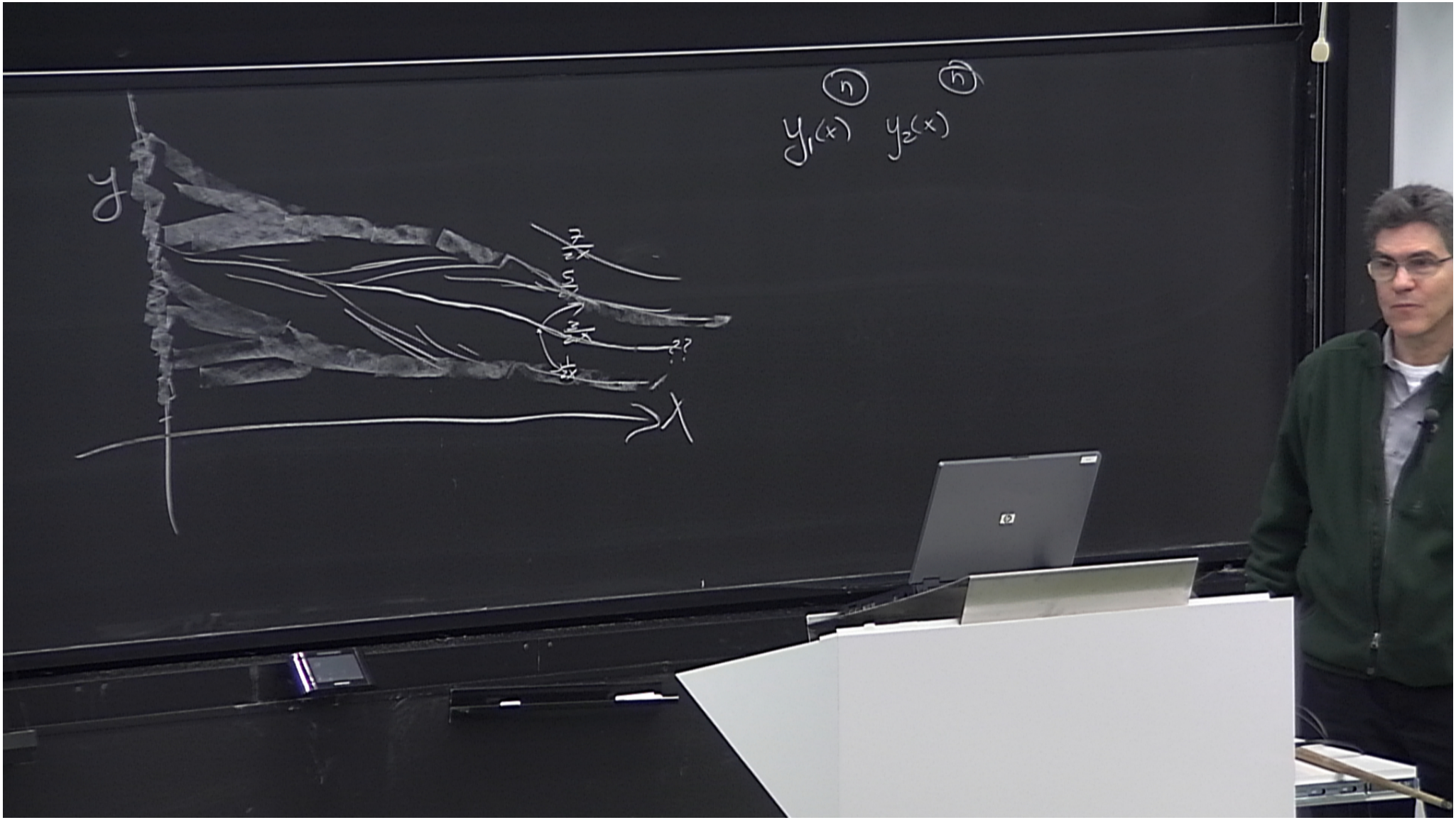
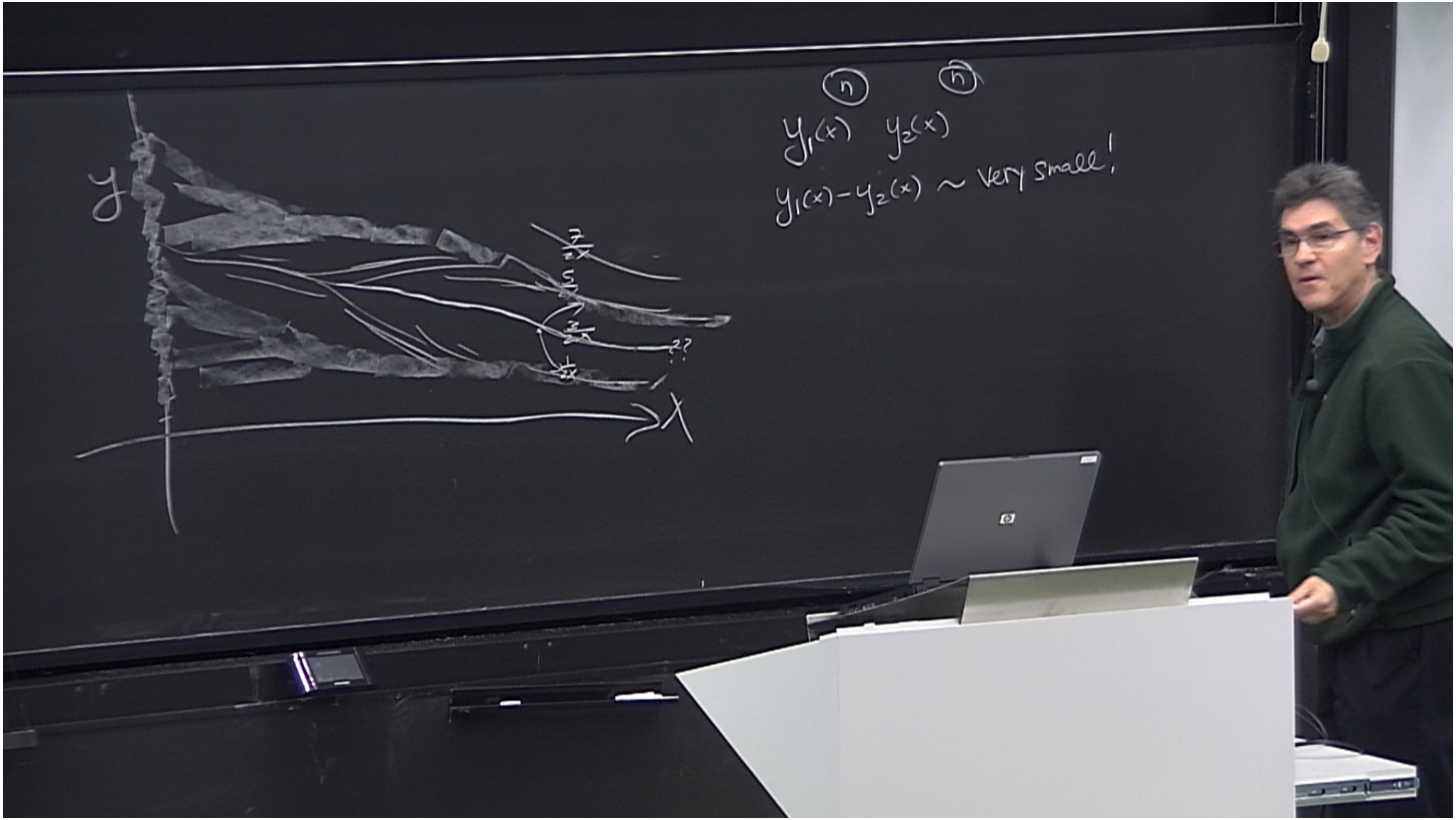


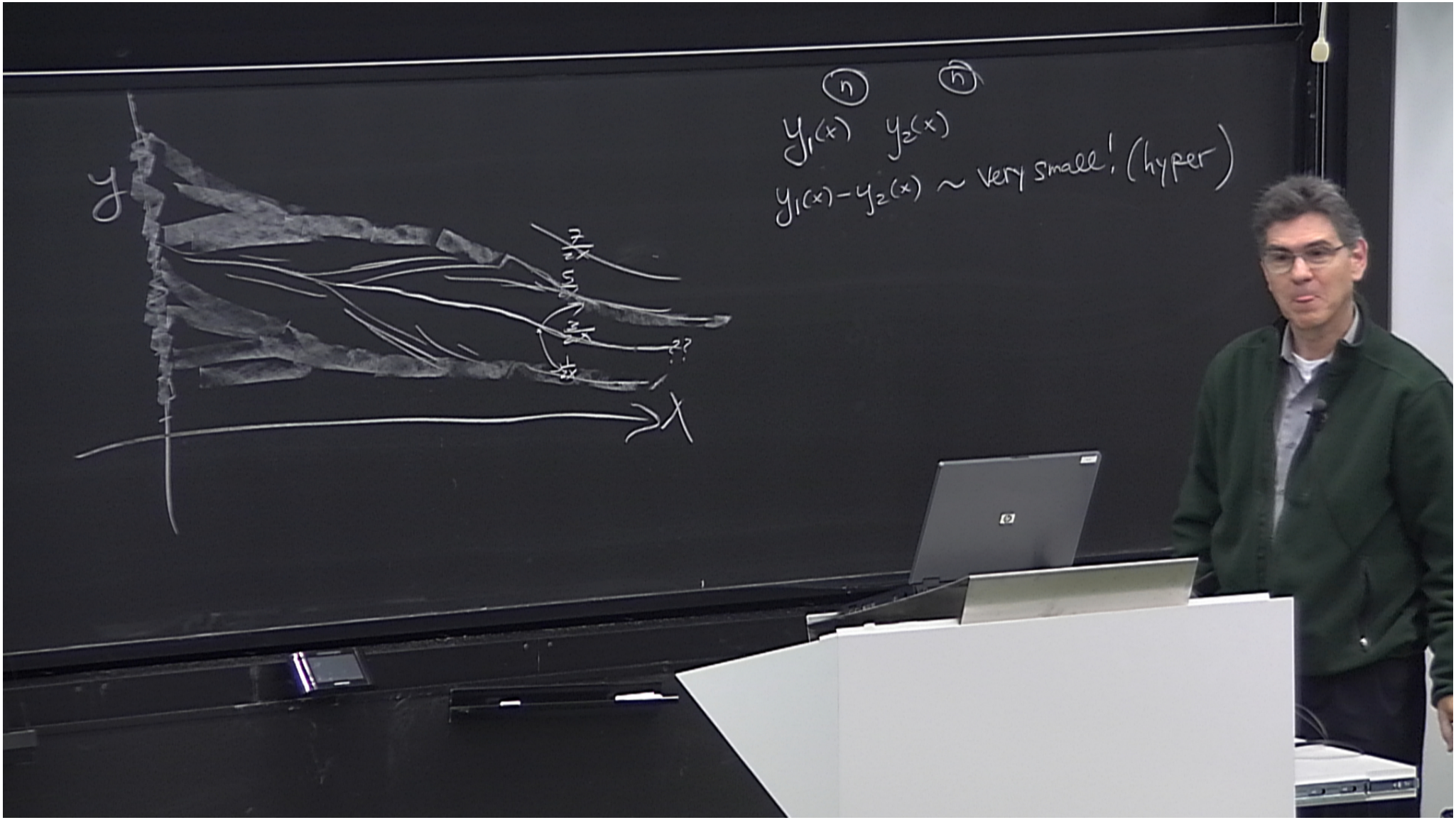
Figure 4.30 Solutions to $y' = \cos(\pi xy)$ for the initial values $y(0) = 0, 0.2, 0.4, 0.6, \ldots, 2.8$. Observe that as x increases, solutions bunch together. In Prob. 4.13 you are asked to explain this bunching phenomenon using asymptotic analysis.

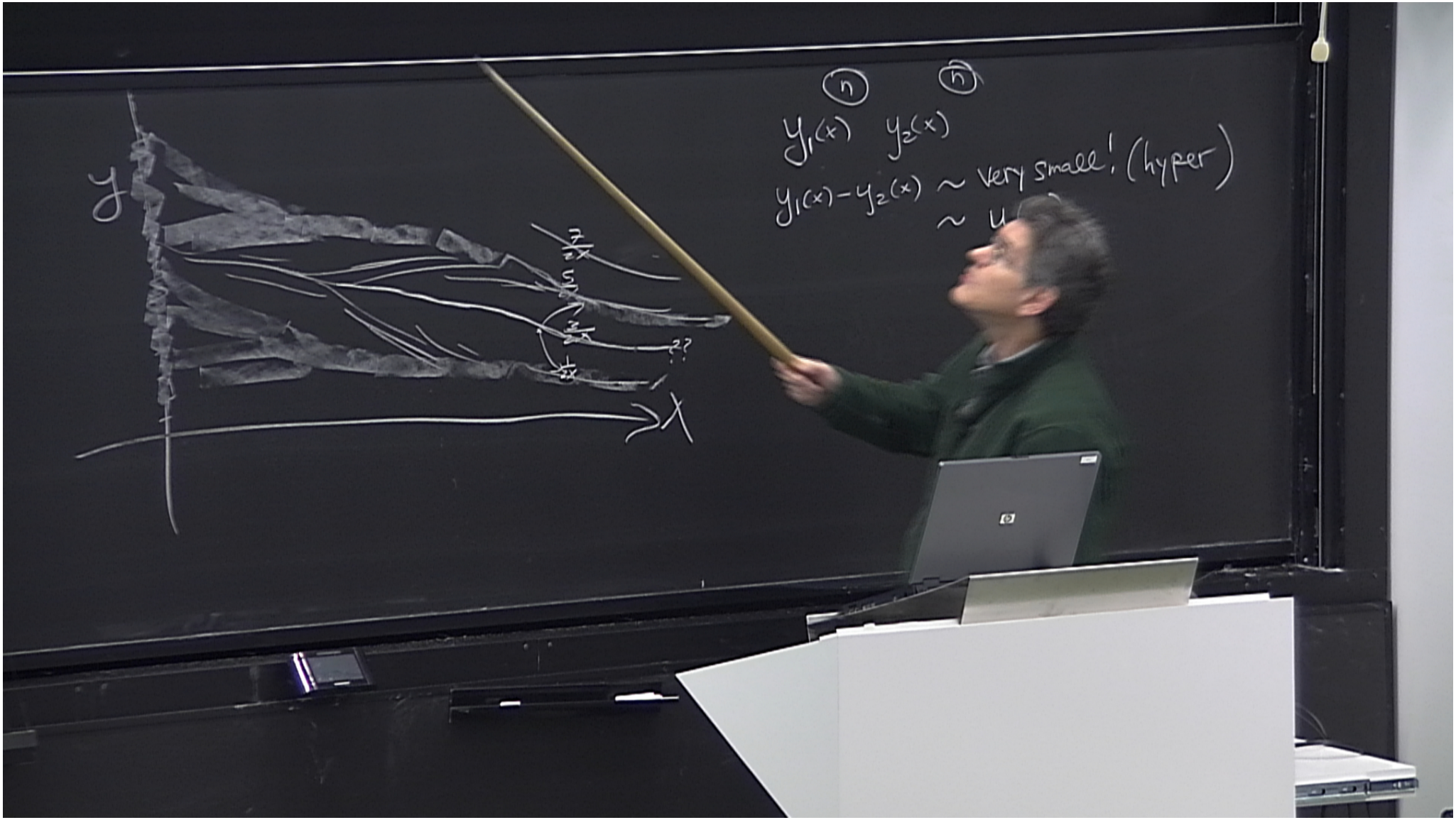

y
x
??

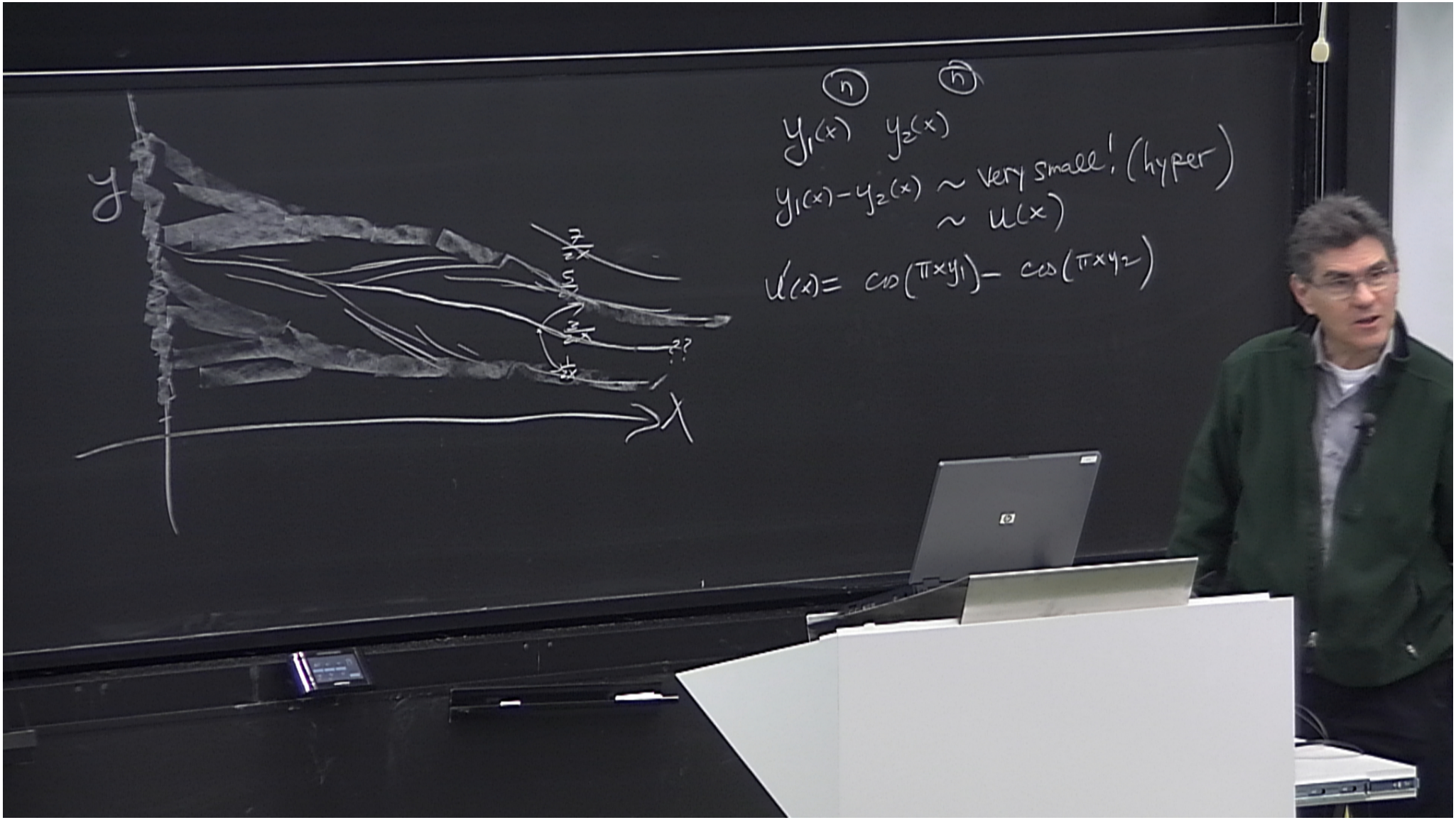

y
x

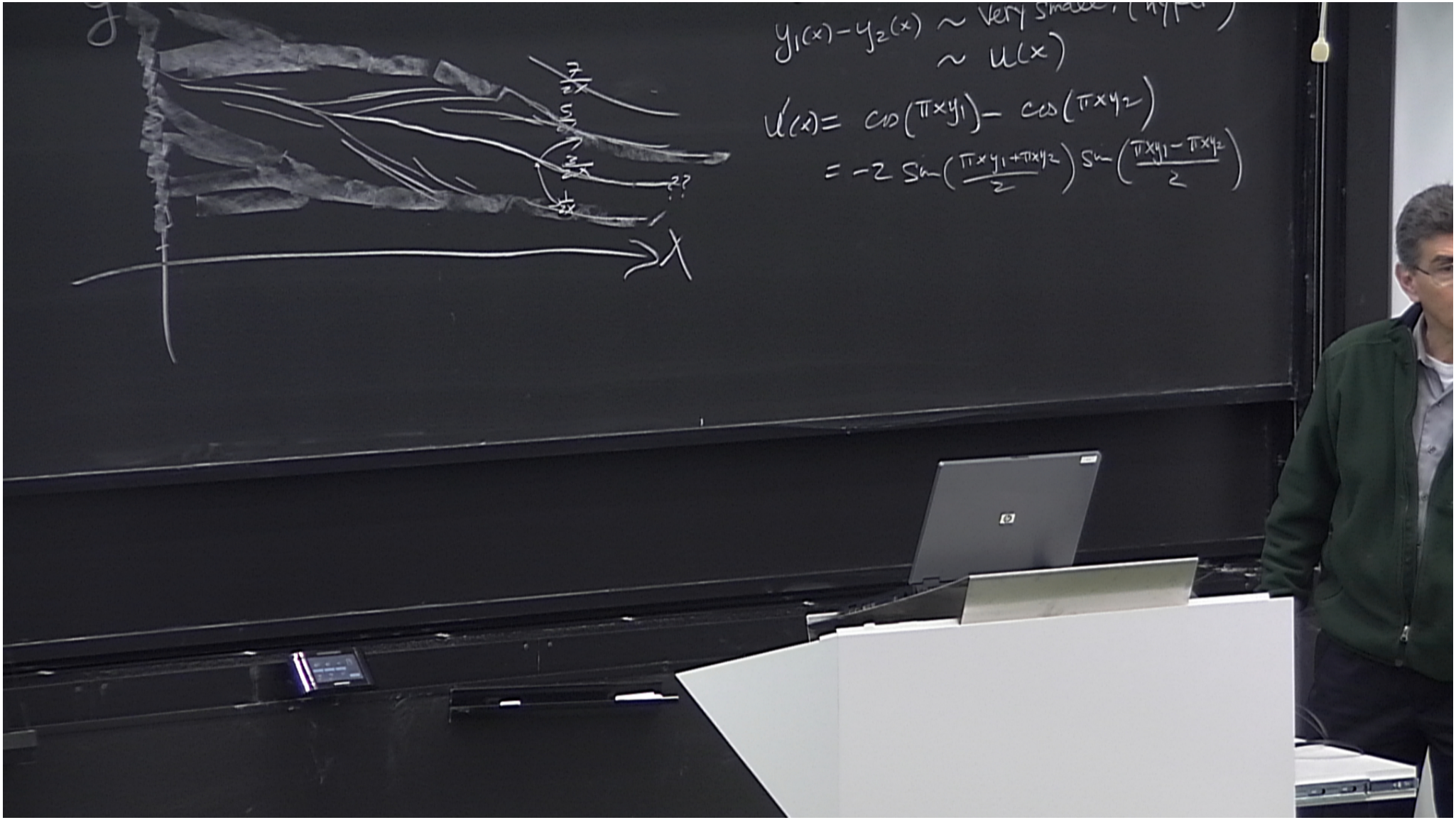

y
x
??
$y_1(x)$ (n)
$y_2(x)$ (n)

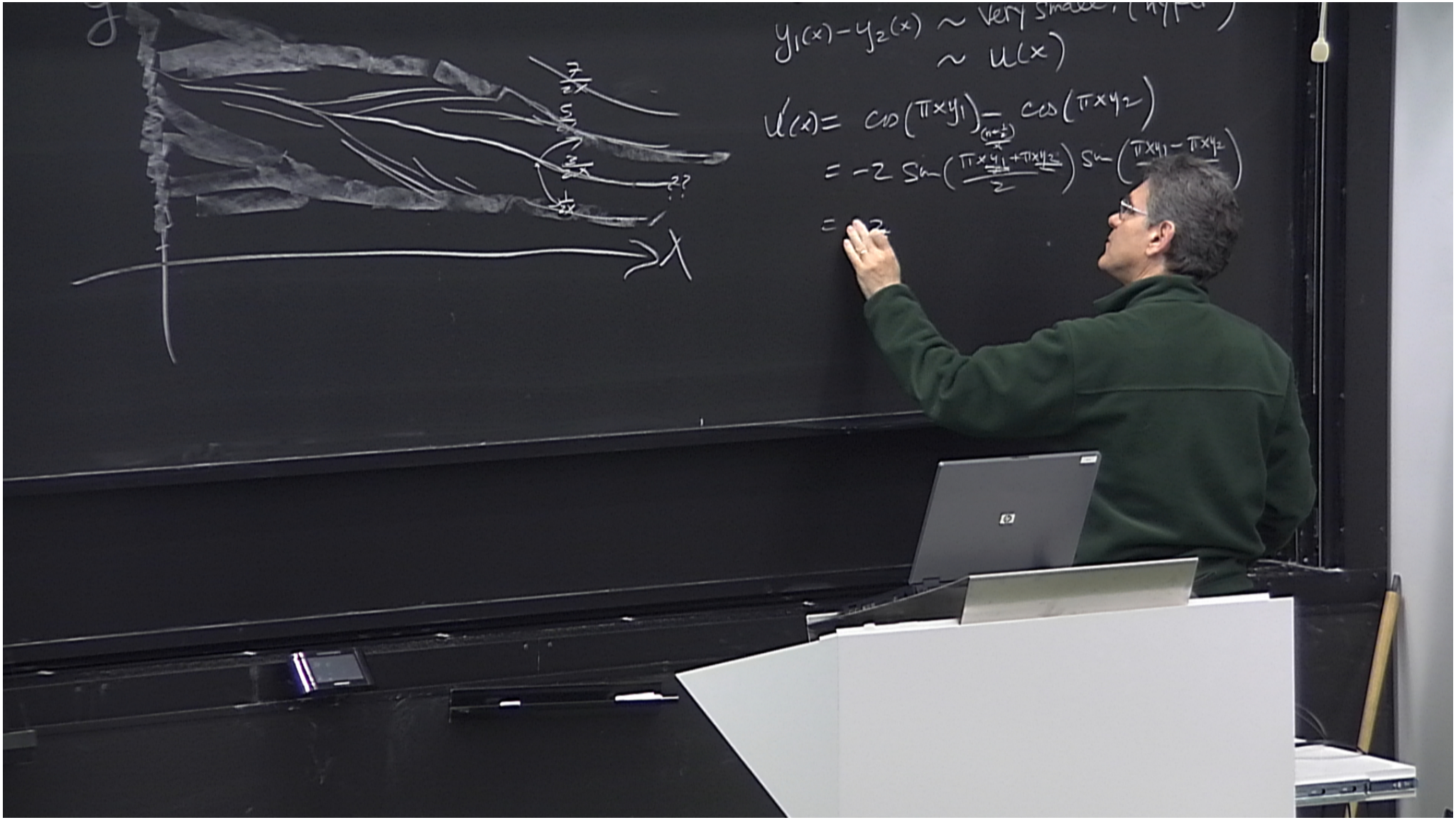

y
x
??
$y_1(x)$ (n) $y_2(x)$ (n)
$y_1(x) - y_2(x) \sim$ very small!


y
x
??
$y_1(x)$ (n) $y_2(x)$ (n)
$y_1(x) - y_2(x) \sim$ very small! (hyper)


n
n
$y_1(x)$ $y_2(x)$
$y_1(x) - y_2(x) \sim$ very small! (hyper)
$\sim u$
y
x
??


y
x
??
(n)
(n)
$y_1(x)$ $y_2(x)$
$y_1(x) - y_2(x) \sim$ very small! (hyper)
$\sim u(x)$
$u(x) = \cos(\pi x y_1) - \cos(\pi x y_2)$


$y_1(x) - y_2(x) \sim$ Very small
$\sim u(x)$
$u(x) = \cos(\pi x y_1) - \cos(\pi x y_2)$
$= -2 \sin\left(\frac{\pi x y_1 + \pi x y_2}{2}\right) \sin\left(\frac{\pi x y_1 - \pi x y_2}{2}\right)$
$\frac{2}{2x}$
$\frac{3}{2x}$
$\frac{4}{2x}$
??
x


$y_1(x) - y_2(x) \sim$ Very small
$\sim u(x)$
$u(x) = \cos(\pi x y_1) - \cos(\pi x y_2)$
$= -2 \sin\left(\frac{\pi x y_1 + \pi x y_2}{2}\right) \sin\left(\frac{\pi x y_1 - \pi x y_2}{2}\right)$
??
x

$y_1(x)$ $y_2(x)$

$y_1(x) - y_2(x) \sim$ very small! (hyper)

$\sim u(x)$

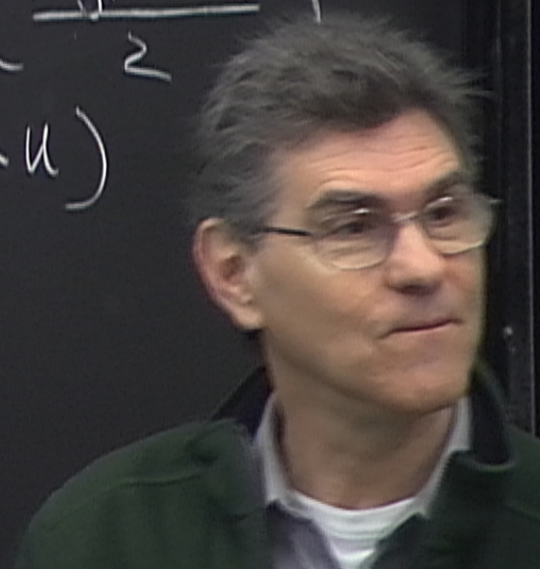
$u(x) = \cos(\pi x y_1) - \cos(\pi x y_2)$

$= -2 \sin\left(\frac{\pi x y_1 + \pi x y_2}{2}\right) \sin\left(\frac{\pi x y_1 - \pi x y_2}{2}\right)$

$\sim -2 \sin\left\{\left(n+\frac{1}{2}\right)\pi\right\} \sin\left(\frac{\pi}{2} x u\right)$

??

$\rightarrow x$



$y_1(x)$ $y_2(x)$

$y_1(x) - y_2(x) \sim$ very small! (hyper)

$\sim u(x)$

$u'(x) = \cos(\pi x y_1) - \cos(\pi x y_2)$

$= -2 \sin\left(\frac{\pi x y_1 + \pi x y_2}{2}\right) \sin\left(\frac{\pi x y_1 - \pi x y_2}{2}\right)$

$\sim -2 \sin\left(\left(n+\frac{1}{2}\right)\pi\right) \sin\left(\frac{\pi}{2} x u\right)$

$(-1)^n$

??

$y_1(x)$ $y_2(x)$

$y_1(x) - y_2(x) \sim$ very small! (hyper)

$\sim u(x)$

$u(x) = \cos(\pi x y_1) - \cos(\pi x y_2)$

$= -2 \sin\left(\frac{\pi x y_1 + \pi x y_2}{2}\right) \sin\left(\frac{\pi x y_1 - \pi x y_2}{2}\right)$

$\sim -2 \sin\left(\left(n+\frac{1}{2}\right)\pi\right) \sin\left(\frac{\pi}{2} x u\right)$

$(-1)^n$ $\frac{\pi}{2} x u$

$y_1(x)$ $y_2(x)$

$y_1(x) - y_2(x) \sim$ very small! (hyper)

$\sim u(x)$

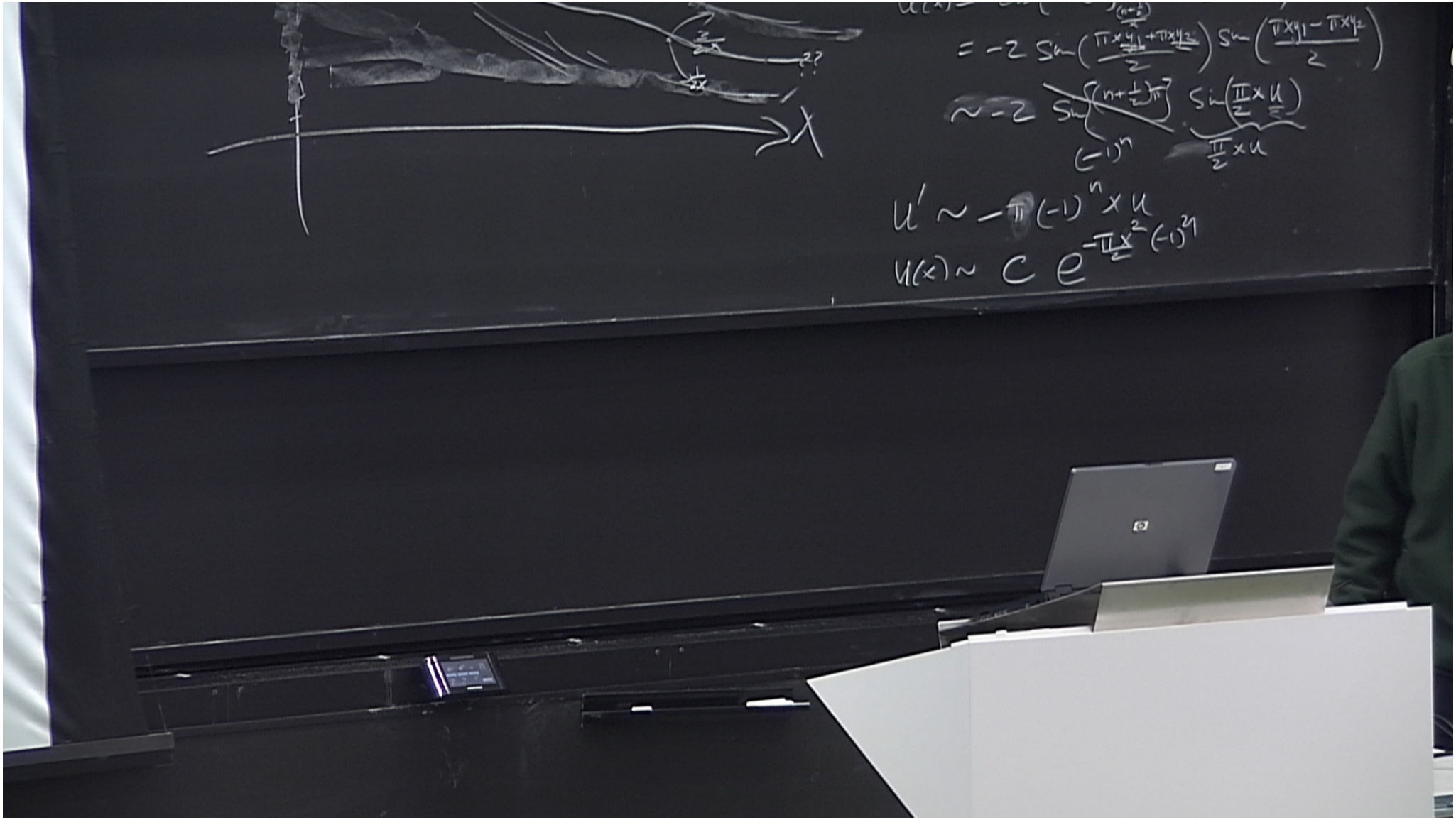
$u'(x) = \cos(\pi x y_1) - \cos(\pi x y_2)$

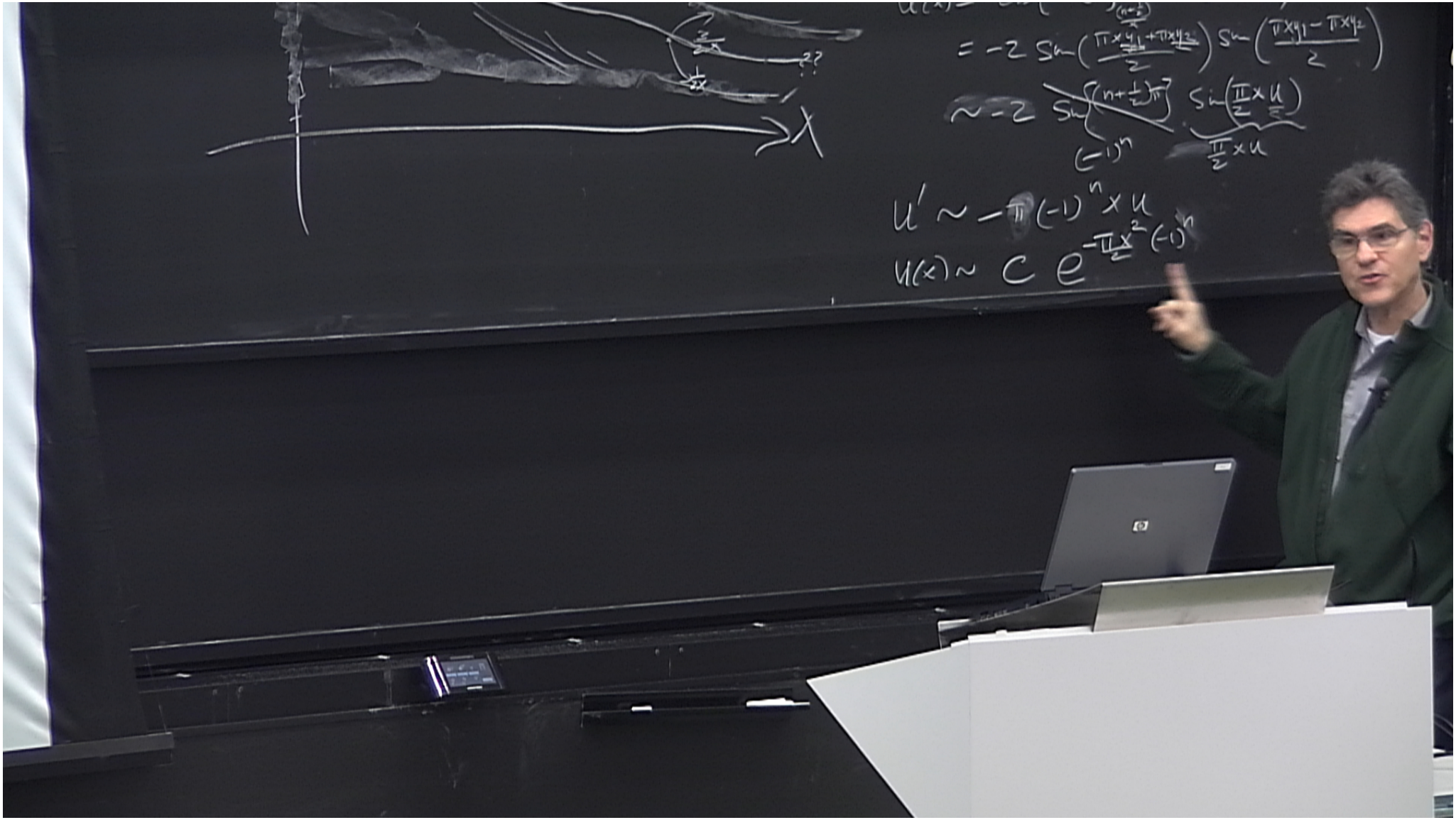
$= -2 \sin\left(\frac{\pi x y_1 + \pi x y_2}{2}\right) \sin\left(\frac{\pi x y_1 - \pi x y_2}{2}\right)$

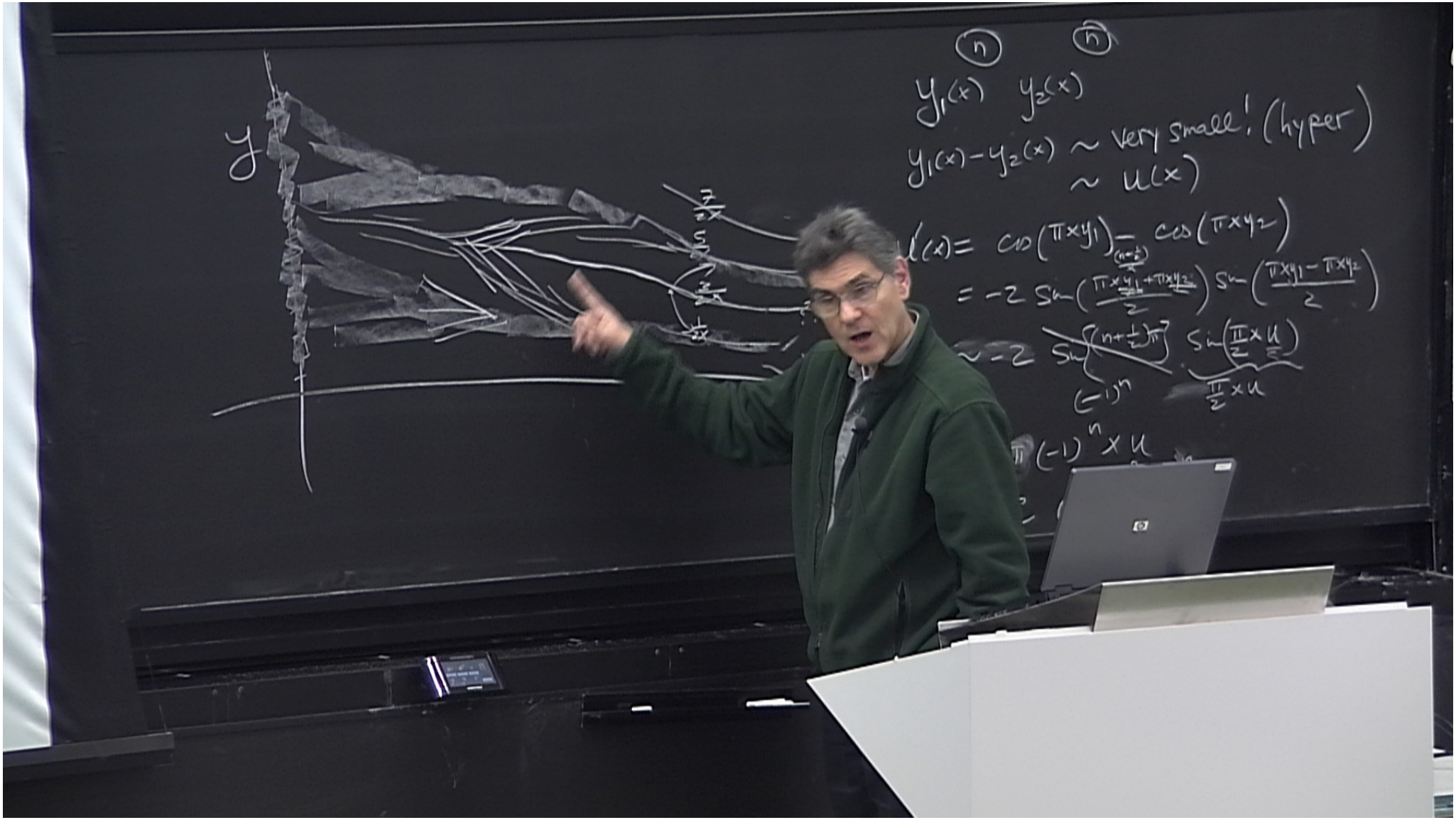
$\sim -2 \sin\left(\left(n+\frac{1}{2}\right)\pi\right) \sin\left(\frac{\pi}{2} x u\right)$

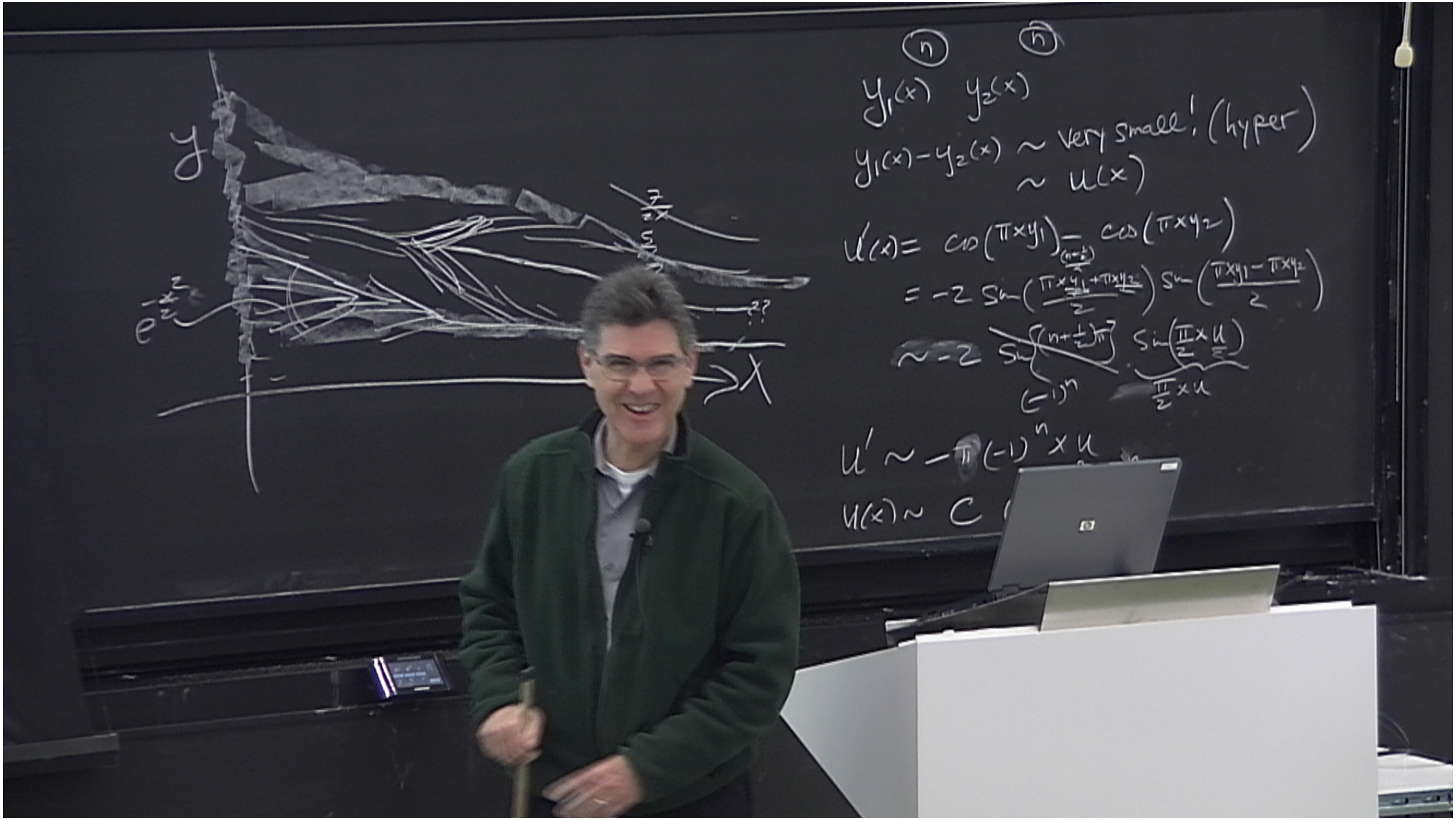
$(-1)^n$ $\quad \frac{\pi}{2} x u$

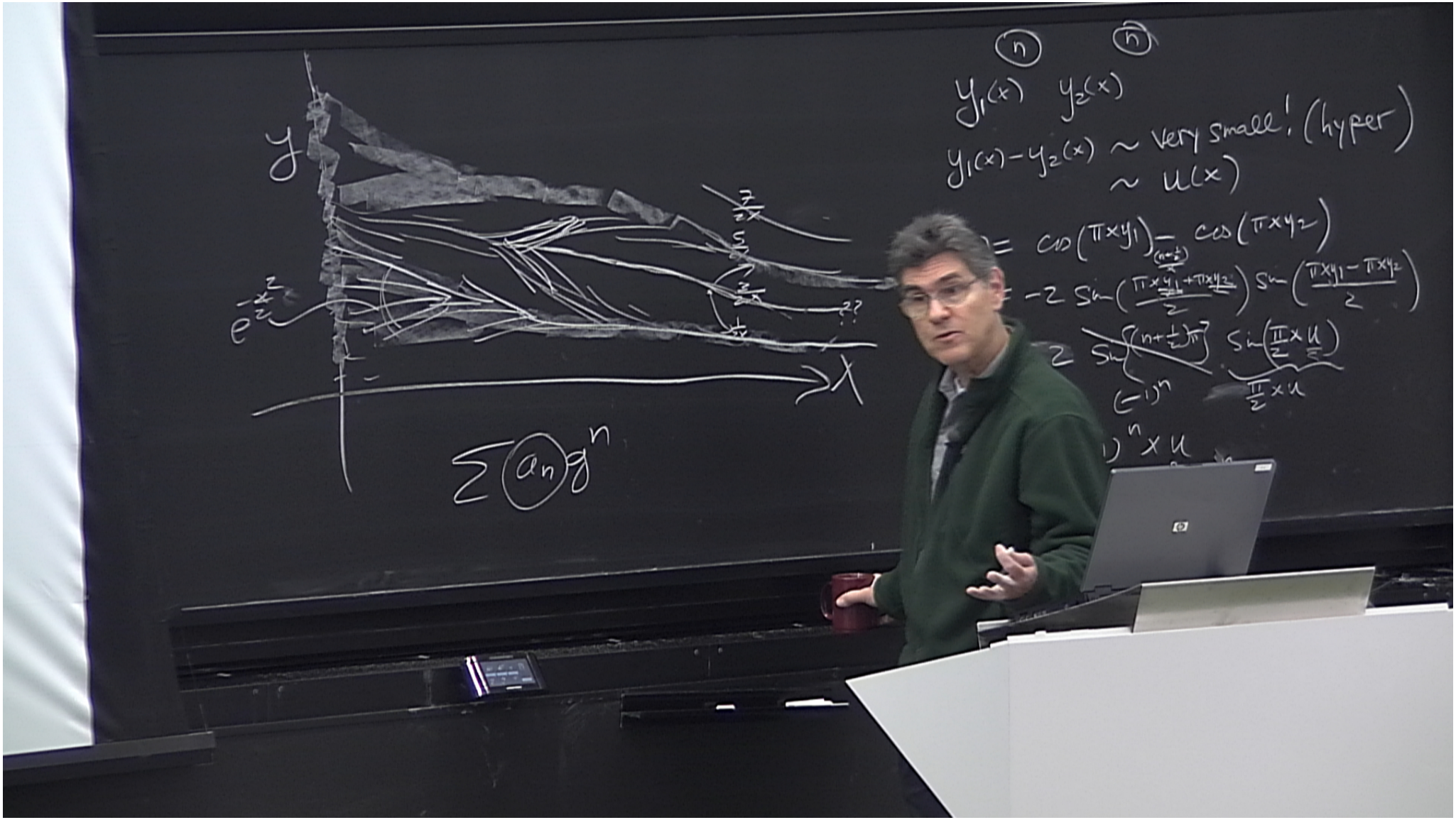
$u' \sim -\pi (-1)^n x u$

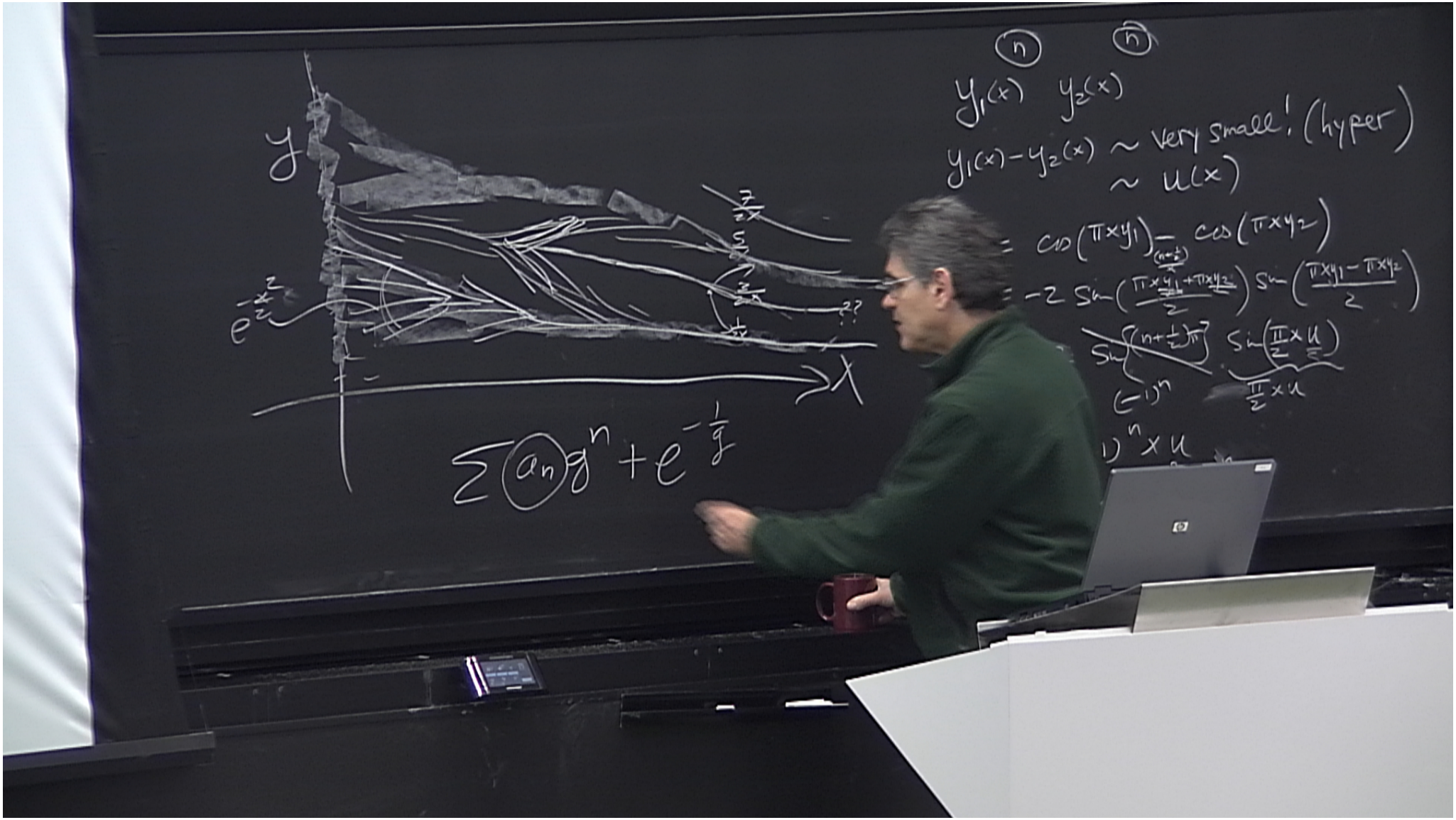

$= -2 \sin\left(\frac{\pi x_1 + \pi x_2}{2}\right) \sin\left(\frac{\pi x_1 - \pi x_2}{2}\right)$
$\sim -2 \sin\left(\left(n+\frac{1}{2}\right)\pi\right) \sin\left(\frac{\pi}{2} x\right)$
$(-1)^n$
$\frac{\pi}{2} x$
$u' \sim -\pi (-1)^n x u$
$u(x) \sim C e^{-\frac{\pi}{2} x^2 (-1)^n}$

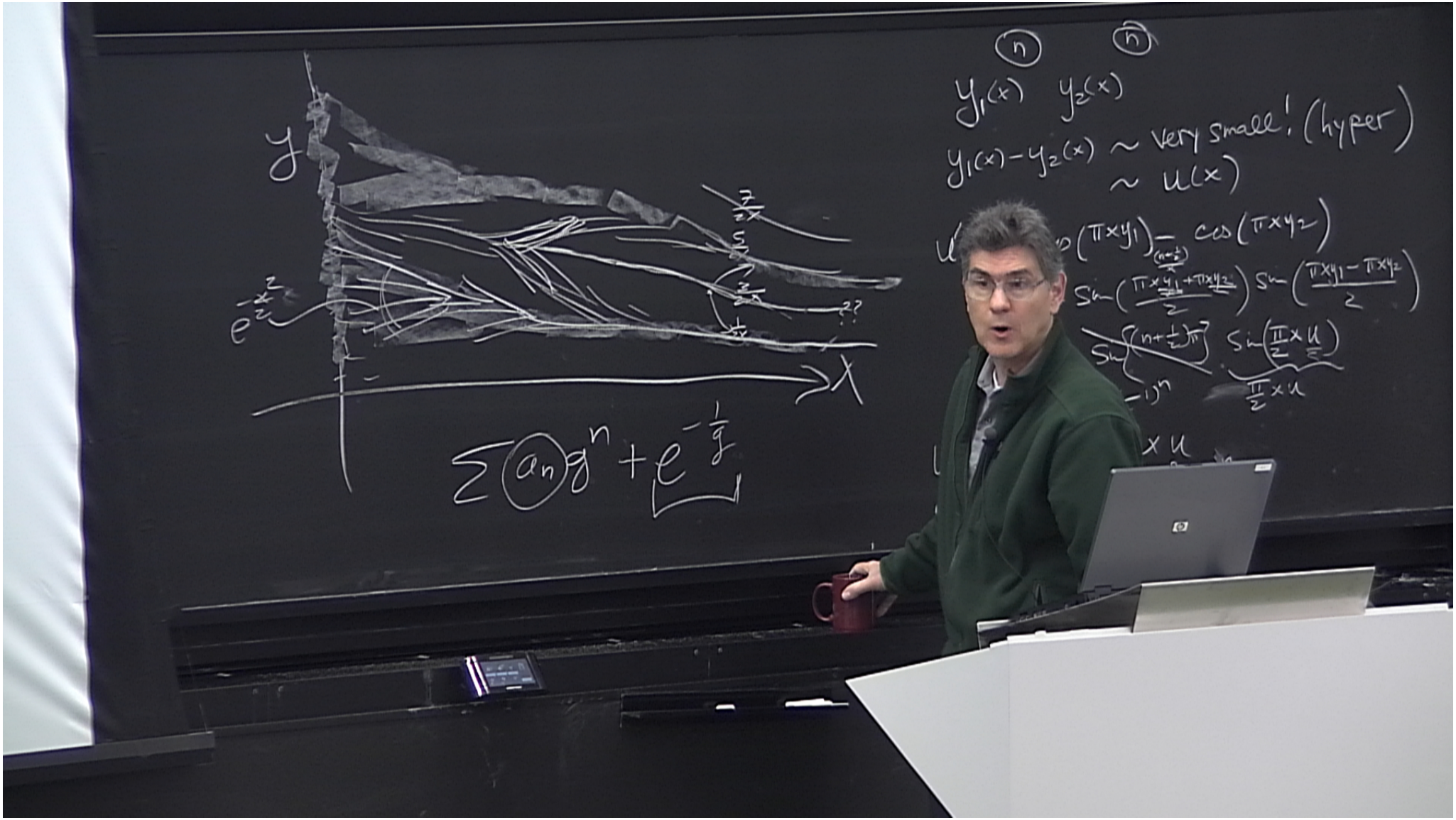

$= -2 \sin\left(\frac{\pi x_1 + \pi x_2}{2}\right) \sin\left(\frac{\pi x_1 - \pi x_2}{2}\right)$
$\sim -2 \sin\left(\left(n+\frac{1}{2}\right)\pi\right) \sin\left(\frac{\pi}{2} x u\right)$
$(-1)^n$
$\frac{\pi}{2} x u$
$u' \sim -\pi (-1)^n x u$
$u(x) \sim C e^{-\frac{\pi}{2} x^2 (-1)^n}$

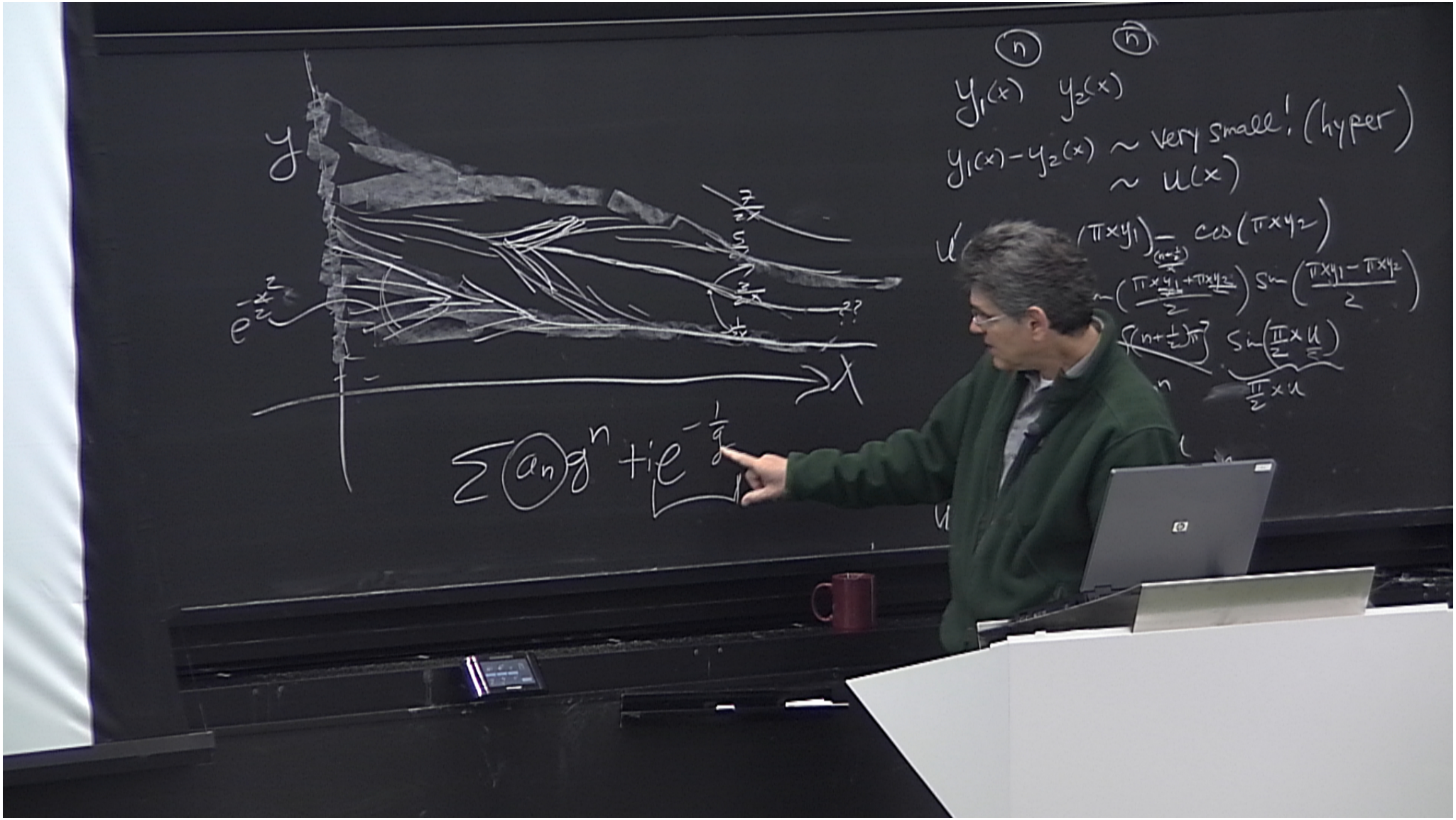

y
$y_1(x)$ $y_2(x)$
$y_1(x) - y_2(x) \sim$ very small! (hyper)
$\sim u(x)$
$\cos(\pi x y_1) - \cos(\pi x y_2)$
$= -2 \sin\left(\frac{\pi x y_1 + \pi x y_2}{2}\right) \sin\left(\frac{\pi x y_1 - \pi x y_2}{2}\right)$
$\sim -2 \sin\left(\left(n+\frac{1}{2}\right)\pi\right) \sin\left(\frac{\pi}{2} x u\right)$
$(-1)^n$
$\frac{\pi}{2} x u$
$(-1)^n x u$


y
x
$y_1(x)$ $y_2(x)$
$y_1(x) - y_2(x) \sim$ very small! (hyper)
$\sim u(x)$
$u'(x) = \cos(\pi x y_1) - \cos(\pi x y_2)$
$= -2 \sin\left(\frac{\pi x y_1 + \pi x y_2}{2}\right) \sin\left(\frac{\pi x y_1 - \pi x y_2}{2}\right)$
$\sim -2 \sin\left(\left(n+\frac{1}{2}\right)\pi\right) \sin\left(\frac{\pi x u}{2}\right)$
$(-1)^n$
$\frac{\pi}{2} x u$
$u' \sim -\pi (-1)^n x u$
$u(x) \sim C$


$y_1(x)$ (n) $y_2(x)$ (n)
$y_1(x) - y_2(x) \sim$ very small! (hyper)
$\sim u(x)$
$\Sigma a_n g^n$


y
x
$e^{-\frac{x^2}{2}}$
$\Sigma a_n g^n + e^{-\frac{1}{g}}$
$y_1(x)$ $y_2(x)$
$y_1(x) - y_2(x) \sim$ very small! (hyper)
$\sim u(x)$
$= \cos(\pi x y_1) - \cos(\pi x y_2)$
$= -2 \sin\left(\frac{\pi x y_1 + \pi x y_2}{2}\right) \sin\left(\frac{\pi x y_1 - \pi x y_2}{2}\right)$
$\sin\left(\frac{\pi}{2} x u\right)$
$\frac{\pi}{2} x u$
$(-1)^n$
$(-1)^n x u$


y
x
$e^{-\frac{x^2}{2}}$
$\Sigma a_n g^n + e^{-\frac{1}{g}}$
$y_1(x)$ $y_2(x)$
$y_1(x) - y_2(x) \sim$ very small! (hyper)
$\sim u(x)$
$\cos(\pi x y_1)$ $\cos(\pi x y_2)$
$\sin\left(\frac{\pi x y_1 + \pi x y_2}{2}\right)$ $\sin\left(\frac{\pi x y_1 - \pi x y_2}{2}\right)$
$\sin\left(\frac{\pi}{2} x u\right)$
$\frac{\pi}{2} x u$


$y_1(x)$ $y_2(x)$
$y_1(x) - y_2(x) \sim$ very small! (hyper)
$\sim u(x)$
$\cos(\pi x y_1)$ $\cos(\pi x y_2)$
$\sin\left(\frac{\pi x y_1 - \pi x y_2}{2}\right)$
$\sin\left(\frac{\pi}{2} x u\right)$
$\frac{\pi}{2} x u$
y
x
$e^{-\frac{x^2}{2}}$
$\sum a_n g^n + i e^{-\frac{1}{g}}$