

Title: Mathematical Physics - Lecture 14

Date: Dec 08, 2011 09:00 AM

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Abstract:

Table 10.1 A comparison of the exact eigenvalues E_n of the Sturm-Liouville problem $y''(x) + E(x + \pi)^4 y(x) = 0$ [$y(0) = y(\pi) = 0$] with the leading-order WKB prediction [see (10.1.34)] for these eigenvalues $E_n \sim 9n^2/49\pi^2$ ($n \rightarrow \infty$)

As expected, this prediction becomes more accurate as n increases. The relative error is defined as (approximate – exact)/(exact)

n	E_n (WKB)	E_n (exact)	Relative error, %
1	0.00188559	0.00174401	8.1
2	0.00754235	0.00734865	2.6
3	0.0169703	0.0167524	1.3
4	0.0301694	0.0299383	0.77
5	0.0471397	0.0469006	0.51
10	0.188559	0.188305	0.13
20	0.754235	0.753977	0.035
40	3.01694	3.01668	0.009

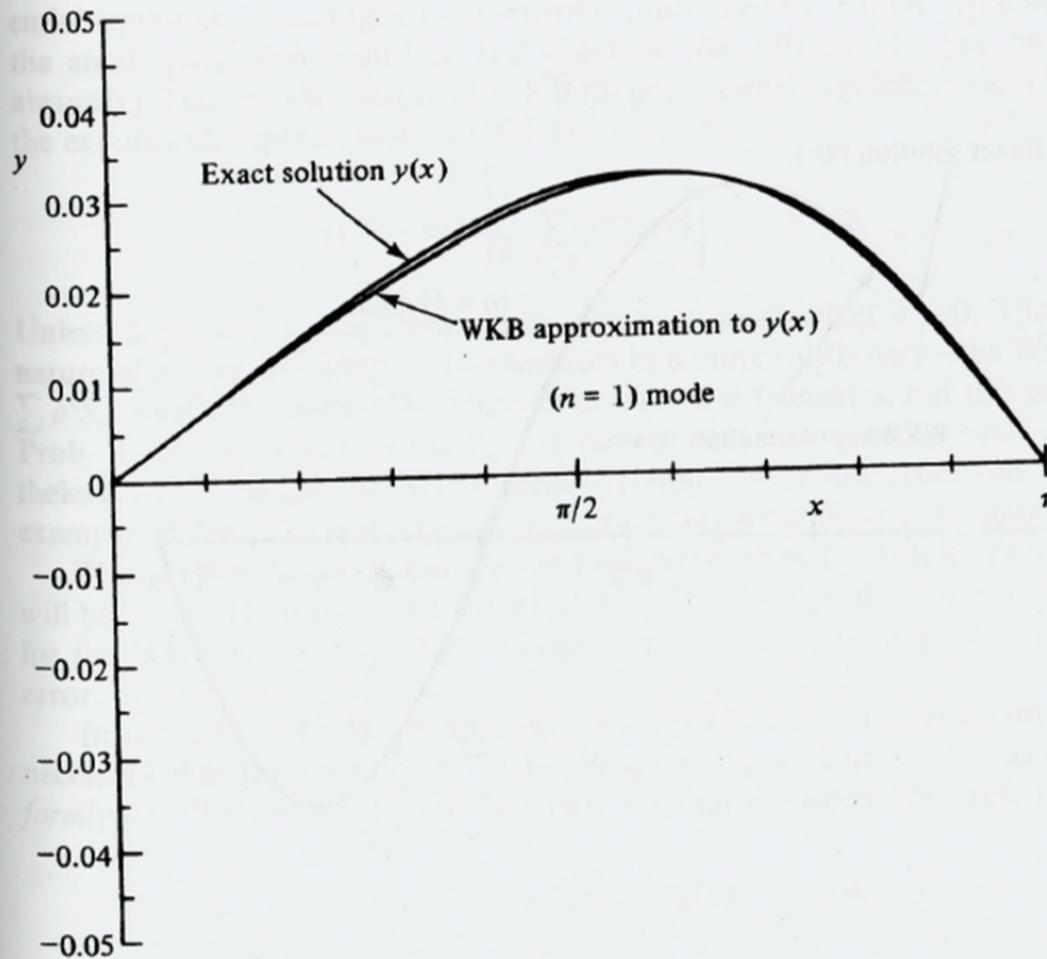


Figure 10.2 Comparison of the exact solution to $y''(x) + E_n(x + \pi)^4 y(x) = 0$ [$y(0) = y(\pi) = 0$], with the WKB approximation to this solution as given in (10.1.35) for the lowest ($n = 1$) mode. Although the WKB becomes exact as $n \rightarrow \infty$, this plot shows that even when $n = 1$ the WKB approximation is extraordinarily accurate.

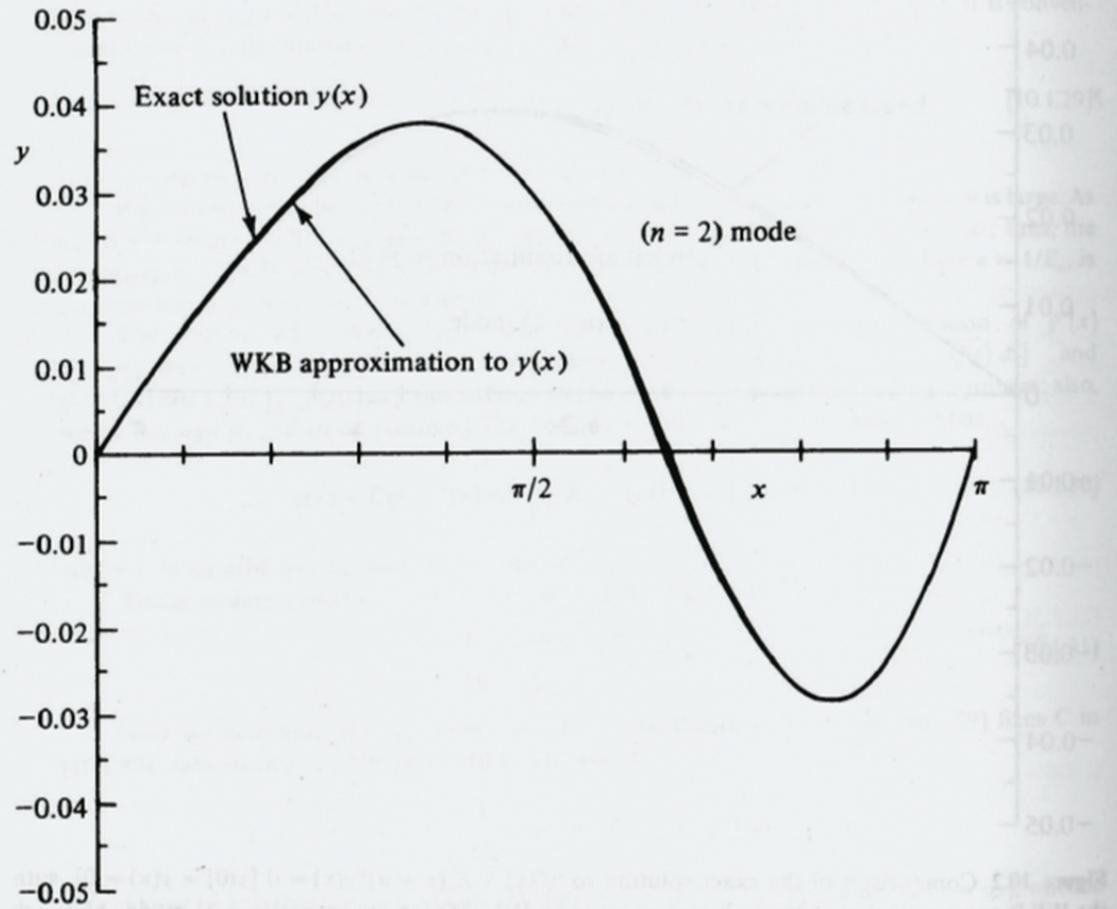
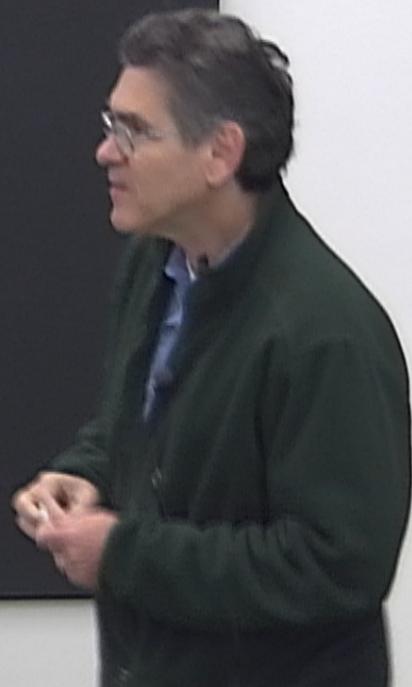


Figure 10.3 Same as in Fig. 10.2 except that $n = 2$. The exact eigenfunction and the WKB approximation are almost indistinguishable.

$$E_n \sim C n^2$$



$$\underbrace{\left(-\frac{d^2}{dx^2} + V(x) \right) \psi = E \psi}_{E_n \sim C n^2}$$



$$\left(-\frac{d^2}{dx^2} + V(x) \right) \psi = E \psi$$

$$E_n \sim C n^2$$

$$\begin{aligned}\psi(x) &\rightarrow 0 \text{ as } x \rightarrow +\infty \\ \psi(x) &\rightarrow 0 \text{ as } x \rightarrow -\infty\end{aligned}$$

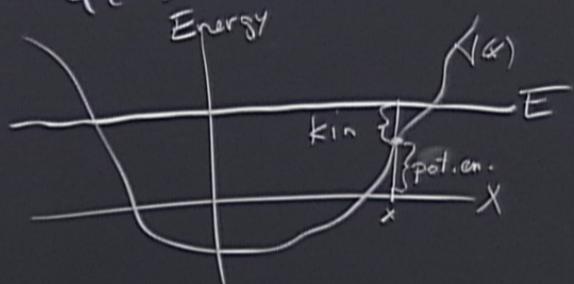


$$\left(-\frac{d^2}{dx^2} + V(x) \right) \psi = E \psi$$

$$E_n \sim C n^2$$

$$\psi(x) \rightarrow 0 \text{ as } x \rightarrow +\infty$$

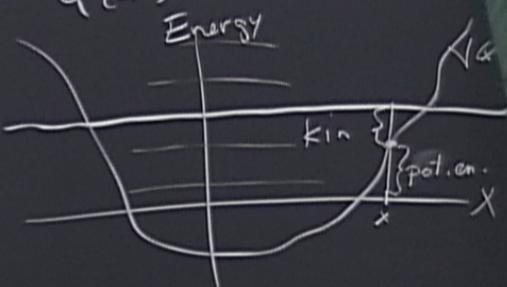
$$\psi(x) \rightarrow 0 \text{ as } x \rightarrow -\infty$$



$$\left(-\frac{d^2}{dx^2} + V(x) \right) \psi = E \psi \quad E_n \sim C n^2$$

$\psi(x) \rightarrow 0$ as $x \rightarrow +\infty$

$\psi(x) \rightarrow 0$ as $x \rightarrow -\infty$

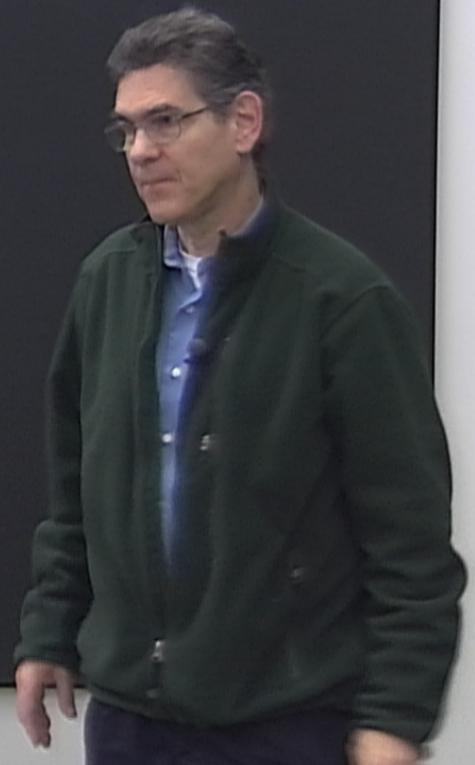
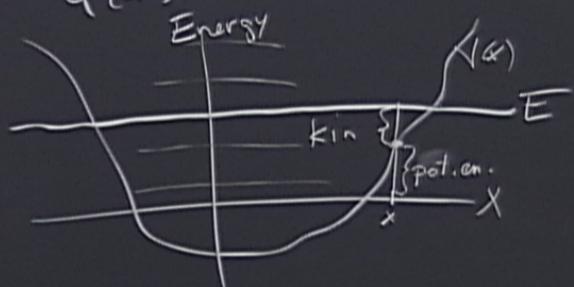


$$\left(-\frac{d^2}{dx^2} + V(x) \right) \psi = E \psi$$

$$E_n \sim C n^2$$

$$\psi(x) \rightarrow 0 \text{ as } x \rightarrow +\infty$$

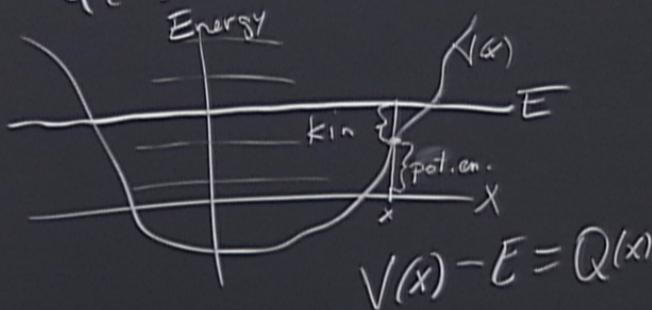
$$\psi(x) \rightarrow 0 \text{ as } x \rightarrow -\infty$$



$$\left(-\frac{d^2}{dx^2} + V(x) \right) \psi = E \psi$$

$$\psi(x) \rightarrow 0 \text{ as } x \rightarrow +\infty$$

$$\psi(x) \rightarrow 0 \text{ as } x \rightarrow -\infty$$

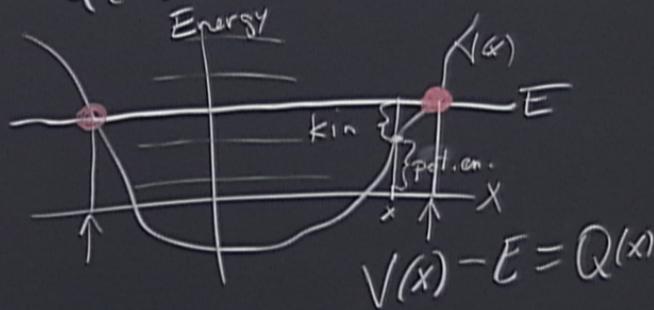


$$\epsilon^z \psi'' = Q(x) \psi$$

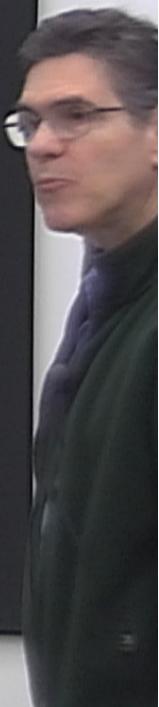
$\epsilon \ll 1$

$$\left(-\frac{d^2}{dx^2} + V(x) \right) \psi = E \psi$$

$$\begin{aligned}\psi(x) &\rightarrow 0 \quad \text{as } x \rightarrow +\infty \\ \psi(x) &\rightarrow 0 \quad \text{as } x \rightarrow -\infty\end{aligned}$$



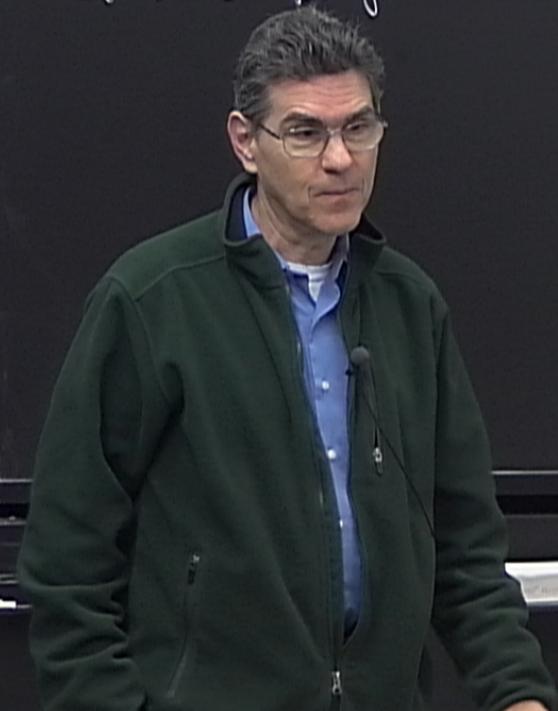
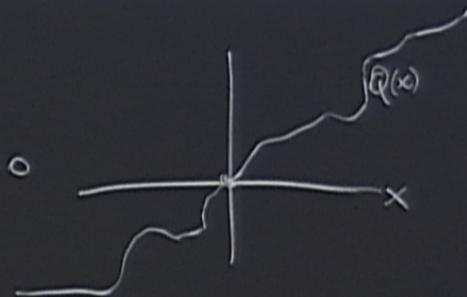
$$\begin{array}{c} \epsilon^z \\ \epsilon^z \psi'' = Q(x) \psi \\ \hline \epsilon \ll 1 \end{array}$$



ONE T-P PROB

$\in^2 y''(x) = Q(x)$
 $Q(x) = 0$ for just 1 value of x , which is $x=0$

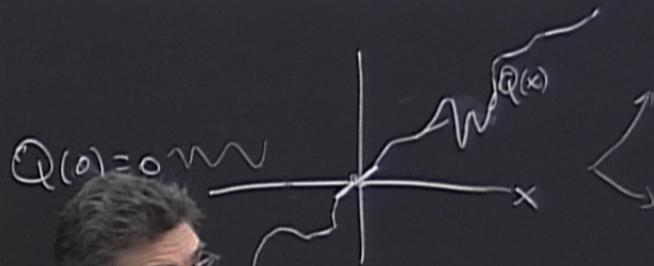
$$Q(0)=0$$



ONE T-P PROB

$$\begin{cases} \text{if } Q(x) = 0 \text{ for just 1 value of } x, \text{ which is } x=0 \\ Q(x) \sim ax \text{ as } x \rightarrow 0, a > 0. \end{cases}$$

$$y(+\infty) = 0$$

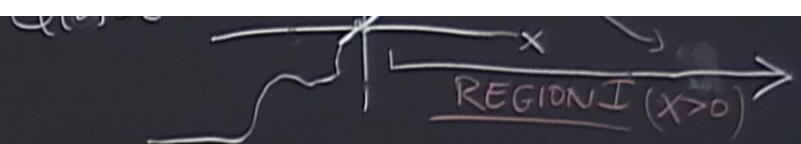


$Q(x) = 0$ for just 1 value of x , which is $x = 0$

$Q(x) \sim \alpha x$ as $x \rightarrow 0$, $\alpha > 0$.

$$y(+\infty) = 0$$

$x \geq 0$ $REG. I$ $y(x) \sim C e^{-\frac{1}{\epsilon} \int_0^x dt / \sqrt{Q(t)}} \quad (\epsilon \rightarrow 0)$



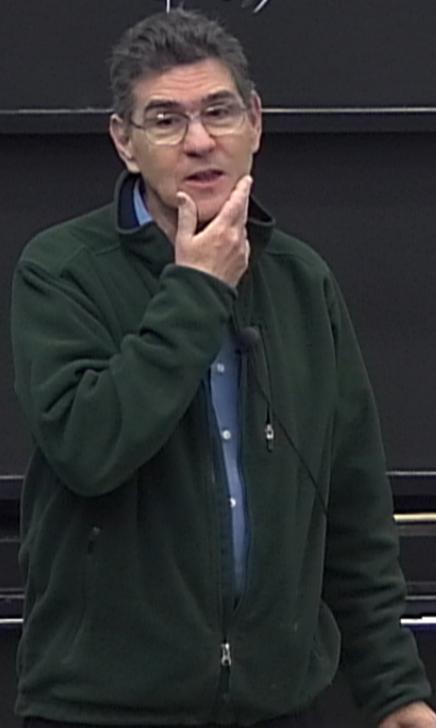
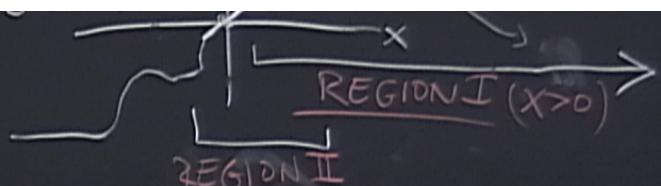
$Q(x) = 0$ for just 1 value of x , which is $x = 0$

$Q(x) \sim \alpha x$ as $x \rightarrow 0$, $\alpha > 0$.

$$y(+\infty) = 0$$

$x \geq 0$ REG. I $y(x) \sim C e^{-\frac{1}{\epsilon} \int_0^x dt \sqrt{Q(t)}} \quad (\epsilon \rightarrow 0)$

REG II $(x \text{ near } 0)$ $x \ll 1$ $\boxed{\epsilon^2 y'' = \alpha x y}$ $\epsilon^2 y'' = (ax + bx^2 + cx^3 + \dots) y$



$$Q^{1/4}(x)$$

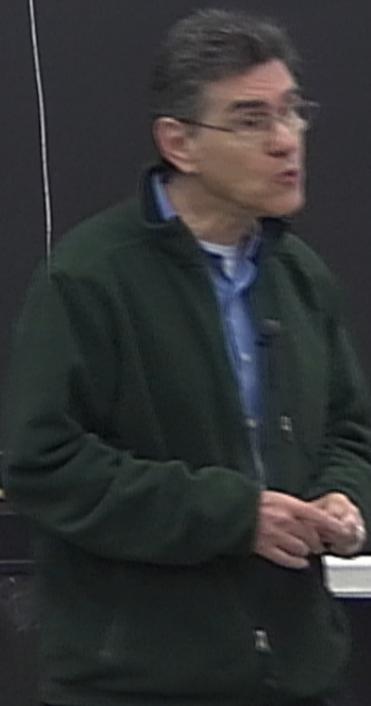
$$\text{Let } x = \gamma t$$

$$\frac{\epsilon^2}{\gamma^2} \frac{d^2}{dt^2} y = \left(\frac{\alpha \gamma^3}{\epsilon^2}\right) t y$$

$$\frac{\alpha \gamma^3}{\epsilon^2} = 1$$

$$\gamma = \frac{\epsilon^{1/3}}{\alpha^{1/3}}$$

$$\frac{d^2 y}{dt^2} = t y$$



$$Q^{\frac{1}{4}}(x)$$

$$\text{Let } x = \gamma t$$

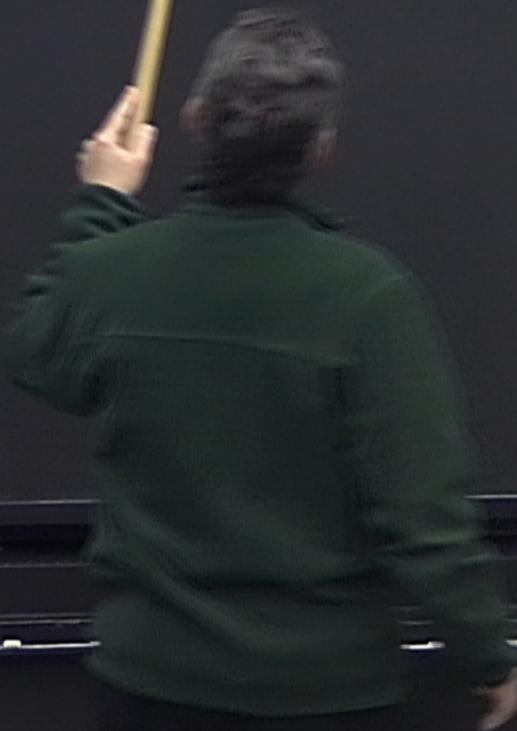
$$\frac{E^2}{\gamma^2} \frac{d^2}{dt^2} y = \left(\frac{\alpha \gamma^3}{E^2}\right) t y$$

$$\frac{\alpha \gamma}{E^2} = 1$$

$$\gamma = \frac{E^2 \beta}{\alpha \beta^3}$$

$$\frac{d^2 y}{dt^2} = t y \quad \text{Airy}$$

$$y(t) = D_1 \underline{A_i(t)} + D_2 \underline{B_i(t)}$$



$$\gamma^2 \sin \gamma = \cos \gamma \quad \gamma = \frac{\pi}{\sqrt{3}}$$

$$\frac{d^2 y}{dt^2} = t y \quad \text{Airy}$$

$$y(t) = D_1 \tilde{A}_i(t) + D_2 \tilde{B}_i(t)$$

$$t = \frac{x}{\delta}$$
$$y(t) \sim D A_i(t) = D A_i(x)$$

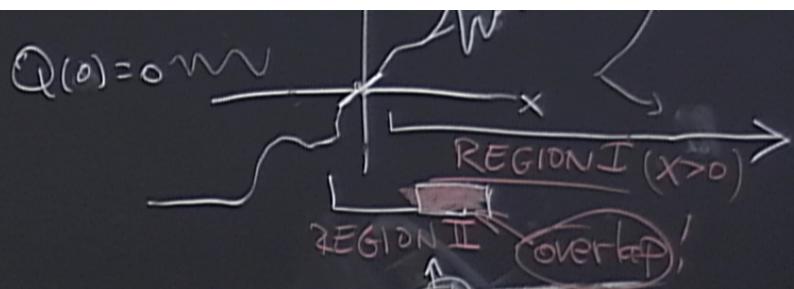


valid EVERWHERE in Reg I

$$\frac{d^2y}{dt^2} = ty \quad \text{Airy}$$
$$y(t) = D_1 A_i(t) + D_2 B_i(t)$$
$$t = \frac{x}{\delta}$$
$$y(t) \sim D A_i(t) = D A_i \left(x \frac{\alpha^{1/3}}{\epsilon^{2/3}} \right)$$

VALID EVERWHERE IN REG II

$$\begin{cases} \epsilon^{-2} y''(x) = Q(x) y \\ Q(x) = 0 \text{ for just 1 value of } x, \text{ which is } x=0 \\ Q(x) \sim ax \text{ as } x \rightarrow 0, a > 0. \end{cases}$$



$$y(+\infty) = 0$$

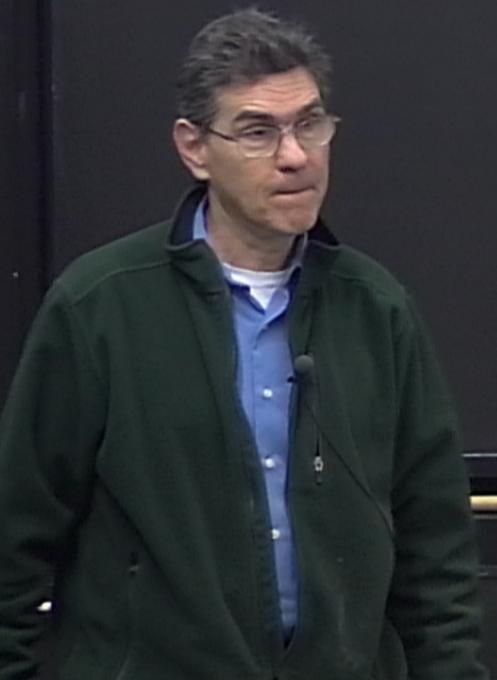
$$\underline{\underline{\text{REG. I}}} \quad y(x) \sim C \frac{e^{-\frac{1}{\epsilon} \int_0^x dt \sqrt{Q(t)}}}{Q^{1/4}(x)} \quad (\epsilon \rightarrow 0)$$

REG. II $x \ll 1$ $\epsilon^2 y'' = a x y$ $\text{Let } x = \gamma t$ $\frac{\epsilon^2}{\gamma^2} \frac{d^2}{dt^2} y = (\frac{a\gamma^3}{\epsilon^2}) y$	$\epsilon^2 y'' = (ax + bx^{\frac{1}{2}} + cx^{\frac{3}{2}} + \dots) y$ $\frac{a\gamma^3}{\epsilon^2} = 1$ $\gamma = \frac{\epsilon^{1/3}}{a^{1/3}}$
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VALID IN REG. II
 $\frac{x}{\epsilon^{1/3}} \gg 1$

\uparrow
 valid EVERWHERE in Reg I
in overlap $x \ll 1$

$$\begin{aligned}
 & \frac{d^2 y}{dt^2} = t y \quad \text{Airy} \\
 & y(t) = D_1 A_i(t) + D_2 B_o(t) \quad t = \frac{x}{\epsilon} \\
 & y(t) \sim D A_i(t) = D A_i \left(x \frac{\alpha^{1/2}}{\epsilon^{2/3}} \right) \\
 & \text{VALID } \nwarrow \text{EVERWHERE IN REG II} \\
 & \frac{x}{\epsilon^{2/3}} \gg 1, \quad (x \gg \epsilon^{2/3}) \Rightarrow \epsilon \rightarrow 0 \\
 & \text{IN OVERLAP} \quad y \sim D \frac{1}{2\sqrt{\pi}} e^{-\frac{2}{3} \frac{x^{3/2}}{\epsilon}} \frac{\sqrt{\alpha}}{x^{1/4} \alpha^{1/2}}
 \end{aligned}$$



$$Q(x) \sim ax \text{ as } x \rightarrow 0, a > 0.$$

$$y(+\infty) = 0$$

$$\underline{\underline{X \geq 0}} \quad \underline{\underline{\text{REG. I}}} \quad y(x) \sim C e^{-\frac{1}{\epsilon} \int_0^x dt \sqrt{Q(t)}} \quad (\epsilon \rightarrow 0)$$



$$\begin{aligned} &\text{REG II (x near 0)} \quad \epsilon^2 y'' = (ax + bx^2 + cx^3 + \dots) y \\ &x \ll 1 \quad \epsilon^2 y'' = ax y \\ &\text{Let } x = \gamma t \quad \frac{a \gamma^3}{\epsilon^2} = 1 \\ &\epsilon^2 \frac{d^2}{dt^2} y = \left(\frac{a \gamma^3}{\epsilon}\right) y \quad \gamma = \frac{\epsilon^{2/3}}{a^{1/3}} \end{aligned}$$

$$y(t) \sim D A_i(t) = D A_i \left(x \frac{a^{1/2}}{\epsilon^{2/3}} \right)$$

VALID EVERWHERE IN REG II

$$\begin{aligned} &\frac{x}{\epsilon^{2/3}} \gg 1, \quad x \gg \epsilon^{2/3} \Rightarrow \epsilon \rightarrow 0 \\ &\text{IN OVERLAP} \quad y \sim D \frac{1}{2\sqrt{\pi}} e^{-\frac{2}{3} x^{3/2} \frac{\sqrt{a}}{\epsilon}} \end{aligned}$$



$$\frac{\epsilon^2}{y^2} \frac{d^2}{dt^2} y = \left(\frac{\alpha\gamma t}{\epsilon}\right) y \quad y = \frac{\epsilon^{2/3}}{\alpha^{1/3}}$$

valid EVERWHERE in Reg I

in overlap $x \ll 1$

$$y(x) \sim C e^{-\frac{1}{\epsilon} \int_0^x dt + \sqrt{at + bt^2 + \dots}}$$

$$\frac{d^2 y}{dt^2} = t y \quad \text{Airy}$$

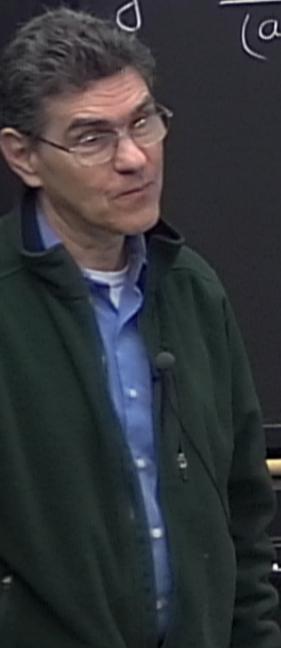
$$y(t) = D_1 A_i(t) + D_2 B_o(t)$$

$$y(t) \sim D A_i(t) = D A_i \left(\frac{x \alpha^{1/2}}{\epsilon^{2/3}} \right)$$

VALID & EVERYWHERE IN REG II

$$\frac{x}{\epsilon^{2/3}} \gg 1, \quad (x \gg \epsilon^{2/3}) \Rightarrow \epsilon \rightarrow 0$$

IN OVERLAP $y \sim D \frac{1}{2\sqrt{\pi}} e^{-\frac{2}{3} \frac{x^{3/2}}{\epsilon}}$



$$\frac{d^2}{dt^2} y = \left(\frac{\alpha \gamma t}{\epsilon}\right) y \quad \gamma = \frac{\epsilon^{2/3}}{\alpha^{1/3}}$$

valid EVERWHERE in Reg I

in overlap $x \ll 1$

$$y(x) \sim e^{-\frac{1}{\epsilon} \int_0^x \sqrt{at+bt^2+\dots}}$$

$$e^{-\frac{1}{\epsilon} \int_0^x \sqrt{at+b\frac{t^2}{2a}} \left(1 + \frac{bt}{2a}\right)^{1/2}} \\ \int_0^x dt \left[\sqrt{at+b\frac{t^2}{2a}} + \frac{b}{2\sqrt{a}} t^{3/2} + \dots \right]$$

$$\sqrt{\frac{2}{3}} t^{3/2} + \frac{b}{2\sqrt{a}} \frac{2}{5} t^{5/2}$$

$$\frac{d^2 y}{dt^2} = t y \quad \text{Airy}$$

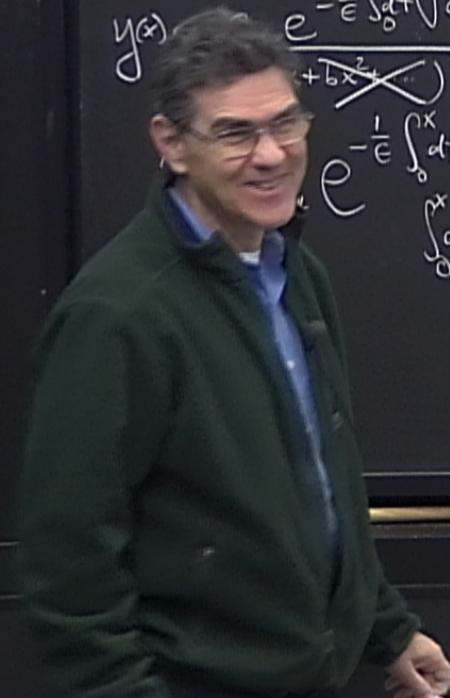
$$y(t) = D_1 A_i(t) + D_2 B_o(t)$$

$$y(t) \sim D A_i(t) = D A_i \left(x \frac{a^{1/2}}{\epsilon^{2/3}} \right)$$

VALID EVERWHERE IN REG II

$$\frac{x}{\epsilon^{2/3}} \gg 1, \quad (x \gg \epsilon^{2/3}) \Rightarrow \epsilon \rightarrow 0$$

IN OVERLAP $y \sim D \frac{1}{2\sqrt{\pi}} e^{-\frac{2}{3} \frac{x^{3/2}}{\epsilon^{1/2}}} \frac{a^{1/2}}{x^{1/4} \epsilon^{1/6}}$



valid ~~EVERWHERE~~ in Reg I

in overlap $x \ll 1$

$$y(x) \sim C e^{-\frac{1}{\epsilon} \int_0^x dt \sqrt{at+bt^2+\dots}}$$

$$\sim \frac{C}{a^{1/4} x^{1/4}} e^{-\frac{1}{\epsilon} \int_0^x dt \sqrt{at} \left(1 + \frac{bt}{a}\right)^{1/2}}$$

$$\sim \frac{C}{a^{1/4} x^{1/4}} e^{-\frac{1}{\epsilon} \int_0^x dt \left[\sqrt{at} t^{1/2} + \frac{b}{2\sqrt{a}} t^{3/2} + \dots \right]}$$

$$\sim \frac{C}{a^{1/4} x^{1/4}} e^{-\frac{1}{\epsilon} \left[\frac{2}{3} \sqrt{a} x^{3/2} + \frac{b}{5\sqrt{a}} x^{5/2} + \dots \right]}$$

$$\frac{d^2 y}{dt^2} = ty \quad \text{Airy}$$

$$y(t) = D_1 \text{Ai}(t) + D_2 \text{Bi}(t)$$

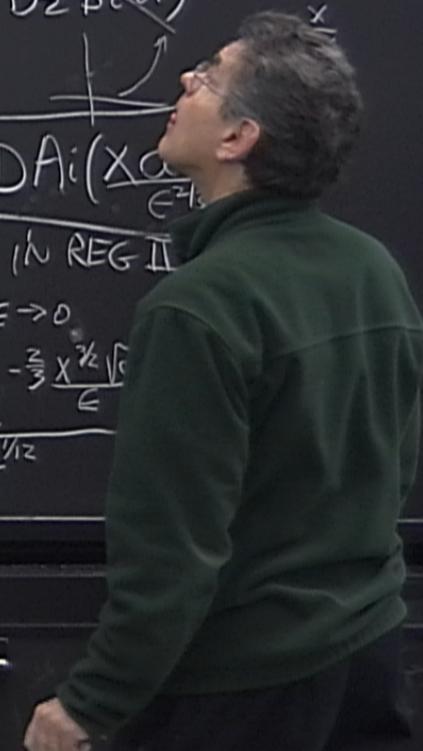
$$y(t) \sim D \text{Ai}(t) = D \text{Ai}\left(\frac{x\alpha}{\epsilon^{2/3}}\right)$$

VALID ~~EVERWHERE~~ IN REG II

$$\frac{x}{\epsilon^{2/3}} \gg 1, \quad x \gg \epsilon^{2/3} \Rightarrow \epsilon \rightarrow 0$$

$$\text{IN OVERLAP } y \sim D \frac{1}{\sqrt{\pi}} e^{-\frac{2}{3} \frac{x^{3/2}}{\epsilon}}$$

$$y(t) \sim D \text{Ai}(t) = D \text{Ai}\left(\frac{x\alpha}{\epsilon^{2/3}}\right)$$

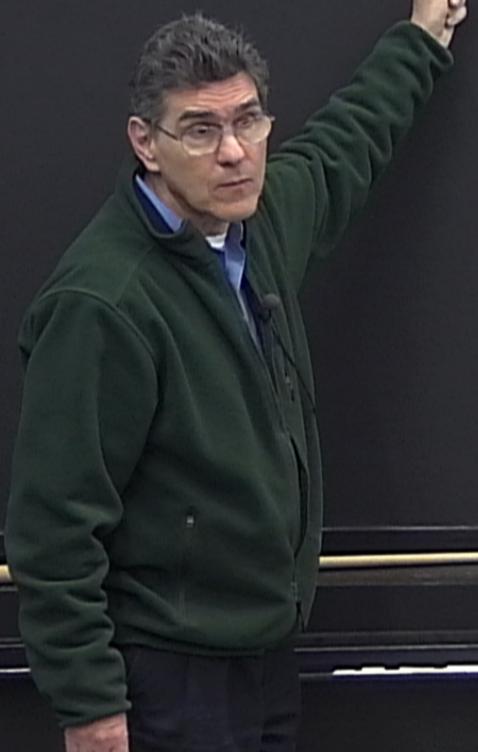


$$\sim \frac{C}{\alpha^{1/4} x^{1/4}} e^{-\frac{1}{\epsilon} \left[\frac{2}{3} \sqrt{\kappa} x^{3/2} + \frac{\sqrt{2} \sqrt{\alpha} \epsilon^{1/2}}{5 \sqrt{\kappa}} x^{5/2} + \dots \right]}$$

IN OVERLAP

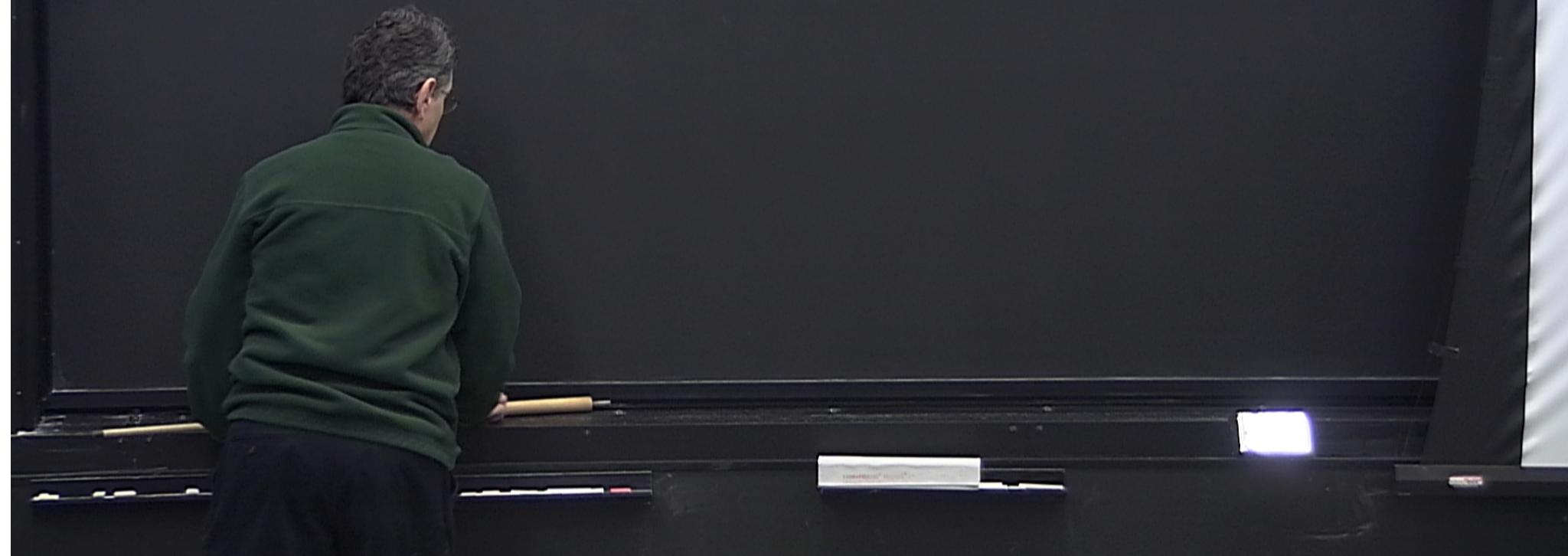
$$y \sim D_{2\sqrt{\pi}} \frac{e^{-\frac{1}{3} x^{12/5}/\epsilon}}{\frac{x^{1/4} \alpha^{1/12}}{\epsilon^{1/6}}}$$

$$\frac{x^{5/2}}{\epsilon} \ll 1 \Rightarrow x^{5/2} \ll \epsilon$$



$$\sim \frac{C}{\alpha^{1/4} x^{1/4}} e^{-\frac{1}{\epsilon} \left[\frac{2}{3} \sqrt{\alpha} x^{3/2} + \frac{b}{5\sqrt{\alpha}} x^{5/2} + \dots \right]} \quad | \text{IN OVERLAP} \quad y \sim D \frac{1}{2\sqrt{\pi}} C \frac{-\frac{1}{3} x^{1/2} \sqrt{\alpha}}{\frac{x^{1/4} \alpha^{1/2}}{\epsilon^{1/6}}}$$

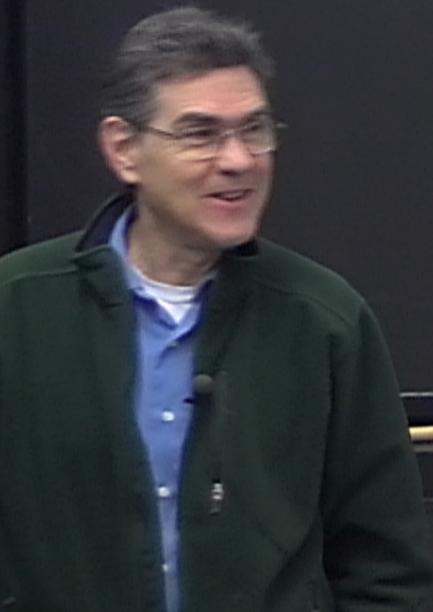
$$\frac{x^{5/2}}{\epsilon} \ll 1 \rightarrow x^{5/2} \ll \epsilon \rightarrow x \ll \epsilon^{1/5}$$



$$\sim \frac{C}{\alpha^{1/4} x^{1/4}} e^{-\frac{1}{\epsilon} \left[\frac{2}{3} \sqrt{\alpha} x^{3/2} + \frac{b}{5\sqrt{\alpha}} x^{5/2} + \dots \right]} \quad | \text{IN OVERLAP} \quad y \sim D_{2\sqrt{\alpha}} \frac{e^{-\frac{2}{3} \frac{x^2}{\epsilon} \sqrt{\alpha}}}{x^{1/4} \alpha^{1/2}}$$

$$\frac{x^{5/2}}{\epsilon} \ll 1 \rightarrow x^{5/2} \ll \epsilon \rightarrow x \ll \epsilon^{1/5}$$

$$y \sim y_{\text{WKB}} \sim \frac{C}{\alpha^{1/4} x^{1/4}} e^{-\frac{1}{\epsilon} \frac{2}{3} \sqrt{\alpha} x^{3/2}}$$



$$\sim \frac{C}{\alpha^{1/4} x^{1/4}} e^{-\frac{1}{\epsilon} \left[\frac{2}{3} \sqrt{\alpha} x^{3/2} + \frac{b}{5 \sqrt{\alpha}} x^{5/2} + \dots \right]}$$

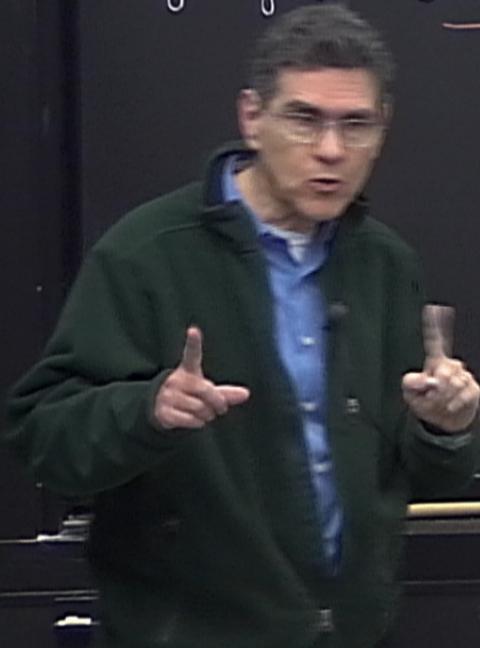
$\epsilon^{1/3} \rightarrow 0$

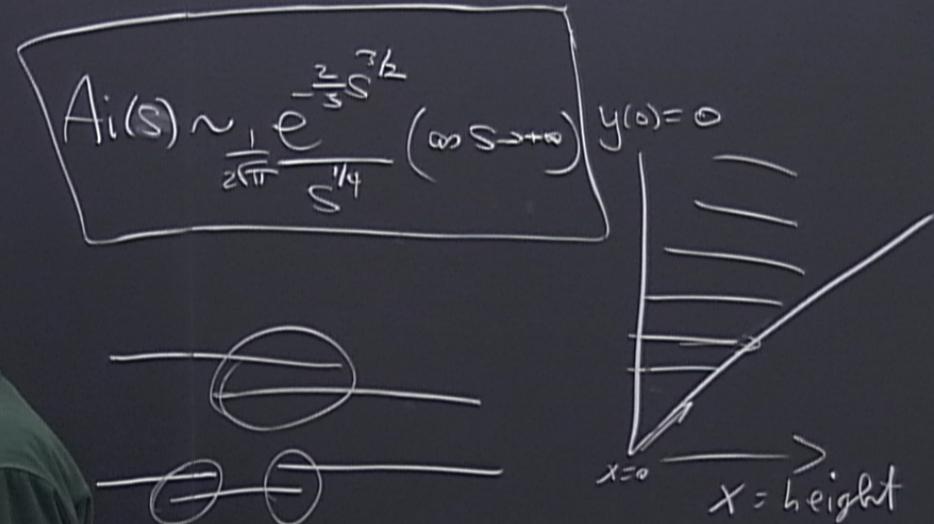
IN OVERLAP

$$y \sim D \frac{1}{2\sqrt{\pi}} C \frac{-\frac{2}{3} x^{1/2} \sqrt{\alpha}}{\frac{x^{1/4} \alpha^{1/2}}{\epsilon^{1/6}}}$$

$$\frac{x^{5/2}}{\epsilon} \ll 1 \rightarrow x^{5/2} \ll \epsilon \rightarrow x \ll \epsilon^{1/5}$$

$$y \sim y_{\text{WKB}} \sim \frac{C}{\alpha^{1/4} x^{1/4}} e^{-\frac{1}{\epsilon} \frac{2}{3} \sqrt{\alpha} x^{3/2}}$$





$$\left(-\frac{d^2}{dx^2} + V(x) \right) \psi = E \psi$$

$\psi(x) \rightarrow 0 \text{ as } x \rightarrow +\infty$

$\psi(x) \rightarrow 0 \text{ as } x \rightarrow -\infty$

Energy

$$\sqrt{V(x)} - E = Q(x)$$

$$Q(x) = Q(z) + V'(z)(x-z) + \frac{V''(z)}{2!}(x-z)^2 + \dots$$

$$\sim \frac{C}{\alpha^{1/4} x^{1/4}} e^{-\frac{1}{\epsilon} \left[\frac{2}{3} \sqrt{\alpha} x^{3/2} + \frac{b}{5\sqrt{\alpha}} x^{5/2} + \dots \right]}$$

$$\text{IN OVERLAP}$$

$$y \sim D \frac{1}{2\sqrt{\pi}} C \frac{-\frac{2}{3} x^{1/2} \sqrt{\alpha}}{\frac{x^{1/4} \alpha^{1/2}}{\epsilon^{1/6}}}$$

$$\frac{x^{5/2}}{\epsilon} \ll 1 \rightarrow x^{5/2} \ll \epsilon \rightarrow x \ll \epsilon^{1/5}$$

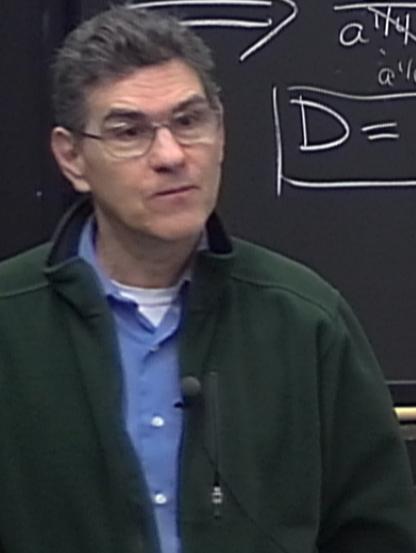
$$y \sim y_{\text{WKB}} \sim \frac{C}{\alpha^{1/4} x^{1/4}} e^{-\frac{1}{\epsilon} \frac{2}{3} \sqrt{\alpha} x^{3/2}}$$

OVERLAP

$$\epsilon^{1/5} \ll x \ll \epsilon^{1/5} \quad \text{as } \epsilon \rightarrow 0$$

$$\Rightarrow \frac{C}{\alpha^{1/4} x^{1/2}} = \frac{D}{2\sqrt{\pi}} \frac{\epsilon^{1/6}}{\alpha^{1/12}}$$

$$D = 2\sqrt{\pi} (\alpha \epsilon)^{-1/6} C$$



$$\sim \frac{C}{\alpha^{1/4} x^{1/4}} e^{-\frac{1}{\epsilon} \left[\frac{2}{3} \sqrt{\alpha} x^{3/2} + \frac{b}{5\sqrt{\alpha}} x^{5/2} + \dots \right]}$$

$\epsilon^{1/3} \rightarrow 0$

IN OVERLAP

$$y \sim D \frac{1}{2\sqrt{\pi}} C \frac{-\frac{2}{3} x^{1/2} \sqrt{\alpha}}{x^{1/4} \alpha^{1/12} \epsilon^{1/6}}$$

$$\frac{x^{5/2}}{\epsilon} \ll 1 \rightarrow x^{5/2} \ll \epsilon \rightarrow x \ll \epsilon^{1/5}$$

$$y \sim y_{\text{WKB}} \sim \frac{C}{\alpha^{1/4} x^{1/4}} e^{-\frac{1}{\epsilon} \frac{2}{3} \sqrt{\alpha} x^{3/2}}$$

OVERLAP

$$\epsilon^{1/3} \ll x \ll \epsilon^{1/5} \quad \text{as } \epsilon \rightarrow 0$$

$$\Rightarrow \frac{C}{\alpha^{1/4} x^{1/2}} \frac{D}{2\sqrt{\pi}} \frac{\epsilon^{1/6}}{\alpha^{1/12}}$$

$$D = \epsilon^{-1/6} C$$

ANSWER

Reg I $y(x) \sim C \frac{1}{[\alpha x]^{1/4}} e^{-\frac{1}{\epsilon} \int_a^x \sqrt{\alpha(t)} dt} \quad (\epsilon \rightarrow 0)$

Reg II $y(x) \sim$

Reg III

$$\sim \frac{C}{\alpha^{1/4} x^{1/4}} e^{-\frac{1}{\epsilon} \left[\frac{2}{3} \sqrt{\alpha} x^{3/2} + \frac{b}{5\sqrt{\alpha}} x^{5/2} + \dots \right]}$$

$$\text{IN OVERLAP } y \sim D \frac{1}{2\sqrt{\pi}} C \frac{-\frac{2}{3} x^{1/2} \sqrt{\alpha}}{x^{1/4} \alpha^{1/12} \epsilon^{1/6}}$$

$$\frac{x^{5/2}}{\epsilon} \ll 1 \rightarrow x^{5/2} \ll \epsilon \rightarrow x \ll \epsilon^{1/5}$$

$$y \sim y_{\text{WKB}} \sim \frac{C}{\alpha^{1/4} x^{1/4}} e^{-\frac{1}{\epsilon} \frac{2}{3} \sqrt{\alpha} x^{3/2}}$$

OVERLAP

$$\epsilon^{2/3} x \ll \epsilon^{2/5} \quad \text{as } \epsilon \rightarrow 0$$

$$\Rightarrow \frac{C}{\alpha} = \frac{D}{2\sqrt{\pi}} \frac{\epsilon^{1/6}}{\alpha^{1/12}} \quad \boxed{(a\epsilon)^{-1/6} C}$$

ANSWER

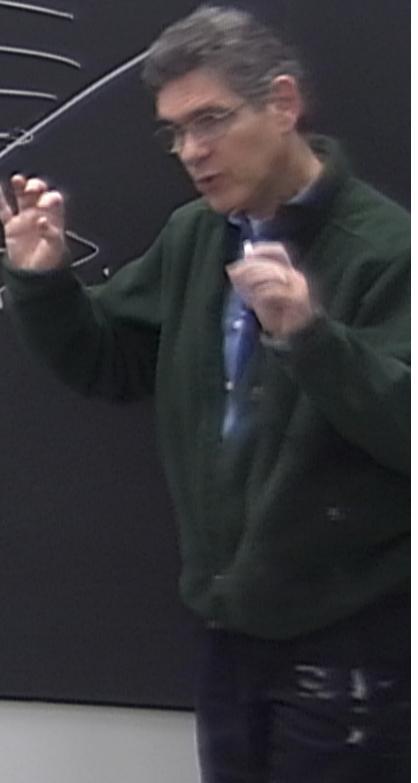
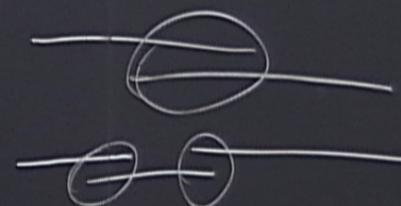
$$\text{Reg I} \quad y(x) \sim C \int_0^x e^{-\frac{1}{\epsilon} \int_0^t \sqrt{\alpha(t')} dt'} \quad (x > 0)$$

$$\text{Reg II} \quad y(x) \sim \frac{2\sqrt{\pi}}{(a\epsilon)^{1/6}} C \text{Ai}\left(\frac{x\alpha^{1/3}}{\epsilon^{2/3}}\right) \quad (x \ll 1)$$

Reg III

$$Ai(s) \sim \frac{1}{\sqrt[4]{(-s)^{1/4}}} s^{1/4} \left[\sum_{n=0}^{\infty} \frac{\frac{2^n}{3} (-s)^{n/4}}{n!} \right] \quad \text{as } s \rightarrow -\infty$$

$$Ai(s) \sim \frac{1}{2\sqrt{\pi}} e^{-\frac{\sqrt{3}}{2}s^{3/2}} \quad \left(\text{as } s \rightarrow +\infty \right)$$



$$\sim \frac{C}{\alpha^{1/4} x^{1/4}} e^{-\frac{1}{\epsilon} \left[\frac{2}{3} \sqrt{\alpha} x^{3/2} + \frac{b}{5\sqrt{\alpha}} x^{5/2} + \dots \right]}$$

$\epsilon^{1/3} \rightarrow 0$

IN OVERLAP

$$y \sim D \frac{1}{2\sqrt{\pi}} C \frac{-\frac{2}{3} x^{1/2} \sqrt{\alpha}}{x^{1/4} \alpha^{1/12} \epsilon^{1/6}}$$

$$\frac{x^{5/2}}{\epsilon} \ll 1 \rightarrow x^{5/2} \ll \epsilon \rightarrow x \ll \epsilon^{1/5}$$

OVERLAP

$$y \sim y_{\text{OVERLAP}} \sim \frac{C}{\alpha^{1/4} x^{1/4}} e^{-\frac{1}{\epsilon} \frac{2}{3} \sqrt{\alpha} x^{3/2}}$$

$$\Rightarrow \frac{C}{\alpha^{1/4} x^{1/2}} = \frac{D}{2\sqrt{\pi}} \frac{\epsilon^{1/6}}{\alpha^{1/12}}$$

$$2\sqrt{\pi} (\alpha \epsilon)^{-1/6} C$$

ANSWER

Reg I	$y(x) \sim C \frac{e^{-\frac{1}{\epsilon} \int_x^0 \sqrt{Q(t)} dt}}{(\alpha x)^{1/4}}$	$(x > 0)$
Reg II	$y(x) \sim \frac{2\sqrt{\pi}}{(\alpha \epsilon)^{1/6}} C \text{Ai}\left(\frac{x \alpha^{1/3}}{\epsilon^{1/3}}\right)$	$(x \ll 1)$
Reg III	$y(x) \sim \frac{2\sqrt{\pi}}{(-Q)^{1/4}} \text{Si} \left[\frac{1}{\epsilon} \int_x^0 \sqrt{ F_Q(t) } dt \right]$	$(x < 0)$

$$\sim \frac{C}{\alpha^{1/4} x^{1/4}} e^{-\frac{1}{\epsilon} \left[\frac{2}{3} \sqrt{\alpha} x^{3/2} + \frac{b}{5\sqrt{\alpha}} x^{5/2} + \dots \right]}$$

$\epsilon^{1/3} \rightarrow 0$ (Reg I)

IN OVERLAP

$$y \sim D \frac{1}{2\sqrt{\pi}} C \frac{-\frac{2}{3} x^{1/2} \sqrt{\alpha}}{\frac{x^{1/4} \alpha^{1/2}}{\epsilon^{1/6}}}$$

$$\frac{x^{5/2}}{\epsilon} \ll 1 \rightarrow x^{5/2} \ll \epsilon \rightarrow x \ll \epsilon^{1/5}$$

$$y \sim y_{\text{near}} \sim \frac{C}{\alpha^{1/4} x^{1/4}} e^{-\frac{1}{\epsilon} \frac{2}{3} \sqrt{\alpha} x^{3/2}}$$

OVERLAP

$$\epsilon^{1/3} \ll x \ll \epsilon^{1/5} \quad \text{as } \epsilon \rightarrow 0$$

$$\Rightarrow \frac{C}{\alpha^{1/4} x^{1/2}} \frac{D}{\alpha^{1/6}} = \frac{\epsilon^{1/6}}{x^{1/12}}$$

$$D = ?$$

ANSWER

Reg I	$y(x) \sim C \frac{-\frac{1}{\epsilon} \int_0^x \sqrt{\Phi(t)} dt}{[\Phi(x)]^{1/4}} \quad (x > 0)$
Reg II	$y(x) \sim \frac{2\sqrt{\pi}}{(\alpha\epsilon)^{1/6}} C \text{Ai}\left(\frac{x\alpha^{1/3}}{\epsilon^{1/3}}\right) \quad (x \ll 1)$
Reg III	$y(x) \sim \frac{C}{(-\Phi)^{1/4}} \text{Si} \left[\frac{1}{\epsilon} \int_x^0 \sqrt{4 \Phi(t) + \frac{\pi}{4}} dt \right] \quad (x < 0)$

$$\sim \frac{C}{\alpha^{1/4} x^{1/4}} e^{-\frac{1}{\epsilon} \left[\frac{2}{3} \sqrt{\alpha} x^{3/2} + \frac{b}{5\sqrt{\alpha}} x^{5/2} + \dots \right]}$$

$\epsilon^{1/3} \rightarrow (\text{Reg I}) \rightarrow \epsilon \rightarrow 0$

IN OVERLAP $y \sim D \frac{1}{2\sqrt{\pi}} C \frac{-\frac{2}{3} x^{1/2} \sqrt{\alpha}}{x^{1/4} \alpha^{1/2} \epsilon^{1/6}}$

$$\frac{x^{5/2}}{\epsilon} \ll 1 \rightarrow x^{5/2} \ll \epsilon \rightarrow x \ll \epsilon^{1/5}$$

$$y \sim y_{\text{WKB}} \sim \frac{C}{\alpha^{1/4} x^{1/4}} e^{-\frac{1}{\epsilon} \frac{2}{3} \sqrt{\alpha} x^{3/2}}$$

OVERLAP

$$\epsilon^{1/3} \ll x \ll \epsilon^{1/5}$$

as $\epsilon \rightarrow 0$

$$\Rightarrow \frac{C}{\alpha^{1/4} \epsilon^{1/2}} = \frac{D}{2\sqrt{\pi}} \frac{\epsilon^{1/6}}{\alpha^{1/12}}$$

$$\boxed{D = 2\sqrt{\pi} (\alpha \epsilon)^{-1/6} C}$$

ANSWER

Reg I	$y(x) \sim C \frac{-\frac{1}{\epsilon} \int_0^x \sqrt{\Phi(t)} dt}{[\Phi(x)]^{1/4}} \quad (x > 0)$
Reg II	$y(x) \sim \frac{2\sqrt{\pi}}{(\alpha \epsilon)^{1/6}} C \text{Ai}\left(\frac{x \alpha^{1/3}}{\epsilon^{1/3}}\right) \quad (x \ll 1)$
Reg III	$y(x) \sim \frac{1}{(-\Phi)^{1/4}} \text{Si} \left[\frac{1}{\epsilon} \int_x^0 \sqrt{ \Phi(t) } dt \right] \quad (x < 0)$

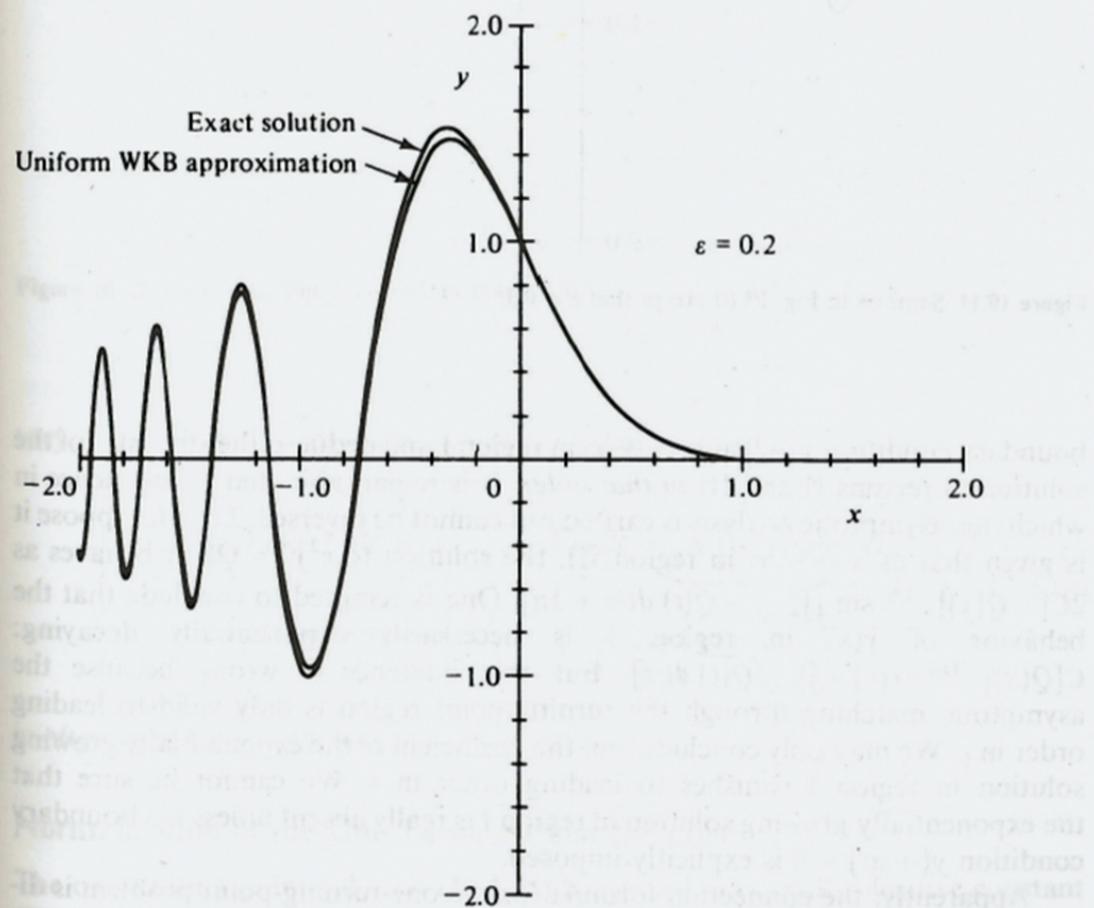


Figure 10.10 A comparison of the exact solution to $\varepsilon^2 y''(x) = \sinh x(\cosh x)^2 y(x)$ [$y(0) = 1$, $y(+\infty) = 0$], with the approximate solution from a one-turning-point WKB analysis. The WKB approximate formulas are given in (10.4.14) and (10.4.15).

$$\sim \frac{C}{\alpha^{1/4} x^{1/4}} e^{-\frac{1}{\epsilon} \left[\frac{2}{3} \sqrt{\alpha} x^{3/2} + \frac{b}{5 \sqrt{\alpha}} x^{5/2} + \dots \right]}$$

$$y \sim D \frac{e^{-\epsilon}}{x^{1/4} \alpha^{1/12}}$$

$$\frac{x^{5/2}}{\epsilon} \ll 1 \rightarrow x^{5/2} \ll \epsilon \rightarrow x \ll \epsilon^{2/5}$$

$$y \sim y_{WKB} \sim \frac{C}{\alpha^{1/4} x^{1/4}} e^{-\frac{1}{\epsilon} \frac{2}{3} \sqrt{\alpha} x^{3/2}}$$

OVERLAP

$$\epsilon^{2/3} x \ll \epsilon^{2/5}$$

as $\epsilon \rightarrow 0$

$$\Rightarrow \frac{C}{\alpha^{1/4} x^{1/2}} = \frac{D}{2\sqrt{\pi}} \frac{\epsilon^{1/6}}{\alpha^{1/12}}$$

$$D = 2\sqrt{\pi} (\alpha \epsilon)^{-1/6} C$$

ANSWER

$$\begin{aligned} \text{Reg I } y(x) &\sim \frac{C}{x^{1/4}} e^{-\frac{2}{3} \sqrt{\alpha} x^{3/2}} & (x > 0) \\ \text{Reg II } y(x) &\sim \frac{C}{x^{1/4}} \left(\frac{x^{1/3}}{\epsilon^{2/3}} \right) & (\epsilon \rightarrow 0) \\ \text{Reg III } y(x) &\sim \frac{C}{x^{1/4}} \left[-\int_0^x \left(\sqrt{Q(t)} + \frac{\pi}{4} \right) dt \right] & (x < 0) \end{aligned}$$

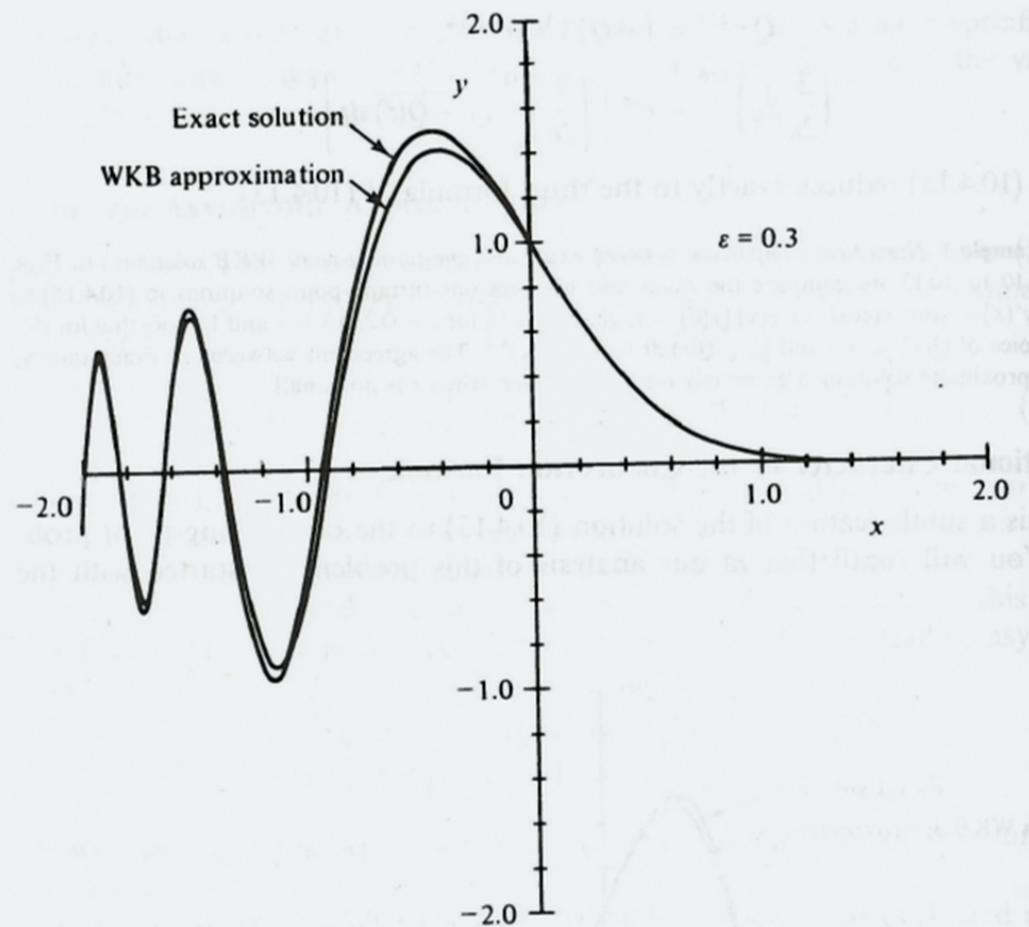


Figure 10.11 Same as in Fig. 10.10 except that $\varepsilon = 0.3$.

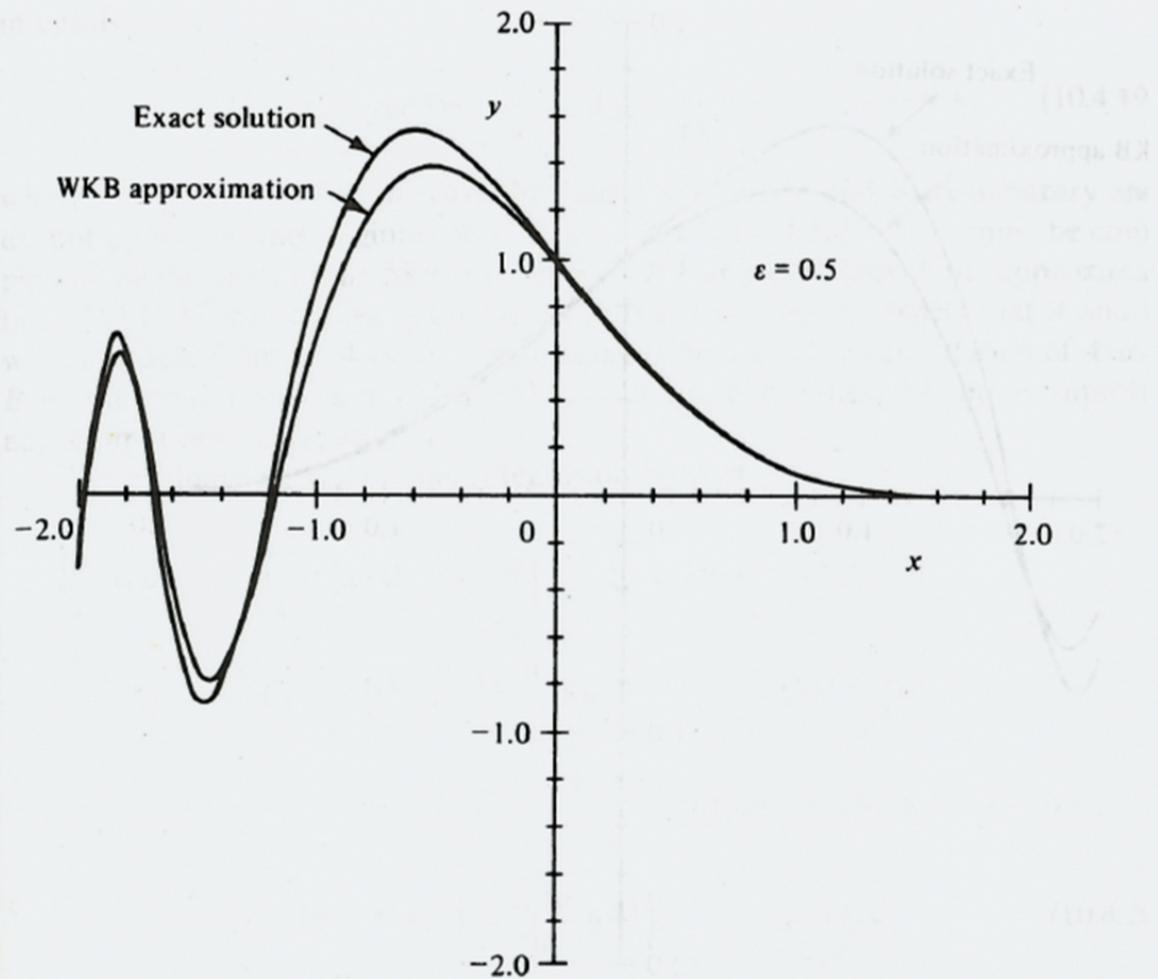


Figure 10.12 Same as in Fig. 10.10 except that $\varepsilon = 0.5$.

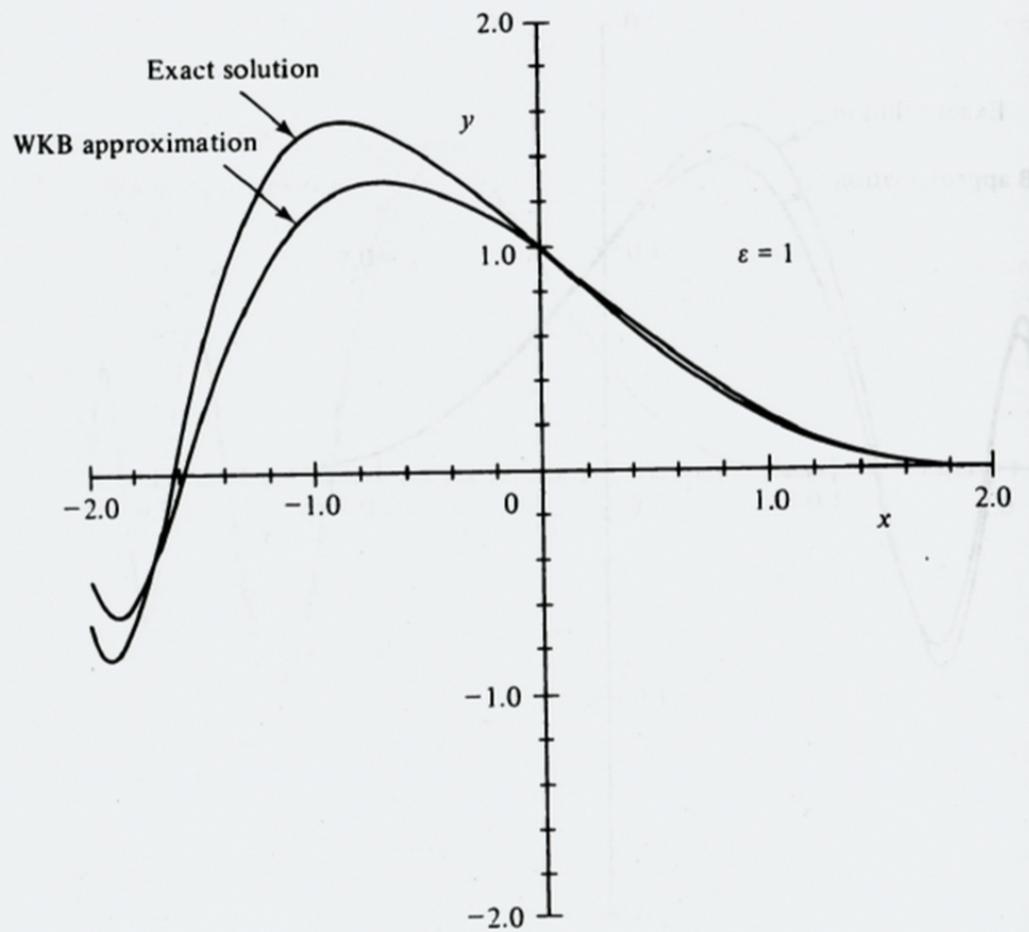


Figure 10.13 Same as in Fig. 10.10 except that $\varepsilon = 1.0$. Even for this large value of ε the agreement between the approximate and exact solutions is impressive.

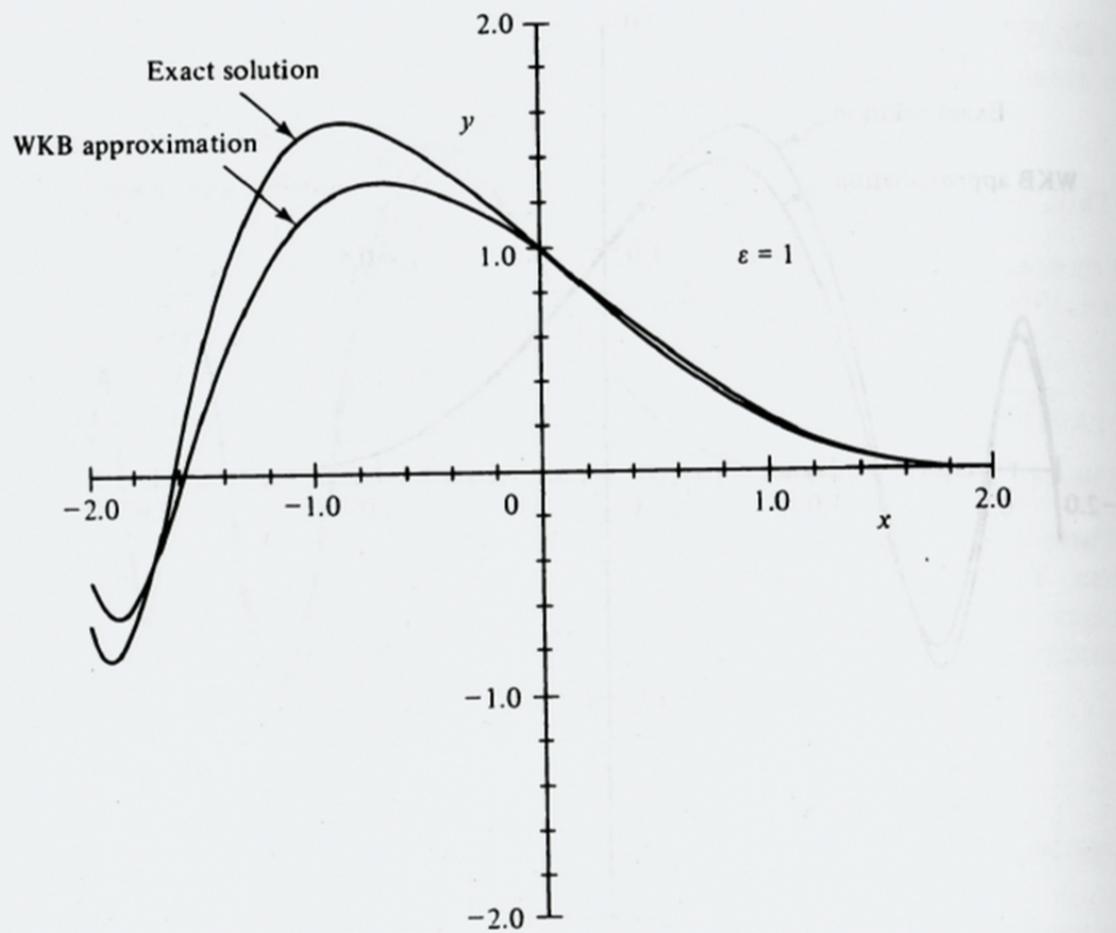


Figure 10.13 Same as in Fig. 10.10 except that $\varepsilon = 1.0$. Even for this large value of ε the agreement between the approximate and exact solutions is impressive.

$$\sim \frac{C}{\alpha^{1/4} x^{1/4}} e^{-\frac{1}{\epsilon} \left[\frac{2}{3} \sqrt{\alpha} x^{3/2} + \frac{b}{5 \sqrt{\alpha}} x^{5/2} + \dots \right]}$$

$$y \sim \frac{D \sqrt{\pi}}{x^{1/4} \alpha^{1/12}} e^{-\frac{x}{\epsilon}}$$

$$\frac{x^{5/2}}{\epsilon} \ll 1 \rightarrow x^{5/2} \ll \epsilon \rightarrow x \ll \epsilon^{2/5}$$

$$y \sim y_{\text{WKB}} \sim \frac{C}{\alpha^{1/4} x^{1/4}} e^{-\frac{1}{\epsilon} \frac{2}{3} \sqrt{\alpha} x^{3/2}}$$

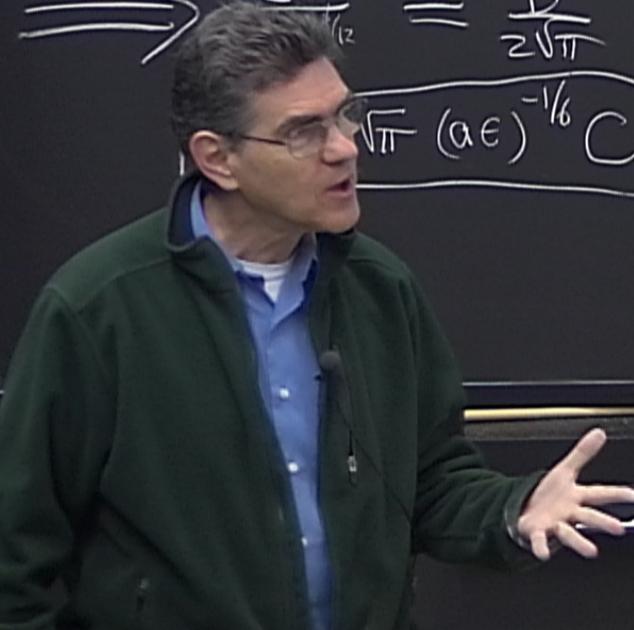
OVERLAP

$$\epsilon^{2/5} x \ll \epsilon^{2/5} \quad \text{as } \epsilon \rightarrow 0$$

$$\Rightarrow \frac{C}{\epsilon^{1/2}} = \frac{D}{2\sqrt{\pi}} \frac{\epsilon^{1/6}}{\alpha^{1/12}} \frac{1}{\sqrt{\pi} (\alpha \epsilon)^{-1/6} C}$$

ANSWER

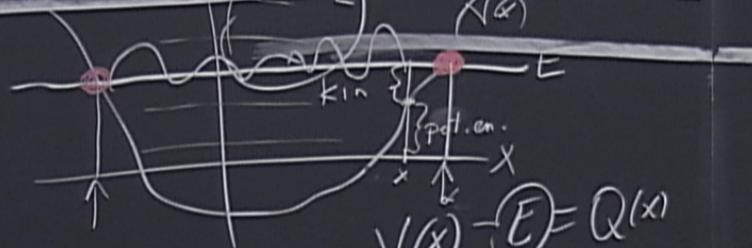
$$\begin{cases} \text{Reg I} & y(x) \sim C \frac{x}{(-Q(x))^{1/4}} e^{-\frac{1}{\epsilon} \int_x^0 \sqrt{Q(t)} dt} \quad (x > 0) \quad (\downarrow \epsilon \rightarrow 0) \\ \text{Reg II} & y(x) \sim \frac{2\sqrt{\pi}}{(\alpha \epsilon)^{1/6}} C \operatorname{Ai}\left(\frac{x \alpha^{1/3}}{\epsilon^{2/3}}\right) \quad (x \ll 1) \\ \text{Reg III} & y(x) \sim C \frac{\sin\left[\frac{1}{\epsilon} \int_x^0 \sqrt{Q(t)} dt + \frac{\pi}{4}\right]}{(-Q(x))^{1/4}} \quad (x < 0) \end{cases}$$



$$\left(-\frac{d^2}{dx^2} + V(x) \right) \psi = E \psi$$

$$\begin{aligned}\psi(x) &\rightarrow 0 \text{ as } x \rightarrow +\infty \\ \psi(x) &\rightarrow 0 \text{ as } x \rightarrow -\infty\end{aligned}$$

Energy



ϵ^2

$$\epsilon^2 \psi'' = Q(x) \psi$$

$\epsilon \ll 1$

$$\begin{array}{c} x < 0 \\ \text{III} \\ \hline \text{II} \end{array}$$

$$\begin{aligned}V(x) - E &= Q(x) \\ Q(x) &= Q(\lambda) + V'(\lambda)(x-\lambda) \\ &\quad + \frac{V''(\lambda)}{2!}(x-\lambda)^2\end{aligned}$$

