Title: Mathematical Physics - Lecture 14

Date: Dec 08, 2011 09:00 AM

URL: http://pirsa.org/11120008

Abstract:

Table 10.1 A comparison of the exact eigenvalues E_n of the Sturm-Liouville problem $y''(x) + E(x+\pi)^4 y(x) = 0$ $[y(0) = y(\pi) = 0]$ with the leading-order WKB prediction [see (10.1.34)] for these eigenvalues $E_n \sim 9n^2/49\pi^2$ $(n \to \infty)$

As expected, this prediction becomes more accurate as n increases. The relative error is defined as (approximate − exact)/(exact)

n	E_n(WKB)	E_n(exact)	Relative error, %
1	0.00188559	0.00174401	8.1
2	0.00754235	0.00734865	2.6
3	0.0169703	0.0167524	1.3
4	0.0301694	0.0299383	0.77
5	0.0471397	0.0469006	0.51
10	0.188559	0.188305	0.13
20	0.754235	0.753977	0.035
40	3.01694	3.01668	0.009

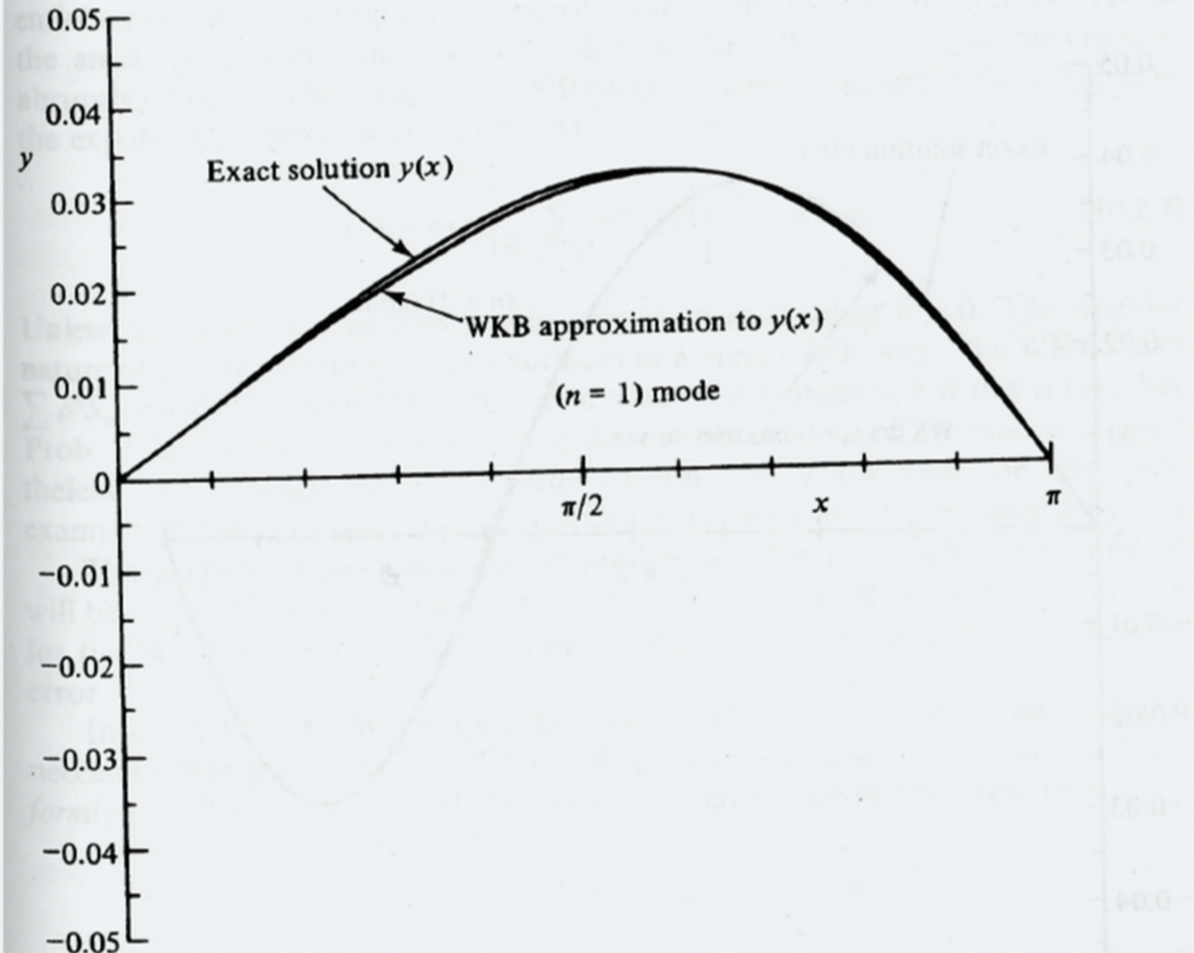


Figure 10.2 Comparison of the exact solution to $y''(x) + E_n(x + \pi)^4 y(x) = 0$ $[y(0) = y(\pi) = 0]$, with the WKB approximation to this solution as given in (10.1.35) for the lowest ($n = 1$) mode. Although WKB becomes exact as $n \to \infty$, this plot shows that even when $n = 1$ the WKB approximation is extraordinarily accurate.

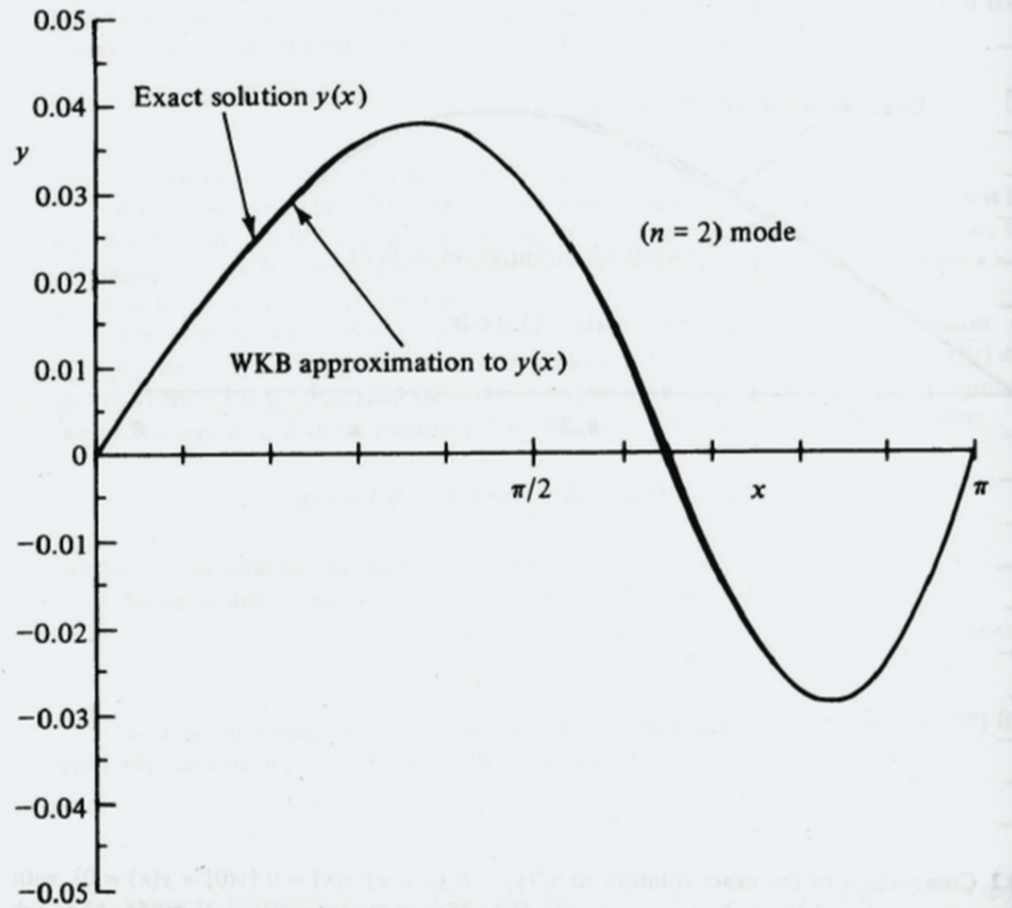
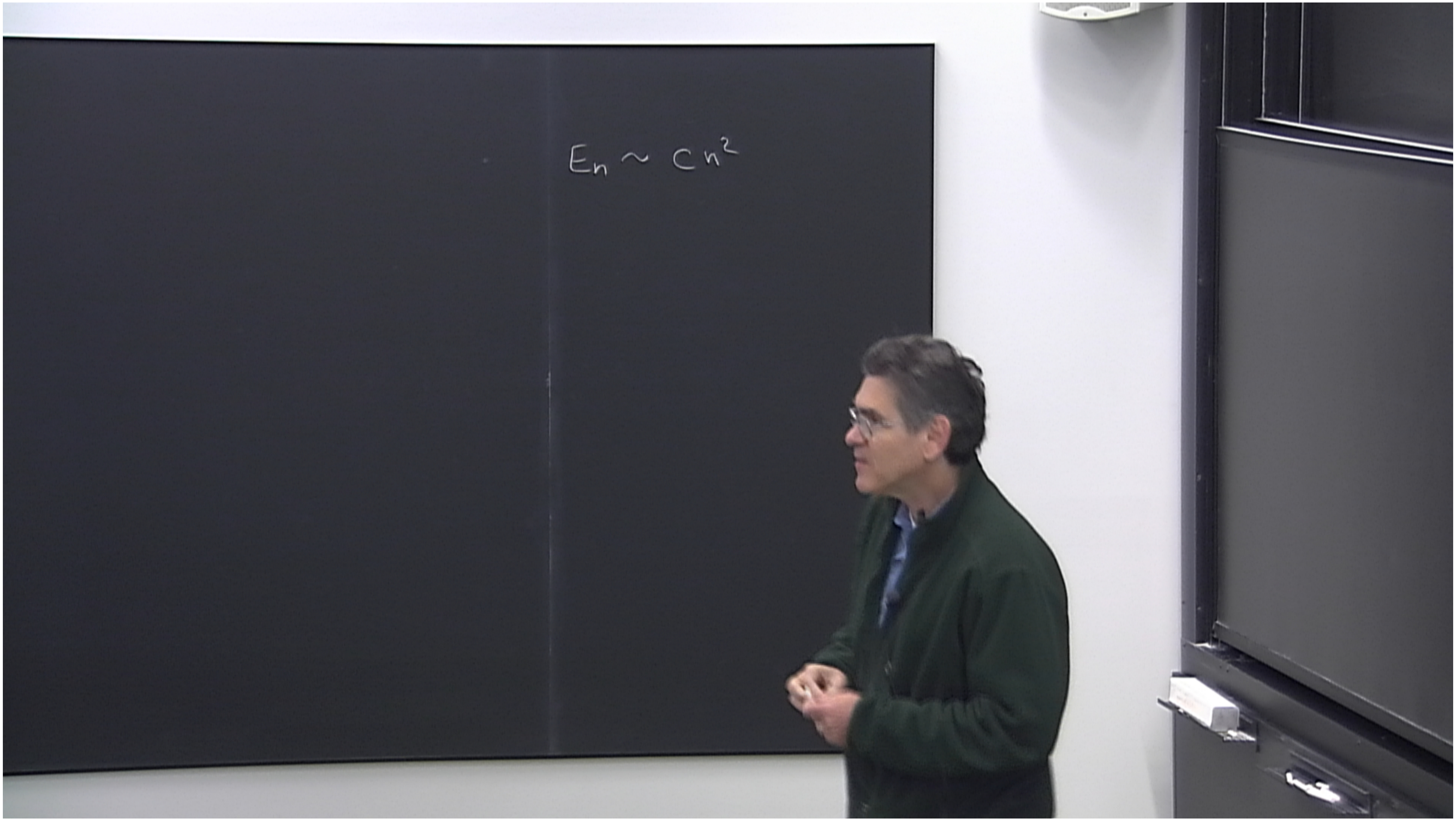
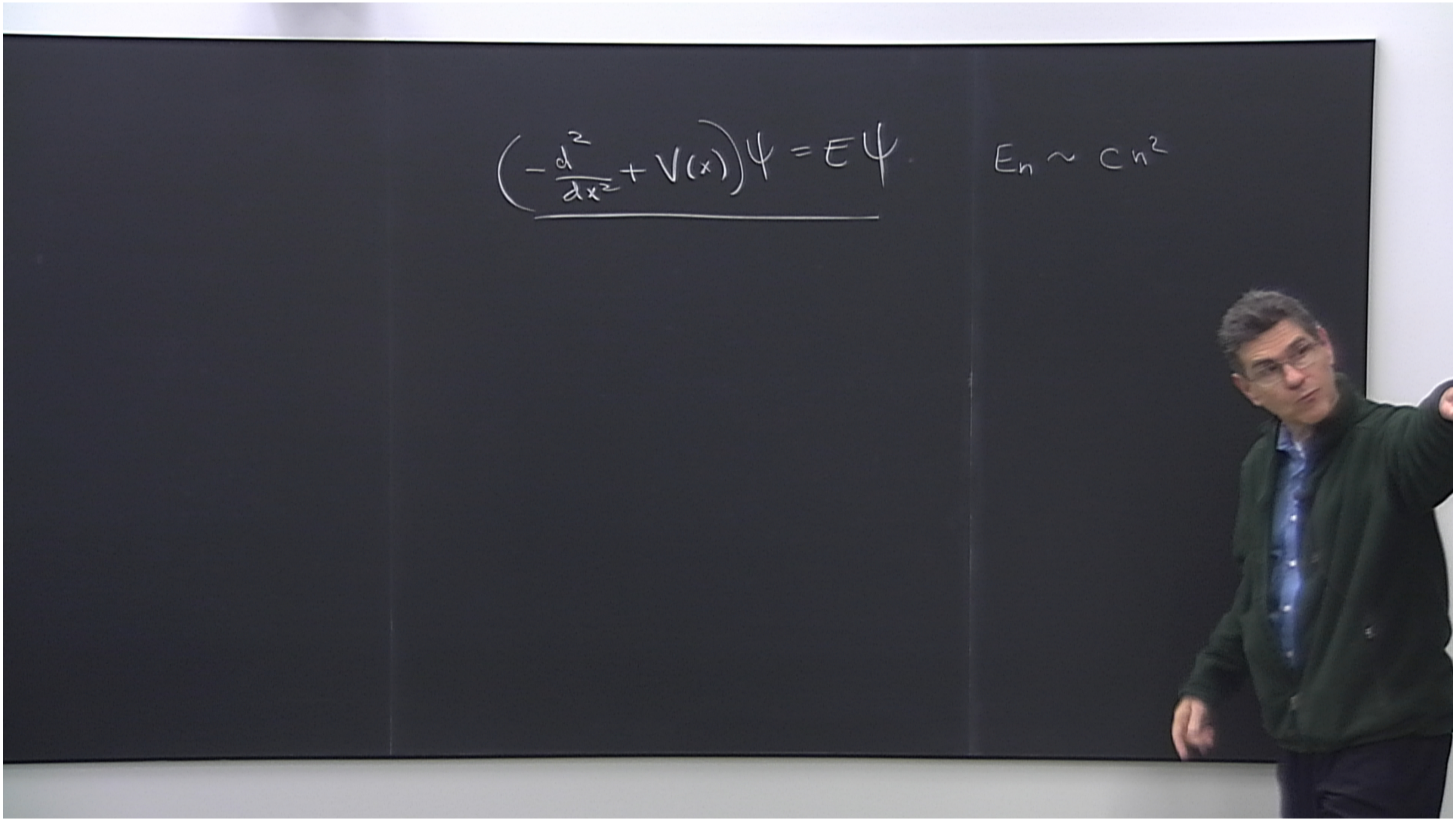
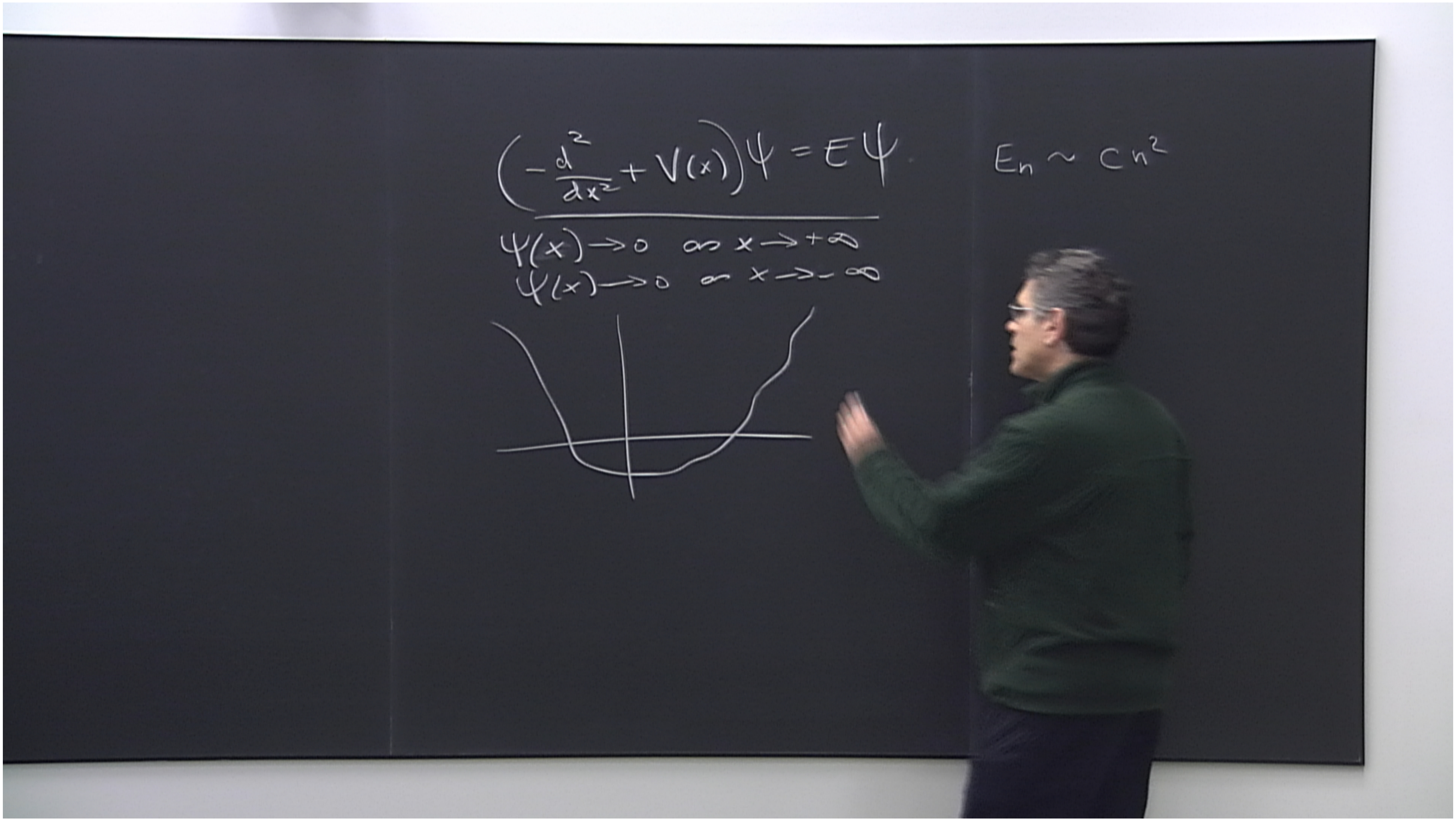
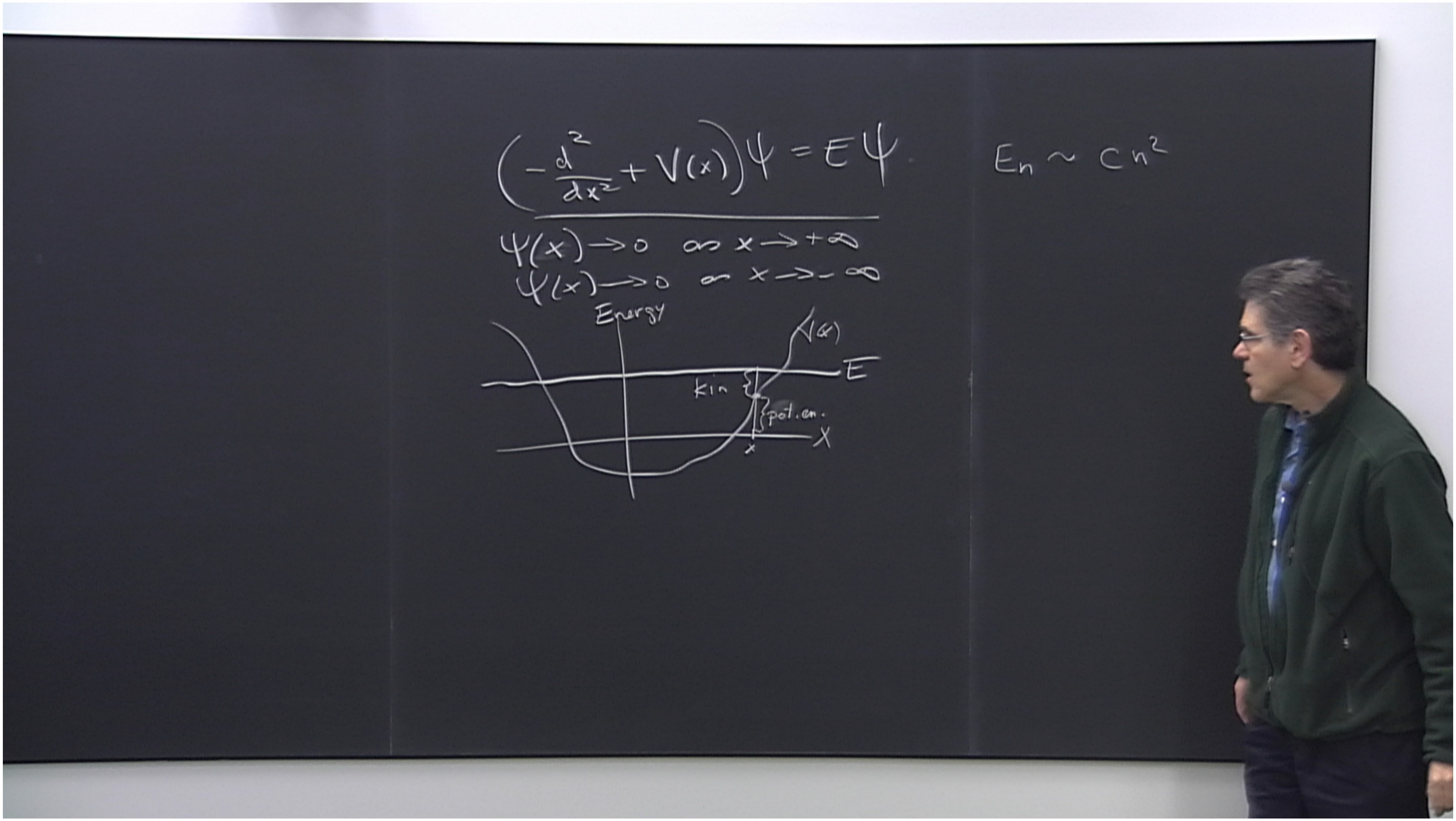


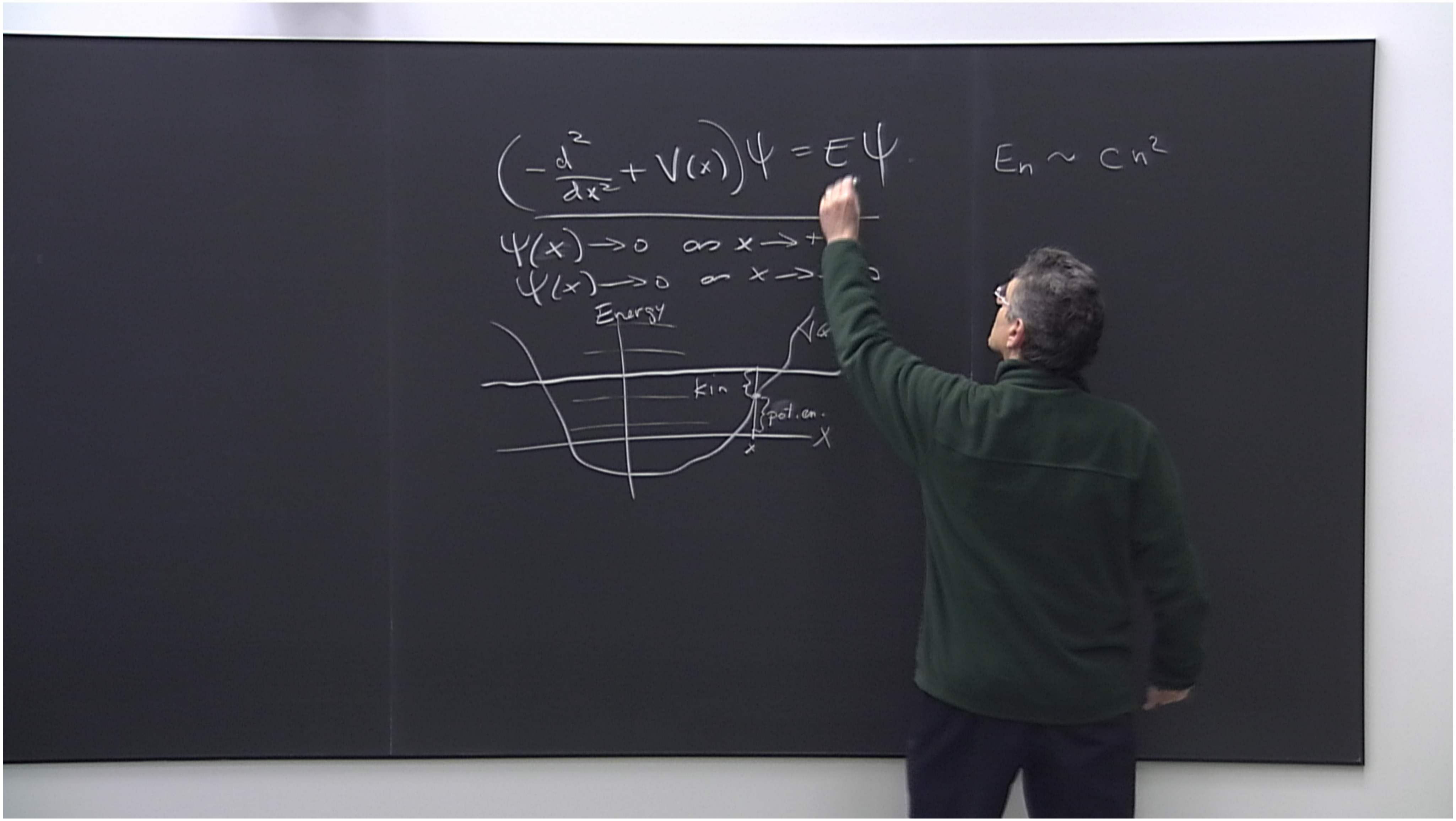
Figure 10.3 Same as in Fig. 10.2 except that $n = 2$. The exact eigenfunction and the WKB approximation are almost indistinguishable.

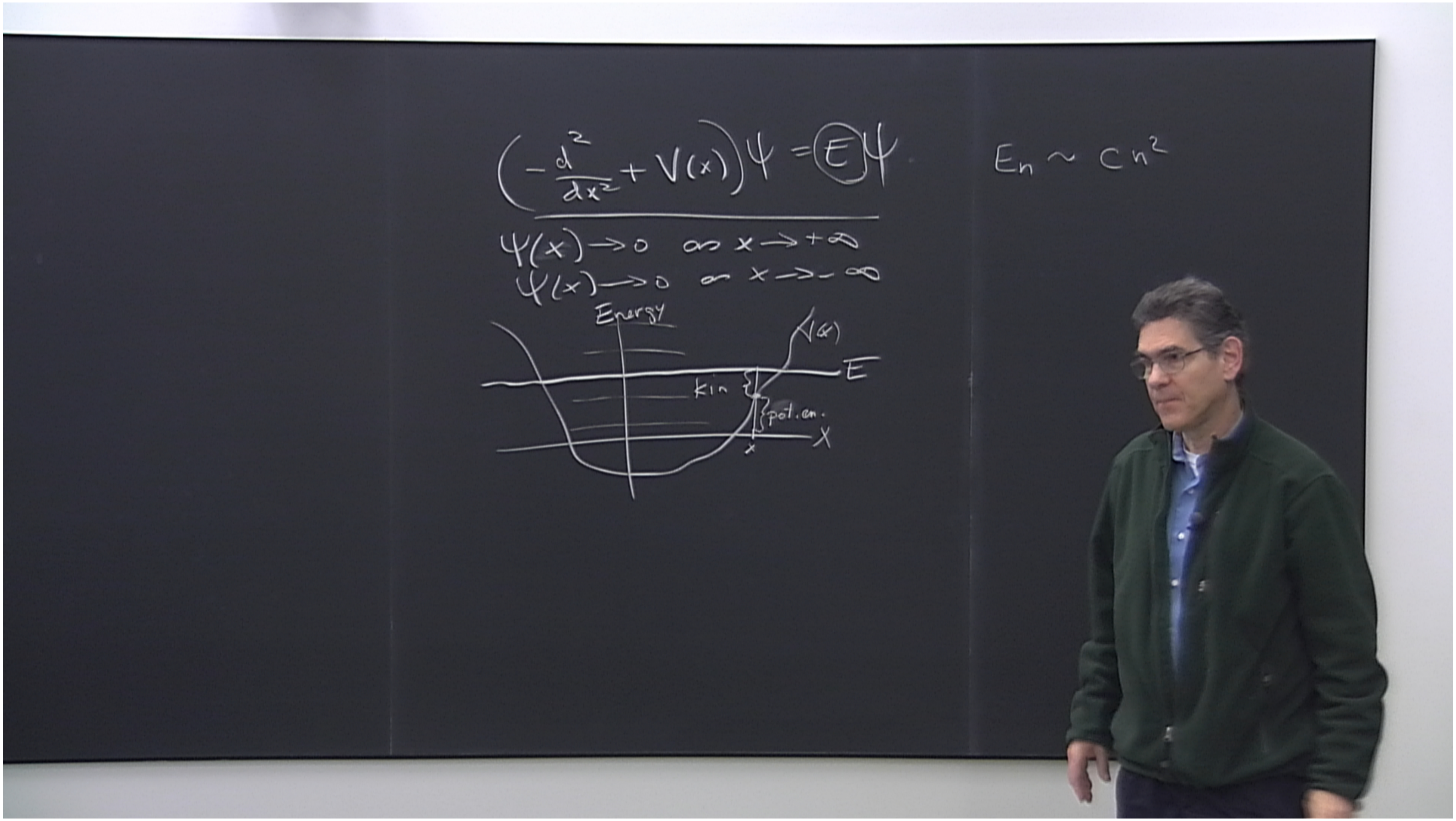

$E_n \sim c n^2$

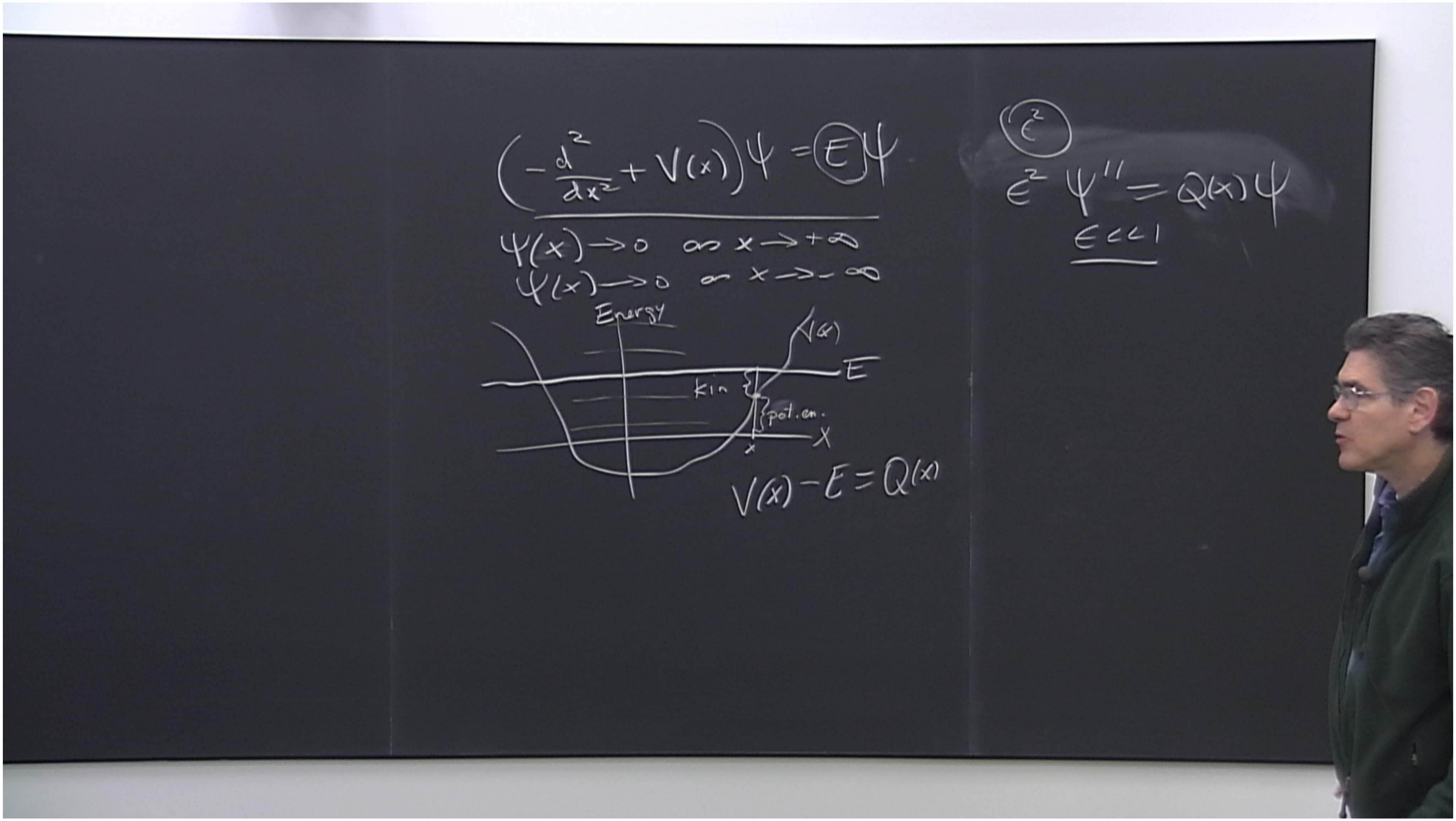

$\left(-\frac{d^2}{dx^2}+V(x)\right)\psi = E\psi$
$E_n \sim Cn^2$

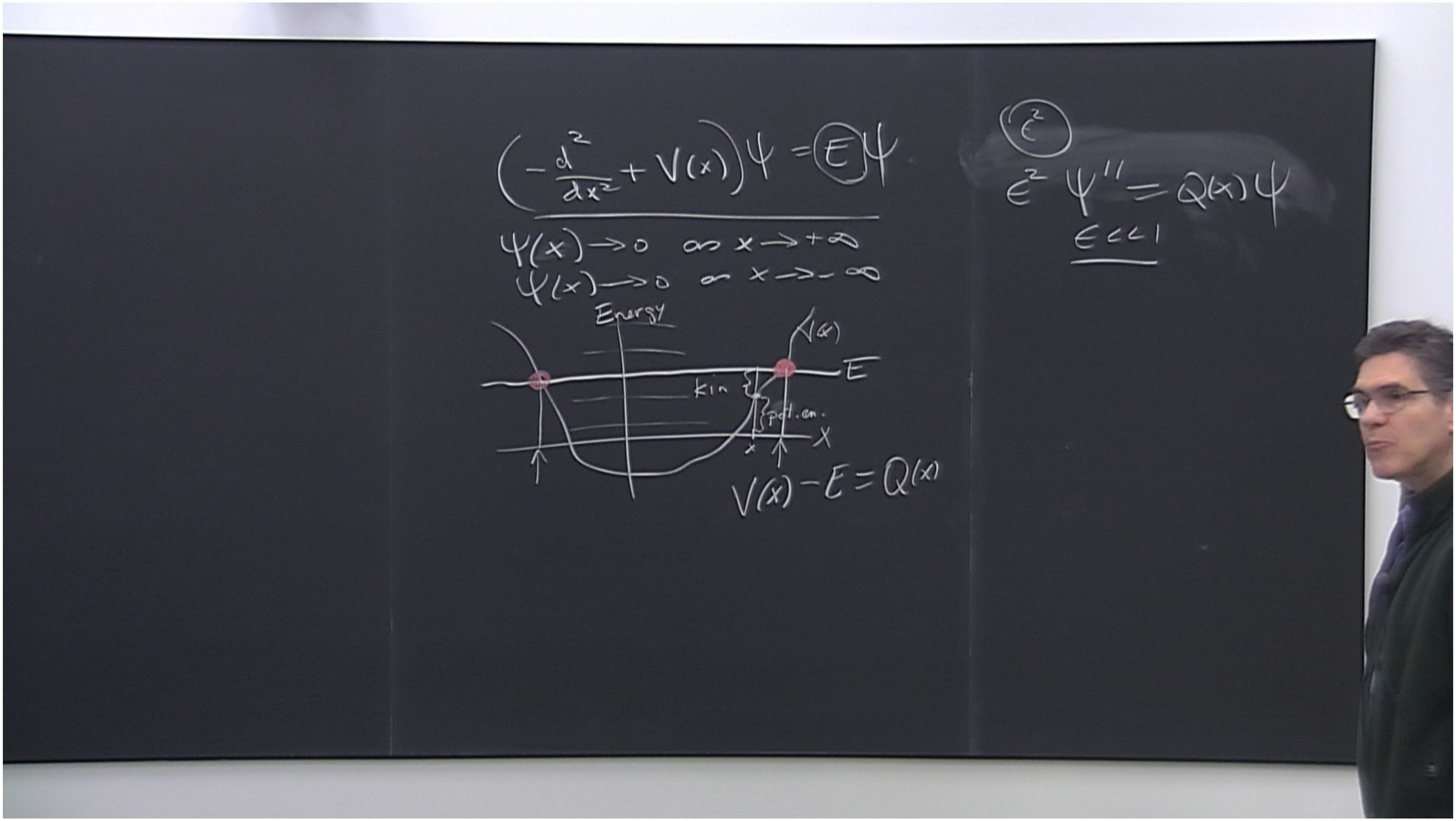

$\left(-\frac{d^2}{dx^2} + V(x)\right)\psi = E\psi$
$E_n \sim c n^2$
$\psi(x) \to 0$ as $x \to +\infty$
$\psi(x) \to 0$ as $x \to -\infty$

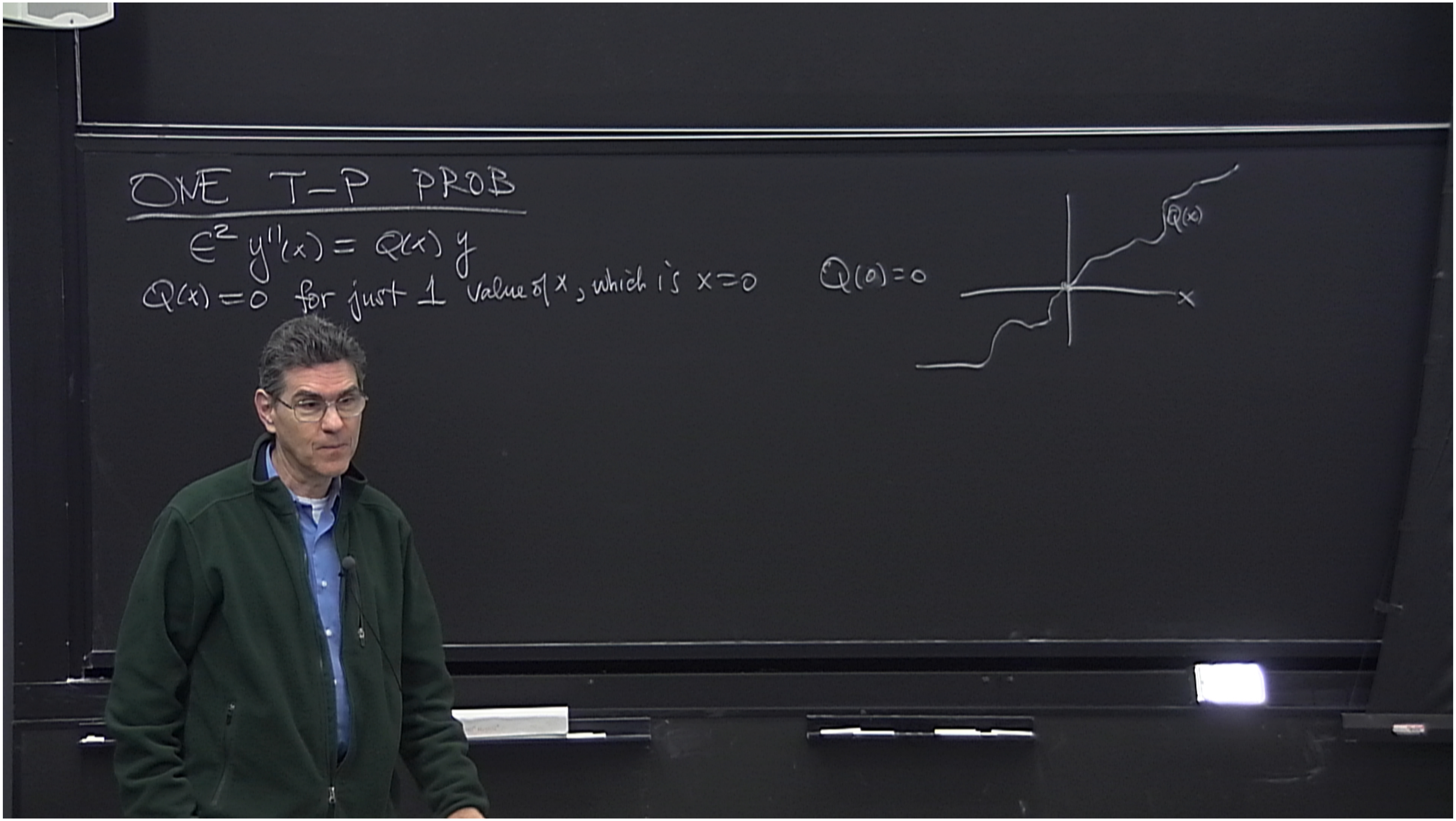

(−d²/dx² + V(x))ψ = Eψ
ψ(x) → 0 as x → +∞
ψ(x) → 0 as x → −∞
Energy
V(x)
E
kin
pot. en.
X
x
Eₙ ~ C n²

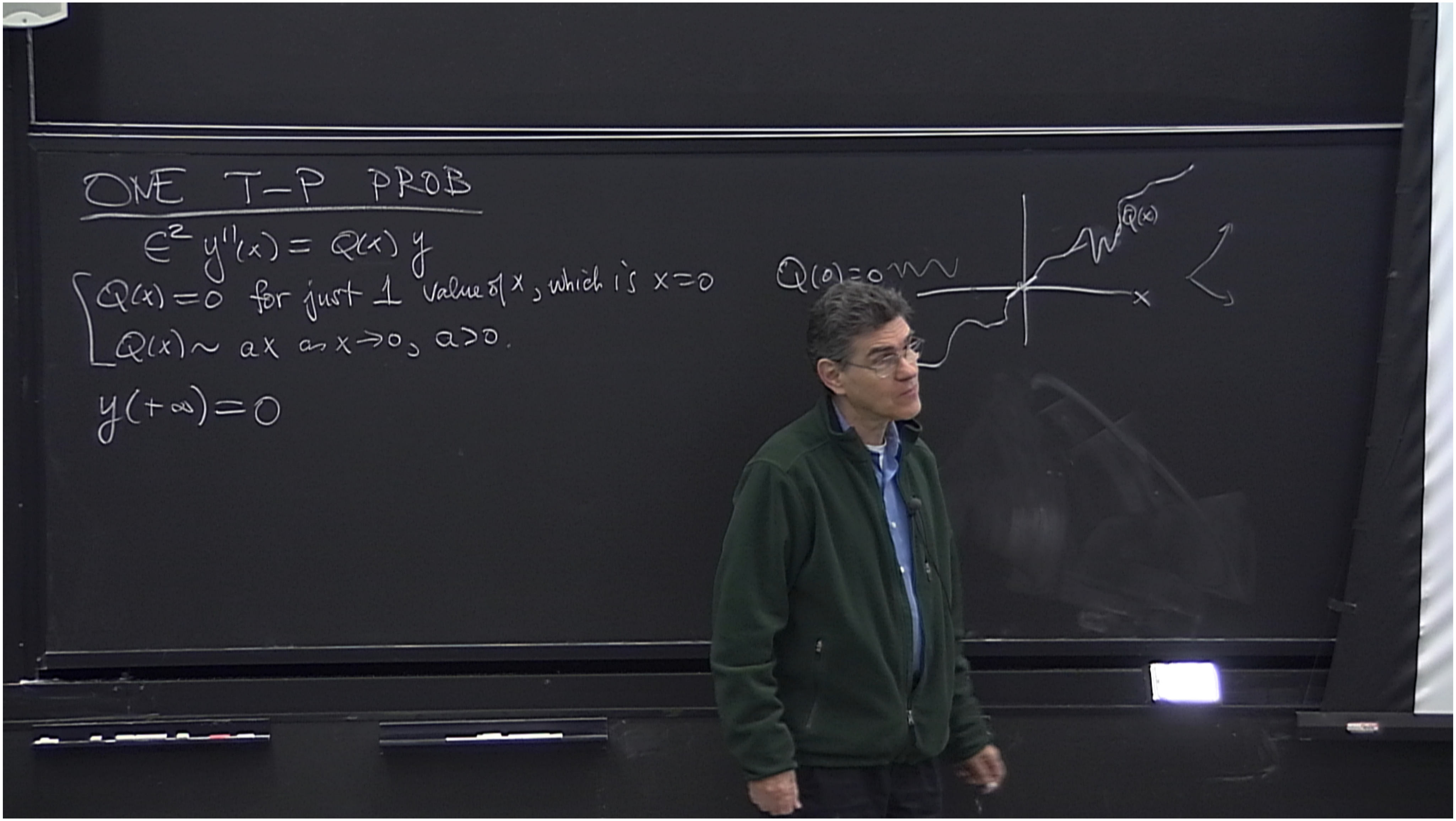

$\left(-\frac{d^2}{dx^2}+V(x)\right)\psi = E\psi$
$E_n \sim c n^2$
$\psi(x) \to 0$ as $x \to +$
$\psi(x) \to 0$ as $x \to$
Energy
kin
pot. en.
X

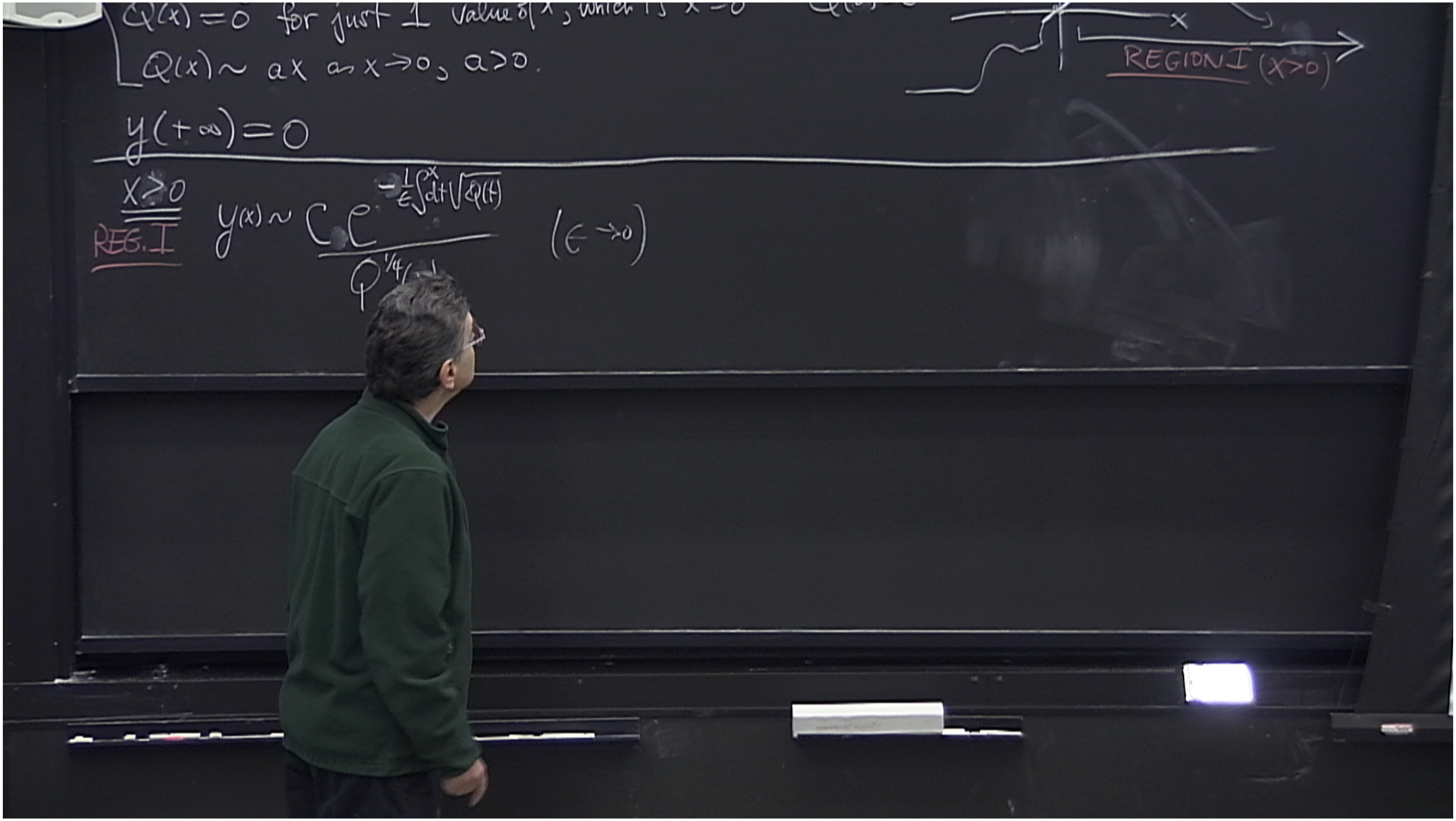

(− d²/dx² + V(x)) ψ = E ψ
En ~ c n²
ψ(x) → 0 as x → +∞
ψ(x) → 0 as x → −∞
Energy
V(x)
E
kin
pot. en.
X
x

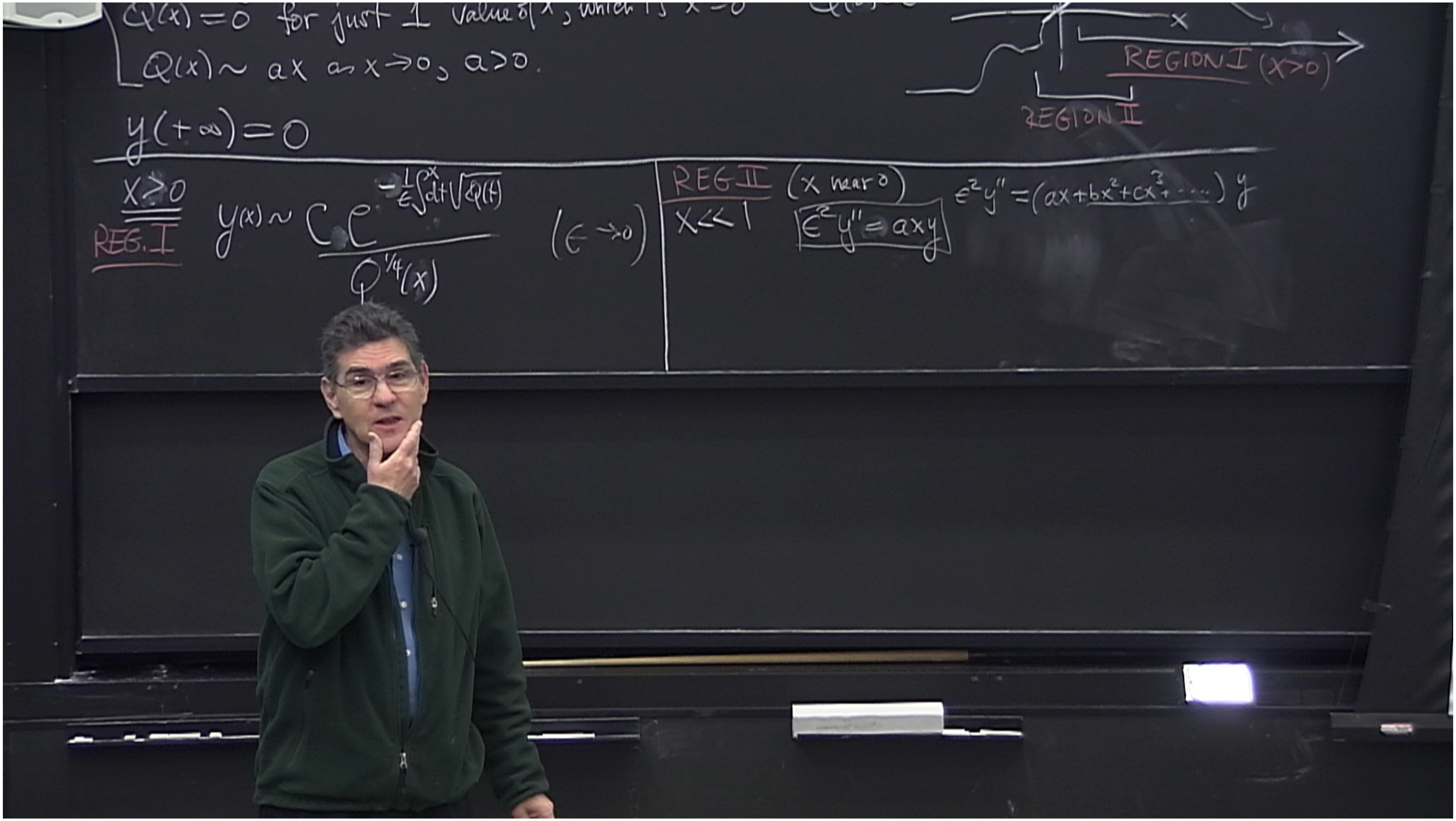

(−d²/dx² + V(x))ψ = Eψ
ψ(x) → 0 as x → +∞
ψ(x) → 0 as x → −∞
Energy
V(x)
E
kin
pot. en.
x
V(x) − E = Q(x)
ε²
ε² ψ'' = Q(x) ψ
ε << 1

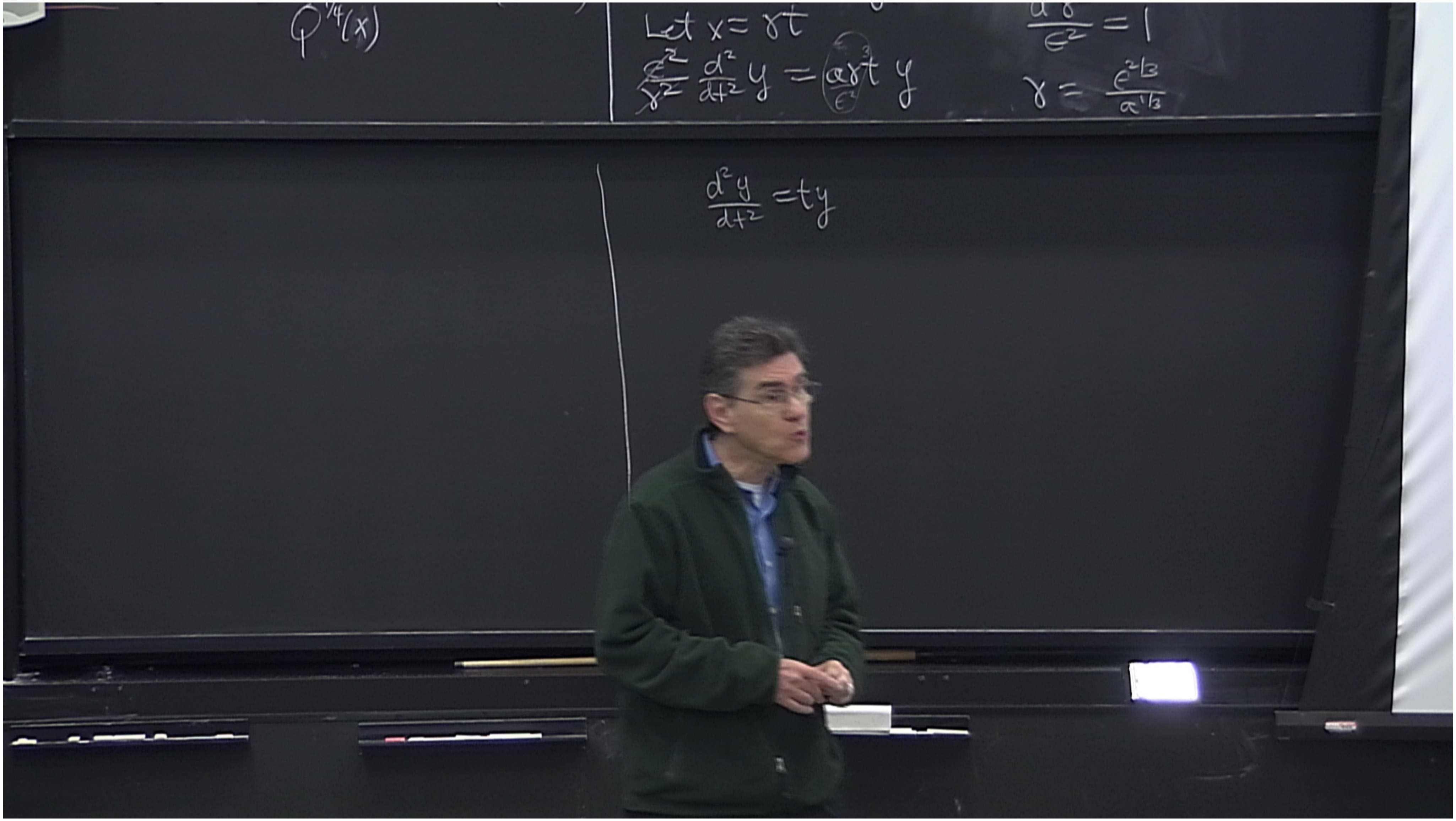

(- d^2/dx^2 + V(x)) ψ = E ψ
ψ(x) → 0 as x → +∞
ψ(x) → 0 as x → -∞
Energy
V(x)
E
kin
pot. en.
x
V(x) - E = Q(x)
ε^2
ε^2 ψ'' = Q(x) ψ
ε << 1

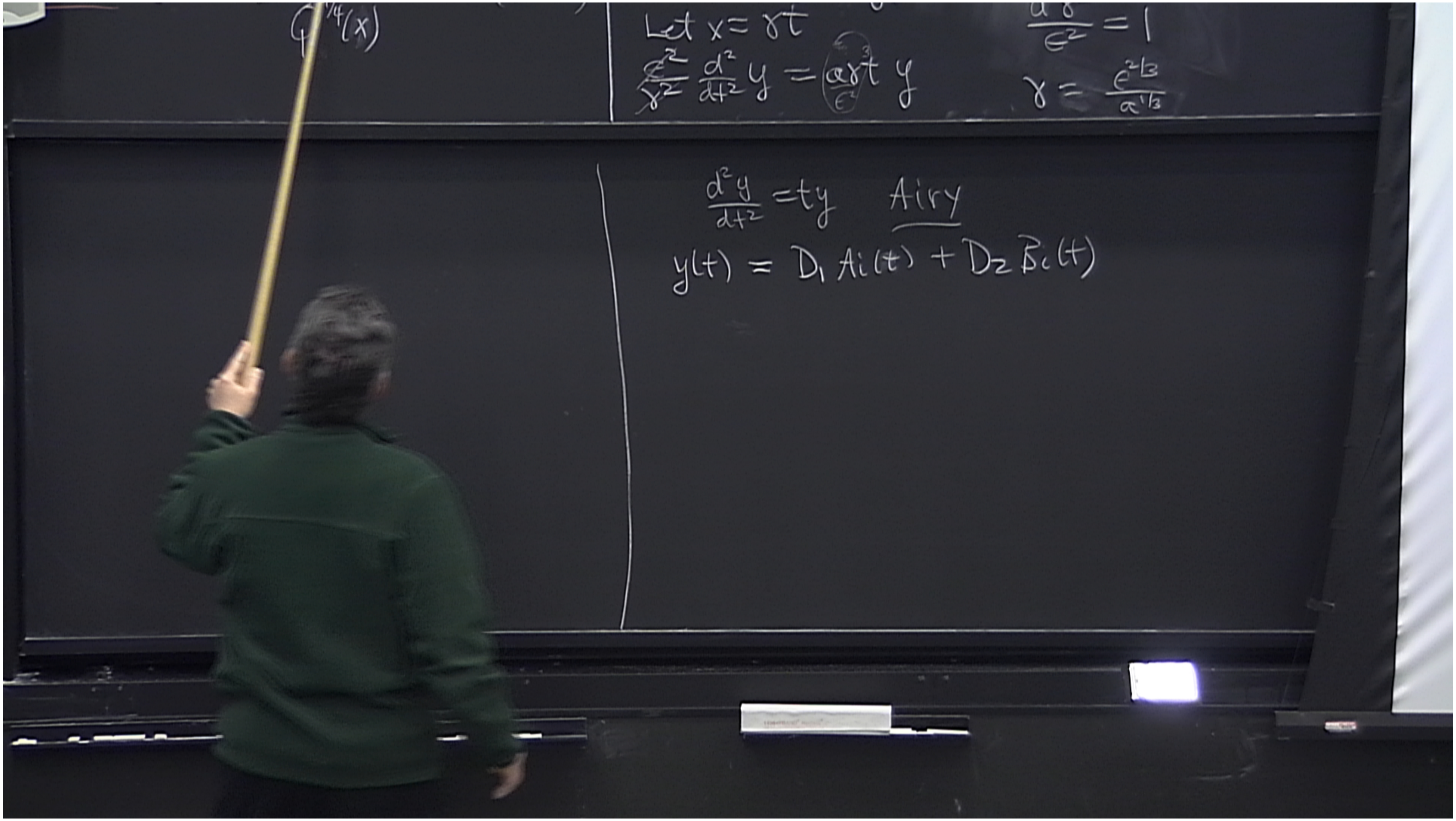

ONE T-P PROB
ε² y''(x) = Q(x) y
Q(x) = 0 for just 1 value of x, which is x=0
Q(0)=0
Q(x)
x


ONE T-P PROB
ε² y''(x) = Q(x) y
Q(x) = 0 for just 1 value of x, which is x=0
Q(x) ~ ax as x→0, a>0.
y(+∞) = 0
Q(0)=0
Q(x)
x

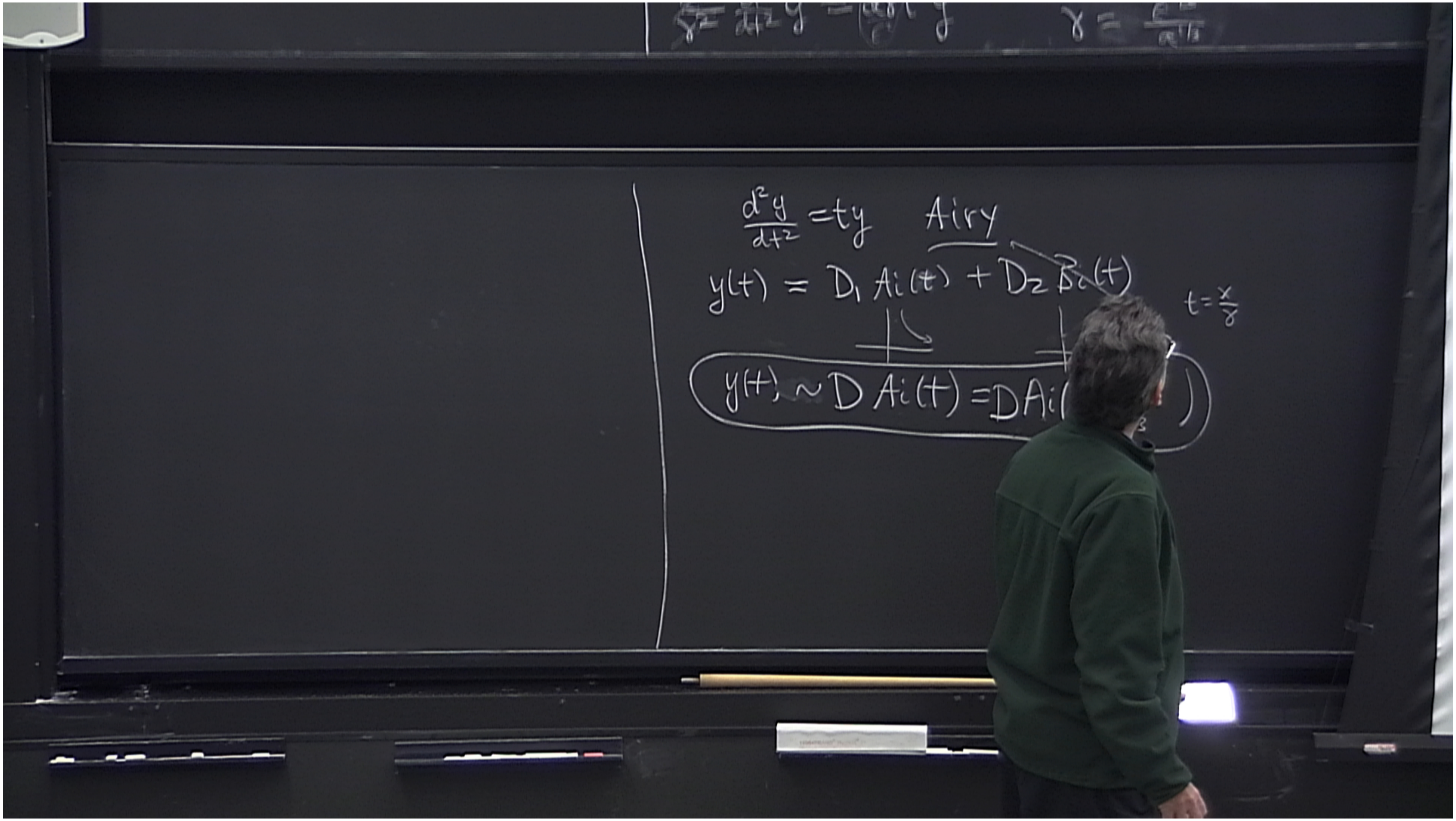

Q(x) = 0 for just 1 value of x, which is x = 0
Q(x) ~ ax as x → 0, a > 0.
y(+∞) = 0
x ≥ 0
REG. I
y(x) ~ C e^{-1/ε ∫^x dt √Q(t)} / Q^{1/4}
(ε → 0)
x
REGION I (x > 0)

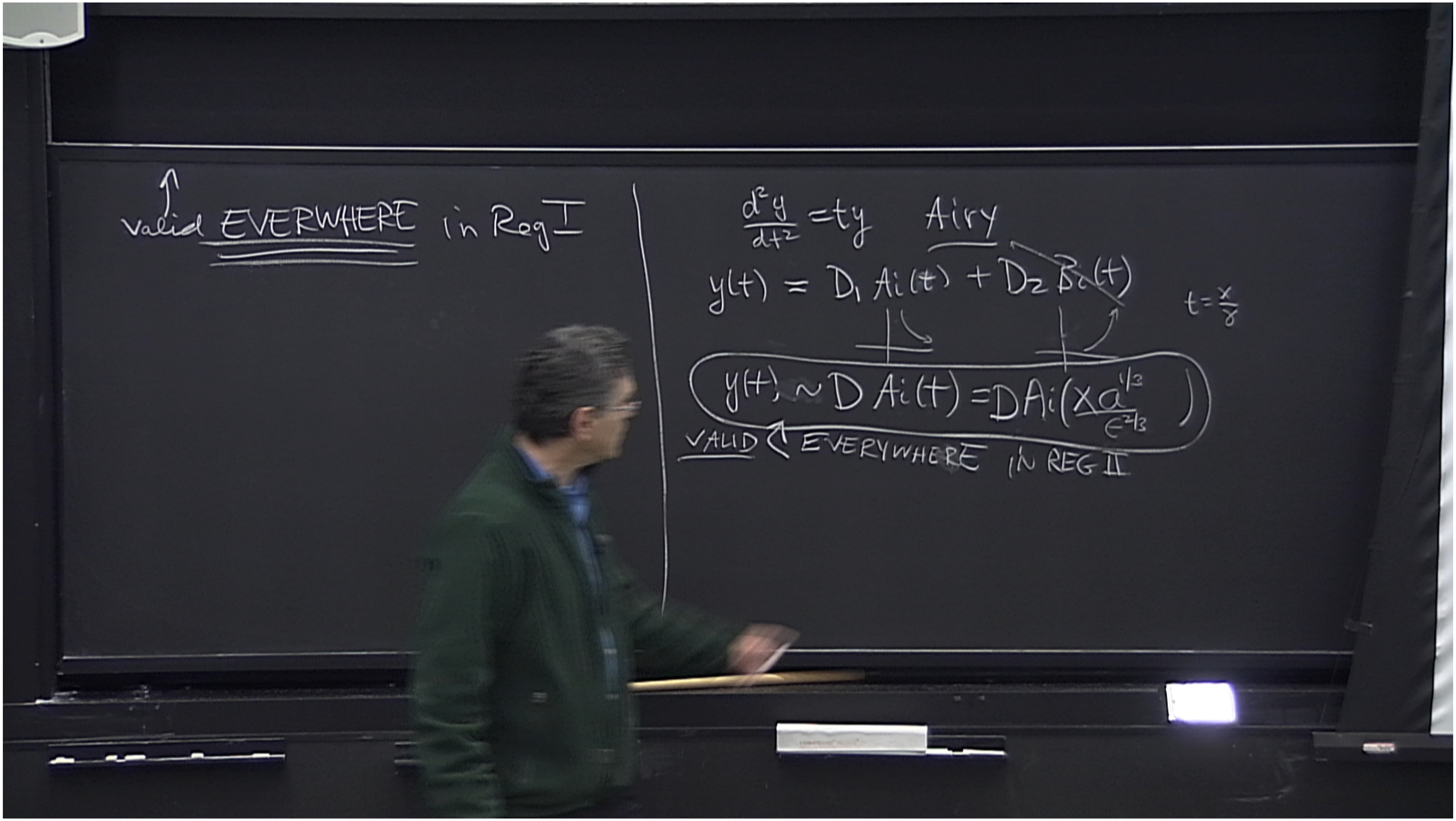

Q(x) ~ ax as x→0, a>0.
y(+∞)=0
REGION I (x>0)
REGION II
x≥0
REG. I
(ε→0)
REG II (x near 0)
x<<1
ε²y''=axy
ε²y''=(ax+bx²+cx³+....) y

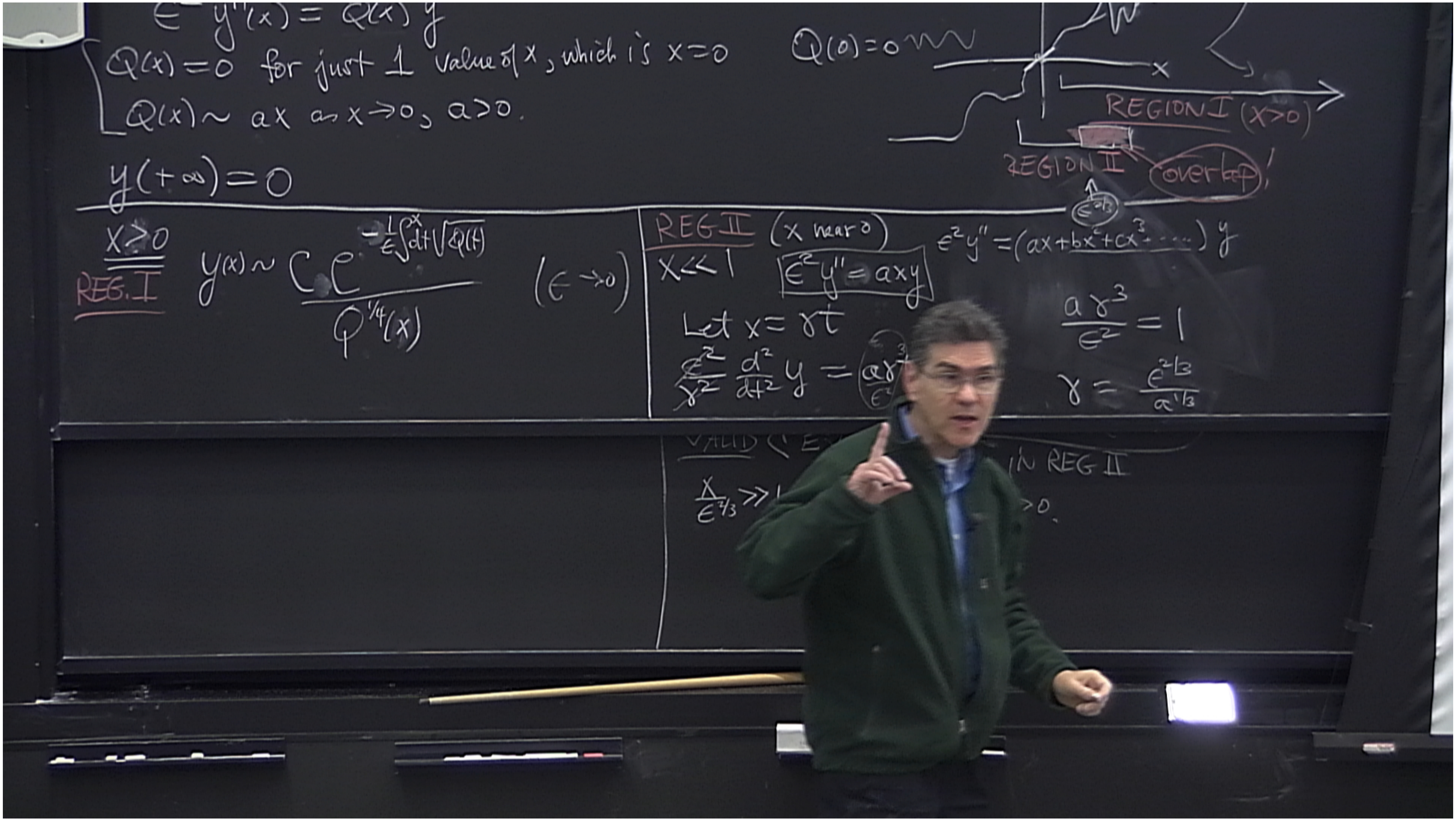

Let x = γt
d²y/dt² = ty

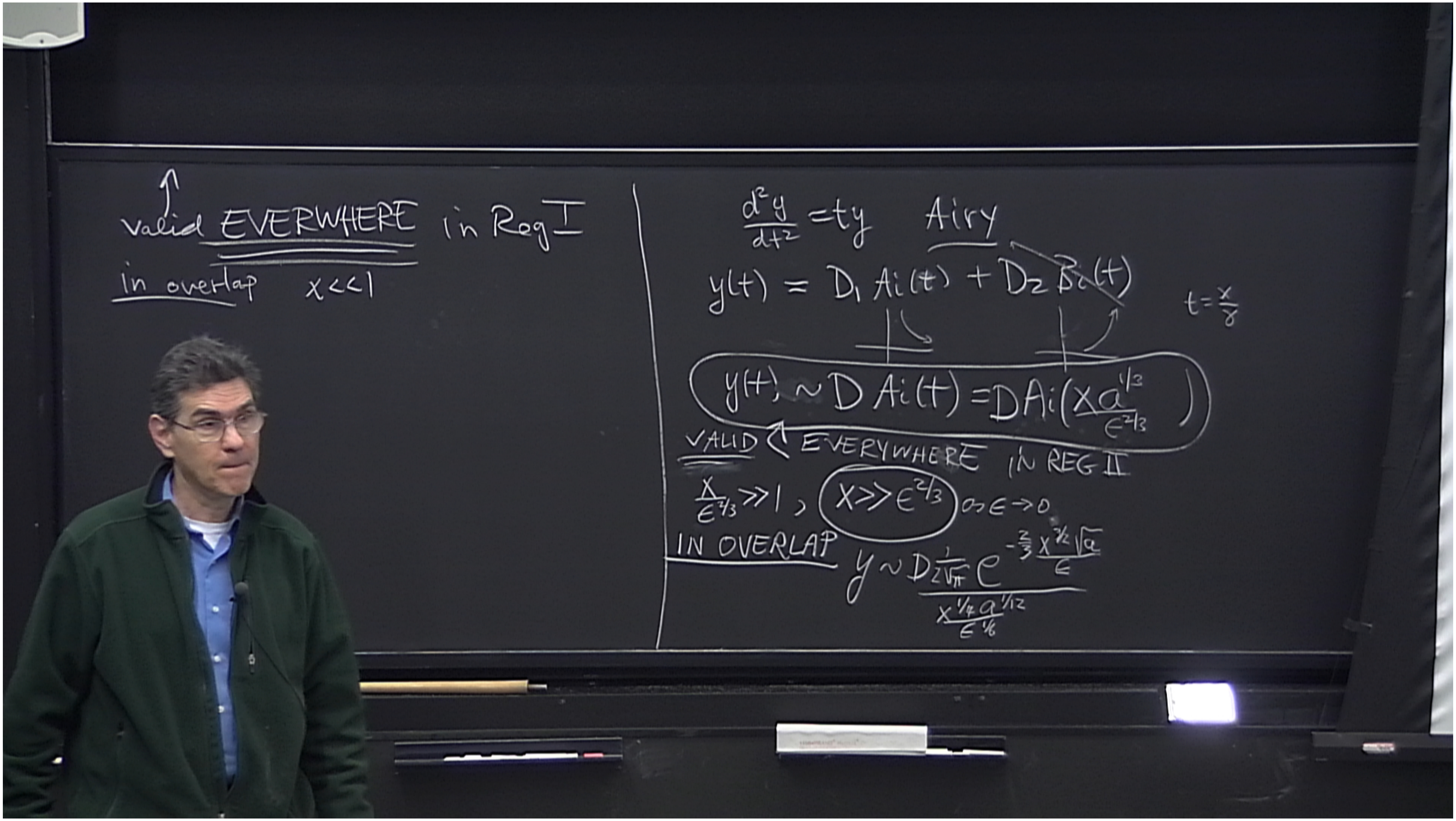


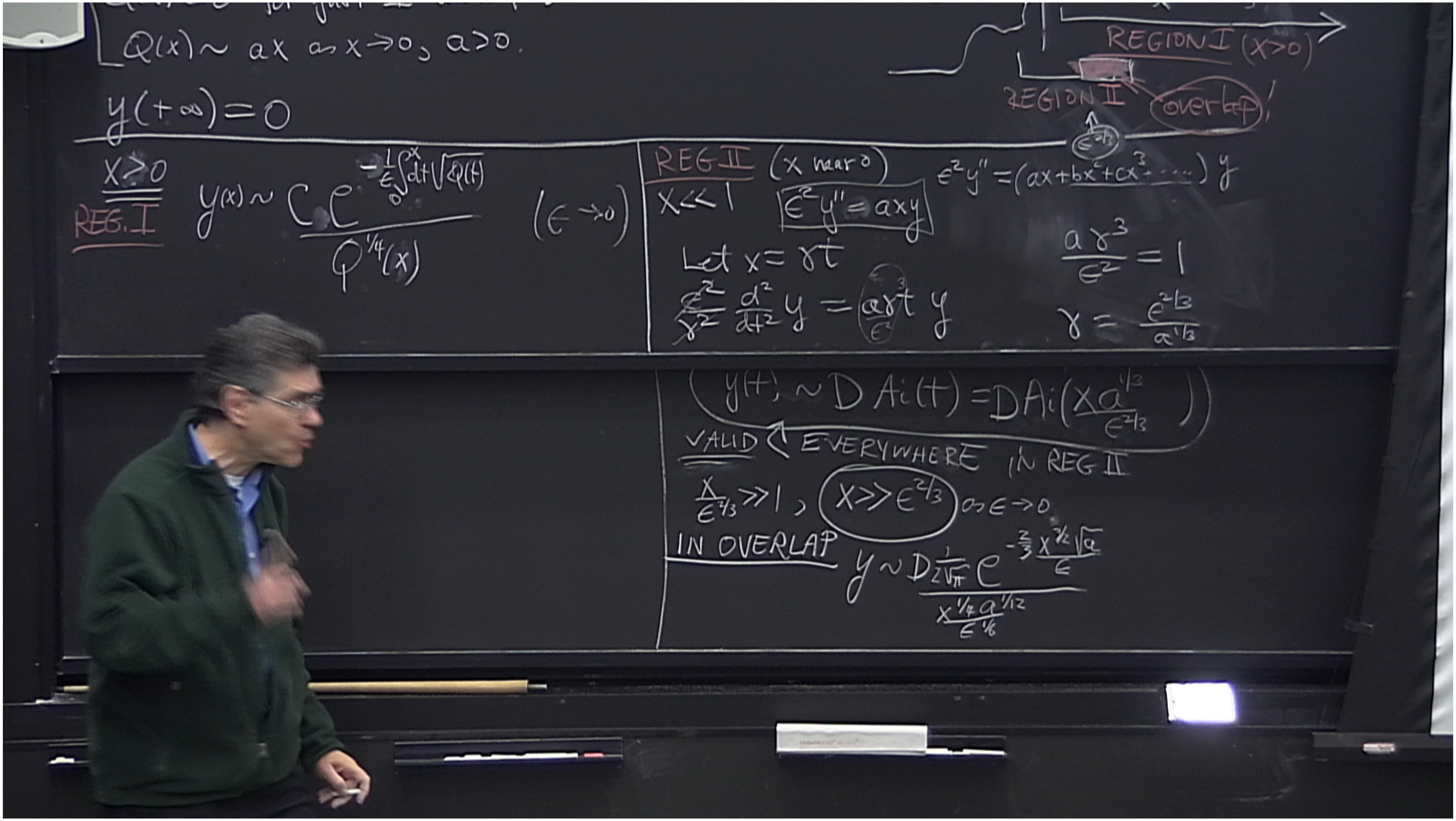
Let x = γt
Airy
y(t) = D1 Ai(t) + D2 Bi(t)

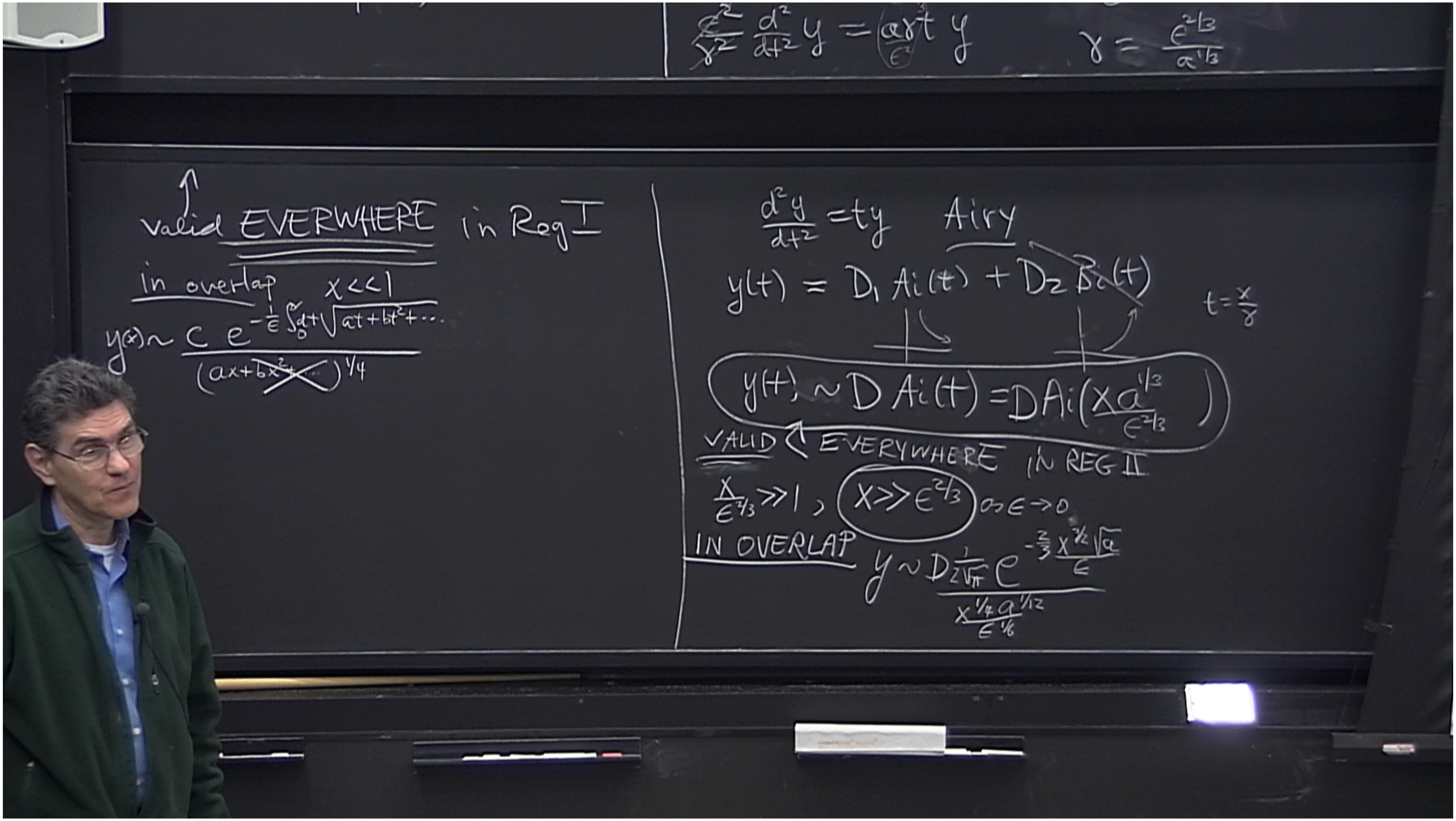

d²y/dt² = ty Airy
y(t) = D₁ Ai(t) + D₂ Bi(t)
t = x/γ
y(t) ~ D Ai(t) = DAi(

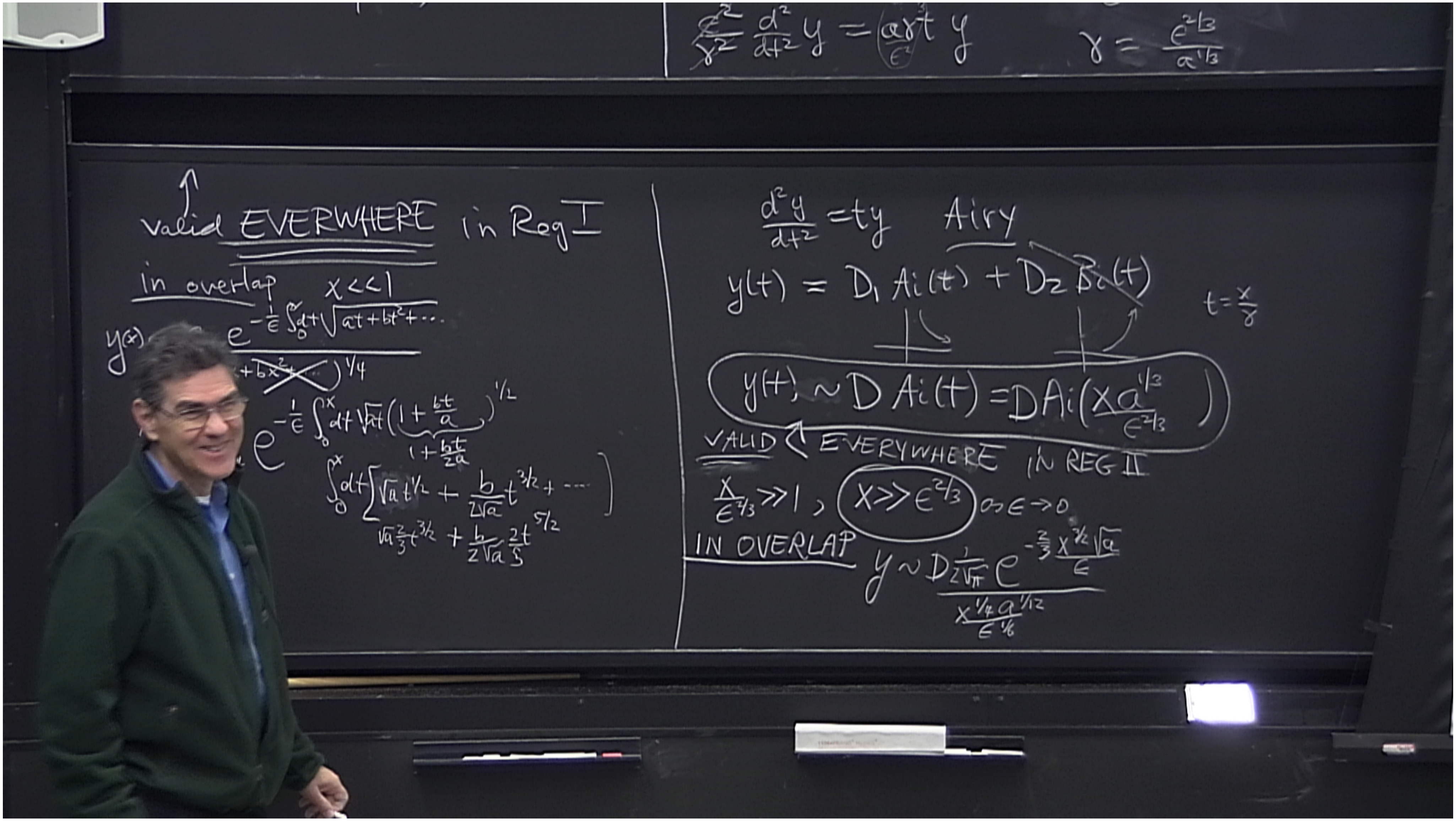

valid EVERWHERE in Reg I
d²y/dt² = ty Airy
y(t) = D1 Ai(t) + D2 Bi(t)
t = x/δ
y(t) ~ D Ai(t) = DAi(xa^1/3 / ε^2/3)
VALID EVERYWHERE IN REG II

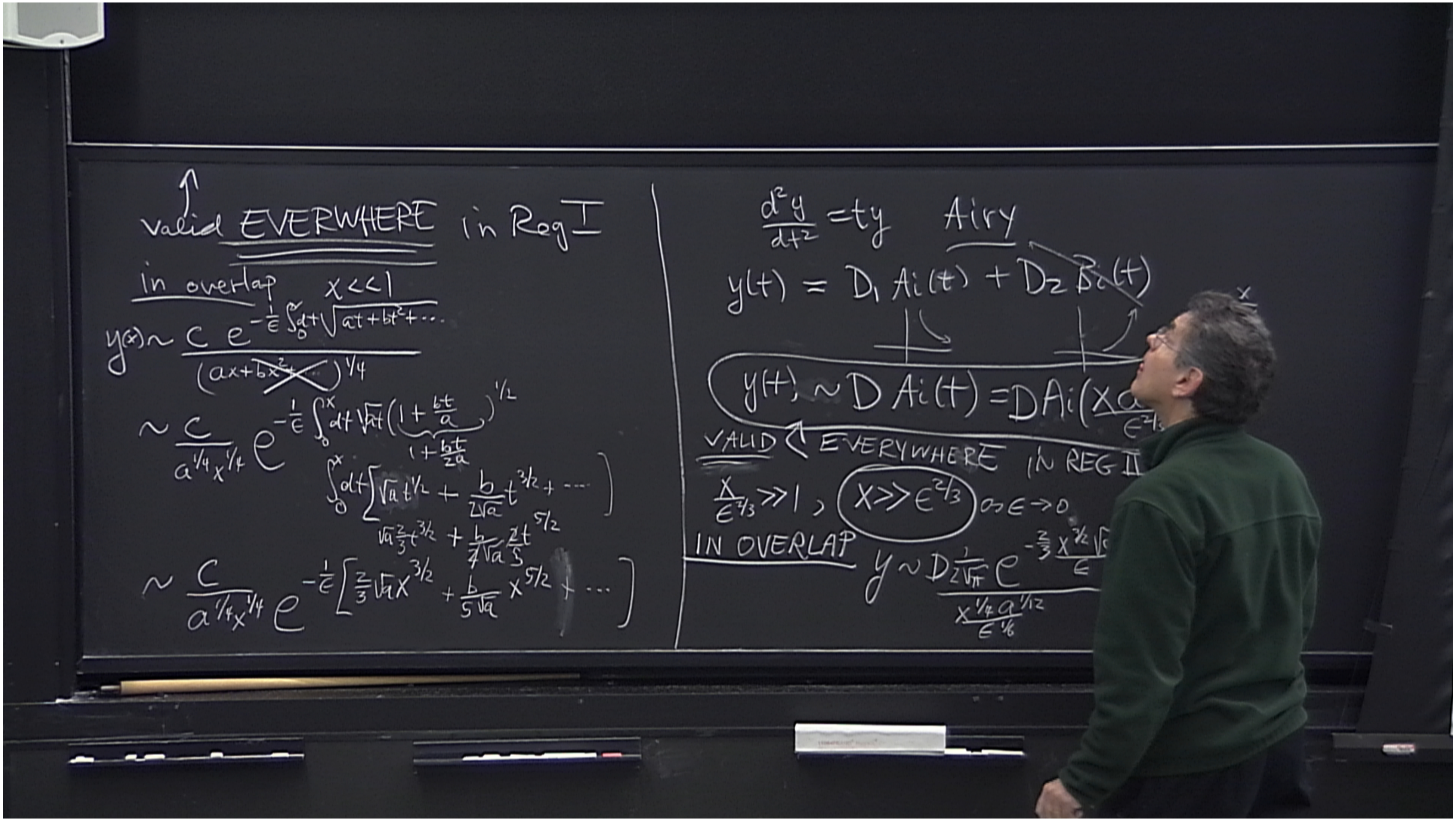

Q(x) = 0 for just 1 value of x, which is x = 0
Q(x) ~ ax as x → 0, a > 0.
y(+∞) = 0
Q(0) = 0
REGION I (x > 0)
REGION II
overlap!
x > 0
REG. I
(ε → 0)
REG II (x near 0)
x << 1
ε²y'' = axy
ε²y'' = (ax + bx² + cx³ + ...) y
Let x = γt
aγ³/ε² = 1
γ = ε^{2/3}/a^{1/3}
IN REG II

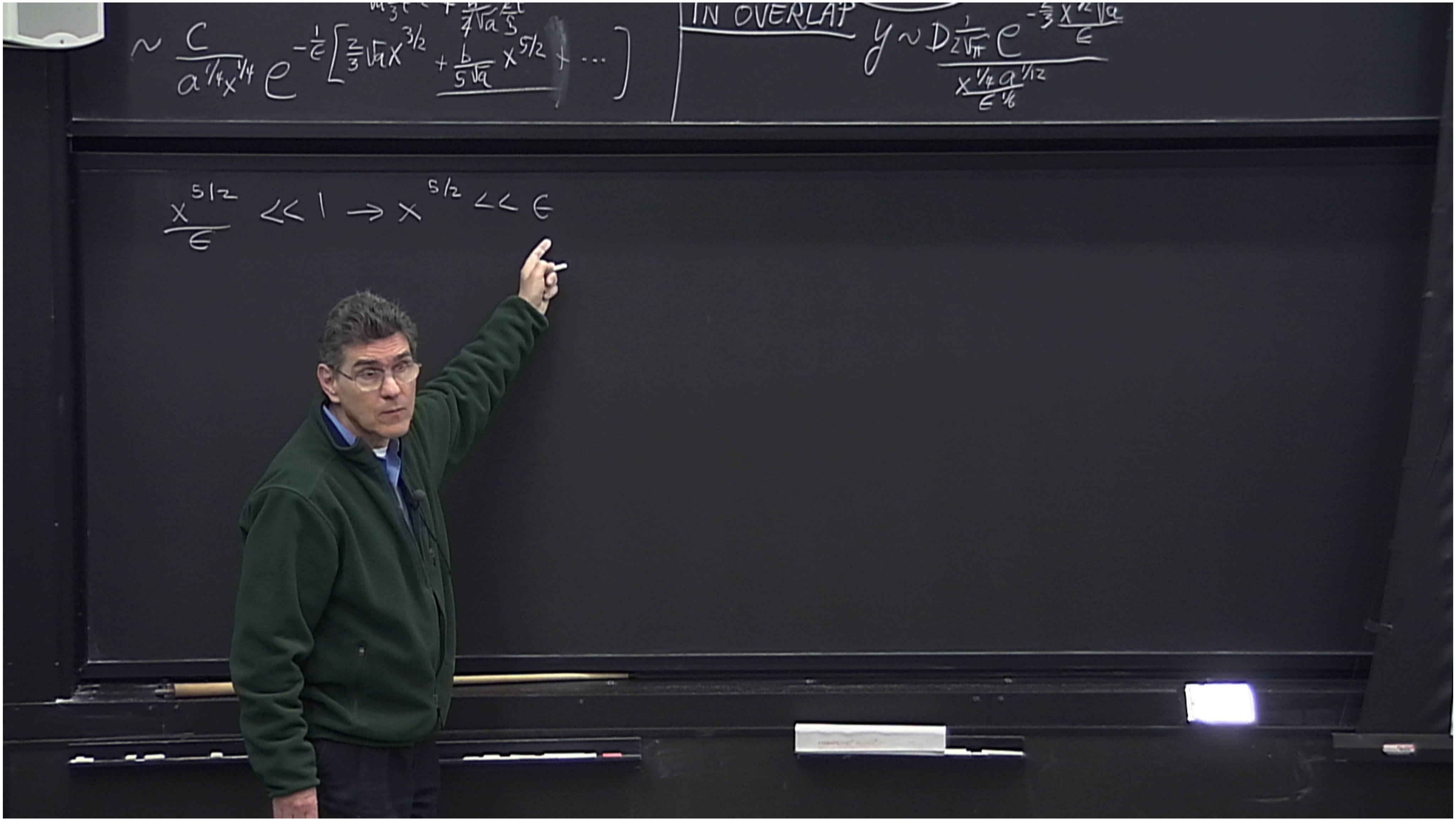

valid EVERWHERE in Reg I
in overlap x<<1
d²y/dt² = ty Airy
y(t) = D1 Ai(t) + D2 Bi(t)
t = x/δ
y(t) ~ D Ai(t) = D Ai(x a^1/3 / ε^2/3)
VALID EVERYWHERE IN REG II
x/ε^2/3 >> 1, x >> ε^2/3 as ε → 0
IN OVERLAP
y ~ D (1/(2√π)) e^(-2/3 x^3/2 √a / ε) / (x^1/4 a^1/12 / ε^1/6)

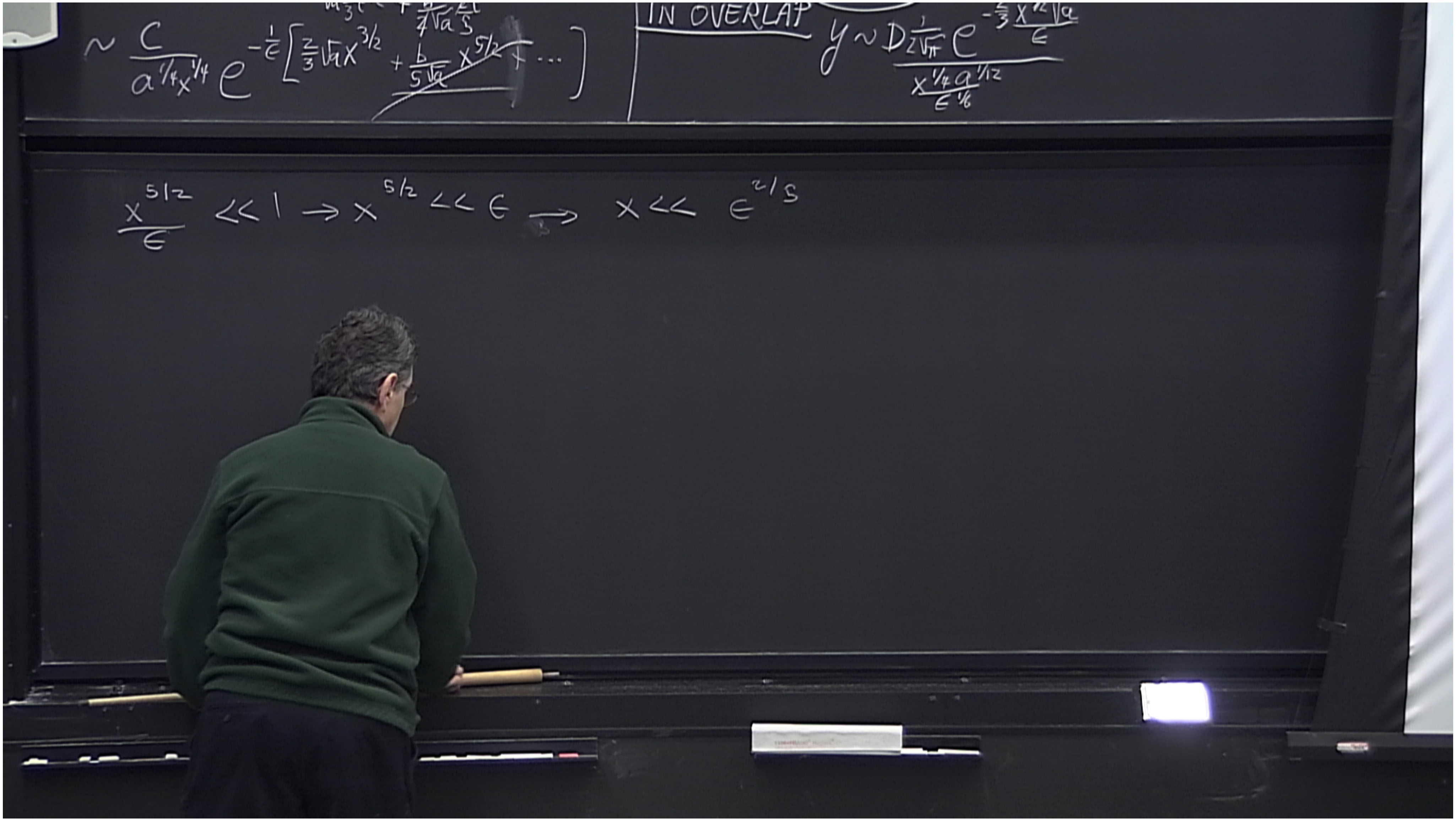

Q(x) ~ ax as x→0, a>0.
y(+∞) = 0
REGION I (x>0)
REGION II
overlap!
x≥0
REG. I
y(x) ~ C e^{-1/ε ∫_0^x dt √Q(t)} / Q^{1/4}(x)
(ε→0)
REG II (x near 0)
x<<1
ε²y'' = axy
ε²y'' = (ax + bx² + cx³ + ...) y
Let x = γt
ε²/γ² d²/dt² y = aγt y
aγ³/ε² = 1
γ = ε^{2/3}/a^{1/3}
y(t) ~ D Ai(t) = D Ai(xa^{1/3}/ε^{2/3})
VALID EVERYWHERE IN REG II
x/ε^{2/3} >> 1, x >> ε^{2/3} as ε→0
IN OVERLAP
y ~ D (1/(2√π)) e^{-(2/3) x^{3/2} √a / ε} / (x^{1/4} a^{1/12} / ε^{1/6})

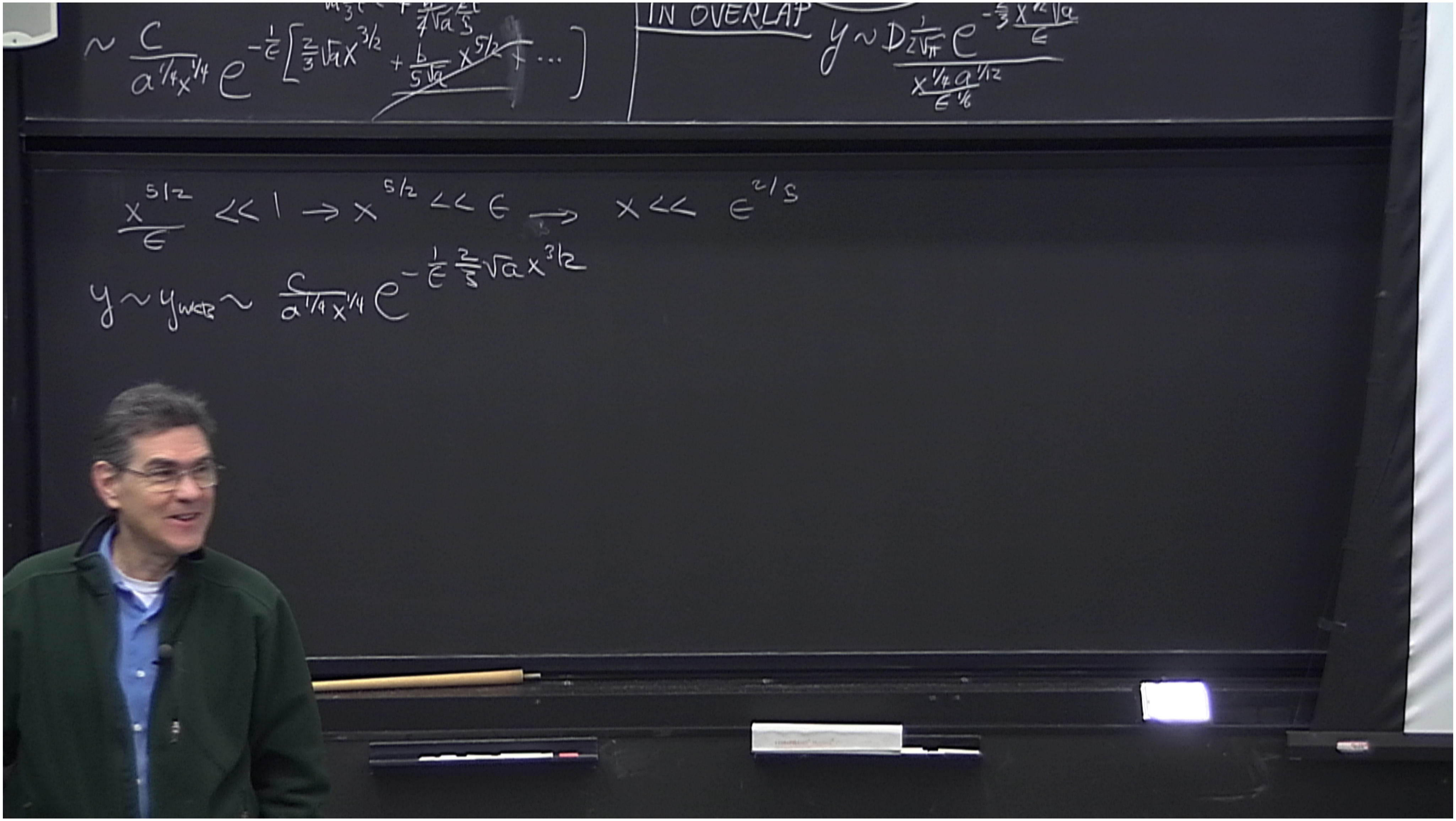

valid EVERWHERE in Reg I
in overlap x<<1
VALID EVERYWHERE IN REG II
IN OVERLAP
Airy
t = x/γ

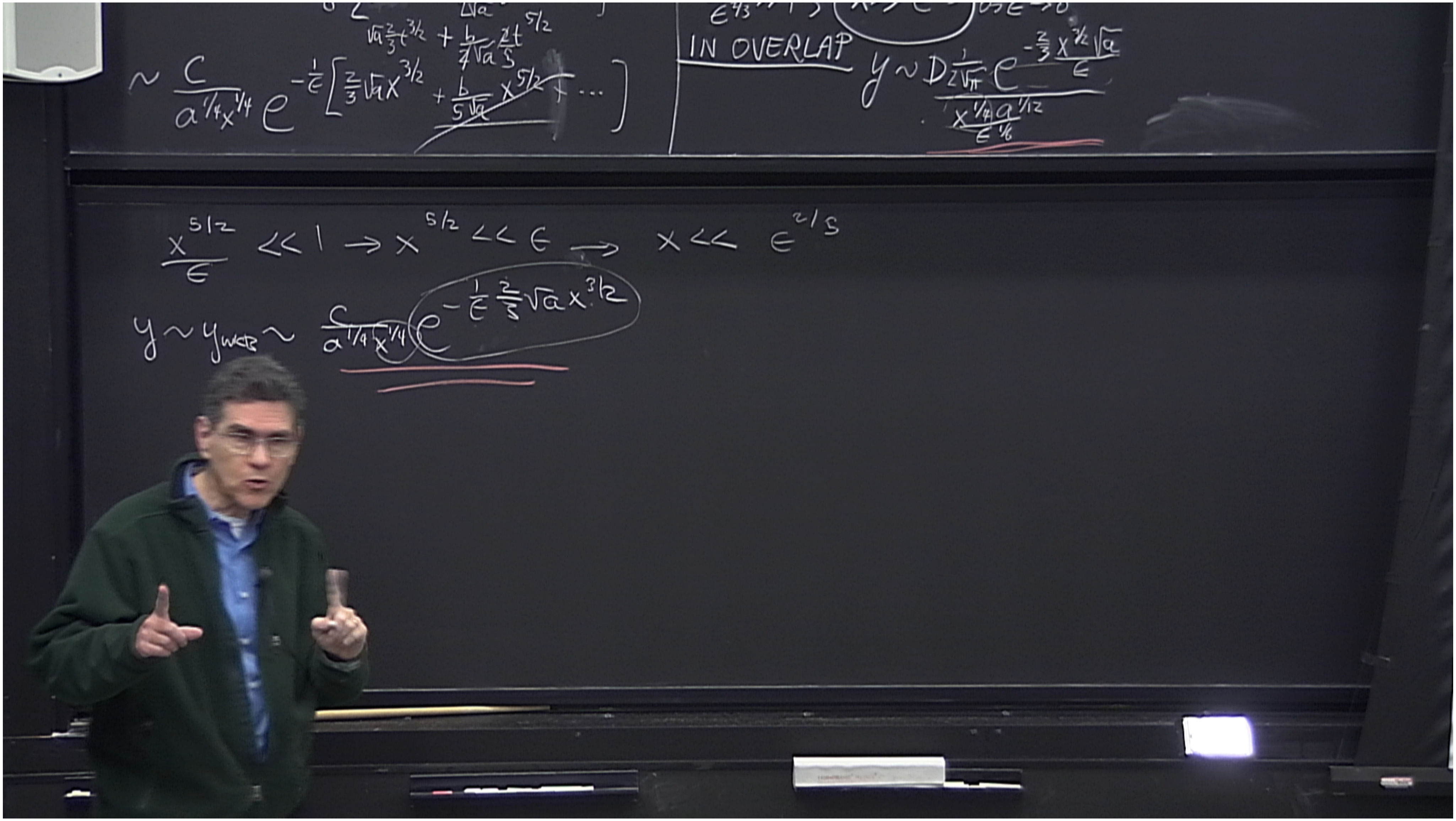

valid EVERWHERE in Reg I
in overlap x<<1
d²y/dt² = ty Airy
y(t) = D₁ Ai(t) + D₂ Bi(t)
t = x/γ
y(t) ~ D Ai(t) = D Ai(x a^{1/3}/ε^{2/3})
VALID EVERYWHERE IN REG II
x/ε^{2/3} >> 1, x >> ε^{2/3} as ε → 0
IN OVERLAP
γ = ε^{2/3}/a^{1/3}

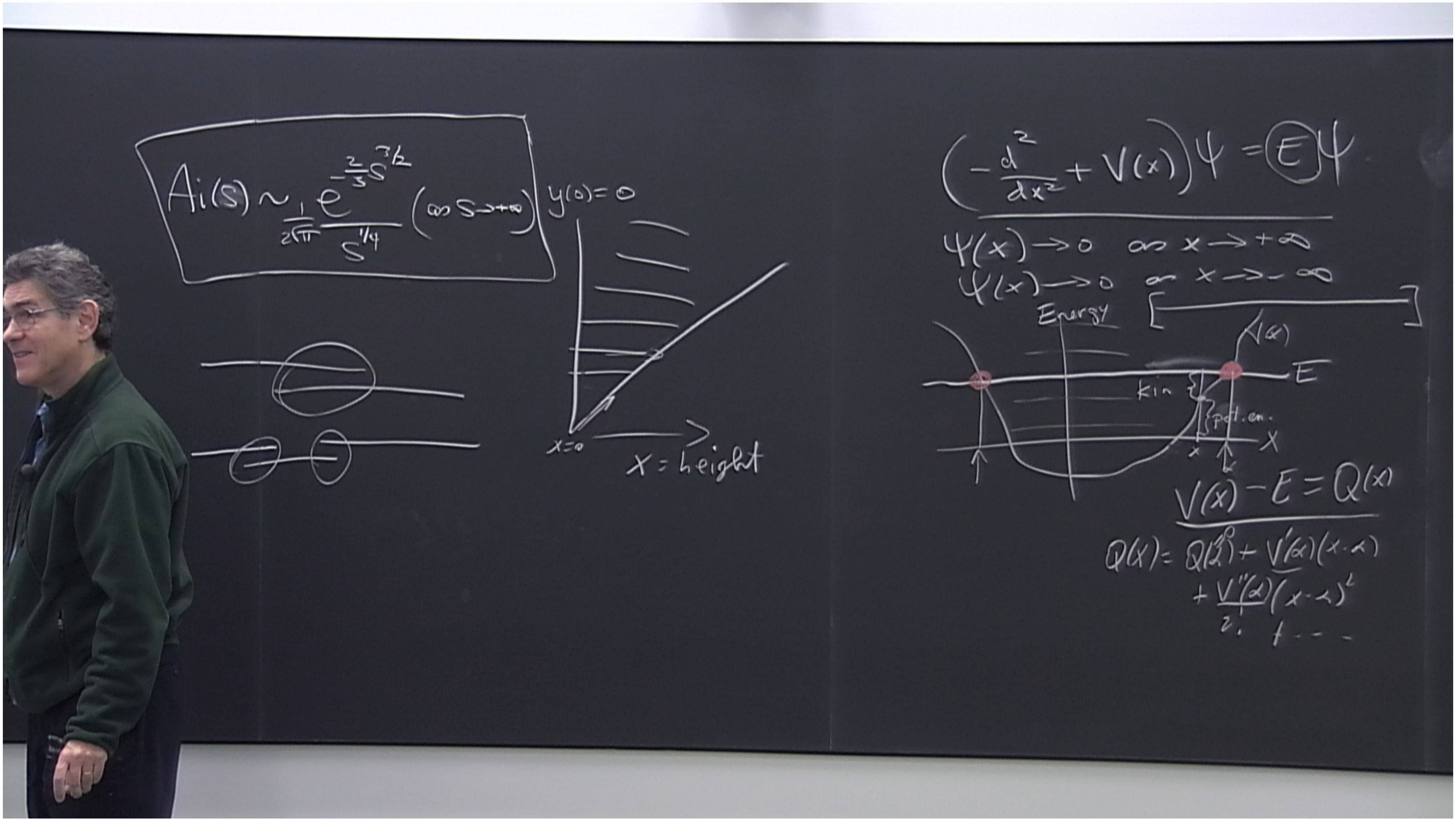

valid EVERWHERE in Reg I
in overlap x<<1
y(x) ~ C e^{-1/ε ∫ dt √(at+bt²+...)} / (ax+bx²+...)^{1/4}
~ C/(a^{1/4} x^{1/4}) e^{-1/ε ∫ dt √(at) (1 + bt/a)^{1/2}}
1 + bt/2a
∫ dt [√a t^{1/2} + b/(2√a) t^{3/2} + ...]
√a 2/3 t^{3/2} + b/(2√a) 2t^{5/2}/5
~ C/(a^{1/4} x^{1/4}) e^{-1/ε [2/3 √a x^{3/2} + b/(5√a) x^{5/2} + ...]}
d²y/dt² = ty Airy
y(t) = D₁ Ai(t) + D₂ Bi(t)
y(t) ~ D Ai(t) = D Ai(x a ... / ε^{2/3})
VALID EVERYWHERE IN REG II
x/ε^{2/3} >> 1, x >> ε^{2/3} as ε → 0
IN OVERLAP
y ~ D 1/(2√π) e^{-2/3 x^{3/2} √a / ε} / (x^{1/4} a^{1/12} / ε^{1/6})

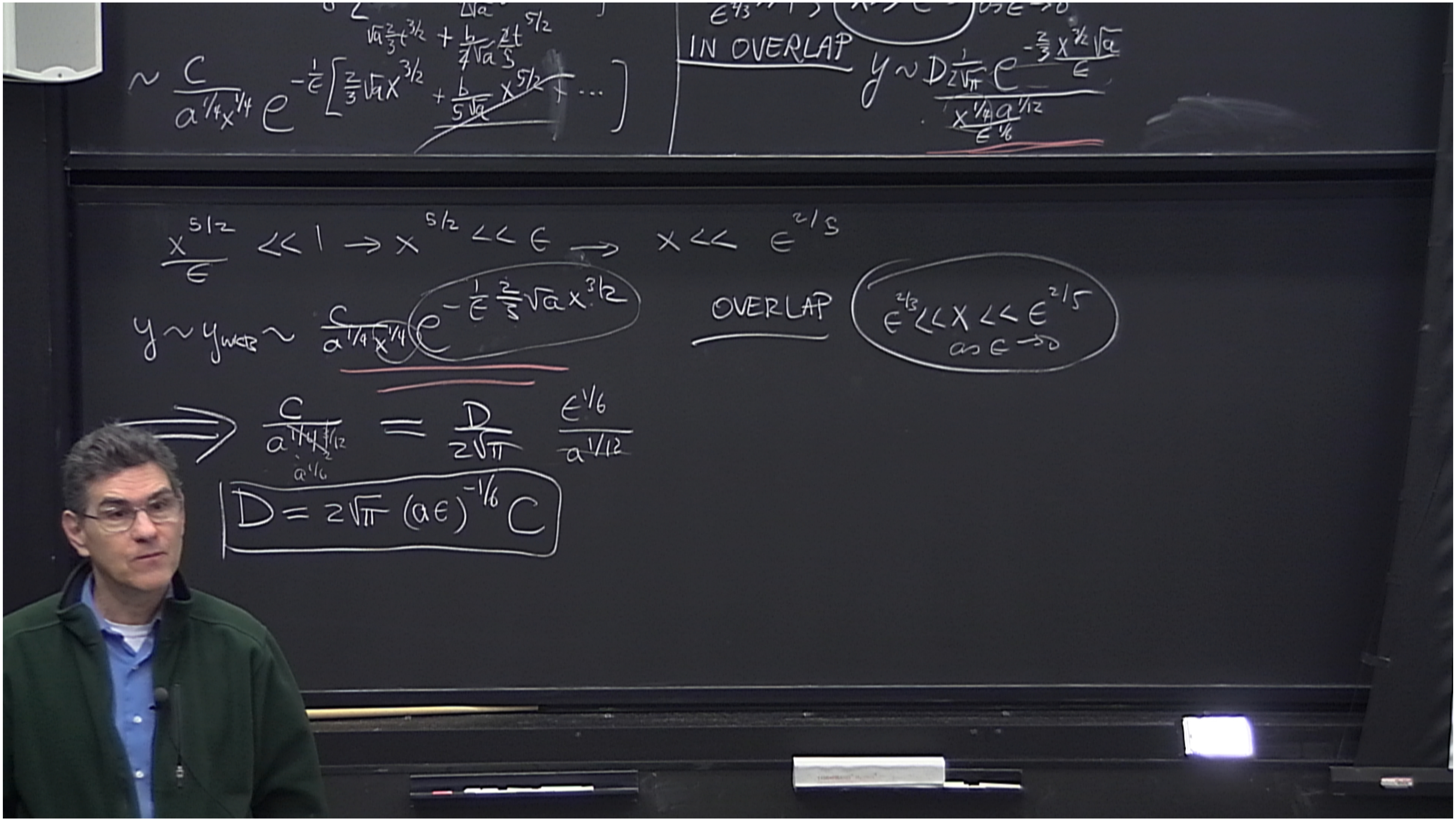

IN OVERLAP
$\frac{x^{5/2}}{\epsilon} \ll 1 \rightarrow x^{5/2} \ll \epsilon$

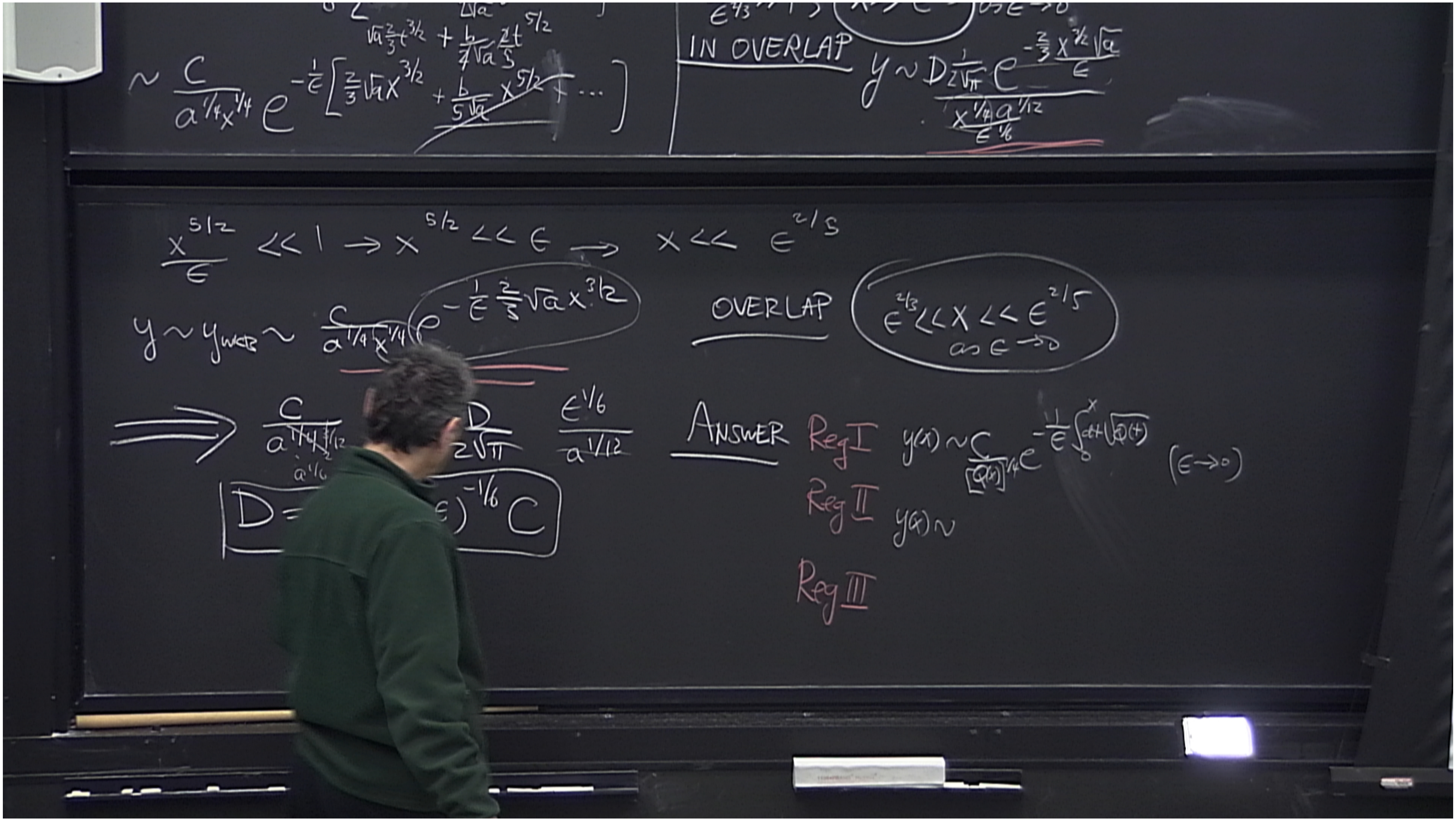

IN OVERLAP
$\frac{x^{5/2}}{\epsilon} \ll 1 \to x^{5/2} \ll \epsilon \to x \ll \epsilon^{2/5}$

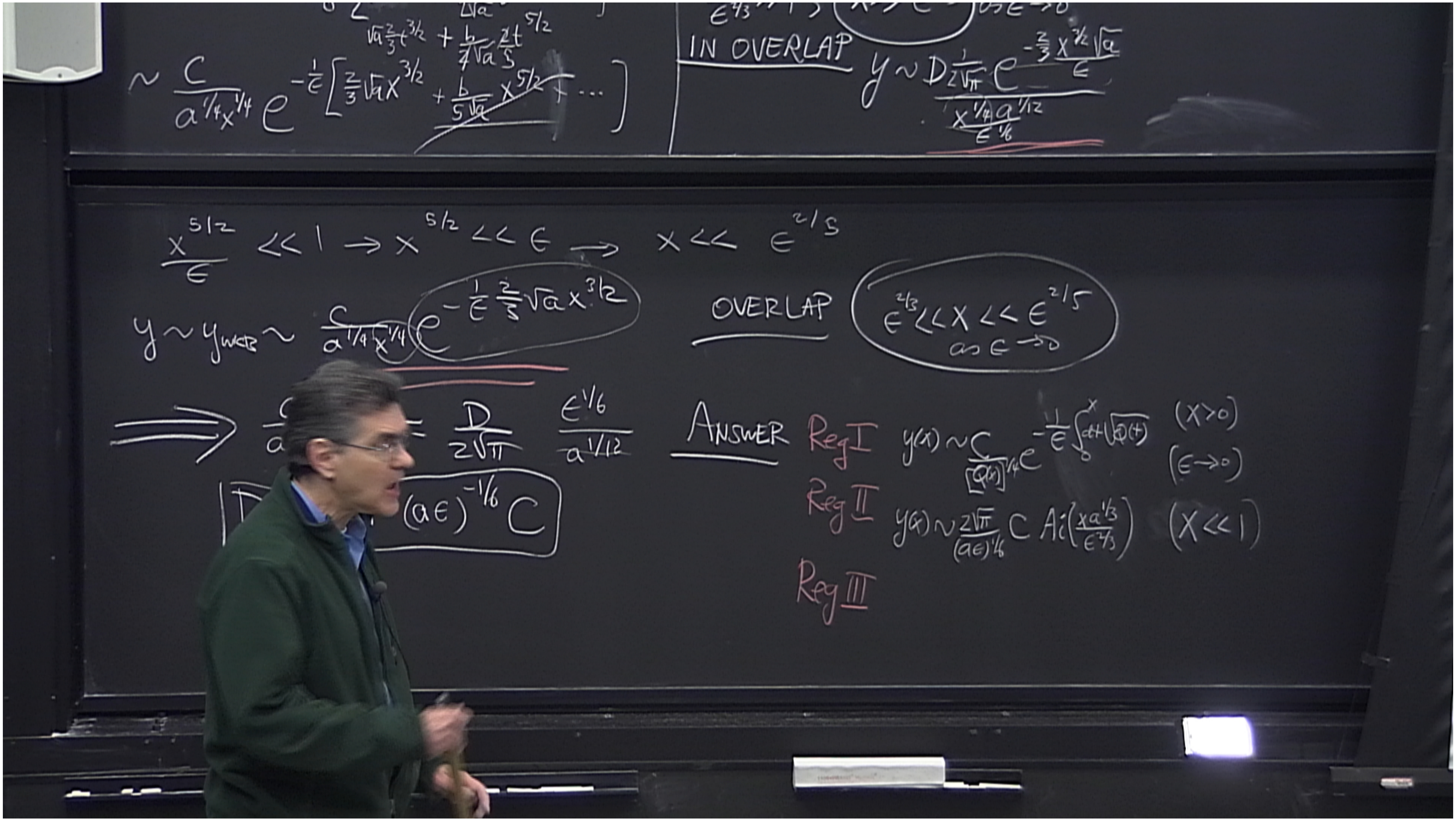

IN OVERLAP
$y \sim D \frac{1}{2\sqrt{\pi}} e^{-\frac{2}{3}\frac{x^{3/2}\sqrt{a}}{\epsilon}} \Big/ \frac{x^{1/4}a^{1/12}}{\epsilon^{1/6}}$
$\sim \frac{c}{a^{1/4}x^{1/4}} e^{-\frac{1}{\epsilon}\left[\frac{2}{3}\sqrt{a}x^{3/2} + \frac{b}{5\sqrt{a}}x^{5/2} + \cdots\right]}$
$\frac{x^{5/2}}{\epsilon} \ll 1 \rightarrow x^{5/2} \ll \epsilon \rightarrow x \ll \epsilon^{2/5}$
$y \sim y_{WKB} \sim \frac{c}{a^{1/4}x^{1/4}} e^{-\frac{1}{\epsilon}\frac{2}{3}\sqrt{a}x^{3/2}}$

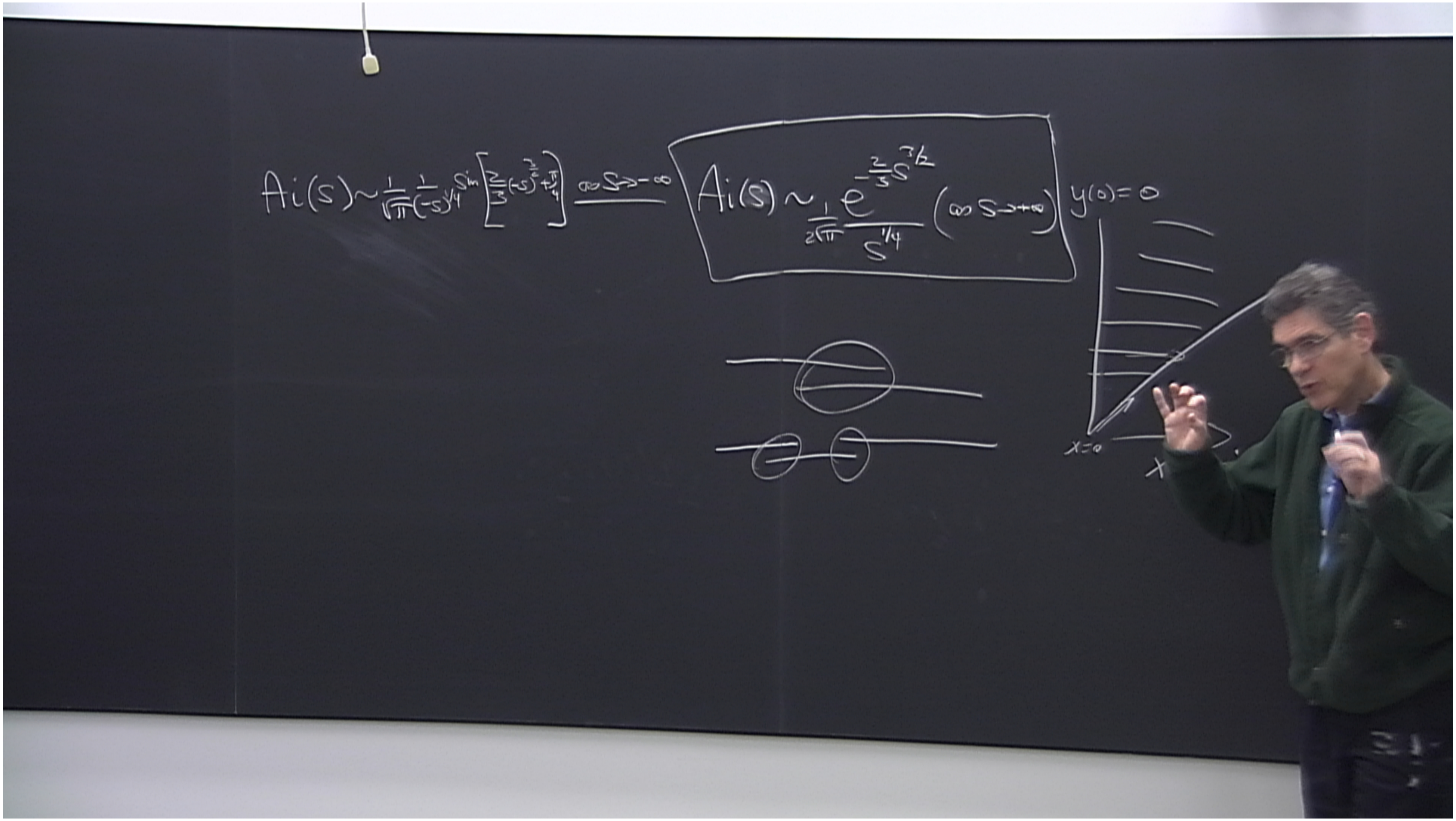

IN OVERLAP
x^{5/2}/ε << 1 → x^{5/2} << ε → x << ε^{2/5}
y ~ y_WKB ~ c/(a^{1/4} x^{1/4}) e^{-(1/ε)(2/3)√a x^{3/2}}

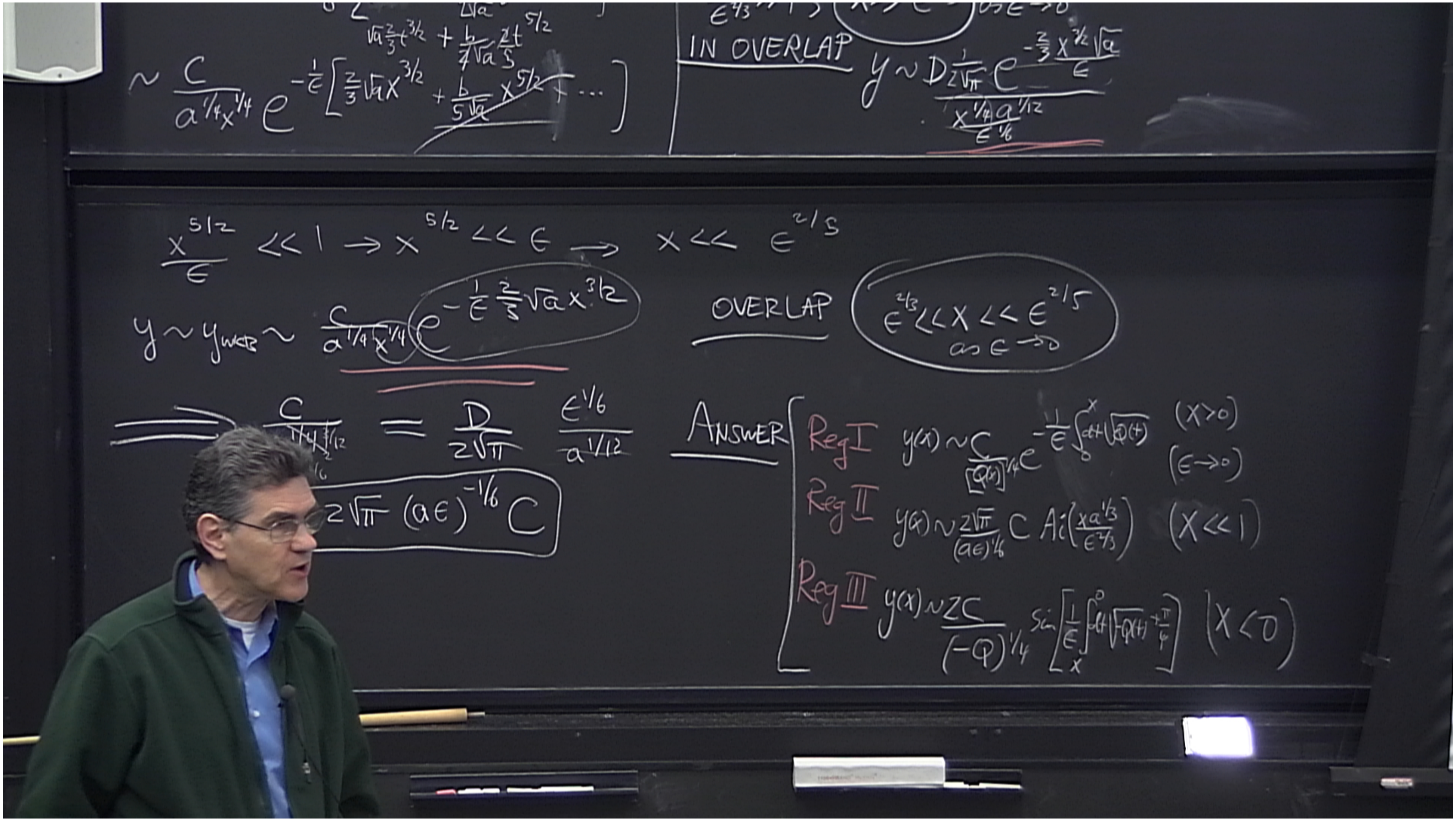

Ai(s) ~ 1/(2√π) e^(−2/3 s^(3/2)) / s^(1/4) (as s → +∞)
y(0) = 0
x=0
x = height
(−d²/dx² + V(x)) ψ = E ψ
ψ(x) → 0 as x → +∞
ψ(x) → 0 as x → −∞
Energy
V(x)
E
kin
pot. en.
X
V(x) − E = Q(x)
Q(x) = Q(α) + V′(α)(x − α)
+ V″(α)/2! (x − α)² + . . .

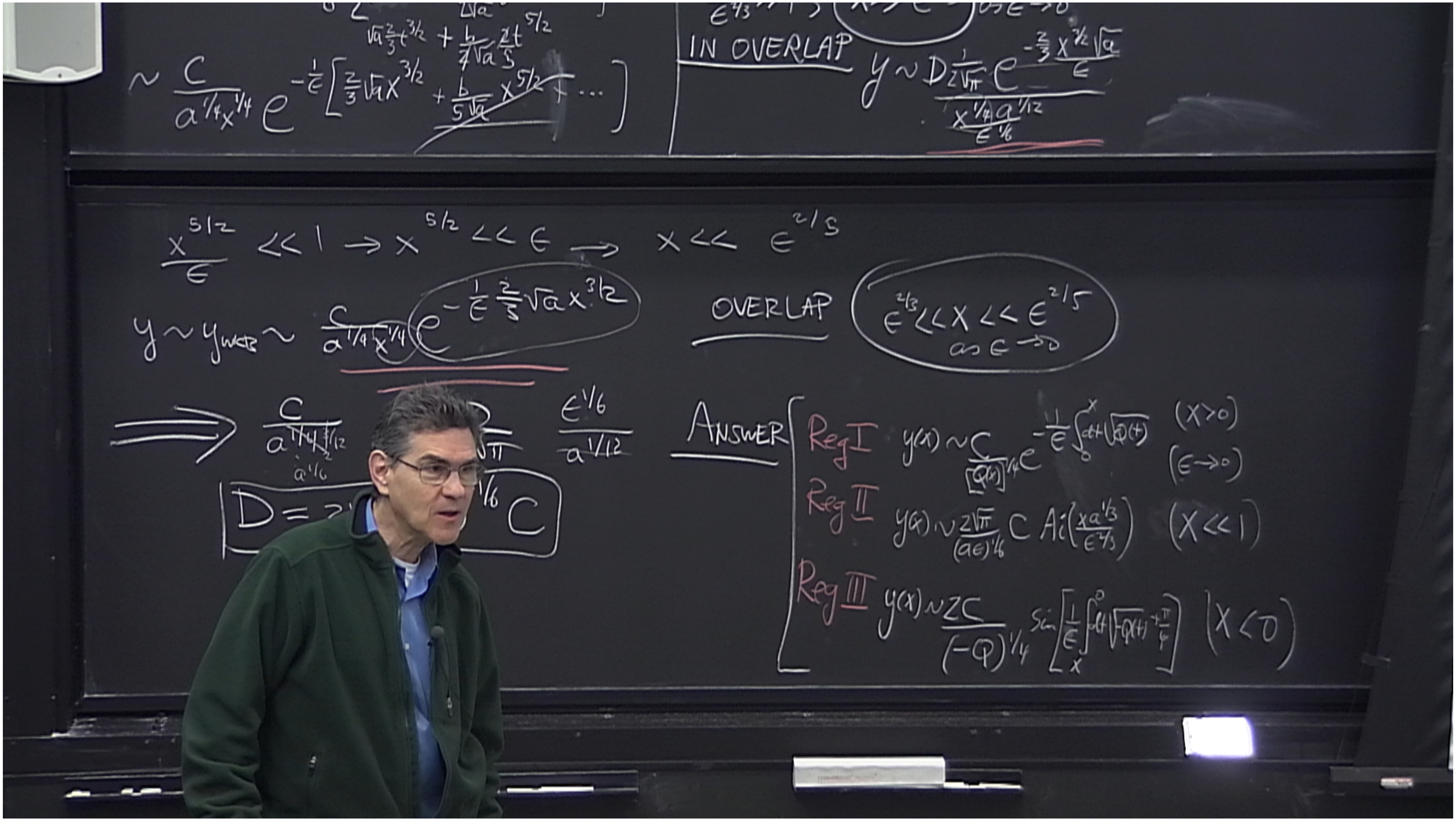

IN OVERLAP
OVERLAP
as ε → 0
D = 2√π (aε)^(-1/6) C

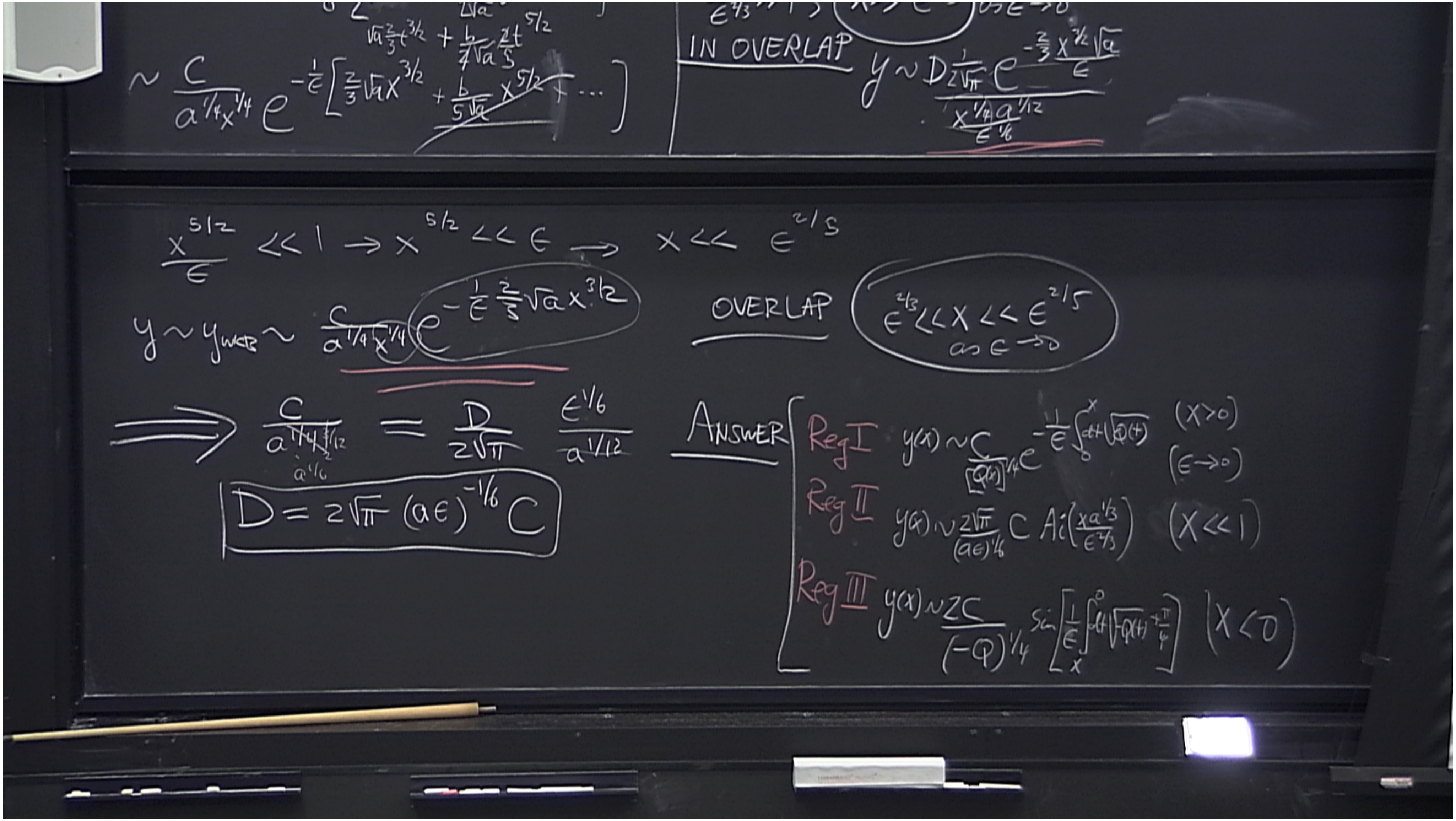

IN OVERLAP
x^{5/2}/ε << 1 → x^{5/2} << ε → x << ε^{2/5}
OVERLAP
ε^{2/3} << x << ε^{2/5}
as ε → 0
ANSWER
Reg I
Reg II
Reg III
(ε → 0)


IN OVERLAP
OVERLAP
ε^{2/3} << X << ε^{2/5}
as ε → 0
ANSWER
Reg I
(X>0)
(ε→0)
Reg II
(X<<1)
Reg III


y(0) = 0
x=0


IN OVERLAP
OVERLAP
as ε → 0
ANSWER
Reg I
(x>0)
(ε→0)
Reg II
(x<<1)
Reg III
(x<0)


IN OVERLAP
OVERLAP
as ε → 0
ANSWER
Reg I
Reg II
Reg III
(x>0)
(ε→0)
(X<<1)
(X<0)


IN OVERLAP
x^{5/2}/ε << 1 → x^{5/2} << ε → x << ε^{2/5}
y ~ y_WKB ~ C/(a^{1/4} x^{1/4}) e^{-(1/ε)(2/3)√a x^{3/2}}
OVERLAP
ε^{2/3} << x << ε^{2/5} as ε → 0
D = 2√π (aε)^{-1/6} C
ANSWER
Reg I y(x) ~ C/[Q(x)]^{1/4} e^{-(1/ε)∫_0^x dt √Q(t)} (x>0) (ε→0)
Reg II y(x) ~ 2√π/(aε)^{1/6} C Ai(x a^{1/3}/ε^{2/3}) (x<<1)
Reg III y(x) ~ 2C/(-Q)^{1/4} sin[(1/ε)∫_x^0 dt √(-Q(t)) + π/4] (x<0)

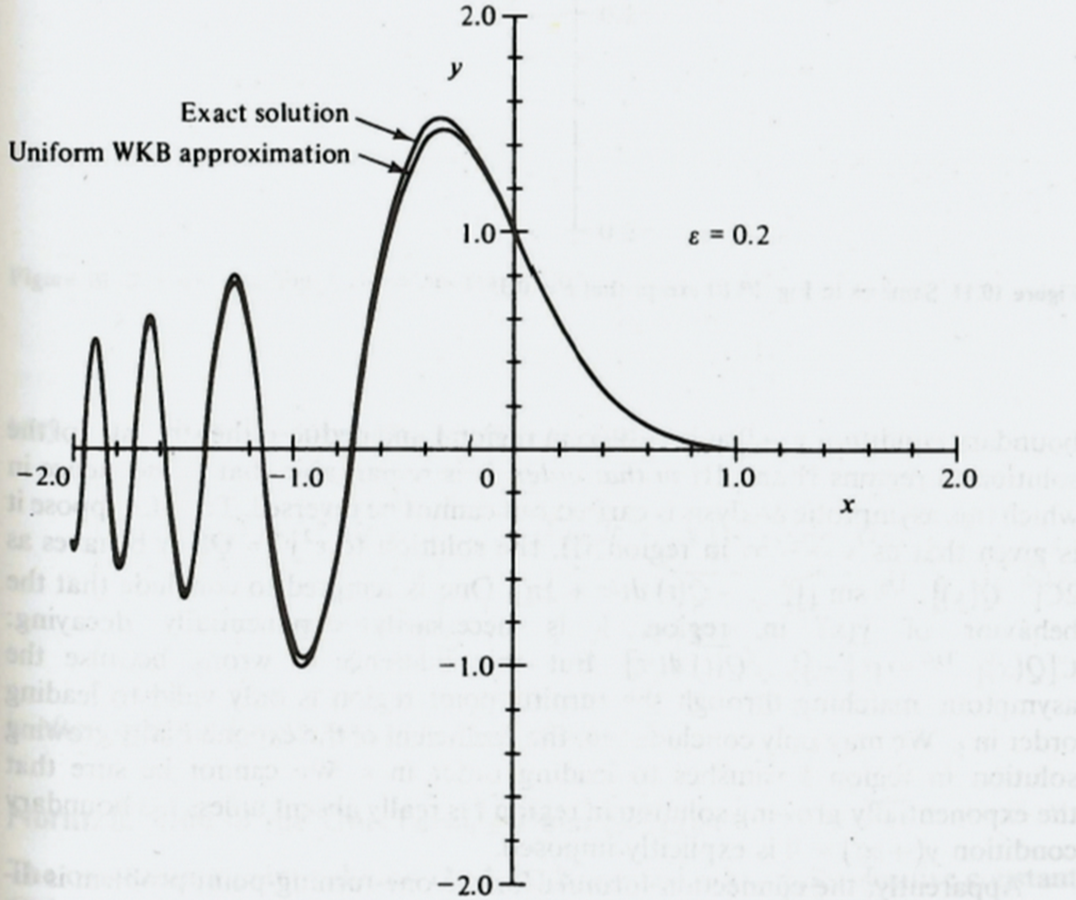
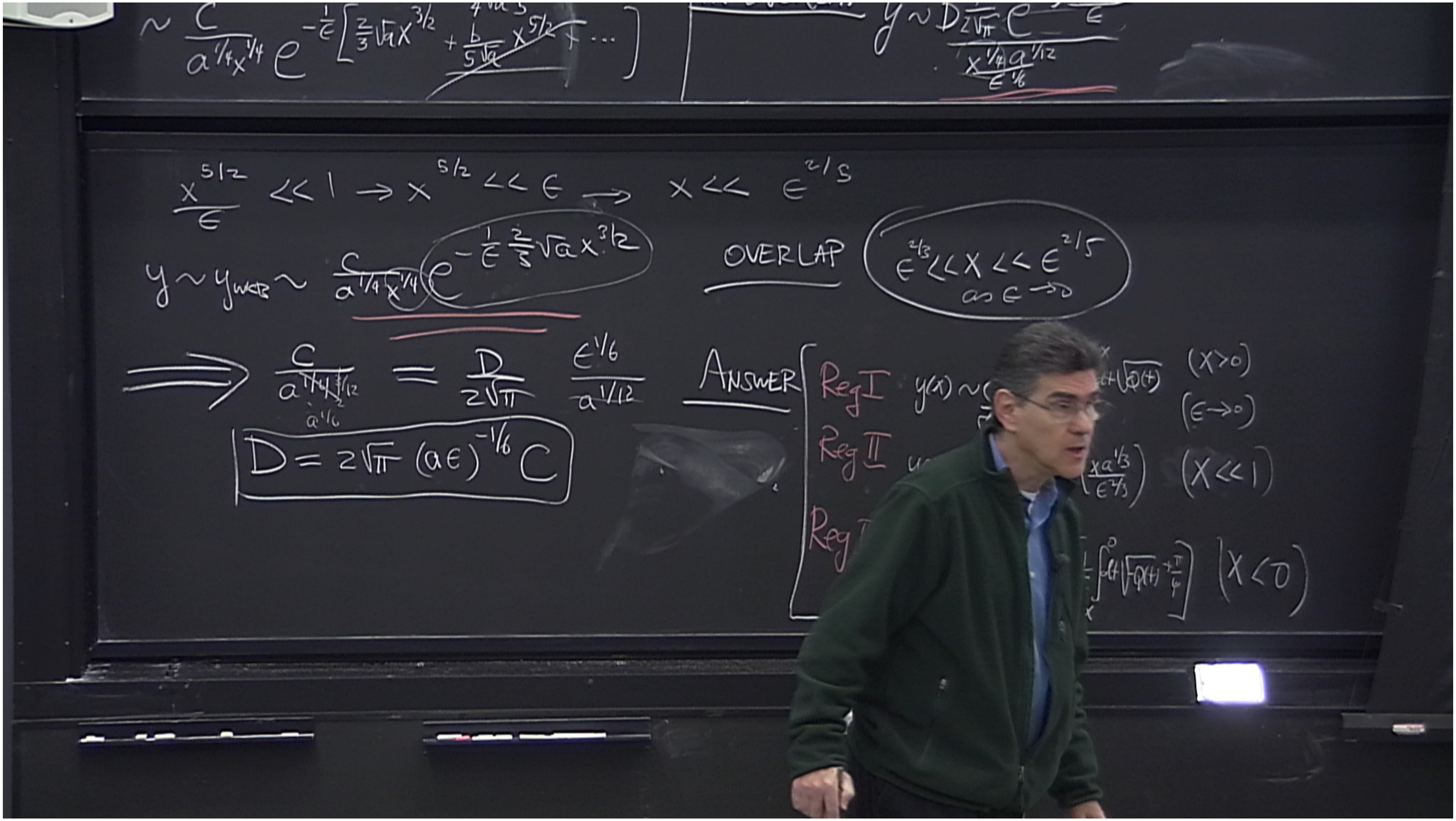


Figure 10.10 A comparison of the exact solution to $\varepsilon^2 y''(x) = \sinh x(\cosh x)^2 y(x)$ $[y(0) = 1,$ $y(+\infty) = 0]$, with the approximate solution from a one-turning-point WKB analysis. The WKB approximate formulas are given in (10.4.14) and (10.4.15).


OVERLAP
ε^{2/3} << x << ε^{2/5}
as ε→0
ANSWER
Reg I
Reg II
(x>0)
(ε→0)
(x<<1)
(x<0)
D = 2√π (aε)^{-1/6} C

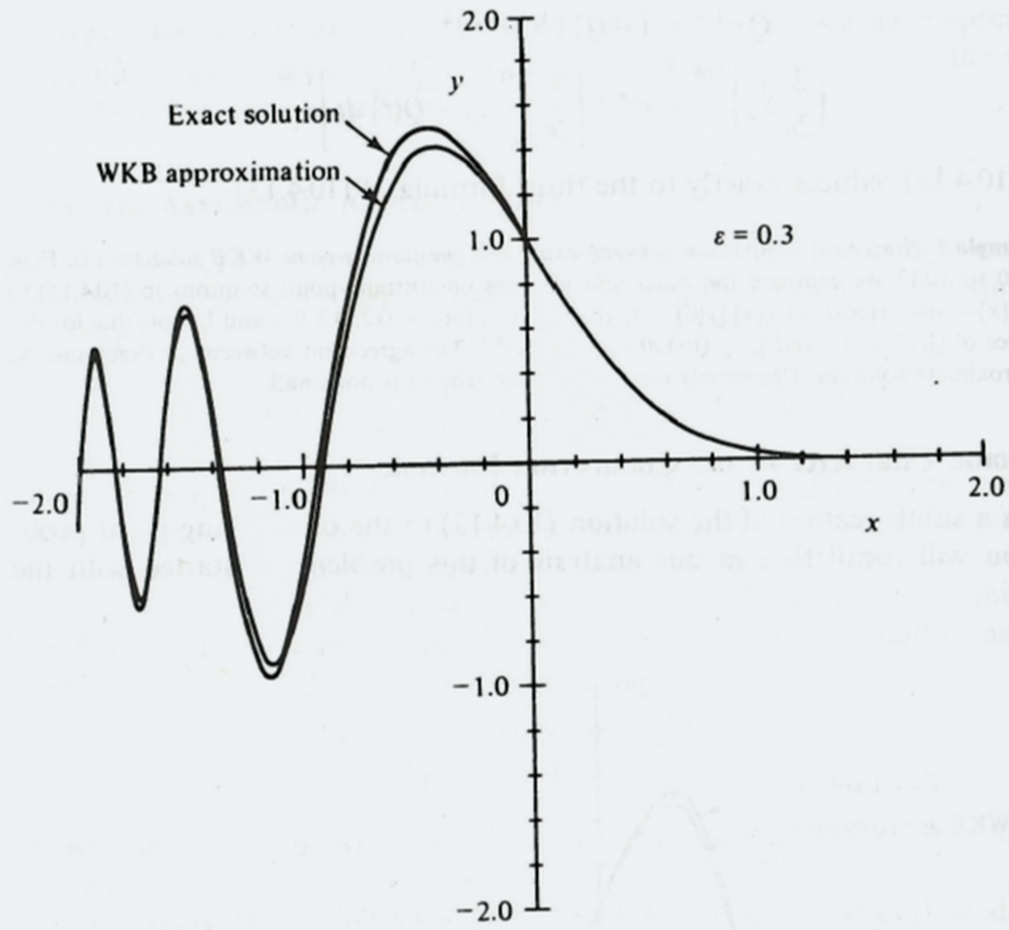


Figure 10.11 Same as in Fig. 10.10 except that $\varepsilon = 0.3$.

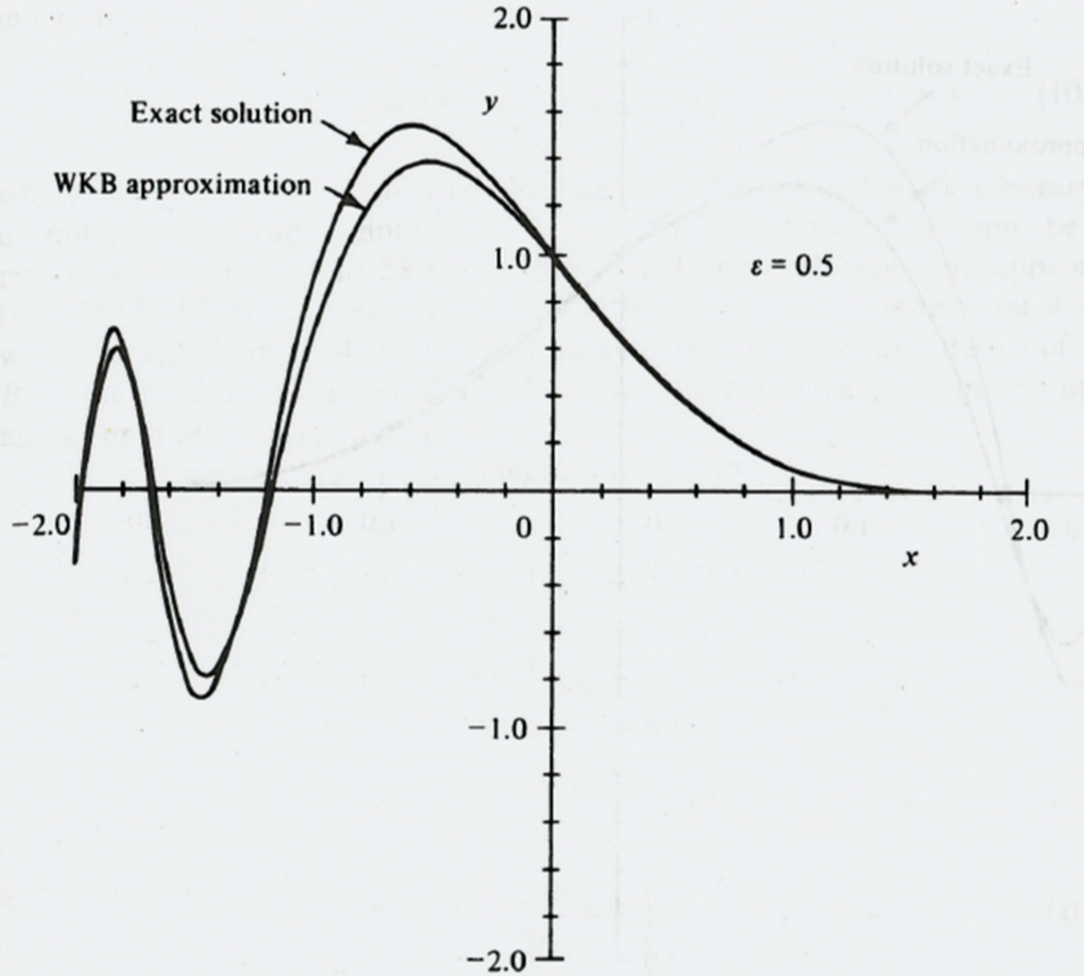


Figure 10.12 Same as in Fig. 10.10 except that $\varepsilon = 0.5$.

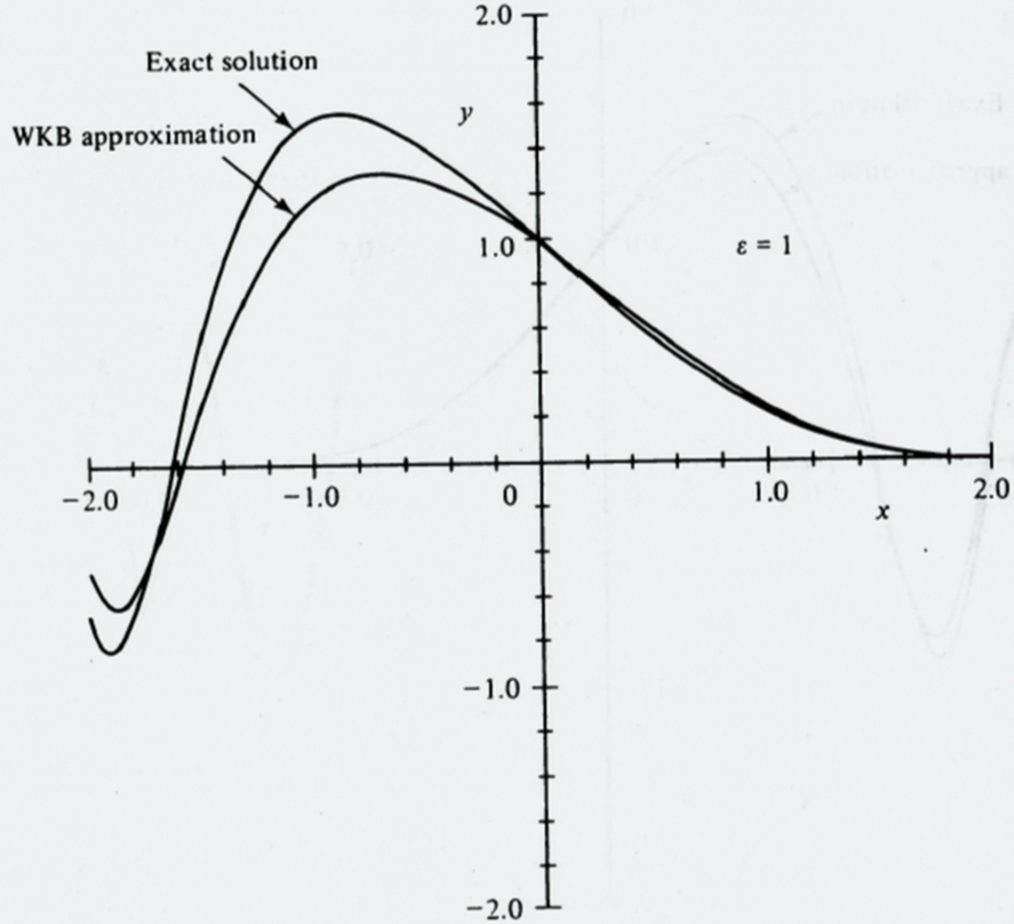


Figure 10.13 Same as in Fig. 10.10 except that $\varepsilon = 1.0$. Even for this large value of ε the agreement between the approximate and exact solutions is impressive.

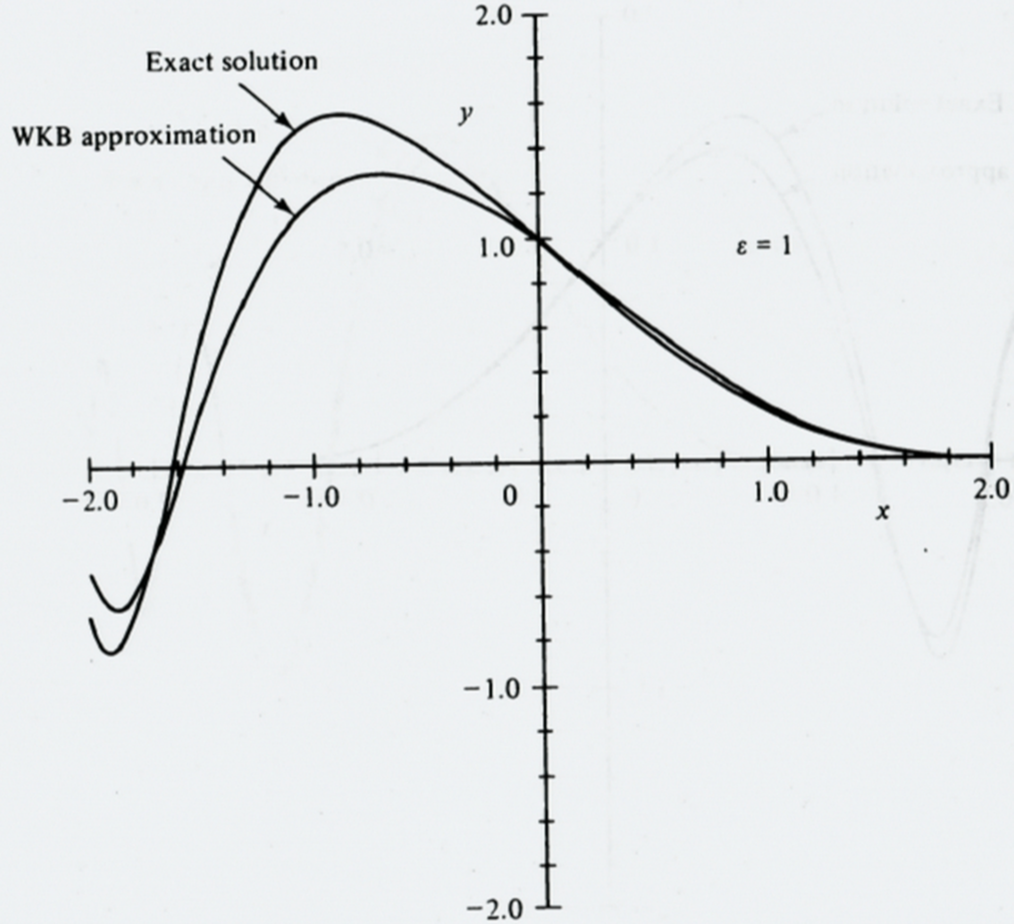
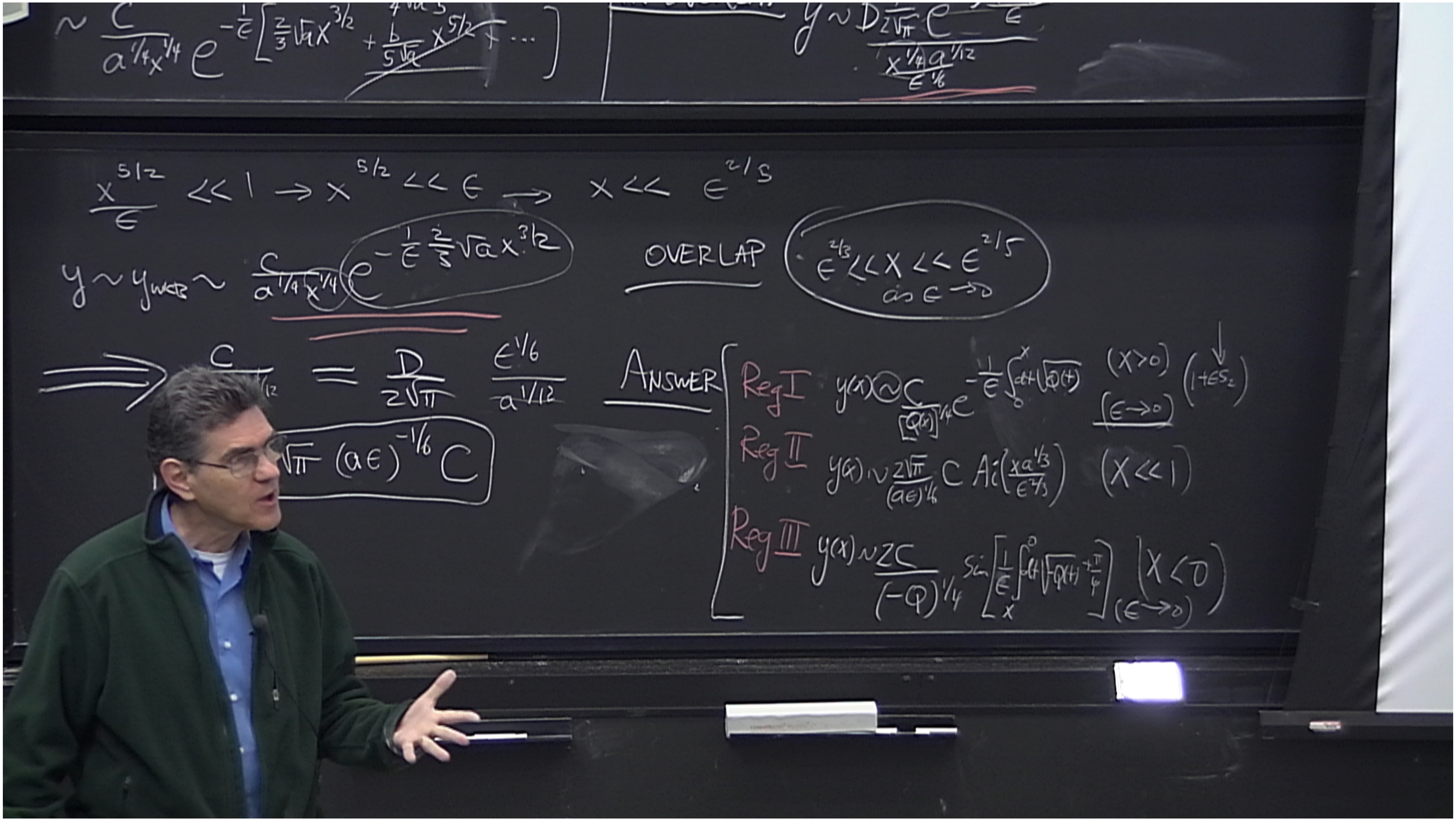
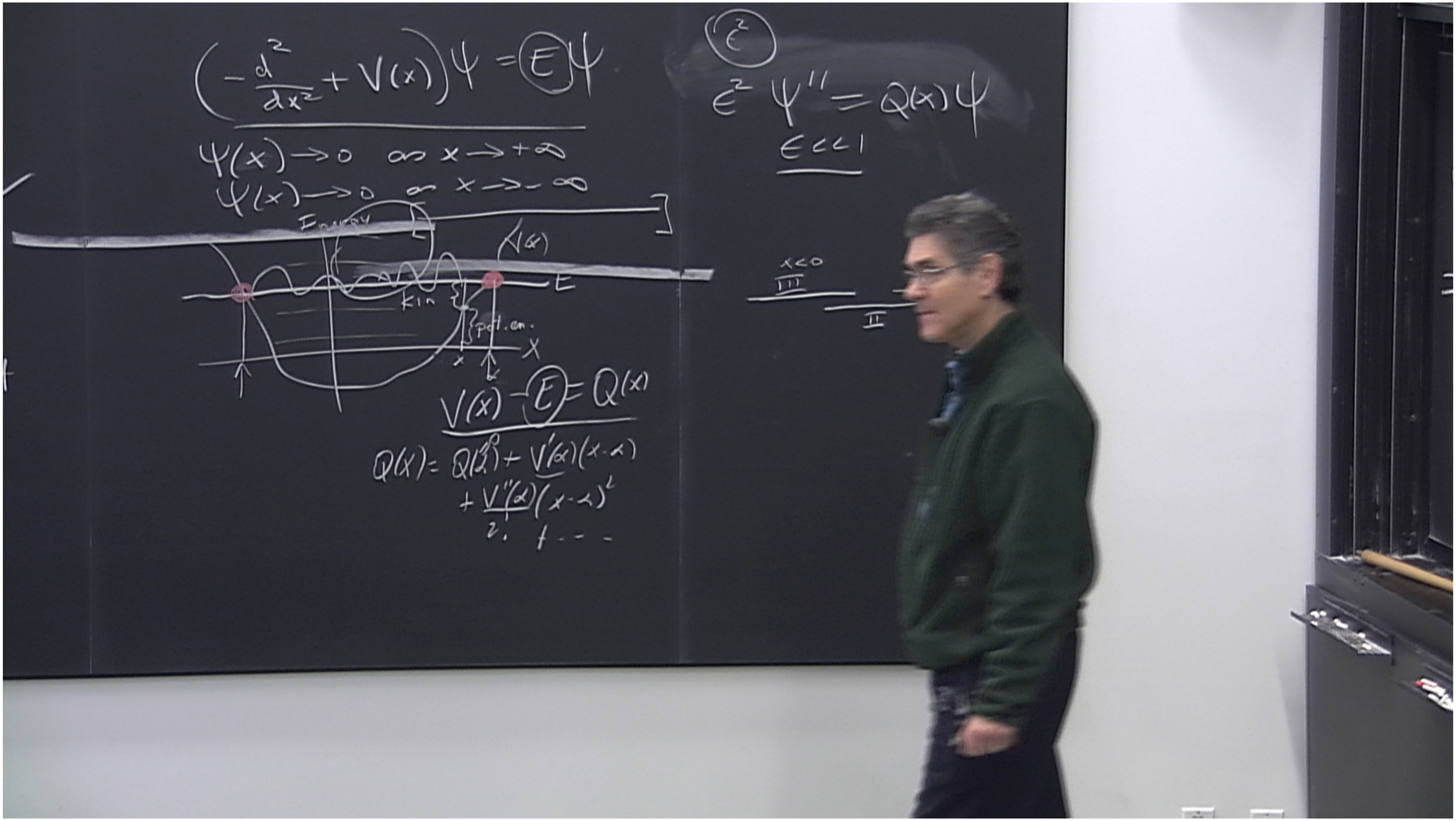


Figure 10.13 Same as in Fig. 10.10 except that $\varepsilon = 1.0$. Even for this large value of ε the agreement between the approximate and exact solutions is impressive.


OVERLAP
ANSWER
Reg I
Reg II
Reg III



$\left(-\frac{d^2}{dx^2}+V(x)\right)\psi = E\psi$
$\psi(x)\to 0$ as $x\to+\infty$
$\psi(x)\to 0$ as $x\to-\infty$
Energy
E
kin
pot. en.
X
$V(x)-E=Q(x)$
$Q(x)=Q(\alpha)+V'(\alpha)(x-\alpha)+\frac{V''(\alpha)}{2!}(x-\alpha)^2+\ldots$
ϵ^2
$\epsilon^2\psi''=Q(x)\psi$
$\epsilon\ll 1$
x<0
III
II