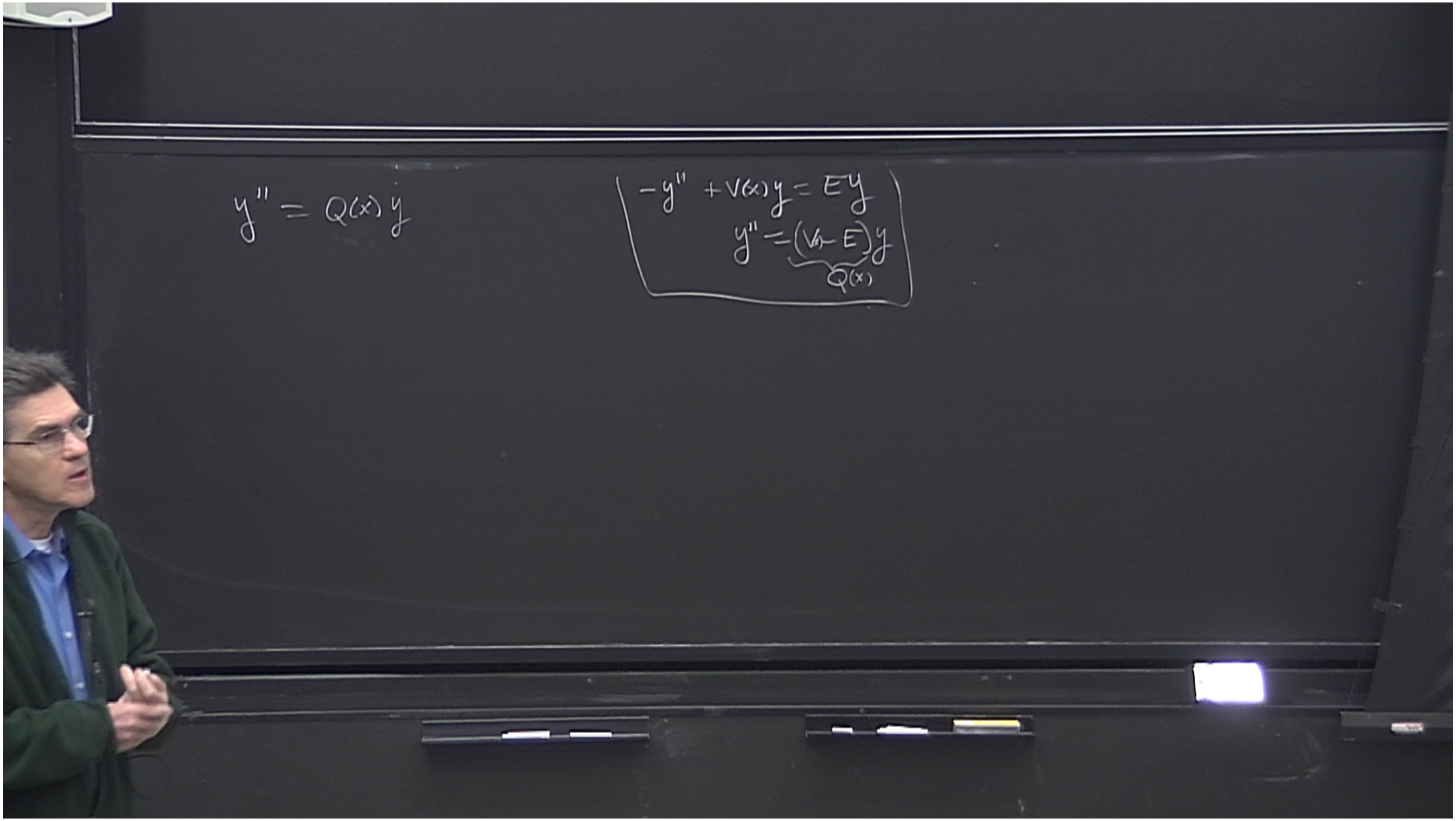
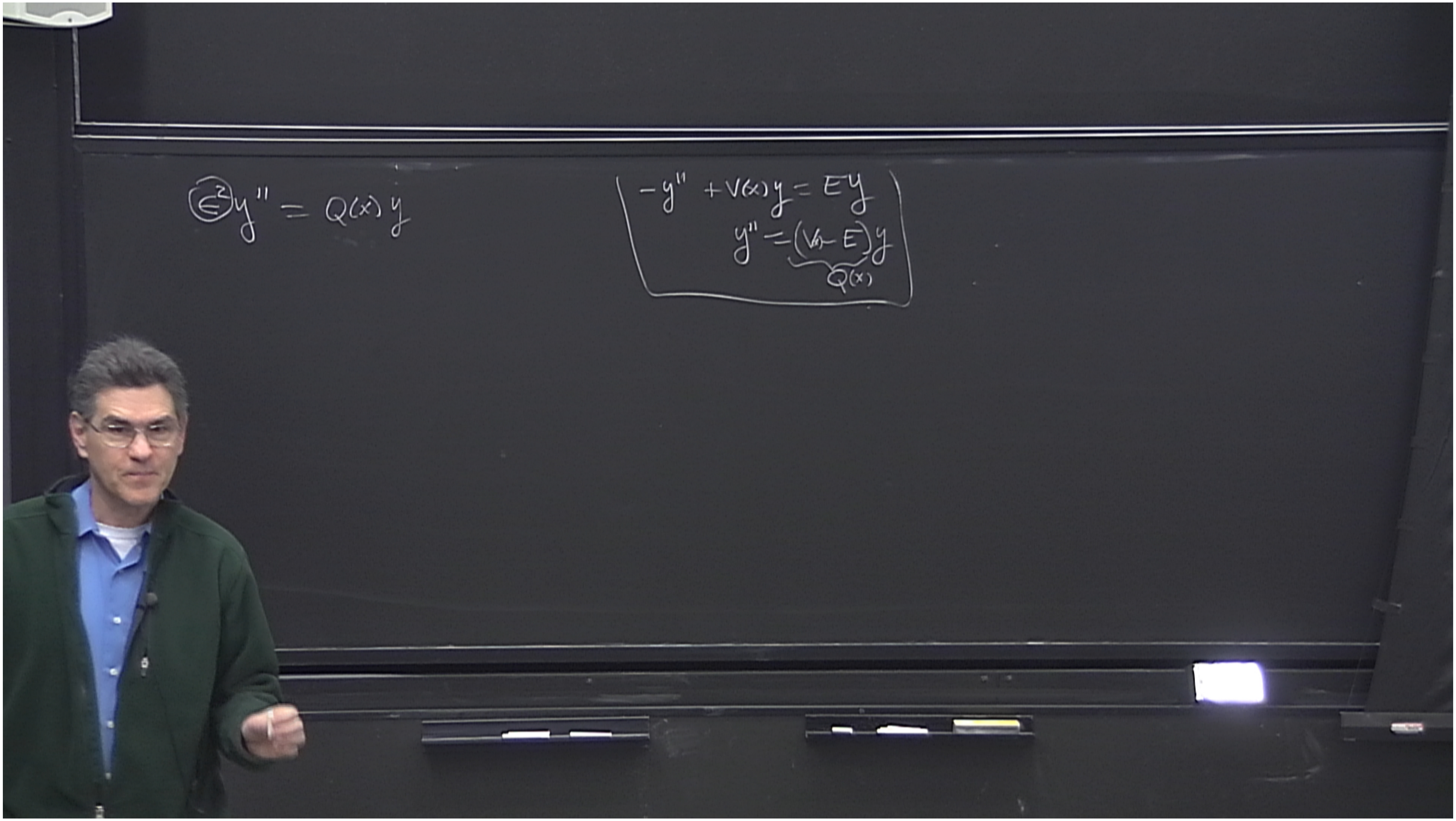
Title: Mathematical Physics - Lecture 13

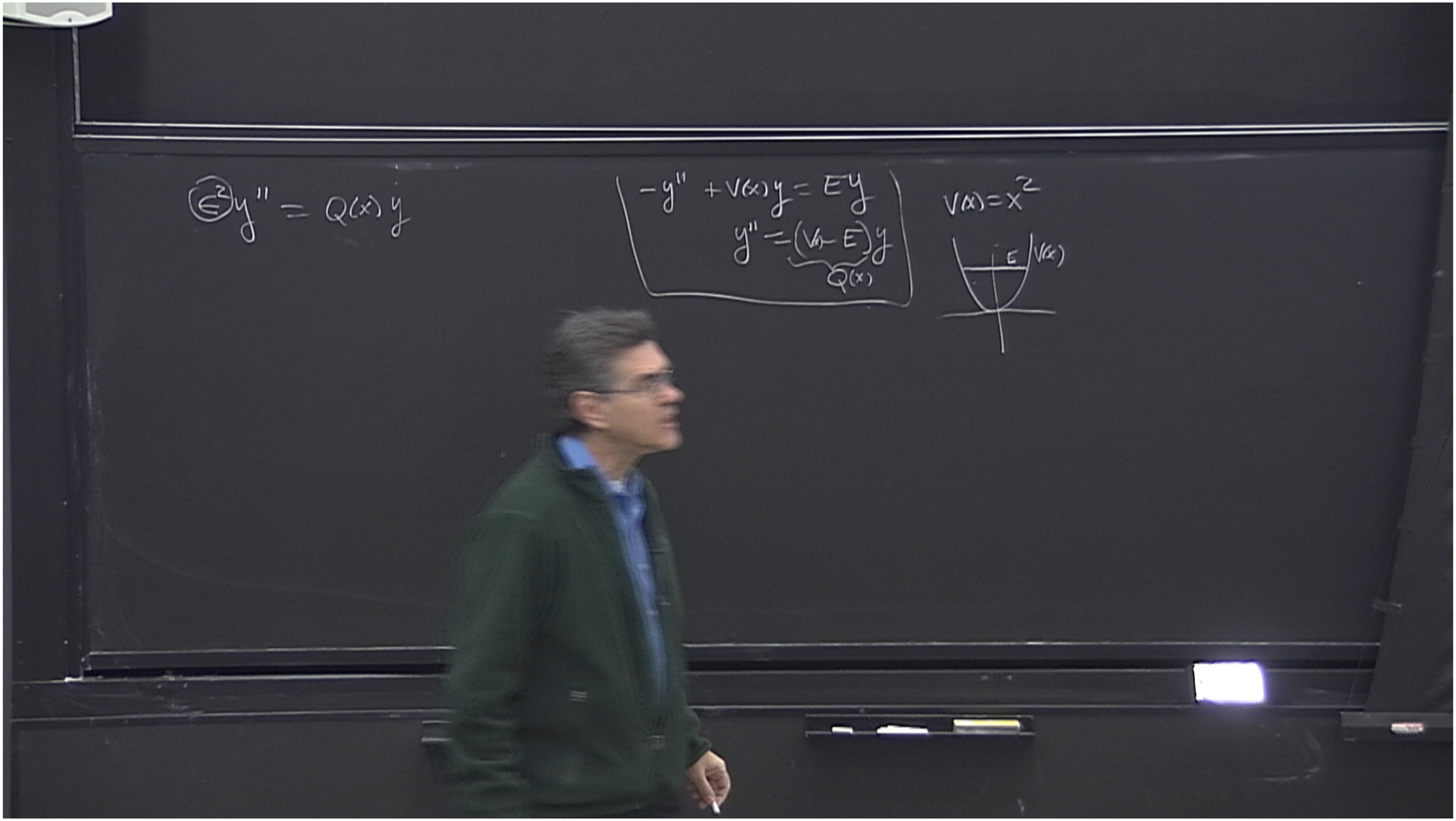
Date: Dec 07, 2011 09:00 AM

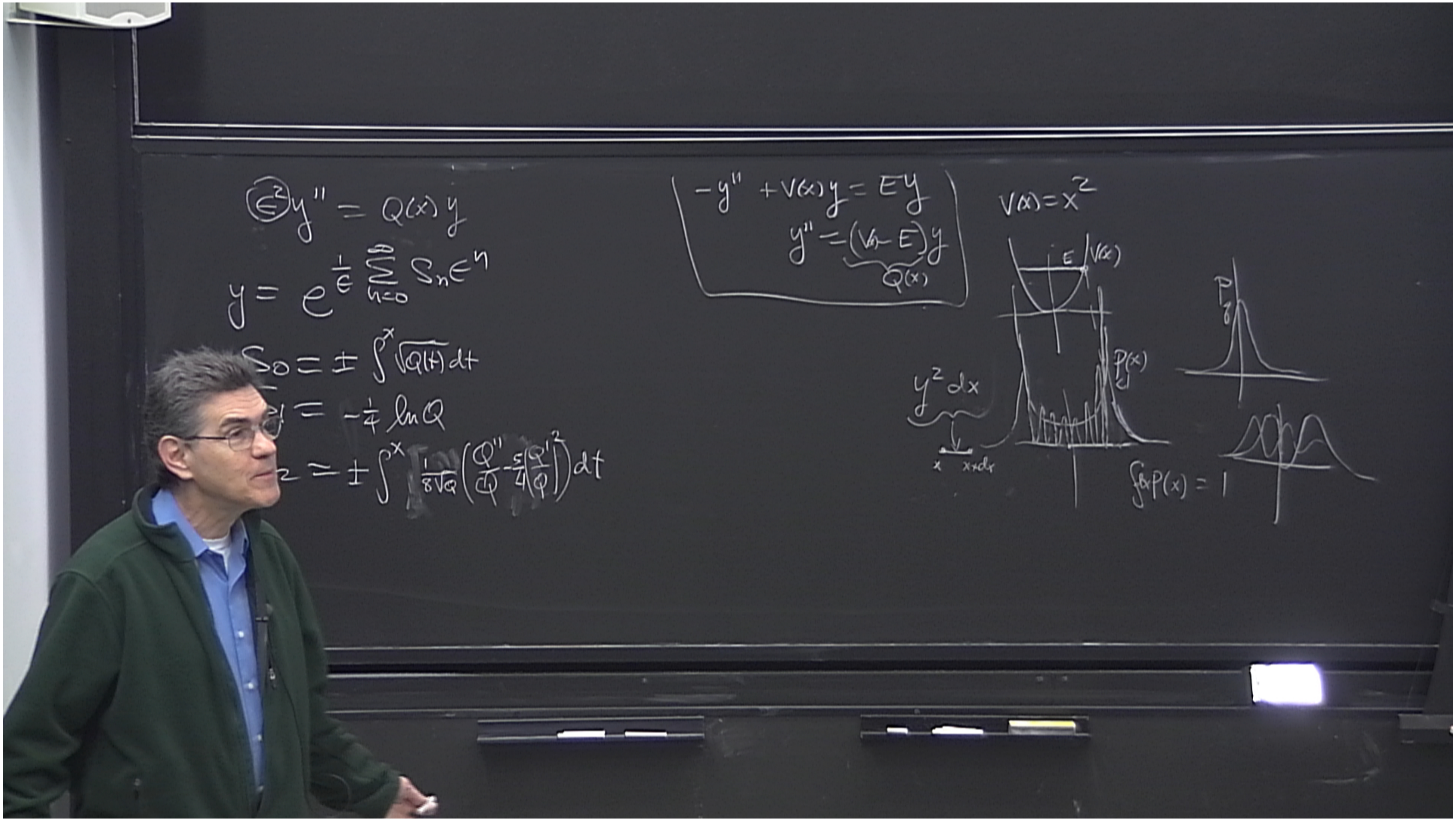
URL: http://pirsa.org/11120007

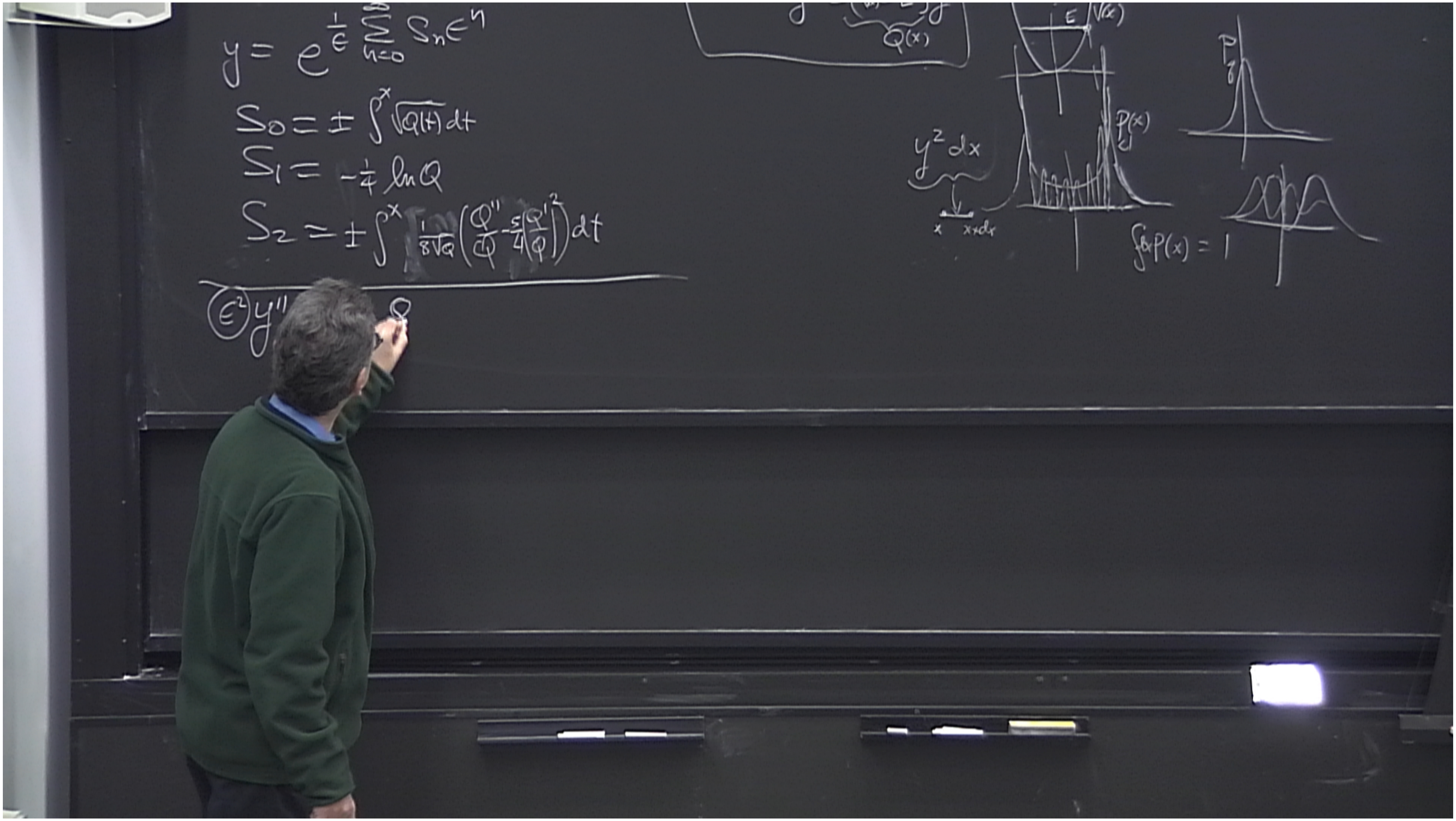
Abstract:

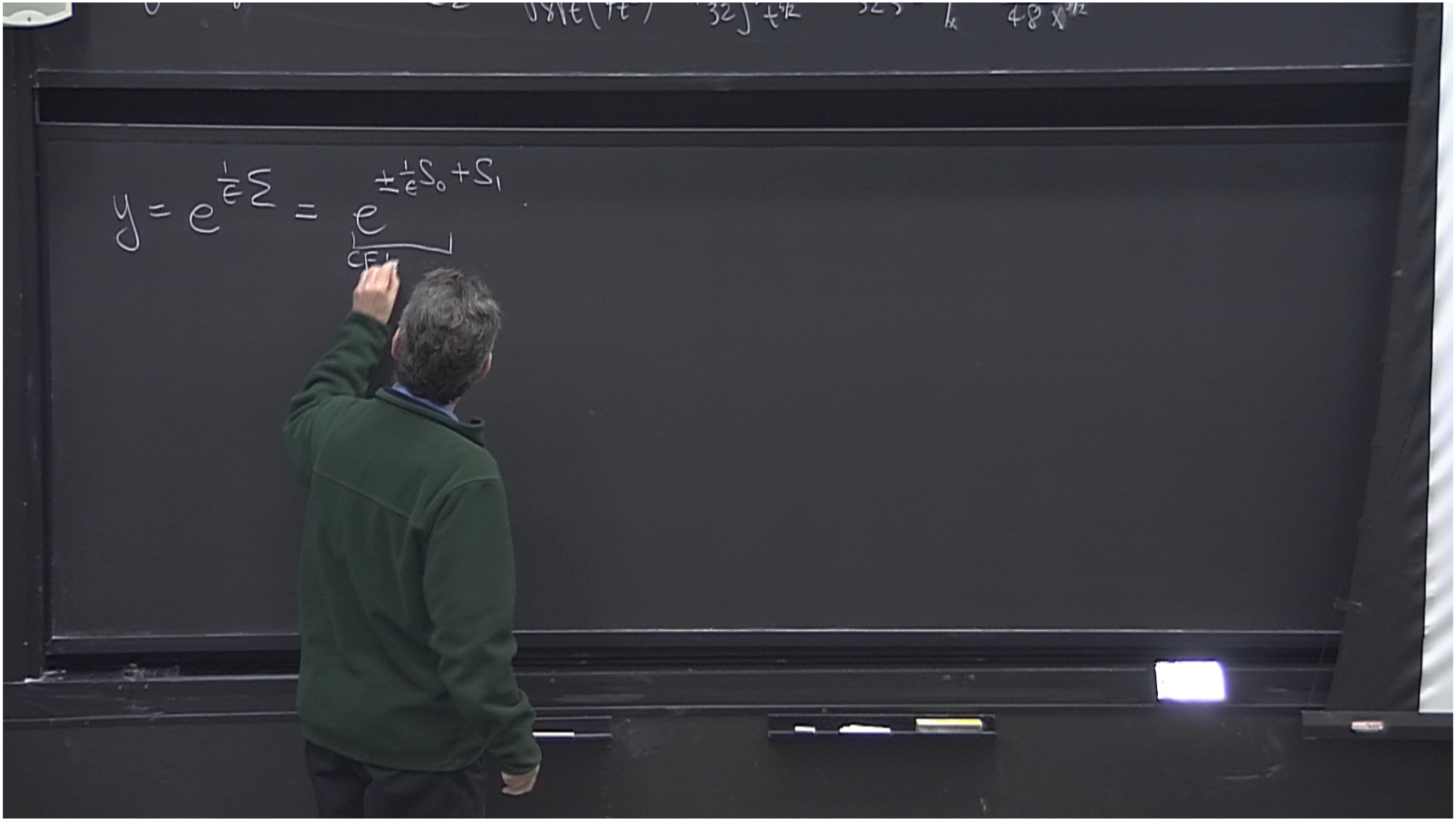

y'' = Q(x) y
-y'' + V(x) y = E y
y'' = (V(x) - E) y
Q(x)

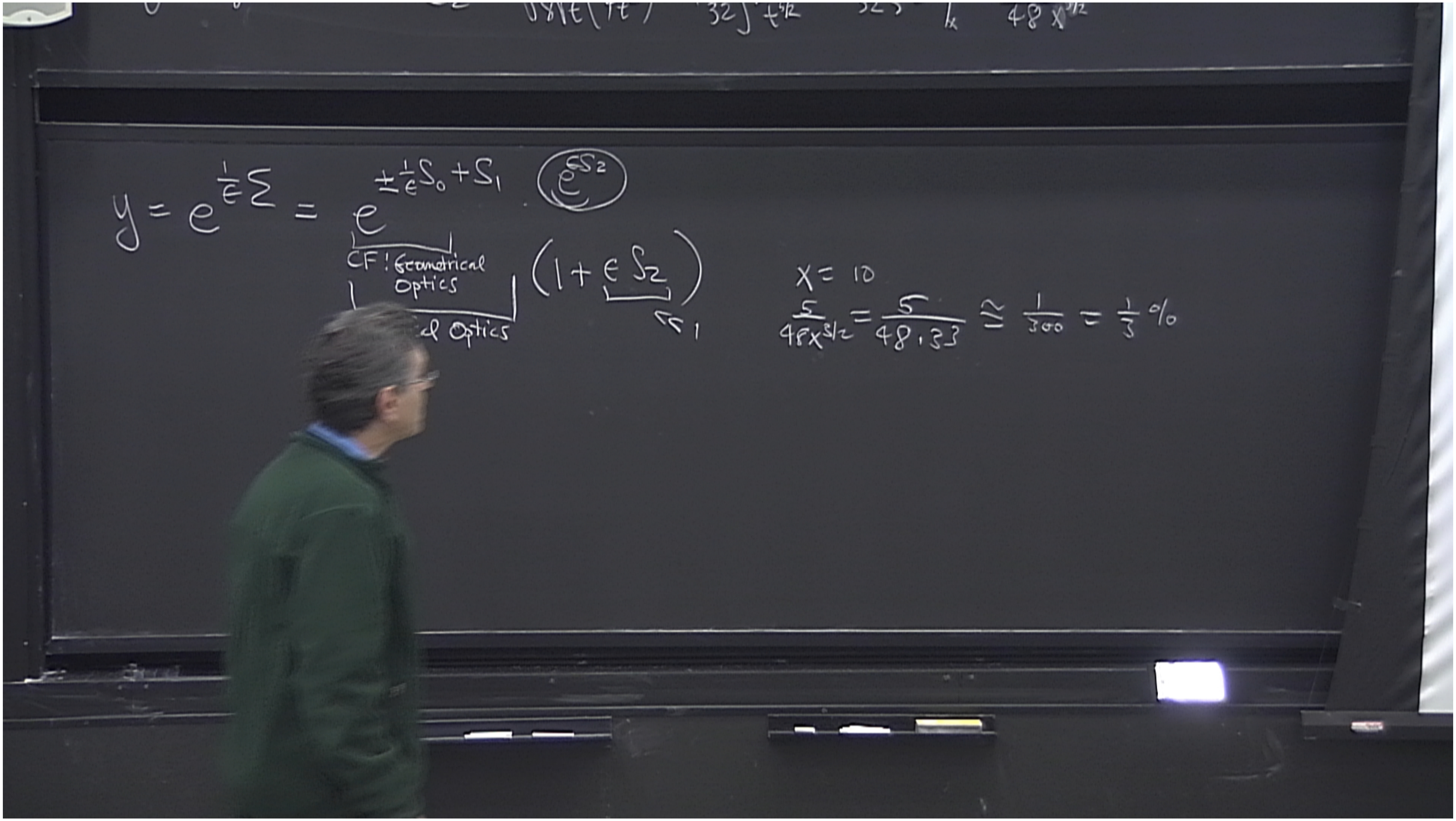

$\epsilon^2 y'' = Q(x)\, y$
$-y'' + V(x)y = Ey$
$y'' = (V(x) - E)y$
$Q(x)$

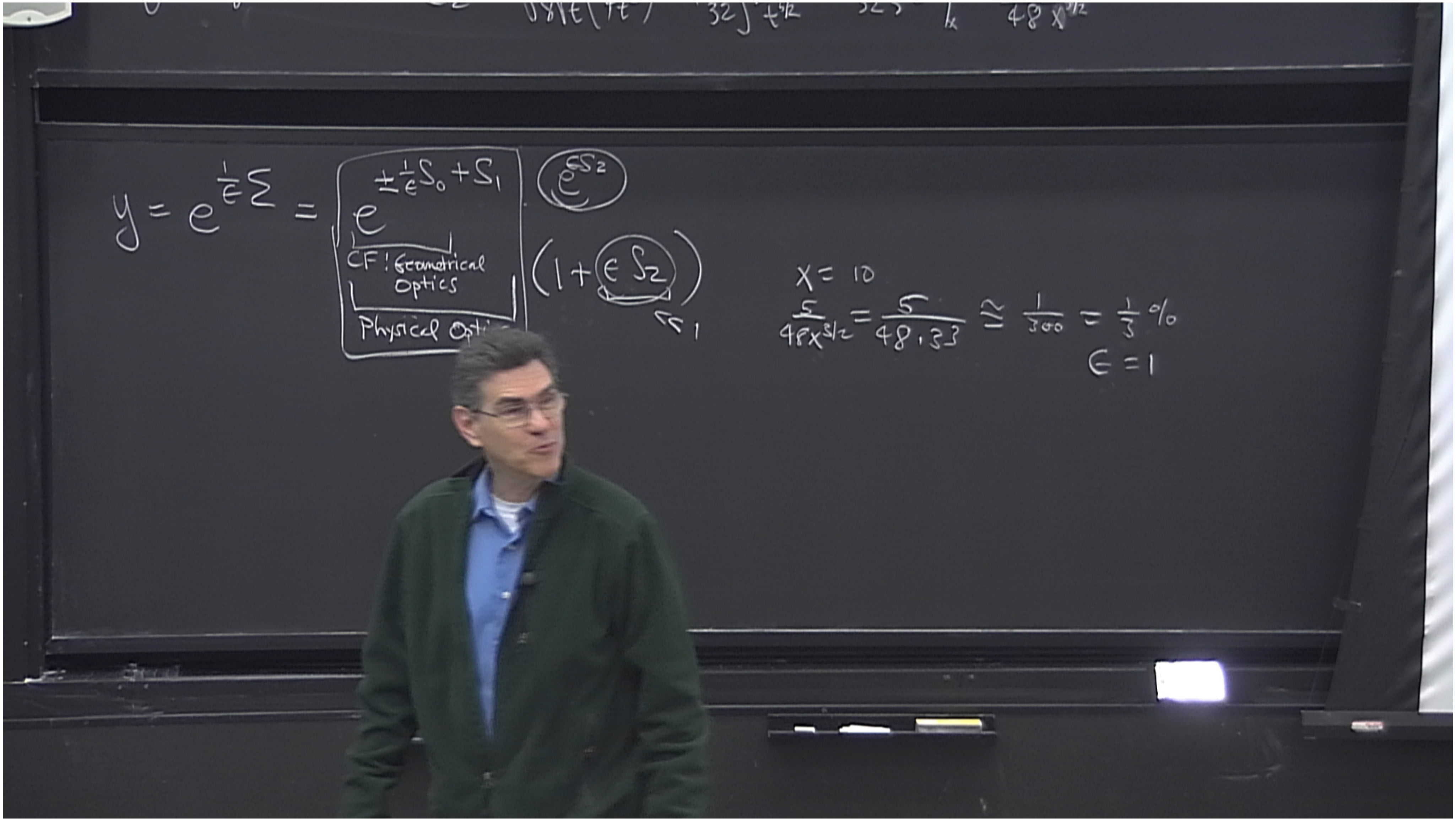

ε²y'' = Q(x) y
-y'' + V(x) y = E y
y'' = (V(x) - E) y
Q(x)
V(x) = x²
E
V(x)


ε²y'' = Q(x) y
-y'' + V(x) y = E y
V(x) = x²
y² dx
P(x)
∫dx P(x) = 1

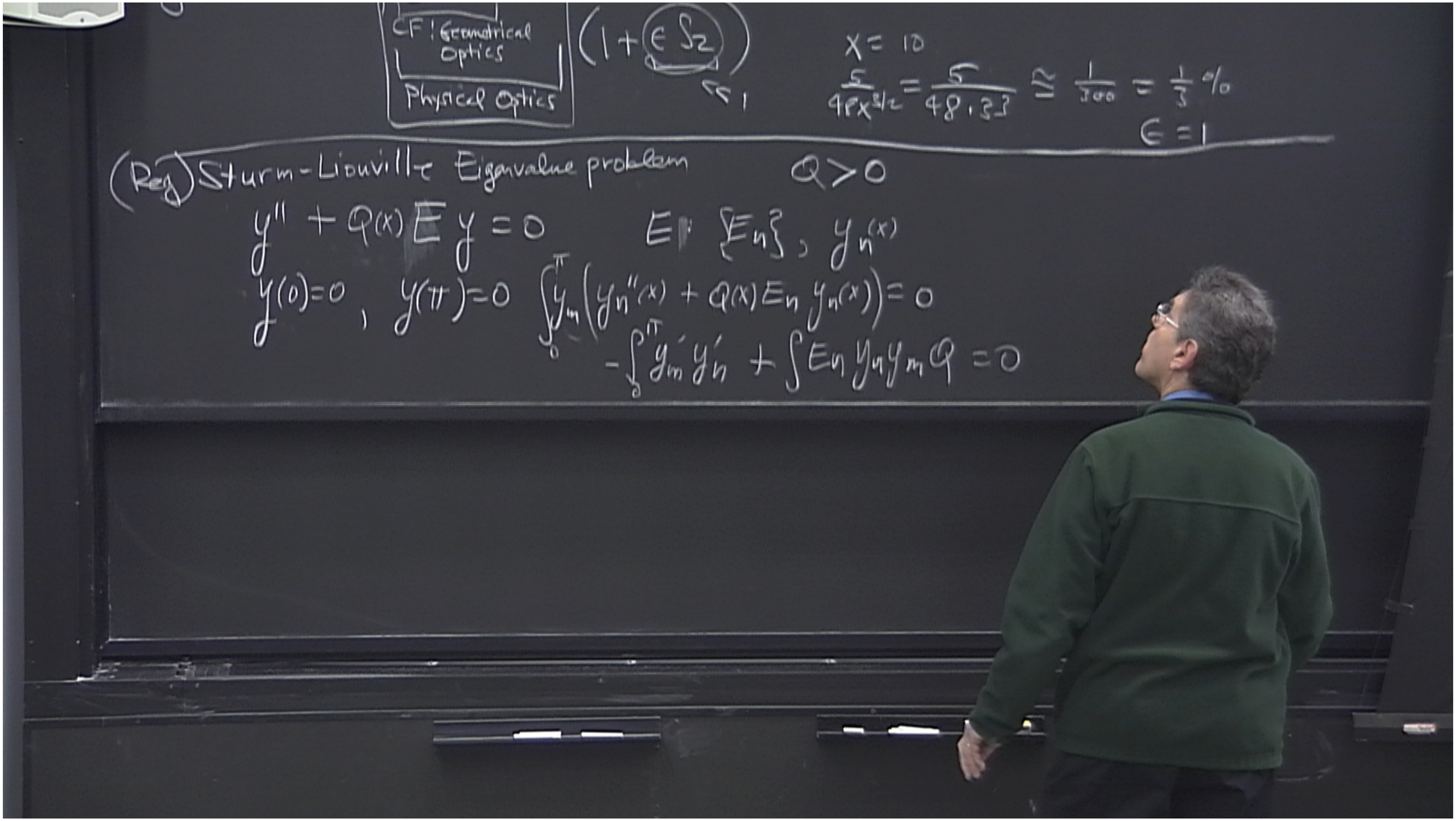

y = e^(1/ε Σ S_n ε^n)
S_0 = ± ∫^x √Q(t) dt
S_1 = −1/4 ln Q
S_2 = ± ∫^x 1/(8√Q) (Q''/Q − 5/4 (Q'/Q)^2) dt
Q(x)
y^2 dx
x
x+dx
P(x)
∫dx P(x) = 1

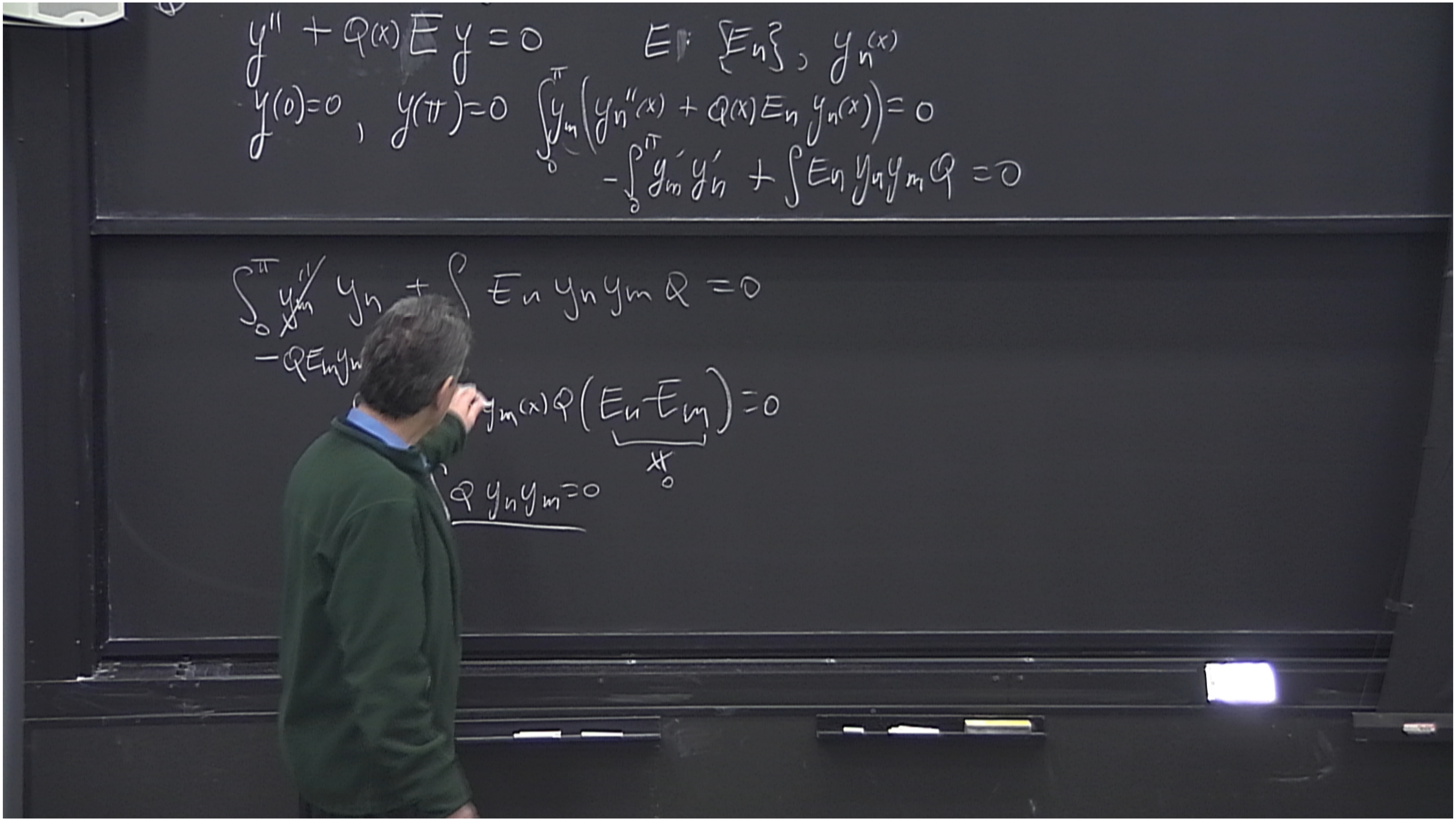

$y = e^{\frac{1}{\epsilon}\Sigma} = e^{\pm\frac{1}{\epsilon}S_0 + S_1}$

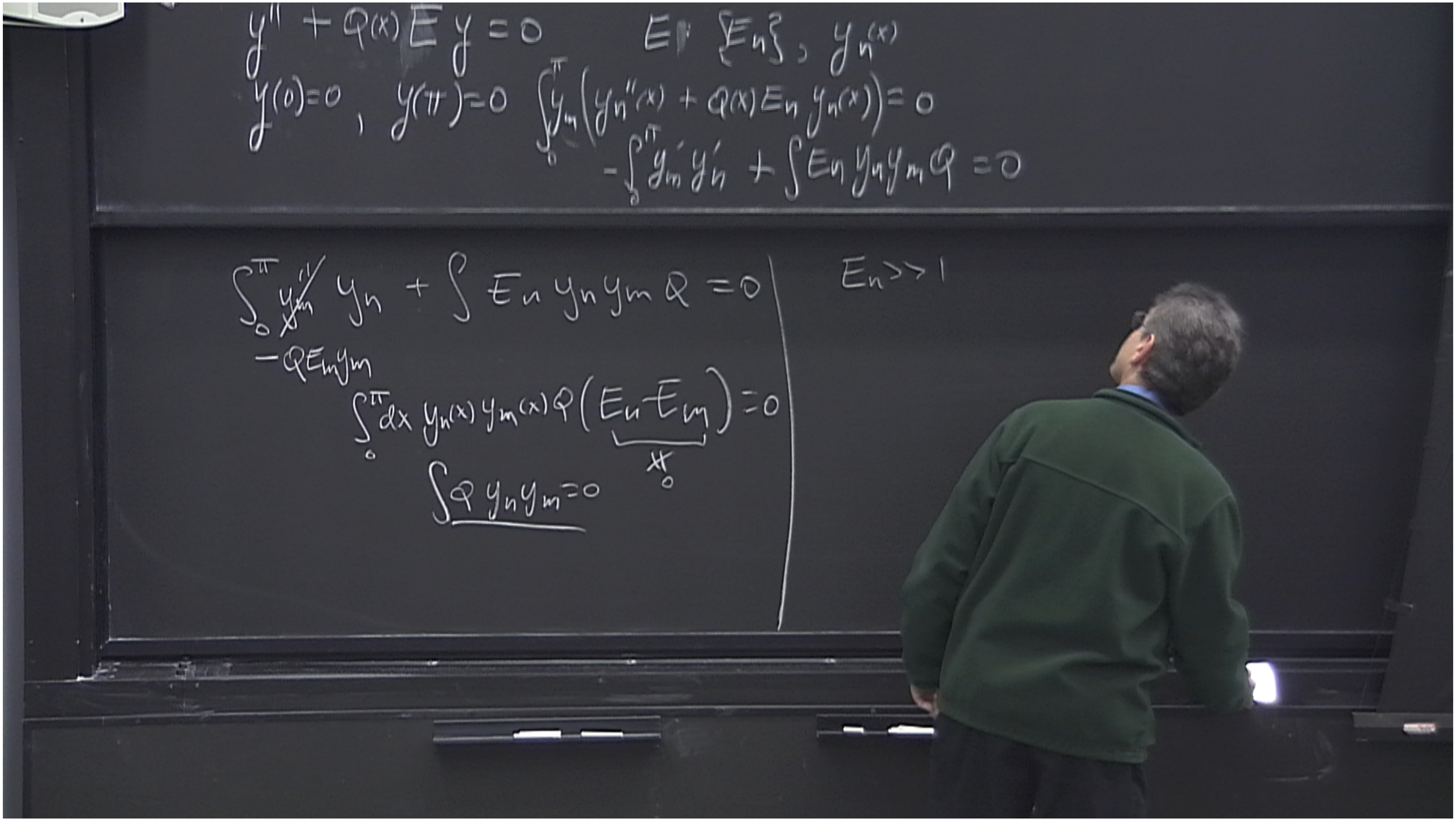

y = e^{\frac{1}{\epsilon}\Sigma} = e^{\pm\frac{1}{\epsilon}S_0 + S_1} e^{\epsilon S_2}
CF! Geometrical Optics
Optics
(1 + \epsilon S_2)
<< 1
x = 10
\frac{5}{48x^{3/2}} = \frac{5}{48 \cdot 33} \cong \frac{1}{300} = \frac{1}{3}\%

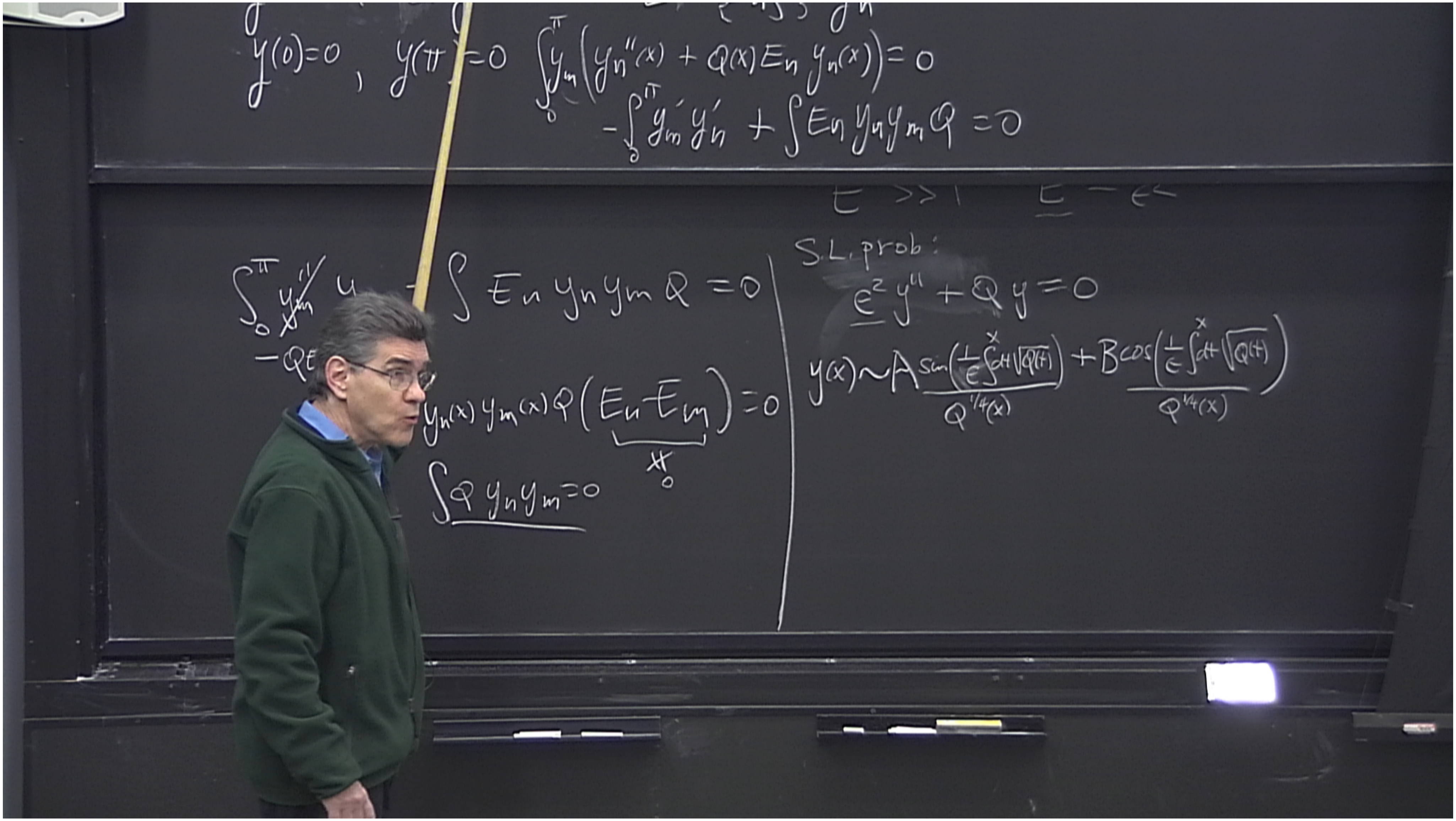


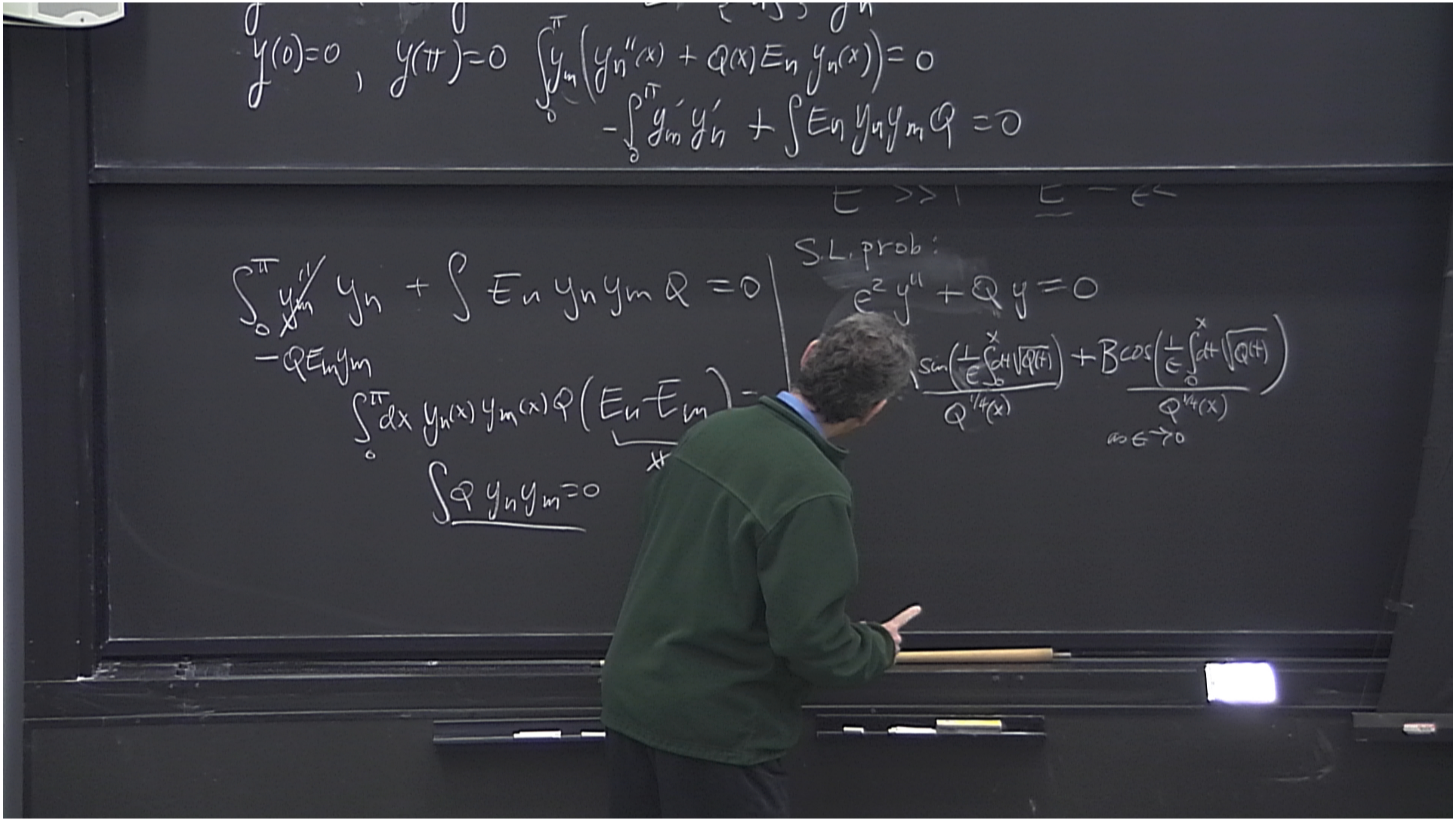
CF! Geometrical optics
Physical Opti
x = 10
ε = 1

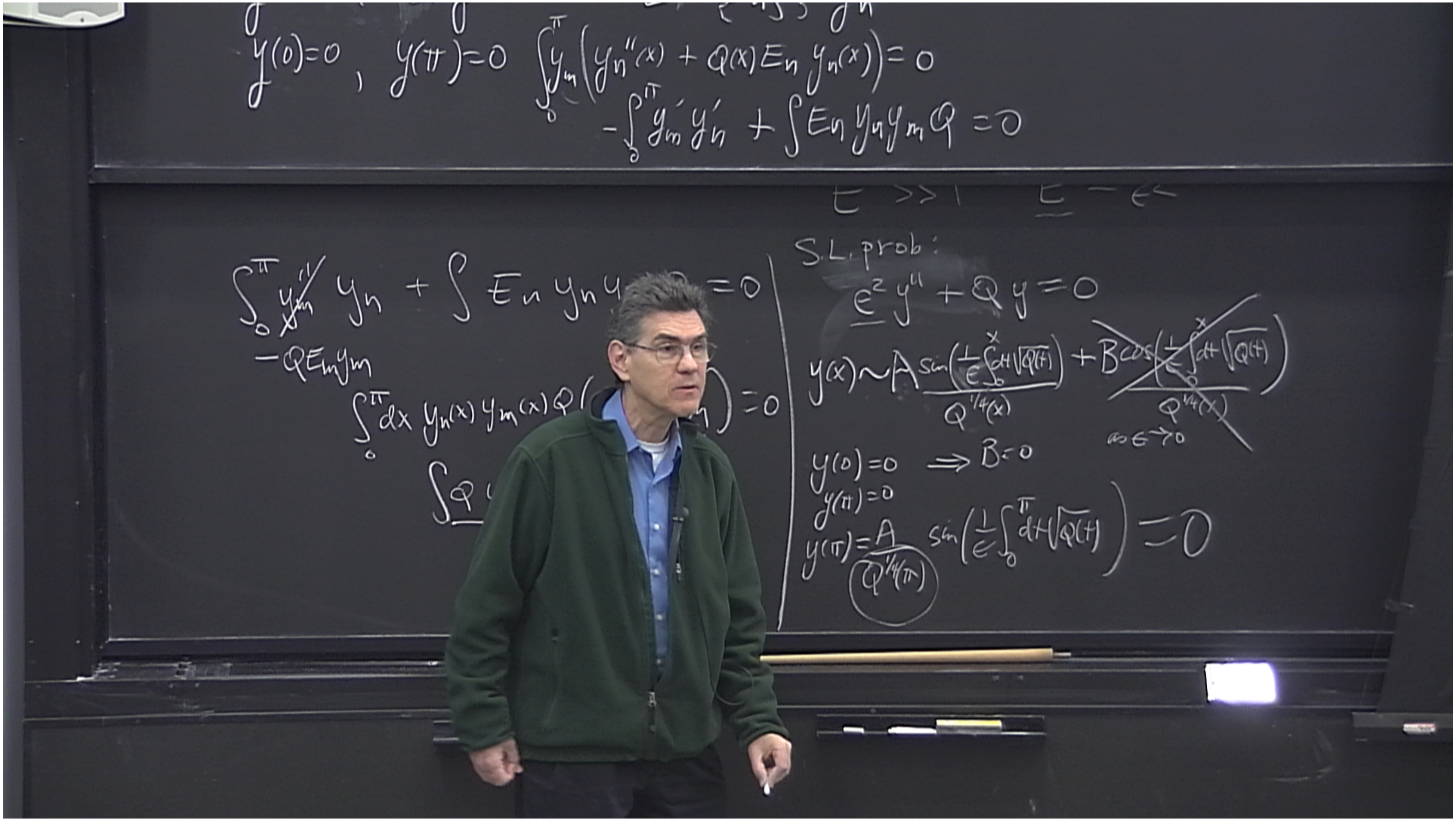

CF: Geometrical optics
Physical Optics
(1 + ε S2)
<< 1
x = 10
5/(48x^{3/2}) = 5/(48·33) ≅ 1/300 = 1/3 %
ε = 1
(Reg) Sturm-Liouville Eigenvalue problem
Q > 0
y'' + Q(x) E y = 0
E: {En}, yn(x)
y(0) = 0 , y(π) = 0
∫ ym (yn''(x) + Q(x) En yn(x)) = 0
-∫ ym' yn' + ∫ En yn ym Q = 0

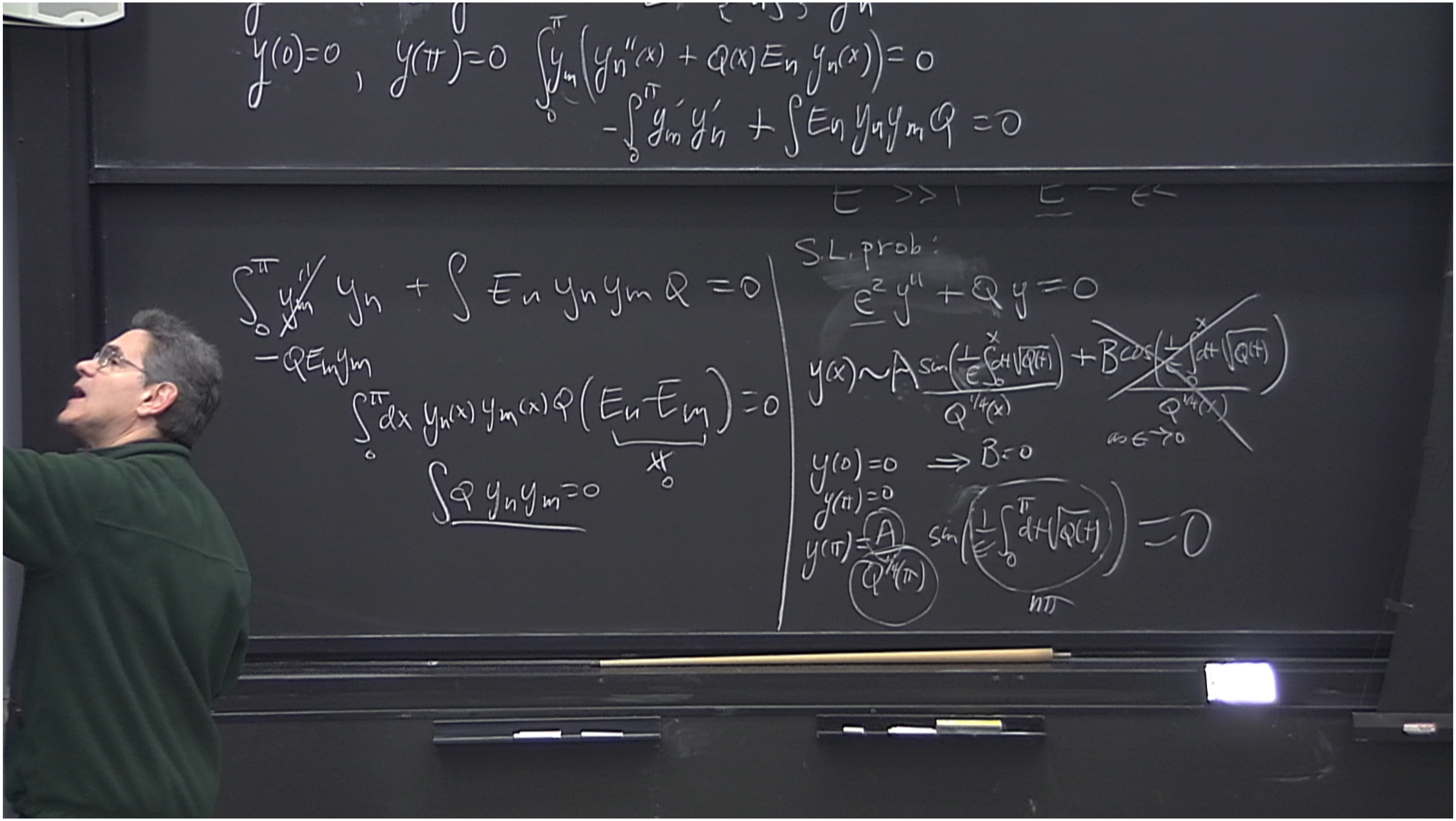

y'' + Q(x) E y = 0
E : {E_n}, y_n(x)
y(0)=0 , y(π)=0
∫_0^π y_m (y_n''(x) + Q(x) E_n y_n(x)) = 0
-∫_0^π y_m' y_n' + ∫ E_n y_n y_m Q = 0
∫_0^π y_m'' y_n + ∫ E_n y_n y_m Q = 0
-QE_m y_m
y_m(x) Q (E_n - E_m) = 0
≠ 0
∫ Q y_n y_m = 0


$y'' + Q(x) E y = 0$
$E: \{E_n\}, y_n(x)$
$y(0)=0, y(\pi)=0$
$\int_0^\pi y_m (y_n''(x) + Q(x) E_n y_n(x)) = 0$
$-\int_0^\pi y_m' y_n' + \int E_n y_n y_m Q = 0$
$\int_0^\pi y_m'' y_n + \int E_n y_n y_m Q = 0$
$-Q E_m y_m$
$\int_0^\pi dx\, y_n(x) y_m(x) Q (E_n - E_m) = 0$
$\neq 0$
$\int Q y_n y_m = 0$
$E_n >> 1$


y(0)=0 , y(π)=0
∫₀^π y_m (y_n''(x) + Q(x) E_n y_n(x)) = 0
-∫₀^π y_m' y_n' + ∫ E_n y_n y_m Q = 0
∫ E_n y_n y_m Q = 0
y_n(x) y_m(x) Q (E_n - E_m) = 0
∫ Q y_n y_m = 0
S.L. prob:
ε² y'' + Q y = 0
y(x) ~ A sin(1/ε ∫^x dt √Q(t)) / Q^{1/4}(x) + B cos(1/ε ∫^x dt √Q(t)) / Q^{1/4}(x)


y(0)=0 , y(π)=0
S.L. prob:
ε²y'' + Qy = 0
∫Q yn ym = 0
as ε→0


y(0)=0 , y(π)=0
S.L. prob:
ε² y'' + Q y = 0
as ε→0
y(0)=0 ⇒ B=0
y(π)=0


y(0)=0 , y(π)=0
S.L. prob:
ε² y'' + Q y = 0
as ε→0
y(0)=0 ⟹ B=0
y(π)=0
=0
nπ

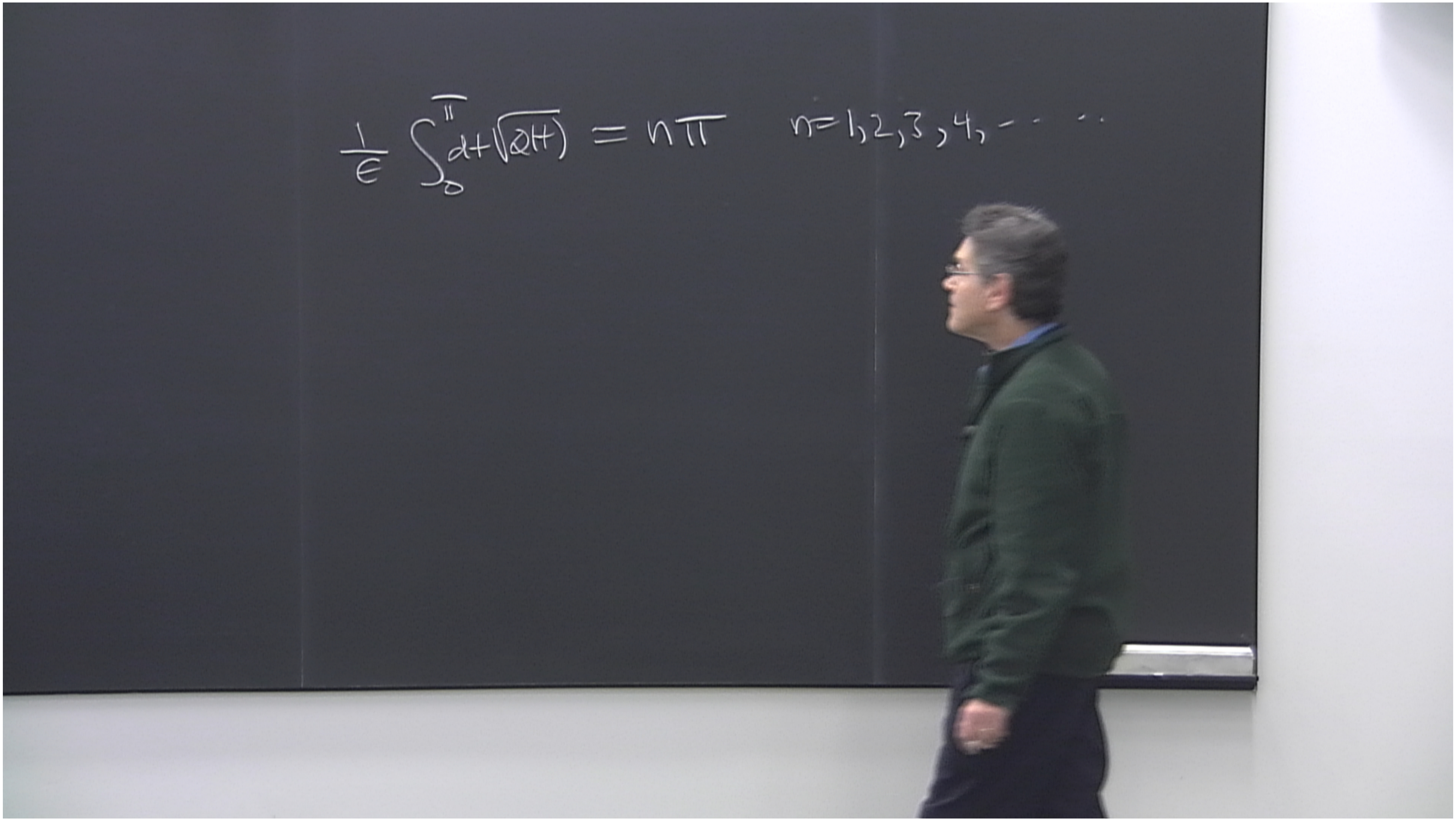

$\frac{1}{\epsilon} \int_0^{\pi} dt \sqrt{Q(t)} = n\pi \quad n = 1, 2, 3, 4, \ldots$

Table 10.1 A comparison of the exact eigenvalues E_n of the Sturm-Liouville problem $y''(x) + E(x+\pi)^4 y(x) = 0$ $[y(0) = y(\pi) = 0]$ with the leading-order WKB prediction [see (10.1.34)] for these eigenvalues $E_n \sim 9n^2/49\pi^2$ $(n \to \infty)$

As expected, this prediction becomes more accurate as n increases. The relative error is defined as (approximate − exact)/(exact)

n	E_n(WKB)	E_n(exact)	Relative error, %
1	0.00188559	0.00174401	8.1
2	0.00754235	0.00734865	2.6
3	0.0169703	0.0167524	1.3
4	0.0301694	0.0299383	0.77
5	0.0471397	0.0469006	0.51
10	0.188559	0.188305	0.13
20	0.754235	0.753977	0.035
40	3.01694	3.01668	0.009

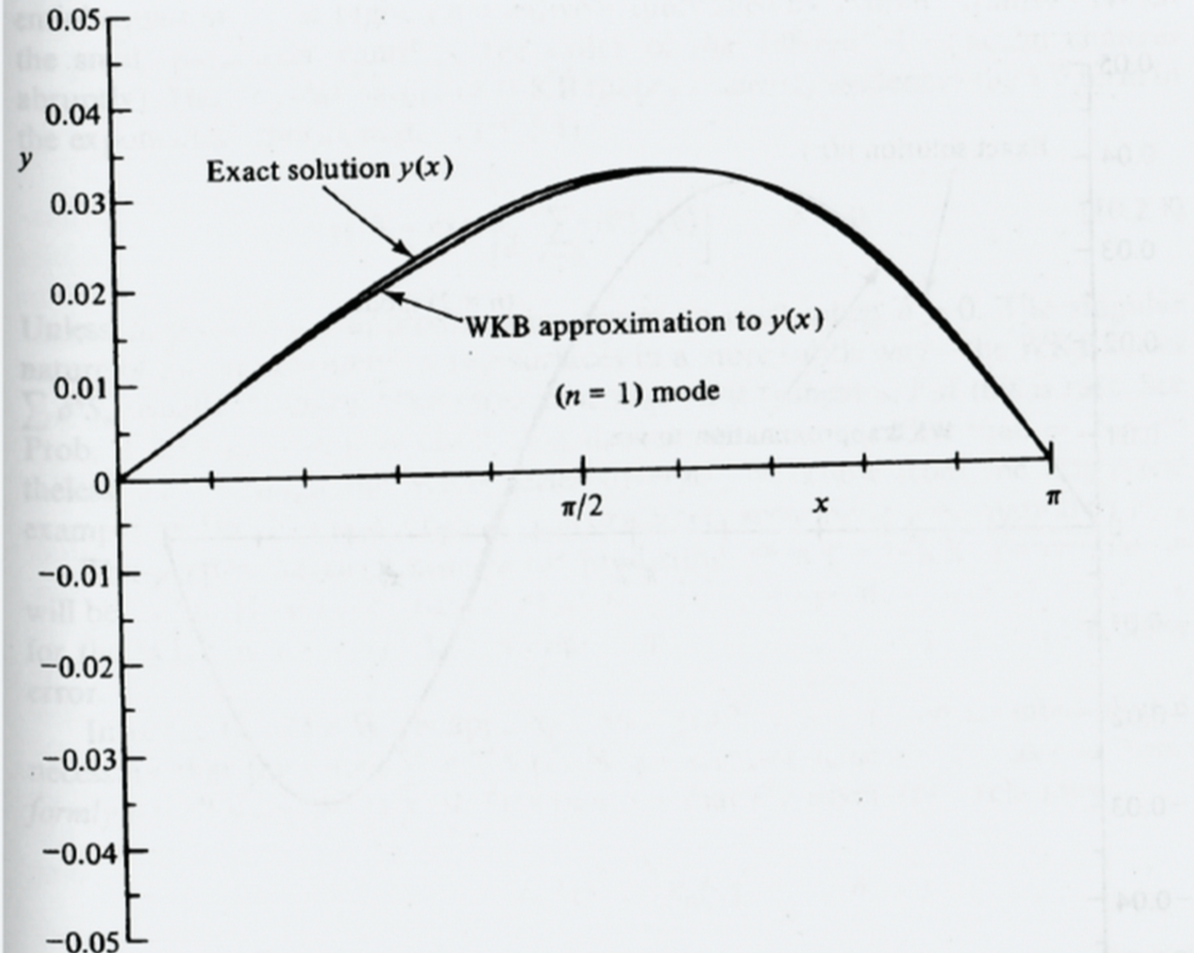
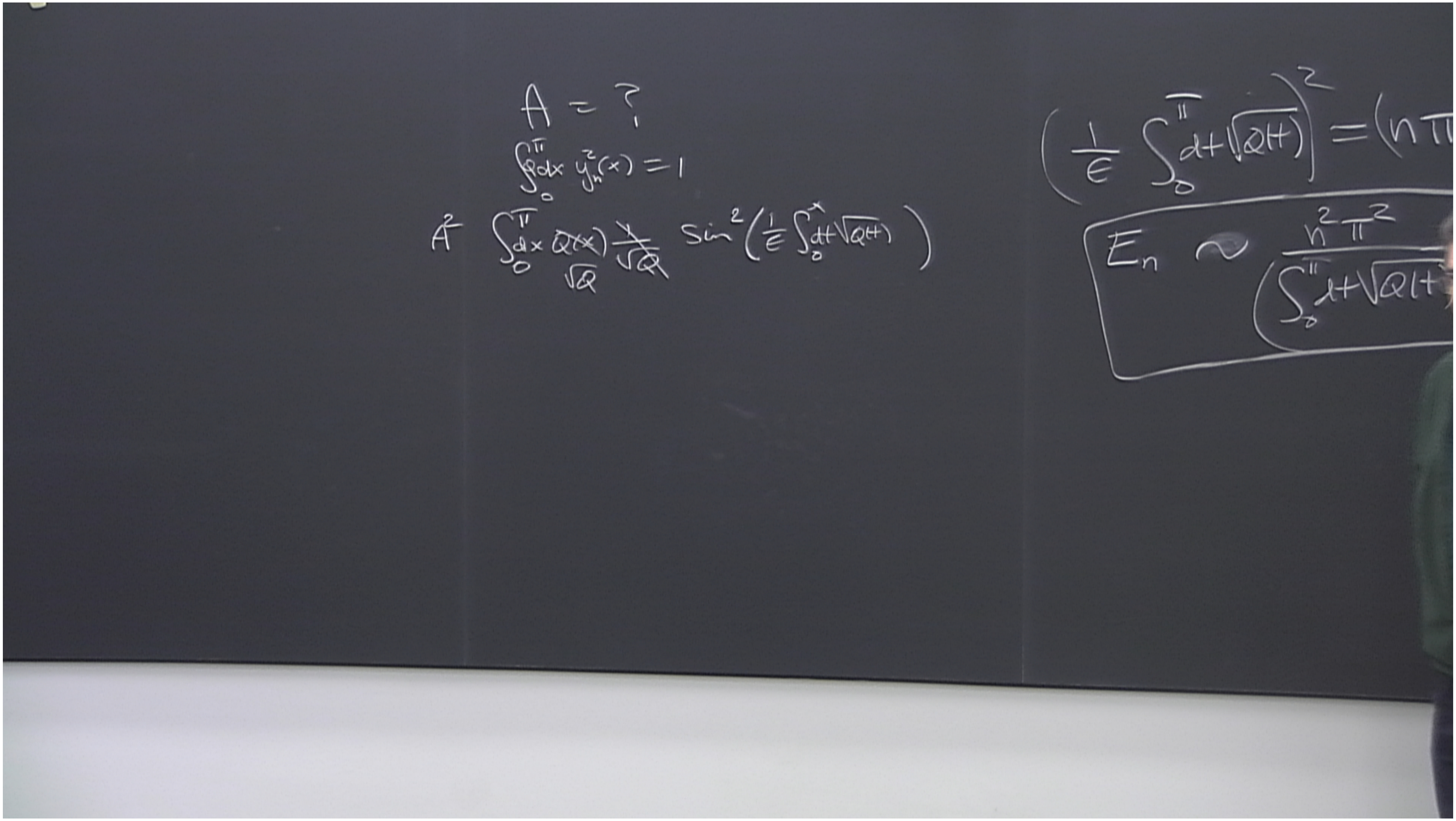
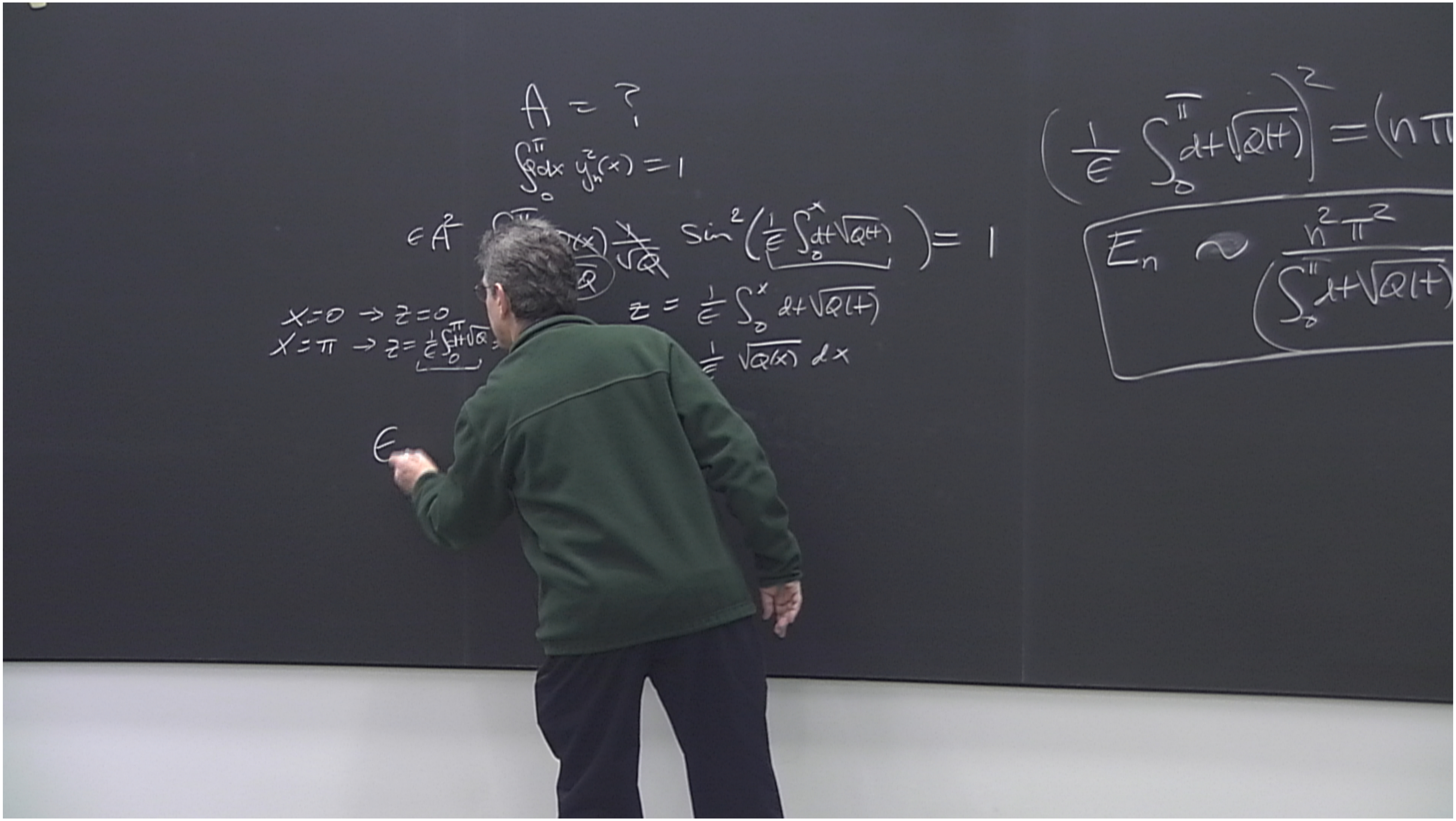
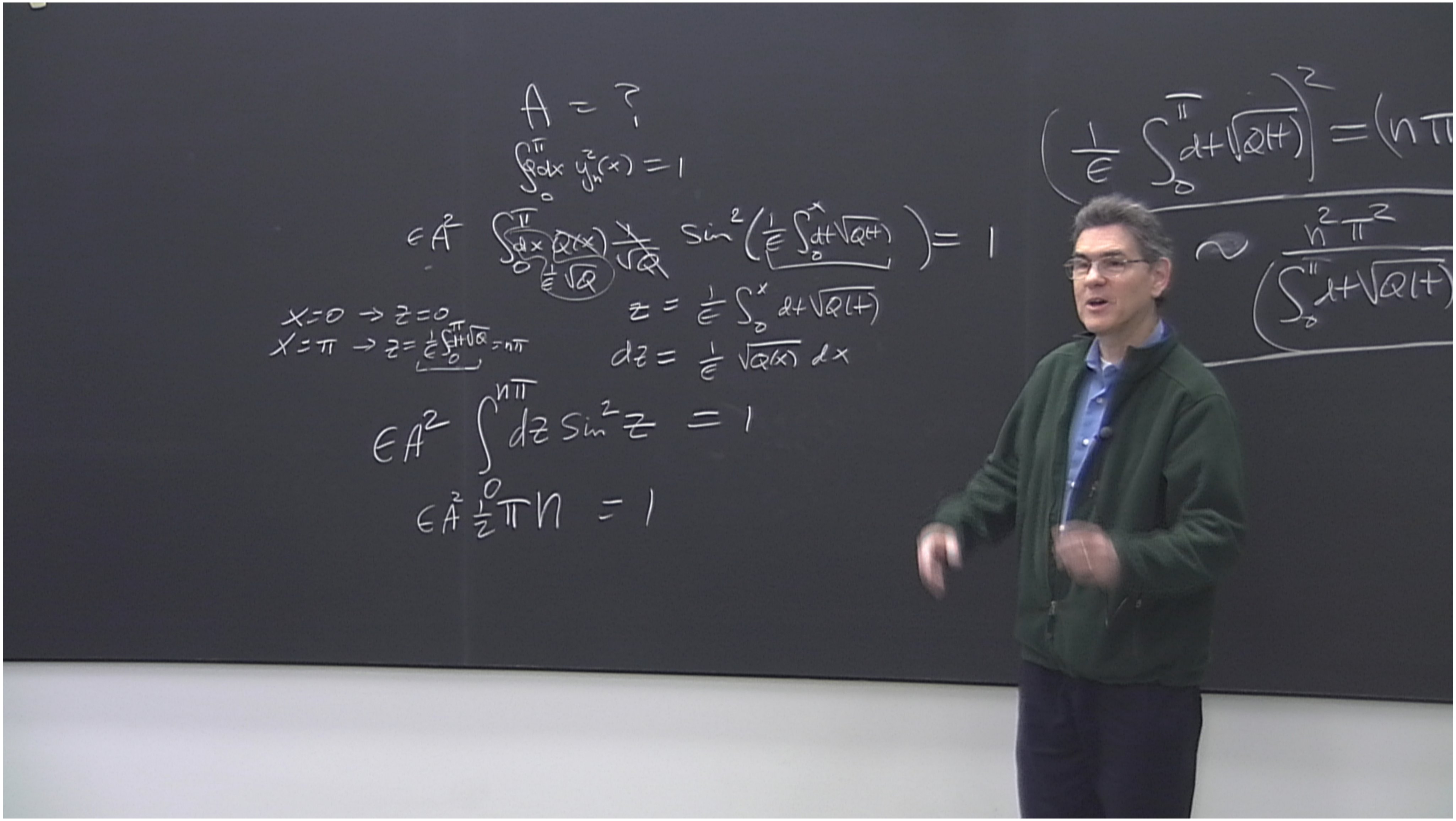


Figure 10.2 Comparison of the exact solution to $y''(x) + E_n(x+\pi)^4 y(x) = 0$ $[y(0) = y(\pi) = 0]$, with the WKB approximation to this solution as given in (10.1.35) for the lowest $(n = 1)$ mode. Although WKB becomes exact as $n \to \infty$, this plot shows that even when $n = 1$ the WKB approximation is extraordinarily accurate.


$A = ?$
$\int_0^\pi dx\, y_n^2(x) = 1$
$A^2 \int_0^\pi dx\, Q(x) \frac{1}{\sqrt{Q}} \frac{1}{\sqrt{Q}} \sin^2\left(\frac{1}{\epsilon}\int_0^x dt \sqrt{Q(t)}\right)$
$\left(\frac{1}{\epsilon}\int_0^\pi dt \sqrt{Q(t)}\right)^2 = (n\pi$
$E_n \sim \frac{n^2\pi^2}{\left(\int_0^\pi dt \sqrt{Q(t}\right.}$




A = ?
$\int_0^\pi dx\, y_n^2(x) = 1$
$\epsilon A^2 \int_0^\pi dx\, Q(x) \frac{1}{\sqrt{Q}} \sin^2\left(\frac{1}{\epsilon}\int_0^x dt \sqrt{Q(t)}\right) = 1$
$z = \frac{1}{\epsilon}\int_0^x dt \sqrt{Q(t)}$
$dz = \frac{1}{\epsilon}\sqrt{Q(x)}\, dx$
$x = 0 \rightarrow z = 0$
$x = \pi \rightarrow z = \frac{1}{\epsilon}\int_0^\pi dt \sqrt{Q} = n\pi$
$\epsilon A^2 \int_0^{n\pi} dz \sin^2 z = 1$
$\epsilon A^2 \frac{1}{2}\pi n = 1$
$\left(\frac{1}{\epsilon}\int_0^\pi dt \sqrt{Q(t)}\right)^2 = (n\pi$
$\sim \frac{n^2\pi^2}{\left(\int_0^\pi dt \sqrt{Q(t)}\right.}$

$$A = ?$$

$$\int_0^\pi dx\, y_n^2(x) = 1$$

$$\epsilon A^2 \int_0^\pi dx\, Q(x) \frac{1}{\sqrt{Q}} \sin^2\left(\frac{1}{\epsilon}\int_0^x dt \sqrt{Q(t)}\right) = 1$$

$$\frac{1}{\epsilon}\sqrt{Q}$$

$$x = 0 \rightarrow z = 0$$

$$x = \pi \rightarrow z = \frac{1}{\epsilon}\int_0^\pi dt \sqrt{Q} = n\pi$$

$$z = \frac{1}{\epsilon}\int_0^x dt \sqrt{Q(t)}$$

$$dz = \frac{1}{\epsilon}\sqrt{Q(x)}\, dx$$

$$\epsilon A^2 \int_0^{n\pi} dz \sin^2 z = 1$$

$$\epsilon A^2 \frac{1}{2}\pi n = 1$$

$$\left(\frac{1}{\epsilon}\int_0^\pi dt \sqrt{Q(t)}\right)^2 = (n\pi$$

$$E_n \sim \frac{n^2\pi^2}{\left(\int_0^\pi dt \sqrt{Q(t)}\right)^2}$$

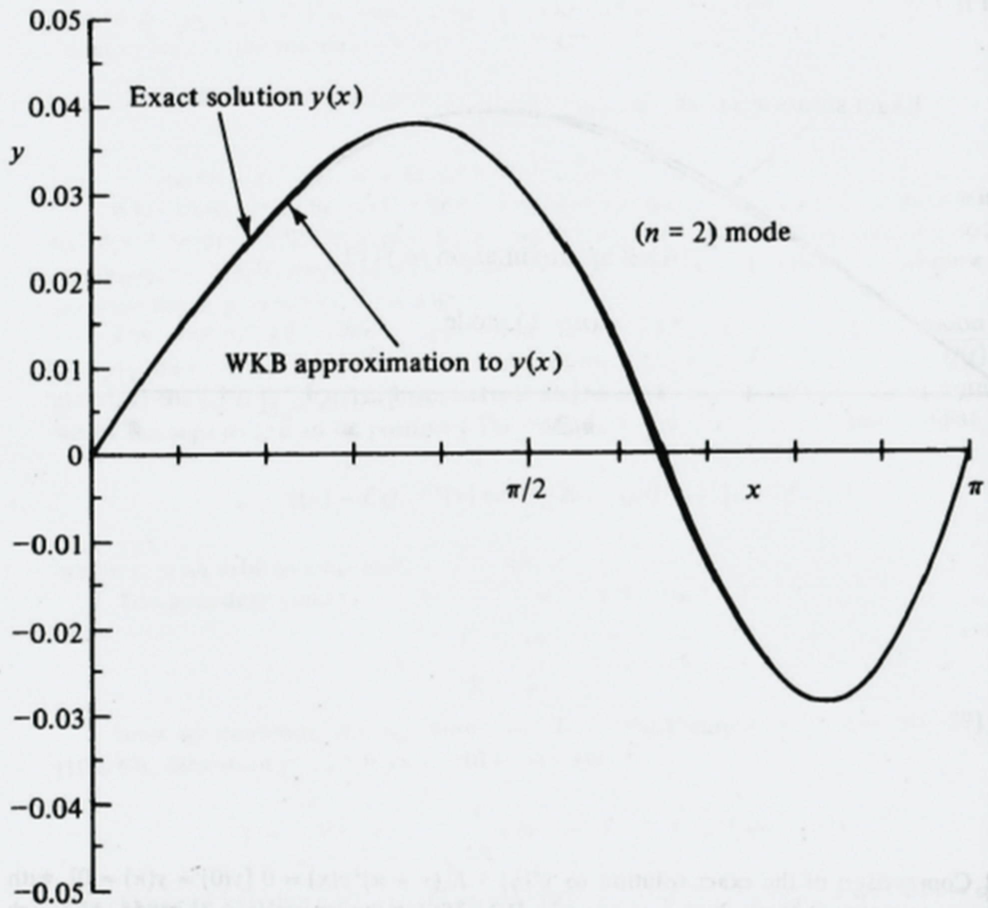
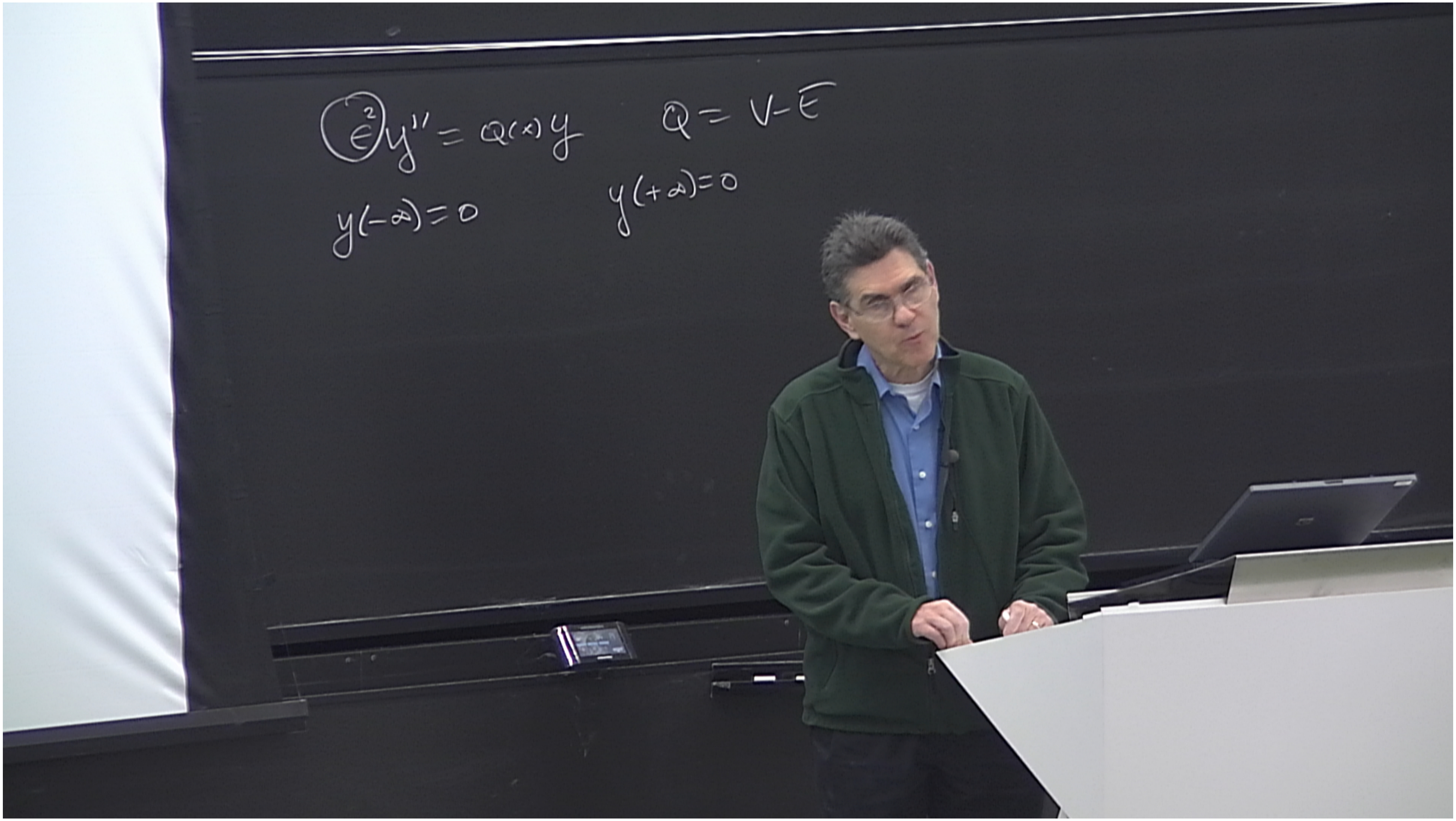
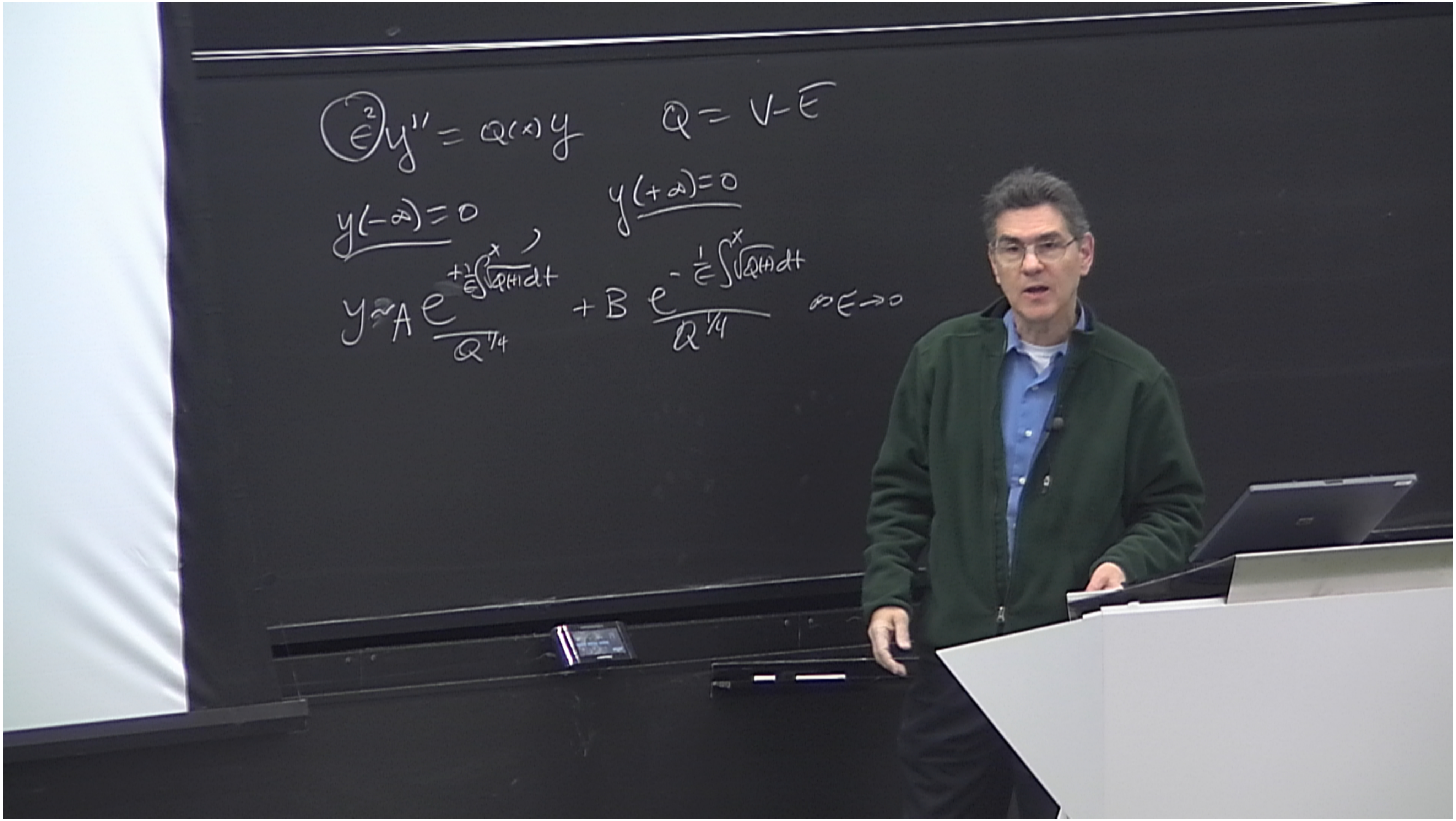
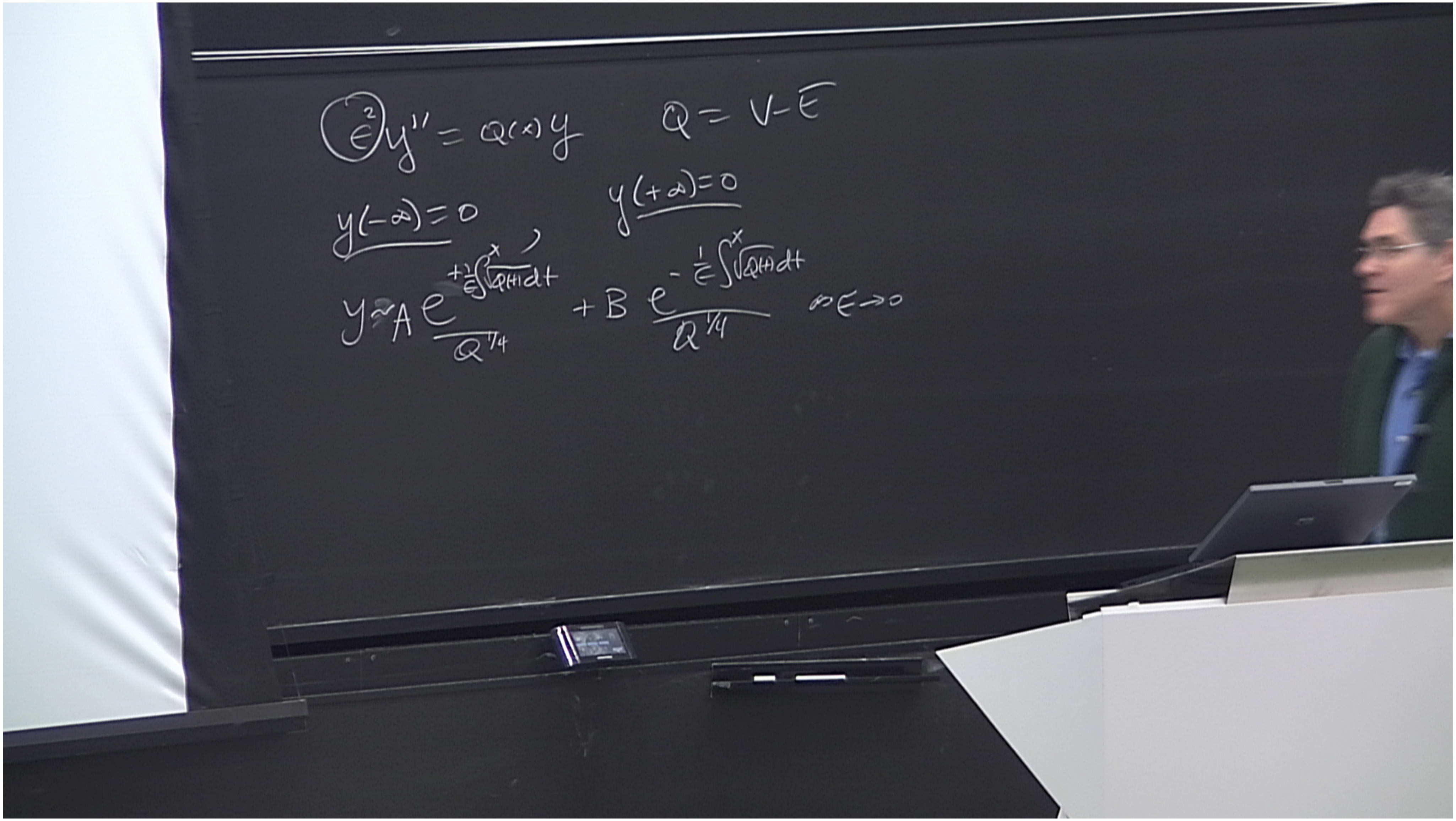
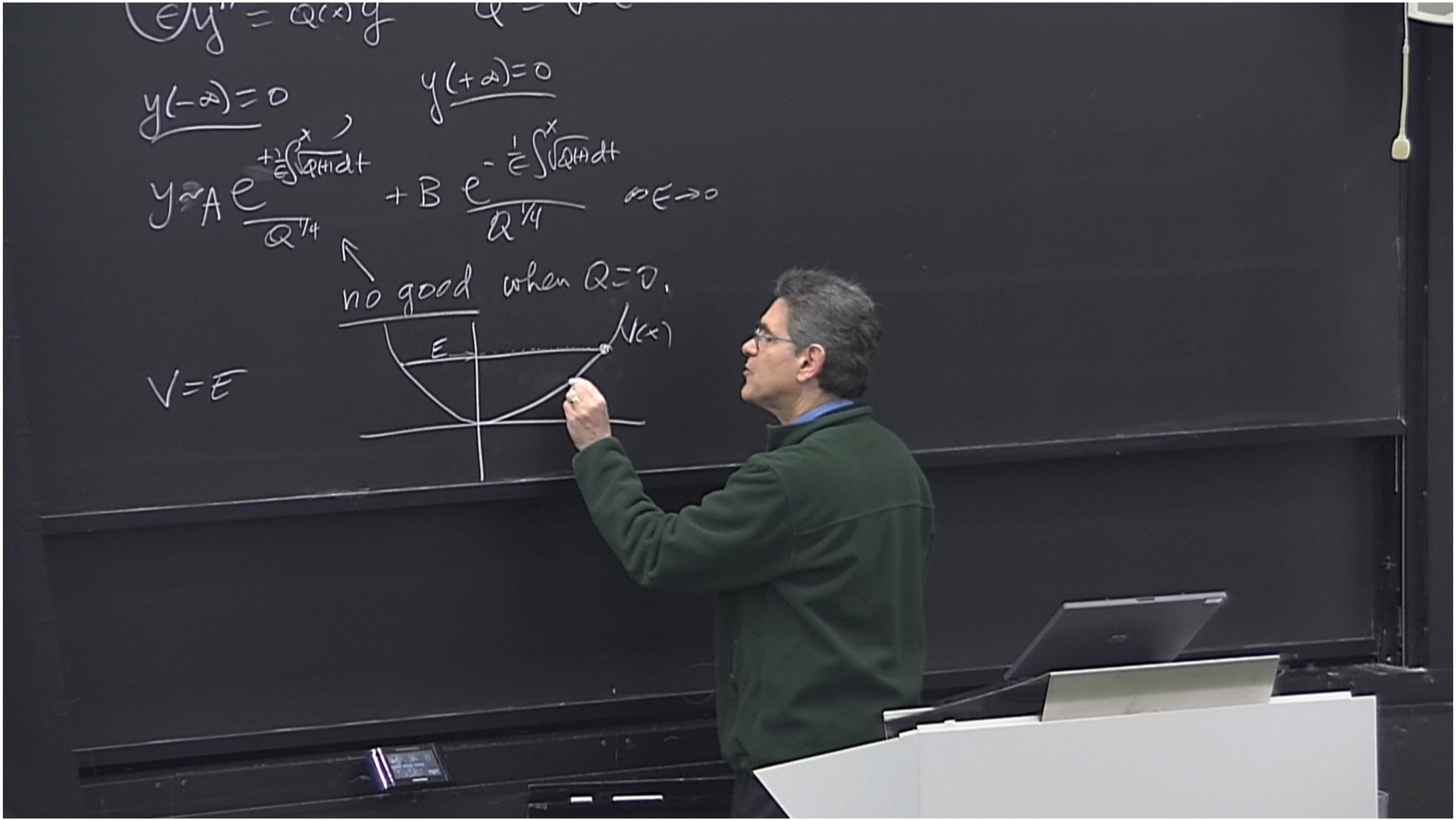


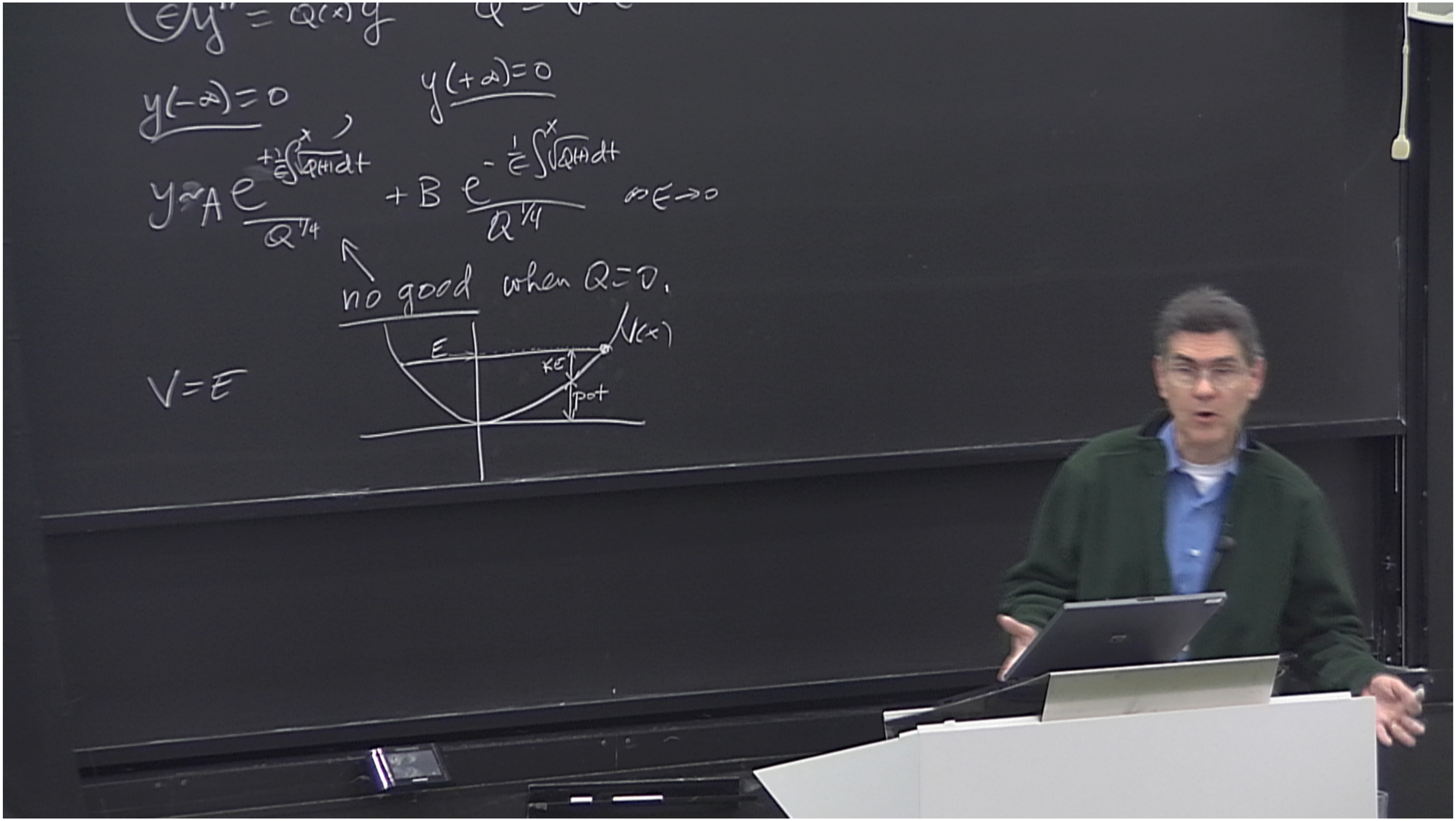
Figure 10.3 Same as in Fig. 10.2 except that $n = 2$. The exact eigenfunction and the WKB approximation are almost indistinguishable.

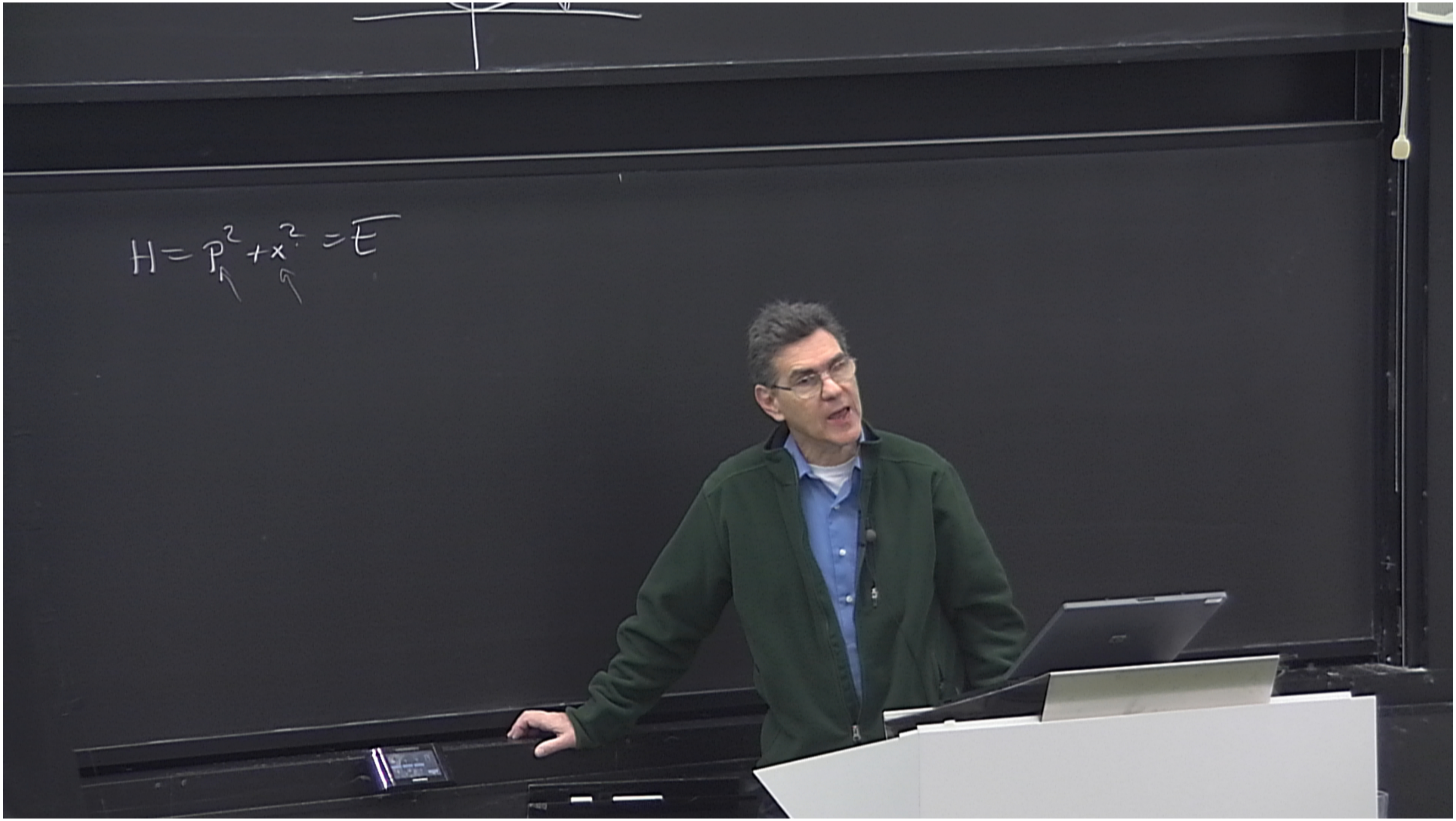

$\epsilon^2 y'' = Q(x)\, y$
$Q = V - E$
$y(-\infty) = 0$
$y(+\infty) = 0$

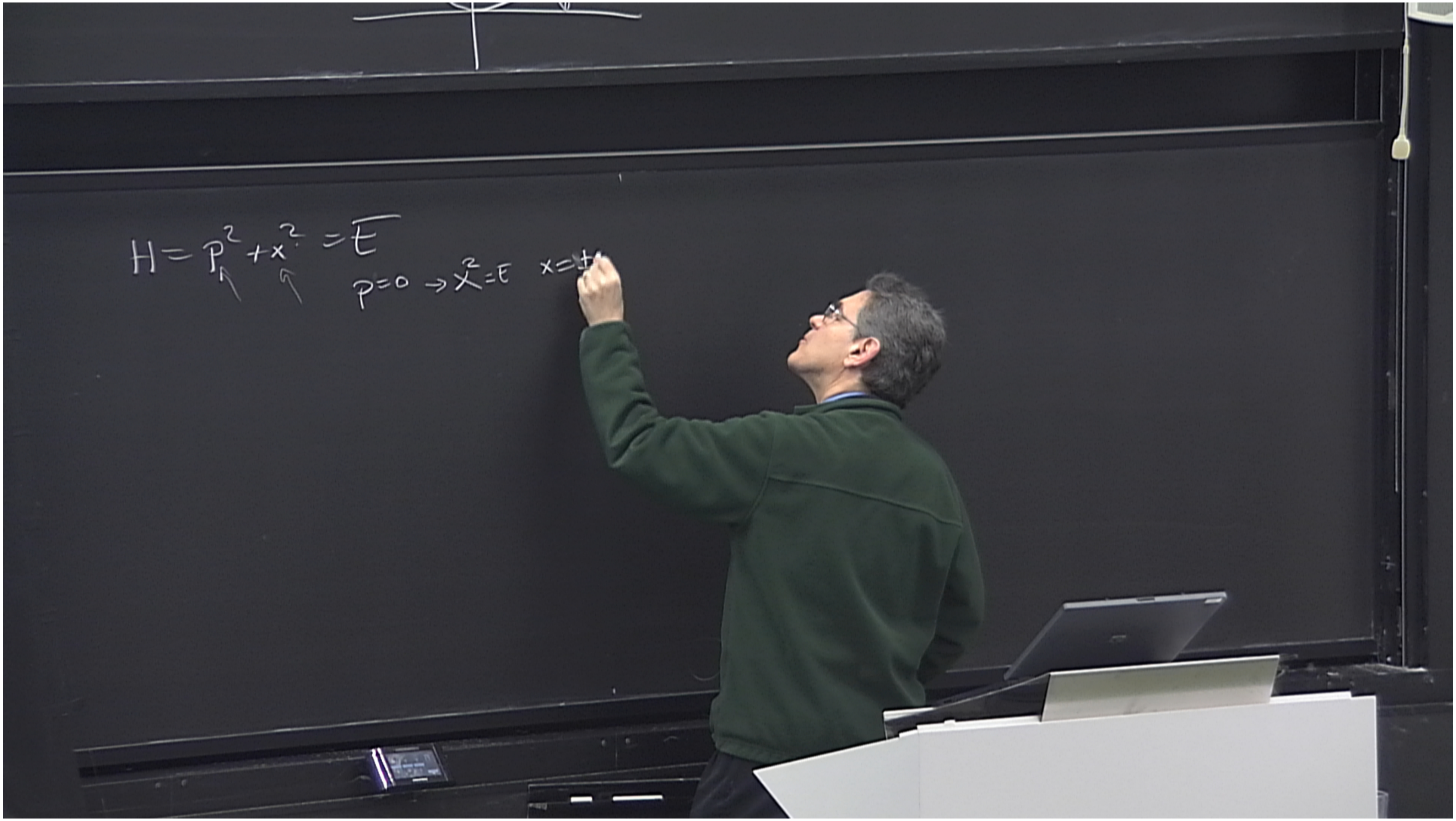

ε² y'' = Q(x) y
Q = V − E
y(−∞) = 0
y(+∞) = 0
y ~ A e^{+1/ε ∫^x √Q(t) dt} / Q^{1/4} + B e^{−1/ε ∫^x √Q(t) dt} / Q^{1/4}
ε → 0

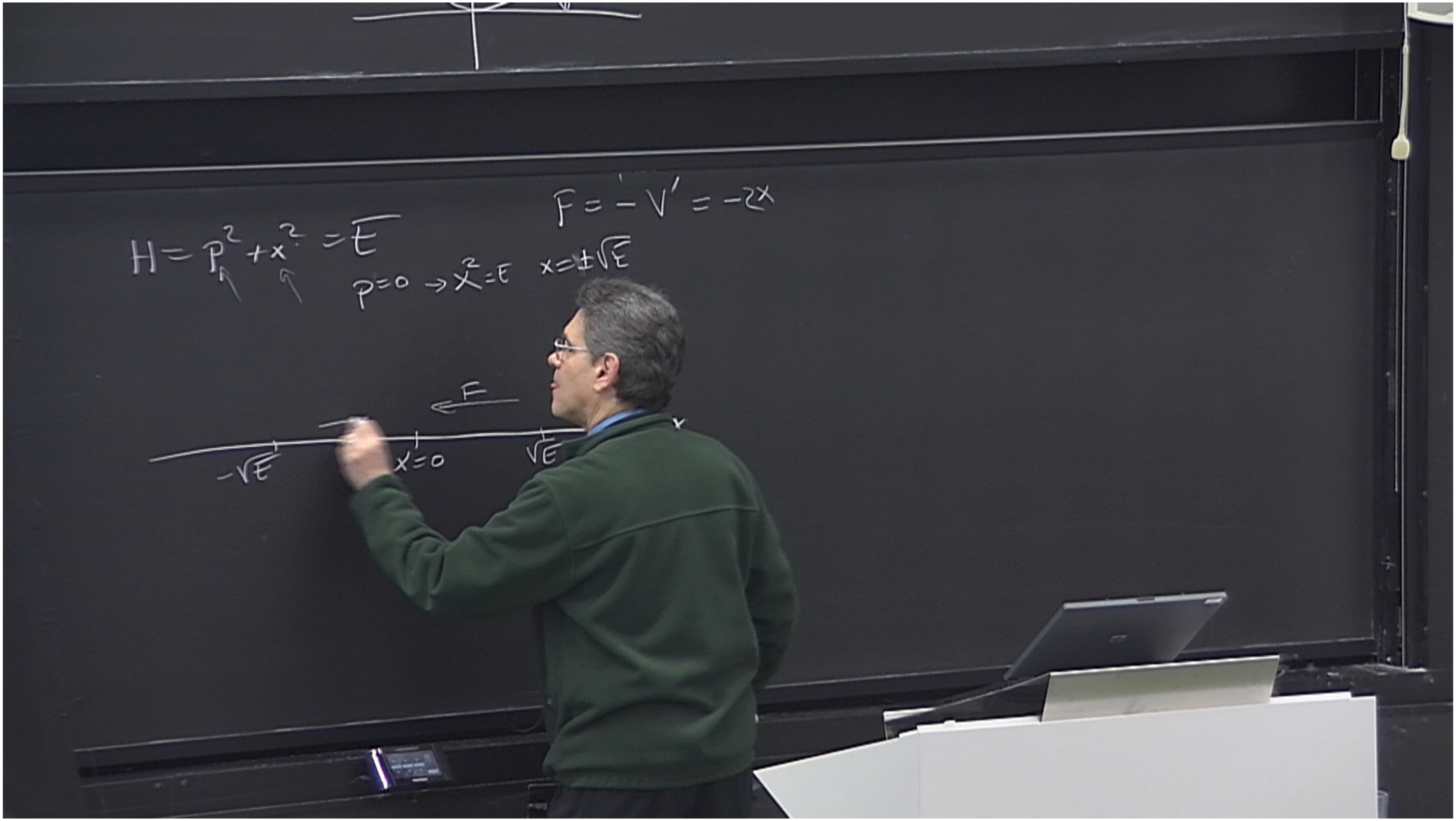

ε² y'' = Q(x) y   Q = V − E
y(−∞) = 0   y(+∞) = 0
y ~ A e^(+1/ε ∫^x √Q(t) dt) / Q^(1/4) + B e^(−1/ε ∫^x √Q(t) dt) / Q^(1/4)   as ε → 0


y(−∞)=0
y(+∞)=0
+B
as ε→0
no good when Q=0.
E
V(x)
V=E

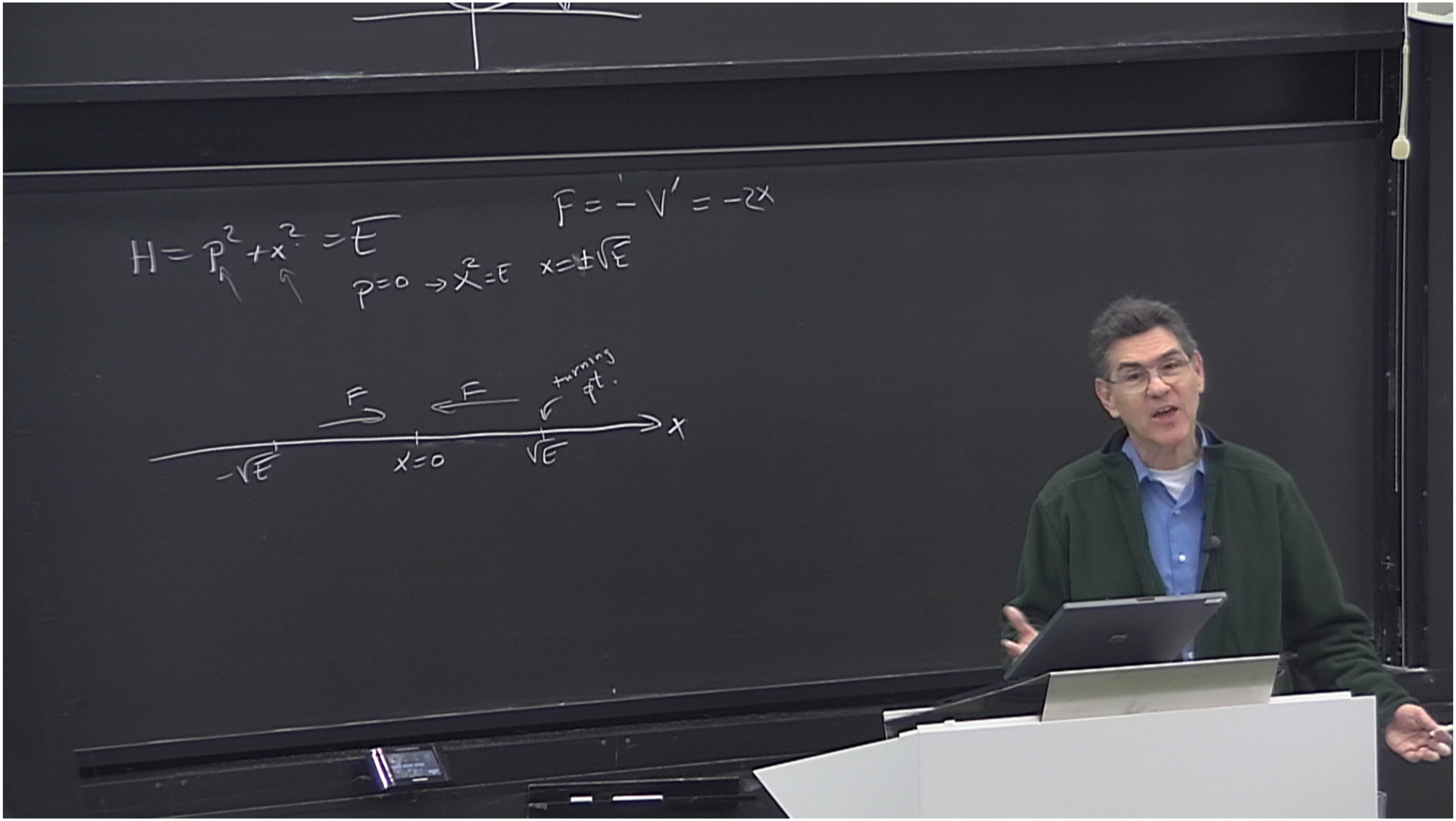

y(−∞)=0
y(+∞)=0
no good when Q=0.
V(x)
E
KE
pot
V=E


$H = p^2 + x^2 = E$

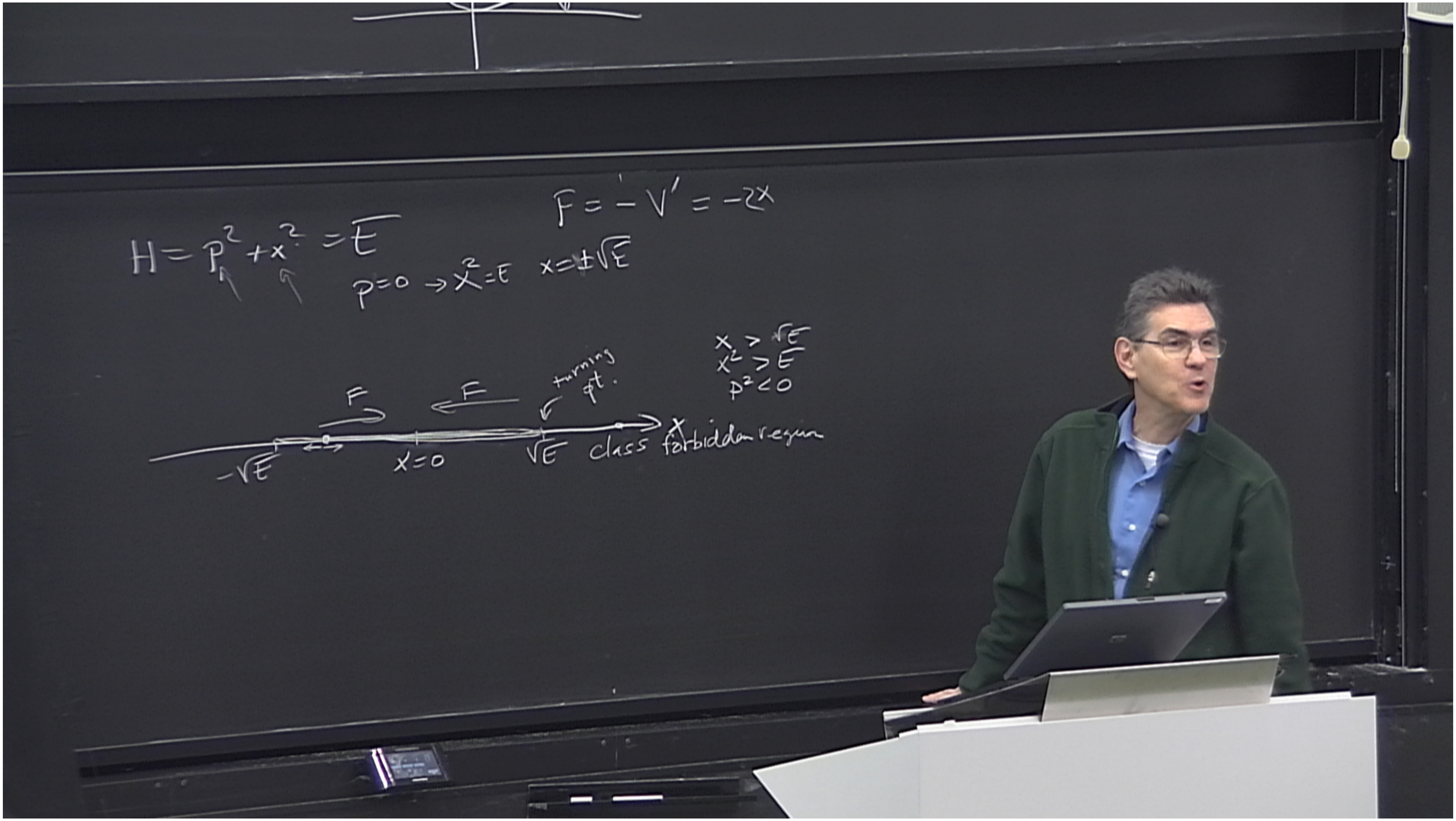

$H = p^2 + x^2 = E$
$p = 0 \to x^2 = E \quad x = \pm$



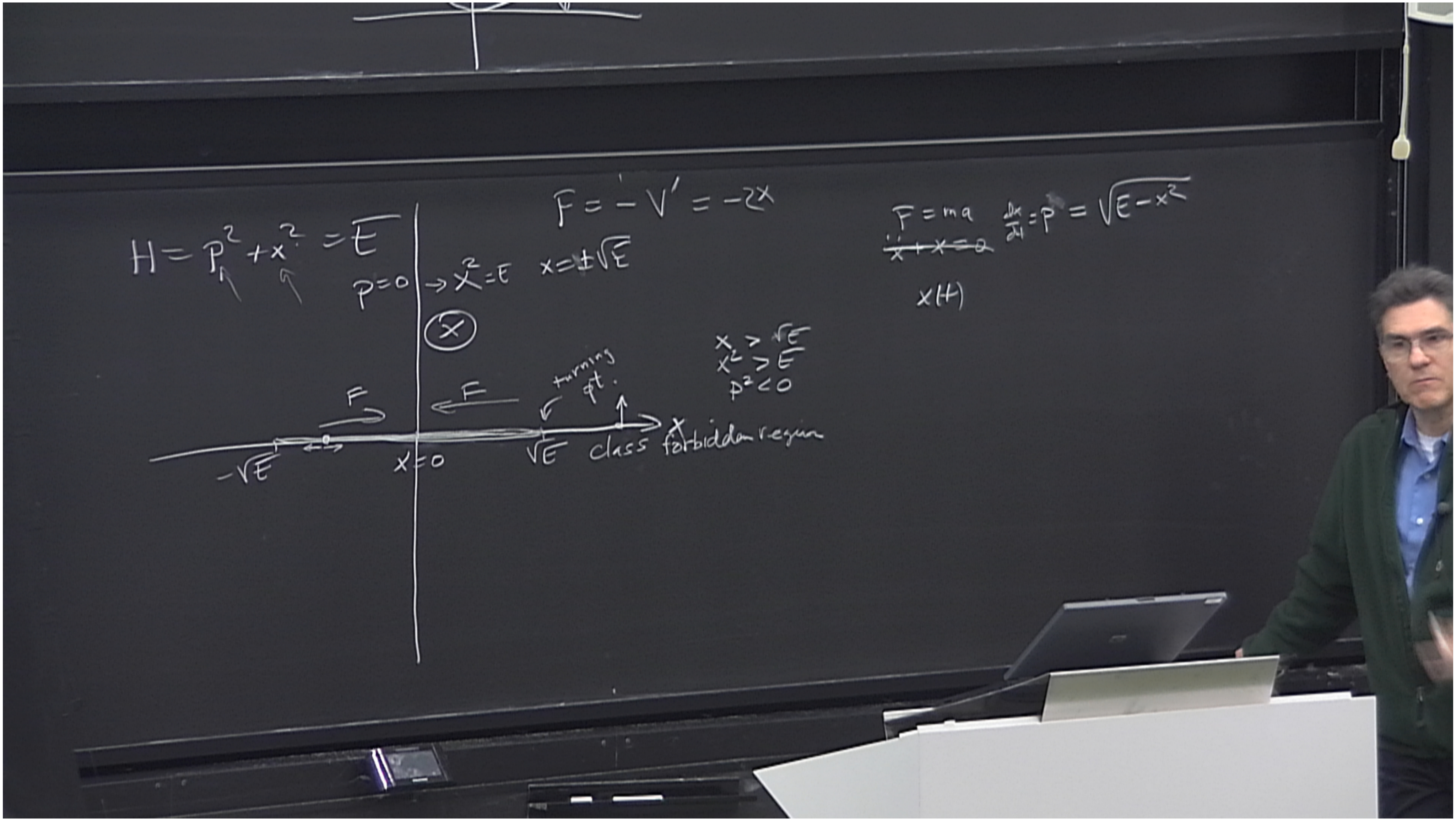
F = −V′ = −2x
H = p² + x² = E
p=0 → x² = E   x = ±√E
F
−√E
x=0
√E

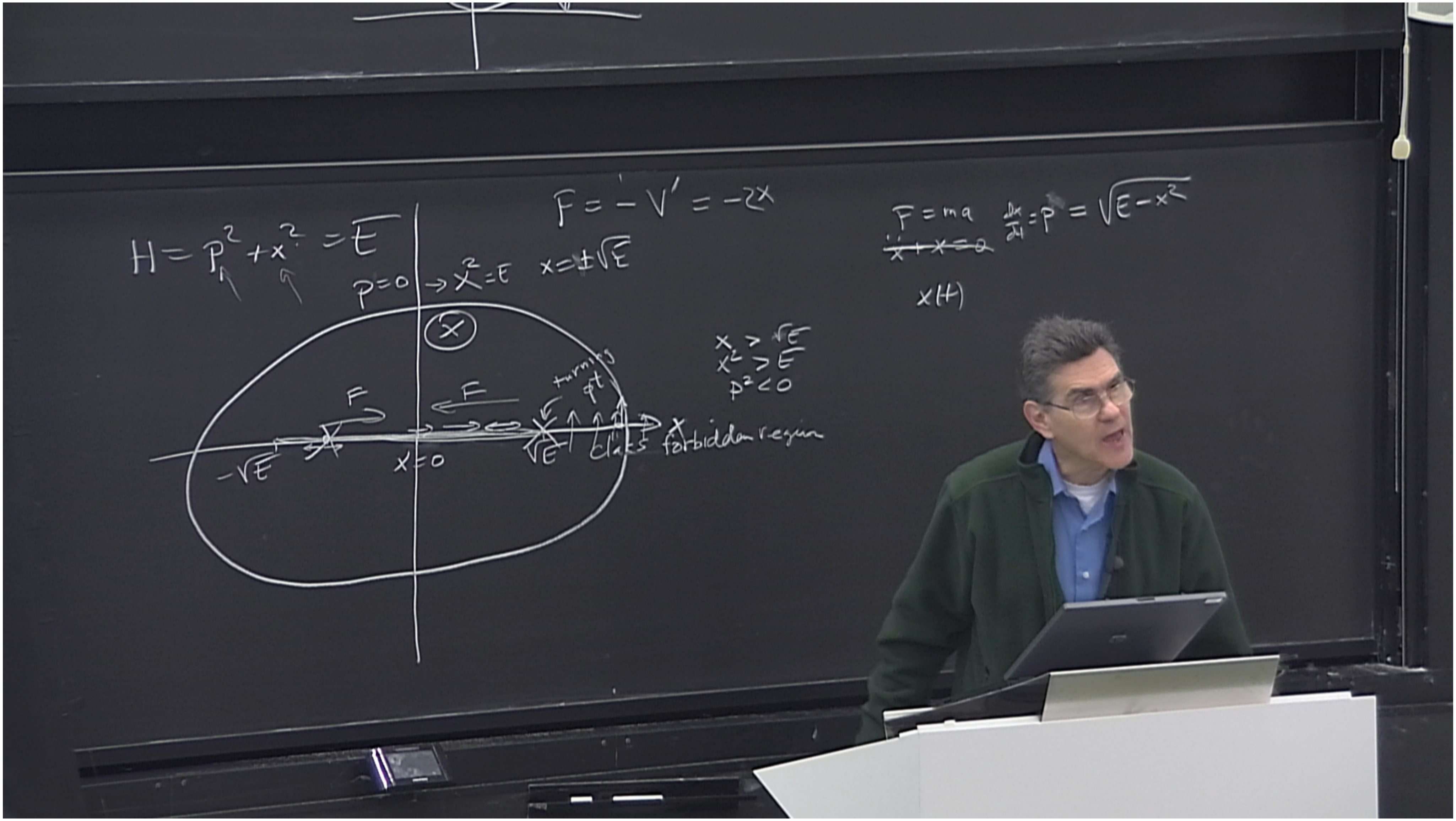


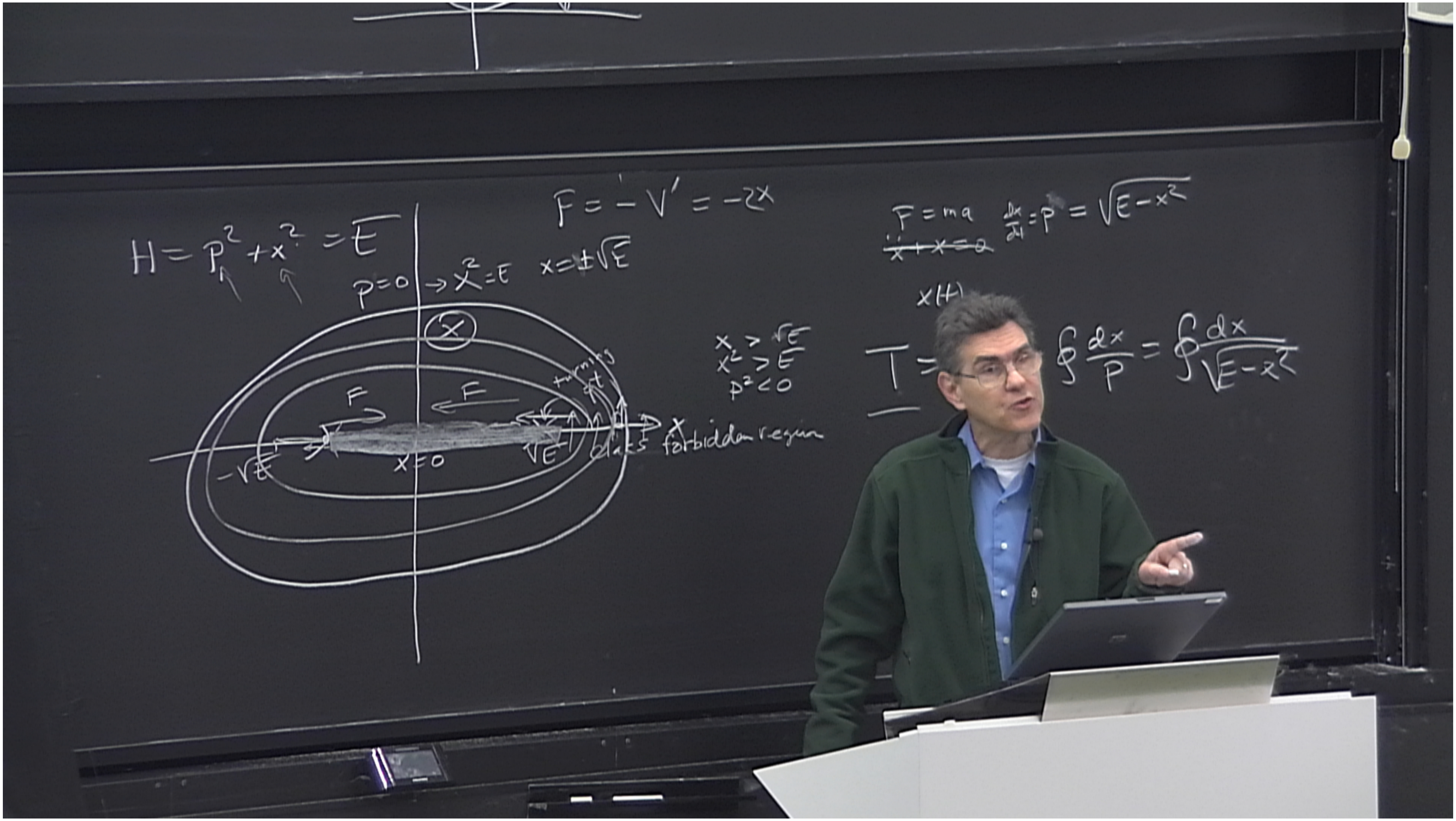
F = −V′ = −2x
H = p² + x² = E
p=0 → x² = E   x = ±√E
F
F
turning pt.
x
−√E
x=0
√E

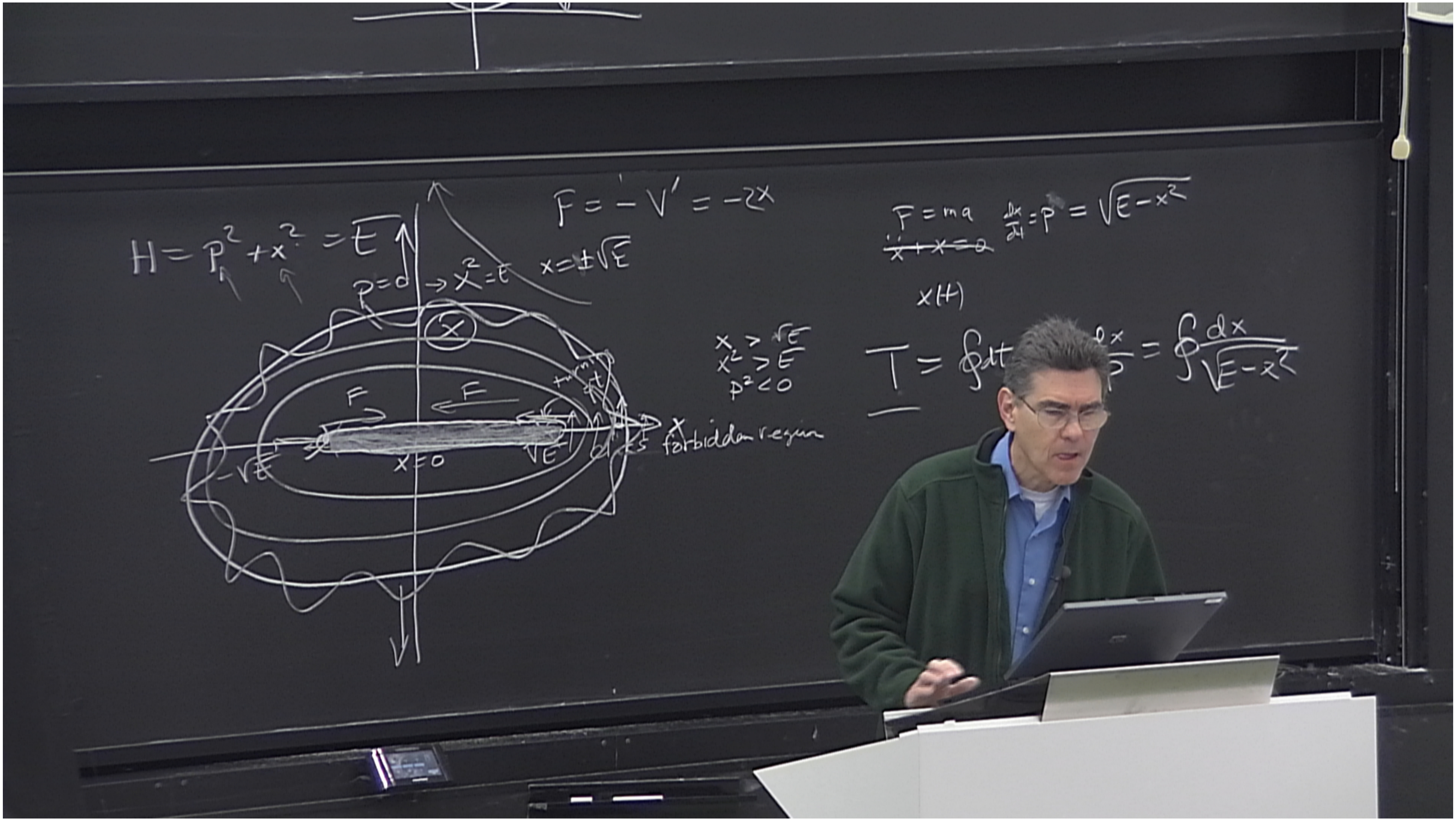


H = p² + x² = E
F = −V′ = −2x
p=0 → x²=E x=±√E
x > √E
x² > E
p² < 0
turning pt.
−√E
x=0
√E
class forbidden region

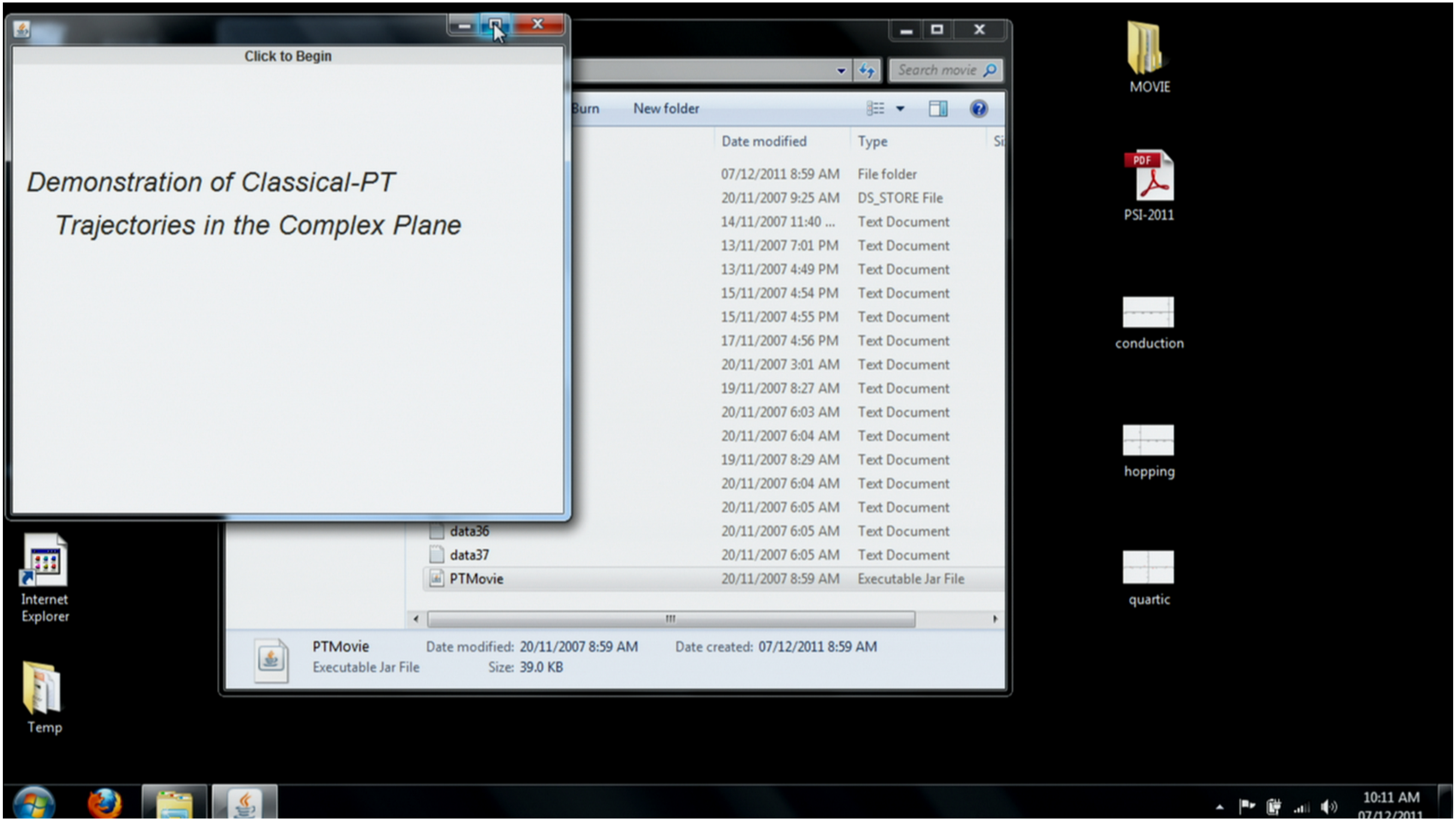

$H = p^2 + x^2 = E$
$F = -V' = -2x$
$p=0 \rightarrow x^2 = E \quad x = \pm\sqrt{E}$
$F = ma$
$\frac{dx}{dt} = p = \sqrt{E - x^2}$
$x(t)$
$x > \sqrt{E}$
$x^2 > E$
$p^2 < 0$
turning pt.
$-\sqrt{E}$
$x=0$
$\sqrt{E}$
class forbidden region

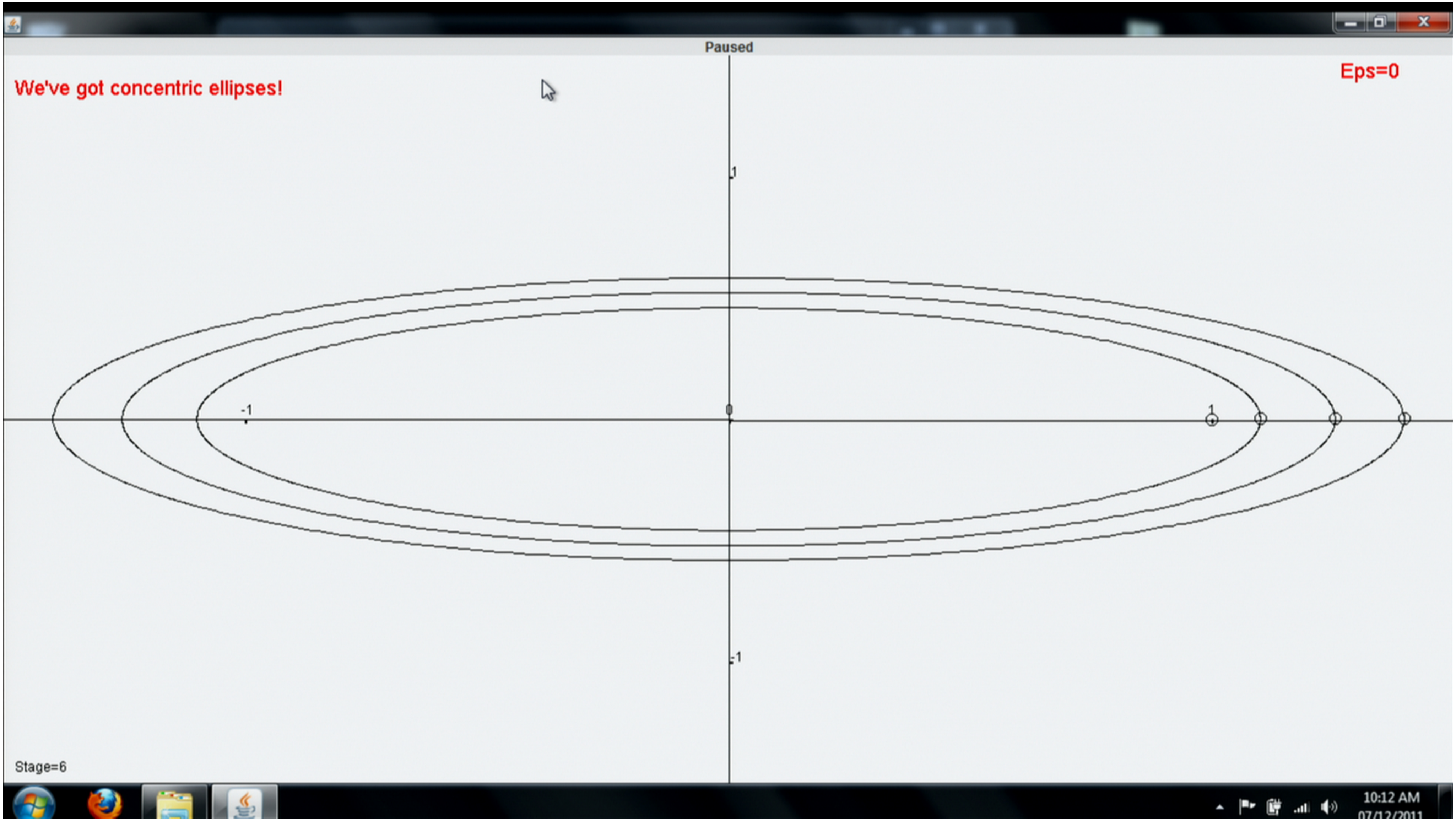

H = p² + x² = E
F = −V′ = −2x
p=0 → x²=E x=±√E
F = ma
dx/dt = p = √(E−x²)
x(t)
x > √E
x² > E
p² < 0
turning pt
forbidden region
x=0
−√E
√E

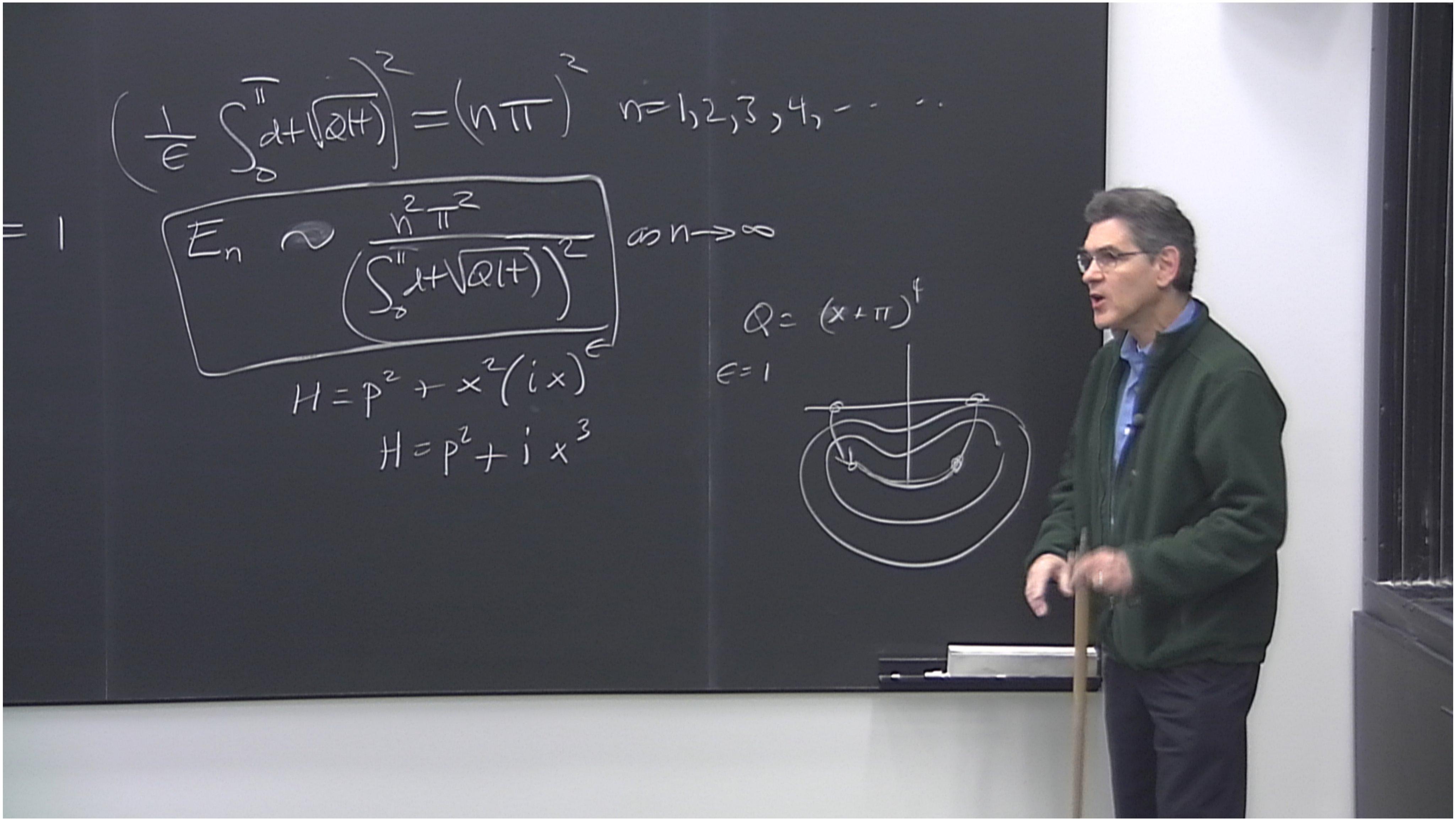

H = p^2 + x^2 = E
F = -V' = -2x
p=0 → x^2=E x=±√E
F = ma
forbidden region
x=0
x(t)

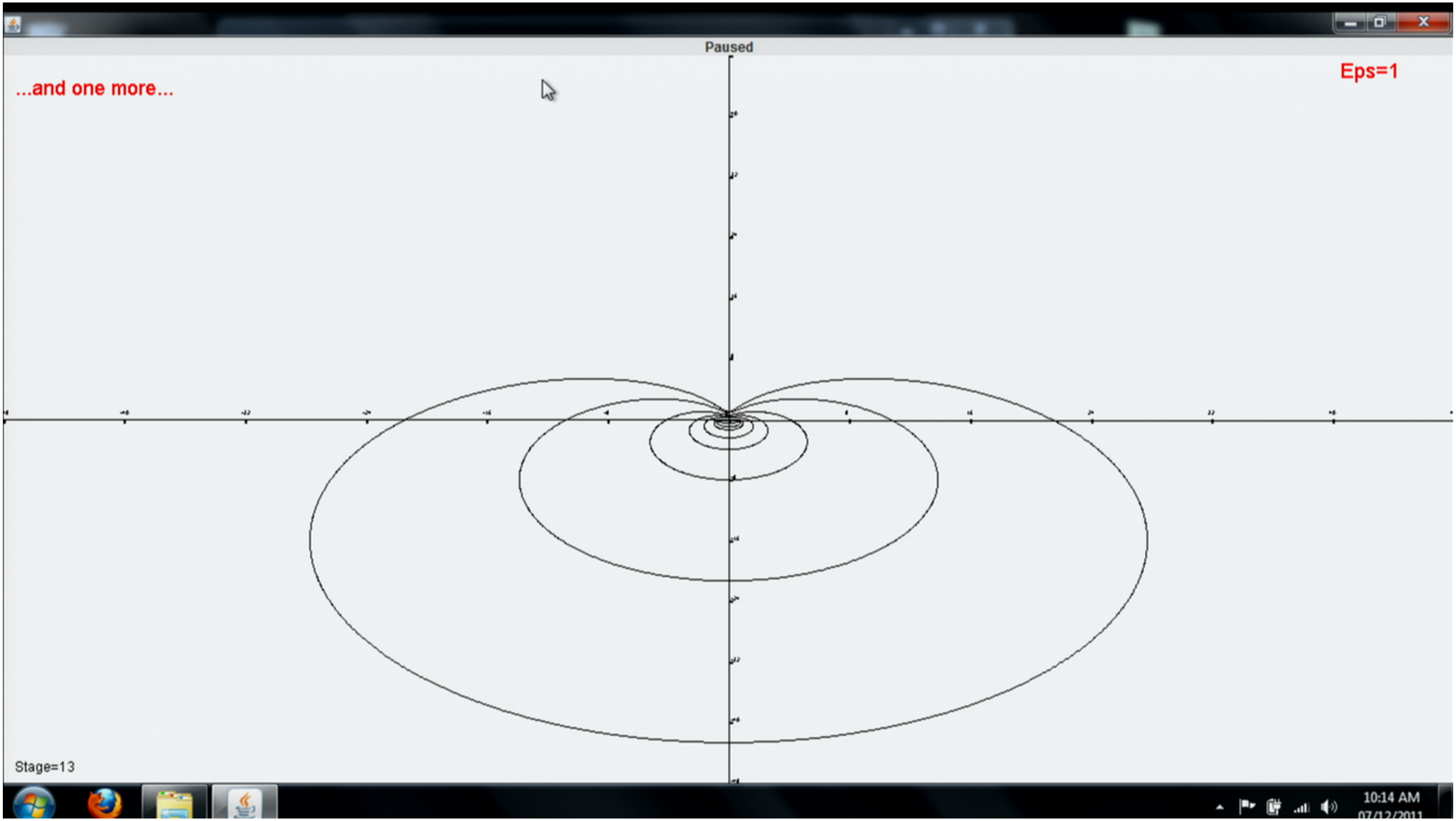


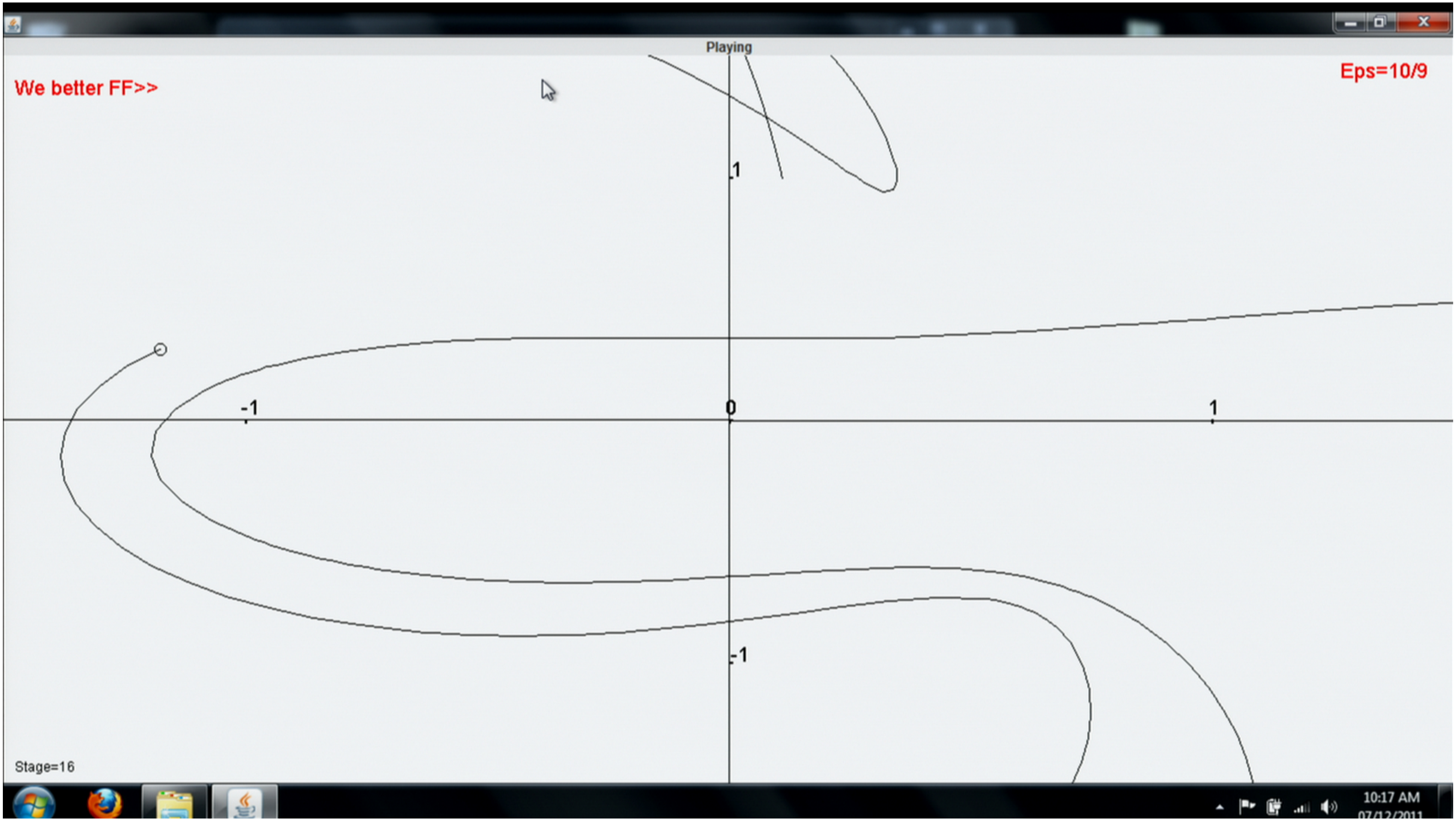
H = p² + x² = E
F = −V′ = −2x
p = 0 → x² = E   x = ±√E
F = ma
x(t)
x > √E
x² > E
p² < 0
T = ∮dt
forbidden region
x = 0
−√E
√E

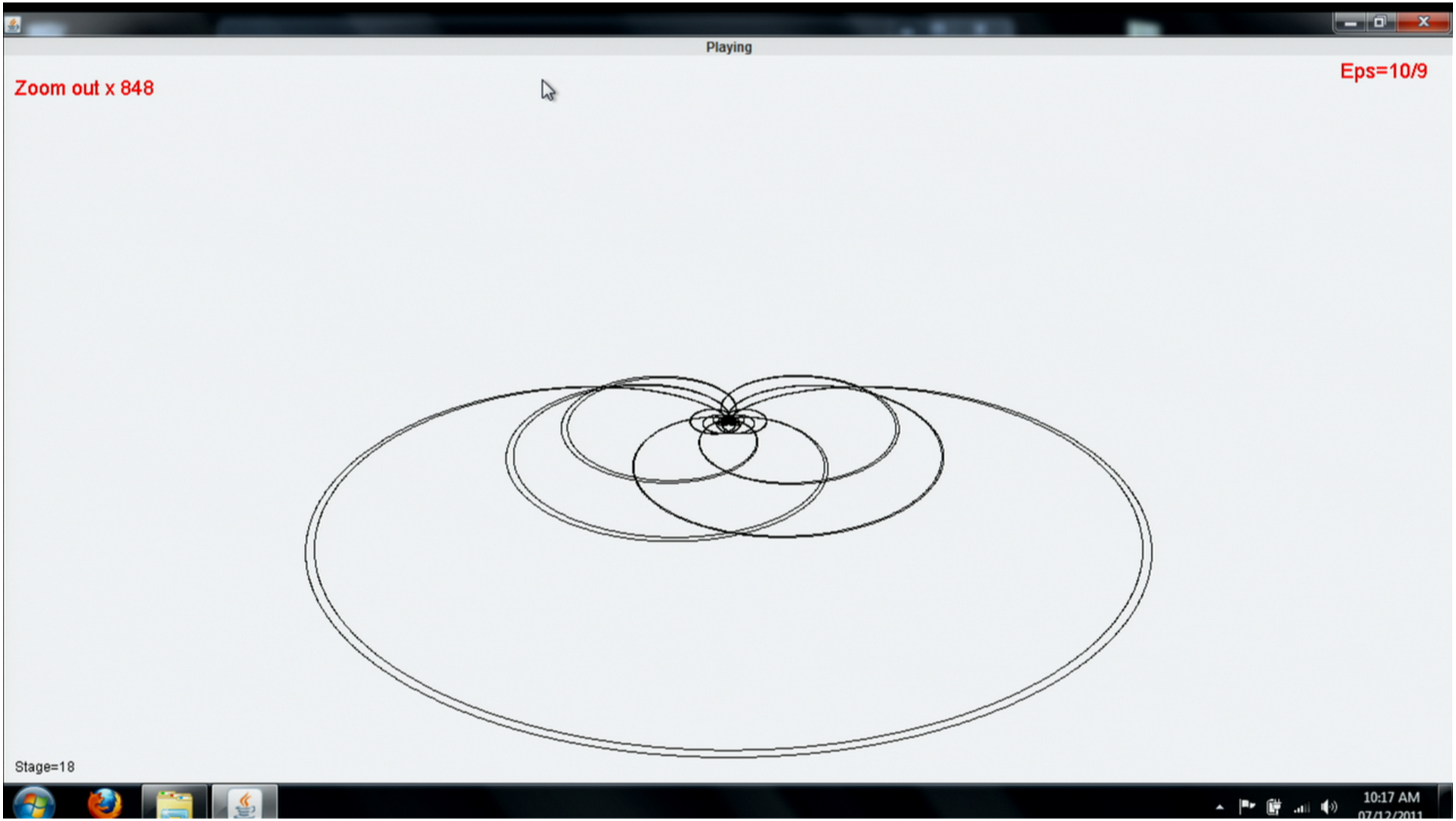

Click to Begin
Demonstration of Classical-PT
Trajectories in the Complex Plane
Burn
New folder
Date modified
Type
07/12/2011 8:59 AM File folder
20/11/2007 9:25 AM DS_STORE File
data36
data37
PTMovie
20/11/2007 8:59 AM Executable Jar File
PTMovie
Executable Jar File
Date modified: 20/11/2007 8:59 AM
Size: 39.0 KB
Date created: 07/12/2011 8:59 AM
MOVIE
PSI-2011
conduction
hopping
quartic
Internet Explorer
Temp
10:11 AM


Paused
We've got concentric ellipses!
Eps=0
1
-1
0
1
-1
Stage=6
10:12 AM


n=1,2,3,4,....
as n→∞
Q= (x+π)⁴
ε=1
H=p²+x²(ix)^ε
H=p²+ix³


Paused
...and one more...
Eps=1
Stage=13
10:14 AM


Playing
We better FF>>
Eps=10/9
1
-1
0
1
-1
Stage=16
10:17 AM


Playing
Zoom out x 848
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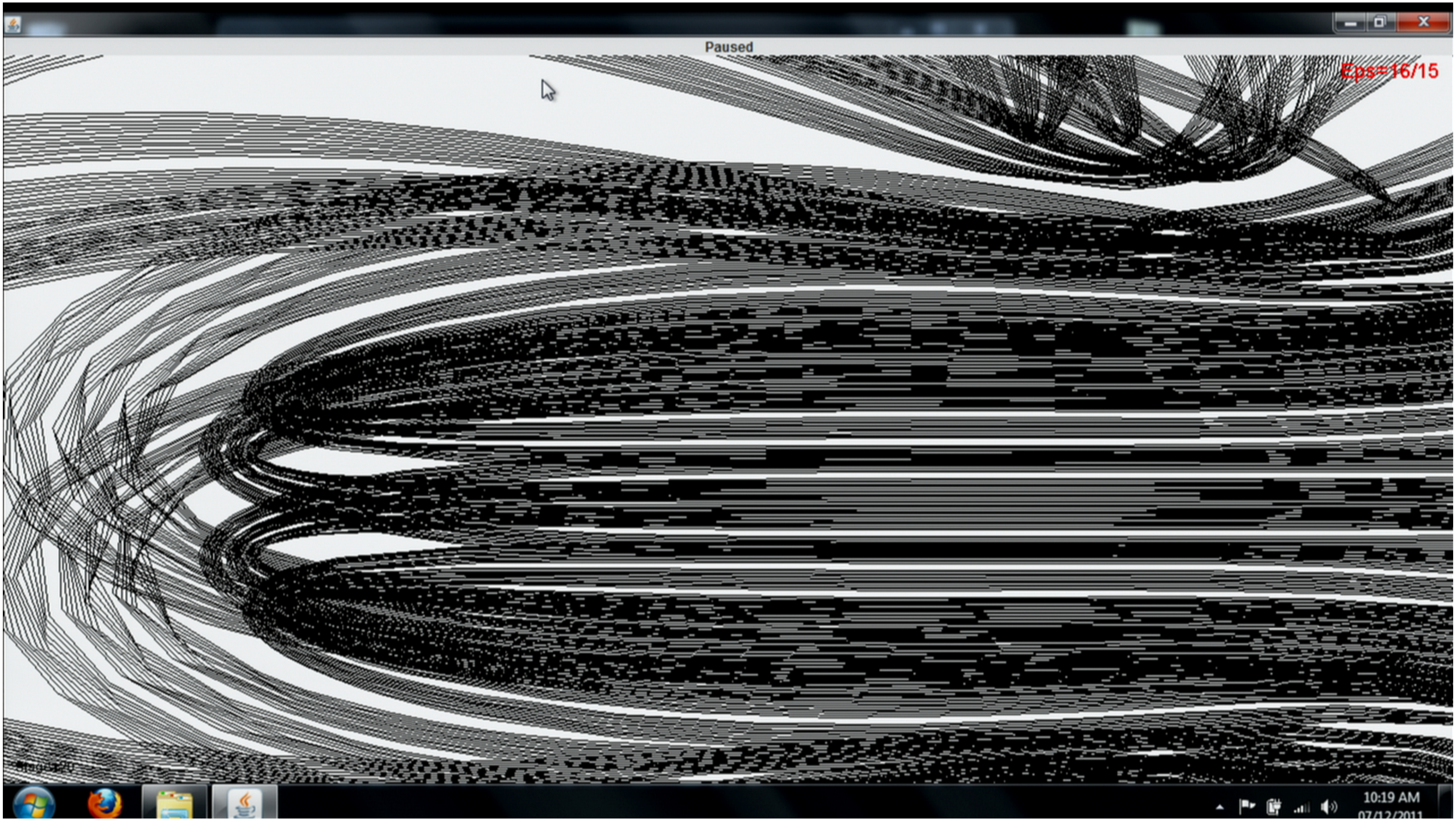
Stage=18
10:17 AM


Paused
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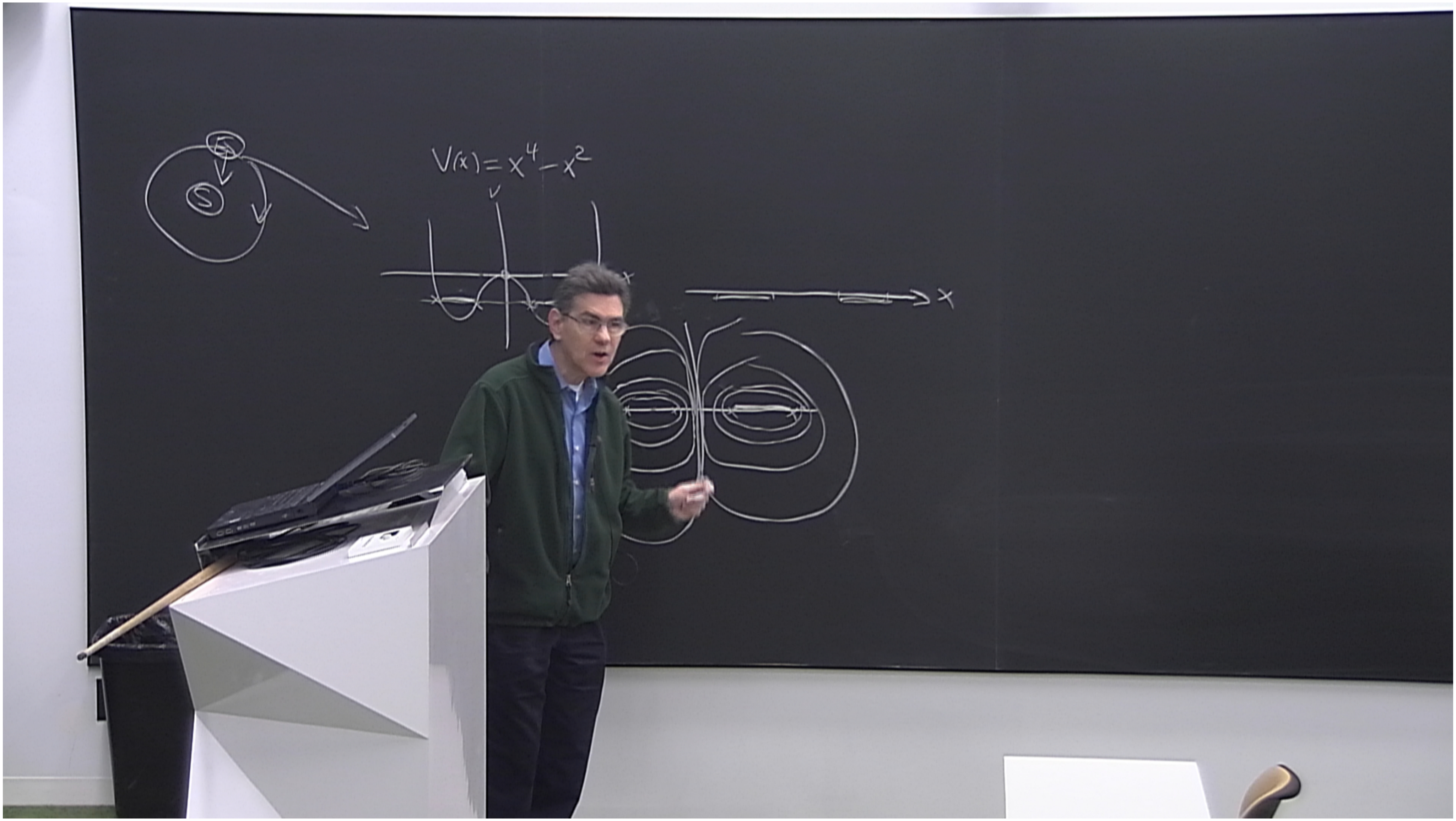
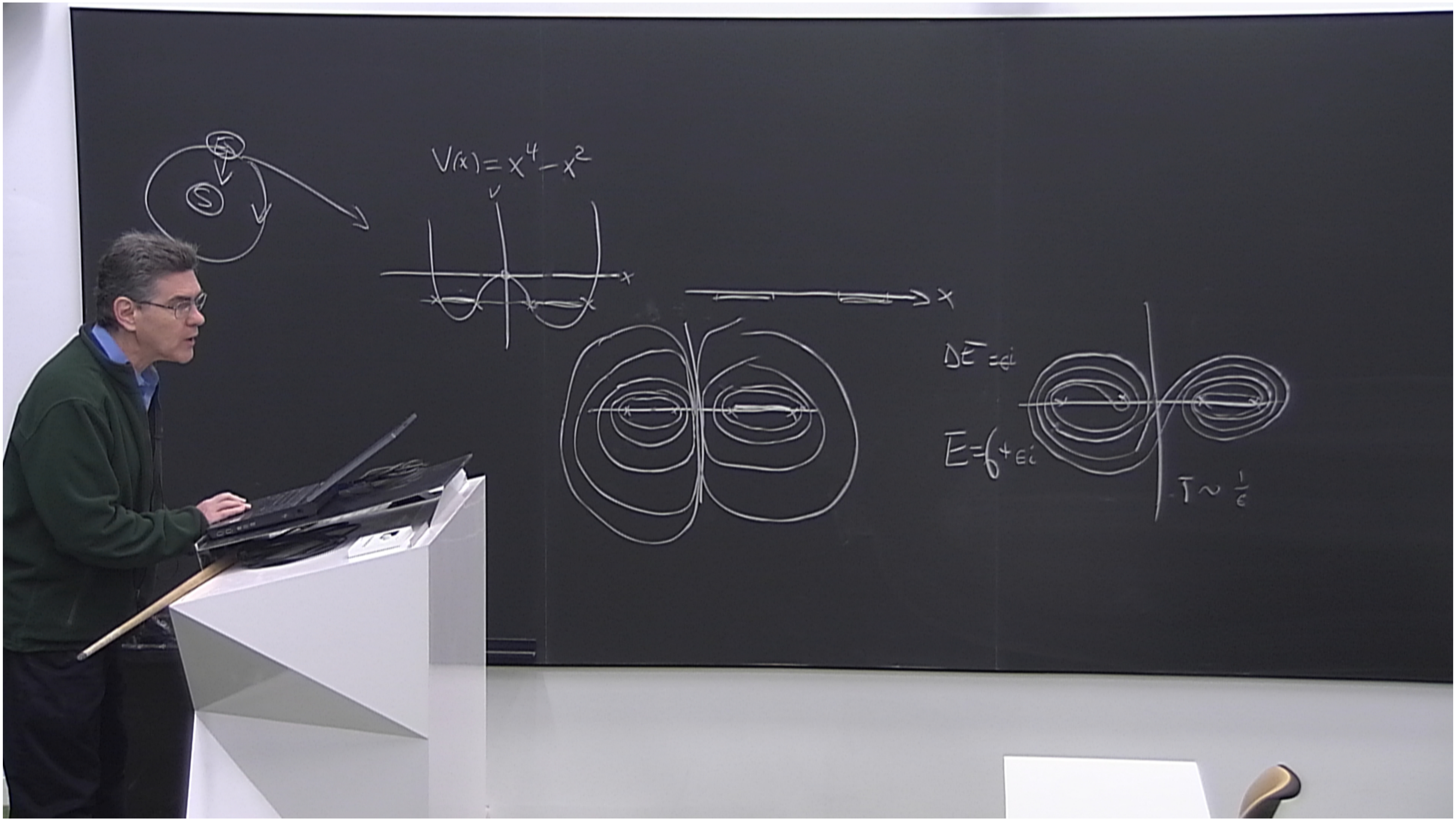
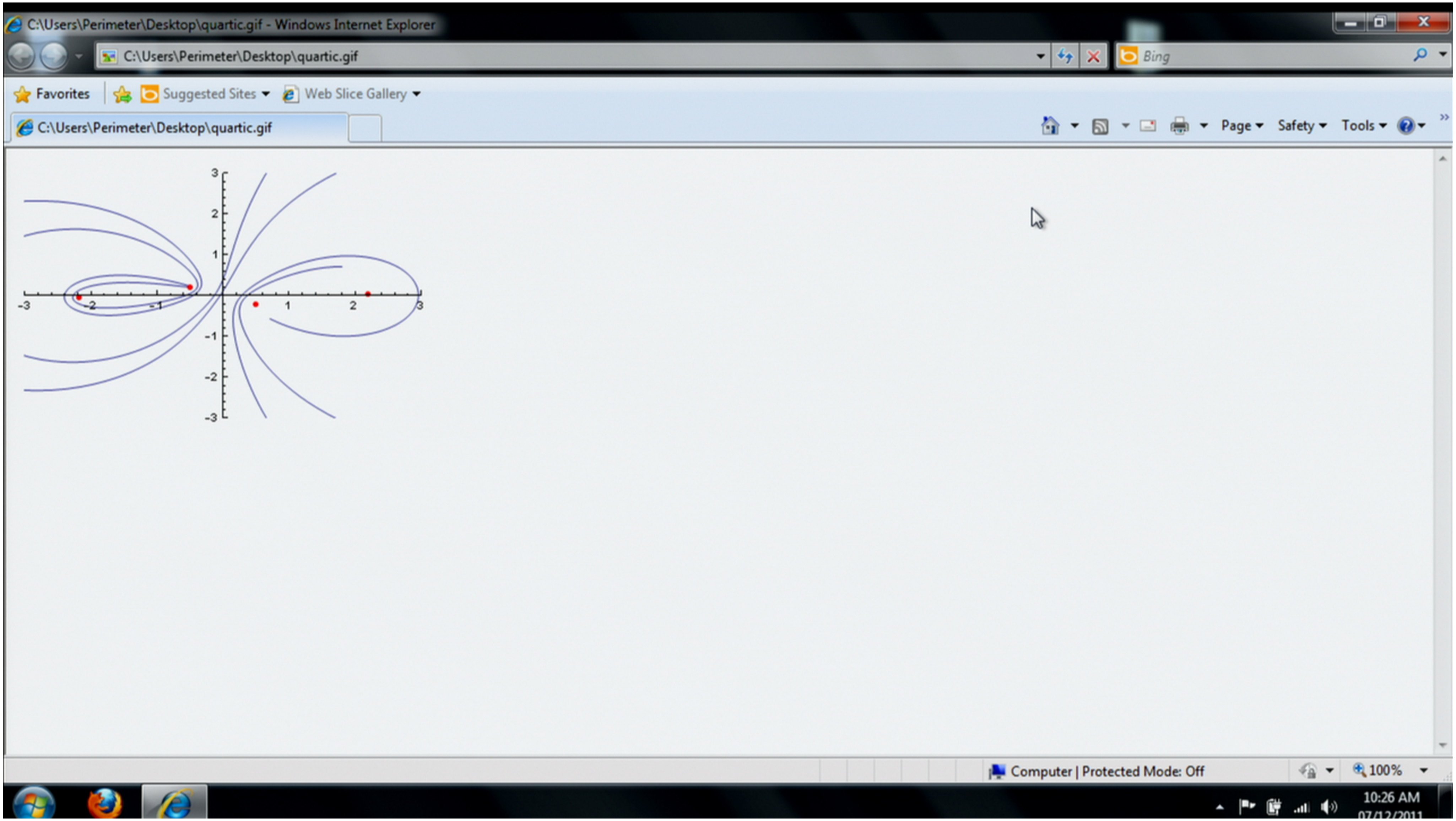


V(x) = x^4 - x^2
x



$V(x) = x^4 - x^2$
$\Delta E = \epsilon i$
$E = 6 + \epsilon i$
$T \sim \frac{1}{\epsilon}$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Bing
Favorites
Suggested Sites
Web Slice Gallery
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Page
Safety
Tools
3
2
1
-3
-2
-1
1
2
3
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Computer | Protected Mode: Off
100%
10:26 AM